Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves I: Inertia Groups and Profinite Dehn Twists

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Abstract.

Let $\Sigma$ be a nonempty set of prime numbers. In the present paper, we continue our study of the pro-$\Sigma$ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields of characteristic zero. Our first main result asserts, roughly speaking, that if an $F$-admissible automorphism [i.e., an automorphism that preserves the fiber subgroups that arise as kernels associated to the various natural projections of the configuration space under consideration to configuration spaces of lower dimension] of a configuration space group arises from an $F$-admissible automorphism of a configuration space group [arising from a configuration space] of strictly higher dimension, then it is necessarily FC-admissible, i.e., preserves the cuspidal inertia subgroups of the various subquotients corresponding to surface groups. After discussing various abstract profinite combinatorial technical tools involving semi-graphs of anabelioids of PSC-type that are motivated by the well-known classical theory of topological surfaces, we proceed to develop a theory of profinite Dehn twists, i.e., an abstract profinite combinatorial analogue of classical Dehn twists associated to cycles on topological surfaces. This theory of profinite Dehn twists leads naturally to comparison results between the abstract combinatorial machinery developed in the present paper and more classical scheme-theoretic constructions. In particular, we obtain a purely combinatorial description of the Galois action associated to a [scheme-theoretic!] degenerating family of hyperbolic curves over a complete equicharacteristic discrete valuation ring of characteristic zero. Finally, we apply the theory of profinite Dehn twists to prove a “geometric version of the Grothendieck Conjecture” for — i.e., put another way, we compute the centralizer of the geometric monodromy associated to — the tautological curve over the moduli stack of pointed smooth curves.

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§ Introduction

Let \( \Sigma \subseteq \mathfrak{Primes} \) be a nonempty subset of the set of prime numbers \( \mathfrak{Primes} \). In the present paper, we continue our study [cf. [SemiAn], [CmbGC], [CmbCsp], [MT], [NodNon]] of the anabelian geometry of semi-graphs of anabelioids of \([\text{pro-} \Sigma] \) PSC-type, i.e., semi-graphs of anabelioids that arise from a pointed stable curve over an algebraically closed field of characteristic zero. Roughly speaking, such a “semi-graph of anabelioids” may be thought of as a slightly modified, Galois category-theoretic formulation of the “graph of profinite groups” associated to such a pointed stable curve that takes into account the cusps [i.e., marked points] of the pointed stable curve, and in which the profinite groups that appear are regarded as being defined only up to inner automorphism. At a more conceptual level, the notion of a semi-graph of anabelioids of PSC-type may be thought of as a sort of abstract profinite combinatorial analogue of the notion of a hyperbolic topological surface of finite type, i.e., the underlying topological surface of a hyperbolic Riemann surface of finite type. One central object of study in this context is the notion of an outer representation of IPSC-type [cf. [NodNon], Definition 2.4, (i)], which may be thought of as an abstract profinite combinatorial analogue of the scheme-theoretic notion of a degenerating family of hyperbolic curves over a complete discrete valuation ring. In [NodNon], we studied a purely combinatorial generalization of this notion, namely, the notion of an outer representation of NN-type [cf. [NodNon], Definition 2.4, (iii)], which may be thought of as an abstract profinite combinatorial analogue of the topological notion of a family of hyperbolic topological surfaces of finite type over a circle.

Here, we recall that such families are a central object of study in the theory of hyperbolic threefolds.

Another central object of study in the combinatorial anabelian geometry of hyperbolic curves [cf. [CmbCsp], [MT], [NodNon]] is the notion of a configuration space group [cf. [MT], Definition 2.3, (i)], i.e., the pro-\( \Sigma \) fundamental group of the configuration space associated to a hyperbolic curve over an algebraically closed field of characteristic zero, where \( \Sigma \) is either equal to \( \mathfrak{Primes} \) or of cardinality one. In [MT], it was shown [cf. [MT], Corollary 6.3] that, if one excludes the case of hyperbolic curves of type \((g, r) \in \{(0,3), (1,1)\}\), then, up to a permutation of the factors of the configuration space under consideration, any automorphism of a configuration space group is necessarily \( F \)-admissible [cf. [CmbCsp], Definition 1.1, (ii)], i.e., preserves the fiber subgroups that arise as kernels associated to the various natural projections of the
In §1, we prove our first main result [cf. Corollary 1.9], by means of techniques that extend the techniques of [MT], §4, i.e., techniques that center around applying the fact that the first Chern class associated to the diagonal divisor in a product of two copies of a proper hyperbolic curve consists, in essence, of the identity matrix [cf. Lemma 1.3, (iii)]. This result asserts, roughly speaking, that if an $F$-admissible automorphism of a configuration space group arises from an $F$-admissible automorphism of a configuration space group [arising from a configuration space] of strictly higher dimension, then it is necessarily FC-admissible [cf. [CmbCsp], Definition 1.1, (ii)], i.e., preserves the cuspidal inertia subgroups of the various subquotients corresponding to surface groups.

**Theorem A (F-admissibility and FC-admissibility).** Let $\Sigma$ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; $n$ a positive integer; $(g, r)$ a pair of nonnegative integers such that $2g - 2 + r > 0$; $X$ a hyperbolic curve of type $(g, r)$ over an algebraically closed field $k$ of characteristic $\not\equiv 2$; $X_n$ the $n$-th configuration space of $X$; $\Pi_n$ the maximal pro-$\Sigma$ quotient of the fundamental group of $X_n$; “$\text{Out}^{FC}(\Pi_n)$”, “$\text{Out}^F(\Pi_n)$” the subgroups of FC- and F-admissible [cf. [CmbCsp], Definition 1.1, (ii)] automorphisms [cf. the discussion entitled “Topological groups” in §0] of “$\Pi_n$”. Then the following hold:

(i) Let $\alpha \in \text{Out}^F(\Pi_{n+1})$. Then $\alpha$ induces the same automorphism of $\Pi_n$ relative to the various quotients $\Pi_{n+1} \twoheadrightarrow \Pi_n$ by fiber subgroups of length 1 [cf. [MT], Definition 2.3, (iii)]. In particular, we obtain a natural homomorphism

$$\text{Out}^F(\Pi_{n+1}) \twoheadrightarrow \text{Out}^F(\Pi_n).$$

(ii) The image of the homomorphism

$$\text{Out}^F(\Pi_{n+1}) \twoheadrightarrow \text{Out}^F(\Pi_n)$$

of (i) is contained in

$$\text{Out}^{FC}(\Pi_n) \subseteq \text{Out}^F(\Pi_n).$$

For the convenience of the reader, we remark that our treatment of Theorem A in §1 does not require any knowledge of the theory of semi-graphs of anabelioids. On the other hand, in a sequel to the present paper, we intend to prove a substantial strengthening of Theorem A, whose
proof makes quite essential use of the theory of [CmbGC], [CmbCsp], and [NodNon] [i.e., in particular, of the theory of semi-graphs of anabelioids of PSC-type].

In §2 and §3, we develop various technical tools that will play a crucial role in the subsequent development of the theory of the present paper. In §2, we study various fundamental operations on semi-graphs of anabelioids of PSC-type. A more detailed description of these operations may be found in the discussion at the beginning of §2, as well as in the various illustrations referred to in this discussion. Roughly speaking, these operations may be thought of as abstract profinite combinatorial analogues of various well-known operations that occur in the theory of “surgery” on topological surfaces — i.e.,

- restriction to a subsurface arising from a decomposition, such as a “pants decomposition”, of the surface or to a [suitably positioned] cycle;
- partially compactifying the surface by adding “missing points”;
- cutting a surface along a [suitably positioned] cycle;
- gluing together two surfaces along [suitably positioned] cycles.

Most of §2 is devoted to the abstract combinatorial formulation of these operations, as well as to the verification of various basic properties involving these operations.

In §3, we develop the local theory of the second cohomology group with compact supports associated to various sub-semi-graphs and components of a semi-graph of anabelioids of PSC-type. Roughly speaking, this theory may be thought of as a sort of abstract profinite combinatorial analogue of the local theory of orientations on a topological surface $S$, i.e., the theory of the locally defined cohomology modules

$$(U, x) \mapsto H^2(U, U \setminus \{x\}; \mathbb{Z}) \quad (\cong \mathbb{Z})$$

— where $U \subseteq S$ is an open subset, $x \in U$. In the abstract profinite combinatorial context of the present paper, the various locally defined second cohomology groups with compact supports give rise to cyclotomes, i.e., copies of quotients of the once-Tate-twisted Galois module $\hat{\mathbb{Z}}(1)$. The main result that we obtain in §3 concerns various canonical synchronizations of cyclotomes [cf. Corollary 3.9], i.e., canonical isomorphisms between these cyclotomes associated to various local
portions of the given semi-graph of anabelioids of PSC-type which are compatible with the various fundamental operations studied in §2.

In §4, we apply the technical tools developed in §2, §3 to define and study the notion of a profinite Dehn [multi-]twist [cf. Definition 4.4; Theorem 4.8, (iv)]. This notion is, needless to say, a natural abstract profinite combinatorial analogue of the usual notion of a Dehn twist in the theory of topological surfaces. On the other hand, it is defined, in keeping with the spirit of the present paper, in a fashion that is purely combinatorial, i.e., without resorting to the “crutch” of considering, for instance, profinite closures of Dehn twists associated to cycles on topological surfaces. Our main results in §4 [cf. Theorem 4.8, (i), (iv); Proposition 4.10, (ii)] assert, roughly speaking, that profinite Dehn twists satisfy a structure theory of the sort that one would expect from the analogy with the topological case, and that this structure theory is compatible, in a suitable sense, with the various fundamental operations studied in §2.

**Theorem B (Properties of profinite Dehn multi-twists).** Let \( \Sigma \) be a nonempty set of prime numbers and \( \mathcal{G} \) a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type. Write

\[
\text{Aut}_{\text{grph}}(\mathcal{G}) \subseteq \text{Aut}(\mathcal{G})
\]

for the group of automorphisms of \( \mathcal{G} \) which induce the identity automorphism on the underlying semi-graph of \( \mathcal{G} \) and

\[
\text{Dehn}(\mathcal{G}) \overset{\text{def}}{=} \{ \alpha \in \text{Aut}_{\text{grph}}(\mathcal{G}) \mid \alpha_{\mathcal{G}|_v} = \text{id}_{\mathcal{G}|_v} \text{ for any } v \in \text{Vert}(\mathcal{G}) \}
\]

— where we write \( \alpha_{\mathcal{G}|_v} \) for the restriction of \( \alpha \) to the semi-graph of anabelioids \( \mathcal{G}|_v \) of pro-\( \Sigma \) PSC-type determined by \( v \in \text{Vert}(\mathcal{G}) \) [cf. Definitions 2.1, (iii); 2.14, (ii); Remark 2.5.1, (ii)]; we shall refer to an element of \( \text{Dehn}(\mathcal{G}) \) as a profinite Dehn multi-twist of \( \mathcal{G} \). Then the following hold:

(i) **(Normality)** \( \text{Dehn}(\mathcal{G}) \) is normal in \( \text{Aut}(\mathcal{G}) \).

(ii) **(Structure of the group of profinite Dehn multi-twists)** Write

\[
\Lambda_{\mathcal{G}} \overset{\text{def}}{=} \text{Hom}_{\mathbb{Z}_\Sigma}(H^2_{\Sigma}(\mathcal{G}, \mathbb{Z}_{\Sigma}), \mathbb{Z}_{\Sigma})
\]

for the cyclotome associated to \( \mathcal{G} \) [cf. Definitions 3.1, (ii), (iv); 3.8, (i)]. Then there exists a natural isomorphism

\[
\mathcal{D}_{\mathcal{G}} : \text{Dehn}(\mathcal{G}) \xrightarrow{\sim} \bigoplus_{\text{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}
\]
that is functorial, in $\mathcal{G}$, with respect to isomorphisms of semi-graphs of anabelioids. In particular, $\text{Dehn}(\mathcal{G})$ is a finitely generated free $\hat{\mathbb{Z}}^\Sigma$-module of rank $\text{Node}(\mathcal{G})$. We shall refer to a nontrivial profinite Dehn multi-twist whose image $\in \bigoplus_{\text{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}$ lies in a direct summand [i.e., in a single “$\Lambda_{\mathcal{G}}$”] as a profinite Dehn twist.

(iii) (Exact sequence relating profinite Dehn multi-twists and glueable automorphisms) Write

$$\text{Glu}(\mathcal{G}) \subseteq \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{grp}\Phi}(\mathcal{G}|_v)$$

for the [closed] subgroup of “glueable” collections of automorphisms of the direct product $\prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{grp}\Phi}(\mathcal{G}|_v)$ consisting of elements $([\alpha_v])_{v \in \text{Vert}(\mathcal{G})}$ such that $\chi_v(\alpha_v) = \chi_w(\alpha_w)$ for any $v, w \in \text{Vert}(\mathcal{G})$ — where we write $\mathcal{G}|_v$ for the semi-graph of anabelioids of pro-$\Sigma$ PSC-type determined by $v \in \text{Vert}(\mathcal{G})$ [cf. Definition 2.1, (iii)] and $\chi_v: \text{Aut}(\mathcal{G}|_v) \to (\hat{\mathbb{Z}}^\Sigma)^*$ for the pro-$\Sigma$ cyclotomic character of $v \in \text{Vert}(\mathcal{G})$ [cf. Definition 3.8, (ii)]. Then the natural homomorphism

$$\text{Aut}^{\text{grp}\Phi}(\mathcal{G}) \to \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{grp}\Phi}(\mathcal{G}|_v)$$

$$\alpha \mapsto ([\alpha_v])_{v \in \text{Vert}(\mathcal{G})}$$

factors through $\text{Glu}(\mathcal{G}) \subseteq \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{grp}\Phi}(\mathcal{G}|_v)$, and, moreover, the resulting homomorphism $\rho^\text{Vert}: \text{Aut}^{\text{grp}\Phi}(\mathcal{G}) \to \text{Glu}(\mathcal{G})$ [cf. (i)] fits into an exact sequence of profinite groups

$$1 \to \text{Dehn}(\mathcal{G}) \to \text{Aut}^{\text{grp}\Phi}(\mathcal{G}) \overset{\rho^\text{Vert}}{\to} \text{Glu}(\mathcal{G}) \to 1.$$
anabelioids of PSC-type under consideration admits a symmetry that permutes the nodes [cf. Fig. 6]; the desired coincidence of synchronizations is then obtained by observing that both the combinatorial and the scheme-theoretic synchronizations are compatible with this symmetry. One way to understand this fundamental coincidence of synchronizations is as a sort of abstract combinatorial analogue of the cyclotomic synchronization given in [GalSct], Theorem 4.3; [AbsHyp], Lemma 2.5, (ii) [cf. Remark 5.9.1, (i)]. Another way to understand this fundamental coincidence of synchronizations is as a statement to the effect that

the Galois action associated to a [scheme-theoretic] degenerating family of hyperbolic curves over a complete equicharacteristic discrete valuation ring of characteristic zero — i.e., “an outer representation of IPSC-type” — admits a purely combinatorial description [cf. Corollary 5.9, (iii)].

That is to say, one central problem in the theory of outer Galois representations associated to hyperbolic curves over arithmetic fields is precisely the problem of giving such a “purely combinatorial description” of the outer Galois representation. Indeed, this point of view plays a central role in the theory of the Grothendieck-Teichmüller group. Thus, although an explicit solution to this problem is well out of reach at the present time in the case of number fields or mixed-characteristic local fields, the theory of §5 yields a solution to this problem in the case of complete equicharacteristic discrete valuation fields of characteristic zero. One consequence of this solution is the following criterion for an outer representation to be of IPSC-type [cf. Corollary 5.10].

**Theorem C** (Combinatorial/group-theoretic nature of scheme-theoreticity). Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(\Sigma\) a nonempty set of prime numbers; \(R\) a complete discrete valuation ring whose residue field \(k\) is separably closed of characteristic \(\neq 2\); \(S^\log\) the log scheme obtained by equipping \(S \defeq \text{Spec } R\) with the log structure determined by the maximal ideal of \(R\); \((\mathcal{M}_g, r)_S\) the moduli stack of \(r\)-pointed stable curves of genus \(g\) over \(S\) whose \(r\) marked points are equipped with an ordering; \((\mathcal{M}_g, r)_S \subseteq (\mathcal{M}_g, r)_S\) the open substack of \((\mathcal{M}_g, r)_S\) parametrizing smooth curves; \((\mathcal{M}_g, r)^{\log}_S\) the log stack obtained by equipping \((\mathcal{M}_g, r)_S\) with the log structure associated to the divisor with normal crossings \((\mathcal{M}_g, r)_S \setminus (\mathcal{M}_g, r)_S \subseteq (\mathcal{M}_g, r)_S\); \(x \in (\mathcal{M}_g, r)_S(k)\) a \(k\)-valued point of \((\mathcal{M}_g, r)_S\); \(\hat{O}\) the completion of the local ring of \((\mathcal{M}_g, r)_S\) at the image of \(x\); \(T^\log\) the log scheme obtained by
equipping \( T \overset{\text{def}}{=} \text{Spec} \hat{O} \) with the log structure induced by the log structure of \((\mathcal{M}_{g,r}^{\log})_S; t^{\log}\) the log scheme obtained by equipping the closed point of \( T \) with the log structure induced by the log structure of \( T^{\log} \); \( X^\log_1 \) the stable log curve over \( t^{\log} \) corresponding to the natural strict \((1-)\)morphism \( t^{\log} \to (\mathcal{M}_{g,r}^{\log})_S; I_{T^{\log}} \) the maximal pro-\( \Sigma \) quotient of the log fundamental group \( \pi_1(T^{\log}) \) of \( T^{\log} \); \( I_{G^{\log}} \) the maximal pro-\( \Sigma \) quotient of the log fundamental group \( \pi_1(S^{\log}) \) of \( S^{\log} \); \( \mathcal{G}_{X^{\log}} \) the semi-graph of anabelioids of pro-\( \Sigma \) PSC-type determined by the stable log curve \( X^{\log}_1 \) [cf. [CmbGC], Example 2.5]; \( \rho_{X^{\log}_1}^{\text{univ}}: I_{T^{\log}} \to \text{Aut}(\mathcal{G}_{X^{\log}}) \) the natural outer representation associated to \( X^{\log}_1 \) [cf. Definition 5.5]; \( I \) a profinite group; \( \rho: I \to \text{Aut}(\mathcal{G}_{X^{\log}}) \) an outer representation of pro-\( \Sigma \) PSC-type [cf. [NodNon], Definition 2.1, (i)]. Then the following conditions are equivalent:

(i) \( \rho \) is of **IPSC-type** [cf. [NodNon], Definition 2.4, (i)].

(ii) There exist a morphism of log schemes \( \phi^{\log}: S^{\log} \to T^{\log} \) over \( S \) and an **isomorphism of outer representations** of pro-\( \Sigma \) PSC-type \( \rho \overset{\sim}{\to} \rho_{X^{\log}_1}^{\text{univ}} \circ I_{\phi^{\log}} \) [cf. [NodNon], Definition 2.1, (i)] — where we write \( I_{\phi^{\log}}: I_{G^{\log}} \to I_{T^{\log}} \) for the homomorphism induced by \( \phi^{\log} \) — i.e., there exist an automorphism \( \beta \) of \( \mathcal{G}_{X^{\log}} \) and an isomorphism \( \alpha: I \overset{\sim}{\to} I_{S^{\log}} \) such that the diagram

\[
\begin{array}{ccc}
I & \overset{\rho}{\longrightarrow} & \text{Aut}(\mathcal{G}_{X^{\log}}) \\
\alpha \downarrow & & \downarrow 1 \\
I_{S^{\log}} & \overset{\rho_{X^{\log}_1}^{\text{univ}} \circ I_{\phi^{\log}}}{\longrightarrow} & \text{Aut}(\mathcal{G}_{X^{\log}})
\end{array}
\]

— where the right-hand vertical arrow is the automorphism of \( \text{Aut}(\mathcal{G}_{X^{\log}}) \) induced by \( \beta \) — commutes.

(iii) There exist a morphism of log schemes \( \phi^{\log}: S^{\log} \to T^{\log} \) over \( S \) and an **isomorphism** \( \alpha: I \overset{\sim}{\to} I_{S^{\log}} \) such that \( \rho = \rho_{X^{\log}_1}^{\text{univ}} \circ I_{\phi^{\log}} \circ \alpha \) — where we write \( I_{\phi^{\log}}: I_{G^{\log}} \to I_{T^{\log}} \) for the homomorphism induced by \( \phi^{\log} \) — i.e., the automorphism “\( \beta \)” of (ii) may be taken to be the identity.

Before proceeding, in this context we observe that one fundamental intrinsic difference between outer representations of IPSC-type and more general outer representations of NN-type is that, unlike the case with outer representations of IPSC-type, the period matrices associated
to outer representations of NN-type may, in general, fail to be nondegenerate — cf. the discussion of Remark 5.9.2.

Here, we remark in passing that in a sequel to the present paper, the theory of §5 will play an important role in the proofs of certain applications to the theory of tempered fundamental groups developed in [André].

Finally, in §6, we apply the theory of profinite Dehn twists developed in §4 to prove a “geometric version of the Grothendieck Conjecture” for — i.e., put another way, we compute the centralizer of the geometric monodromy associated to — the tautological curve over the moduli stack of pointed smooth curves [cf. Theorems 6.13; 6.14].

Theorem D (Centralizers of geometric monodromy groups arising from moduli stacks of pointed curves). Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(\Sigma\) a nonempty set of prime numbers; \(k\) an algebraically closed field of characteristic zero. Write \((M_{g, r})_k\) for the moduli stack of \(r\)-pointed smooth curves of genus \(g\) over \(k\) whose \(r\) marked points are equipped with an ordering; \(C_{g, r} \to M_{g, r}\) for the tautological curve over \(M_{g, r}\) [cf. the discussion entitled “Curves” in §0]; \(\Pi_{M_{g, r}} \overset{\text{def}}{=} \pi_1((M_{g, r})_k)\) for the étale fundamental group of the moduli stack \((M_{g, r})_k\); \(\Pi_{g, r}\) for the maximal pro-\(\Sigma\)-quotient of the kernel \(N_{g, r}\) of the natural surjection \(\pi_1((C_{g, r})_k) \to \pi_1((M_{g, r})_k) = \Pi_{M_{g, r}}; \Pi_{C_{g, r}}\) for the quotient of the étale fundamental group \(\pi_1((C_{g, r})_k)\) of \((C_{g, r})_k\) by the kernel of the natural surjection \(N_{g, r} \to \Pi_{g, r}; \text{Out}^C(\Pi_{g, r})\) for the group of automorphisms [cf. the discussion entitled “Topological groups” in §0] of \(\Pi_{g, r}\) which induce bijections on the set of cuspidal inertia subgroups of \(\Pi_{g, r}\). Thus, we have a natural exact sequence of profinite groups

\[
1 \to \Pi_{g, r} \to \Pi_{C_{g, r}} \to \Pi_{M_{g, r}} \to 1,
\]

which determines an outer representation

\[
\rho_{g, r}: \Pi_{M_{g, r}} \to \text{Out}(\Pi_{g, r}).
\]

Then the following hold:

(i) Let \(H \subseteq \Pi_{M_{g, r}}\) be an open subgroup of \(\Pi_{M_{g, r}}\). Suppose that one of the following two conditions is satisfied:

(a) \(2g - 2 + r > 1\), i.e., \((g, r) \notin \{(0, 3), (1, 1)\}\).

(b) \((g, r) = (1, 1), 2 \in \Sigma, \text{ and } H = \Pi_{M_{g, r}}\).
Then the composite of natural homomorphisms
\[ \text{Aut}_{(M,g,r)_k}((C,g,r)_k) \to \text{Aut}_{\Pi(M,g,r)}(\Pi(C,g,r))/\text{Im}(\Pi(g,r)) \]
\[ \to Z_{\text{Out}^{C}(\Pi(g,r))}(\text{Im}(\rho_{g,r})) \subseteq Z_{\text{Out}^{C}(\Pi_{g,r})}(\rho_{g,r}(H)) \]
[cf. the discussion entitled “Topological groups” in §0] determines an isomorphism
\[ \text{Aut}_{(M,g,r)_k}((C,g,r)_k) \to Z_{\text{Out}^{C}(\Pi(g,r))}(\rho_{g,r}(H)). \]

Here, we recall that \( \text{Aut}_{(M,g,r)_k}((C,g,r)_k) \) is isomorphic to
\[
\begin{cases}
\mathbb{Z}/2\mathbb{Z} & \text{if } (g,r) = (0,4); \\
\mathbb{Z}/2\mathbb{Z} & \text{if } (g,r) \in \{(1,1),(1,2),(2,0)\}; \\
\{1\} & \text{if } (g,r) \not\in \{(0,4),(1,1),(1,2),(2,0)\}. 
\end{cases}
\]

(ii) Let \( H \subseteq \text{Out}^{C}(\Pi_{g,r}) \) be a closed subgroup of \( \text{Out}^{C}(\Pi_{g,r}) \) that contains an open subgroup of \( \text{Im}(\rho_{g,r}) \subseteq \text{Out}(\Pi_{g,r}) \). Suppose that
\[ 2g - 2 + r > 1, \text{ i.e., } (g,r) \not\in \{(0,3),(1,1)\}. \]

Then \( H \) is almost slim [cf. the discussion entitled “Topological groups” in §0]. If, moreover,
\[ 2g - 2 + r > 2, \text{ i.e., } (g,r) \not\in \{(0,3),(0,4),(1,1),(1,2),(2,0)\}, \]
then \( H \) is slim [cf. the discussion entitled “Topological groups” in §0].

§0. Notations and Conventions

Sets: If \( S \) is a set, then we shall denote by \( 2^S \) the power set of \( S \) and by \( S^\# \) the cardinality of \( S \).

Numbers: The notation \( \mathfrak{P} \text{rimes} \) will be used to denote the set of all prime numbers. The notation \( \mathbb{N} \) will be used to denote the set or [additive] monoid of nonnegative rational integers. The notation \( \mathbb{Z} \) will be used to denote the set, group, or ring of rational integers. The notation \( \mathbb{Q} \) will be used to denote the set, group, or field of rational numbers. The notation \( \hat{\mathbb{Z}} \) will be used to denote the profinite completion of \( \mathbb{Z} \). If \( p \in \mathfrak{P} \text{rimes} \), then the notation \( \mathbb{Z}_p \) (respectively, \( \mathbb{Q}_p \)) will be used to denote the \( p \)-adic completion of \( \mathbb{Z} \) (respectively, \( \mathbb{Q} \)). If \( \Sigma \subseteq \mathfrak{P} \text{rimes} \), then the notation \( \hat{\mathbb{Z}}^\Sigma \) will be used to denote the pro-\( \Sigma \) completion of \( \mathbb{Z} \).
Monoids: We shall write $M^{gp}$ for the groupification of a monoid $M$.

Topological groups: Let $G$ be a topological group and $P$ a property of topological groups [e.g., “abelian” or “pro-$\Sigma$” for some $\Sigma \subseteq \text{Primes}$]. Then we shall say that $G$ is almost $P$ if there exists an open subgroup of $G$ that is $P$.

Let $G$ be a topological group and $H \subseteq G$ a closed subgroup of $G$. Then we shall denote by $Z_G(H)$ (respectively, $N_G(H)$; $C_G(H)$) the centralizer (respectively, normalizer; commensurator) of $H$ in $G$, i.e.,

$$Z_G(H) \overset{\text{def}}{=} \{ g \in G \mid g h g^{-1} = h \text{ for any } h \in H \},$$

(respectively, $N_G(H) \overset{\text{def}}{=} \{ g \in G \mid g \cdot H \cdot g^{-1} = H \}$;

$C_G(H) \overset{\text{def}}{=} \{ g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1} \}$);

we shall refer to $Z(G) \overset{\text{def}}{=} Z_G(G)$ as the center of $G$. It is immediate from the definitions that

$$Z_G(H) \subseteq N_G(H) \subseteq C_G(H) ; \ H \subseteq N_G(H).$$

We shall say that the closed subgroup $H$ is centrally terminal (respectively, normally terminal; commensurably terminal) in $G$ if $H = Z_G(H)$ (respectively, $H = N_G(H)$; $H = C_G(H)$). We shall say that $G$ is slim if $Z_G(U) = \{1\}$ for any open subgroup $U$ of $G$.

Let $G$ be a topological group. Then we shall write $G^{ab}$ for the abelianization of $G$, i.e., the quotient of $G$ by the closure of the commutator subgroup of $G$.

Let $G$ be a topological group. Then we shall write $\text{Aut}(G)$ for the group of continuous automorphisms of $G$, $\text{Inn}(G) \subseteq \text{Aut}(G)$ for the group of inner automorphisms of $G$, and $\text{Out}(G) \overset{\text{def}}{=} \text{Aut}(G)/\text{Inn}(G)$. We shall refer to an element of $\text{Out}(G)$ as an automorphism of $G$. Now suppose that $G$ is center-free [i.e., $Z(G) = \{1\}$]. Then we have an exact sequence of groups

$$1 \rightarrow G(\widetilde{\rightarrow} \text{Inn}(G)) \rightarrow \text{Aut}(G) \rightarrow \text{Out}(G) \rightarrow 1.$$

If $J$ is a group and $\rho: J \rightarrow \text{Out}(G)$ is a homomorphism, then we shall denote by

$$G \overset{\text{out}}{\rtimes} J$$

the group obtained by pulling back the above exact sequence of profinite groups via $\rho$. Thus, we have a natural exact sequence of groups

$$1 \rightarrow G \rightarrow G \overset{\text{out}}{\rtimes} J \rightarrow J \rightarrow 1.$$
Suppose further that $G$ is profinite and topologically finitely generated. Then one verifies easily that the topology of $G$ admits a basis of characteristic open subgroups, which thus induces a profinite topology on the groups $\text{Aut}(G)$ and $\text{Out}(G)$ with respect to which the above exact sequence relating $\text{Aut}(G)$ and $\text{Out}(G)$ determines an exact sequence of profinite groups. In particular, one verifies easily that if, moreover, $J$ is profinite and $\rho: J \to \text{Out}(G)$ is continuous, then the above exact sequence involving $G^\text{out} \rtimes J$ determines an exact sequence of profinite groups.

Let $G$, $J$ be profinite groups. Suppose that $G$ is center-free and topologically finitely generated. Let $\rho: J \to \text{Out}(G)$ be a continuous homomorphism. Write $\text{Aut}_J(G^\text{out} \rtimes J)$ for the group of [continuous] automorphisms of $G^\text{out} \rtimes J$ that preserve and induce the identity automorphism on the quotient $J$. Then one verifies easily that the operation of restricting to $G$ determines an isomorphism of profinite groups

$$\text{Aut}_J(G^\text{out} \rtimes J)/\text{Inn}(G) \xrightarrow{\sim} \text{Z}_{\text{Out}(G)(\text{Im}(\rho))}.$$ 

Let $G$ and $H$ be topological groups. Then we shall refer to a homomorphism of topological groups $\phi: G \to H$ as a split injection (respectively, split surjection) if there exists a homomorphism of topological groups $\psi: H \to G$ such that $\psi \circ \phi$ (respectively, $\phi \circ \psi$) is the identity automorphism of $G$ (respectively, $H$).

**Log schemes:** When a scheme appears in a diagram of log schemes, the scheme is to be understood as the log scheme obtained by equipping the scheme with the trivial log structure. If $X^\log$ is a log scheme, then we shall refer to the largest open subscheme of the underlying scheme of $X^\log$ over which the log structure is trivial as the interior of $X^\log$. Fiber products of fs log schemes are to be understood as fiber products taken in the category of fs log schemes.

**Curves:** We shall use the terms "hyperbolic curve", "cusp", "stable log curve", "smooth log curve", and "tripod" as they are defined in [CmbGC], §0; [Hsh], §0. If $(g; r)$ is a pair of nonnegative integers such that $2g - 2 + r > 0$, then we shall denote by $\bar{M}_{g, r}$ the moduli stack of $r$-pointed stable curves of genus $g$ over $\mathbb{Z}$ whose $r$ marked points are equipped with an ordering, by $M_{g, r} \subseteq \bar{M}_{g, r}$ the open substack of $\bar{M}_{g, r}$ parametrizing smooth curves, by $M_{g, r}^\log$ the log stack obtained by equipping $\bar{M}_{g, r}$ with the log structure associated to the divisor with normal crossings $\bar{M}_{g, r} \setminus$
\[ \mathcal{M}_{g,k} \subseteq \overline{\mathcal{M}}_{g,r}, \] by \( \mathcal{C}_{g,r} \rightarrow \overline{\mathcal{M}}_{g,r} \) the tautological curve over \( \overline{\mathcal{M}}_{g,r} \), and by \( \mathcal{D}_{g,r} \subseteq \mathcal{C}_{g,r} \) the corresponding tautological divisor of marked points of \( \mathcal{C}_{g,r} \rightarrow \overline{\mathcal{M}}_{g,r} \). Then the divisor given by the union of \( \mathcal{D}_{g,r} \) with the inverse image in \( \mathcal{C}_{g,r} \) of the divisor \( \overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r} \) determines a log structure on \( \mathcal{C}_{g,r} \); denote the resulting log stack by \( \mathcal{C}_{g,r}^{\log} \). Thus, we obtain a (1-)morphism of log stacks \( \mathcal{C}_{g,r}^{\log} \rightarrow \overline{\mathcal{M}}_{g,r}^{\log} \). We shall denote by \( \mathcal{C}_{g,r} \subseteq \mathcal{C}_{g,r} \) the interior of \( \mathcal{C}_{g,r} \). Thus, we obtain a (1-)morphism of stacks \( \mathcal{C}_{g,r} \rightarrow \mathcal{M}_{g,r} \). Let \( S \) be a scheme. Then we shall write \( (\mathcal{M}_{g,r})_S \defeq \mathcal{M}_{g,r} \times_{\Spec \mathbb{Z}} S \), \( (\mathcal{M}_{g,r})_S \defeq \mathcal{M}_{g,r} \times_{\Spec \mathbb{Z}} S \), \( (\mathcal{M}_{g,r})_S \defeq \mathcal{M}_{g,r} \times_{\Spec \mathbb{Z}} S \), \( (\mathcal{M}_{g,r})_S \defeq \mathcal{M}_{g,r} \times_{\Spec \mathbb{Z}} S \), \( (\mathcal{C}_{g,r})_S \defeq \mathcal{C}_{g,r} \times_{\Spec \mathbb{Z}} S \), and \( (\mathcal{C}_{g,r})_S \defeq \mathcal{C}_{g,r} \times_{\Spec \mathbb{Z}} S \).

Let \( n \) be a positive integer and \( X^{\log} \) a stable log curve of type \((g,r)\) over a log scheme \( \mathcal{S}^{\log} \). Then we shall refer to the log scheme obtained by pulling back the (1-)morphism \( \overline{\mathcal{M}}_{g,r+n}^{\log} \rightarrow \overline{\mathcal{M}}_{g,r}^{\log} \) given by forgetting the last \( n \) points via the classifying (1-)morphism \( \mathcal{S}^{\log} \rightarrow \overline{\mathcal{M}}_{g,r}^{\log} \) of \( X^{\log} \) as the \( n \)-th log configuration space of \( X^{\log} \).

\section{F-admissibility and FC-admissibility}

In the present \( \S \), we consider the FC-admissibility [cf. [CmbCsp], Definition 1.1, (ii)] of F-admissible automorphisms [cf. [CmbCsp], Definition 1.1, (ii)] of configuration space groups [cf. [MT], Definition 2.3, (i)]. Roughly speaking, we prove that if an F-admissible automorphism of a configuration space group arises from an F-admissible automorphism of a configuration space group [arising from a configuration space] of strictly higher dimension, then it is necessarily FC-admissible, i.e., preserves the cuspidal inertia subgroups of the various subquotients corresponding to surface groups [cf. Theorem 1.8, Corollary 1.9 below].

**Lemma 1.1 (Representations arising from certain families of hyperbolic curves).** Let \((g,r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \( l \) a prime number; \( k \) an algebraically closed field of characteristic \( \neq l \); \( B \) and \( C \) hyperbolic curves over \( k \) of type \((g,r)\); \( n \) a positive integer. Suppose that \((r,n) \neq (0,1)\). For \( i = 1, \cdots, n \), let \( f_i : B \longrightarrow C \) be an isomorphism over \( k \); \( s_i \) the section of \( B \times_k C \overset{\text{pr}_1}{\longrightarrow} B \) determined by the isomorphism \( f_i \). Suppose that, for any \( i \neq j \),
\[ \text{Im}(s_i) \cap \text{Im}(s_j) = \emptyset. \] Write

\[ Z \overset{\text{def}}{=} B \times_k C \setminus \bigcup_{i=1,\ldots,n} \text{Im}(s_i) \subseteq B \times_k C \]

for the complement of the images of the \( s_i \)'s, where \( i \) ranges over the integers such that \( 1 \leq i \leq n; \) pr for the composite \( Z \twoheadrightarrow B \times_k C \twoheadrightarrow B \) [thus, pr: \( Z \twoheadrightarrow B \) is a family of hyperbolic curves of type \( (g,r + n) \)]. \( \Pi_B \) (respectively, \( \Pi_C; \Pi_Z \)) the maximal pro-\( l \) quotient of the étale fundamental group \( \pi_1(B) \) (respectively, \( \pi_1(C); \pi_1(Z) \)) of \( B \) (respectively, \( C; Z \)); \( \text{pr}: \Pi_Z \twoheadrightarrow \Pi_B \) for the surjection induced by \( \text{pr} \); \( \Pi_Z/B \) for the kernel of \( \text{pr} : \Pi_B \to \text{Out}(\Pi_Z/B) \) for the outer representation of \( \Pi_B \) on \( \Pi_Z/B \) determined by the exact sequence

\[ 1 \to \Pi_{Z/B} \to \Pi_Z \overset{\text{pr}}{\to} \Pi_B \to 1. \]

Let \( \overline{b} \) be a geometric point of \( B \) and \( Z_{\overline{b}} \) the geometric fiber of \( \text{pr}: Z \to B \) at \( \overline{b} \). For \( i = 1, \ldots, n \), fix an inertia subgroup [among its various conjugates] of the étale fundamental group \( \pi_1(Z_{\overline{b}}) \) of \( Z_{\overline{b}} \) associated to the cusp of \( Z_{\overline{b}} \) determined by the section \( s_i \), and denote by

\[ I_{s_i} \subseteq \Pi_{Z/B} \]

the image in \( \Pi_{Z/B} \) of this inertia subgroup of \( \pi_1(Z_{\overline{b}}) \). Then the following hold:

(i) (Fundamental groups of fibers) The quotient \( \Pi_{Z/B} \) of the étale fundamental group \( \pi_1(Z_{\overline{b}}) \) of the geometric fiber \( Z_{\overline{b}} \) coincides with the maximal pro-\( l \) quotient of \( \pi_1(Z_{\overline{b}}) \).

