Étale theta functions and mono-theta environments II

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Introduction

In IUT, we consider the following setting [k is an MLF; $q \in \mathcal{O}_k \setminus \mathcal{O}_k^\times$]:

$$^\dagger\begin{pmatrix}\text{a mathematical setting}\\ \text{arising from sch-theory}/k\end{pmatrix} \xrightarrow{\text{link}} {}^\ddagger\begin{pmatrix}\text{a mathematical setting}\\ \text{arising from sch-theory}/k\end{pmatrix}$$

where the link is not arising from sch-/ring- theory like a "frobenius" $q \mapsto q^N$ ($q \neq 0$, $N > 1$). To relate "†-tools" to "‡-tools", we use

- a coric object "$=$" an object arising from a "math. setting$/k$" which we can "share" via the link

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{}^{\dagger}\begin{pmatrix}\text{“tools” which we need to} \\ \text{compute “}\deg(\mathcal{L})\text{”}\end{pmatrix} & & {}^{\ddagger}\begin{pmatrix}\text{“tools” which we need to} \\ \text{compute “}\deg(\mathcal{L})\text{”}\end{pmatrix} \\
\Big\uparrow {\scriptstyle {}^{\dagger}\Xi\text{:functorial algorithm}} & & \Big\uparrow {\scriptstyle {}^{\ddagger}\Xi\text{:functorial algorithm}} \\
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where the link is not arising from sch-/ring- theory like a “frobenius” $q \mapsto q^N$ ($q \neq 0$, $N > 1$). To relate “†-tools” to “‡-tools”, we use

- a coric object “=” an object arising from a “math. setting$/k$” which we can “share” via the link

For instance, since the action $G_k \curvearrowright \mathcal{O}_k \ni q,\ q^N$ is trivial,

$$G_k$$

may be regarded as a coric object. Then, for instance, can the pair

$$(G_k \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}} \overset{\text{def}}{=} \mathcal{O}_{\overline{k}} \setminus \{0\}) \quad [\mathcal{O}^{\rhd}_{\overline{k}} : \text{a multiplicative monoid}]$$

be regarded as a coric object? The answer is "no". In fact,

$$({}^{\dagger}G_k \curvearrowright {}^{\dagger}\mathcal{O}^{\rhd}_{\overline{k}}) \xrightarrow{\sim} ({}^{\ddagger}G_k \curvearrowright {}^{\ddagger}\mathcal{O}^{\rhd}_{\overline{k}})$$

$$\Longrightarrow \quad {}^{\dagger}\mathcal{O}^{\rhd}_{k} \xrightarrow{\sim} {}^{\ddagger}\mathcal{O}^{\rhd}_{k}$$

$$\Longrightarrow \quad {}^{\dagger}\mathbb{N} \xrightarrow{\sim} {}^{\dagger}\mathcal{O}^{\rhd}_{k}/{}^{\dagger}\mathcal{O}^{\times}_{k} \xrightarrow{\sim} {}^{\ddagger}\mathcal{O}^{\rhd}_{k}/{}^{\ddagger}\mathcal{O}^{\times}_{k} \xrightarrow{\sim} {}^{\ddagger}\mathbb{N}\ ;\ 1 \ \mapsto\ 1$$

hence, we can not consider a link such as $q \mapsto q^N$. In IUT, we often use

$$(G_k \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}} \overset{\text{def}}{=} \mathcal{O}^{\times}_{\overline{k}}/(\mathcal{O}^{\times}_{\overline{k}})_{\text{tor}})$$

as a coric object.

- an example of "tools" $\cdots$ a cyclotomic rigidity isomorphism

Let $(G \curvearrowright M) \cong (G_k \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}})$. Write $\Lambda(M) \stackrel{\mathrm{def}}{=} \varprojlim_n (M_{\mathrm{tor}}[n])$.

Note: We can reconstruct a monoid $\mathcal{O}^{\rhd}_{\overline{k}}(G) \cong \mathcal{O}^{\rhd}_{\overline{k}}$ from G.

Write $\Lambda(G) \stackrel{\mathrm{def}}{=} \varprojlim_n (\mathcal{O}^{\rhd}_{\overline{k}}(G)_{\mathrm{tor}}[n])$.

Theorem (Cyc. Rig. Isom. via Local Class Field Theory)

$^{\exists}$functorial algorithm

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$$(G \curvearrowright M)$$

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$$(G \curvearrowright M) \quad \longmapsto \quad \text{a natural isom. } \Lambda(M) \stackrel{\sim}{\to} \Lambda(G)$$

But, at the moment, since $G_k \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}}$ may not be a coric object, this cyclotomic rigidity isomorphism [via LCT] is not "good".