(ii) (Abelianizations of the fundamental groups of fibers) For \( i = 1, \ldots, n \), write \( J_{s_i} \subseteq \Pi_{Z/B}^{ab} \) for the image of \( I_{s_i} \subseteq \Pi_{Z/B} \) in \( \Pi_{Z/B}^{ab} \). Then the composite \( I_{s_i} \hookrightarrow \Pi_{Z/B} \twoheadrightarrow \Pi_{Z/B}^{ab} \) determines an isomorphism \( I_{s_i} \cong J_{s_i} \); moreover, the inclusions \( J_{s_i} \hookrightarrow \Pi_{Z/B}^{ab} \) determine an exact sequence

\[ 1 \to \bigoplus_{i=1}^n J_{s_i}/J_r \to \Pi_{Z/B}^{ab} \to \Pi_C^{ab} \to 1 \]

where

\[ J_r \subseteq \bigoplus_{i=1}^n J_{s_i} \]
is a $\mathbb{Z}_l$-submodule such that

$$J_r \simeq \begin{cases} \mathbb{Z}_l & \text{if } r = 0, \\ 0 & \text{if } r \neq 0, \end{cases}$$

and, moreover, if $r = 0$ and $i = 1, \ldots, n$, then the composite

$$J_r \hookrightarrow \bigoplus_{i=1}^{n} J_{s_i} \xrightarrow{pr_i} J_{s_i}$$

is an isomorphism.

(iii) (Unipotency of a certain natural representation) The action of $\Pi_B$ on $\Pi_{Z/B}^{ab}$ determined by $\rho_{Z/B}$ preserves the exact sequence

$$1 \longrightarrow (\bigoplus_{i=1}^{n} J_{s_i})/J_r \longrightarrow \Pi_{Z/B}^{ab} \longrightarrow \Pi_{C}^{ab} \longrightarrow 1$$

[cf. (ii)] and induces the identity automorphisms on the subquotients $(\bigoplus_{i=1}^{n} J_{s_i})/J_r$ and $\Pi_{C}^{ab}$; in particular, the natural homomorphism $\Pi_B \rightarrow \text{Aut}_{\mathbb{Z}_l}(\Pi_{Z/B}^{ab})$ factors through a uniquely determined homomorphism

$$\Pi_B \rightarrow \text{Hom}_{\mathbb{Z}_l}\left(\Pi_{Z/B}^{ab}, (\bigoplus_{i=1}^{n} J_{s_i})/J_r \right).$$

Proof. Assertion (i) follows immediately from the [easily verified] fact that the natural action of $\pi_1(B)$ on $\pi_1(Z_B)^{ab} \otimes_{\mathbb{Z}_l} \mathbb{Z}_l$ is unipotent — cf., e.g., [Hsh], Proposition 1.4, (i), for more details. [Note that although [Hsh], Proposition 1.4, (i), is only stated in the case where the hyperbolic curves corresponding to $B$ and $C$ are proper, the same proof may be applied to the case where these hyperbolic curves are affine.] Assertion (ii) follows immediately, in light of our assumption that $(r, n) \neq (0, 1)$, from assertion (i), together with the well-known structure of the maximal pro-$l$ quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\neq l$. Finally, we verify assertion (iii). The fact that the action of $\Pi_B$ on $\Pi_{Z/B}^{ab}$ preserves the exact sequence appearing in the statement of assertion (iii) follows immediately from the fact that the surjection $\Pi_{Z/B}^{ab} \twoheadrightarrow \Pi_{C}^{ab}$ is induced by the open immersion $Z \rightarrow B \times_k C$ over $B$. The fact that the action in question induces the identity automorphism on $(\bigoplus_{i=1}^{n} J_{s_i})/J_r$
(respectively, \( \Pi_C^{ab} \)) follows immediately from the fact that the \( f_i \)'s are isomorphisms (respectively, the fact that the surjection \( \Pi_{Z/B}^{ab} \to \Pi_{Z/B}^{ab} \) is induced by the open immersion \( Z \hookrightarrow B \times_k C \) over \( B \)). Q.E.D.

**Lemma 1.2 (Maximal cuspidally central quotients of certain fundamental groups).** In the notation of Lemma 1.1, for \( i = 1, \cdots, n \), write

\[
\Pi_{Z/B} \twoheadrightarrow \Pi_{(Z/B)[i]} \twoheadrightarrow \Pi_C
\]

for the quotient of \( \Pi_{Z/B} \) by the normal closed subgroup topologically normally generated by the \( I_{s_i} \)'s, where \( j \) ranges over the integers such that \( 1 \leq j \leq n \) and \( j \neq i \);

\[
\Pi_{(Z/B)[i]} \twoheadrightarrow E_{(Z/B)[i]}
\]

for the maximal cuspidally central quotient [cf. [AbsCsp], Definition 1.1, (i)] relative to the surjection \( \Pi_{(Z/B)[i]} \twoheadrightarrow \Pi_{C} \) determined by the natural open immersion \( Z \hookrightarrow B \times_k C \);

\[
I_{s_i}^E \subseteq E_{(Z/B)[i]}
\]

for the kernel of the natural surjection \( E_{(Z/B)[i]} \twoheadrightarrow \Pi_C \); and

\[
E_{Z/B} \overset{\text{def}}{=} E_{(Z/B)[1]} \times_{\Pi_C} \cdots \times_{\Pi_C} E_{(Z/B)[n]}.
\]

Then the following hold:

(i) **(Cuspidal inertia subgroups)** Let \( 1 \leq i, j \leq n \) be integers. Then the homomorphism \( I_{s_i} \to I_{s_j}^E \) determined by the composite \( I_{s_i} \hookrightarrow \Pi_{Z/B} \to E_{(Z/B)[j]} \) is an isomorphism (respectively, trivial) if \( i = j \) (respectively, \( i \neq j \)).

(ii) **(Surjectivity)** The homomorphism \( \Pi_{Z/B} \to \mathbb{E}_{Z/B} \) determined by the natural surjections \( \Pi_{Z/B} \to E_{(Z/B)[i]} \) — where \( i \) ranges over the integers such that \( 1 \leq i \leq n \) — is surjective.

(iii) **(Maximal cuspidally central quotients and abelianizations)** The quotient \( \Pi_{Z/B} \twoheadrightarrow \mathbb{E}_{Z/B} \) of \( \Pi_{Z/B} \) [cf. (ii)] coincides with the maximal cuspidally central quotient [cf. [AbsCsp], Definition 1.1, (i)] relative to the surjection \( \Pi_{Z/B} \twoheadrightarrow \Pi_{C} \) determined by the natural open immersion \( Z \hookrightarrow B \times_k C \). In particular, the natural surjection \( \Pi_{Z/B} \twoheadrightarrow \Pi_{Z/B}^{ab} \) factors
through the surjection $\Pi_{Z/B} \twoheadrightarrow E_{Z/B}$, and the resulting surjection $E_{Z/B} \twoheadrightarrow \Pi_{Z/B}^b$ fits into a commutative diagram

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \bigoplus_{i=1}^n I_{s_i}^Z & \longrightarrow & E_{Z/B} & \longrightarrow & \Pi_C & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & (\bigoplus_{i=1}^n J_{s_i})/J_r & \longrightarrow & \Pi_{Z/B}^b & \longrightarrow & \Pi_C^b & \longrightarrow & 1 \\
\end{array}
\]

where the horizontal sequences are exact, and the vertical arrows are surjective. Moreover, the left-hand vertical arrow coincides with the surjection induced by the natural isomorphisms $I_{s_i} \cong J_{s_i}$ [cf. Lemma 1.1, (ii)] and $I_{s_i} \cong \bigoplus_{i=1}^n I_{s_i}$ [cf. (i)]. Finally, if $r \neq 0$, then the right-hand square is cartesian.

**Proof.** Assertion (i) follows immediately from the definition of the quotient $E_{(Z/B)[j]}$ of $\Pi_{Z/B}$, together with the well-known structure of the maximal pro-$l$ quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\neq l$ [cf., e.g., [MT], Lemma 4.2, (iv), (v)]. Assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately from assertions (i), (ii) [cf. [AbsCsp], Proposition 1.6, (iii)]. Q.E.D.

**Lemma 1.3 (The kernels of representations arising from certain families of hyperbolic curves).** In the notation of Lemmas 1.1, 1.2, suppose that $r \neq 0$. Then the following hold:

(i) (Unipotency of a certain natural outer representation)

Consider the action of $\Pi_B$ on $E_{Z/B}$ determined by the natural isomorphism

\[E_{Z/B} \cong \Pi_{Z/B}^b \times \Pi_C^b\]

[cf. Lemma 1.2, (iii)], together with the natural action of $\Pi_B$ on $\Pi_{Z/B}^b$ induced by $p_{Z/B}$ and the trivial action of $\Pi_B$ on $\Pi_C$. Then the outer action of $\Pi_B$ on $E_{Z/B}$ induced by this action coincides with the natural outer action of $\Pi_B$ on $E_{Z/B}$ induced by $p_{Z/B}$. In particular, relative to the natural identification $I_{s_i} \cong J_{s_i}^Z$ [cf. Lemma 1.2, (i)], the above action of $\Pi_B$ on $E_{Z/B}$ factors through the homomorphism

\[\Pi_B \longrightarrow \text{Hom}_{Z/l} \left( \Pi_C, \bigoplus_{i=1}^n I_{s_i} \right) \cong \text{Hom}_{Z/l} \left( \Pi_C^b, \bigoplus_{i=1}^n I_{s_i} \right)\]
obtained in Lemma 1.1, (iii).

(ii) \textbf{(Homomorphisms arising from a certain extension)} For \(i = 1, \ldots, n\), write \(\phi_i\) for the composite

\[
\Pi_B \longrightarrow \text{Hom}_{\mathbb{Z}_l}\left(\Pi_{C}^{ab}, \bigoplus_{j=1}^{n} I_{s_j}\right) 
\longrightarrow \text{Hom}_{\mathbb{Z}_l}\left(\Pi_{C}^{ab}, I_{s_i}\right)
\]

— where the first arrow is the homomorphism of (i), and the second arrow is the homomorphism determined by the projection \(\text{pr}_i: \bigoplus_{j=1}^{n} I_{s_j} \twoheadrightarrow I_{s_i}\). Then the homomorphism \(\phi_i\) \textbf{coincides} with the image of the element of \(H^2(\Pi_B \times \Pi_C, I_{s_i})\) determined by the extension

\[
1 \longrightarrow I_{s_i} \longrightarrow \Pi_{\mathbb{Z}_l}^{[i]} \longrightarrow \Pi_B \times \Pi_C \longrightarrow 1
\]

— where we write \(\Pi_{\mathbb{Z}_l}^{[i]} \overset{\text{def}}{=} \Pi_{\mathbb{Z} / \mathbb{Z}_l} / \text{Ker}(\Pi_{\mathbb{Z} / \mathbb{B}} \twoheadrightarrow \mathbb{E}_{\mathbb{Z} / \mathbb{B}[i]})\) — of \(\Pi_B \times \Pi_C\) by \(I_{s_i} \cong I_{s_i}^{\mathbb{Z}_l}\) [cf. Lemma 1.2, (i)] via the composite

\[
H^2(\Pi_B \times \Pi_C, I_{s_i}) \cong H^1(\Pi_B, H^1(\Pi_C, I_{s_i})) \cong \text{Hom}(\Pi_B, \text{Hom}(\Pi_C, I_{s_i}))
\]

— where the first arrow is the isomorphism determined by the Hochschild-Serre spectral sequence relative to the surjection \(\Pi_B \times \Pi_C \twoheadrightarrow \Pi_B\).

(iii) \textbf{(Factorization)} Write \(\overline{B}\) (respectively, \(\overline{C}\)) for the compactification of \(B\) (respectively, \(C\)) and \(\Pi_{\overline{B}}\) (respectively, \(\Pi_{\overline{C}}\)) for the maximal pro-\(l\) quotient of the étale fundamental group \(\pi_1(\overline{B})\) (respectively, \(\pi_1(\overline{C})\)) of \(\overline{B}\) (respectively, \(\overline{C}\)). Then the homomorphism \(\phi_i\) of (ii) \textbf{factors} as the composite

\[
\Pi_B \twoheadrightarrow \Pi_{\overline{B}} \cong \Pi_{\mathbb{Z}_l}^{[i]} \cong \Pi_{\mathbb{Z}_l}^{[i]} \cong \text{Hom}_{\mathbb{Z}_l}\left(\Pi_{\mathbb{Z}_l}^{[i]}, I_{s_i}\right) 
\rightarrow \text{Hom}_{\mathbb{Z}_l}\left(\Pi_{\mathbb{Z}_l}^{[i]}, I_{s_i}\right)
\]

— where the first (respectively, second; fourth) arrow is the homomorphism induced by \(B \hookrightarrow \overline{B}\) (respectively, \(f_i: B \hookrightarrow \overline{C}; C \to \overline{C}\)), and the third arrow is the isomorphism determined by the Poincaré duality isomorphism in étale cohomology, relative to the natural isomorphism \(I_{s_i} \cong \mathbb{Z}_l(1)\). [Here, the “(1)” denotes a “Tate twist.”]

(iv) \textbf{(Kernel of a certain natural representation)} The kernel of the homomorphism \(\Pi_B \to \text{Aut}_{\mathbb{Z}_l}(\Pi_{\mathbb{Z}_l}^{[i]})\) determined by \(\rho_{\mathbb{Z}_l/B}\) \textbf{coincides} with the kernel of the natural surjection \(\Pi_B \twoheadrightarrow \Pi_{\overline{B}}^{[i]}\).
Proof. Assertions (i), (ii) follow immediately from the various definitions involved. Next, we verify assertion (iii). It follows from assertion (ii), together with [MT], Lemma 4.2, (ii), (v) [cf. also the discussion surrounding [MT], Lemma 4.2], that, relative to the natural isomorphism $I_{s_i} \sim \mathbb{Z}_l(1)$, the image of $\phi_i \in \text{Hom}(\Pi_B, \text{Hom}_{\mathbb{Z}_l}(\Pi_{C_i}^{ab}, I_{s_i}))$ via the isomorphisms

$$\text{Hom}(\Pi_B, \text{Hom}_{\mathbb{Z}_l}(\Pi_{C_i}^{ab}, I_{s_i})) \xrightarrow{\sim} H^2(\Pi_B \times \Pi_C, \mathbb{Z}_l(1)) \xrightarrow{\sim} H^2(B \times_k C, \mathbb{Z}_l(1))$$

— where the first (respectively, second) isomorphism is the isomorphism induced by the above isomorphism $I_{s_i} \sim \mathbb{Z}_l(1)$ (respectively, the Hochschild-Serre spectral sequence relative to the surjection $\Pi_B \times \Pi_C \xrightarrow{pr_1} \Pi_B$) — is the first Chern class of the invertible sheaf associated to the divisor determined by the scheme-theoretic image of $s_i: B \to B \times_k C$. Thus, since the section $s_i$ extends uniquely to a section $s_i: B \to B \times_k C$, whose scheme-theoretic image we denote by $\text{Im}(s_i)$, it follows that the homomorphism $\phi_i \in \text{Hom}(\Pi_B, \text{Hom}_{\mathbb{Z}_l}(\Pi_{C_i}^{ab}, I_{s_i}))$ coincides with the image of the first Chern class of the invertible sheaf on $B \times_k C$ associated to the divisor $\text{Im}(s_i)$ via the composite

$$H^2(B \times_k C, \mathbb{Z}_l(1)) \xrightarrow{\sim} H^2(B \times_k C, I_{s_i}) \to H^2(B \times_k C, I_{s_i})$$

— where the first arrow is the isomorphism induced by the above isomorphism $I_{s_i} \sim \mathbb{Z}_l(1)$, and the second arrow is the homomorphism induced by the natural open immersion $B \times_k C \hookrightarrow B \times_k C$. In particular, assertion (iii) follows immediately from [Mln], Chapter VI, Lemma 12.2 [cf. also the argument used in the proof of [MT], Lemma 4.4]. Finally, we verify assertion (iv). To this end, we recall that by Lemma 1.1, (iii), the homomorphism $\Pi_B \to \text{Aut}_{\mathbb{Z}_l}(\Pi_{C_i}^{ab})$ factors through the homomorphism $\Pi_B \to \text{Hom}_{\mathbb{Z}_l}(\Pi_{C_i}^{ab}, \bigoplus_{s_i} \mathbb{Z}_l)$ of assertion (i). Thus, assertion (iv) follows immediately from assertion (iii). This completes the proof of assertion (iv).

Q.E.D.

Definition 1.4. For $\square \in \{\circ, \bullet\}$, let $\Sigma^\square$ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$; $X^\square$ a hyperbolic curve of type $(g^\square, r^\square)$ over an algebraically closed field of
characteristic \( \not\in \Sigma \); \( d \) a positive integer; \( X_{d,\Sigma} \) the \( d \)-th configuration space of \( X_{\Sigma} \) [cf. [MT], Definition 2.1, (i)]; \( \Pi_{d,\Sigma}^* \) the pro-\( \Sigma \)-configuration space group [cf. [MT], Definition 2.3, (i)] obtained by forming the maximal pro-\( \Sigma \) quotient of the étale fundamental group \( \pi_1(X_{d,\Sigma}) \) of \( X_{d,\Sigma} \).

\( \text{(i) We shall say that an isomorphism of profinite groups } \alpha : \Pi_{d,\Sigma}^* \rightarrow \Pi_{d,\Sigma}^* \text{ is PF-admissible [i.e., “permutation-fiber-admissible”] if } \alpha \text{ induces a bijection between the set of fiber subgroups [cf. [MT], Definition 2.3, (iii)] of } \Pi_{d,\Sigma}^*, \text{ and the set of fiber subgroups of } \Pi_{d,\Sigma}^*. \)

\( \text{We shall say that an outer isomorphism } \Pi_{d,\Sigma}^* \rightarrow \Pi_{d,\Sigma}^* \text{ is PF-admissible if it is determined by a PF-admissible isomorphism.} \)

\( \text{(ii) We shall say that an isomorphism of profinite groups } \alpha : \Pi_{d,\Sigma}^* \rightarrow \Pi_{d,\Sigma}^* \text{ is PC-admissible [i.e., “permutation-cusp-admissible”] if the following condition is satisfied: Let} \)

\[ \{1\} = K_0 \subseteq K_{d-1} \subseteq \cdots \subseteq K_m \subseteq \cdots \subseteq K_2 \subseteq K_1 \subseteq K_0 = \Pi_{d,\Sigma}^* \]

be the standard fiber filtration of \( \Pi_{d,\Sigma}^* \) [cf. [CmbCsp], Definition 1.1, (i)]; then for any integer \( 1 \leq a \leq d \), the image \( \alpha(K_a) \subseteq \Pi_{d,\Sigma}^* \) is a fiber subgroup of \( \Pi_{d,\Sigma}^* \) of length \( d^a - a \) [cf. [MT], Definition 2.3, (iii)], and, moreover, the isomorphism \( K_{a-1}/K_a \rightarrow \alpha(K_{a-1})/\alpha(K_a) \) determined by \( \alpha \) induces a bijection between the set of cuspidal inertia subgroups of \( K_{a-1}/K_a \) and the set of cuspidal inertia subgroups of \( \alpha(K_{a-1})/\alpha(K_a) \). [Note that it follows immediately from the various definitions involved that the profinite group \( K_{a-1}/K_a \) (respectively, \( \alpha(K_{a-1})/\alpha(K_a) \)) is equipped with a natural structure of pro-\( \Sigma \) (respectively, pro-\( \Sigma^* \)) surface group [cf. [MT], Definition 1.2].] We shall say that an outer isomorphism \( \Pi_{d,\Sigma}^* \rightarrow \Pi_{d,\Sigma}^* \) is PC-admissible if it is determined by a PC-admissible isomorphism.

\( \text{(iii) We shall say that an isomorphism of profinite groups } \alpha : \Pi_{d,\Sigma}^* \rightarrow \Pi_{d,\Sigma}^* \text{ is PFC-admissible [i.e., “permutation-fiber-cusp-admissible”] if } \alpha \text{ is PF-admissible and PC-admissible. We shall say that an outer isomorphism } \Pi_{d,\Sigma}^* \rightarrow \Pi_{d,\Sigma}^* \text{ is PFC-admissible if it is determined by a PFC-admissible isomorphism.} \)
We shall say that an isomorphism of profinite groups \( \alpha : \Pi^{d^e}_{d^e} \rightarrow \Pi^{\bullet d^e}_{d^e} \) is PF-cuspidalizable if there exists a commutative diagram

\[
\begin{array}{ccc}
\Pi^{d^e+1}_{d^e} & \sim & \Pi^{\bullet d^e+1}_{d^e} \\
\downarrow & & \downarrow \\
\Pi^{d^e}_{d^e} & \sim & \Pi^{\bullet d^e}_{d^e}
\end{array}
\]

where the upper horizontal arrow is a PF-admissible isomorphism, and the left-hand (respectively, right-hand) vertical arrow is the surjection obtained by forming the quotient by a fiber subgroup of length 1 [cf. [MT], Definition 2.3, (iii)] of \( \Pi^{d^e+1}_{d^e} \) (respectively, \( \Pi^{\bullet d^e+1}_{d^e} \)). We shall say that an outer isomorphism \( \alpha : \Pi^{d^e}_{d^e} \rightarrow \Pi^{\bullet d^e}_{d^e} \) is PF-cuspidalizable if it is determined by a PF-cuspidalizable isomorphism.

\[\text{Proposition 1.5 (Properties of PF-admissible isomorphisms).}\] In the notation of Definition 1.4, let \( \alpha : \Pi^{d^e}_{d^e} \rightarrow \Pi^{\bullet d^e}_{d^e} \) be an isomorphism. Then the following hold:

(i) \( \Sigma^\circ = \Sigma^\bullet \).
(ii) Suppose that the isomorphism $\alpha$ is PF-admissible. Let $1 \leq n \leq d^o$ be an integer and $H \subseteq \Pi^o_{d^o}$ a fiber subgroup of length $n$ of $\Pi^o_{d^o}$. Then the subgroup $\alpha(H) \subseteq \Pi^o_{d^o}$ is a fiber subgroup of length $n$ of $\Pi^o_{d^o}$. In particular, it holds that $d^o = d^o$.

(iii) Write $\Xi^o \subseteq \Pi^o_{d^o}$ (respectively, $\Xi^* \subseteq \Pi^*_{d^o}$) for the normal closed subgroup of $\Pi^o_{d^o}$ (respectively, $\Pi^*_{d^o}$) obtained by taking the intersection of the various fiber subgroups of length $d^o - 1$ (respectively, $d^* - 1$). Then the isomorphism $\alpha$ is PF-admissible if and only if $\alpha$ induces an isomorphism $\Xi^o \cong \Xi^*$. 

Proof. Assertion (i) follows immediately from the [easily verified] fact that $\Sigma^o$ may be characterized as the smallest set of primes $\Sigma^o$ for which $\Pi^o_{d^o}$ is pro-$\Sigma^o$. Assertion (ii) follows immediately from the various definitions involved. Finally, we verify assertion (iii). The necessity of the condition follows immediately from assertion (ii). The sufficiency of the condition follows immediately from a similar argument to the argument used in the proof of [CmbCsp], Proposition 1.2, (i). This completes the proof of assertion (iii). Q.E.D.

Lemma 1.6 (C-admissibility of certain isomorphisms). In the notation of Definition 1.4, let $\alpha_2: \Pi^o_2 \to \Pi^o_2$, $\alpha^1_1: \Pi^o_1 \to \Pi^o_1$, $\alpha^2_1: \Pi^o_1 \to \Pi^o_1$ be isomorphisms of profinite groups which, for $i = 1, 2$, fit into a commutative diagram

\[
\begin{array}{ccc}
\Pi^o_2 & \xrightarrow{\alpha_2} & \Pi^o_2 \\
pr^i_2 & & pr^i_2 \\
\Pi^o_1 & \xrightarrow{\alpha^1_1} & \Pi^o_1
\end{array}
\]

— where the vertical arrow “$pr^i_2$” is the surjection induced by the projection “$X^o_2 \to X^o_1$” obtained by projecting to the $i$-th factor. Then the isomorphism $\alpha^1_1$ is C-admissible. In particular, $(g^o, r^o) = (g^*, r^*)$.

Proof. Write $\Sigma \overset{\text{def}}{=} \Sigma^o = \Sigma^*$ [cf. Proposition 1.5, (i)]. Now it follows from the well-known structure of the maximal pro-$\Sigma$ quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $p \not\equiv 0$ that $\Pi^o_1$ is a free pro-$\Sigma$ group if and only if $r^o \neq 0$ [cf. [CmbGC], Remark 1.1.3]. Thus, if $r^o = r^* = 0$, then it is immediate that $\alpha^1_1$ is C-admissible; moreover, it follows, by considering the rank of the abelianization of $\Pi^o_1$ [cf. [CmbGC], Remark 1.1.3], that $g^o = g^*$. In particular, to verify Lemma 1.6, we may assume without loss
of generality that \( r^\circ, r^* \neq 0 \). Then it follows from [CmbGC], Theorem 1.6, (i), that, to verify Lemma 1.6, it suffices to show that \( \alpha^1_1 \) is numerically cuspidal [cf. [CmbGC], Definition 1.4, (ii)], i.e., to show that the following assertion holds:

Let \( \Pi_Y^0 \subseteq \Pi_Y^1 \) be an open subgroup of \( \Pi_Y^1 \). Write

\[
\Pi_Y^0 \defeq \alpha_1^1(\Pi_Y^0) \subseteq \Pi_Y^1, \quad Y^\circ \to X^\circ \text{ (respectively, } Y^* \to X^*)
\]

for the connected finite étale covering of \( X^\circ \) (respectively, \( X^* \)) corresponding to the open subgroup \( \Pi_Y^0 \subseteq \Pi_Y^1 \) (respectively, \( \Pi_Y^* \subseteq \Pi_Y^1 \)), and \( (g_Y^r, r_Y^r) \) (respectively, \( (g_Y^*, r_Y^*) \)) for the type of \( Y^\circ \) (respectively, \( Y^* \)). Then it holds that \( r_Y^r = r_Y^* \).

On the other hand, in the notation of the above assertion, one verifies easily that for any \( l \in \Sigma \) and \( \square \in \{\circ, \bullet\} \), if \( \Pi_{Y^\square}^0 \subseteq \Pi_{Y^\square}^1 \) is an open subgroup of \( \Pi_{Y^\square}^1 \) contained in \( \Pi_{Y^\square}^2 \), then the natural inclusion \( \Pi_{Y^\square}^2 \hookrightarrow \Pi_{Y^\square}^1 \) induces a surjection

\[
\text{Ker}((\Pi_{Y^\square}^2)^{ab}) \to (\Pi_{Y^\square}^1)^{ab}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \text{Ker}((\Pi_{Y^\square}^2)^{ab}) \to (\Pi_{Y^\square}^1)^{ab}) \otimes_{\mathbb{Z}} \mathbb{Q}_l
\]

where we write \( (\Pi_{Y^\square}^1), (\Pi_{Y^\square}^2) \) for the maximal pro-\( \Sigma \)-quotients of the étale fundamental groups of the compactifications \( Y, Y^\square \) of \( Y, Y^\square \), respectively. Thus, since any open subgroup of \( \Pi_Y^0 \) contains a characteristic open subgroup of \( \Pi_Y^1 \), it follows immediately from the well-known fact that for \( \square \in \{\circ, \bullet\} \), \( (\Pi_{Y^\square}^1)^{ab} \) (respectively, \( (\Pi_{Y^\square}^2)^{ab} \)) is a free \( \mathbb{Z}_{\Sigma} \)-module of rank \( 2g_Y^\square + r_Y^\square - 1 \) (respectively, \( 2g_Y^\square \)) [cf., e.g., [CmbGC], Remark 1.1.3] that to verify the above assertion, it suffices to verify that if \( \Pi_Y^0 \subseteq \Pi_Y^1 \) in the above assertion is characteristic, then the isomorphism \( \Pi_Y^0 \simeq \Pi_Y^1 \) determined by \( \alpha_1^1 \) induces an isomorphism of \( \text{Ker}((\Pi_{Y^\square}^2)^{ab}) \to (\Pi_{Y^\square}^1)^{ab}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \) with \( \text{Ker}((\Pi_{Y^\square}^1)^{ab}) \to (\Pi_{Y^\square}^2)^{ab}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \) for some \( l \in \Sigma \).

To this end, for \( \square \in \{\circ, \bullet\} \), write \( \Pi_{Y^\square}^2 \subseteq \Pi_{Y^\square}^3 \) for the normal open subgroup of \( \Pi_{Y^\square}^3 \) obtained by forming the inverse image via the surjection

\[
(pr_{\{1\}}^\square, pr_{\{2\}}^\square) : \Pi_Y^2 \to \Pi_{Y^\square}^2 \times \Pi_{Y^\square}^1
\]

of the image of the natural inclusion \( \Pi_{Y^\square}^2 \times \Pi_{Y^\square}^1 \hookrightarrow \Pi_{Y^\square}^1 \times \Pi_{Y^\square}^1 \); \( Z^\square \to X_{Y^\square}^\square \) for the connected finite étale covering corresponding to this normal open subgroup \( \Pi_{Y^\square}^2 \subseteq \Pi_{Y^\square}^3 \); \( \Pi_{Z/Y}^2 \) for the kernel of the natural surjection \( \Pi_{Y^\square}^2 \to \Pi_{Y^\square}^3 \) induced by the composite \( Z^\square \to X_{Y^\square}^\square \hookrightarrow X^\square \times_k X^\square \to X^\square \). Then the natural surjection \( \Pi_{Y^\square}^3 \to \Pi_{Y^\square}^2 \) determines a representation

\[
\Pi_{Y^\square}^2 \to \text{Aut}((\Pi_{Z/Y}^2)^{ab})
\]
moreover, the isomorphisms $\alpha_2$, $\alpha_1^1$, and $\alpha_1^2$ determine a commutative diagram

$$
\begin{array}{ccc}
\Pi^2_Y & \longrightarrow & \text{Aut}(\Pi^2_{Z/Y})^{ab} \\
\downarrow & & \downarrow \\
\Pi^1_\bullet & \longrightarrow & \text{Aut}(\Pi^1_{Z/Y})^{ab}
\end{array}
$$

— where the vertical arrows are isomorphisms. [Here, we note that since $\Pi^2_Y$ is a characteristic subgroup of $\Pi^1_\bullet$, and the composite $\alpha_1^1 \circ (\alpha_1^1)^{-1}$ is an automorphism of $\Pi^1_\bullet$, it follows that $\Pi^1_\bullet = \alpha_1^1(\Pi^2_Y)$, hence that $\alpha_2$ induces an isomorphism $\Pi^2_Z \cong \Pi^1_Z$.] On the other hand, it follows from the definition of $Z$ that $Z$ is isomorphic to the open subscheme of $Y^\boxtimes_k Y^\boxtimes$ obtained by forming the complement of the graphs of the various elements of $\text{Aut}(Y^\boxtimes / X^\boxtimes)$. Thus, it follows from Lemma 1.3, (iv) — by replacing the various profinite groups involved by their maximal pro-$l$ quotients for some $l \in \Sigma$ — that the isomorphism $\Pi^1_\bullet \cong \Pi^1_Y$ determined by $\alpha_1^1$ induces an isomorphism of $\text{Ker}(\Pi^1_Y^{ab} \rightarrow (\Pi^1_Y)^{ab}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ for some $l \in \Sigma$. This completes the proof of Lemma 1.6. Q.E.D.

**Lemma 1.7 (PFC-admissibility of certain PF-admissible isomorphisms).** In the notation of Definition 1.4, let $\alpha : \Pi^2_{\triangleleft \bullet} \rightarrow \Pi^1_{\triangleleft \bullet}$ be a PF-admissible isomorphism. Then the following condition implies that the isomorphism $\alpha$ is PFC-admissible:

Let $H^\circ \subseteq \Pi^2_{\triangleleft \bullet}$ be a fiber subgroup of length 1 [cf. [MT], Definition 2.3, (iii)]. Write $H^\bullet \overset{\text{def}}{=} \alpha(H^\circ) \subseteq \Pi^1_{\triangleleft \bullet}$, for the fiber subgroup of length 1 obtained as the image of $H^\circ$ via $\alpha$ [cf. Proposition 1.5, (ii)].

[Thus, it follows immediately from the various definitions involved that $H^\circ$ (respectively, $H^\bullet$) is equipped with a natural structure of pro-$\Sigma^\circ$ (respectively, pro-$\Sigma^\bullet$) surface group.] Then the isomorphism $H^\circ \cong H^\bullet$ induced by $\alpha$ is C-admissible.

**Proof.** Let $\triangle \in \{\circ, \bullet\}$. Then one may verify easily that the following fact holds:

Let $1 \leq a \leq d^\triangle$ be an integer and $F' \subseteq F \subseteq \Pi^\triangle_{d^\triangle}$ fiber subgroups of $\Pi^\triangle_{d^\triangle}$ such that $F$ is of length $a$, and $F'$ is of length $a - 1$. Then there exists a fiber
subgroup $H \subseteq F \subseteq \Pi^\square_d$ of length 1 such that the composite

$$H \twoheadrightarrow F \twoheadrightarrow F/F'$$

arises from a natural open immersion of a hyperbolic curve of type $(g^\square, r^\square + d^\square - 1)$ into a hyperbolic curve of type $(g^\square, r^\square + d^\square - \alpha)$. [Note that it follows immediately from the various definitions involved that $H$ (respectively, $F/F'$) is equipped with a natural structure of pro-$\Sigma^\square$ surface group.] In particular, the composite is a surjection whose kernel is topologically normally generated by suitable cuspidal inertia subgroups of $H$; moreover, any cuspidal inertia subgroup of $F/F'$ may be obtained as the image of a cuspidal inertia subgroup of $H$.

On the other hand, one may verify easily that Lemma 1.7 follows immediately from the above fact. This completes the proof of Lemma 1.7.

Q.E.D.

Theorem 1.8 (PFC-admissibility of certain isomorphisms).
For $\square \in \{\circ, \bullet\}$, let $\Sigma^\square$ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; $(g^\square, r^\square)$ a pair of nonnegative integers such that $2g^\square - 2 + r^\square > 0$; $X^\square$ a hyperbolic curve of type $(g^\square, r^\square)$ over an algebraically closed field of characteristic $\neq 2$; $d^\square$ a positive integer; $\Pi^\square_d$ the pro-$\Sigma^\square$ configuration space group [cf. [MT], Definition 2.3, (i)] obtained by forming the maximal pro-$\Sigma^\square$ quotient of the étale fundamental group of the $d^\square$-th configuration space of $X^\square$;

$$\alpha: \Pi^\square_{d^\circ} \twoheadrightarrow \Pi^\square_{d^\bullet},$$

an isomorphism of [abstract] groups. If

$$\{(g^\circ, r^\circ), (g^\bullet, r^\bullet)\} \cap \{(0, 3), (1, 1)\} \neq \emptyset,$$

then we suppose further that the isomorphism $\alpha$ is PF-admissible [cf. Definition 1.4, (i)]. Then the following hold:

(i) $\Sigma^\circ = \Sigma^\bullet$.

(ii) The isomorphism $\alpha$ is an isomorphism of profinite groups.

(iii) The isomorphism $\alpha$ is PF-admissible. In particular, $d^\circ = d^\bullet$. 

(iv) If $\alpha$ is PF-cuspidalizable [cf. Definition 1.4, (iv)], then $\alpha$ is PFC-admissible [cf. Definition 1.4, (iii)]. In particular, $(g^*, r^*) = (g^*, r^*)$.

Proof. Assertion (ii) follows from [NS], Theorem 1.1. In light of assertion (ii), assertion (i) follows from Proposition 1.5, (i). Assertion (iii) follows from Proposition 1.5, (ii); [MT], Corollary 6.3, together with the assumption appearing in the statement of Theorem 1.8. Assertion (iv) follows immediately from Lemmas 1.6, 1.7. Q.E.D.

Corollary 1.9 (F-admissibility and FC-admissibility). Let $\Sigma$ be a set of prime numbers which is either of cardinality one or equal to the set of all prime numbers; $n$ a positive integer; $(g, r)$ a pair of nonnegative integers such that $2g + 2 + r > 0$; $X$ a hyperbolic curve of type $(g, r)$ over an algebraically closed field $k$ of characteristic $\not= 2$; $X_n$ the $n$-th configuration space of $X$; $\Pi_n$ the maximal pro-$\Sigma$ quotient of the fundamental group of $X_n$; “Out$^F (\cdot )$” the subgroups of FC- and F-admissible [cf. [CmbCsp], Definition 1.1, (ii)] outomorphisms [cf. the discussion entitled “Topological groups” in §7 of “(\cdot )”]. Then the following hold:

(i) Let $\alpha \in \text{Out}^F (\Pi_{n+1})$. Then $\alpha$ induces the same outomorphism of $\Pi_n$ relative to the various quotients $\Pi_{n+1} \twoheadrightarrow \Pi_n$ by fiber subgroups of length 1 [cf. [MT], Definition 2.3, (iii)]. In particular, we obtain a natural homomorphism

$$\text{Out}^F (\Pi_{n+1}) \rightarrow \text{Out}^F (\Pi_n).$$

(ii) The image of the homomorphism

$$\text{Out}^F (\Pi_{n+1}) \rightarrow \text{Out}^F (\Pi_n)$$

of (i) is contained in

$$\text{Out}^F (\Pi_n) \leq \text{Out}^F (\Pi_n).$$

Proof. First, we verify assertion (i). Let $H^1, H^2 \subseteq \Pi_{n+1}$ be two distinct fiber subgroups of $\Pi_{n+1}$ of length 1. Observe that the normal closed subgroup $H \subseteq \Pi_{n+1}$ of $\Pi_{n+1}$ topologically generated by $H^1$ and $H^2$ is a fiber subgroup of $\Pi_{n+1}$ of length 2 [cf. [MT], Proposition 2.4, (iv)], hence is equipped with a natural structure of pro-$\Sigma$ configuration space group, with respect to which $H^1 \leq H$ may be regarded as a fiber
subgroup of length 1 [cf. [MT], Proposition 2.4, (ii)]. Moreover, it follows immediately from the scheme-theoretic definition of the various configuration space groups involved that one has natural outer isomorphisms $\Pi_{n+1}/H_i \cong \Pi_n$ and $H/H_i \cong H/H_1^i$. Thus, since for $i \in \{1, 2\}$, we have natural outer isomorphisms

$$\Pi_n \cong \Pi_{n+1}/H_i \cong (H/H_i)^{\text{out}} \cong \Pi_{n+1}/H$$

[cf. the discussion entitled “Topological groups” in §0] which are compatible with the various natural outer isomorphisms discussed above, one verifies easily [cf. the argument given in the first paragraph of the proof of [CmbCsp], Theorem 4.1] that to complete the proof of assertion (i), by replacing $n+1$ by $H_i$, it suffices to verify assertion (i) in the case where $n = 1$. The rest of the proof of assertion (i) is devoted to verifying assertion (i) in the case where $n = 1$.

Let $e \in \text{Aut}^F(\Pi_2)$ be an $F$-admissible automorphism of $\Pi_2$; $e^1$, $e^2 \in \text{Aut}(\Pi_1)$ the automorphisms of $\Pi_1$ induced by $e$ relative to the quotients $\Pi_2 \twoheadrightarrow \Pi_2/H_i \twoheadrightarrow \Pi_1$, $\Pi_2 \twoheadrightarrow \Pi_2/H_1 \twoheadrightarrow \Pi_1$, respectively. Now it is immediate that to complete the proof of assertion (i), it suffices to verify that the difference $e^1 \circ (e^2)^{-1} \in \text{Aut}(\Pi_1)$ is $\Pi_1$-inner. Therefore, it follows immediately from [JR], Theorem B, that to complete the proof of assertion (i), it suffices to verify that

$$(*)_1: \text{for any normal open subgroup } N \subseteq \Pi_1 \text{ of } \Pi_1,$$

it holds that $\tilde{\alpha}^1(N) = \tilde{\alpha}^2(N)$. To this end, let $N \subseteq \Pi_1$ be a normal open subgroup of $\Pi_1$. Write $\Pi_N \overset{\text{def}}{=} \Pi_2 \times_{\Pi_1} N$ for the fiber product of $\Pi_2 \to \Pi_2/H_i \twoheadrightarrow \Pi_1$ and $N \twoheadrightarrow \Pi_1$ and for the kernel of the composite $\Pi_N = \Pi_2 \times_{\Pi_1} N \overset{pr_2}{\twoheadrightarrow} \Pi_2 \twoheadrightarrow \Pi_2/H_1 \twoheadrightarrow \Pi_1$. Then the surjection $\Pi_N \twoheadrightarrow N \times \Pi_1$ determined by the natural surjection $\Pi_N \twoheadrightarrow \Pi_N/F_N \twoheadrightarrow \Pi_1$ and the second projection $\Pi_N = \Pi_2 \times_{\Pi_1} N \overset{pr_2}{\twoheadrightarrow} N$ fits into a commutative diagram of profinite groups

$$
\begin{array}{cccc}
1 & \rightarrow & F_N & \rightarrow & \Pi_N & \rightarrow & \Pi_1 & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & N & \rightarrow & N \times \Pi_1 & \overset{pr_2}{\rightarrow} & \Pi_1 & \rightarrow & 1
\end{array}
$$

— where the horizontal sequences are exact, and the vertical arrows are surjective. Write $\rho_N: \Pi_1 \rightarrow \text{Aut}(F_N^{ab})$ for the natural action determined by the upper horizontal sequence and $V_N \subseteq F_N^{ab}$ for the kernel of the natural surjection $F_N^{ab} \twoheadrightarrow N^{ab}$ induced by the left-hand vertical arrow.

Now we claim that
(*2): the action $\rho_N$ of $\Pi_1$ on $F_N^{ab}$ preserves $V_N \subseteq F_N^{ab}$, and, moreover, the resulting action $\rho_N^V : \Pi_1 \to \text{Aut}(V_N)$ factors as the composite

$$\Pi_1 \to \Pi_1/N \hookrightarrow \text{Aut}(V_N)$$

— where the second arrow is injective.

Indeed, the fact that the action $\rho_N$ of $\Pi_1$ on $F_N^{ab}$ preserves $V_N \subseteq F_N^{ab}$ follows immediately from the definition of $\rho_N$ [cf. also the above commutative diagram]. Next, let us observe that it follows immediately from the various definitions involved that if we write $f : Y \to X$ for the connected finite étale Galois covering of $X$ corresponding to $N \subseteq \Pi_1$, then the right-hand square of the above diagram arises from a commutative diagram of schemes

$$\begin{array}{ccc}
(Y \times_k X) \setminus \Gamma_f & \to & X \\
\downarrow & & \parallel \\
Y \times_k X & \to & X
\end{array}$$

— where we write $\Gamma_f \subseteq Y \times_k X$ for the graph of $f$, and the left-hand vertical arrow is the natural open immersion. Thus, it follows immediately from a similar argument to the argument used in the proof of Lemma 1.1, (i) [cf. also [Hsh], Proposition 1.4, (i)], that $F_N, N$ are naturally isomorphic to the maximal pro-$\Sigma$ quotients of the étale fundamental groups of geometric fibers of the families of hyperbolic curves $Y \times_k X \setminus \Gamma_f, Y \times_k X \to X$ over $X$, respectively. Therefore, by the well-known structure of the maximal pro-$\Sigma$ quotient of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\not\in \Sigma$, we conclude — by considering the natural action of $\Pi_1$ on the set of cusps of the family of hyperbolic curves $Y \times_k X \setminus \Gamma_f \to X$ — that the resulting action $\rho_N^V : \Pi_1 \to \text{Aut}(V_N)$ factors as the composite $\Pi_1 \to \Pi_1/N \to \text{Aut}(V_N)$, and that if $X$ is affine (respectively, proper), then for any $l \in \Sigma$, the resulting representation $\Pi_1/N \to \text{Aut}(V_N \otimes \mathbb{Z}_l \otimes \mathbb{Q}_l)$ is isomorphic to

- the regular representation of $\Pi_1/N$ over $\mathbb{Q}_l$, respectively,
- the quotient of the regular representation of $\Pi_1/N$ over $\mathbb{Q}_l$ by the trivial subrepresentation (of dimension 1).

In particular, as is well-known, the homomorphism $\Pi_1/N \to \text{Aut}(V_N \otimes \mathbb{Z}_l \otimes \mathbb{Q}_l)$, hence also the homomorphism $\Pi_1/N \to \text{Aut}(V_N)$, is injective. This completes the proof of the claim (*2).
Next, let us observe that since $\tilde{\alpha}$ is $F$-admissible, it follows immediately from the definition of $\rho^V_N$ that the automorphism $\tilde{\alpha}$ induces a commutative diagram

\[
\begin{array}{ccc}
\Pi_1 & \xrightarrow{\rho^V_N} & \text{Aut}(V_N) \\
\tilde{\alpha}^2 \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\Pi_1 & \xrightarrow{\rho^V_{\tilde{\alpha}^2(N)}} & \text{Aut}(V_{\tilde{\alpha}^2(N)})
\end{array}
\]

— where the vertical arrows are isomorphisms that are induced by $\tilde{\alpha}$. Thus, by considering the kernels of $\rho^V_N$, $\rho^V_{\tilde{\alpha}^2(N)}$, one concludes from the claim $(\ast_2)$ that $\tilde{\alpha}^1(N) = \tilde{\alpha}^2(N)$. This completes the proof of $(\ast_1)$, hence also of assertion (i).

Assertion (ii) follows immediately from Theorem 1.8, (iv) [cf. also Remark 1.4.1]. This completes the proof of Corollary 1.9. Q.E.D.

Remark 1.9.1. The discrete versions of Theorem 1.8, Corollary 1.9 will be discussed in a sequel to the present paper.

§2. Various operations on semi-graphs of anabelioids of PSC-type

In the present §, we study various operations on semi-graphs of anabelioids of PSC-type. These operations include the following:

(Op1) the operation of restriction to a sub-semi-graph [satisfying certain conditions] of the underlying semi-graph [cf. Definition 2.2, (ii); Fig. 2 below],

(Op2) the operation of partial compactification [cf. Definition 2.4, (ii); Fig. 3 below],

(Op3) the operation of resolution of a given set [satisfying certain conditions] of nodes [cf. Definition 2.5, (ii); Fig. 4 below], and

(Op4) the operation of generization [cf. Definition 2.8; Fig. 5 below].

A basic reference for the theory of semi-graphs of anabelioids of PSC-type is [CmbGC]. We shall use the terms "semi-graph of anabelioids of PSC-type", "PSC-fundamental group of a semi-graph of anabelioids of
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Let \( \Sigma \) be a nonempty set of prime numbers and \( G \) a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type. Write \( \tilde{G} \) for the underlying semi-graph of \( G \), \( \Pi_G \) for the [pro-\( \Sigma \)] fundamental group of \( G \), and \( \tilde{G} \to G \) for the universal covering of \( \tilde{G} \) corresponding to \( \Pi_G \). Then since the fundamental group \( \Pi_G \) of \( G \) is topologically finitely generated, the profinite topology of \( \Pi_G \) induces [profinite] topologies on \( \text{Aut}(\Pi_G) \) and \( \text{Out}(\Pi_G) \) [cf. the discussion entitled “Topological groups” in \( \leq 0 \)]. If, moreover, we write \( \text{Aut}(\tilde{G}) \) for the automorphism group of \( \tilde{G} \), then by the discussion preceding [CmbGC], Lemma 2.1, the natural homomorphism

\[
\text{Aut}(\tilde{G}) \to \text{Out}(\Pi_G)
\]

is an injection with closed image. [Here, we recall that an automorphism of a semi-graph of anabelioids consists of an automorphism of the underlying semi-graph, together with a compatible system of isomorphisms between the various anabelioids at each of the vertices and edges of the underlying semi-graph which are compatible with the various morphisms of anabelioids associated to the branches of the underlying semi-graph — cf. [SemiAn], Definition 2.1; [SemiAn], Remark 2.4.2.] Thus, by equipping \( \text{Aut}(\tilde{G}) \) with the topology induced via this homomorphism by the topology of \( \text{Out}(\Pi_G) \), we may regard \( \text{Aut}(\tilde{G}) \) as being equipped with the structure of a profinite group.

**Definition 2.1.**

(i) For \( z \in \text{VCN}(\tilde{G}) \) such that \( z \in \text{Vert}(\tilde{G}) \) (respectively, \( z \in \text{Edge}(\tilde{G}) \)), we shall say that a closed subgroup of \( \Pi_G \) is a \( \text{VCN-subgroup} \) of \( \Pi_G \) associated to \( z \in \text{VCN}(\tilde{G}) \) if the closed subgroup is a verticial (respectively, an edge-like) subgroup of
\( \Pi_G \) associated to \( z \in VCN(\mathcal{G}) \). For \( \tilde{z} \in VCN(\tilde{\mathcal{G}}) \) such that \( \tilde{z} \in \text{Vert}(\tilde{\mathcal{G}}) \) (respectively, \( \tilde{z} \in \text{Edge}(\tilde{\mathcal{G}}) \)), we shall say that a closed subgroup of \( \Pi_G \) is the \textit{VCN-subgroup} of \( \Pi_G \) associated to \( \tilde{z} \in VCN(\tilde{\mathcal{G}}) \) if the closed subgroup is the vertical (respectively, edge-like) subgroup of \( \Pi_G \) associated to \( \tilde{z} \in VCN(\tilde{\mathcal{G}}) \) [cf. [NodNon], Definition 1.1, (vi)].

(ii) For \( z \in VCN(\mathcal{G}) \), we shall write

\[ \mathcal{G}_z \]

for the anabelioid corresponding to \( z \in VCN(\mathcal{G}) \).

(iii) For \( v \in \text{Vert}(\mathcal{G}) \), we shall write

\[ \mathcal{G}|_v \]

for the semi-graph of anabelioids of pro-\( \Sigma \) PSC-type defined as follows [cf. Fig. 1 below]: We take \( \text{Vert}(\mathcal{G}|_v) \) to consist of the single element “\( v \)”, \( \text{Cusp}(\mathcal{G}|_v) \) to be the set of branches of \( \mathcal{G} \) which abut to \( v \), and \( \text{Node}(\mathcal{G}|_v) \) to be the empty set. We take the anabelioid of \( \mathcal{G}|_v \) corresponding to the unique vertex “\( v \)” to be \( \mathcal{G}_v \) [cf. (ii)]. For each edge \( e \in \mathcal{E}(v) \) of \( \mathcal{G} \) and each branch \( b \) of \( e \) that abuts to the vertex \( v \), we take the anabelioid of \( \mathcal{G}|_v \) corresponding to the branch \( b \) to be a copy of the anabelioid \( \mathcal{G}_b \) [cf. (ii)]. For each edge \( e \in \mathcal{E}(v) \) of \( \mathcal{G} \) and each branch \( b \) of \( e \) that abuts, relative to \( \mathcal{G} \), to the vertex \( v \), we take the morphism of anabelioids \( (\mathcal{G}|_v)_e \to (\mathcal{G}|_v)_b \) of \( \mathcal{G}|_v \) — where we write \( e_b \) for the cusp of \( \mathcal{G}|_v \) corresponding to \( b \) — to be the morphism of anabelioids \( \mathcal{G}_e \to \mathcal{G}_b \) associated, relative to \( \mathcal{G} \), to the branch \( b \). Thus, one has a \textit{natural morphism}

\[ \mathcal{G}|_v \to \mathcal{G} \]

of semi-graphs of anabelioids.

\textbf{Remark 2.1.1.} Let \( v \in \text{Vert}(\mathcal{G}) \) be a vertex of \( \mathcal{G} \) and \( \Pi_v \subseteq \Pi_G \) a vertical subgroup of \( \Pi_G \) associated to \( v \in \text{Vert}(\mathcal{G}) \). Then it follows immediately from the various definitions involved that the fundamental group of \( \mathcal{G}|_v \) is \textit{naturally isomorphic} to \( \Pi_v \), and that we have a natural identification

\[ \text{Aut}(\mathcal{G}|_v) \simeq \text{Out}(\Pi_v) \]
and a natural injection

\[ \text{Aut}(\mathcal{G}_v) \hookrightarrow \text{Aut}(\mathcal{G}_v). \]

![Diagram](image)

**Figure 1:** $\mathcal{G}_v$

**Definition 2.2** (cf. the operation (Op1) discussed at the beginning the present §2).

(i) Let $K$ be a [not necessarily finite] semi-graph and $H$ a sub-semi-graph of $K$ [cf. [SemiAn], the discussion following the figure entitled “A Typical Semi-graph”]. Then we shall say that $H$ is of $PSC$-type if the following three conditions are satisfied:

1. $H$ is finite [i.e., the set consisting of vertices and edges of $H$ is finite] and connected.
2. $H$ has at least one vertex.
3. If $v$ is a vertex of $H$, and $e$ is an edge of $K$ that abuts to $v$, then $e$ is an edge of $H$. [Thus, if $e$ abuts both to a vertex lying in $H$ and to a vertex not lying in $H$, then the resulting edge of $H$ is a “cusp”, i.e., an open edge.]

Thus, a sub-semi-graph of $PSC$-type $H$ is completely determined by the set of vertices that lie in $H$.

(ii) Let $H$ be a sub-semi-graph of $PSC$-type [cf. (i)] of $G$. Then one may verify easily that the semi-graph of anabelioids obtained by restricting $G$ to $H$ [cf. the discussion preceding [SemiAn], Definition 2.2] is of $pro-$-$\Sigma$ $PSC$-type. Here, we recall that the semi-graph of anabelioids obtained by restricting $G$ to $H$ is the semi-graph of anabelioids such that the underlying semi-graph is $H$; for each vertex $v$ (respectively, edge $e$) of $H$, the anabelioid corresponding to $v$ (respectively, $e$) is $G_v$ (respectively,
\( G_e \) [cf. Definition 2.1, (ii)]; for each branch \( b \) of an edge \( e \) of \( \mathbb{H} \) that abuts to a vertex \( v \) of \( \mathbb{H} \), the morphism associated to \( b \) is the morphism \( G_e \to G_v \) associated to the branch of \( G \) corresponding to \( b \). We shall write

\[ G|_\mathbb{H} \]

for this semi-graph of anabelioids of pro-\( \Sigma \) PSC-type and refer to \( G|_\mathbb{H} \) as the semi-graph of anabelioids of pro-\( \Sigma \) PSC-type obtained by restricting \( G \) to \( \mathbb{H} \) [cf. Fig. 2 below]. Thus, one has a natural morphism

\[ G|_\mathbb{H} \to G \]

of semi-graphs of anabelioids.

\[ \mathbb{H}: \text{the sub-semi-graph of PSC-type whose set of vertices} = \{v\} \]

\( \downarrow \)

\[ G|_\mathbb{H} \]

Figure 2: Restriction
**Definition 2.3.** Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\).

(i) We shall say that \(G\) is of type \((g, r)\) if \(G\) arises from a stable log curve of type \((g, r)\) over an algebraically closed field of characteristic \(\neq 2\), i.e., \(\text{Cusp}(G)\) is of cardinality \(r\), and, moreover,

\[
\text{rank}_{\mathbb{Z}}(\Pi_g^{\text{ab}}) = 2g + \text{Cusp}(G)^2 - c_G
\]

— where

\[
c_G \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } \text{Cusp}(G) = \emptyset, \\
1 & \text{if } \text{Cusp}(G) \neq \emptyset.
\end{cases}
\]

[Here, we recall that it follows from the discussion of [CmbGC], Remark 1.1.3, that \(\Pi_g^{\text{ab}}\) is a free \(\mathbb{Z}\)-module of finite rank.]

(ii) Let \(\mathcal{H}\) be a sub-semi-graph of PSC-type [cf. Definition 2.2, (i)]. Then we shall say that \(\mathcal{H}\) is of type \((g, r)\) if the semi-graph of anabelioids \(G_{|\mathcal{H}}\), which is of pro-\(\Sigma\) PSC-type [cf. Definition 2.2, (ii)], is of type \((g, r)\) [cf. (i)].

(iii) Let \(v \in \text{Vert}(G)\) be a vertex. Then we shall say that \(v\) is of type \((g, r)\) if the semi-graph of anabelioids \(G_{|v}\), which is of pro-\(\Sigma\) PSC-type [cf. Definition 2.1, (iii)], is of type \((g, r)\) [cf. (i)].

(iv) We shall say that \(G\) is totally degenerate if each vertex of \(G\) is of type \((0, 3)\) [cf. (iii)].

(v) One may verify easily that there exists a unique, up to isomorphism, semi-graph of anabelioids of pro-\(\Sigma\) PSC-type that is of type \((g, r)\) [cf. (i)] and has no node. We shall write

\[G_{g,r}^{\text{model}}\]

for this semi-graph of anabelioids of pro-\(\Sigma\) PSC-type.

**Remark 2.3.1.** It follows immediately from the various definitions involved that there exists a unique pair \((g, r)\) of nonnegative integers such that \(G\) is of type \((g, r)\) [cf. Definition 2.3, (i)].

**Definition 2.4** (cf. the operation (Op2) discussed at the beginning the present §2).
We shall say that a subset $S \subseteq \text{Cusp}(\mathcal{G})$ of $\text{Cusp}(\mathcal{G})$ is \textit{omittable} if the following condition is satisfied: For each vertex $v \in \text{Vert}(\mathcal{G})$ of $\mathcal{G}$, if $v$ is of type $(g, r)$ [cf. Definition 2.3, (iii); Remark 2.3.1], then it holds that $2g - 2 + r - (E(v) \cap S)^2 > 0$.

Let $S \subseteq \text{Cusp}(\mathcal{G})$ be a subset of $\text{Cusp}(\mathcal{G})$ which is omittable [cf. (i)]. Then by eliminating the cusps [i.e., the open edges] contained in $S$, and, for each vertex $v$ of $\mathcal{G}$, replacing the anabelioid $\mathcal{G}_v$ corresponding to $v$ by the anabelioid of finite étale coverings of $\mathcal{G}_v$ that restrict to a trivial covering over the cusps contained in $S$ that abut to $v$, we obtain a semi-graph of anabelioids $\mathcal{G}_S$ of pro-$\Sigma$ PSC-type. We shall refer to $\mathcal{G}_S$ as the partial compactification of $\mathcal{G}$ with respect to $S$ [cf. Fig. 3 below]. Thus, for each $v \in \text{Vert}(\mathcal{G}) = \text{Vert}(\mathcal{G}_S)$, the pro-$\Sigma$ fundamental group of the anabelioid $(\mathcal{G}_S)_v$ corresponding to $v \in \text{Vert}(\mathcal{G}) = \text{Vert}(\mathcal{G}_S)$ may be naturally identified, up to inner automorphism, with the quotient of a vertical subgroup $v'$ of $\mathcal{G}$ associated to $v \in \text{Vert}(\mathcal{G}) = \text{Vert}(\mathcal{G}_S)$ by the subgroup of topologically normally generated by the $e \in E(v)$ for $e \in E(v) \cap S$.

If, moreover, we write $\mathcal{G}_S$ for the [pro-$\Sigma$] fundamental group of $\mathcal{G}_S$ and $N_S \subseteq \Pi_G$ for the normal closed subgroup of $\Pi_G$ topologically normally generated by the cuspidal subgroups of $\Pi_G$ associated to elements of $S$, then we have a natural outer isomorphism

$$
\Pi_G/N_S \xrightarrow{\sim} \Pi_{\mathcal{G}_S}.
$$

\textbf{Remark 2.4.1.}

(i) Let $S_1 \subseteq S_2 \subseteq \text{Cusp}(\mathcal{G})$ be subsets of $\text{Cusp}(\mathcal{G})$. Then it follows immediately from the various definitions involved that the \textit{omittability} of $S_2$ [cf. Definition 2.4, (i)] implies the \textit{omittability} of $S_1$.

(ii) If $\mathcal{G}$ is \textit{sturdy}, then it follows from the various definitions involved that $\text{Cusp}(\mathcal{G})$, hence also any subset of $\text{Cusp}(\mathcal{G})$ [cf. (i)], is \textit{omittable}. Moreover, the partial compactification of $\mathcal{G}$ with
respect to Cusp(\(G\)) coincides with the compactification of \(G\) [cf. [CmbGC], Remark 1.1.6; [NodNon], Definition 1.11].

\[\begin{align*}
\mathcal{G} \\
\downarrow \\
\mathcal{G}_{\{c_1, c_2\}}
\end{align*}\]

Figure 3: Partial compactification

**Definition 2.5** (cf. the operation (Op3) discussed at the beginning the present §2). Let \(S \subseteq \text{Node}(\mathcal{G})\) be a subset of \(\text{Node}(\mathcal{G})\).

(i) We shall say that \(S\) is of separating type if the semi-graph obtained by removing the closed edges corresponding to the elements of \(S\) from \(\mathcal{G}\) is disconnected. Moreover, for each node \(e \in \text{Node}(\mathcal{G})\), we shall say that \(e\) is of separating type if \(\{e\} \subseteq \text{Node}(\mathcal{G})\) is of separating type.

(ii) Suppose that \(S\) is not of separating type [cf. (i)]. Then one may define a semi-graph of anabelioids of pro-\(\Sigma\) PSC-type as follows: We take the underlying semi-graph \(G_{\succ S}\) to be the semi-graph obtained by replacing each node \(e\) of \(G\) contained in \(S\) such that \(V(e) = \{v_1, v_2\} \subseteq \text{Vert}(\mathcal{G})\) — where \(v_1, v_2\) are not necessarily distinct — by two cusps that abut to \(v_1, v_2 \in \text{Vert}(\mathcal{G})\), respectively. We take the anabelioid corresponding to a vertex \(v\) (respectively, node \(e\)) of \(G_{\succ S}\) to be \(G_v\) (respectively, \(G_e\)). [Note that the set of vertices (respectively, nodes]
of $G_{\succ S}$ may be naturally identified with $\text{Vert}(G)$ (respectively, $\text{Node}(G) \setminus S$). We take the anabelioid corresponding to a cusp of $G_{\succ S}$ arising from a cusp $e$ of $G$ to be $G'_e$. We take the anabelioid corresponding to a cusp of $G_{\succ S}$ arising from a node $e$ of $G$ to be $G_e$. For each branch $b$ of $G_{\succ S}$ that abuts to a vertex $v$ of a node $e$ (respectively, of a cusp $e$ that does not arise from a node of $G$), we take the morphism associated to $b$ to be the morphism $G_e \to G_v$ associated to the branch of $G$ corresponding to $b$. For each branch $b$ of $G_{\succ S}$ that abuts to a vertex $v$ of a cusp of $G_{\succ S}$ that arises from a node $e$ of $G$, we take the morphism associated to $b$ to be the morphism $G_e \to G_v$ associated to the branch of $G$ corresponding to $b$. We shall denote the resulting semi-graph of anabelioids of pro-$\Sigma$ PSC-type by $G_{\succ S}$

and refer to $G_{\succ S}$ as the semi-graph of anabelioids of pro-$\Sigma$ PSC-type obtained from $G$ by resolving $S$ [cf. Fig. 4 below]. Thus, one has a natural morphism

$$G_{\succ S} \to G$$

of semi-graphs of anabelioids.

Remark 2.5.1.

(i) Let $S_1 \subseteq S_2 \subseteq \text{Node}(G)$ be subsets of $\text{Node}(G)$. Then it follows immediately from the various definitions involved that if $S_2$ is not of separating type [cf. Definition 2.5, (i)], then $S_1$ is not of separating type.

(ii) Let $v \in \text{Vert}(G)$ be a vertex of $G$. Then one may verify easily that there exists a unique sub-semi-graph of PSC-type [cf. Definition 2.2, (i)] $G_v$ of $G$ such that the set of vertices of $G_v$ is equal to $\{v\}$. Moreover, one may also verify easily that $\text{Node}(G|_{G_v})$ [cf. Definition 2.2, (ii)] is not of separating type [cf. Definition 2.5, (i)], relative to $G|_{G_v}$, and that the semi-graph of anabelioids of pro-$\Sigma$ PSC-type

$$(G|_{G_v})_{\succ \text{Node}(G|_{G_v})}$$

[cf. Definition 2.5, (ii)] is naturally isomorphic to $G|_v$ [cf. Definition 2.1, (iii)].
Figure 4: Resolution
Definition 2.6.

(i) Let \( S \subseteq \text{VCN}(\mathcal{G}) \) be a subset of \( \text{VCN}(\mathcal{G}) \). Then we shall denote by
\[
\text{Aut}^S(\mathcal{G}) \subseteq \text{Aut}(\mathcal{G})
\]
the [closed] subgroup of \( \text{Aut}(\mathcal{G}) \) consisting of automorphisms \( \alpha \) of \( \mathcal{G} \) such that the automorphism of the underlying semi-graph \( \mathcal{G} \) of \( \mathcal{G} \) induced by \( \alpha \) preserves \( S \) and by
\[
\text{Aut}^{|S|}(\mathcal{G}) \subseteq \text{Aut}^S(\mathcal{G})
\]
the [closed] subgroup of \( \text{Aut}(\mathcal{G}) \) consisting of automorphisms \( \alpha \) of \( \mathcal{G} \) such that the automorphism of the underlying semi-graph \( \mathcal{G} \) of \( \mathcal{G} \) induced by \( \alpha \) preserves and induces the identity automorphism of \( S \). Moreover, we shall write
\[
\text{Aut}^{\text{graph}}(\mathcal{G}) \triangleq \text{Aut}^{\text{VCN}(\mathcal{G})}(\mathcal{G})
\]

(ii) Let \( H \subseteq \Pi_\mathcal{G} \) be a closed subgroup of \( \Pi_\mathcal{G} \). Then we shall denote by
\[
\text{Out}^H(\Pi_\mathcal{G}) \subseteq \text{Out}(\Pi_\mathcal{G})
\]
the [closed] subgroup of \( \text{Out}(\Pi_\mathcal{G}) \) consisting of automorphisms \[\text{cf. the discussion entitled “Topological groups” in §0}\] of \( \Pi_\mathcal{G} \) which preserve the \( \Pi_\mathcal{G} \)-conjugacy class of \( H \subseteq \Pi_\mathcal{G} \). Moreover, we shall denote by
\[
\text{Aut}^H(\mathcal{G}) \triangleq \text{Aut}(\mathcal{G}) \cap \text{Out}^H(\Pi_\mathcal{G})
\]

(iii) Let \( \mathbb{H} \) be a sub-semi-graph of \( \text{PSC-type \[\text{cf. Definition 2.2, (i)}\] of \( \mathcal{G} \). Then \( \text{VCN}(\mathcal{G}[\mathbb{H}]) \) \[\text{cf. Definition 2.2, (ii)}\] may be regarded as a subset of \( \text{VCN}(\mathcal{G}) \). We shall write
\[
\text{Aut}^{\mathbb{H}}(\mathcal{G}) \triangleq \text{Aut}^{\text{VCN}(\mathcal{G}[\mathbb{H}])}(\mathcal{G}) \subseteq \text{Aut}^S(\mathcal{G}) \triangleq \text{Aut}^{\text{VCN}(\mathcal{G}[\mathbb{H}])}(\mathcal{G}) \subseteq \text{Aut}^{\text{Vert}(\mathcal{G}[\mathbb{H}])}(\mathcal{G})
\]

Proposition 2.7 (Subgroups determined by sets of components). Let \( S \subseteq \text{VCN}(\mathcal{G}) \) be a nonempty subset of \( \text{VCN}(\mathcal{G}) \). Then:
(i) It holds that
\[ \text{Aut}^{[S]}(G) = \bigcap_{z \in S} \text{Aut}^{\Pi_z}(G) \]
where we use the notation \( \Pi_z \) to denote a VCN-subgroup [cf. Definition 2.1, (i)] of \( \Pi_G \) associated to \( z \in \text{VCN}(G) \).

(ii) It holds that
\[ \text{Aut}^{[\text{grph}]}(G) = \bigcap_{z \in \text{VCN}(G)} \text{Out}^{\Pi_z}(\Pi_G) \]
where we use the notation \( \Pi_z \) to denote a VCN-subgroup of \( G \) associated to \( z \in \text{VCN}(G) \).

(iii) The closed subgroups \( \text{Aut}^{[S]}(G) \), \( \text{Aut}^{\text{S}}(G) \subseteq \text{Aut}(G) \) are open in \( \text{Aut}(G) \). Moreover, the closed subgroup \( \text{Aut}^{[S]}(G) \subseteq \text{Aut}^{\text{S}}(G) \) is normal in \( \text{Aut}^{\text{S}}(G) \). In particular, \( \text{Aut}^{[\text{grph}]}(G) \subseteq \text{Aut}(G) \) is normal in \( \text{Aut}(G) \).

Proof. Assertion (i) follows immediately from [CmbGC], Proposition 1.2, (i). Next, we verify assertion (ii). It follows immediately from [CmbGC], Proposition 1.5, (ii), that the right-hand side of the equality in the statement of assertion (ii) is contained in \( \text{Aut}(G) \). Thus, assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately from the finiteness of the semi-graph \( G \), together with the various definitions involved.

Q.E.D.

Definition 2.8 (cf. the operation (Op4) discussed at the beginning the present §2). Let \( S \subseteq \text{Node}(G) \) be a subset of \( \text{Node}(G) \). Then we define the semi-graph of anabelioids of pro-\( \Sigma \) PSC-type
\[ G_{\rightarrow S} \]
as follows:

(i) We take \( \text{Cusp}(G_{\rightarrow S}) \stackrel{\text{def}}{=} \text{Cusp}(G) \).

(ii) We take \( \text{Node}(G_{\rightarrow S}) \stackrel{\text{def}}{=} \text{Node}(G) \setminus S \).

(iii) We take \( \text{Vert}(G_{\rightarrow S}) \) to be the set of connected components of the semi-graph obtained from \( G \) by omitting the edges \( e \in \text{Edge}(G) \setminus S \). Alternatively, one may take \( \text{Vert}(G_{\rightarrow S}) \) to be
the set of equivalence classes of elements of $\text{Vert}(\mathcal{G})$ with respect to the equivalence relation “$\sim$” defined as follows: for $v, w \in \text{Vert}(\mathcal{G})$, $v \sim w$ if either $v = w$ or there exist $n$ elements $e_1, \ldots, e_n \in S$ of $S$ and $n + 1$ vertices $v_0, v_1, \ldots, v_n \in \text{Vert}(\mathcal{G})$ of $\mathcal{G}$ such that $v_0 \overset{\text{def}}{=} v$, $v_n \overset{\text{def}}{=} w$, and, for $1 \leq i \leq n$, it holds that $\mathcal{V}(e_i) = \{v_{i-1}, v_i\}$.

(iv) For each branch $b$ of an edge $e \in \text{Edge}(\mathcal{G} \to S)$ ($= \text{Edge}(\mathcal{G}) \setminus S$ — cf. (i), (ii)) and each vertex $v \in \text{Vert}(\mathcal{G} \to S)$ of $\mathcal{G} \to S$, $b$ abuts, relative to $\mathcal{G} \to S$, to $v$ if $b$ abuts, relative to $\mathcal{G}$, to an element of the equivalence class $v$ [cf. (iii)].

(v) For each edge $e \in \text{Edge}(\mathcal{G} \to S)$ ($= \text{Edge}(\mathcal{G}) \setminus S$ — cf. (i), (ii)) of $\mathcal{G} \to S$, we take the anabelioid of $\mathcal{G} \to S$ corresponding to $e \in \text{Edge}(\mathcal{G} \to S)$ to be $\mathcal{G}_e$ [cf. Definition 2.1, (ii)].

(vi) Let $v \in \text{Vert}(\mathcal{G} \to S)$ be a vertex of $\mathcal{G} \to S$. Then one verifies easily that there exists a unique sub-semi-graph of $\text{PSC-type}$ [cf. Definition 2.2, (i)] $\mathbb{H}_v$ of $\mathcal{G}$ such that the set of vertices of $\mathbb{H}_v$ consists of the elements of the equivalence class $v$ [cf. (iii)]. Write

$$T_v \overset{\text{def}}{=} \text{Node}(\mathcal{G}|_{\mathbb{H}_v}) \setminus (S \cap \text{Node}(\mathcal{G}|_{\mathbb{H}_v}))$$

[cf. Definition 2.2, (ii)]. Then we take the anabelioid of $\mathcal{G} \to S$ corresponding to $v \in \text{Vert}(\mathcal{G} \to S)$ to be the anabelioid determined by the finite étale coverings of

$$\mathcal{G}|_{\mathbb{H}_v} \to (\mathcal{G}|_{\mathbb{H}_v})_{T_v}$$

[cf. Definition 2.5, (ii)] of degree a product of primes $\in \Sigma$.

(vii) Let $b$ be a branch of an edge $e \in \text{Edge}(\mathcal{G} \to S)$ ($= \text{Edge}(\mathcal{G}) \setminus S$ — cf. (i), (ii)) that abuts to a vertex $v \in \text{Vert}(\mathcal{G} \to S)$. Then since $b$ abuts to $v$, one verifies easily that there exists a unique vertex $w$ of $\mathcal{G}$ which belongs to the equivalent class $v$ [cf. (iii)] such that $b$ abuts to $w$ relative to $\mathcal{G}$. We take the morphism of anabelioids associated to $b$, relative to $\mathcal{G} \to S$, to be the morphism naturally determined by the morphism of anabelioids

$$\mathcal{G}_e \to \mathcal{G}_w$$

corresponding to the branch $b$ relative to $\mathcal{G}$ and the morphism of semi-graphs of anabelioids of pro-$\Sigma$ PSC-type

$$\mathcal{G}|_w \to (\mathcal{G}|_{\mathbb{H}_v})_{T_v}.$$
[cf. (vi); Definition 2.1, (iii)]. Here, we recall that the anabelioid obtained by considering the connected finite étale coverings of $G_w$ may be naturally identified with $G_w$ [cf. Remark 2.1.1].

We shall refer to this semi-graph of anabelioids of pro-$\Sigma$ PSC-type $G_{\rightarrow S}$ as the generalization of $G$ with respect to $S$ [cf. Fig. 5 below].

![Figure 5: Generization](image)

**Remark 2.8.1.** It follows immediately from the various definitions involved that if $G$ is of type $(g,r)$ [cf. Definition 2.3, (i)], then the generalization $G_{\rightarrow \text{Node}(G)}$ of $G$ with respect to $\text{Node}(G)$ is isomorphic to $G_{g,r}^{\text{model}}$ [cf. Definition 2.3, (v)].

**Proposition 2.9 (Specialization outer isomorphisms).** Let $S \subseteq \text{Node}(G)$ be a subset of $\text{Node}(G)$. Write $\Pi_{G_{\rightarrow S}}$ for the [pro-$\Sigma$] fundamental group of the generalization $G_{\rightarrow S}$ of $G$ with respect to $S$ [cf. Definition 2.8]. Then the following hold:
(i) There exists a **natural outer isomorphism** of profinite groups

\[ \Phi_{G \to S} : \Pi_{G \to S} \sim \to \Pi_G \]

which satisfies the following three conditions:

1. \( \Phi_{G \to S} \) induces a bijection between the set of cuspidal subgroups of \( \Pi_{G \to S} \) and the set of cuspidal subgroups of \( \Pi_G \).

2. \( \Phi_{G \to S} \) induces a bijection between the set of nodal subgroups of \( \Pi_{G \to S} \) and the set of nodal subgroups of \( \Pi_G \) associated to the elements of \( \text{Node}(G) \setminus S \).

3. Let \( v \in \text{Vert}(G \to S) \) be a vertex of \( G \to S \); \( \mathbb{H}_v, T_v \) as in Definition 2.8, (vi). Then \( \Phi_{G \to S} \) induces a bijection between the \( \Pi_{G \to S} \)-conjugacy class of any vertical subgroup \( \Pi_v \subseteq \Pi_{G \to S} \) of \( \Pi_{G \to S} \) associated to \( v \in \text{Vert}(G \to S) \) and the \( \Pi_G \)-conjugacy class of subgroups determined by the image of the outer homomorphism

\[ \Pi(G|_{\mathbb{H}_v}) \to_{T_v} \to \Pi_G \]

induced by the natural morphism \( (G|_{\mathbb{H}_v}) \to_{T_v} \to G \) [cf. Definitions 2.2, (ii); 2.5, (ii)] of semi-graphs of anabelioids of pro-\( \Sigma \) PSC-type.

Moreover, any two outer isomorphisms \( \Pi_{G \to S} \sim \to \Pi_G \) that satisfy the above three conditions differ by composition with a **graphic** [cf. [CmbGC], Definition 1.4, (i)] automorphism [cf. the discussion entitled “Topological groups” in §0] of \( \Pi_{G \to S} \).

(ii) The isomorphism

\[ \text{Out}(\Pi_G) \sim \to \text{Out}(\Pi_{G \to S}) \]

induced by the natural outer isomorphism of (i) determines an injection

\[ \text{Aut}^S(G) \hookrightarrow \text{Aut}(G \to S) \]

[cf. Definition 2.6, (i)].

**Proof.** First, we verify assertion (i). An outer isomorphism that satisfies the three conditions of assertion (i) may be obtained by observing that, after sorting through the various definitions involved, a finite étale covering of \( G \to S \) amounts to the same data as a finite étale covering of \( G \). The final portion of assertion (i) follows immediately, in light of the three conditions in the statement of assertion (i), from
This completes the proof of assertion (i). Assertion (ii) follows immediately from [CmbGC], Proposition 1.5, (ii), together with the three conditions in the statement of assertion (i). This completes the proof of Proposition 2.9. Q.E.D.