$\Longrightarrow$ We want another version of cyclotomic rigidity isomorphism.

Cyclotomic Rigidity of a Mono-theta Env.

$\mathbb{M}^{\Theta}$: a mod N mono-theta environment, i.e., a triple

$$(\Pi_{\mathbb{M}^{\Theta}},\ \mathcal{D}_{\Pi_{\mathbb{M}^{\Theta}}},\ s^{\Theta}_{\Pi_{\mathbb{M}^{\Theta}}}) \ \cong\ (\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N],\ \mathcal{D}_{\underline{Y}},\ \{\gamma \cdot \mathrm{Im}(s^{\Theta}_{\underline{\underline{\ddot{Y}}}}) \cdot \gamma^{-1}\}_{\gamma \in \boldsymbol{\mu}_N})$$

[where $\boldsymbol{\mu}_N \stackrel{\mathrm{def}}{=} \boldsymbol{\mu}_N(\overline{k})$, $\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N] \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}} \times_{G_k} (\boldsymbol{\mu}_N \rtimes G_k)$]

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$$\mathbb{M}^{\Theta} \quad \longmapsto \quad \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}) \ \subseteq\ \Pi_{\mathbb{M}^{\Theta}}$$

for constructing the subgp corresp. to the subgp $\boldsymbol{\mu}_N \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$.

Definition

We refer to $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta})$ as the exterior cyclotome assoc. to $\mathbb{M}^{\Theta}$.

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We refer to $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta})$ as the interior cyclotome assoc. to $\mathbb{M}^{\Theta}$.

Theorem (Cyclotomic Rigidity — cf. [EtTh], Cor 2.19, (i))

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for constructing a natural isom. corresp. to a natural isom.

$$(l \cdot \Delta_{\Theta}) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \ \stackrel{\sim}{\to}\ \boldsymbol{\mu}_N(\overline{k})$$

arising from scheme theory.

Constant Multiple Rigidity of a Mono-theta Env.

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for constructing a subset of H^1 corresp. to $\mathcal{O}_k^{\times} \cdot \underline{\underline{\ddot{\eta}}}^{\Theta, l\cdot\underline{\mathbb{Z}}\times\boldsymbol{\mu}_2}$, i.e.,

$$(l \cdot \underline{\mathbb{Z}} \times \boldsymbol{\mu}_2)\text{–orbit of } \mathcal{O}_k^{\times} \cdot \ddot{\Theta}^{\frac{1}{l}}.$$

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Now we want to reduce the indeterminacy from $\mathcal{O}_k^{\times}$ to $\boldsymbol{\mu}_l$.

To reduce the indet., we use a theta function of standard type $\ddot{\Theta}_{\mathrm{st}}$.

(Review of $\ddot{\Theta}_{\mathrm{st}}$)

For an evaluation pt $\xi_0 \in \ddot{Y}(k)$ labeled by $0 \in \underline{\mathbb{Z}}$, write

$$\ddot{\Theta}_{\mathrm{st}} \ \stackrel{\mathrm{def}}{=}\ \ddot{\Theta}(\xi_0)^{-1} \cdot \ddot{\Theta}.$$

In particular, we have

$$\ddot{\Theta}_{\mathrm{st}}^{\frac{1}{l}}(\xi_0) \ \in \ \boldsymbol{\mu}_{2l}.$$

More precisely, by substituting ξ_0, we can see, up to $\boldsymbol{\mu}_{2l}$-multiple, whether or not an $\in \mathcal{O}_k^{\times} \cdot \ddot{\Theta}^{\frac{1}{l}}$ coincides with $\ddot{\Theta}_{\mathrm{st}}^{\frac{1}{l}}$.

[Note: $\boldsymbol{\mu}_{2l} \ \cong\ \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_l \ \cong\ \mathrm{Gal}(\ddot{Y}^{\log}/Y^{\log}) \times \mathrm{Gal}(\ddot{\underline{\underline{Y}}}^{\log}/\ddot{Y}^{\log})$]

Note: gp-theoretic interpretation of "substituting ξ_0"

$$= \text{ considering the image via } H^1(\Pi_{\underline{\underline{\ddot{Y}}}}, l \cdot \Delta_\Theta) \stackrel{\text{res.}}{\to} H^1(D, l \cdot \Delta_\Theta)$$

$D \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}$: the decomposition group of ξ_0 [well-defined up to conj.]