**Definition 2.10.** Let \( S \subseteq \text{Node}(G) \) be a subset of \( \text{Node}(G) \). Write \( \Pi_{G \to S} \) for the \([\text{pro-\Sigma}]\) fundamental group of the generization \( G \to_S \) of \( G \) with respect to \( S \) [cf. Definition 2.8]. Then we shall refer to the natural outer isomorphism

\[
\Phi_{G \to S} : \Pi_{G \to S} \to \Pi_G
\]

obtained in Proposition 2.9, (i), as the *specialization outer isomorphism* with respect to \( S \).

**Proposition 2.11 (Commensurable terminality of closed subgroups determined by certain semi-graphs).** Let \( H \) be a sub-semi-graph of PSC-type [cf. Definition 2.2, (i)] of \( G \) and \( S \subseteq \text{Node}(G|H) \) [cf. Definition 2.2, (ii)] a subset of \( \text{Node}(G|H) \) that is **not of separating type** [cf. Definition 2.5, (i)]. Then the natural morphism \( (G|H) \to_S G \) [cf. Definitions 2.2, (ii); 2.5, (ii)] of semi-graphs of anabelioids of pro-\( \Sigma \) PSC-type determines an **outer injection** of profinite groups

\[
\Pi_{(G|H) \to S} \hookrightarrow \Pi_G.
\]

Moreover, the image of this outer injection is **commensurably terminal** in \( \Pi_G \) [cf. the discussion entitled “Topological groups” in §0].

**Proof.** Write \( H \overset{\text{def}}{=} (G|H) \to_S \) and \( T \overset{\text{def}}{=} \text{Node}(G|H) \setminus S \). Note that it follows from the definition of \( G|H \) that \( T \) may be regarded as the subset of \( \text{Node}(G) \) determined by \( \text{Node}(H) \); for simplicity, we shall identify \( T \) with \( \text{Node}(H) \). Now it follows immediately from the definition of \( “G \to_T” \) that the composite

\[
\Pi_H \xrightarrow{\Phi_{H \to_S}} \Pi_G \xrightarrow{\Phi_{G \to_T}^{-1}} \Pi_{G \to T}
\]

factors through a vertical subgroup \( \Pi_v \subseteq \Pi_{G \to T} \) of \( \Pi_{G \to T} \) associated to a vertex \( v \in \text{Vert}(G_{\to T}) \), and that the composite

\[
\Pi_H \to \Pi_{(G \to T)_v}
\]

of the resulting outer homomorphism \( \Pi_H \to \Pi_v \) [which is well-defined in light of the **commensurable terminality** of \( \Pi_v \) in \( \Pi_{G \to S} \) — cf. [CmbGC],
Proposition 1.2, (ii)] and the natural outer isomorphism $\Pi_{v} \simeq \Pi_{(\varphi_{v}^{-1} \tau)} \mid_{v}$ [cf. Remark 2.1.1] may be identified with "$\Phi_{H \rightarrow \tau}^{-1}$" [cf. Definition 2.10]. Thus, Proposition 2.11 follows immediately from the fact that $\Phi_{H \rightarrow \tau}$ is an outer isomorphism, together with the fact that $\Pi_{v} \subseteq \Pi_{\varphi_{v}^{-1} \tau}$ is commensurably terminal in $\Pi_{\varphi_{v}^{-1} \tau}$ [cf. [CmbGC], Proposition 1.2, (ii)]. This completes the proof of Proposition 2.11.

Q.E.D.

Lemma 2.12 (Restrictions of automorphisms). Let $H \subseteq \Pi_{G}$ be a closed subgroup of $\Pi_{G}$ which is normally terminal [cf. the discussion entitled "Topological groups" in §0] and $\alpha \in \text{Out}^{H}(\Pi_{G})$ [cf. Definition 2.6, (ii)]. Then the following hold:

(i) There exists a lifting $\tilde{\alpha} \in \text{Aut}(\Pi_{G})$ of $\alpha$ such that $\tilde{\alpha}$ preserves the closed subgroup $H \subseteq \Pi_{G}$. Moreover, such a lifting $\tilde{\alpha}$ is uniquely determined up to composition with an $H$-inner automorphism of $\Pi_{G}$.

(ii) Write $\alpha_{H}$ for the automorphism [cf. the discussion entitled "Topological groups" in §0] of $H$ determined by the restriction of a lifting $\tilde{\alpha}$ as obtained in (i) to the closed subgroup $H \subseteq \Pi_{G}$. Then the map

$$\text{Out}^{H}(\Pi_{G}) \longrightarrow \text{Out}(H)$$

given by assigning $\alpha \mapsto \alpha_{H}$ is a homomorphism.

(iii) The homomorphism

$$\text{Out}^{H}(\Pi_{G}) \longrightarrow \text{Out}(H)$$

obtained in (ii) depends only on the conjugacy class of the closed subgroup $H \subseteq \Pi_{G}$, i.e., if we write $H^{\gamma} \overset{\text{def}}{=} \gamma \cdot H \cdot \gamma^{-1}$ for $\gamma \in \Pi_{G}$, then the diagram

$$\begin{array}{ccc}
\text{Out}^{H}(\Pi_{G}) & \longrightarrow & \text{Out}(H) \\
\| & & \| \\
\text{Out}^{H}(\Pi_{G}) & \longrightarrow & \text{Out}(H^{\gamma})
\end{array}$$

— where the upper (respectively, lower) horizontal arrow is the homomorphism given by mapping $\alpha \mapsto \alpha_{H}$ (respectively, $\alpha \mapsto \alpha_{H^{\gamma}}$), and the right-hand vertical arrow is the isomorphism.
obtained by mapping $\phi \in \text{Out}(H)$ to

$$H^\gamma \xrightarrow{\text{Inn}(\gamma^{-1})} H \xrightarrow{\phi} H \xrightarrow{\text{Inn}(\gamma)} H^\gamma$$

— commutes.

**Proof.** Assertion (i) follows immediately from the normal terminality of $H$ in $\Pi_G$. Assertion (ii) follows immediately from assertion (i). Assertion (iii) follows immediately from the various definitions involved. Q.E.D.

**Definition 2.13.** Let $H \subseteq \Pi_G$ be a [closed] subgroup of $\Pi_G$ which is normally terminal [cf. the discussion entitled “Topological groups” in §0]. Then we shall write

$$\text{Out}^{[H]}(\Pi_G) \subseteq \text{Out}^H(\Pi_G)$$

for the closed subgroup of $\text{Out}^H(\Pi_G)$ consisting of automorphisms [cf. the discussion entitled “Topological groups” in §0] $\alpha$ of $\Pi_G$ such that the image $\alpha_H$ of $\alpha$ via the homomorphism $\text{Out}^H(\tilde{\mathcal{G}}) \to \text{Out}(H)$ obtained in Lemma 2.12, (ii), is trivial. Also, we shall write

$$\text{Aut}^{[H]}(\tilde{\mathcal{G}}) \overset{\text{def}}{=} \text{Out}^{[H]}(\Pi_G) \cap \text{Aut}(\tilde{\mathcal{G}}).$$

**Definition 2.14.**

(i) Let $T \subseteq \text{Cusp}(\mathcal{G})$ be an ommittable [cf. Definition 2.4, (ii)] subset of $\text{Cusp}(\mathcal{G})$. Write $\Pi_{G^T}$ for the [pro-$\Sigma$] fundamental group of $G_0^T$ [cf. Definition 2.4, (ii)] and $N_T \subseteq \Pi_G$ for the normal closed subgroup of $\Pi_G$ topologically normally generated by the cuspidal subgroups of $\Pi_G$ associated to elements of $T$. Then one verifies easily that the natural outer isomorphism $\Pi_G/N_T \xrightarrow{\sim} \Pi_{G^T}$ [cf. Definition 2.4, (ii)] induces a homomorphism $\text{Out}^{N_T}(\Pi_G) \to \text{Out}(\Pi_{G^T})$ that fits into a commutative diagram

$$\begin{array}{ccc}
\text{Aut}^T(\mathcal{G}) & \longrightarrow & \text{Aut}(G_0^T) \\
\downarrow & & \downarrow \\
\text{Out}^{N_T}(\Pi_G) & \longrightarrow & \text{Out}(\Pi_{G^T})
\end{array}$$
—— where the vertical arrows are the natural injections. For \( \alpha \in \text{Out}^{N_T}(\Pi_G) \), we shall write

\[
\alpha_{G,T} \in \text{Out}(\Pi_{G,T})
\]

for the image of \( \alpha \) via the lower horizontal arrow in the above commutative diagram. If, moreover, \( \alpha \in \text{Aut}^T(G) \), then, in light of the injectivity of the right-hand vertical arrow in the above diagram, we shall write [by abuse of notation]

\[
\alpha_{G,T} \in \text{Aut}(G_T)
\]

for the image of \( \alpha \) via the upper horizontal arrow in the above commutative diagram.

(ii) Let \( H \) be a sub-semi-graph of PSC-type [cf. Definition 2.2, (i)] of \( G \) and \( S \subseteq \text{Node}(G_H) \) [cf. Definition 2.2, (ii)] a subset of \( \text{Node}(G_H) \) that is not of separating type [cf. Definition 2.5, (i)]. Write \( \Pi_{\langle G \rangle,S} \) for the [pro-\( \Sigma \)] fundamental group of \( (G_H \rangle,S) \) [cf. Definition 2.5, (ii)]. Then the natural outer homomorphism \( \Pi_{\langle G \rangle,S} \to \Pi_G \) is an outer injection whose image is commensurably terminal [cf. Proposition 2.11]. Thus, it follows from Lemma 2.12, (iii), that we have a homomorphism \( \text{Out}^{\Pi_{\langle G \rangle,S}}(\Pi_G) \to \text{Out}(\Pi_{\langle G \rangle,S}) \) that fits into a commutative diagram

\[
\begin{array}{ccc}
\text{Aut}^{H,S}(G) & \cong & \text{Aut}^{H}(G) \cap \text{Aut}^S(G) \\
& \Downarrow & \\
\text{Out}^{\Pi_{\langle G \rangle,S}}(\Pi_G) & \longrightarrow & \text{Out}(\Pi_{\langle G \rangle,S})
\end{array}
\]

—— where the vertical arrows are the natural injections. For \( \alpha \in \text{Out}^{\Pi_{\langle G \rangle,S}}(\Pi_G) \), we shall write

\[
\alpha_{\langle G \rangle,S} \in \text{Out}(\Pi_{\langle G \rangle,S})
\]

for the image of \( \alpha \) via the lower horizontal arrow in the above commutative diagram. If, moreover, \( \alpha \in \text{Aut}^{H,S}(G) \), then, in light of the injectivity of the right-hand vertical arrow in the above diagram, we shall write [by abuse of notation]

\[
\alpha_{\langle G \rangle,S} \in \text{Aut}(\langle G \rangle,S)
\]

for the image of \( \alpha \) via the upper horizontal arrow in the above commutative diagram. Finally, if \( T \subseteq \text{Cusp}(\langle G \rangle,S) \) is an
omittable subset of \( \text{Cusp}(\mathcal{G}^H_{[\mathbb{H}]-S}) \), then we shall write
\[
\text{Aut}^{H>-S\star T}(\mathcal{G}) \subseteq \text{Aut}^{H>-S}(\mathcal{G})
\]
for the inverse image of the closed subgroup \( \text{Aut}^{T}(\mathcal{G}^H_{[\mathbb{H}]-S}) \subseteq \text{Aut}(\mathcal{G}^H_{[\mathbb{H}]-S}) \) of \( \text{Aut}(\mathcal{G}^H_{[\mathbb{H}]-S}) \) in \( \text{Aut}^{H>-S\star T}(\mathcal{G}) \) via the upper horizontal arrow \( \text{Aut}^{H>-S\star T}(\mathcal{G}) \rightarrow \text{Aut}(\mathcal{G}^H_{[\mathbb{H}]-S}) \) of the above commutative diagram; thus, we have a natural homomorphism [cf. (i)]
\[
\text{Aut}^{H>-S\star T}(\mathcal{G}) \longrightarrow \text{Aut}(\mathcal{G}^H_{[\mathbb{H}]-S})_{\star T}
\]
\( \alpha \mapsto \alpha_{(\mathcal{G}^H_{[\mathbb{H}]-S})_{\star T}} \).

(iii) Let \( z \in \text{VCN}(\mathcal{G}) \) be an element of \( \text{VCN}(\mathcal{G}) \) and \( \Pi_z \subseteq \Pi_G \) a \( \text{VCN} \)-subgroup of \( \Pi_G \) associated to \( z \in \text{VCN}(\mathcal{G}) \). Then it follows from [CmbGC], Proposition 1.2, (ii), that the closed subgroup \( \Pi_z \subseteq \Pi_G \) is commensurably terminal. Thus, it follows from Lemma 2.12, (iii), that we obtain a homomorphism \( \text{Out}^{H>-S\star T}(\Pi_G) \rightarrow \text{Out}(\Pi_z) \) that fits into a commutative diagram
\[
\begin{array}{ccc}
\text{Aut}^{H^{-1}}(\mathcal{G}) & \longrightarrow & \text{Aut}(\mathcal{G}_z) \\
\downarrow & & \downarrow \\
\text{Out}^{H^{-1}}(\Pi_G) & \longrightarrow & \text{Out}(\Pi_z)
\end{array}
\]
where the left-hand vertical arrow is injective, and the right-hand vertical arrow is an isomorphism. For \( \alpha \in \text{Out}^{H^{-1}}(\Pi_G) \), we shall write
\[
\alpha_z \in \text{Out}(\Pi_z)
\]
for the image of \( \alpha \) via the lower horizontal arrow in the above commutative diagram.

§3. Synchronization of cyclotomes

In the present §, we introduce and study the notion of the second cohomology group with compact supports of a semi-graph of anabelioids of PSC-type [cf. Definition 3.1, (ii), (iii) below]. In particular, we show that such cohomology groups are compatible with graph-theoretic localization [cf. Definition 3.4, Lemma 3.5 below]. This leads naturally
to a discussion of the phenomenon of *synchronization* among the various cyclotomes [cf. Definition 3.8 below] arising from a semi-graph of anabelioids of PSC-type [cf. Corollary 3.9 below].

Let \( \Sigma \) be a nonempty set of prime numbers and \( \mathcal{G} \) a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type. Write \( \mathcal{G} \) for the underlying semi-graph of \( \mathcal{G} \), \( \Pi_G \) for the [pro-\( \Sigma \)] fundamental group of \( \mathcal{G} \), and \( \tilde{\mathcal{G}} \to \mathcal{G} \) for the universal covering of \( \mathcal{G} \) corresponding to \( \Pi_G \).

**Definition 3.1.** Let \( M \) be a finitely generated \( \mathbb{Z} \)-module and \( v \in \text{Vert}(\mathcal{G}) \) a vertex of \( \mathcal{G} \).

(i) We shall write

\[
H^2(\mathcal{G}, M) \overset{\text{def}}{=} H^2(\Pi_G, M)
\]

— where we regard \( M \) as being equipped with the trivial action of \( \Pi_G \) — and refer to \( H^2(\mathcal{G}, M) \) as the *second cohomology group* of \( \mathcal{G} \).

(ii) Let \( s \) be a section of the natural surjection \( \text{Cusp}(\tilde{\mathcal{G}}) \twoheadrightarrow \text{Cusp}(\mathcal{G}) \). Given a central extension of profinite groups

\[
1 \longrightarrow M \longrightarrow E \longrightarrow \Pi_G \longrightarrow 1,
\]

and a cusp \( e \in \text{Cusp}(\mathcal{G}) \), we shall refer to a section of this extension over the edge-like subgroup \( \Pi_{s(e)} \subseteq \Pi_G \) of \( \Pi_G \) determined by \( s(e) \in \text{Cusp}(\tilde{\mathcal{G}}) \) as a *trivialization of this extension at the cusp \( e \).* We shall write

\[
H^2_e(\mathcal{G}, M)
\]

for the set of equivalence classes

\[
[E, (\iota_e : \Pi_{s(e)} \to E)_{e \in \text{Cusp}(\mathcal{G})}]
\]

of collections of data \( (E, (\iota_e : \Pi_{s(e)} \to E)_{e \in \text{Cusp}(\mathcal{G})}) \) as follows:

(a) \( E \) is a central extension of profinite groups

\[
1 \longrightarrow M \longrightarrow E \longrightarrow \Pi_G \longrightarrow 1;
\]

(b) for each \( e \in \text{Cusp}(\mathcal{G}) \), \( \iota_e \) is a trivialization of this extension at the cusp \( e \). The equivalence relation “\( \sim \)” is then defined as follows: for two collections of data \( (E, (\iota_e)) \) and \( (E', (\iota'_e)) \), we shall write \( (E, (\iota_e)) \sim (E', (\iota'_e)) \) if there exists an isomorphism
of profinite groups $\alpha \colon E \to E'$ over $\Pi_{\mathcal{G}}$ which induces the identity automorphism of $M$, and, moreover, for each $e \in \text{Cusp}(\mathcal{G})$, maps $\iota_e$ to $\iota'_e$. We shall refer to $H^2_e(\mathcal{G}, M)$ as the second cohomology group with compact supports of $\mathcal{G}$.

(iii) We shall write

$$H^2_e(v, M) \overset{\text{def}}{=} H^2_e(\mathcal{G}|_v, M)$$

[cf. (ii); Definition 2.1, (iii)] and refer to $H^2_e(v, M)$ as the second cohomology group with compact supports of $v$.

(iv) The set $H^2_e(\mathcal{G}, M)$ is equipped with a natural structure of $\mathbb{Z}$-module defined as follows:

- Let $[E, (\iota_e)], [E', (\iota'_e)] \in H^2_e(\mathcal{G}, M)$. Then the fiber product $E \times_{\Pi_{\mathcal{G}}} E'$ of the surjections $E \to \Pi_{\mathcal{G}}, E' \to \Pi_{\mathcal{G}}$ is an extension of $\Pi_{\mathcal{G}}$ by $M \times M$. Thus, the quotient $S$ of $E \times_{\Pi_{\mathcal{G}}} E'$ by the image of the composite

$$M \leftarrow M \times M \rightarrow E \times_{\Pi_{\mathcal{G}}} E'$$

$$m \mapsto (m, -m)$$

is an extension of $\Pi_{\mathcal{G}}$ by $M$. On the other hand, it follows from the definition of $S$ that for each $e \in \text{Cusp}(\mathcal{G})$, the sections $\iota_e$ and $\iota'_e$ naturally determine a section $\iota^S_e \colon \Pi_{\mathcal{G}(e)} \to S$ over $\Pi_{\mathcal{G}(e)}$. Thus, we define

$$[E, (\iota_e)] + [E', (\iota'_e)] \overset{\text{def}}{=} [S, (\iota^S_e)].$$

Here, one may verify easily that the equivalence class $[S, (\iota^S_e)]$ depends only on the equivalence classes $[E, (\iota_e)], [E', (\iota'_e)]$, and that this definition of $\overset{\text{def}}{+}$ determines a module structure on $H^2_e(\mathcal{G}, M)$.

- Let $[E, (\iota_e)] \in H^2_e(\mathcal{G}, M)$ be an element of $H^2_e(\mathcal{G}, M)$ and $a \in \mathbb{Z}$. Now the composite $E \times M \overset{pr}{\to} E \to \Pi_{\mathcal{G}}$ determines an extension of $\Pi_{\mathcal{G}}$ by $M \times M$. Thus, the quotient $P$ of $E \times M$ by the image of the composite

$$M \leftarrow M \times M \rightarrow E \times M$$

$$m \mapsto (m, -am)$$

is an extension of $\Pi_{\mathcal{G}}$ by $M$. On the other hand, it follows from the definition of $P$ that for each $e \in \text{Cusp}(\mathcal{G})$, the
section \( \iota_e \) and the zero homomorphism \( \Pi_{s(e)} \to M \) naturally determine a section \( \iota_e^P : \Pi_{s(e)} \to P \) over \( \Pi_{s(e)} \). Thus, we define

\[
\alpha \cdot [E, (\iota_e)] \overset{\text{def}}{=} [P, (\iota_e^P)].
\]

Here, one may verify easily that the equivalence class \([P, (\iota_e^P)]\) depends only on the equivalence class \([E, (\iota_e)]\) and \(\alpha \in \hat{\mathbb{Z}}^\Sigma\), and that this definition of \(\alpha\) determines a \(\hat{\mathbb{Z}}^\Sigma\)-module structure on \(H^2_c(\mathcal{G}, M)\).

Finally, we note that it follows from Lemma 3.2 below that the \(\hat{\mathbb{Z}}^\Sigma\)-module \(H^2_c(\mathcal{G}, M)\) does not depend on the choice of the section \(s\). More precisely, the \(\hat{\mathbb{Z}}^\Sigma\)-module \(H^2_c(\mathcal{G}, M)\) is uniquely determined by \(\mathcal{G}\) and \(M\) up to the natural isomorphism obtained in Lemma 3.2.

Lemma 3.2 (Independence of the choice of section). Let \(M\) be a finitely generated \(\hat{\mathbb{Z}}^\Sigma\)-module and \(s, s'\) sections of the natural surjection \(\text{Cusp}(\hat{\mathcal{G}}) \twoheadrightarrow \text{Cusp}(\mathcal{G})\). Write \(H^2_c(\mathcal{G}, M, s), H^2_c(\mathcal{G}, M, s')\) for the \(\hat{\mathbb{Z}}^\Sigma\)-modules \(H^2_c(\mathcal{G}, M)\) defined in Definition 3.1 by means of the sections \(s, s'\), respectively. Then there exists a natural isomorphism of \(\hat{\mathbb{Z}}^\Sigma\)-modules

\[
H^2_c(\mathcal{G}, M, s) \xrightarrow{\sim} H^2_c(\mathcal{G}, M, s').
\]

Proof. Let \([E, (\iota_e)] \in H^2_c(\mathcal{G}, M, s)\) be an element of \(H^2_c(\mathcal{G}, M, s)\). Now it follows from the various definitions involved that, for each \(e \in \text{Cusp}(\hat{\mathcal{G}})\), there exists an element \(\gamma_e \in \Pi_{\mathcal{G}}\) such that \(\Pi_{s'(e)} = \gamma_e \cdot \Pi_{s(e)} \cdot \gamma_e^{-1}\). For each \(e \in \text{Cusp}(\mathcal{G})\), fix a lifting \(\overline{\gamma}_e \in E\) of \(\gamma_e \in \Pi_{\mathcal{G}}\) and write \(\iota'_e : \Pi_{s'(e)} \to E\) for the section given by

\[
\Pi_{s'(e)} = \gamma_e \cdot \Pi_{s(e)} \cdot \gamma_e^{-1} \quad \mapsto \quad E \quad \mapsto \quad \overline{\gamma}_e \iota'_e(a) \overline{\gamma}_e^{-1}.
\]

Then it follows immediately from the fact that \(M \subseteq E\) is contained in the center \(Z(E)\) of \(E\) that this section \(\iota'_e\) does not depend on the choice of the lifting \(\overline{\gamma}_e \in E\) of \(\gamma_e \in \Pi_{\mathcal{G}}\). Moreover, it follows immediately from the various definitions involved that the assignment \(\overline{[E, (\iota_e)]} \mapsto [E, (\iota'_e)]\) determines an isomorphism of \(\hat{\mathbb{Z}}^\Sigma\)-modules

\[
H^2_c(\mathcal{G}, M, s) \xrightarrow{\sim} H^2_c(\mathcal{G}, M, s').
\]

This completes the proof of Lemma 3.2. Q.E.D.
Lemma 3.3 (Exactness of certain sequences). Let $M$ be a finitely generated $\mathbb{Z}$-module. Suppose that $\text{Cusp}(\mathcal{G}) \neq \emptyset$. Then the natural inclusions $\Pi_e \hookrightarrow \Pi_G$ — where $e$ ranges over the cusps of $\mathcal{G}$, and, for each cusp $e \in \text{Cusp}(\mathcal{G})$, we use the notation $\Pi_e$ to denote an edge-like subgroup of $\Pi_G$ associated to the cusp $e$ — determine an exact sequence of $\mathbb{Z}$-modules

$$\text{Hom}_{\mathbb{Z}}(\Pi_G^{ab}, M) \rightarrow \bigoplus_{e \in \text{Cusp}(\mathcal{G})} \text{Hom}_{\mathbb{Z}}(\Pi_e, M) \rightarrow H^2(\mathcal{G}, M) \rightarrow 0.$$ 

Proof. Let $s$ be a section of the natural surjection $\text{Cusp}(\mathcal{G}) \twoheadrightarrow \text{Cusp}(\mathcal{G})$. Then given an element $(\phi_e : \Pi_e \rightarrow M)_{e \in \text{Node}(\mathcal{G})} \in \bigoplus_{e \in \text{Cusp}(\mathcal{G})} \text{Hom}_{\mathbb{Z}}(\Pi_e, M),$ one may construct an element $[M \times \Pi_G \xrightarrow{pr_2} \Pi_G, (\iota_e : \Pi_{s(e)} \rightarrow M \times \Pi_G)_{e \in \text{Node}(\mathcal{G})}]$ — where we write $\iota_e : \Pi_{s(e)} \rightarrow M \times \Pi_G$ for the section determined by $\phi_e : \Pi_{s(e)} \rightarrow M$ and the natural inclusion $\Pi_{s(e)} \hookrightarrow \Pi_G$ — of $H^2(\mathcal{G}, M)$. In particular, we obtain a map $\bigoplus_{e \in \text{Cusp}(\mathcal{G})} \text{Hom}_{\mathbb{Z}}(\Pi_e, M) \rightarrow H^2(\mathcal{G}, M)$, which, as is easily verified, is a homomorphism of $\hat{\mathbb{Z}}\Sigma$-modules. Now the exactness of the sequence in question follows immediately from the fact that $\Pi_G$ is free pro-$\Sigma$ [cf. [CmbGC], Remark 1.1.3]. This completes the proof of Lemma 3.3. Q.E.D.

Definition 3.4. Let $M$ be a finitely generated $\hat{\mathbb{Z}}\Sigma$-module.

(i) Let $\mathcal{E}$ be a semi-graph of anabelioids. Denote by $\text{VCN}(\mathcal{E})$ the set of components of $\mathcal{E}$ [i.e., the set of vertices and edges of $\mathcal{E}$] and, for each $z \in \text{VCN}(\mathcal{E})$, by $\Pi_{\mathcal{E}_z}$ the fundamental group of the anabelioid $\mathcal{E}_z$ of $\mathcal{E}$ corresponding to $z \in \text{VCN}(\mathcal{E})$. Then we define a central extension of $\mathcal{G}$ by $M$ to be a collection of data

$$(\mathcal{E}, \alpha = (\alpha_z : M \hookrightarrow \Pi_{\mathcal{E}_z})_{z \in \text{VCN}(\mathcal{E})}, \beta : \mathcal{E} / \alpha \rightarrow \mathcal{G})$$

as follows:

(a) For each $z \in \text{VCN}(\mathcal{E})$, $\alpha_z : M \hookrightarrow \Pi_{\mathcal{E}_z}$ is an injective homomorphism of profinite groups whose image is contained
in the center $Z(\Pi_{E_z})$ of $\Pi_{E_z}$. [Thus, the image of $\alpha_z$ is a normal closed subgroup of $\Pi_{E_z}$.]

(b) For each branch $b$ of an edge $e$ that abuts to a vertex $v$ of $\mathcal{E}$, we assume that the outer homomorphism $\Pi_{E_e} \to \Pi_{E_v}$ associated to $b$ is injective and fits into a commutative diagram of [outer] homomorphisms of profinite groups

\[
\begin{array}{ccc}
M & \longrightarrow & M \\
\alpha_e & \downarrow & \alpha_v \\
\Pi_{E_e} & \longrightarrow & \Pi_{E_v}
\end{array}
\]

i.e., where the lower horizontal arrow is the outer injection associated to $b$.

(c) Write $\mathcal{E}/\alpha$ for the semi-graph of anabelioids defined as follows: We take the underlying semi-graph of $\mathcal{E}/\alpha$ to be the underlying semi-graph of $\mathcal{E}$; for each $z \in \text{VCN}(\mathcal{E})$, we take the anabelioid $(\mathcal{E}/\alpha)_z$ of $\mathcal{E}/\alpha$ corresponding to $z \in \text{VCN}(\mathcal{E})$ to be the anabelioid determined by the profinite group $\Pi_{E_e}/\text{Im}(\alpha_e)$ [cf. condition (a)]; for each branch $b$ of an edge $e$ that abuts to a vertex $v$ of $\mathcal{E}$, we take the associated morphism of anabelioids $(\mathcal{E}/\alpha)_e \to (\mathcal{E}/\alpha)_v$ to be the morphism of anabelioids naturally determined by the morphism $E_e \to E_v$ associated, relative to $\mathcal{E}$, to $b$ [cf. condition (b)].

(d) $\beta : \mathcal{E}/\alpha \to \mathcal{G}$ is an isomorphism of semi-graphs of anabelioids.

There is an evident notion of isomorphisms of central extensions of $\mathcal{G}$ by $M$. Also, given a central extension of $\mathcal{G}$ by $M$, and a section $s$ of the natural surjection $\text{Cusp}(\mathcal{G}) \twoheadrightarrow \text{Cusp}(\mathcal{G})$, there is an evident notion of trivialization of the given central extension of $\mathcal{G}$ by $M$ at a cusp of $\mathcal{G}$ [cf. the discussion of Definition 3.1, (ii), (iv)].

(ii) Let

\[
1 \longrightarrow M \longrightarrow E \longrightarrow \Pi_{\mathcal{G}} \longrightarrow 1
\]

be a central extension of $\Pi_{\mathcal{G}}$ by $M$. Then we shall define a semi-graph of anabelioids

$\mathcal{G}_E$
— which we shall refer to as the **semi-graph of anabelioids associated to the central extension** $E$ — as follows: We take the underlying semi-graph of $G_E$ to be the underlying semi-graph of $G$. We take the anabeloid of $G_E$ corresponding to $z \in \text{VCN}(G)$ to be the anabeloid determined by the fiber product $E \times_{\Pi_z} \Pi_z$ of the surjection $E \to \Pi_G$ and a natural inclusion $\Pi_z \hookrightarrow \Pi_G$ — where we use the notation $\Pi_z \subseteq \Pi_G$ to denote a VCN-subgroup [cf. Definition 2.1, (i)] of $\Pi_G$ associated to $z \in \text{VCN}(G)$; for each branch $b$ of an edge $e$ that abuts to a vertex $v$ of $G$, if we write $(G_E)_v$, $(G_E)_e$ for the anabelioids of $G_E$ corresponding to $v$, $e$, respectively, then we take the morphism of anabelioids $(G_E)_e \to (G_E)_v$ associated to the branch $b$ to be the morphism naturally determined by the morphism of anabelioids $G_e \to G_v$ associated, relative to $G$, to $b$.

(iii) In the notation of (ii), one may verify easily that the semi-graph of anabelioids $G_E$ associated to the central extension $E$ is equipped with a **natural structure of central extension of $G$ by $M$**. More precisely, for each $z \in \text{VCN}(G)$, if we denote by $\alpha_z : M \hookrightarrow \Pi_{(G_E)_z} = E \times_{\Pi_z} \Pi_z$ the homomorphism determined by the natural inclusion $M \hookrightarrow E$ and the trivial homomorphism $M \to \Pi_z$, then there exists a natural isomorphism $\beta : \overline{G}_E/(\alpha_z)_{z \in \text{VCN}(G)} \cong \hat{G}$ such that the collection of data

$$(G_E, (\alpha_z)_{z \in \text{VCN}(G)}, \beta)$$

forms a central extension of $G$ by $M$, which we shall refer to as the **central extension of $G$ by $M$ associated to the central extension $E$**.

**Lemma 3.5 (Graph-theoretic localizability of central extensions of fundamental groups).** Let $M$ be a finitely generated $\mathbb{Z}^\Sigma$-module. Then the following hold:

(i) **(Exactness and centrality)** Let

$$(\mathcal{E}, \alpha = (\alpha_z : M \hookrightarrow \Pi_{(G_E)_z})_{z \in \text{VCN}(\mathcal{E})}, \beta : \mathcal{E} / \alpha \rightarrow \hat{G}) \quad (\dagger_1)$$
be a central extension of $\mathcal{G}$ by $M$ [cf. Definition 3.4, (i)]. Write $\Pi_{\mathcal{E}}$ for the pro-$\Sigma$ fundamental group of $\mathcal{E}$, i.e., the maximal pro-$\Sigma$ quotient of the fundamental group of $\mathcal{E}$ [cf. the discussion preceding [SemiAn], Definition 2.2]. Then the composite $\mathcal{E} \to \mathcal{E}/\alpha \to \mathcal{G}$ determines an exact sequence of profinite groups

$$1 \to M \to \Pi_{\mathcal{E}} \to \Pi_{\mathcal{G}} \to 1 \quad (4.2)$$

which is central.

(ii) (Natural isomorphism I) In the notation of (i), the central extension of $\mathcal{G}$ by $M$ associated to the central extension $(4.2)$ [cf. Definition 3.4, (iii)] is naturally isomorphic, as a central extension of $\mathcal{G}$ by $M$, to $(4.1)$.

(iii) (Natural isomorphism II) Let

$$1 \to M \to E \to \Pi_{\mathcal{G}} \to 1$$

be a central extension of $\Pi_{\mathcal{G}}$ by $M$. Then the pro-$\Sigma$ fundamental group of the semi-graph of anabelioids $\mathcal{G}_{\mathcal{E}}$ associated to the central extension $E$ [cf. Definition 3.4, (ii)] — i.e., the maximal pro-$\Sigma$ quotient of the fundamental group of $\mathcal{G}_{\mathcal{E}}$ — is naturally isomorphic, over $\Pi_{\mathcal{G}}$, to $E$.

(iv) (Equivalence of categories) The correspondences of (i), (ii), (iii) determine a natural equivalence of categories between the category of central extensions of $\mathcal{G}$ by $M$ and the category of central extensions of $\Pi_{\mathcal{G}}$ by $M$. [Here, we take the morphisms in both categories to be the isomorphisms of central extensions of the sort under consideration.] Moreover, this equivalence extends to a similar natural equivalence of categories between categories of central extensions equipped with trivializations at the cusps of $\mathcal{G}$ [cf. Definitions 3.1, (ii); 3.4, (i)].

Proof First, we verify assertion (i). If Node($\mathcal{G}$) = $\emptyset$, then assertion (i) is immediate; thus, suppose that Node($\mathcal{G}$) $\neq \emptyset$. For each connected finite étale covering $\mathcal{E}' \to \mathcal{E}$ of $\mathcal{E}$, denote by $\Pi_{\mathcal{E}'}$ the pro-$\Sigma$ fundamental group of $\mathcal{E}'$, by VCN($\mathcal{E}'$) the set of components of $\mathcal{E}'$ [i.e., the set of vertices and edges of $\mathcal{E}'$], and by Vert($\mathcal{E}'$) the set of vertices of $\mathcal{E}'$; for each $z \in$ VCN($\mathcal{E}'$), denote by $\mathcal{E}'_z$ the anabelioid of $\mathcal{E}'$ corresponding to $z \in$ VCN($\mathcal{E}'$) and by $\Pi_{\mathcal{E}'_z}$ the fundamental group of $\mathcal{E}'_z$. Now we claim that
The composite in question \( E \to \mathcal{E}/\alpha \sim \mathcal{G} \) induces an isomorphism between the underlying semi-graphs, as well as an outer surjection \( \Pi_E \to \Pi_G \).

Indeed, the fact that the composite in question determines an isomorphism between the underlying semi-graphs follows from conditions (c), (d) of Definition 3.4, (i). In particular, we obtain a bijection \( \text{VCN}(\mathcal{E}) \sim \text{VCN}(\mathcal{G}) \). Now for each \( z \in \text{VCN}(\mathcal{E}) \), the composite \( E \to \mathcal{E}/\alpha \to \mathcal{G} \) induces an outer surjection \( E_z \to \mathcal{G}_z \), where we use the notation \( \mathcal{G}_z \) to denote a VCN-subgroup [cf. Definition 2.1, (i)] of \( \mathcal{G} \) associated to \( z \in \text{VCN}(\mathcal{G}) \). Therefore, in light of the isomorphism verified above between the semi-graphs of \( E \) and \( \mathcal{G} \), one may verify easily that the natural outer homomorphism \( E \to \mathcal{G} \) is surjective. This completes the proof of the claim \((*)_1\).

For each vertex \( v \in \text{Vert}(\mathcal{E}) \) [cf. claim \((*)_1\)], it follows from the assumption that \( \text{Node}(\mathcal{G}) \neq \emptyset \) that any vertical subgroup \( \Pi_v \subseteq \Pi_G \) of \( \Pi_G \) associated to a vertex \( v \in \text{Vert}(\mathcal{G}) \) is a free pro-\( \Sigma \) group [cf. [CmbGC], Remark 1.1.3]; thus, there exists a section of the natural surjection \( \Pi_{E_v} \to \Pi_v \). Now for each vertex \( v \in \text{Vert}(\mathcal{G}) \), let us fix such a section of the natural surjection \( \Pi_{E_v} \to \Pi_v \) hence also — since the extension \( \Pi_{E_v} \) of \( \Pi_v \) by \( M \) is central [cf. condition (a) of Definition 3.4, (i)] — an isomorphism \( t_v : M \times \Pi_v \to \Pi_{E_v} \). Let \( \mathcal{G}_1 \to \mathcal{G} \) be a connected finite étale Galois covering of \( \mathcal{G} \) and write \( \mathcal{E}_1 \buildrel \text{def} \over = \mathcal{E} \times_{\mathcal{G}} \mathcal{G}_1 \). Then it follows from the claim \((*)_1\) that \( \mathcal{E}_1 \) is connected; moreover, one may verify easily that the structure of central extension of \( \mathcal{G} \) by \( M \) on \( \mathcal{E} \) naturally determines a structure of central extension of \( \mathcal{G}_1 \) by \( M \) on \( \mathcal{E}_1 \), and that for each vertex \( v \in \text{Vert}(\mathcal{E}) \) \( \to \text{Vert}(\mathcal{G}) \) and each vertex \( w \in \text{Vert}(\mathcal{E}_1) \to \text{Vert}(\mathcal{G}_1) \) that lies over \( v \), the normal closed subgroup \( \Pi_{E_{(\mathcal{E}_1)_w}} \subseteq \Pi_{E_v} \) corresponds to \( M \times \Pi_w \subseteq M \times \Pi_v \) relative to the isomorphism \( t_v : M \times \Pi_v \to \Pi_{E_v} \) fixed above, i.e., we obtain an isomorphism \( t_w : M \times \Pi_w \to \Pi_{(\mathcal{E}_1)_w} \).

Now for a finite quotient \( M \twoheadrightarrow Q \) of \( M \) and a connected finite étale Galois covering \( \mathcal{G}_1 \to \mathcal{G} \) of \( \mathcal{G} \), we shall say that a connected finite étale covering \( \mathcal{E}_2 \to \mathcal{E} \) of \( \mathcal{E} \) satisfies the condition \((\dagger_{Q,\mathcal{G}_1})\) if the following two conditions are satisfied:

\((\dagger_{Q,\mathcal{G}_1})\) \( \mathcal{E}_2 \to \mathcal{E} \) factors through \( \mathcal{E}_1 \buildrel \text{def} \over = \mathcal{E} \times_{\mathcal{G}} \mathcal{G}_1 \to \mathcal{E} \), the resulting covering \( \mathcal{E}_2 \to \mathcal{E}_1 \) is Galois, and for each vertex \( v \in \text{VCN}(\mathcal{E}_1) \), the composite

\[
M \twoheadrightarrow \Pi_{(\mathcal{E}_1)_w} \twoheadrightarrow \Pi_{\mathcal{E}_1} \twoheadrightarrow \Pi_{\mathcal{E}_1}/\Pi_{\mathcal{E}_2}
\]
is surjective, with kernel equal to the kernel of $M \rightarrow Q$.

$$(\mathfrak t^2_{Q, G_1}) \mathcal E_2 \rightarrow \mathcal E$$ is Galois.

Then we claim that

\((*2): \) for any finite quotient $M \rightarrow Q$ of $M$ and any connected finite étale Galois covering $G_1 \rightarrow G$, there exists — after possibly replacing $G_1 \rightarrow G$ by a connected finite étale Galois covering of $G$ that factors through $G_1 \rightarrow G$ — a connected finite étale covering of $\mathcal E$ which satisfies the condition $$(\mathfrak t^1_{Q, G_1})$$.

Indeed, let $M \rightarrow Q$ be a finite quotient of $M$, $G_1 \rightarrow G$ a connected finite étale Galois covering of $G$, and $\mathcal E_1 \overset{\text{def}}{=} \mathcal E \times_G G_1$. For each vertex $v \in \text{Vert}(\mathcal E_1) \rightarrow \text{Vert}(G_1)$ [cf. the above discussion], denote by $\Pi_{(\mathcal E_1)_v} \rightarrow Q_v$ the quotient of $\Pi_{(\mathcal E_1)_v}$ obtained by forming the composite

$$\Pi_{(\mathcal E_1)_v} \overset{t_v}{\twoheadrightarrow} M \times \Pi_v \overset{pr_v}{\rightarrow} M \rightarrow Q.$$ 

Thus, we have a natural isomorphism $Q \overset{\sim}{\rightarrow} Q_v$. Next, let $e$ be a node of $\mathcal E_1$; $b, b'$ the two distinct branches of $e, v, v'$ the [not necessarily distinct] vertices of $\mathcal E_1$ to which $b, b'$ abut. Then since the quotient $Q \overset{\sim}{\rightarrow} Q_v$ is finite, one may verify easily that — after possibly replacing $G_1 \rightarrow G$ by a connected finite étale Galois covering of $G$ that factors through $G_1 \rightarrow G$ — the kernels of the two composites $\Pi_{(\mathcal E_1)_v} \rightarrow \Pi_{(\mathcal E_1)_v} \rightarrow Q_v$, $\Pi_{(\mathcal E_1)_v} \rightarrow \Pi_{(\mathcal E_1)_v} \rightarrow Q_v$ — where $\Pi_{(\mathcal E_1)_v} \rightarrow \Pi_{(\mathcal E_1)_v}$, $\Pi_{(\mathcal E_1)_v} \rightarrow \Pi_{(\mathcal E_1)_v}$ are the natural outer injections corresponding to $b, b'$, respectively — coincide. Moreover, if we write $N_v \subseteq \Pi_{(\mathcal E_1)_v}$ for this kernel, then it follows immediately from condition (b) of Definition 3.4, (i), that the actions of $Q$ induced by the natural isomorphisms $Q \overset{\sim}{\rightarrow} Q_v \overset{\sim}{\rightarrow} \Pi_{(\mathcal E_1)_v}/N_v$, $Q \overset{\sim}{\rightarrow} Q_v \overset{\sim}{\rightarrow} \Pi_{(\mathcal E_1)_v}/N_v$ on the connected finite étale Galois covering of $(\mathcal E_1)_v$ corresponding to $N_v \subseteq \Pi_{(\mathcal E_1)_v}$ coincide. Therefore, since the underlying semi-graph of $\mathcal E_1$ is finite, by applying this argument to the various nodes of $\mathcal E_1$ and then gluing the connected finite étale Galois coverings of the various $(\mathcal E_1)_v$’s corresponding to the quotients $\Pi_{(\mathcal E_1)_v} \rightarrow Q_v$ to one another by means of $Q$-equivariant isomorphisms, we obtain a connected finite étale Galois covering $\mathcal E_2 \rightarrow \mathcal E_1$ which satisfies the condition $$(\mathfrak t^1_{Q, G_1})$$.

Write $\mathcal E^n_2 \rightarrow \mathcal E$ for the Galois closure of the connected finite étale covering $\mathcal E_2 \rightarrow \mathcal E$; thus, since $\mathcal E_1$ is Galois over $\mathcal E$, we have connected finite étale Galois coverings $\mathcal E^n_2 \rightarrow \mathcal E_2 \rightarrow \mathcal E_1$ of $\mathcal E_1$. Now it follows
immediately from the condition \((1_{Q_1,G_1})\) that \(E_2 \to E_1\) induces an isomorphism between the underlying semi-graphs. In particular, it follows from Lemma 3.6 below, in light of the claim \((*1)\), that the natural outer homomorphisms \(\Pi E_2 \to \Pi E_1 \to \Pi G_1\) induce outer isomorphisms \(\Pi E_2 / \Pi E_2 \iso \Pi E_1 / \Pi E_1 \iso \Pi G_1 / \Pi G_1 \iso \pi_1^{\text{top}}(G_1)^2\), where we write “\(\Pi(-) \subseteq \Pi(-)^*\)” for the normal closed subgroup of “\(\Pi(-)^*\)” topologically normally generated by the vertical subgroups and \(\pi_1^{\text{top}}(G_1)^2\) for the pro-\(\Sigma\) completion of the (discrete) topological fundamental group of the underlying semi-graph \(G_1\) of \(G_1\). On the other hand, since for each vertex \(v \in \text{Vert}(E) \iso \text{Vert}(G)\) and each vertex \(w \in \text{Vert}(E_1) \iso \text{Vert}(G_1)\) that lies over \(v\), the isomorphism \(t_w: M \times \Pi w \iso \Pi(\varepsilon_1)_w\) arises from the isomorphism \(\varepsilon_0: M \times \Pi w \iso \Pi E_v\), one may verify easily that the closed subgroup \(\Pi(\varepsilon_1)_w \subseteq \Pi E_v\) is normal. [Here, we regard \(w \in \text{Vert}(E_1)\) as an element of \(\text{Vert}(E_2)\) by the bijection \(\text{Vert}(E_2) \iso \text{Vert}(E_1)\) induced by \(E_2 \to E_1\).] In particular, it follows immediately that the connected finite étale Galois covering \(E_2^2 \to E_2\) arises from a normal open subgroup of the quotient \(\Pi E_2 \to \Pi E_2 / \Pi E_2 \iso \pi_1(G_1)^2\). Therefore, there exists a connected finite étale Galois covering \(G_1^i \to G\) that factors through \(G_1 \to G\) (and arises from a normal open subgroup of the quotient \(\Pi G_1 \to \pi_1^{\text{top}}(G_1)^2\)) such that the connected finite étale covering \(E_2 \times G, G_1^i\) of \(E\) is Galois. Now it follows immediately from the fact that \(E_2 \to E\) satisfies the condition \((1_{Q_1,G_1})\) that \(E_2 \times G, G_1^i\) to \(E\) satisfies both conditions \((1_{Q_1,G_1})\) and \((1_{Q_1,G_1})\), as desired. This completes the proof of the claim \((*2)\).

Next, we claim that

\[(**3)\): the composite \(E \to E / \alpha \iso G\), together with the composites

\[
M \mapsto \Pi E_v \mapsto \Pi E
\]

for \(v \in \text{Vert}(E)\), determine an exact sequence of profinite groups

\[
1 \to M \to \Pi E \to \Pi G \to 1.
\]

Indeed, it follows immediately from the claim \((*2)\) — by arguing as in the final portion of the proof of \((*2)\) — that any connected finite étale Galois covering of \(E\) is a subcovering of a covering of \(E\) which satisfies the condition \((1_{Q_1,G_1})\) for some finite quotient \(M \to Q\) of \(M\) and some connected finite étale Galois covering \(G_1\) of \(G\). Therefore, the exactness of the sequence in question follows immediately from the various definitions involved, together with the claim \((*1)\). This completes the proof of the claim \((**3)\).
Finally, we claim that

\((*)_4\): the exact sequence of profinite groups

\[1 \rightarrow M \rightarrow \Pi_E \rightarrow \Pi_G \rightarrow 1\]

of \((*)_3\) is central, i.e., if we write \(\rho: \Pi_G \rightarrow \text{Aut}(M)\) for the representation of \(\Pi_G\) on \(M\) determined by this extension \(\Pi_E\), then \(\rho\) is trivial.

Indeed, it follows immediately from condition (a) of Definition 3.4, (i), that \(\Pi_G^{\text{vert}} \subseteq \text{Ker}(\rho)\), where we write \(\Pi_G^{\text{vert}} \subseteq \Pi_G\) for the normal closed subgroup of \(\Pi_G\) topologically normally generated by the vertical subgroups of \(\Pi_G\). On the other hand, it follows immediately from condition (b) of Definition 3.4, (i), by “parallel transporting” along loops on \(G\), that the restriction to \(\pi_1^{\text{top}}(G) \subseteq \pi_1^{\text{top}}(G)^{\Sigma}\) of the representation \([\Pi_G \rightarrow \Pi_G^{\text{vert}} \xrightarrow{\sim} \pi_1^{\text{top}}(G)^{\Sigma} \rightarrow \text{Aut}(M)\) [cf. Lemma 3.6 below] induced by \(\rho\) — where we write \(\pi_1^{\text{top}}(G)\) for the [discrete] topological fundamental group of the semi-graph \(G\) and \(\pi_1^{\text{top}}(G)^{\Sigma}\) for the pro-\(\Sigma\) completion of \(\pi_1^{\text{top}}(G)\) — is trivial. In particular, since the subgroup \(\pi_1^{\text{top}}(G) \subseteq \pi_1^{\text{top}}(G)^{\Sigma}\) is dense, the representation \(\rho\) is trivial, as desired. This completes the proof of the claim \((*)_4\), hence also the proof of assertion (i).

Assertion (ii) follows immediately from the various definitions involved. Next, we verify assertion (iii). It follows immediately from assertion (i), together with Definition 3.4, (iii), that if we write \(\Pi_{GE}\) for the pro-\(\Sigma\) fundamental group of \(G_E\), then we have a natural exact sequence of profinite groups

\[1 \rightarrow M \rightarrow \Pi_{GE} \rightarrow \Pi_G \rightarrow 1.\]

On the other hand, it follows immediately from the definition of \(G_E\) that one may construct a tautological profinite covering of \(G_E\) [i.e., a pro-object of the category \(\mathcal{B}(G_E)\) that appears in the discussion following [SemiAn], Definition 2.1] equipped with a tautological action by \(E\). In particular, one obtains an outer surjection \(\Pi_{GE} \rightarrow E\) that is compatible with the respective outer surjections to \(\Pi_G\). Thus, one concludes from the “Five Lemma” that this outer surjection \(\Pi_{GE} \rightarrow E\) is an outer isomorphism, as desired. This completes the proof of assertion (iii). Assertion (iv) follows immediately, in light of assertions (i), (ii), (iii), from the various definitions involved. This completes the proof of Lemma 3.5.

Q.E.D.
Lemma 3.6 (Quotients by vertical subgroups). Let \( \mathcal{H} \) be a semi-graph of anabelioids. Write \( \Pi_\mathcal{H} \) for the pro-\( \Sigma \) fundamental group of \( \mathcal{H} \) [i.e., the pro-\( \Sigma \) quotient of the fundamental group of \( \mathcal{H} \)] and \( \Pi_\mathcal{H}^{\text{vert}} \subseteq \Pi_\mathcal{H} \) for the normal closed subgroup of \( \Pi_\mathcal{H} \) topologically normally generated by the vertical subgroups of \( \Pi_\mathcal{H} \). Then the natural injection \( \Pi_\mathcal{H}^{\text{vert}} \hookrightarrow \Pi_\mathcal{H} \) determines an exact sequence of profinite groups

\[
1 \longrightarrow \Pi_\mathcal{H}^{\text{vert}} \longrightarrow \Pi_\mathcal{H} \longrightarrow \pi_1^{\text{top}}(\mathcal{H}) \longrightarrow 1
\]

— where we write \( \pi_1^{\text{top}}(\mathcal{H})^\Sigma \) for the pro-\( \Sigma \) completion of the [discrete] topological fundamental group \( \pi_1^{\text{top}}(\mathcal{H}) \) of the underlying semi-graph \( \mathcal{H} \).

Proof. This follows immediately from the various definitions involved. Q.E.D.

Theorem 3.7 (Properties of the second cohomology group with compact supports). Let \( \Sigma \) be a nonempty set of prime numbers, \( \mathcal{G} \) a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type, and \( M \) a finitely generated \( \mathbb{Z} \)-module. Then the following hold:

(i) (Change of coefficients) There exists a natural isomorphism of \( \widehat{\mathbb{Z}}^\Sigma \)-modules

\[
H^2_c(\mathcal{G}, M) \sim H^2(\mathcal{G}, \widehat{\mathbb{Z}}^\Sigma) \otimes_{\widehat{\mathbb{Z}}^\Sigma} M
\]

that is functorial with respect to isomorphisms of the pair \((\mathcal{G}, M)\). If, moreover, \( \text{Cusp}(\mathcal{G}) = \emptyset \), then there exists a natural isomorphism of \( \widehat{\mathbb{Z}}^\Sigma \)-modules

\[
H^2(\mathcal{G}, M) \sim H^2(\mathcal{G}, M)
\]

that is functorial with respect to isomorphisms of the pair \((\mathcal{G}, M)\).

(ii) (Structure as an abstract profinite group) The second cohomology group with compact supports \( H^2_c(\mathcal{G}, M) \) of \( \mathcal{G} \) is [non-canonically] isomorphic to \( M \).

(iii) (Synchronization with respect to generization) Let \( S \subseteq \text{Node}(\mathcal{G}) \) be a subset of \( \text{Node}(\mathcal{G}) \). Then the specialization outer isomorphism \( \Phi_{\mathcal{G} \rightarrow S} : \Pi_{\mathcal{G} \rightarrow S} \sim \Pi_\mathcal{G} \) with respect to \( S \) [cf. Definition 2.10] determines a natural isomorphism

\[
H^2_c(\mathcal{G}, M) \sim H^2_c(\mathcal{G} \rightarrow S, M)
\]
that is functorial with respect to isomorphisms of the triple $(\mathcal{G}, S, M)$.

(iv) (Synchronization with respect to “surgery”) Let $\mathcal{H}$ be a sub-semi-graph of PSC-type [cf. Definition 2.2, (i)] of $\mathcal{G}$, $S \subseteq \text{Node}(\mathcal{G}_{\text{HS}})$ [cf. Definition 2.2, (ii)] a subset of Node($\mathcal{G}_{\text{HS}}$) that is not of separating type [cf. Definition 2.5, (i)], and $T \subseteq \text{Cusp}((\mathcal{G}_{\text{HS}})_S)$ [cf. Definition 2.5, (ii)] an omittable [cf. Definition 2.4, (i)] subset of Cusp((\mathcal{G}_{\text{HS}})_S). Then there exists a natural isomorphism — given by “extension by zero” —

\[ H^2_c((\mathcal{G}_{\text{HS}})_S \rtimes T, M) \sim H^2_c(\mathcal{G}, M) \]

[cf. Definition 2.4, (ii)] that is functorial with respect to isomorphisms of the quintuple $(\mathcal{G}, \mathcal{H}, S, T, M)$. In particular, for each vertex $v \in \text{Vert}(\mathcal{G})$ of $\mathcal{G}$, there exists a natural isomorphism of $\mathbb{Z}^\Sigma$-modules

\[ H^2_c(v, M) \sim H^2_c(\mathcal{G}, M) \]

[cf. Remark 2.5.1, (ii)] that is functorial with respect to isomorphisms of the triple $(\mathcal{G}, v, M)$.

(v) (Homomorphisms induced by finite étale coverings) Let $\mathcal{H} \to \mathcal{G}$ be a connected finite étale covering of $\mathcal{G}$. Then the image of the natural homomorphism

\[ H^2_c(\mathcal{G}, M) \to H^2_c(\mathcal{H}, M) \]

is given by

\[ [\Pi_\mathcal{G} : \Pi_\mathcal{H}] \cdot H^2_c(\mathcal{H}, M). \]

Proof. Assertion (iii) follows immediately from condition (1) of Proposition 2.9, (i).

Next, we verify assertions (i), (ii) in the case where Cusp($\mathcal{G}$) $\neq \emptyset$. The existence of a natural isomorphism $H^2_c(\mathcal{G}, M) \sim H^2_c(\mathcal{G}, \mathbb{Z}^\Sigma) \oplus \mathbb{Z}^\Sigma$ $M$ follows immediately from Lemma 3.3. On the other hand, the fact that $H^2_c(\mathcal{G}, M)$ is [noncanonically] isomorphic to $M$ follows immediately from Lemma 3.3, together with the following well-known facts [cf. [CmbGC], Remark 1.1.3]:

(A) $\Pi_\mathcal{G}$ is a free pro-$\Sigma$ group.
(B) For any cusp $e_0 \in \text{Cusp}(G)$ of $G$, the natural homomorphism of $\mathbb{Z}^\Sigma$-modules

$$\bigoplus_{e \in \text{Cusp}(G) \setminus \{e_0\}} \Pi_e \to \Pi^\text{ab}_G$$

is a split injection of free $\mathbb{Z}^\Sigma$-modules [cf. the discussion entitled “Topological groups” in §0], and its image contains the image of $\Pi_{e_0}$ in $\Pi^\text{ab}_G$.

This completes the proof of assertions (i), (ii) in the case where $\text{Cusp}(G) \neq \emptyset$.

Next, we verify assertions (i), (ii) in the case where $\text{Cusp}(G) = \emptyset$. The existence of a natural isomorphism $H^2_c(G, M)\cong H^2_c(G, \mathbb{Z}^\Sigma) \otimes \mathbb{Z}_G$ and the fact that $H^2_c(G, M)$ is non-canonical isomorphic to $M$ follow immediately from the existence of a natural isomorphism $H^2_c(G, M)\cong H^2_c(G, \mathbb{Z}_G)$ and the fact that any compact Riemann surface of genus $g \neq 0$ is a $K(\pi, 1)$ space [i.e., its universal covering is contractible], together with the well-known structure of the second cohomology group of a compact Riemann surface. This completes the proof of assertions (i), (ii) in the case where $\text{Cusp}(G) = \emptyset$.

Next, we verify assertion (iv) in the case where $H = G$ and $S = \emptyset$, i.e., $(G_{[\Sigma]})_s = G_{\Sigma}$. Thus, suppose that $H = G$ and $S = \emptyset$. Now define a homomorphism of $\mathbb{Z}^\Sigma$-modules

$$H^2_c(G_{\Sigma}, M) \to H^2_c(G, M)$$

as follows: Let $G_{\Sigma} \to G_{\Sigma}$ be a universal covering of $G_{\Sigma}$, which is compatible [in the evident sense], with the universal covering $\tilde{G} \to G$ of $G$, $s^\bullet$ a section of the natural surjection $\text{Cusp}(G_{\Sigma}) \to \text{Cusp}(G)$, and $[E^\bullet, (\iota_e^\bullet : \Pi_e \to E^\bullet)_{e \in \text{Cusp}(G_{\Sigma})}] \in H^2_c(G_{\Sigma}, M)$ an element of $H^2_c(G_{\Sigma}, M)$. Write $E$ for the fiber product of the surjection $E^\bullet \to \Pi_{G_{\Sigma}}$ and the natural surjection $\Pi_G \to \Pi_{G_{\Sigma}}$ arising from the compatibility of the respective universal coverings]. Next, we introduce notation as follows:

- for $e \in \text{Cusp}(G_{\Sigma})$ (= $\text{Cusp}(G) \setminus T \subseteq \text{Cusp}(G)$), denote by $\iota_e : \Pi_e \to E$ — where we use the notation $\Pi_e \subseteq \Pi_G$ to denote an edge-like subgroup of $\Pi_G$ associated to $e$ such that the
composite $\Pi_e \hookrightarrow \Pi_G \twoheadrightarrow \Pi_{G_{\mathfrak{r}}}^e$ determines an isomorphism of $\Pi_e$ with $\Pi_{s(e)}^e \subseteq \Pi_{G_{\mathfrak{r}}}^e$ — the section over $\Pi_e$ naturally determined by the composite

$$\Pi_e \sim \to \Pi_{s(e)}^e \sim \to E^\bullet,$$

and

- for $e \in \text{Cusp}(G) \setminus \text{Cusp}(G_{T})$ ($= T \subseteq \text{Cusp}(G)$), denote by $\iota_e : \Pi_e \to E$ — where we use the notation $\Pi_e \subseteq \Pi_G$ to denote an edge-like subgroup of $\Pi_G$ associated to $e$ — the section over $\Pi_e$ naturally determined by the trivial homomorphism $\Pi_e \to E^\bullet$.

Then it follows immediately from the various definitions involved that the assignment $\left[E^\bullet, (\iota_e^e)_{e \in \text{Cusp}(G_{T})}\right] \mapsto (E, (\iota_e)_{e \in \text{Cusp}(G)})$ determines a homomorphism of $\hat{\mathbb{Z}}^\Sigma$-modules

$$H^2_\ell(G_{T}, M) \longrightarrow H^2_\ell(G, M),$$

as desired.

Next, we verify that this homomorphism $H^2_\ell(G_{T}, M) \to H^2_\ell(G, M)$ is an isomorphism. First, let us observe that it follows from assertion (ii) that, to verify that the homomorphism in question is an isomorphism, it suffices to verify that it is surjective. The rest of the proof of assertion (iv) in the case where $H = \mathcal{G}$ and $S = \emptyset$ is devoted to verifying this surjectivity. To verify the desired surjectivity, by induction on the cardinality $T^2$ of the finite set $T$, we may assume without loss of generality that $T^2 = 1$, i.e., $T = \{e_0\}$ for some $e_0 \in \text{Cusp}(G)$.

To verify the desired surjectivity, let $[E, (\iota_e)_{e \in \text{Cusp}(G)}] \in H^2_\ell(G, M)$ be an element of $H^2_\ell(G, M)$. Then since $\Pi_G$ is a free pro-$\Sigma$ group, there exists a continuous section $\Pi_G \to E$ of the surjection $E \twoheadrightarrow \Pi_G$, hence also — since the extension $E$ of $\Pi_G$ is central — an isomorphism $M \times \Pi_G \to E$. Write $\Pi_G \to \Pi$ for the maximal cuspidally central quotient [cf. [AbsCsp], Definition 1.1, (i)] relative to the surjection $\Pi_G \to \Pi_{G_{\mathfrak{r}}}$, $E_{\Pi}$ for the quotient of $E$ by the normal closed subgroup of $E$ corresponding to $\{1\} \times \ker(\Pi_G \to \Pi) \subseteq M \times \Pi_G$ [thus, $E_{\Pi} \sim M \times \Pi$], and $N \subseteq E_{\Pi}$ for the image of the composite

$$\Pi_{s(e_0)}^e \longleftarrow E \twoheadrightarrow E_{\Pi}.$$

Now we claim that $N \subseteq E_{\Pi}$ is contained in the center $Z(E_{\Pi})$ of $E_{\Pi}$, hence also normal in $E_{\Pi}$. Indeed, since the composite

$$\Pi_{s(e_0)}^e \hookrightarrow \Pi_G \twoheadrightarrow \Pi$$
is injective, and its image coincides with the kernel of the natural surjection \( \Pi \to \Pi_{G^sT} \); it holds that the image of the composite

\[
\Pi_{s(e)} \overset{\iota_e} \to E \to E_{\Pi} \overset{\sim} \to M \times \Pi
\]

is contained in \( M \times \text{Ker}(\Pi \to \Pi_{G^sT}) \). On the other hand, since the extension \( E \) of \( \Pi_G \) is central, it follows from the definition of the quotient \( \Pi \) of \( \Pi_G \) that the image of \( M \times \text{Ker}(\Pi \to \Pi_{G^sT}) \) in \( E_{\Pi} \) via \( M \times \Pi \overset{\sim} \to E_{\Pi} \) is contained in the center \( Z(E_{\Pi}) \) of \( E_{\Pi} \). This completes the proof of the above claim.

Now it follows from the definition of \( N \subseteq E_{\Pi} \), together with the above claim, that we obtain a commutative diagram of profinite groups

\[
\begin{array}{ccc}
1 & \to & M & \to & E & \to & \Pi_G & \to & 1 \\
\| & & \| & & \| & & \| \end{array}
\]

\[
\begin{array}{ccc}
1 & \to & M & \to & E_{\Pi}/N & \to & \Pi_{G^sT} & \to & 1 \\
\| & & \| & & \| & & \| \end{array}
\]

where the horizontal sequences are exact, and the vertical arrows are surjective. In particular, we obtain an extension \( E_{\Pi}/N \) of \( \Pi_{G^sT} \) by \( M \), which is central since the extension \( E \) is central. For \( e \in \text{Cusp}(G^*T) = \text{Cusp}(G) \setminus \{e_0\} \), write \( \Pi^*_e \subseteq \Pi_{G^sT} \) for the edge-like subgroup of \( \Pi_{G^sT} \) [associated to \( e \in \text{Cusp}(G^*T) \)] determined by the image of \( \Pi_{e(e)} \subseteq \Pi_G \) and \( \iota_e \) for the section \( \Pi^*_e \to E_{\Pi}/N \) over \( \Pi^*_e \) determined by \( \iota_e \); \( \Pi_{e(e)} \to E \).

Then it follows immediately from the various definitions involved that the image of

\[
[E_{\Pi}/N, (\iota^*_e)_{e \in \text{Cusp}(G^*T)}] \in H^2_\pi(G^*T, M)
\]

in \( H^2_\pi(G, M) \) is \([E, (\iota_e)_{e \in \text{Cusp}(G)}] \in H^2_\pi(G, M)\). This completes the proof of the desired surjectivity and hence of assertion (iv) in the case where \( \mathbb{H} = G \) and \( S = \emptyset \).

Next, to complete the proof of assertion (iv) in the general case, one verifies immediately that it suffices to verify assertion (iv) in the case where \( T = \emptyset \), i.e., \( ((G_{[\mathbb{H}]})_\pi)_S = (G_{[\mathbb{H}]})_\pi \). Thus, suppose that \( T = \emptyset \). Write \( \mathcal{H} \overset{\text{def}}{=} (G_{[\mathbb{H}]})_\pi \). To define a natural homomorphism of \( \mathbb{Z} \)-modules \( H^2_\pi(\mathcal{H}, M) \to H^2_\pi(G, M) \), let \( \mathcal{H} \to \mathcal{H} \) be a universal covering of \( \mathcal{H} \) which is compatible [in the evident sense] with the universal covering \( \mathcal{G} \to G \), \( s_{\mathcal{H}} \) a section of the natural surjection \( \text{Cusp}(\mathcal{H}) \to \text{Cusp}(\mathcal{H}) \), and \([E^H, (\iota^H_e)_{e \in \text{Cusp}(\mathcal{H})}] \in H^2_\pi(\mathcal{H}, M) \) an element of \( H^2_\pi(\mathcal{H}, M) \). Since the extension \( E^H \) of \( \Pi_H \) by \( M \) is central, the section \( \iota^H_e : \Pi_{s_{\mathcal{H}}(c)} \to E^H \) naturally determines an isomorphism

\[
M \times \Pi_{s_{\mathcal{H}}(e)} \overset{\sim} \to E^H \times_{\Pi_H} \Pi_{s_{\mathcal{H}}(c)}
\]
of the direct product $M \times \Pi_{s\mathcal{H}(e)}$ with the fiber product $E^H \times_{\Pi_H} \Pi_{s\mathcal{H}(e)}$ of the surjection $E^H \twoheadrightarrow \Pi_H$ and the natural inclusion $\Pi_{s\mathcal{H}(e)} \hookrightarrow \Pi_H$. Write $\mathcal{G}_{EN}$ for the semi-graph of anabelioids associated to the central extension $E^H$ (cf. Definition 3.4, (ii)). Then one may define a central extension of $\mathcal{G}$ by $M$

$(\mathcal{E}, \alpha, \beta : \mathcal{E}/\alpha \xrightarrow{\sim} \mathcal{G})$

[cf. Definition 3.4, (i)] whose restriction to $\mathcal{H}$, relative to the isomorphism $\beta : \mathcal{E}/\alpha \xrightarrow{\sim} \mathcal{G}$, is isomorphic to the semi-graph of anabelioids $\mathcal{G}_{EN}$ as follows: We take the underlying semi-graph of $\mathcal{E}$ to be the underlying semi-graph of $\mathcal{G}$; for each vertex $v \in \text{Vert}(\mathcal{G}[E])$, we take the anabelioid $\mathcal{E}_v$ of $\mathcal{E}$ corresponding to the vertex $v \in \text{Vert}(\mathcal{G}[E])$ to be the anabelioid $(\mathcal{G}_{EN})_v$ of $\mathcal{G}_{EN}$ corresponding to the vertex $v$; for each vertex $v \in \text{Vert}(\mathcal{G}) \setminus \text{Vert}(\mathcal{G}[E])$, we take the anabelioid $\mathcal{E}_v$ of $\mathcal{E}$ corresponding to $v \in \text{Vert}(\mathcal{G}) \setminus \text{Vert}(\mathcal{G}[E])$ to be the anabelioid associated to the profinite group $M \times \Pi_v$. Then the above isomorphisms $M \times \Pi_{s\mathcal{H}(e)} \xrightarrow{\sim} E^H \times_{\Pi_H} \Pi_{s\mathcal{H}(e)}$ induced by the various $\iota^H_e$’s naturally determine the remaining data [i.e., consisting of anabelioids associated to edges and morphisms of anabelioids associated to branches] necessary to define a semi-graph of anabelioids $\mathcal{E}$ which is naturally equipped with a structure of central extension of $\mathcal{G}$ by $M$ whose restriction to $\mathcal{H}$ is naturally isomorphic to the semi-graph of anabelioids $\mathcal{G}_{EN}$, as desired.

Now it follows from Lemma 3.5, (i), that if we denote by $\Pi_{\mathcal{E}}$ the pro-$\Sigma$ fundamental group of $\mathcal{E}$ — i.e., the maximal pro-$\Sigma$ quotient of the fundamental group of $\mathcal{E}$ — then $\Pi_{\mathcal{E}}$ is a central extension of $\Pi_{\mathcal{G}}$ by $M$. Thus, it follows from the equivalences of categories of Lemma 3.5, (iv), that the sections $\iota^H_e$ — where $e$ ranges over the cusps of $\mathcal{G}$ that abut to a vertex of $\mathcal{G}[E]$ — and the tautological sections $\Pi_{\mathcal{E}}' \hookrightarrow M \times \Pi_{\mathcal{E}}' = \Pi_{\mathcal{E}}$, — where $e'$ ranges over the cusps of $\mathcal{G}$ that do not abut to a vertex of $\mathcal{G}[E]$ — naturally determine an equivalence class $[\Pi_{\mathcal{E}}', (\iota_e')_{e \in \text{Cusp}(\mathcal{G})}] \in H^2_c(\mathcal{G}, M)$. In particular, we obtain a map

$$H^2_c(\mathcal{H}, M) \longrightarrow H^2_c(\mathcal{G}, M)$$

by assigning $[E^H, (\iota^H_e)_{e \in \text{Cusp}(\mathcal{H})}] \mapsto [\Pi_{\mathcal{E}}', (\iota_e')_{e \in \text{Cusp}(\mathcal{G})}]$. Moreover, it follows immediately from the various definitions involved that this map is a homomorphism of $\mathbb{G}$-modules, as desired.

Next, we verify that this homomorphism $H^2_c(\mathcal{H}, M) \rightarrow H^2_c(\mathcal{G}, M)$ is an isomorphism. Since, for any vertex $v \in \text{Vert}(\mathcal{G}[E])$, the natural morphism $\mathcal{G}_v \rightarrow \mathcal{G}$ factors through $(\mathcal{G}[E]_{/S} = \mathcal{H} \rightarrow \mathcal{G}$, by replacing $\mathcal{H}$ by $\mathcal{G}_v$ [cf. Remark 2.5.1, (ii)], we may assume without loss of generality that $\mathcal{H} = \mathcal{G}_v$. Moreover, if Node($\mathcal{G}$) = $\emptyset$, then assertion (iv) in the case where $T = \emptyset$ is immediate; thus, we may assume without loss
of generality that Node(G) \neq \emptyset. On the other hand, it follows from assertion (ii) that to verify that the homomorphism in question is an isomorphism, it suffices to verify that it is surjective. The rest of the proof of assertion (iv) in the case where T = \emptyset is devoted to verifying the surjectivity of the homomorphism H^2_v(v, M) \to H^2_v(G, M).

Let J be a semi-graph of anabelioids of pro-S PSC-type such that there exist a vertex w \in \text{Vert}(J) and an "omittable" cusp e \in C(w) [i.e., a cusp that abuts to w such that \{e\} is omittable] such that J_{\bullet\{e\}} \to G induces an isomorphism of (J|_w)_{\bullet\{e\}} \to G|_v. [Note that one may verify easily that such a semi-graph of anabelioids of pro-S PSC type always exists.] Then it follows immediately from assertion (iv) in the case where H = G and S = \emptyset, together with the various definitions involved, that we have a commutative diagram

\[
\begin{array}{ccc}
H^2_v(v, M) & \longrightarrow & H^2_v((J|_w)_{\bullet\{e\}}, M) & \longrightarrow & H^2_v(w, M) \\
\downarrow & & & & \downarrow & \\
H^2_v(G, M) & \longrightarrow & H^2_v(J_{\bullet\{e\}}, M) & \longrightarrow & H^2_v(J, M)
\end{array}
\]

where the left-hand horizontal arrows are isomorphisms induced by the isomorphisms (J|_w)_{\bullet\{e\}} \to G|_v, J_{\bullet\{e\}} \to G, respectively, and the right-hand horizontal arrows are isomorphisms obtained by applying assertion (iv) in the case where H = G and S = \emptyset. In particular, to verify the desired surjectivity of the homomorphism H^2_v(v, M) \to H^2_v(G, M), by replacing G (respectively, v) by J (respectively, w), we may assume without loss of generality that C(v) \neq \emptyset.

To verify the desired surjectivity of the homomorphism H^2_v(v, M) \to H^2_v(G, M) in the case where C(v) \neq \emptyset, let [E, (t_e)e\in\text{Cusp}(G)] \in H^2_v(G, M) be an element of H^2_v(G, M). Now it follows from Lemma 3.3, together with the assumption that C(v) \neq \emptyset, that we have two exact sequences of \hat{Z}^\Sigma-modules

\[
\begin{align*}
\text{Hom}_{\hat{Z}^\Sigma}(\Pi G, M) & \longrightarrow \bigoplus_{e\in\text{Cusp}(G)} \text{Hom}_{\hat{Z}^\Sigma}(\Pi e, M) & \longrightarrow & H^2_v(G, M) & \longrightarrow & 0; \\
\text{Hom}_{\hat{Z}^\Sigma}(\Pi G|_v, M) & \longrightarrow \bigoplus_{e\in\text{Cusp}(G|_v)} \text{Hom}_{\hat{Z}^\Sigma}(\Pi e, M) & \longrightarrow & H^2_v(v, M) & \longrightarrow & 0 .
\end{align*}
\]

Let c_0 \in C(v) be a cusp of G that abuts to v. Here, note that it follows immediately from the definition of G|_v that c_0 may be regarded as a cusp of G|_v. Then it follows immediately from the facts (A), (B) used in the proof of assertions (i), (ii) in the case where Cusp(G) \neq \emptyset.

\[
\begin{array}{ccc}
\text{Hom}_{\hat{Z}^\Sigma}(\Pi G, M) & \longrightarrow & \bigoplus_{e\in\text{Cusp}(G)} \text{Hom}_{\hat{Z}^\Sigma}(\Pi e, M) \\
\text{Hom}_{\hat{Z}^\Sigma}(\Pi G|_v, M) & \longrightarrow & \bigoplus_{e\in\text{Cusp}(G|_v)} \text{Hom}_{\hat{Z}^\Sigma}(\Pi e, M)
\end{array}
\]
that there exists a lifting \((\phi_e)_{e \in \text{Cusp}(G)} \in \bigoplus_{e \in \text{Cusp}(G)} \text{Hom}\mathcal{M}(\Pi_e, M)\) of \([E, (t_e)_{e \in \text{Cusp}(G)}] \in H^2_c(G, M)\) [with respect to the first exact sequence of the above display] such that if \(e \neq e_0\), then \(\phi_e = 0\). Write \((\psi_e)_{e \in \text{Cusp}(G)} \in \bigoplus_{e \in \text{Cusp}(G)} \text{Hom}\mathcal{M}(\Pi_e, M)\) for the element such that \(\psi_{e_0} = \phi_{e_0}, \psi_e = 0\) for \(e \neq e_0\). Then it follows immediately from the definitions of the above exact sequences and the homomorphism \(H^2(v, M) \rightarrow H^2_c(G, M)\) in question that the image of \((\psi_e)_{e \in \text{Cusp}(G)} \in \bigoplus_{e \in \text{Cusp}(G)} \text{Hom}\mathcal{M}(\Pi_e, M)\) in \(H^2(v, M)\) is mapped to \([E, (t_e)_{e \in \text{Cusp}(G)}]\) \(\in H^2_c(G, M)\) via the homomorphism \(H^2(v, M) \rightarrow H^2_c(G, M)\) in question. This completes the proof of assertion (iv) in the case where \(T = \emptyset\), hence also of assertion (iv) in the general case.

Finally, we verify assertion (v). If \(\text{Cusp}(G) = \emptyset\), then it follows immediately from a similar argument to the argument used in the proof of assertions (i), (ii) in the case where \(\text{Cusp}(G) = \emptyset\), together with the well-known structure of the second cohomology group of a compact Riemann surface, that assertion (v) holds. Next, suppose that \(\text{Cusp}(G) \neq \emptyset\).