Thus, by applying

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for constructing the subgp of $\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^\Theta)$ corr. to the decomp. gp of ξ_0.

we have:

Theorem (Constant Multiple Rigidity — cf. [EtTh], Cor 2.19, (iii))

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Discrete Rigidity of a Mono-theta Env.

By means of the natural surjections $\boldsymbol{\mu}_{M'} \twoheadrightarrow \boldsymbol{\mu}_{M}$ [where $M|M'$], we can define a natural proj. system

$$\cdots \to \ \mathbb{M}_{M'} \ \to \ \mathbb{M}_{M} \ \to \cdots$$

of model mono-theta env.

Theorem (Discrete Rigidity — cf. [EtTh], Cor 2.19, (ii))

$^{\forall}$proj. system

$$\cdots \to \ \mathbb{M}^{\bullet}_{M'} \ \to \ \mathbb{M}^{\bullet}_{M} \ \to \cdots$$

of mono-theta env. is isomorphic to the above natural one.

Consequences of the Three Rigidities

$\mathbb{M}^{\Theta}_{*} = \{\mathbb{M}^{\Theta}_{N}\}_N$: a proj. system of mono-theta environments

By discrete rigidity, such a proj. system is uniquely determined, up to isom.

$$\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\text{def}}{=} \varprojlim \left(\cdots \stackrel{\sim}{\to} \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{M'}) \stackrel{\sim}{\to} \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{M}) \stackrel{\sim}{\to} \cdots \right)$$

Also, we have:

- $\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*}) \subseteq \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$: an open subgp corresp. to $\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}$
- $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*})$: a subquot. of $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ corresp. to $l \cdot \Delta_{\Theta}$.

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\text{def}}{=} \varprojlim \left(\cdots \twoheadrightarrow \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{M'}) \twoheadrightarrow \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{M}) \twoheadrightarrow \cdots \right)$$

By cyclotomic rigidity, we have a natural isom. between "$\widehat{\mathbb{Z}}(1)$"

$$(l \cdot \Delta_\Theta)(\mathbb{M}_*^\Theta) \stackrel{\sim}{\longrightarrow} \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^\Theta).$$

In particular, by constant multiple rigidity, we conclude:

Theorem (Exterior Cyclotome Ver. of $\underline{\underline{\theta}}(\mathbb{M}^\Theta)$ — cf. [IUTchII], Prop 1.5)

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$^\exists$functorial algorithm

$$\mathbb{M}_*^\Theta$$

By cyclotomic rigidity, we have a natural isom. between "$\widehat{\mathbb{Z}}(1)$"

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for constructing an exterior cyclotome version of $\underline{\underline{\theta}}(\mathbb{M}^\Theta)$.

Write

$$_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}_*^\Theta) \quad \subseteq \quad \varinjlim_{J \subseteq \Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*^\Theta):\ \text{fin index, open}} H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*^\Theta)|_J, \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^\Theta))$$

for the subset $\{ \eta \in \varinjlim_J H^1 \mid n \cdot \eta \in \underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}_*^\Theta)$ for $^\exists n \geq 1 \}$.

Rough Sketch of the Multiradiality

Consider the following situation:

$$^{\dagger}\begin{pmatrix}\text{a mathematical setting}\\ \text{arising from sch-theory}/k\end{pmatrix} \xrightarrow{\text{link}} {}^{\ddagger}\begin{pmatrix}\text{a mathematical setting}\\ \text{arising from sch-theory}/k\end{pmatrix}$$

Rough Sketch of the Multiradiality

Consider the following situation:

$$\begin{array}{ccc}
{}^{\dagger}\begin{pmatrix}\text{“tools” which we need to}\\ \text{compute “}\deg(\mathcal{L})\text{”}\end{pmatrix} & & {}^{\ddagger}\begin{pmatrix}\text{“tools” which we need to}\\ \text{compute “}\deg(\mathcal{L})\text{”}\end{pmatrix} \\
\Big\uparrow {}^{\dagger}\Xi\text{:functorial algorithm} & & \Big\uparrow {}^{\ddagger}\Xi\text{:functorial algorithm} \\
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- If the link is “isom.” arising from sch-/ring- theory, then by the functoriality of the algorithms, we have:

$${}^{\dagger}\begin{pmatrix}\text{“tools” which we need to} \\ \text{compute “}\deg(\mathcal{L})\text{”}\end{pmatrix} \xrightarrow{\sim} {}^{\ddagger}\begin{pmatrix}\text{“tools” which we need to} \\ \text{compute “}\deg(\mathcal{L})\text{”}\end{pmatrix}$$