Write \(\text{Cusp}(G)\) for the double of \(\text{Cusp}(G)\) \([\text{cf. [CmbGC], Proposition 2.2, (i)}]\) — i.e., the analogue in the theory of semi-graphs of anabelioids of pro-$\Sigma$ PSC-type to the well-known “double” of a Riemann surface with boundary. Write \(\text{H}^\circ\) for the double of \(\text{H}\). Then it follows from the various definitions involved that the connected finite étale covering \(\text{H} \rightarrow \text{G}\) determines a connected finite étale covering \(\text{H}^\circ \rightarrow \text{G}^\circ\) of degree \([\Pi_G : \Pi_H]\). Next, let us observe \(\text{G}\) (respectively, \(\text{H}\)) may be naturally identified with the restriction \([\text{cf. Definition 2.2, (ii)}]\) of \(\text{G}^\circ\) (respectively, \(\text{H}^\circ\)) to a suitable sub-semi-graph of PSC-type of the underlying semi-graph of \(\text{G}^\circ\) (respectively, \(\text{H}^\circ\)). Thus, it follows from assertion (iv) that we have a commutative diagram of \(\mathbb{Z}^\Sigma\)-modules

\[
\begin{array}{ccc}
H^2_c(G, M) & \xrightarrow{\sim} & H^2_c(G^\circ, M) \\
\downarrow & & \downarrow \\
H^2_c(H, M) & \xrightarrow{\sim} & H^2_c(H^\circ, M)
\end{array}
\]

where the horizontal arrows are the isomorphisms of assertion (iv), and the vertical arrows are the homomorphisms induced by the connected finite étale coverings \(\text{H} \rightarrow \text{G}, \text{H}^\circ \rightarrow \text{G}^\circ\), respectively — and hence that assertion (v) in the case where \(\text{Cusp}(G) \neq \emptyset\) follows immediately from assertion (v) in the case where \(\text{Cusp}(G) = \emptyset\). This completes the proof of assertion (v).

Q.E.D.
Definition 3.8.

(i) We shall write
\[ \Lambda_G \overset{\text{def}}{=} \text{Hom}_{\mathbb{Z}_l}(H^2_c(G, \hat{\mathbb{Z}}^\Sigma), \hat{\mathbb{Z}}^\Sigma) \]
and refer to \( \Lambda_G \) as the cyclotome associated to \( G \). For a vertex \( v \in \text{Vert}(G) \) of \( G \), we shall write
\[ \Lambda_v \overset{\text{def}}{=} \text{Hom}_{\mathbb{Z}_l}(H^2_c(v, \hat{\mathbb{Z}}^\Sigma), \hat{\mathbb{Z}}^\Sigma) \]
and refer to \( \Lambda_v \) as the cyclotome associated to \( v \in \text{Vert}(G) \). Note that it follows from Theorem 3.7, (ii), that the cyclotomes \( \Lambda_G \) and \( \Lambda_v \) are free \( \hat{\mathbb{Z}}^\Sigma \)-modules of rank 1.

(ii) We shall write
\[ \chi_G : \text{Aut}(G) \longrightarrow \text{Aut}(\Lambda_G) \cong (\hat{\mathbb{Z}}^\Sigma)^* \]
for the natural homomorphism induced by the natural action of \( \text{Aut}(G) \) on \( H^2_c(G, \hat{\mathbb{Z}}^\Sigma) \) and refer to \( \chi_G \) as the pro-\( \Sigma \) cyclotomic character of \( G \). For a vertex \( v \in \text{Vert}(G) \) of \( G \), we shall write
\[ \chi_v \overset{\text{def}}{=} \chi_{G|_v} : \text{Aut}(G|_v) \longrightarrow \text{Aut}(\Lambda_v) \cong (\hat{\mathbb{Z}}^\Sigma)^* \]
and refer to \( \chi_v \) as the pro-\( \Sigma \) cyclotomic character of \( v \).

Remark 3.8.1. One verifies easily that if \( l \in \Sigma \), then the composite
\[ \text{Aut}(G) \overset{\chi}{\longrightarrow} (\hat{\mathbb{Z}}^\Sigma)^* \overset{l}{\longrightarrow} \mathbb{Z}_l^* \]
coincides with the pro-\( l \) cyclotomic character of \( \text{Aut}(G) \) defined in the statement of [CmbGC], Lemma 2.1.

Corollary 3.9 (Synchronization of cyclotomes). Let \( \Sigma \) be a nonempty set of prime numbers and \( G \) a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type. Then the following hold:

(i) (Synchronization with respect to generization) Let \( S \subseteq \text{Node}(G) \) be a subset of \( \text{Node}(G) \). Then the specialization outer
isomorphism $\Phi_{G \rightarrow S}: \Pi_{G \rightarrow S} \xrightarrow{\sim} \Pi_G$ with respect to $S$ [cf. Definition 2.10] determines a natural isomorphism

$$
\Lambda_{G \rightarrow S} \xrightarrow{\sim} \Lambda_G
$$

that is functorial with respect to isomorphisms of the pair $(G, S)$.

(ii) (Synchronization with respect to “surgery”) Let $H$ be a sub-semi-graph of PSC-type [cf. Definition 2.2, (i)] of $G$, $S \subset \text{Node}(G|_H)$ [cf. Definition 2.2, (ii)] a subset of $\text{Node}(G|_H)$ that is not of separating type [cf. Definition 2.5, (i)], and $T \subset \text{Cusp}((G|_H), S)$ [cf. Definition 2.5, (ii)] an omittable [cf. Definition 2.4, (i)] subset of $\text{Cusp}((G|_H), S)$. Then there exists a natural isomorphism — given by “extension by zero”

$$
\Lambda_G \xrightarrow{\sim} \Lambda_{((G|_H) \setminus S) \setminus T}
$$

[cf. Definition 2.4, (ii)] that is functorial with respect to isomorphisms of the quadruple $(G, H, S, T)$. In particular, by taking the inverse of this isomorphism we obtain, for each vertex $v \in \text{Vert}(G)$ of $G$, a natural isomorphism of $\mathbb{Z}$-modules

$$
\kappa : \Lambda_v \xrightarrow{\sim} \Lambda_G
$$

that is functorial with respect to isomorphisms of the pair $(G, v)$.

(iii) (Synchronization with respect to finite étale coverings) Let $H \rightarrow G$ be a connected finite étale covering of $G$. Then there exists a natural isomorphism

$$
\Lambda_H \xrightarrow{\sim} \Lambda_G
$$

that is functorial with respect to isomorphisms of the pair $(G, H)$.

(iv) (Synchronization of cyclotomic characters) Let $v \in \text{Vert}(G)$ be a vertex of $G$ and $\alpha \in \text{Aut}^{(v)}(G)$ [cf. Definition 2.6, (i)]. Then it holds that

$$
\chi_G(\alpha) = \chi_v(\alpha|_v)
$$

[cf. Definitions 2.14, (ii); 3.8, (ii); Remark 2.5.1, (ii)].
(v) (Synchronization associated to branches) Let \( e \in \text{Edge}(G) \) be an edge of \( G \), \( b \) a branch of \( e \) that abuts to a vertex \( v \in \mathcal{V}(e) \), and \( \Pi_e \subseteq \Pi_G \) an edge-like subgroup of \( \Pi_G \) associated to \( e \in \text{Edge}(G) \). Then there exists a **natural isomorphism**
\[
\text{syn}_b : \Pi_e \xrightarrow{\sim} \Lambda_v
\]
that is **functorial** with respect to isomorphisms of the quadruple \((G, b, e, v)\).

\[
\text{(vi) (Difference between two synchronizations associated to the two branches of a node)}
\]
Let \( e \in \text{Node}(G) \) be a node of \( G \) with branches \( b_1 \neq b_2 \) that abut to vertices \( v_1, v_2 \in \text{Vert}(G) \), respectively. Then the two composites
\[
\Pi_e \xrightarrow{\text{syn}_{b_1}} \Lambda_{v_1} \xrightarrow{\text{syn}_{v_1}} \Lambda_G ; \quad \Pi_e \xrightarrow{\text{syn}_{b_2}} \Lambda_{v_2} \xrightarrow{\text{syn}_{v_2}} \Lambda_G
\]
differ by the automorphism of \( \Lambda_G \) given by multiplication by 
\[ -1 \in \hat{\mathbb{Z}}. \]

**Proof.** Assertion (i) (respectively, (ii)) follows immediately from Theorem 3.7, (iii) (respectively, Theorem 3.7, (iv)). Assertion (iv) follows immediately from assertion (ii).

Next, we verify assertion (iii). It follows immediately from Theorem 3.7, (v), that the homomorphism of \( \hat{\mathbb{Z}}\Sigma \)-modules \( \Lambda_H \to \Lambda_G \) obtained by applying the functor \( \text{Hom}_{\hat{\mathbb{Z}}\Sigma}(\cdot, \hat{\mathbb{Z}}\Sigma) \) to the induced homomorphism \( H^2_c(G, \hat{\mathbb{Z}}\Sigma) \to H^2_c(H, \hat{\mathbb{Z}}\Sigma) \) and dividing by the index \([\Pi_G : \Pi_H] \) is an **isomorphism**. This completes the proof of assertion (iii).

Next, we verify assertion (v). First, we observe that to verify assertion (v), by replacing \( G \) by \( G|_v \) and \( e \in \text{Edge}(G) \) by the cusp of \( G|_v \) corresponding to \( b \), we may assume without loss of generality that \( e \in \text{Cusp}(G) \). Then we have homomorphisms of \( \hat{\mathbb{Z}}\Sigma \)-modules
\[
\text{Hom}_{\hat{\mathbb{Z}}\Sigma}(\Pi_e, \Pi_L) \xrightarrow{\sim} \bigoplus_{e' \in \text{Cusp}(G)} \text{Hom}_{\hat{\mathbb{Z}}\Sigma}(\Pi_{e'}, \Pi_e)
\]
\[
\to H^2_c(G, \Pi_e) \xrightarrow{\sim} \text{Hom}_{\hat{\mathbb{Z}}\Sigma}(\Lambda_G, \Pi_e)
\]
— where the first arrow is the natural inclusion into the component indexed by \( e \), and the second arrow is the surjection appearing in the exact sequence of Lemma 3.3 in the case where \( M = \Pi_e \). Here, we note that it follows immediately from the facts (A), (B) used in the proof of Theorem 3.7, (i), (ii), that the composite of these homomorphisms is an
isomorphism. Therefore, we obtain a natural isomorphism

$$\text{syn}_G : \Pi_e \overset{\sim}{\rightarrow} \Lambda_G$$

by forming the inverse of the image of the identity automorphism of \(\Pi_e\) via the composite of the homomorphisms of the above display. This completes the proof of assertion (v).

Finally, we verify assertion (vi). First, we observe that one may verify easily that there exist

- a semi-graph of anabelioids of pro-\(\Sigma\) PSC-type \(\mathcal{H}^1\),

- a sub-semi-graph of PSC-type \(\mathcal{K}^1\) of the underlying semi-graph of \(\mathcal{H}^1\),

- an ommittable subset \(S^1 \subseteq \text{Cusp}(\mathcal{H}^1|_{\mathcal{K}^1})\), and

- an isomorphism

$$((\mathcal{H}^1|_{\mathcal{K}^1})_{S^1}) \overset{\sim}{\rightarrow} \mathcal{G}$$

such that the node \(e_{\mathcal{K}^1} \in \text{Node}(\mathcal{H}^1)\) of \(\mathcal{H}^1\) corresponding, relative to the isomorphism \(((\mathcal{H}^1|_{\mathcal{K}^1})_{S^1}) \overset{\sim}{\rightarrow} \mathcal{G}\), to the node \(e \in \text{Node}(\mathcal{G})\) is not of separating type. [Note that it follows immediately from the various definitions involved that \(\text{Node}(\mathcal{G}) \leftarrow \text{Node}(\mathcal{H}^1|_{\mathcal{K}^1})\) may be regarded as a subset of \(\text{Node}(\mathcal{H}^1)\).] Thus, it follows immediately from assertions (i), (ii) — by replacing \(\mathcal{G}\) (respectively, \(e\)) by \((\mathcal{H}^1)_{\rightarrow \text{Node}(\mathcal{H}^1) \setminus \{e_{\mathcal{H}^1}\}}\) (respectively, \(e_{\mathcal{H}^1}\)) — that to verify assertion (vi), we may assume without loss of generality that \(\text{Node}(\mathcal{G}) = \{e\}\), and that \(e\) is not of separating type.

Next, we observe that one may verify easily that there exists a semi-graph of anabelioids of pro-\(\Sigma\) PSC-type \(\mathcal{H}^1\) such that

- \(\text{Node}(\mathcal{H}^1)\) consists of precisely two elements \(e_{\mathcal{H}^1}, e'_{\mathcal{H}^1}\);

- \(V(e_{\mathcal{H}^1})\) consists of precisely one element \(v_{\mathcal{H}^1}\) of type \((0, 3)\) [cf. Definition 2.3, (iii)].

- \(e'_{\mathcal{H}^1}\) is of separating type;

- \((\mathcal{H}^1)_{\rightarrow \{e'_{\mathcal{H}^1}\}}\) is isomorphic to \(\mathcal{G}\).

Thus, if we write \(\mathcal{K}^1\) for the unique sub-semi-graph of PSC-type of the underlying semi-graph of \(\mathcal{H}^1\) whose set of vertices = \(\{v_{\mathcal{H}^1}\}\), then it follows immediately from assertions (i), (ii) — by replacing \(\mathcal{G}\) (respectively, \(e\)) by \(\mathcal{H}^1|_{\mathcal{K}^1}\) (respectively, \(e_{\mathcal{H}^1}\)) — that to verify assertion (vi), we may assume without loss of generality that \(\text{Node}(\mathcal{G}) = \{e\}\), that \(e\) is not of
separating type [so \(\text{Vert}(\mathcal{G})\) consists of precisely one element], and that \(\mathcal{G}\) is of type \((1,1)\).

Write \(v \in \text{Vert}(\mathcal{G})\) for the unique vertex of \(\mathcal{G}\). Note that it follows immediately from the various assumptions on \(\mathcal{G}\) that \(\mathcal{G}|_v\) is of type \((0,3)\). Write \(e_1, e_2 \in \text{Cusp}(\mathcal{G}|_v)\) for the cusps of \(\mathcal{G}|_v\) corresponding, respectively, to the two branches \(b_1, b_2\) of the node \(e\); write \(e_3 \in \text{Cusp}(\mathcal{G}|_v)\) for the unique element of \(\text{Cusp}(\mathcal{G}|_v) \setminus \{e_1, e_2\}\). Then since \(\mathcal{G}|_v\) is of type \((0,3)\), there exists a graphic isomorphism of \(\mathcal{G}|_v\) with the semi-graph of anabelioids of pro-\(\Sigma\) PSC-type \([\text{without nodes}]\) determined by the tripod \(\ldots\) cf. the discussion entitled “Curves” in \(\ldots\) over an algebraically closed field \(k\) of characteristic \(\not= 2\) such that the cusps \(e_1, e_2\) of \(\mathcal{G}|_v\) correspond to the cusps 0, \(\infty\) of \(\mathbb{P}^1_k \setminus \{0,1,\infty\}\), respectively, relative to the graphic isomorphism. Thus, by considering the automorphism of \(\mathbb{P}^1_k \setminus \{0,1,\infty\}\) over \(k\) given by \(t \mapsto 1/t\), we obtain an automorphism \(\tau_v \in \text{Aut}(\mathcal{G}|_v)\) of \(\mathcal{G}|_v\) that maps \(e_1 \mapsto e_2, e_2 \mapsto e_1\). Moreover, since this automorphism of \(\mathbb{P}^1_k \setminus \{0,1,\infty\}\) induces an automorphism of the stable log curve of type \((1,1)\) obtained by identifying the cusps 0 and \(\infty\) of \(\mathbb{P}^1_k \setminus \{0,1,\infty\}\), we also obtain an automorphism \(\tau_\mathcal{G} \in \text{Aut}(\mathcal{G})\) of \(\mathcal{G}\). Note that it follows immediately from the definition of \(\tau_v\), together with the well-known structure of the étale fundamental group of the tripod \(\mathbb{P}^1_k \setminus \{0,1,\infty\}\), that the automorphism \(\tau_v\) induces the identity automorphism of the anabeloid \((\mathcal{G}|_v)e_3\) corresponding to \(e_3\).

Next, let us observe that it follows immediately from the definition of \(\mathcal{G}|_v\), together with the proof of assertion (v), that for \(i = 1, 2\), there exists a natural isomorphism \(\Pi_e \rightarrow \Pi_{e_i}\) — where we use the notations \(\Pi_e, \Pi_{e_i}\) to denote edge-like subgroups of \(\Pi_\mathcal{G}, \Pi_{\mathcal{G}|_v}\) associated to \(e, e_i\), respectively — such that the composite

\[
\Pi_e \xrightarrow{\sim} \Pi_{e_i} \xrightarrow{\sim} \Lambda_v \xrightarrow{\sim} \Lambda_\mathcal{G}
\]

— where we write \(b_i^\prime\) for the [unique] branch of \(e_i\) — coincides with the composite in question

\[
\Pi_e \xrightarrow{\sim} \Lambda_v \xrightarrow{\sim} \Lambda_\mathcal{G}.
\]

Next, let us observe that it follows immediately from the functoriality portion of assertion (v) that the automorphisms \(\tau_v, \tau_\mathcal{G}\) induce a
commutative diagram of $\overline{E}\Sigma$-modules

\[
\begin{array}{ccc}
\Pi_e & \xrightarrow{\sim} & \Pi_{e_1} \\
\downarrow & & \downarrow \\
\Pi_e & \xrightarrow{\sim} & \Pi_{e_2}
\end{array}
\begin{array}{ccc}
\xrightarrow{\text{sgn}_v'} & \Lambda_v \text{ [= } \Lambda_{G|v}] & \xrightarrow{\text{sgn}_v'} & \Lambda_{G|v} \\
\downarrow & & \downarrow & & \downarrow \\
\Lambda_v \text{ [= } \Lambda_{G|v}] & \xrightarrow{\text{sgn}_v'} & \Lambda_{G|v} & \xrightarrow{\text{sgn}_v'} & \Lambda_{G|v}
\end{array}
\]

— where the vertical arrows are the isomorphisms induced by the automorphisms $\tau_v, \tau_G$. Now by considering the well-known local structure of a stable log curve in a neighborhood of a node, one may verify easily that the left-hand vertical arrow in the above diagram is the automorphism of $\Pi_e$ given by multiplication by $-1 \in \overline{E}$. Thus, to complete the verification of assertion (vi), it suffices, in light of the commutativity of the above diagram, to verify that $\tau_v \in \text{Aut}(G|v)$ induces the identity automorphism of $\Lambda_{G|v} = \Lambda_v$. On the other hand, this follows immediately from assertion (v), applied to the cusp $e_3$, together with the fact that the automorphism $\tau_v$ induces the identity automorphism of $(G|v)_{e_3}$. This completes the proof of assertion (vi).

Q.E.D.

§4. Profinite Dehn multi-twists

In the present §, we introduce and discuss the notion of a profinite Dehn multi-twist. Although our definition of this notion [cf. Definition 4.4 below] is entirely group-theoretic in nature, our main result concerning this notion [cf. Theorem 4.8 below] asserts, in effect, that this group-theoretic notion coincides with the usual geometric notion of a “Dehn multi-twist”.

Let $\Sigma$ be a nonempty set of prime numbers and $G$ a semi-graph of anabelioids of pro-$\Sigma$ PSC-type. Write $G$ for the underlying semi-graph of $G$, $\Pi_G$ for the [pro-$\Sigma$] fundamental group of $G$, and $\Pi_G \to G$ for the universal covering of $G$ corresponding to $\Pi_G$.

Definition 4.1. We shall say that $G$ is cyclically primitive (respectively, noncyclically primitive) if $\text{Node}(G) = 1$, and the unique node of $G$ is not of separating type (respectively, is of separating type) [cf. Definition 2.5, (i)].
Remark 4.1.1. If $G$ is cyclically primitive (respectively, noncyclically primitive), then $\text{Vert}(G)^r = 1$ (respectively, 2), and the [discrete] topological fundamental group $\pi_1^\text{top}(G)$ of the underlying semi-graph $G$ of $G$ is noncanonically isomorphic to $\mathbb{Z}$ (respectively, is trivial).

Lemma 4.2 (Structure of the fundamental group of a noncyclically primitive semi-graph of anabelioids of PSC-type). Suppose that $G$ is noncyclically primitive [cf. Definition 4.1]. Let $v, w \in \text{Vert}(G)$ be the two distinct vertices of $G$ [cf. Remark 4.1.1]; $\tilde{v}, \tilde{w} \in \text{VCN}(\tilde{G})$ elements of $\text{VCN}(\tilde{G})$ such that $\tilde{v}(\tilde{G}) = v, \tilde{w}(\tilde{G}) = w$, and, moreover, $\tilde{e} \in N(\tilde{v}) \cap N(\tilde{w})$. Then the natural inclusions $\Pi_{\tilde{e}}, \Pi_{\tilde{v}}, \Pi_{\tilde{w}} \hookrightarrow \Pi_G$ determine an isomorphism of pro-$\Sigma$ groups

$$\lim_{\longrightarrow} (\Pi_{\tilde{e}} \leftarrow \Pi_{\tilde{v}} \leftarrow \Pi_{\tilde{w}}) \rightarrow \Pi_G$$

— where the inductive limit is taken in the category of pro-$\Sigma$ groups.

Proof. This may be thought of as a consequence of the “van Kampen Theorem” in elementary algebraic topology. At a more combinatorial level, one may reason as follows: It follows immediately from the simple structure of the underlying semi-graph $G$ that there is a natural equivalence of categories between

- the category of finite sets with continuous $\Pi_G$-action [and $\Pi_G$-equivariant morphisms] and
- the category of finite sets with continuous actions of $\Pi_{\tilde{e}}, \Pi_{\tilde{w}}$ which restrict to the same action on $\Pi_{\tilde{v}}$ [and $\Pi_{\tilde{v}}$-, $\Pi_{\tilde{w}}$-equivariant morphisms].

The isomorphism between $\Pi_G$ and the inductive limit appearing in the statement of Lemma 4.2 now follows formally from this equivalence of categories.

Q.E.D.

Lemma 4.3 (Infinite cyclic tempered covering of a cyclically primitive semi-graph of anabelioids of PSC-type). Suppose that $G$ is cyclically primitive [cf. Definition 4.1]. Denote by $\pi_1^\text{temp}(G)$ the tempered fundamental group of $G$ [cf. the discussion preceding [SemiAn], Proposition 3.6], by $\pi_1^\text{top}(G) \cong \mathbb{Z}$ — cf. Remark 4.1.1] the [discrete] topological fundamental group of the underlying semi-graph $G$ of $G$, and by $G_\infty \rightarrow G$ the connected tempered covering of $G$ corresponding to the natural surjection $\pi_1^\text{temp}(G) \rightarrow \pi_1^\text{top}(G)$ [where we refer to [SemiAn], §3,
concerning tempered coverings of a semi-graph of anabelioids. Then the following hold:

(i) **(Exact sequence)** The natural morphism $\mathcal{G}_\infty \to \mathcal{G}$ induces an exact sequence

\[ 1 \to \pi_1^{\text{temp}}(\mathcal{G}_\infty) \to \pi_1^{\text{temp}}(\mathcal{G}) \to \pi_1^{\text{top}}(\mathcal{G}) \to 1. \]

Moreover, the subgroup $\pi_1^{\text{temp}}(\mathcal{G}_\infty) \subseteq \pi_1^{\text{temp}}(\mathcal{G})$ of $\pi_1^{\text{top}}(\mathcal{G})$ is characteristic.

(ii) **(Automorphism groups)** There exist natural injective homomorphisms

\[ \text{Aut}^{\text{graph}}(\mathcal{G}) \hookrightarrow \text{Aut}^{\text{graph}}(\mathcal{G}_\infty), \quad \pi_1^{\text{top}}(\mathcal{G}) \hookrightarrow \text{Aut}(\mathcal{G}_\infty) \]

where we write $\text{Aut}^{\text{graph}}(\mathcal{G}_\infty)$ for the group of automorphisms of $\mathcal{G}_\infty$ that induce the identity automorphism of the underlying semi-graph of $\mathcal{G}_\infty$. Moreover, the centralizer of $\pi_1^{\text{top}}(\mathcal{G})$ in $\text{Aut}^{\text{graph}}(\mathcal{G}_\infty)$ satisfies the equality

\[ Z_{\text{Aut}^{\text{graph}}(\mathcal{G}_\infty)}(\pi_1^{\text{top}}(\mathcal{G})) = \text{Aut}^{\text{graph}}(\mathcal{G}). \]

(iii) **(Action of the fundamental group of the underlying semi-graph)** Let $\gamma_\infty \in \pi_1^{\text{top}}(\mathcal{G}) \subseteq \text{Aut}(\mathcal{G}_\infty)$ [cf. (ii)] be a generator of $\pi_1^{\text{top}}(\mathcal{G}) \simeq \mathbb{Z}$. Write $\text{Vert}(\mathcal{G}_\infty)$, $\text{Node}(\mathcal{G}_\infty)$, and $\text{Cusp}(\mathcal{G}_\infty)$ for the sets of vertices, nodes [i.e., closed edges], and cusps [i.e., open edges] of $\mathcal{G}_\infty$, respectively. Then there exist bijections

\[ V : \mathbb{Z} \xrightarrow{\sim} \text{Vert}(\mathcal{G}_\infty), \quad N : \mathbb{Z} \xrightarrow{\sim} \text{Node}(\mathcal{G}_\infty), \]

\[ C : \mathbb{Z} \times \text{Cusp}(\mathcal{G}) \xrightarrow{\sim} \text{Cusp}(\mathcal{G}_\infty) \]

such that, for each $a \in \mathbb{Z},$

- the set of edges that abut to the vertex $V(a)$ is equal to the disjoint union of $\{N(a), N(a+1)\}$ and $\{C(a, z) \mid z \in \text{Cusp}(\mathcal{G})\};$

- the automorphism of $\text{Vert}(\mathcal{G}_\infty)$ (respectively, $\text{Node}(\mathcal{G}_\infty)$; $\text{Cusp}(\mathcal{G}_\infty)$) induced by $\gamma_\infty \in \text{Aut}(\mathcal{G}_\infty)$ maps $V(a)$ (respectively, $N(a)$; $C(a, z)$) to $V(a+1)$ (respectively, $N(a+1)$; $C(a+1, z)$).
(iv) (Restriction to a finite sub-semi-graph) Let \( a \leq b \in \mathbb{Z} \) be integers. Denote by \( G_{[a,b]} \) the \( \text{uniquely determined} \) sub-semi-graph of \( \text{PSC-type} \) \( \text{cf. Definition 2.2, (i)} \) of the underlying semi-graph of \( \mathcal{G}_\infty \) such that the set of vertices of \( G_{[a,b]} \) is equal to \( \{ V(a), V(a+1), \ldots, V(b) \} \) \( \text{cf. (iii)} \); denote by \( G_{[a,b]} \) the semi-graph of anabelioids obtained by restricting \( \mathcal{G}_\infty \) to \( G_{[a,b]} \) \( \text{cf. the discussion preceding [SemiAn], Definition 2.2} \). Then \( G_{[a,b]} \) is a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type. Moreover, \( G_{[a,a+1]} \) is noncyclically primitive.

(v) (Restriction to a sub-semi-graph having precisely one vertex) Let \( a \leq c \leq b \in \mathbb{Z} \) be integers. Then the natural morphism of semi-graphs of anabelioids \( G_{[c,c]} \rightarrow G_{[a,b]} \) \( \text{cf. (iv)} \) determines an isomorphism \( G_{[c,c]} \rightarrow G_{[a,b]} \) \( \text{cf. (iv)} \) determines an isomorphism \( G_{[c,c]} \rightarrow G_{[a,b]} \) \( \text{cf. Remark 4.1.1} \), then the composite of natural morphisms of semi-graphs of anabelioids \( G_{[c,c]} \rightarrow G_{[a,b]} \) \( \text{cf. Remark 4.1.1} \), then the composite of natural morphisms of semi-graphs of anabelioids \( G_{[c,c]} \rightarrow G_{[a,b]} \) determines an isomorphism of \( G_{[c,c]} \) with \( G_{v} \).

(vi) (Natural isomorphisms between restrictions to finite sub-semi-graphs) Let \( a \leq b \in \mathbb{Z} \) be integers and \( \gamma_\infty \in \pi_1^{\text{top}}(G) \subseteq \text{Aut}(\mathcal{G}_\infty) \) the automorphism of \( \mathcal{G}_\infty \) appearing in (iii). Then \( \gamma_\infty \) determines an isomorphism \( \mathcal{G}_{[a,b]} \rightarrow \mathcal{G}_{[a+1,b+1]} \).

Proof. First, we verify assertion (i). To show that the natural morphism \( \mathcal{G}_\infty \rightarrow \mathcal{G} \) induces an exact sequence

\[
1 \rightarrow \pi_1^{\text{temp}}(\mathcal{G}_\infty) \rightarrow \pi_1^{\text{temp}}(\mathcal{G}) \rightarrow \pi_1^{\text{top}}(\mathcal{G}) \rightarrow 1,
\]

it suffices to verify that every tempered covering of \( \mathcal{G}_\infty \) determines, via the morphism \( \mathcal{G}_\infty \rightarrow \mathcal{G} \), a tempered covering of \( \mathcal{G} \). But this follows immediately, in light of the definition of a tempered covering, from the finiteness of the underlying semi-graph \( \mathcal{G} \) and the topologically finitely generated nature of the vertical subgroups of the tempered fundamental group \( \pi_1^{\text{temp}}(\mathcal{G}_\infty) \) of \( \mathcal{G}_\infty \). On the other hand, the fact that the subgroup \( \pi_1^{\text{temp}}(\mathcal{G}_\infty) \subseteq \pi_1^{\text{temp}}(\mathcal{G}) \) is characteristic follows immediately from the observation that the quotient \( \pi_1^{\text{temp}}(\mathcal{G}) \rightarrow \pi_1^{\text{temp}}(\mathcal{G})/\pi_1^{\text{temp}}(\mathcal{G}_\infty) \) may be characterized as the maximal discrete free quotient of \( \pi_1^{\text{temp}}(\mathcal{G}) \) \( \text{cf. the argument of [Andre], Lemma 6.1.1} \). This completes the proof of assertion (i).

Next, we verify assertion (ii). The existence of a natural injection \( \pi_1^{\text{top}}(\mathcal{G}) \hookrightarrow \text{Aut}(\mathcal{G}_\infty) \) follows immediately from the definition of
the connected tempered covering $G_{\infty} \to G$, together with the fact that $\pi_1^{\text{top}}(G)$ is abelian. On the other hand, it follows immediately from assertion (i), together with the various definitions involved, that any element of $\text{Aut}^{\text{grph}}(G)$ determines — up to composition with an element of $\pi_1^{\text{top}}(G) \leq \text{Aut}(G_{\infty})$ — an automorphism of $G_{\infty}$. Therefore, by composing with a suitable element of $\pi_1^{\text{top}}(G) \subseteq \text{Aut}(G_{\infty})$, one obtains a uniquely determined element of $\text{Aut}^{\text{grph}}(G_{\infty})$, hence also a natural injective homomorphism $\text{Aut}^{\text{grph}}(G) \to \text{Aut}^{\text{grph}}(G_{\infty})$. Next, to verify the equality $Z_{\text{Aut}^{\text{grph}}(G)}(\pi_1^{\text{top}}(G)) = \text{Aut}^{\text{grph}}(G)$, observe that $\pi_1^{\text{temp}}(G_{\infty})$ is center-free [cf. [SemiAn], Example 2.10; [SemiAn], Proposition 3.6, (iv)]; this implies that we have a natural isomorphism $\pi_1^{\text{temp}}(G_{\infty}) \simeq \pi_1^{\text{temp}}(G_{\infty}) \rtimes \pi_1^{\text{top}}(G)$ [cf. the discussion entitled “Topological groups” in §0]. Thus, in light of the [easily verified] inclusion $\text{Aut}^{\text{grph}}(G) \subseteq Z_{\text{Aut}^{\text{grph}}(G_{\infty})}(\pi_1^{\text{top}}(G))$, the desired equality follows immediately from [CmbGC], Proposition 1.5, (ii). This completes the proof of assertion (ii).

Assertions (iii), (iv), (v), and (vi) follow immediately from the definition of the connected tempered covering $G_{\infty} \to G$. Q.E.D.

**Definition 4.4.** We shall write

$$\text{Dehn}(G) \overset{\text{def}}{=} \{ \alpha \in \text{Aut}^{\text{grph}}(G) | \alpha_{g|v} = \text{id}_{g|v}, \text{ for any } v \in \text{Vert}(G) \}$$

— where we refer to Definitions 2.1, (iii); 2.14, (ii); Remark 2.5.1, (ii), concerning “$\alpha_{g|v}$”. We shall refer to an element of $\text{Dehn}(G)$ as a profinite Dehn multi-twist of $G$.

**Proposition 4.5 (Equalities concerning the group of profinite Dehn multi-twists).** It holds that

$$\text{Dehn}(G) = \bigcap_{v \in \text{Vert}(G)} \text{Aut}^{\text{II}_{v}}(G) = \bigcap_{v \in VCN(G)} \text{Aut}^{\text{II}_{v}}(G)$$

$$\subseteq \bigcap_{v \in VCN(G)} \text{Out}^{\text{II}_{v}}(\Pi_{v})$$

[cf. Definitions 2.13; 2.6, (i); [CmbGC], Proposition 1.2, (ii)] — where we use the notation “$\Pi_{(-)}$” to denote a VCN-subgroup [cf. Definition 2.1, (i)] of $\Pi_{G}$ associated to “$(-)$” $\in VCN(G)$.

**Proof.** The first equality follows immediately from the various definitions involved [cf. also [CmbGC], Proposition 1.2, (i)]. The second
equality follows immediately from the fact that any edge-like subgroup is contained in a vertical subgroup. The third equality follows immediately from Proposition 2.7, (ii). This completes the proof of Proposition 4.5. Q.E.D.

**Lemma 4.6 (Construction of certain homomorphisms).** Let \( \tilde{c} \in \text{Node}(\tilde{G}) \), \( e \overset{\text{def}}{=} \tilde{c}(G) \in \text{Node}(G) \). Then the following hold:

(i) Let \( \alpha \in \text{Dehn}(G) \) be a profinite Dehn multi-twist of \( G \) and \( \tilde{v} \in V(\tilde{c}) \subseteq \text{Vert}(\tilde{G}) \). Write \( \tilde{w} \) for the unique element of the complement \( V(\tilde{c}) \setminus \{\tilde{v}\} \) [cf. [NodNon], Remark 1.2.1, (iii)]. Then there exists a unique lifting \( \alpha[\tilde{v}] \in \text{Aut}(\Pi_G) \) of \( \alpha \) which preserves the vertical subgroup \( \Pi_{\tilde{v}} \subseteq \Pi_G \) of \( \Pi_G \) associated to \( \tilde{v} \in \text{Vert}(\tilde{G}) \) and induces the identity automorphism of \( \Pi_{\tilde{v}} \). Moreover, this lifting \( \alpha[\tilde{v}] \) preserves the vertical subgroup \( \Pi_{\tilde{w}} \subseteq \Pi_G \) of \( \Pi_G \) associated to \( \tilde{w} \in \text{Vert}(\tilde{G}) \), and there exists a unique element \( \delta_{\tilde{v}, \tilde{w}} \in \Pi_{\tilde{v}} \) of the edge-like subgroup \( \Pi_{\tilde{e}} \subseteq \Pi_G \) of \( \Pi_G \) associated to \( \tilde{v} \in \text{Node}(\tilde{G}) \) such that the restriction of \( \alpha[\tilde{v}] \) to \( \Pi_{\tilde{w}} \) is the inner automorphism determined by \( \delta_{\tilde{v}, \tilde{w}} \).

(ii) For \( \tilde{v} \in V(\tilde{c}) \), denote by \( \mathcal{D}_{\tilde{v}, \tilde{w}} : \text{Dehn}(G) \rightarrow \Lambda_G \) the composite of the map

\[
\text{Dehn}(G) \rightarrow \Pi_{\tilde{v}}
\]

given by assigning \( \alpha \mapsto \delta_{\tilde{v}, \tilde{w}} \in \Pi_{\tilde{v}} \) [cf. (i)] and the isomorphism

\[
\Pi_{\tilde{v}} \overset{\text{isom}}{\rightarrow} \Lambda_{\tilde{v}} \overset{\text{isom}}{\rightarrow} \Lambda_G
\]

[cf. Corollary 3.9, (ii), (v)] — where we write \( v \overset{\text{def}}{=} \tilde{v}(G) \) and \( b \) for the branch of \( e \) determined by the unique branch of \( \tilde{c} \) which abuts to \( \tilde{v} \). Then the map \( \mathcal{D}_{\tilde{v}, \tilde{w}} : \text{Dehn}(G) \rightarrow \Lambda_G \) is a homomorphism of profinite groups which does not depend on the choice of the element \( \tilde{v} \in V(\tilde{c}) \), i.e., if \( \tilde{w} \in V(\tilde{c}) \setminus \{\tilde{v}\} \), then \( \mathcal{D}_{\tilde{v}, \tilde{w}} = \mathcal{D}_{\tilde{v}, \tilde{w}} \). Moreover, the homomorphism \( \mathcal{D}_{\tilde{v}, \tilde{w}} \) (= \( \mathcal{D}_{\tilde{v}, \tilde{w}} \)) depends only on \( e \in \text{Node}(G) \), i.e., it does not depend on the choice of the element \( \tilde{c} \in \text{Node}(\tilde{G}) \) such that \( \tilde{c}(G) = e \).

**Proof.** First, we verify assertion (i). The fact that there exists a unique lifting \( \alpha[\tilde{v}] \in \text{Aut}(\Pi_G) \) of \( \alpha \) which preserves \( \Pi_{\tilde{v}} \) and induces the
identity automorphism of $\Pi_{\overline{v}}$ follows immediately, in light of the \textit{slimness} of $\Pi_{\overline{v}}$ [cf. [CmbGC], Remark 1.1.3] and the \textit{commensurable terminality} of $\Pi_{\overline{v}}$ in $\Pi_G$ [cf. [CmbGC], Proposition 1.2, (ii)], from the fact that $\alpha \in \text{Out}^{[\Pi_{\overline{v}}]}(\Pi_G)$ [cf. Proposition 4.5]. The fact that $\alpha[\overline{v}]$ \textit{preserves} $\Pi_{\overline{v}}$ follows immediately, in light of the \textit{graphicity} of $\alpha[\overline{v}]$, from the fact that $\Pi_{\overline{v}}$ is the \textit{unique vertical subgroup} $H$ of $\Pi_G$ such that $H \neq \Pi_{\overline{v}}$ and $\Pi_{\overline{v}} \subseteq H$ [cf. [NodNon], Remark 1.2.1, (iii); [NodNon], Lemma 1.7], together with the fact that $\alpha[\overline{v}]$ preserves $\Pi_{\overline{v}}, \Pi_{\overline{v}} \subseteq \Pi_G$. The fact that there exists a \textit{unique} element $\delta_{\overline{v},\overline{v}} \in \Pi_{\overline{v}}$ of $\Pi_{\overline{v}}$ such that the restriction of $\alpha[\overline{v}]$ to $\Pi_{\overline{v}}$ is the inner automorphism determined by $\delta_{\overline{v},\overline{v}}$ follows immediately, in light of the \textit{slimness} of $\Pi_{\overline{v}}$ [cf. [CmbGC], Remark 1.1.3] and the \textit{commensurable terminality} of $\Pi_{\overline{v}}$ [cf. [CmbGC], Proposition 1.2, (ii)], from the fact that $\alpha \in \text{Out}^{[\Pi_{\overline{v}}]}(\Pi_G)$ [cf. Proposition 4.5].

This completes the proof of assertion (i). Next, we verify assertion (ii).

The fact that the map $\mathcal{D}_{\overline{v},\overline{v}}$ is a \textit{homomorphism} follows immediately from the various \textit{uniqueness} properties discussed in assertion (i). The fact that the map $\mathcal{D}_{\overline{v},\overline{v}}$ does \textit{not depend} on the choice of the element $\overline{v} \in \mathcal{V}(\overline{v})$ follows immediately from Corollary 3.9, (vi). The fact that the \textit{homomorphism} $\mathcal{D}_{\overline{v},\overline{v}}$ does not depend on the choice of the element $\overline{v} \in \text{Node}(\overline{G})$ such that $\overline{v}(\overline{G}) = e$ follows immediately from the definition of the map $\mathcal{D}_{\overline{v},\overline{v}}$. This completes the proof of assertion (ii). Q.E.D.

\textbf{Definition 4.7.} For each node $e \in \text{Node}(\mathcal{G})$ of $\mathcal{G}$, we shall write

$$\mathcal{D}_e \overset{\text{def}}{=} \mathcal{D}_{\overline{v},\overline{v}} : \text{Dehn}(\mathcal{G}) \longrightarrow \Lambda_{\mathcal{G}}$$

for the homomorphism obtained in Lemma 4.6, (ii). [Note that it follows from Lemma 4.6, (ii), that this homomorphism depends only on $e \in \text{Node}(\mathcal{G})$.] We shall write

$$\mathcal{D}_\mathcal{G} \overset{\text{def}}{=} \bigoplus_{e \in \text{Node}(\mathcal{G})} \mathcal{D}_e : \text{Dehn}(\mathcal{G}) \longrightarrow \bigoplus_{\text{Node}(\mathcal{G})} \Lambda_{\mathcal{G}}.$$

\textbf{Theorem 4.8 (Properties of profinite Dehn multi-twists).} Let $\Sigma$ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of an-abelioids of pro-$\Sigma$ PSC-type. Then the following hold:

(i) \textbf{(Normality)} $\text{Dehn}(\mathcal{G})$ is \textbf{normal} in $\text{Aut}(\mathcal{G})$. 

(ii) **(Compatibility with generalization)** Let $S \subseteq \text{Node}(\mathcal{G})$. Then relative to the inclusion $\text{Aut}^S(\mathcal{G}) \subseteq \text{Aut}(\mathcal{G}_{\sim S})$ [cf. Definition 2.8] induced by the specialization outer isomorphism $\Pi_\mathcal{G} \sim \Pi_{\mathcal{G}_{\sim S}}$ with respect to $S$ [cf. Proposition 2.9, (ii)] we have a diagram of inclusions

$$\begin{align*}
\text{Dehn}(\mathcal{G}) & \leftrightarrow \text{Dehn}(\mathcal{G}_{\sim S}) \\
\cap & \quad \cap \\
\text{Aut}^S(\mathcal{G}) & \leftrightarrow \text{Aut}(\mathcal{G}_{\sim S}).
\end{align*}$$

Moreover, if we regard $\text{Node}(\mathcal{G}_{\sim S})$ as a subset of $\text{Node}(\mathcal{G})$, then the above inclusion $\text{Dehn}(\mathcal{G}_{\sim S}) \leftrightarrow \text{Dehn}(\mathcal{G})$ fits into a commutative diagram of profinite groups

$$\begin{align*}
\text{Dehn}(\mathcal{G}_{\sim S}) & \longrightarrow \text{Dehn}(\mathcal{G}) \\
\oplus_{\text{Node}(\mathcal{G}_{\sim S})} \Lambda_\mathcal{G} & \longrightarrow \oplus_{\text{Node}(\mathcal{G})} \Lambda_\mathcal{G}
\end{align*}$$

where the lower horizontal arrow is the natural inclusion determined by the inclusion $\text{Node}(\mathcal{G}_{\sim S}) \hookrightarrow \text{Node}(\mathcal{G})$ and the natural isomorphism $\Lambda_{\mathcal{G}_{\sim S}} \sim \Lambda_\mathcal{G}$ [cf. Corollary 3.9, (i)].

(iii) **(Compatibility with “surgery”)** Let $\mathbb{H}$ be a sub-semi-graph of PSC-type [cf. Definition 2.2, (i)] of $\mathcal{G}$, $S \subseteq \text{Node}(\mathcal{G}_{\mathbb{H}})$ [cf. Definition 2.2, (ii)] a subset of $\text{Node}(\mathcal{G}_{\mathbb{H}})$ that is **not** of separating type [cf. Definition 2.5, (i)], and $T \subseteq \text{Cusp}(\mathcal{G}_{\mathbb{H}_{\sim S}})$ [cf. Definition 2.5, (ii)] an omittable [cf. Definition 2.4, (i)] subset of $\text{Cusp}(\mathcal{G}_{\mathbb{H}_{\sim S}})$. Then the natural homomorphism

$$\begin{align*}
\text{Aut}^{\mathbb{H}_{\sim S} \bullet T}(\mathcal{G}) & \longrightarrow \text{Aut}((\mathcal{G}_{\mathbb{H}_{\sim S}})_{\bullet T}) \\
\alpha & \mapsto \alpha((\mathcal{G}_{\mathbb{H}_{\sim S}})_{\bullet T})
\end{align*}$$

[cf. Definitions 2.4, (ii); 2.14, (ii)] induces a homomorphism

$$\text{Dehn}(\mathcal{G}) \longrightarrow \text{Dehn}((\mathcal{G}_{\mathbb{H}_{\sim S}})_{\bullet T}).$$

Moreover, if we regard $\text{Node}((\mathcal{G}_{\mathbb{H}_{\sim S}})_{\bullet T})$ as a subset of $\text{Node}(\mathcal{G})$, then the above homomorphism $\text{Dehn}(\mathcal{G}) \rightarrow$
Dehn\(((G[|3]|)\succ S)\bullet T\) fits into a commutative diagram of profinite groups

\[
\begin{array}{ccc}
\text{Dehn}(G) & \longrightarrow & \text{Dehn}\(((G[|3]|)\succ S)\bullet T) \\
\oplus_{\text{Node}(G)} \Lambda G & \longrightarrow & \oplus_{\text{Node}((G[|3]|)\succ S)\bullet T} \Lambda G \\
\end{array}
\]

where the lower horizontal arrow is the natural projection, and we apply the natural isomorphism \(\Lambda G \rightarrow \Lambda((G[|3]|)\succ S)\bullet T\) [cf. Corollary 3.9, (ii)].

(iv) **(Structure of the group of profinite Dehn multi-twists)**

The homomorphism defined in Definition 4.7

\[\mathcal{D}_G : \text{Dehn}(G) \longrightarrow \bigoplus_{\text{Node}(G)} \Lambda G\]

is an isomorphism of profinite groups that is functorial, in \(G\), with respect to isomorphisms of semi-graphs of anabelioids of pro-\(\Sigma\) PSC-type. In particular, \(\text{Dehn}(G)\) is a finitely generated free \(\mathbb{Z}[\Sigma]\)-module of rank \(\text{Node}(G)\). We shall refer to a nontrivial profinite Dehn multi-twist whose image \(\in \bigoplus_{\text{Node}(G)} \Lambda G\) lies in a direct summand [i.e., in a single “\(\Lambda G\)’] as a profinite Dehn twist.

(v) **(Conjugation action on the group of profinite Dehn multi-twists)**

The action of \(\text{Aut}(G)\) on \(\bigoplus_{\text{Node}(G)} \Lambda G\)

\[\text{Aut}(G) \longrightarrow \text{Aut}(\text{Dehn}(G)) \longrightarrow \text{Aut}(\bigoplus_{\text{Node}(G)} \Lambda G)\]

determined by conjugation by elements of \(\text{Aut}(G)\) [cf. (i)] and the isomorphism of (iv) coincides with the action of \(\text{Aut}(G)\) on \(\bigoplus_{\text{Node}(G)} \Lambda G\) determined by the action \(\chi_G\) of \(\text{Aut}(G)\) on \(\Lambda G\) and the natural action of \(\text{Aut}(G)\) on the finite set \(\text{Node}(G)\).

**Proof.** Assertions (i), (ii), and (iii) follow immediately from the various definitions involved. Next, we verify assertion (iv). The functoriality of the homomorphism \(\mathcal{D}_G\) follows immediately from the various definitions involved. The rest of the proof of assertion (iv) is devoted to verifying that the homomorphism \(\mathcal{D}_G\) is an isomorphism. First, we claim that
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(1): if $G$ is noncyclically primitive [cf. Definition 4.1], then the homomorphism $D_G$ is injective.

Indeed, this follows immediately from Lemma 4.2, together with the definition of the homomorphism $D_G$. This completes the proof of the above claim (1).

Next, we claim that

(2): if $G$ is cyclically primitive [cf. Definition 4.1], then the homomorphism $D_G$ is injective.

Indeed, let $\alpha \in \text{Ker}(D_G) \subseteq \text{Out}(G)$ be an element of $\text{Ker}(D_G)$. Since we are in the situation of Lemma 4.3, we shall apply the notational conventions established in Lemma 4.3. Denote by $\alpha_\infty \in \text{Aut}^{\text{grph}}(G_\infty)$ the automorphism of $G_\infty$ determined by $\alpha$ [cf. Lemma 4.3, (ii)]; for integers $a \leq b \in \mathbb{Z}$, denote by $\alpha_{[a,b]} \in \text{Aut}^{\text{grph}}(G_{[a,b]})$ the automorphism of $G_{[a,b]}$ obtained by restricting $\alpha_\infty \in \text{Aut}^{\text{grph}}(G_\infty)$. Then since $\alpha$ is a profinite Dehn multi-twist, one may verify easily that $\alpha_{[a,b]}$ is a profinite Dehn multi-twist of $G_{[a,b]}$. Thus, since $G_{[a,a+1]}$ is noncyclically primitive [cf. Lemma 4.3, (iv)], it follows immediately from the fact that $\alpha \in \text{Ker}(D_G)$, together with the claim (1), that $\alpha_{[a,a+1]}$ is trivial. Moreover, for any $a < b \in \mathbb{Z}$, it follows — by applying induction on $b-a$ and considering, in light of the claim (1), the various generalizations [cf. assertion (ii)] of $G_{[a,b]}$ with respect to sets of the form “Node($G_{[a,b]}$) \ {e}” — that the profinite Dehn multi-twist $\alpha_{[a,b]}$, hence also the automorphism $\alpha_\infty$, is trivial. In particular, it holds that $\alpha$ is trivial [cf. Lemma 4.3, (ii)], as desired. This completes the proof of the above claim (2).

Next, we claim that

(3): for arbitrary $G$, the homomorphism $D_G$ is injective.

We verify this claim (3) by induction on $\text{Node}(G)^\#$. If $\text{Node}(G)^\# \leq 1$, then the claim (3) follows formally from the claims (1) and (2). Now suppose that $\text{Node}(G)^\# > 1$, and that the induction hypothesis is in force. Let $e \in \text{Node}(G)$ be a node of $G$. Write $H$ for the unique sub-semi-graph of PSC-type of $G$ whose set of vertices is $V(e)$. Then one may verify easily that $S \overset{\text{def}}{=} \text{Node}(G_{[e]} \ {e})$ is not of separating type as a subset of $\text{Node}(G_{[e]})$. Thus, since $(G_{[e]})_>_S$ has precisely one node, it follows immediately from assertion (iii), together with the claims (1) and (2), that the profinite Dehn multi-twist $\alpha_{(G_{[e]})_>_S}$ of $(G_{[e]})_>_S$ determined by $\alpha \in \text{Dehn}(G)$ is trivial. In particular, it follows immediately from the definition of a generalization [cf., especially, the definition of the anabelioids corresponding to the vertices of a generalization given in Definition 2.8, (vi)], together with the definition of a profinite Dehn multi-twist, that
the automorphism $\alpha_{G_{\{e\}}}$ of the generization $G_{\{e\}}$ determined by $\alpha$ [cf. Proposition 2.9, (ii)] is a profinite Dehn multi-twist. Therefore, since Node($G_{\{e\}}$)$^2$ < Node($G$)$^2$, it follows immediately from assertion (ii), together with the induction hypothesis, that $\alpha_{G_{\{e\}}}$ is profinite Dehn multi-twist, hence also $\alpha$ is profinite Dehn multi-twist, is trivial. This completes the proof of the claim (\#3).

Next, we claim that

(\#4): if $G$ is noncyclically primitive [cf. Definition 4.1],
then the homomorphism $\mathcal{D}_G$ is surjective.

Indeed, this follows immediately from Lemma 4.2, together with the various definitions involved. This completes the proof of the claim (\#4).

Next, we claim that

(\#5): if $G$ is cyclically primitive [cf. Definition 4.1],
then the homomorphism $\mathcal{D}_G$ is surjective.

Indeed, let $\lambda \in \Lambda_G$ be an element of $\Lambda_G$. Since we are in the situation of Lemma 4.3, we shall apply the notational conventions established in Lemma 4.3. Then it follows immediately from Corollary 3.9, (ii), together with Lemma 4.3, (v), that for any integers $a \leq b \in \mathbb{Z}$, the natural morphisms $G_{[a, b]} \to G_{[a, b]}$ and $G_{[0, 0]} \to G_{[0, 0]}$ induce isomorphisms $\Lambda_G \cong \Lambda_G$ for the element of $\Lambda_G$ corresponding to $\lambda \in \Lambda_G$. By abuse of notation, write $\lambda \in \Lambda_G$ for $\lambda \in \Lambda_G$ such that $\Lambda_G \cong \Lambda_G$.

Next, we claim that

(\dagger): for any $a \leq b \in \mathbb{Z}$, there exists a [necessarily unique] profinite Dehn multi-twist $\alpha_{G_{[a, b]}}$ such that $\mathcal{D}_G(\alpha_{G_{[a, b]}}) = \lambda$ for every node $e \in \text{Node}(G_{[a, b]})$.

We verify this claim (\dagger) by induction on $b-a$. If $b-a = 1$, or equivalently, $[a, b] = [0, 1]$, then we have already shown the existence of a profinite Dehn multi-twist $\alpha_{[0, 1]} \in \text{Dehn}(G_{[0, 1]})$ of the desired type. Now suppose that $1 < b-a$, and that for $I \in \{[a, b-1], [a+1, b]\}$, there exists a profinite Dehn multi-twist $\lambda_I \in \text{Dehn}(G_I)$ such that $\mathcal{D}_G(\lambda_I) = \lambda$ for every node $e \in \text{Node}(G_I)$. Then one may verify easily that $\text{Node}(G_I)$ may be regarded as a subset of $\text{Node}(G_{[a, b]})$, that $H_{[a, b]} \stackrel{\text{def}}{=} (\text{Node}(G_I))_{\text{Node}(G_I)}$ is noncyclically primitive, and that, if one allows $v$ to range over the [two] vertices of $H_{[a, b]}$, then the resulting semi-graphs of anabelioids $H_{[a, b]}(v)$ are naturally isomorphic to $H_{[a, b]}(v)$, where we write $c_I$ for $b$ (respectively, $a$) if $I = [a, b-1]$ (respectively, $I = [a+1, b]$).
Let $\Pi_{e_I} \subseteq \Pi_H$ be a cuspidal subgroup of $\Pi_H$, corresponding to the cusp $e_I$ determined by the unique node of $H_{[a,b]}$; $\Pi_{e_{[e_I,e_I]}} \subseteq \Pi_{G_{[e_I,e_I]}}$ a cuspidal subgroup of $\Pi_{G_{[e_I,e_I]}}$ corresponding to the cusp $e_{[e_I,e_I]}$ determined by the unique node of $H_{[a,b]}$; $\tilde{\lambda}_I \in \operatorname{Aut}(\Pi_{H_I})$ a lifting of the automorphism of $\Pi_{H_I}$ determined by $\lambda_I \in \operatorname{Dehn}(G_I) \hookrightarrow \operatorname{Aut}(H_I)$ [cf. Proposition 2.9, (ii)] which preserves $\Pi_{e_I}$ and induces the identity automorphism of $\Pi_{e_I}$. [Note that since $\lambda_I \in \operatorname{Dehn}(G_I)$, one may verify easily that such a lifting $\tilde{\lambda}_I \in \operatorname{Aut}(\Pi_{H_I})$ exists.] Then for any element $\delta \in \Pi_{e_{[e_I,e_I]}}$ of $\Pi_{e_{[e_I,e_I]}}$, it follows immediately from Lemma 4.2 that by gluing — by means of the natural isomorphism $\Pi_{e_I} \sim \Pi_{e_{[e_I,e_I]}}$ — the automorphism $\tilde{\lambda}_I \in \operatorname{Aut}(\Pi_{H_I})$ to the inner automorphism of $\Pi_{G_{[e_I,e_I]}}$ by $\delta \in \Pi_{e_{[e_I,e_I]}}$, we obtain an automorphism $\lambda_{[a,b]}[\delta]$ of $\Pi_{H_{[a,b]}}$, which — in light of [CmbGC], Proposition 1.5, (ii), together with the fact that $\lambda_I \in \operatorname{Dehn}(G_I)$ — is contained in

$$\operatorname{Dehn}(G_{[a,b]}) \subseteq \operatorname{Aut}^{\grph}(G_{[a,b]}) \hookrightarrow \operatorname{Aut}^{\grph}(H_{[a,b]}) \subseteq \operatorname{Out}(\Pi_{H_{[a,b]}})$$

[cf. Proposition 2.9, (ii)]. Now it follows immediately from the definition of the homomorphism “$\mathcal{D}_e$” that the assignment $\delta \mapsto \mathcal{D}_{c_{[a,b]}}(\lambda_{[a,b]}[\delta])$ — where we write $c_{[a,b]}$ for the node of $G_{[a,b]}$ corresponding to the unique node of $H_{[a,b]}$ — determines a bijection $\Pi_{e_{[e_I,e_I]}} \sim \Lambda_G$. Thus, since $\mathcal{D}_e(\lambda_I) = \lambda$ for every node $e \in \operatorname{Node}(G_I)$, we conclude that there exists a unique element $\delta \in \Pi_{e_{[e_I,e_I]}}$ of $\Pi_{e_{[e_I,e_I]}}$, such that $\mathcal{D}_e(\lambda_{[a,b]}[\delta]) = \lambda$ for every node $e \in \operatorname{Node}(G_{[a,b]})$. This completes the proof of the claim (i).

Write $\lambda_\infty \in \operatorname{Aut}^{\grph}(G_\infty)$ for the automorphism of $G_\infty$ determined by the $\lambda_{[a,b]}$’s of the claim (i). Now since $\mathcal{D}_e(\lambda_{[a,b]}) = \lambda$ for arbitrary $a < b \in \mathbb{Z}$ and $e \in \operatorname{Node}(G_{[a,b]})$, one may verify easily, by applying the claim (*3), that the automorphism $\lambda_\infty$ commutes with the natural action of $\pi_1^{\grph}(G) \simeq \mathbb{Z}$ on $G_\infty$. Thus, the automorphism $\lambda_\infty$ determines an automorphism $\lambda_\mathcal{G} \in \operatorname{Aut}^{\grph}(G)$ of $G$ [cf. Lemma 4.3, (ii)]. Moreover, it follows immediately from the definition of $\lambda_\mathcal{G}$, together with the fact that $\mathcal{D}_e(\lambda_{[a,b]}) = \lambda$ for arbitrary $a < b \in \mathbb{Z}$ and $e \in \operatorname{Node}(G_{[a,b]})$, that $\lambda_\mathcal{G}$ is a profinite Dehn multi-twist such that $\mathcal{D}_e(\lambda_\mathcal{G}) = \lambda \in \Lambda_G$. This completes the proof of the claim (*5).

Finally, we claim that

(*6): for arbitrary $G$, the homomorphism $\mathcal{D}_G$ is surjective.
For each node \( e \in \text{Node}(G) \) of \( G \), it follows from assertion (ii) that we have a commutative diagram of profinite groups

\[
\begin{array}{ccc}
\text{Dehn}(G_{\rightarrow \text{Node}(G) \setminus \{e\}}) & \longrightarrow & \text{Dehn}(G) \\
\n\n\text{Dehn}(G_{\rightarrow \text{Node}(G) \setminus \{e\}}) & \downarrow & \text{Dehn}(G) \\
\n\Lambda_G & \longrightarrow & \bigoplus_{e' \in \text{Node}(G)} \Lambda_{G_{e'}}
\end{array}
\]

where the lower horizontal arrow is the natural inclusion into the component indexed by \( e \). Now since \( \text{Node}(G_{\rightarrow \text{Node}(G) \setminus \{e\}})^2 = 1 \), it follows from the claims \((\ast_4), (\ast_5)\) that the left-hand vertical arrow \( \text{Dehn}(G_{\rightarrow \text{Node}(G) \setminus \{e\}}) \) in the above commutative diagram is surjective. Therefore, by allowing “\( e \)” to vary among the elements of \( \text{Node}(G) \), we conclude that \( \text{Dehn}(G) \) is surjective. This completes the proof of the claim \((\ast_6)\) — hence also, in light of the claim \((\ast_3)\) — of assertion (iv).

Finally, assertion (v) follows immediately from the various definitions involved, together with assertion (iv). This completes the proof of Theorem 4.8. Q.E.D.

Remark 4.8.1. In the notation of Theorem 4.8, denote by \( \pi_1^{\text{temp}}(G) \) the \textit{tempered fundamental group} of \( G \) [cf. the discussion preceding [SemiAn], Proposition 3.6], by \( \pi_1^{\text{top}}(G) \) the \textit{discrete topological fundamental group} of the underlying semi-graph \( G \) of \( G \), by \( G_{\infty} \rightarrow G \) the \textit{connected tempered covering} of \( G \) corresponding to the natural surjection \( \pi_1^{\text{temp}}(G) \twoheadrightarrow \pi_1^{\text{top}}(G) \) [where we refer to [SemiAn], \S3, concerning tempered coverings of a semi-graph of anabelioids], by \( \text{Aut}^{\text{grph}}(G_{\infty}) \) the group of automorphisms of \( G_{\infty} \) that induce the identity automorphism of the underlying semi-graph of \( G_{\infty} \), and by \( \text{Dehn}(G_{\infty}) \subseteq \text{Aut}^{\text{grph}}(G_{\infty}) \) the group of “\textit{profinite Dehn multi-twists}” of \( G_{\infty} \) — i.e., automorphisms of \( G_{\infty} \) which induce the identity automorphism on the underlying semi-graph of \( G_{\infty} \), as well as on the anabelioids of \( G_{\infty} \) corresponding to the vertices of \( G_{\infty} \).

Then the following hold:

(i) The natural morphism \( G_{\infty} \rightarrow G \) induces an exact sequence

\[
1 \longrightarrow \pi_1^{\text{temp}}(G_{\infty}) \longrightarrow \pi_1^{\text{temp}}(G) \longrightarrow \pi_1^{\text{top}}(G) \longrightarrow 1.
\]

Moreover, the subgroup \( \pi_1^{\text{temp}}(G_{\infty}) \leq \pi_1^{\text{temp}}(G) \) of \( \pi_1^{\text{temp}}(G) \) is characteristic.

(ii) There exist natural injections

\[
\text{Aut}^{\text{grph}}(G) \hookrightarrow \text{Aut}^{\text{grph}}(G_{\infty}) , \hspace{1cm} \text{Dehn}(G) \hookrightarrow \text{Dehn}(G_{\infty}) ,
\]
where the third injection is determined up to composition with a \( \pi_1^{\text{top}}(\mathcal{G}) \)-inner automorphism — which satisfy the equalities

\[
Z_{\text{Aut}^{\text{grph}}(\mathcal{G}_\infty)}(\pi_1^{\text{top}}(\mathcal{G})) = \text{Aut}^{\text{grph}}(\mathcal{G});
\]

\[
\text{Dehn}(\mathcal{G}) = \text{Aut}^{\text{grph}}(\mathcal{G}) \cap \text{Dehn}(\mathcal{G}_\infty).
\]

(iii) There exists a natural isomorphism

\[
\text{Dehn}(\mathcal{G}_\infty) \simeq \prod_{\text{Node}(\mathcal{G}_\infty)} A\mathcal{G}.
\]

Indeed, assertion (i) (respectively, (ii)) follows immediately from a similar argument to the proof of Lemma 4.3, (i) (respectively, Lemma 4.3, (ii)), together with the various definitions involved. On the other hand, the existence of the natural isomorphism asserted in assertion (iii) follows immediately from the fact that the various homomorphisms \( \mathcal{D}_{(\mathcal{G}_\infty)|_H} \) — where \( H \) ranges over the sub-semi-graphs of \( \text{PSC-type} \) [cf. Definition 2.2, (i)] of the underlying semi-graph of \( \mathcal{G}_\infty \), and we write \( (\mathcal{G}_\infty)|_H \) for the semi-graph of anabelioids obtained by restricting \( \mathcal{G}_\infty \) to \( H \) [cf. the discussion preceding [SemiAn], Definition 2.2], which [as is easily verified] is of \( \text{pro-\Sigma PSC-type} \) — are isomorphisms. [Note that since \( (\mathcal{G}_\infty)|_H \) is of \( \text{pro-\Sigma PSC-type} \), the fact that \( \mathcal{D}_{(\mathcal{G}_\infty)|_H} \) is an isomorphism is a consequence of Theorem 4.8, (iv). However, since \( H \) is a tree, it follows from the \textit{simple structure of} \( H \) that one may verify that \( \mathcal{D}_{(\mathcal{G}_\infty)|_H} \) is an isomorphism in a fairly \textit{direct} fashion, by arguing as in the proofs of the claims \((\ast_1), (\ast_2)\) that appear in the proof of Theorem 4.8, (iv).]

In particular, it follows immediately from assertions (ii), (iii) that one may recover the natural isomorphism

\[
\text{Dehn}(\mathcal{G}) \simeq Z_{\prod_{\text{Node}(\mathcal{G}_\infty)} A\mathcal{G}}(\pi_1^{\text{top}}(\mathcal{G})) \simeq \prod_{\text{Node}(\mathcal{G})} A\mathcal{G}
\]

of Theorem 4.8, (iv).

**Definition 4.9.** We shall write

\[
\text{Glu}(\mathcal{G}) \subseteq \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{grph}}(\mathcal{G}|_v).
\]
for the [closed] subgroup of “glueable” collections of automorphisms of the direct product $\prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{graph}}(\mathcal{G}_v)$ consisting of elements $(\alpha_v)_{v \in \text{Vert}(\mathcal{G})}$ such that $\chi_v(\alpha_v) = \chi_w(\alpha_w)$ [cf. Definition 3.8, (ii)] for any $v, w \in \text{Vert}(\mathcal{G})$.

**Proposition 4.10 (Properties of automorphisms that fix the underlying semi-graph).**

(i) **(Factorization)** The natural homomorphism

$$\text{Aut}^{\text{graph}}(\mathcal{G}) \rightarrow \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{graph}}(\mathcal{G}_v)$$

$\alpha \mapsto (\alpha_{\mathcal{G}_v})_{v \in \text{Vert}(\mathcal{G})}$

[cf. Definition 2.14, (ii); Remark 2.5.1, (ii)] factors through the closed subgroup $\text{Glu}(\mathcal{G}) \subseteq \prod_{v \in \text{Vert}(\mathcal{G})} \text{Aut}^{\text{graph}}(\mathcal{G}_v)$.

(ii) **(Exact sequence relating profinite Dehn multi-twists and glueable automorphisms)** The resulting homomorphism $\rho^{\text{Vert}}_{\mathcal{G}} : \text{Aut}^{\text{graph}}(\mathcal{G}) \rightarrow \text{Glu}(\mathcal{G})$ [cf. (i)] fits into an exact sequence of profinite groups

$$1 \rightarrow \text{Dehn}(\mathcal{G}) \rightarrow \text{Aut}^{\text{graph}}(\mathcal{G}) \xrightarrow{\rho^{\text{Vert}}_{\mathcal{G}}} \text{Glu}(\mathcal{G}) \rightarrow 1.$$  

(iii) **(Surjectivity of cyclotomic characters)** The restriction of the pro-$\Sigma$ cyclotomic character $\chi_{\mathcal{G}}$ of $\mathcal{G}$ [cf. Definition 3.8, (ii)] to $\text{Aut}^{\text{graph}}(\mathcal{G}) \subseteq \text{Aut}(\mathcal{G})$

$$\chi_{\mathcal{G}}|_{\text{Aut}^{\text{graph}}(\mathcal{G})} : \text{Aut}^{\text{graph}}(\mathcal{G}) \rightarrow (\hat{\mathbb{Z}}^\Sigma)^*$$  

— hence also $\chi_{\mathcal{G}}$ — is surjective.

(iv) **(Liftability of automorphisms)** Let $\mathbb{H}$ be a sub-semi-graph of PSC-type [cf. Definition 2.2, (i)] of $\mathcal{G}$ and $S \subseteq \text{Node}(\mathcal{G}_{\mathbb{H}})$ [cf. Definition 2.2, (ii)] a subset of $\text{Node}(\mathcal{G}_{\mathbb{H}})$ that is not of separating type [cf. Definition 2.5, (i)]. Then the homomorphism

$$\text{Aut}^{\text{graph}}(\mathcal{G}) \rightarrow \text{Aut}^{\text{graph}}((\mathcal{G}_{\mathbb{H}})_{> S})$$

$\alpha \mapsto \alpha_{(\mathcal{G}_{\mathbb{H}})_{> S}}$

[cf. Definitions 2.5, (ii); 2.14, (ii)] is surjective.
Proof. Assertion (i) follows immediately from Corollary 3.9, (iv). Next, we verify assertion (ii). It follows immediately from the various definitions involved that \( \ker(\rho_G^{\text{vert}}) = \text{Dehn}(G) \subseteq \text{Aut}_{\text{graph}}(G) \). Thus, to complete the proof of assertion (ii), it suffices to verify that the homomorphism \( \rho_G^{\text{vert}} \) is surjective.

Now we claim that

\[ (*)_1: \text{if } G \text{ is noncyclically primitive [cf. Definition 4.1], then the homomorphism } \rho_G^{\text{vert}} \text{ is surjective.} \]

Indeed, this follows immediately from Corollary 3.9, (v); Lemma 4.2, together with the various definitions involved. This completes the proof of the claim \((*)_1\).

Next, we claim that

\[ (*)_2: \text{if } G \text{ is cyclically primitive [cf. Definition 4.1], then the homomorphism } \rho_G^{\text{vert}} \text{ is surjective.} \]

Indeed, since we are in the situation of Lemma 4.3, we shall apply the notational conventions established in Lemma 4.3. Then it follows immediately from the fact that \( \text{Vert}(G)^t = 1 \) [cf. Remark 4.1.1], together with Lemma 4.3, (v), that the composite of natural morphisms \( G_{[0,0]} \to G_\infty \to G \) determines a natural identification \( \text{Glu}(G) \to \text{Aut}_{\text{graph}}(G_{[0,0]}). \)

Let \( \alpha = \alpha_{[0,0]} \in \text{Glu}(G) \to \text{Aut}_{\text{graph}}(G_{[0,0]}) \) be an element of \( \text{Glu}(G) \to \text{Aut}_{\text{graph}}(G_{[0,0]}). \) For each \( a \in \mathbb{Z} \), denote by \( \alpha_{[a,a]} \in \text{Aut}(G_{[a,a]}) \) the automorphism of \( G_{[a,a]} \) determined by conjugating the automorphism \( \alpha \) of \( G_{[0,0]} \) by the isomorphism \( \gamma^a_{[a,a]} : G_{[0,0]} \to G_{[a,a]} \) [cf. Lemma 4.3, (iii), (vi)]. Then for any \( c < b \in \mathbb{Z} \), it follows from the various definitions involved that the various \( \alpha_{[a,a]} \)'s satisfy the gluing condition necessary to apply the claim \((*)_1\), hence that we may glue them together [cf. the proof of the claim \((*)_2\) below for more details concerning this sort of gluing argument] to obtain a natural identification \( \text{Glu}(G) \to \text{Aut}_{\text{graph}}(G_{[0,0]}). \)

Thus, by allowing \( c < b \in \mathbb{Z} \) to vary, we obtain a natural identification \( \text{Glu}(G_\infty) \to \text{Aut}_{\text{graph}}(G_\infty). \) Now it follows immediately from the definition of \( \alpha_\infty \) that for any \( \gamma \in \pi_1^\text{top}(G) \), the automorphism \( [\alpha_\infty, \gamma] \) of \( G_\infty \) is a “profinite Dehn multi-twist” of \( G_\infty \), i.e., \( [\alpha_\infty, \gamma] \in \text{Dehn}(G_\infty) \) [cf. Remark 4.8.1]. Moreover, one may verify easily that the assignment \( \gamma \mapsto [\alpha_\infty, \gamma] \) determines a 1-cocycle \( \pi_1^\text{top}(G) \to \text{Dehn}(G_\infty) \). Thus, by Remark 4.8.1, (iii), together with the [easily verified] fact that

\[ H^1(\mathbb{Z}, \prod_{\mathbb{Z}} \hat{\mathbb{Z}}^\mathbb{Z}) = \{0\} \]
where we take the action of \( \mathbb{Z} \) on \( \prod_{j} \tilde{Z}^{j} \) to be the action determined by the trivial action of \( \mathbb{Z} \) on \( \tilde{Z}^{j} \) and the action of \( \mathbb{Z} \) on the index set \( \mathbb{Z} \) given by addition — we conclude that there exists an element \( \beta \in \text{Dehn}(\mathcal{G}) \) such that the automorphism \( \beta \circ \alpha_{\infty} \) commutes with the natural action of \( \pi_{1}^{\top}(\mathcal{G}) \) on \( \mathcal{G} \). In particular, it follows from Lemma 4.3, (ii), that \( \beta \circ \alpha_{\infty} \) determines an element \( \alpha_{\mathcal{G}} \in \text{Aut}_{\text{grph}}(\mathcal{G}) \) of \( \text{Aut}_{\text{grph}}(\mathcal{G}) \). Now since \( \beta \in \text{Dehn}(\mathcal{G}) \), it follows immediately from the various definitions involved that \( \rho_{\mathcal{G}}^{\text{Vert}}(\alpha_{\mathcal{G}}) = \alpha \in \text{Glu}(\mathcal{G}) \rightarrow \text{Aut}_{\text{grph}}(\mathcal{G}[0,0]) \). This completes the proof of the claim (\( *_{2} \)).

Finally, we claim that

(\( *_{3} \)): for arbitrary \( \mathcal{G} \), the homomorphism \( \rho_{\mathcal{G}}^{\text{Vert}} \) is surjective.

We verify this claim (\( *_{3} \)) by induction on \( \text{Node}(\mathcal{G}) \). If \( \text{Node}(\mathcal{G}) \leq 1 \), then this follows immediately from the claims (\( *_{1} \), (\( *_{2} \)). Now suppose that \( \text{Node}(\mathcal{G}) > 1 \), and that the induction hypothesis is in force. Let \( e \in \text{Node}(\mathcal{G}) \) be a node of \( \mathcal{G} \). Write \( \mathcal{H} \) for the unique sub-semi-graph of \( \text{PSC-type} \) of \( \mathcal{G} \) whose set of vertices is \( V(e) \). Then one may verify easily that \( S \triangleq \text{Node}(\mathcal{G}) \setminus \{ e \} \) is not of separating type as a subset of \( \text{Node}(\mathcal{G}) \). Thus, since \( \mathcal{G}[S] \) has precisely one node, and \( (\alpha_{e})_{e \in V(e)} \) may be regarded as an element of \( \text{Glu}(\mathcal{G}[S]) \), it follows from the claims (\( *_{1} \), (\( *_{2} \)) that there exists an automorphism \( \beta \in \text{Aut}_{\text{grph}}(\mathcal{G}[S]) \) of \( \mathcal{G}[S] \) such that \( \rho_{\mathcal{G}[S]}^{\text{Vert}}(\beta) = (\alpha_{e})_{e \in V(e)} \in \text{Glu}(\mathcal{G}[S]) \). Write \( \beta_{e} \in \text{Aut}_{\text{grph}}(\mathcal{G}[S]) \) for the automorphism of \( \mathcal{G}[S] \) determined by \( \beta \in \text{Aut}_{\text{grph}}(\mathcal{G}[S]) \) [cf. Proposition 2.9, (ii)]. Then it follows immediately from Corollary 3.9, (i), together with the definition of a generalization [cf., especially, the definition of the anabelioids corresponding to the vertices of a generalization given in Definition 2.8, (vi)], that the element

\[
\gamma \triangleq (\beta_{e}, (\alpha_{e})_{e \in V(e)}) \in \text{Aut}_{\text{grph}}(\mathcal{G}[S]) \times \prod_{e \in V(e)} \text{Aut}_{\text{grph}}(\mathcal{G}[e])
\]

may be regarded as an element of \( \text{Glu}(\mathcal{G}). \) Now since \( \text{Node}(\mathcal{G}) \leq 1 \), it follows from the induction hypothesis that there exists an automorphism \( \alpha_{\mathcal{G}}(e) \in \text{Aut}_{\text{grph}}(\mathcal{G}) \) of \( \mathcal{G}(e) \) such that \( \rho_{\mathcal{G}}^{\text{Vert}}(\alpha_{\mathcal{G}}(e)) = \gamma \in \text{Glu}(\mathcal{G}(e)). \) On the other hand, since \( \beta_{e} \) arises from an element \( \beta \) of \( \text{Aut}_{\text{grph}}(\mathcal{G}[S]) \), it follows immediately from [CmbGC], Proposition 1.5, (ii), that \( \alpha_{\mathcal{G}}(e) \in \text{Aut}_{\text{grph}}(\mathcal{G}[S]) \) is contained in the image of \( \text{Aut}_{\text{grph}}(\mathcal{G}) \rightarrow \text{Aut}_{\text{grph}}(\mathcal{G}(e)) \) [cf. Proposition 2.9, (ii)]. Moreover, since \( \rho_{\mathcal{G}[S]}^{\text{Vert}}(\beta) = (\alpha_{e})_{e \in V(e)} \in \text{Glu}(\mathcal{G}[S]), \)
it follows immediately from our original characterization of $\alpha_{\rightarrow(e)}$ that $\rho^\chi_{\text{Vert}}(\alpha_{\rightarrow(e)}) = (\alpha_v)_{v \in \text{Vert}(\mathcal{G})} \in \text{Glu}(\mathcal{G})$. Thus, we conclude that $\rho^\chi_{\text{Vert}}$ is surjective, as desired. This completes the proof of the claim ($\ast_3$), hence also of assertion (ii).