- In IUT, consider the link [e.g., a "frobenius" $q \mapsto q^N$] not arising from sch-/ring- theory, so, a priori:

$$^{\dagger}\left(\begin{array}{c}\text{"tools" which we need to}\\ \text{compute "}\deg(\mathcal{L})\text{"}\end{array}\right) \overset{\text{relation?}}{\longleftrightarrow} {}^{\ddagger}\left(\begin{array}{c}\text{"tools" which we need to}\\ \text{compute "}\deg(\mathcal{L})\text{"}\end{array}\right)$$

Now suppose that:

For $\Box \in \{\dagger, \ddagger\}$, $^{\exists}$functorial algorithm

$$^{\Box}\left(\begin{array}{c}\text{a mathematical setting}\\ \text{arising from sch-theory}/k\end{array}\right) \longmapsto \begin{array}{c}\text{a } \Box\text{-coric object } {}^{\Box}C\\ \text{[e.g., a top. gp } {}^{\Box}G \cong G_k]\end{array}$$

Then the multiradiality of the functorial algorithms ${}^{\dagger}\Xi$, ${}^{\ddagger}\Xi$ implies:

$$
\begin{tikzcd}[column sep=large, row sep=huge]
{}^{\dagger}\begin{pmatrix} \text{“tools” which we need to} \\ \text{compute “}\deg(\mathcal{L})\text{”} \end{pmatrix} & {}^{\ddagger}\begin{pmatrix} \text{“tools” which we need to} \\ \text{compute “}\deg(\mathcal{L})\text{”} \end{pmatrix} \\
{}^{\dagger}\begin{pmatrix} \text{a mathematical setting} \\ \text{arising from sch-theory}/k \end{pmatrix} \arrow[u, mapsto, "{}^{\dagger}\Xi\text{:multiradial functorial algorithm}" description] \arrow[r, "\text{link}"] & {}^{\ddagger}\begin{pmatrix} \text{a mathematical setting} \\ \text{arising from sch-theory}/k \end{pmatrix} \arrow[u, mapsto, "{}^{\ddagger}\Xi\text{:multiradial functorial algorithm}" description]
\end{tikzcd}
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\end{tikzcd}
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{}^{\dagger}\left(\begin{matrix}\text{a mathematical setting} \\ \text{arising from sch-theory}/k\end{matrix}\right) \arrow[r, "\text{link}"] \arrow[u, mapsto, "{}^{\dagger}\Xi\text{:multiradial functorial algorithm}"] \arrow[d, mapsto, "\text{functorial algorithm}"'] & {}^{\ddagger}\left(\begin{matrix}\text{a mathematical setting} \\ \text{arising from sch-theory}/k\end{matrix}\right) \arrow[u, mapsto, "{}^{\ddagger}\Xi\text{:multiradial functorial algorithm}"'] \arrow[d, mapsto, "\text{functorial algorithm}"] \\
{}^{\dagger}C \arrow[r, "\sim"] & {}^{\ddagger}C
\end{tikzcd}$$

Radial Environment

Definition

We refer to a triple $(\mathcal{R},\ \mathcal{C},\ \Phi : \mathcal{R} \to \mathcal{C})$ consisting of

- a category $\mathcal{R}$ — whose objects is a specific collection of radial data, and each of whose morphism is an isomorphism
- a category $\mathcal{C}$ — whose objects is a specific collection of coric data, and each of whose morphism is an isomorphism
- a functor Φ — an essentially surj. functor, which is a "functorial algorithm" whose "input data" is $\in \mathcal{R}$ and whose "output data" is $\in \mathcal{C}$

as a radial environment. If Φ is full, then we say that the radial env. is multiradial. We refer to a radial env. which is not multiradial as uniradial.

Example (a multiradial environment)

- $\mathcal{R}$ — Obj: a triple $(\Pi,\ G,\ \alpha)$ consist. of
  - Π: a topological gp $\cong \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$;
  - G: a topological gp $\cong G_k$;
  - $\alpha : \Pi/\Delta \xrightarrow{\sim} G$: the full poly-isomorphism, i.e., the collection of all isomorphisms, where we write $\Delta \subseteq \Pi$ for the gp-theoretic subgroup corresp. to $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}}$.

  Hom: a pair of isom. of top. gps $\Pi \xrightarrow{\sim} \Pi^*$, $G \xrightarrow{\sim} G^*$.

- $\mathcal{C}$ — Obj: a top. gp $\cong G_k$ Hom: an isom. of top. gps $G \xrightarrow{\sim} G^*$

- Φ — the functor $\mathcal{R} \to \mathcal{C}$ defined to be $(\Pi,\ G,\ \alpha) \ \mapsto\ G$

Multiradial Algorithm

Definition

$(\mathcal{R},\ \mathcal{C},\ \Phi)$: a radial environment

Suppose: we have a commutative diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\Xi"] \arrow[d, "\Phi"'] & \mathcal{F} \arrow[dl] \\ \mathcal{C} & \end{tikzcd}$$

where $\mathcal{F}$ is a category, and all arrows are functors. We say that the functor [i.e., "functorial algorithm"] Ξ is multiradial if $(\mathcal{R},\ \mathcal{C},\ \Phi)$ is multiradial.

Note: If we have a multiradial env. $(\mathcal{R},\ \mathcal{C},\ \Phi)$ and a diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\Xi"] \arrow[d, "\Phi"'] & \mathcal{F} \\ \mathcal{C} & \end{tikzcd}$$

then we can always get a multiradial functor as follows:

Let $\mathcal{R}^{\blacklozenge}$ be the category whose object is a pair $(R,\ \Xi(R))$ and whose morphism is a pair $(f : R \xrightarrow{\sim} R^*,\ \Xi(f) : \Xi(R) \xrightarrow{\sim} \Xi(R^*))$. Then the functor $\Psi_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}^{\blacklozenge}; R \mapsto (R,\ \Xi(R))$ satisfies the comm. diag.

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\Psi_{\mathcal{R}}"] \arrow[d, "\Phi"'] & \mathcal{R}^{\blacklozenge} \arrow[d] & \ni & (R,\ \Xi(R)) \arrow[d, mapsto] \\ \mathcal{C} & \mathcal{R} \arrow[l, "\Phi"] & \ni & R \end{tikzcd}$$

so $\Psi_{\mathcal{R}}$ is a multiradial functor.

Multiadial Algorithm Related to Étale Theta Functions

Define a multiradial environment $(\mathcal{R},\ \mathcal{C},\ \Phi)$ as follows:

- $\mathcal{R}$ — Obj: $(\ \Pi \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z},\ G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G),\ \alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}}\ )$
  - $\Pi \ \cong\ \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$,
  - $G \ \cong\ G_k \ \ (\Rightarrow\ (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)) \cong (G_k \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}})\ )$,
  - $\alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}}$ is the pair of the full-poly isomorphism $\Pi/\Delta \xrightarrow{\sim} G$ and $\Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z} \ \overset{\text{zero}}{\to}\ \mathcal{O}^{\times\boldsymbol{\mu}}(G)$

  Hom: $(\Pi \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} (\Pi^* \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}(\Pi^*)) \otimes \mathbb{Q}/\mathbb{Z})$: the isom. induced by $\Pi \xrightarrow{\sim} \Pi^*$; $(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)) \xrightarrow{\sim} (G^* \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G^*))$: an "$\mathrm{Ism}$"-multiple of the isom. induced by $G \xrightarrow{\sim} G^*$.

- $\mathcal{C}$ — Obj: $(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$ Hom: the same as in $\mathcal{R}$

- Φ — the functor $\mathcal{R} \to \mathcal{C}$ defined by "forgetting".

Theorem (A Multiradial Algorithm Related to Étale Theta Functions — cf. [IUTchII], Cor 1.12)

$(\mathcal{R},\ \mathcal{C},\ \Phi)$: the multiradial env. defined at the previous page

Then the functorial algorithm

$$\Pi \longmapsto {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}_*^{\Theta}(\Pi)) \ \ [\subseteq \ \varinjlim_{J} H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*^{\Theta}(\Pi))|_J, \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}(\Pi)))\ ]$$

determines a functor

$$\Xi : \mathcal{R} \longrightarrow \mathcal{F}$$

where $\mathcal{F}$ is the category whose objects are $\{{}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}_*^{\Theta}(\Pi))\}$.
In particular, the functor

$$\Psi_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathcal{R}^{\blacklozenge}$$

[where $\mathcal{R}^{\blacklozenge}$ denotes the "graph of Ξ"] is multiradial.