Next, we verify assertion (iii). First, let us observe that one may verify easily that there exist a semi-graph of anabelioids of pro-$\Sigma$ PSC-type $\mathcal{H}$ that is totally degenerate [cf. Definition 2.3, (iv)], a subset $S \subseteq \text{Node}(\mathcal{H})$, and an isomorphism of semi-graphs of anabelioids $\mathcal{H}_{\rightarrow S} \rightarrow \mathcal{G}$. Now since we have a natural injection $\text{Aut}^{\text{grph}}(\mathcal{H}) \hookrightarrow \text{Aut}^{\text{grph}}(\mathcal{H}_{\rightarrow S}) \cong \text{Aut}^{\text{grph}}(\mathcal{G})$ [cf. Proposition 2.9, (ii)], it follows immediately from Corollary 3.9, (i), that to verify assertion (iii), by replacing $\mathcal{G}$ by $\mathcal{H}_v$, we may assume without loss of generality that $\mathcal{G}$ is totally degenerate. On the other hand, it follows immediately from assertion (ii), together with Corollary 3.9, (ii), that to verify assertion (iii), it suffices to verify the surjectivity of $\chi^{\mathcal{G}}_v$ for each $v \in \text{Vert}(\mathcal{G})$. Thus, to verify assertion (iii), by replacing $\mathcal{G}$ by $\mathcal{G}_v$, we may assume without loss of generality that $\mathcal{G}$ is of type $(0, 3)$ [cf. Definition 2.3, (i)]. But assertion (iii) in the case where $\mathcal{G}$ is of type $(0, 3)$ follows immediately by considering the natural outer action of the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of the field of rational numbers $\mathbb{Q}$ — where we use the notation $\bar{\mathbb{Q}}$ to denote an algebraic closure of $\mathbb{Q}$ — on the semi-graph of anabelioids of pro-$\Sigma$ PSC-type associated to the tripod $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ over $\mathbb{Q}$. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). Write $\mathcal{H} \overset{\text{def}}{=} (\mathcal{G}_v)_{v \in S}$. Then it follows immediately from assertion (ii), together with Theorem 4.8, (iii), that the homomorphism $\text{Aut}^{\text{grph}}(\mathcal{G}) \rightarrow \text{Aut}^{\text{grph}}(\mathcal{H})$ in question fits into a commutative diagram of pro-finite groups

$$
\begin{align*}
1 & \longrightarrow \text{Dehn}(\mathcal{G}) \longrightarrow \text{Aut}^{\text{grph}}(\mathcal{G}) \longrightarrow \text{Glu}(\mathcal{G}) \longrightarrow 1 \\
1 & \longrightarrow \text{Dehn}(\mathcal{H}) \longrightarrow \text{Aut}^{\text{grph}}(\mathcal{H}) \longrightarrow \text{Glu}(\mathcal{H}) \longrightarrow 1
\end{align*}
$$

— where the horizontal sequences are exact. Now since the left-hand vertical arrow is surjective [cf. Theorem 4.8, (iii), (iv)], to verify assertion (iv), it suffices to verify the surjectivity of the right-hand vertical arrow. But this follows immediately from assertion (iii), together with the definition of “Glu(−)”. This completes the proof of assertion (iv). Q.E.D.
§5. Comparison with scheme theory

In the present §, we discuss [cf. Proposition 5.6; Theorem 5.7; Corollaries 5.9, 5.10 below] the relationship between intrinsic, group-theoretic properties of pro-
finite Dehn multi-twists [such as length, nondegeneracy, and positive definiteness—cf. Definitions 5.1; 5.8, (ii), (iii) below] and scheme-theoretic characterizations of properties of outer representations of pro-\Sigma PSC-type [such as length, strict nodal nondegeneracy, and IPSC-ness—cf. Definition 5.3, (ii) below; [NodNon], Definition 2.4, (i), (iii)]. The resulting theory leads naturally to a proof of the graphicity of \( C \)-admissible automorphisms contained in the commensurator of the group of pro-
finite Dehn multi-twists [cf. Theorem 5.14 below].

Let \( \Sigma \) be a nonempty set of prime numbers and \( G \) a semi-graph of
anabelioids of pro-\( \Sigma \) PSC-type. Write \( G \) for the underlying semi-graph
of \( G \), \( \Pi_G \) for the [pro-] fundamental group of \( G \), and \( \epsilon_G ! G \to \tilde{G} \) is an
open injection. Write \( \Pi_I \) def = \( \Pi_G \times I \) [cf. the discussion entitled
"Topological groups" in §0]; \( \tilde{v}, \tilde{w} \in \text{Vert}(G) \) for the two distinct elements
of \( \text{Vert}(\tilde{G}) \) such that \( \mathcal{V}(\tilde{e}) = \{ \tilde{v}, \tilde{w} \} \) [cf. [NodNon], Remark 1.2.1, (iii)];
\( I_{\tilde{e}}, I_{\tilde{v}}, I_{\tilde{w}} \subseteq \Pi_I \) for the inertia subgroups of \( \Pi_I \) associated to \( \tilde{e}, \tilde{v}, \tilde{w} \),
respectively, i.e., the centralizers of \( \Pi_{\tilde{e}}, \Pi_{\tilde{v}}, \Pi_{\tilde{w}} \subseteq \Pi_I \) in \( \Pi_I \), respectively
[cf. [NodNon], Definition 2.2]. Then it follows from condition (3) of
[NodNon], Definition 2.4, that the natural homomorphism \( I_{\tilde{e}} \times I_{\tilde{w}} \to I_{\tilde{v}} \)
is an open injection. Write

\[
\text{lng}_{\tilde{G}}(\tilde{e}, \rho) \defeq [I_{\tilde{e}} : I_{\tilde{w}}]
\]

for the index of \( I_{\tilde{e}} \times I_{\tilde{w}} \) in \( I_{\tilde{v}} \); we shall refer to \( \text{lng}_{\tilde{G}}(\tilde{e}, \rho) \) as the \( \Sigma \)-length of \( \tilde{e} \) with respect to \( \rho \). Note that it follows immediately from the various
definitions involved that the \( \Sigma \)-length of \( \tilde{e} \) with respect to \( \rho \) depends only
on \( e \defeq \tilde{e}(G) \in \text{Node}(G) \) and \( \rho \). Write

\[
\text{lng}_{\tilde{G}}(e, \rho) \defeq \text{lng}_{\tilde{G}}(\tilde{e}, \rho);
\]

we shall refer to \( \text{lng}_{\tilde{G}}(e, \rho) \) as the \( \Sigma \)-length of \( e \in \text{Node}(G) \) with respect
to \( \rho \).
Lemma 5.2 (Outer representations of SVA-type and profinite Dehn multi-twists). Let $\rho: I \to \text{Aut}(\mathcal{G}) \subseteq \text{Out}(\Pi_{\mathcal{G}})$ be an outer representation of pro-$\Sigma$ PSC-type which is of SVA-type [cf. [NodNon], Definition 2.4, (ii)] and $\tilde{e} \in \text{Node}(\tilde{\mathcal{G}})$ an element of Node($\tilde{\mathcal{G}}$). Write $\Pi_I \defeq \Pi_{\mathcal{G}} \text{ out} \times I$ [cf. the discussion entitled “Topological groups” in §$\theta$]; $\tilde{v}, \tilde{w} \in \text{Vert}(\tilde{\mathcal{G}})$ for the two distinct elements of Vert($\tilde{\mathcal{G}}$) such that $V(\tilde{e}) = \{\tilde{v}, \tilde{w}\}$ [cf. [NodNon], Remark 1.2.1, (iii)]; $I_{\tilde{v}}, I_{\tilde{w}} \subseteq \Pi_I$ for the inertia subgroups of $\Pi_I$ associated to $\tilde{e}, \tilde{v}, \tilde{w}$, respectively; $e \defeq \tilde{e}(\mathcal{G})$; $v \defeq \tilde{v}(\mathcal{G})$. Then the following hold:

(i) (Outer representations of SVA-type and profinite Dehn multi-twists) The outer representation $\rho$ factors through the closed subgroup $\text{Dehn}(\tilde{\mathcal{G}}) \subseteq \text{Aut}(\tilde{\mathcal{G}})$. By abuse of notation, write $\rho$ for the resulting homomorphism $I \to \text{Dehn}(\tilde{\mathcal{G}})$.

(ii) (Outer representations of SVA-type and homomorphisms of Dehn coordinates) The natural inclusions $I_{\tilde{v}}, I_{\tilde{w}} \hookrightarrow I_{\tilde{e}}$ and the composite $I_{\tilde{e}} \hookrightarrow \Pi_I \twoheadrightarrow I$ determine a diagram of profinite groups

$$
I_{\tilde{e}} \times I_{\tilde{w}}

\downarrow

1 \longrightarrow \Pi_{\tilde{e}} \longrightarrow I_{\tilde{e}} \longrightarrow I \longrightarrow 1
$$

— where the lower horizontal sequence is exact, and the closed subgroups $I_{\tilde{e}}, I_{\tilde{w}} \subseteq I_{\tilde{e}}$ determine sections of the surjection $I_{\tilde{e}} \twoheadrightarrow I$, respectively — hence also homomorphisms

$$
I \overset{\sim}{\leftarrow} I_{\tilde{e}} \twoheadrightarrow I_{\tilde{e}}/I_{\tilde{w}} \overset{\sim}{\leftarrow} \Pi_{\tilde{e}} \overset{\text{SNN}_v}{\sim} \Lambda_v \overset{\text{SNN}_v}{\sim} \Lambda_{\mathcal{G}}
$$

— where the first “$\overset{\sim}{\leftarrow}$” denotes the isomorphism given by the composite $I_{\tilde{e}} \twoheadrightarrow \Pi_I \twoheadrightarrow I$, and $b_v$ denotes the branch of $e$ determined by the [unique] branch of $\tilde{e}$ that abuts to $\tilde{v}$. Moreover, the composite of these homomorphisms

$$
I \to \Lambda_{\mathcal{G}}
$$

coincides with the composite

$$
I \overset{\rho}{\twoheadrightarrow} \text{Dehn}(\mathcal{G}) \overset{\mathcal{D}}{\twoheadrightarrow} \Lambda_{\mathcal{G}}
$$

[cf. (i); Definition 4.7]. In particular, if $\rho$ is of SNN-type [cf. [NodNon], Definition 2.4, (iii)], then the image of the
composite $I \to \text{Dehn}(\mathcal{G}) \overset{\Delta G}{\to} \Lambda_G$ coincides with $\text{ln}_{\Sigma}(e, \rho) \cdot \Lambda_G \subseteq \Lambda_G$.

(iii) (Centralizers and cyclotomic characters) Suppose that $\rho$ is of SNN-type [cf. [NodNon], Definition 2.4, (iii)]. Let $e \in \text{Node}(\mathcal{G})$ be a node of $\mathcal{G}$. Then $\chi_G(\alpha) = 1$ [cf. Definition 3.8, (ii)] for any $\alpha \in Z_{\text{Aut}^{(\cdot)}(\mathcal{G})}(\text{Im}(\rho)) \subseteq \text{Aut}^{(\cdot)}(\mathcal{G})$ [cf. Definition 2.6, (i)].

Proof. Assertion (i) follows immediately from condition (2′) of [NodNon], Definition 2.4. Next, we verify assertion (ii). The fact that the natural inclusions $I_e, I_{e'} \hookrightarrow I_\overline{e}$ and the composite $I_{\overline{e}} \hookrightarrow \Pi_I \twoheadrightarrow I$ give rise to the diagram and homomorphisms of the first and second displays in the statement of assertion (ii) follows immediately from [NodNon], Lemma 2.5, (iv); condition (2′) of [NodNon], Definition 2.4. On the other hand, it follows immediately from the various definitions involved that the image of each $\gamma \in I$ via the composite of $I_e \hookrightarrow \text{Aut}(\Pi_I)$ given by conjugation coincides with the $\text{ln}_{\Sigma}(e, \rho)$ of Lemma 4.6, (i), in the case where one takes $\alpha$ to be $\rho(\beta)$. Thus, it follows immediately from the definition of $I_{\overline{e}}$ that the image of $\beta \in I$ via the composite $I \overset{\overline{e}}{\hookrightarrow} I_\overline{e} \twoheadrightarrow \Pi_{\overline{e}}$ coincides with the $\text{ln}_{\Sigma}(e, \rho)$ of Lemma 4.6, (i), in the case where one takes $\alpha$ to be $\rho(\beta)$. Therefore, it follows immediately from the definition of $D_e$ that the homomorphisms of the final two displays of assertion (ii) coincide. Thus, the final portion of assertion (ii) concerning $\rho$ of SNN-type follows immediately from the definition of $\Sigma$-length. This completes the proof of assertion (ii). To verify assertion (iii), let us first observe that, by Theorem 4.8, (v), the conjugation action of $\alpha \in \text{Aut}^{(\cdot)}(\mathcal{G})$ on the $\Lambda_G \subseteq \bigoplus_{\text{Node}(\mathcal{G})} \Lambda_G \overset{\Delta G}{\twoheadrightarrow} \text{Dehn}(\mathcal{G})$ indexed by $e \in \text{Node}(\mathcal{G})$ is given by multiplication by $\chi_G(\alpha)$. On the other hand, since $\mathbb{N} \ni \text{ln}_{\Sigma}(e, \rho) \neq 0$, it follows from the final portion of assertion (ii) that the projection of $\text{Im}(\rho)$ to the coordinate indexed by $e$ is open. Thus, the fact that $\alpha$ lies in the centralizer $Z_{\text{Aut}^{(\cdot)}(\mathcal{G})}(\text{Im}(\rho))$ implies that $\chi_G(\alpha) = 1$, as desired. This completes the proof of assertion (iii). Q.E.D.

Definition 5.3. Let $R$ be a complete discrete valuation ring whose residue field is separably closed of characteristic $\not\in \Sigma$; $\pi \in R$ a prime element of $R$; $v_R$ the discrete valuation of $R$ such that $v_R(\pi) = 1$; $S^{\log}$ the log scheme obtained by equipping $S \overset{\text{def}}{=} \text{Spec} R$ with the log structure defined by the maximal ideal $(\pi) \subseteq R$ of $R$; $s^{\log}$ the log scheme
obtained by equipping the spectrum \( s \) of the residue field of \( R \) with the log structure induced by the log structure of \( S^{\log} \) via the natural closed immersion \( s \hookrightarrow S; \ X^{\log} \) a stable log curve over \( S^{\log}; \ G_{X^{\log}} \) the semi-graph of anabelioids of pro-\( \Sigma \) PSC-type determined by the special fiber \( X_s^{\log} \equiv X^{\log} \times_{S^{\log}} s^{\log} \) of the stable log curve \( X^{\log} \) [cf. [CmbGC], Example 2.5]; \( I_{S^{\log}} \ (\simeq \widehat{\mathbb{Z}}) \) the maximal pro-\( \Sigma \) completion of the log fundamental group \( \pi_1(S^{\log}) \) of \( S^{\log} \).

(i) One may verify easily that the natural outer representation \( I_{S^{\log}} \to \text{Aut}(G_{X^{\log}}) \) associated to the stable log curve \( X^{\log} \) over \( S^{\log} \) factors through \( \text{Dehn}(G_{X^{\log}}) \). We shall write

\[
\rho_{X^{\log}}: I_{S^{\log}} \to \text{Dehn}(G_{X^{\log}})
\]

for the resulting homomorphism.

(ii) It follows from the well-known local structure of a stable log curve in a neighborhood of a node that for each node \( e \) of the special fiber of \( X^{\log} \), there exists a nonzero element \( a_e \neq 0 \) of the maximal ideal \( \mathfrak{m} \subseteq R \) such that the completion \( \hat{O}_{X,e} \) of the local ring \( O_{X,e} \) at \( e \) is isomorphic to \( R[[s_1, s_2]]/(s_1s_2 - a_e) \) — where \( s_1, s_2 \) denote indeterminates. Write

\[
\ln g_{X^{\log}}(e) \equiv v_R(a_e); \quad \ln g_{X^{\log}}^\Sigma(e) \equiv [\widehat{\mathbb{Z}} : \ln g_{X^{\log}}(e) \cdot \widehat{\mathbb{Z}}].
\]

We shall refer to \( \ln g_{X^{\log}}(e) \) as the length of \( e \) and to \( \ln g_{X^{\log}}^\Sigma(e) \) as the \( \Sigma \)-length of \( e \). One verifies easily that \( \ln g_{X^{\log}}(e) \), hence also \( \ln g_{X^{\log}}^\Sigma(e) \), depends only on \( e \), i.e., is independent of the choice of the isomorphism \( \hat{O}_{X,e} \simeq R[[s_1, s_2]]/(s_1s_2 - a_e) \).

**Lemma 5.4 (Local geometric universal outer representations).** In the notation of Definition 5.3, suppose that \( G_{X^{\log}} \) is of type \((g, r) \) [cf. Definition 2.3, (i); Remark 2.3.1]. Write \( N \equiv \text{Node}(G_{X^{\log}})^\Sigma \) and \( \sigma^{\log}: S^{\log} \to \mathcal{M}_{g,r}^{\log} \) [cf. the discussion entitled “Curves” in §30 for the classifying morphism of the stable log curve \( X^{\log} \) over \( S^{\log} \). Then the following hold:

(i) (Local structure of the moduli stack of pointed stable curves) Write \( \hat{\mathcal{O}} \) for the completion of the local ring of
(\(\overline{\mathcal{M}}_{g,r}\))_S at the image of the closed point of \(S\) via the underlying
(1-)morphism of stacks \(\sigma_\log\) and \(T_\log\) for the
\([fs]\) log scheme obtained by equipping \(T \cong \text{Spec} \hat{\mathcal{O}}\) with the log
structure induced by the log structure of \((\overline{\mathcal{M}}_{g,r})_S\). [Thus, we
have a tautological strict [cf. [Illu], 1.2] (1-)morphism \(T_\log \to
(\overline{\mathcal{M}}_{g,r})_S\).] Then there exists an isomorphism of \(R\)-algebras
\(R[[t_1, \ldots, t_{3g-3+r}]] \sim \hat{\mathcal{O}}\) such that the following hold:

- The log structure of the log scheme \(T_\log\) is given by the
  following chart:

\[
\bigoplus_{e \in \text{Node}(\mathcal{G}_{X_{\log}})} N_e \rightarrow R[[t_1, \ldots, t_{3g-3+r}]] \sim \hat{\mathcal{O}}
\]

where we write \(N_e\) for the copy of \(N\) indexed by \(e \in \text{Node}(\mathcal{G}_{X_{\log}})\).

- For \(1 \leq i \leq N\), the homomorphism of \(R\)-algebras \(\mathcal{O} \rightarrow R\)
  induced by the morphism \(\sigma\) maps \(t_i\) to \(a_{o_i}\) [cf. Definition 5.3, (ii)].

(ii) (Log-scheme-theoretic description of log fundamental
groups) Write \(I_{T_\log}\) for the maximal pro-\(\Sigma\) quotient of the
log fundamental group \(\pi_1(T_\log)\) of \(T_\log\). Then we have natural
isomorphisms

\[
I_{S_\log} \sim \text{Hom}\left(N_{\mathbb{E}^P}, \hat{\mathbb{Z}}^E(1)\right);
\]

\[
I_{T_\log} \sim \text{Hom}\left(\bigoplus_{e \in \text{Node}(\mathcal{G}_{X_{\log}})} N_e, \hat{\mathbb{Z}}^E(1)\right);
\]

\[
\sim \bigoplus_{e \in \text{Node}(\mathcal{G}_{X_{\log}})} \text{Hom}\left(N_{\mathbb{E}^P}, \hat{\mathbb{Z}}^E(1)\right),
\]

and the homomorphism \(I_{S_\log} \rightarrow I_{T_\log}\) induced by the classifying
morphism \(\sigma_\log\) is the homomorphism obtained by applying
the functor “\(\text{Hom}_{\mathbb{Z}^E}((-)^{\mathbb{E}^P}, \hat{\mathbb{Z}}^E(1))\)” to the homomorphism of
monoids

\[
\bigoplus_{e \in \text{Node}(\mathcal{G}_{X_{\log}})} N_e \rightarrow N
\]

\[
(n_{e_1}, \ldots, n_{e_N}) \rightarrow \sum_{i=1}^{N} n_{e_i} \text{log}\_{X_{\log}}(e_i)
\]

(iii) (Local geometric universal outer representations) The
natural outer representation \(I_{T_\log} \rightarrow \text{Aut}(\mathcal{G}_{X_{\log}})\) associated to the stable log curve over \(T_\log\) determined
by the tautological strict morphism $T^\log \to (\mathcal{M}^\log_{g,r})_S$ factors through $\text{Dehn}(G_{X^\log}) \subseteq \text{Aut}(G_{X^\log})$; thus, we have a homomorphism $I_{T^\log} \to \text{Dehn}(G_{X^\log})$. Moreover, the homomorphism $\rho_{X^\log} : I_{S^\log} \to \text{Dehn}(G_{X^\log})$ factors as the composite of the homomorphism $I_{S^\log} \to I_{T^\log}$ induced by $\sigma^\log$ and this homomorphism $I_{T^\log} \to \text{Dehn}(G_{X^\log})$.

**Proof.** Assertion (i) follows immediately from the well-known local structure of the log stack $(\mathcal{M}^\log_{g,r})_S$ [cf. [Knud], Theorem 2.7]. Assertion (ii) follows immediately from assertion (i), together with the well-known structure of the log fundamental groups of $S^\log$ and $T^\log$. Assertion (iii) follows immediately from the various definitions involved. Q.E.D.

**Definition 5.5.** In the notation of Definition 5.3, Lemma 5.4, we shall write $t^\log$ for the log scheme obtained by equipping the closed point $t$ of $T$ with the log structure naturally induced by the log structure of $T^\log$; $X^\log_t$ for the stable log curve over $t^\log$ corresponding to the natural strict morphism $t^\log (\to T^\log) \to (\mathcal{M}^\log_{g,r})_S$;

$$\rho^\text{univ}_{X^\log_t}, I_{T^\log} \longrightarrow \text{Dehn}(G_{X^\log})$$

for the homomorphism obtained in Lemma 5.4, (iii).

**Proposition 5.6 (Outer representations arising from stable log curves).** In the notation of Definition 5.3, Lemma 5.4, the following hold:

(i) **(Compatibility of $\Sigma$-lengths)** For each node $e \in \text{Node}(G_{X^\log})$ of $G_{X^\log}$, it holds that

$$\ln_\Sigma G_{X^\log}(e, \rho_{X^\log}) = \ln_\Sigma X^\log(e)$$

[cf. Definitions 5.1; 5.3, (ii)].

(ii) **(Isomorphicity of local geometric universal outer representations)** The homomorphism

$$\rho^\text{univ}_{X^\log_t}, I_{T^\log} \longrightarrow \text{Dehn}(G_{X^\log})$$

[cf. Definition 5.5] is an isomorphism of profinite groups.
(iii) **(Compatibility with generalization)** Let \( Q \subseteq \text{Node}(\mathcal{G}_{X^\log}) \) be a subset of \( \text{Node}(\mathcal{G}_{X^\log}) \). Then there exist a stable log curve \( Y^\log \) over \( S^\log \) and an isomorphism of semi-graphs of anabelioids \( (\mathcal{G}_{X^\log})_{Q} \rightarrow \mathcal{G}_{Y^\log} \) that fit into a commutative diagram of profinite groups

\[
\begin{array}{ccc}
I_{T_X^\log} & \xrightarrow{\rho_{Y^\log}} & \text{Dehn}(\mathcal{G}_{Y^\log}) \\
\downarrow & & \downarrow \\
I_{T_Y^\log} & \xrightarrow{\rho_{X^\log}} & \text{Dehn}(\mathcal{G}_{X^\log})
\end{array}
\]

— where we write \( I_{T_X^\log}, I_{T_Y^\log} \) for the “\( I_{T^\log} \)” associated to \( X^\log, Y^\log \), respectively; the right-hand vertical arrow is the natural inclusion induced, via the isomorphism \( (\mathcal{G}_{X^\log})_{Q} \rightarrow \mathcal{G}_{Y^\log} \), by the natural inclusion of Theorem 4.8, (ii); the left-hand vertical arrow is the injection induced, via the [relevant] isomorphism of Lemma 5.4, (ii), by the natural projection of monoids

\[
\bigoplus_{e \in \text{Node}(\mathcal{G}_{X^\log})} N_e \rightarrow \bigoplus_{e \in \text{Node}(\mathcal{G}_{Y^\log})} N_e.
\]

[Note that it follows immediately from the various definitions involved that \( \text{Node}(\mathcal{G}_{Y^\log}) \rightarrow \text{Node}((\mathcal{G}_{X^\log})_{Q}) \) may be regarded as a subset of \( \text{Node}(\mathcal{G}_{X^\log}) \).]

(iv) **(Compatibility with specialization)** Let \( \mathcal{H} \) be a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type, \( Q \subseteq \text{Node}(\mathcal{H}) \), and \( \mathcal{H} \rightarrow \mathcal{G}_{X^\log} \) an isomorphism of semi-graphs of anabelioids. Then there exist a stable log curve \( Y^\log \) over \( S^\log \) and an isomorphism of semi-graphs of anabelioids \( \mathcal{H} \rightarrow \mathcal{G}_{Y^\log} \) that fit into a commutative diagram of profinite groups

\[
\begin{array}{ccc}
I_{T_X^\log} & \xrightarrow{\rho_{Y^\log}} & \text{Dehn}(\mathcal{G}_{X^\log}) \\
\downarrow & & \downarrow \\
I_{T_Y^\log} & \xrightarrow{\rho_{X^\log}} & \text{Dehn}(\mathcal{G}_{Y^\log})
\end{array}
\]

— where we write \( I_{T_X^\log}, I_{T_Y^\log} \) for the “\( I_{T^\log} \)” associated to \( X^\log, Y^\log \), respectively; the right-hand vertical arrow is the natural
inclusion induced, via the isomorphisms $\mathcal{H}_{\sim Q} \xrightarrow{\sim} \mathcal{G}_{X^{\log}}$ and $\mathcal{H} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$, by the natural inclusion of Theorem 4.8, (ii); the left-hand vertical arrow is the injection induced, via the [relevant] isomorphism of Lemma 5.4, (ii), by the natural projection of monoids
\[ \bigoplus_{e \in \text{Node}(\mathcal{G}_{Y^{\log}})} N_e \rightarrow \bigoplus_{e \in \text{Node}(\mathcal{G}_{X^{\log}})} N_e. \]

[Note that it follows immediately from the various definitions involved that $\text{Node}(\mathcal{G}_{X^{\log}}) \xleftarrow{\text{natural}} \text{Node}(\mathcal{H}_{\sim Q})$ may be regarded as a subset of $\text{Node}(\mathcal{G}_{Y^{\log}})$.

(v) (Input compatibility with “surgery”) Let $\mathfrak{I}$ be a sub-semi-graph of PSC-type [cf. Definition 2.2, (i)] of the underlying semi-graph of $\mathcal{G}_{X^{\log}}$, $Q \subseteq \text{Node}((\mathcal{G}_{X^{\log}})[\mathfrak{I}])$ [cf. Definition 2.2, (ii)] a subset of $\text{Node}((\mathcal{G}_{X^{\log}})[\mathfrak{I}])$ that is not of separating type [cf. Definition 2.5, (i)], and $U \subseteq \text{Cusp}((\mathcal{G}_{X^{\log}})[\mathfrak{I}])_{\sim Q}$ [cf. Definition 2.5, (ii)] an omittable [cf. Definition 2.4, (i)] subset of $\text{Cusp}((\mathcal{G}_{X^{\log}})[\mathfrak{I}])_{\sim Q}$. Then there exist a stable log curve $Y^{\log}$ over $S^{\log}$ and an isomorphism $((\mathcal{G}_{X^{\log}})[\mathfrak{I}])_{\sim Q} \xrightarrow{\sim} \mathcal{G}_{X^{\log}}$ [cf. Definition 2.4, (ii)] that fit into a commutative diagram of profinite groups
\[
\begin{array}{ccc}
I_{S^{\log}} & \longrightarrow & I_{T^{\log}} \xrightarrow{\rho_{\text{max}}^{X^{\log}}} \text{Dehn}(\mathcal{G}_{X^{\log}}) \\
\| & & \downarrow \\
I_{S^{\log}} & \longrightarrow & I_{T^{\log}} \xrightarrow{\rho_{\text{max}}^{Y^{\log}}} \text{Dehn}(\mathcal{G}_{Y^{\log}})
\end{array}
\]
where we write $I_{T^{\log}}$, $I_{T^{\log}}$ for the “$I_{T^{\log}}$” associated to $X^{\log}$, $Y^{\log}$, respectively; the left-hand horizontal arrows are the homomorphisms induced by the classifying morphisms associated to $X^{\log}$, $Y^{\log}$, respectively; the right-hand vertical arrow is the natural surjection induced, via the isomorphism $((\mathcal{G}_{X^{\log}})[\mathfrak{I}])_{\sim Q} \xrightarrow{\sim} \mathcal{G}_{Y^{\log}}$, by the natural surjection of Theorem 4.8, (iii); the middle vertical arrow is the surjection induced, via the [relevant] isomorphism of Lemma 5.4, (ii), by the natural inclusion of monoids
\[ \bigoplus_{e \in \text{Node}(\mathcal{G}_{Y^{\log}})} N_e \rightarrow \bigoplus_{e \in \text{Node}(\mathcal{G}_{X^{\log}})} N_e. \]
(vi) \textbf{(Output compatibility with “surgery”) Let $\mathcal{H}$ be a semi-graph of anabelioids of pro-$\Sigma$ PSC-type, $\mathbb{K}$ a sub-semi-graph of PSC-type \cite[Definition 2.2, (i)]{existing_text} of the underlying semi-graph of $\mathcal{H}$, $Q \subseteq \text{Node}(\mathcal{H}|_X)$ \cite[Definition 2.2, (ii)]{existing_text} a subset of $\text{Node}(\mathcal{H}|_X)$ that is \textbf{not of separating type} \cite[Definition 2.5, (i)]{existing_text}, $U \subseteq \text{Cusp}(\mathcal{H}|_X)$ \cite[Definition 2.5, (ii)]{existing_text} an \textbf{omittable} \cite[Definition 2.4, (i)]{existing_text} subset of $\text{Cusp}(\mathcal{H}|_X)$, and $((\mathcal{H}|_X)\setminus \mathbb{K}) \setminus U \to \mathcal{G}_{X_{\log}}$ \cite[Definition 2.4, (i)]{existing_text} an isomorphism of semi-graphs of anabelioids. Then there exist a stable log curve $Y_{\log}$ over $S_{\log}$ and an isomorphism of semi-graphs of anabelioids $\mathcal{H} \to \mathcal{G}_{Y_{\log}}$ that fit into a commutative diagram of profinite groups

$$
\begin{array}{ccc}
I_{Y_{\log}} & \xrightarrow{\rho_{Y_{\log}}^{\text{univ}}} & \text{Dehn}(\mathcal{G}_{Y_{\log}}) \\
\parallel & & \\
I_{X_{\log}} & \xrightarrow{\rho_{X_{\log}}^{\text{univ}}} & \text{Dehn}(\mathcal{G}_{X_{\log}})
\end{array}
$$

where we write $I_{X_{\log}}$, $I_{Y_{\log}}$ for the “$I_{T_{\log}}$” associated to $X_{\log}$, $Y_{\log}$, respectively; the left-hand horizontal arrows are the homomorphisms induced by the classifying morphisms associated to $Y_{\log}$, $X_{\log}$, respectively; the right-hand vertical arrow is the natural surjection induced, via the isomorphisms $((\mathcal{H}|_X)\setminus \mathbb{K}) \setminus U \to \mathcal{G}_{X_{\log}}$ and $\mathcal{H} \to \mathcal{G}_{Y_{\log}}$, by the natural surjection of Theorem 4.8, (iii); the middle vertical arrow is the surjection induced, via the [relevant] isomorphism of Lemma 5.4, (ii), by the natural inclusion of monoids

$$
\bigoplus_{e \in \text{Node}(\mathcal{G}_{X_{\log}})} \mathbb{N}_e \hookrightarrow \bigoplus_{e \in \text{Node}(\mathcal{G}_{Y_{\log}})} \mathbb{N}_e.
$$

(Note that it follows immediately from the various definitions involved that $\text{Node}(\mathcal{G}_{X_{\log}}) \subseteq \text{Node}(\mathcal{G}_{Y_{\log}})$ may be regarded as a subset of $\text{Node}(\mathcal{G}_{Y_{\log}})$.)

\textbf{Proof.} Assertion (i) follows immediately from the well-known local structure of a stable log curve in a neighborhood of a node. Next,
we verify assertion (ii). By allowing \("\rho_{X_{\log}}^{\text{univ}}\) to vary among the natural outer representations \(I_{S_{\log}} \rightarrow \text{Dehn}(G_{X_{\log}})\) associated to stable log curves \(X_{\log}^{\text{univ}}\) over \(S_{\log}\) whose classifying morphisms \("\sigma"\) coincide with the given \(\sigma\) on the closed point \(s\) of \(S\), one concludes that the surjectivity of \(\rho_{X_{\log}}^{\text{univ}}\) follows immediately from the final portion of Lemma 5.2, (ii), concerning \(\rho\) of SNN-type [cf. also assertion (i); Theorem 4.8, (iv)].

[Here, we recall that \(X_{\log}\) is of IPSC-type [cf. [NodNon], Definition 2.4, (i)], hence also of SNN-type [cf. [NodNon], Remark 2.4.2].] On the other hand, since both \(\text{Dehn}(G_{X_{\log}})\) and \(I_{T_{\log}}\) are free \(\mathbb{Z}\)-modules of rank \(\text{Node}(G_{X_{\log}})\) [cf. Theorem 4.8, (iv); Lemma 5.4, (ii)], assertion (ii) follows immediately from this surjectivity of \(\rho_{X_{\log}}^{\text{univ}}\). This completes the proof of assertion (ii).

Assertion (iii) (respectively, (iv)) follows immediately, in light of the well-known structure of \((\mathcal{M}_{g,r}^{\log})_S\) [cf. also the discussion entitled \("The Étale Fundamental Group of a Log Scheme"\) in [CmbCsp], §0, concerning the specialization isomorphism on fundamental groups, as well as Remark 5.6.1 below], by considering a lifting to \(S_{\log}\) of a stable log curve over \(S_{\log}\) obtained by \(\text{deforming}\) the nodes of the special fiber \(X_{\log}^{\text{def}} := X_{\log} \times_{S_{\log}} S_{\log}^{\text{univ}}\) corresponding to the nodes contained in \(Q\) (respectively, \(\text{degenerating}\) the moduli of \(X_{\log}^{\text{univ}}\) so as to obtain nodes corresponding to the nodes contained in \(Q\)) [cf. also Proposition 4.10, (iii)].

Next, we verify assertion (v). First, we observe that one may verify easily that if \(\mathcal{H}\) is the underlying semi-graph of \(G_{X_{\log}}\), and \(Q = 0\), then the stable log curve \(Y_{\log}^{\text{univ}}\) over \(S_{\log}\) obtained by omitting the cusps of \(X_{\log}^{\text{univ}}\) contained in \(U\) and the resulting natural isomorphism \((G_{X_{\log}})_U \simto G_{Y_{\log}}\) satisfy the conditions given in the statement of assertion (v). Thus, one verifies immediately that to verify assertion (v), we may assume without loss of generality that \(U = \emptyset\).

Write \(\mathcal{H} \overset{\text{def}}{=} ([G_{X_{\log}}]_H)_J\) and \(V \overset{\text{def}}{=} \text{Vert}(G_{X_{\log}}) \setminus \text{Vert}([G_{X_{\log}}]_H) \subseteq \text{Vert}(G_{X_{\log}})\). Denote by \((g_H, r_H)\) the type of \(\mathcal{H}\), and, for each \(v \in V\), by \((g_v, r_v)\) the type of \(v\) [cf. Definition 2.3, (i), (iii); Remark 2.3.1]. Then it follows immediately from the general theory of stable log curves that there exists a \("clutching (1-)morphism"\) corresponding to the operations \("(\cdot)_{\mathcal{H}}"\) and \("(\cdot)_{\mathcal{G}}\"\) [i.e., obtained by forming appropriate composites of the clutching morphisms discussed in [Knud], Definition 3.6]

\[
\mathcal{N} \overset{\text{def}}{=} (\mathcal{M}_{g_H, r_H})_s \times_s \left( \prod_{v \in V} (\mathcal{M}_{g_v, r_v})_s \right) \rightarrow (\mathcal{M}_{g, r})_s
\]
— where the fiber product \( \prod_{v \in V} \) is taken over \( s \) — that satisfies the following condition: write \( N^{\log} \) for the log stack obtained by equipping the stack \( \mathcal{N} \) with the log structure induced by the log structure of \( (\mathcal{M}_{g,r})_s \) via the above clutching morphism; then there exists an \( s^{\log} \)-valued point \( \sigma_{N_s}^{\log} \in \mathcal{N}^{\log}(s^{\log}) \) of \( N^{\log} \) such that the image of \( \sigma_{N_s}^{\log} \) via the natural strict \((1-)\)morphism \( N^{\log} \to (\mathcal{M}_{g,r})_s \) coincides with the \( s^{\log} \)-valued point of \( (\mathcal{M}_{g,r})_s \) obtained by restricting the classifying morphism \( \sigma^{\log} \in (\mathcal{M}_{g,r})_S(S^{\log}) \) of \( X^{\log} \to S^{\log} \). If, moreover, we write \( Y^{\log} \) for the stable log curve over \( s^{\log} \) corresponding to the image of \( \sigma_{N_s}^{\log} \) via the composite of \((1-)\)morphisms

\[
\mathcal{N}^{\log} \to N^{\log} \overset{\text{def}}{=} (\mathcal{M}_{g,r})_s \times_s \left( \prod_{v \in V} (\mathcal{M}_{g_r})_s \right) \overset{\text{pr}^{\log}_s}{\to} (\mathcal{M}_{g,r})_s
\]

— where the first arrow is the \((1-)\)morphism of log stacks obtained by “forgetting” the portion of the log structure of \( \mathcal{N}^{\log} \) that arises from \([\text{the portion of the log structure of } (\mathcal{M}_{g,r})_s \text{ determined by}] \) the irreducible components of the divisor \( (\mathcal{M}_{g,r})_s \) which contain the image of \( N \to (\mathcal{M}_{g,r})_s \) — then one verifies immediately that, for any stable log curve \( Y^{\log} \) over \( S^{\log} \) that lifts \( Y^{\log} \), there exists a natural identification isomorphism \( H = (\mathcal{G}_{X^{\log}})|_{H^{\log} \to Q} \to \mathcal{G}_{Y^{\log}} \).

Next, let us observe that by applying the various definitions involved, together with the fact that the \((1-)\)morphism \( N^{\log} \to (\mathcal{M}_{g,r})_S \) is strict, one may verify easily that the restrictions of the natural \((1-)\)morphisms of log stacks

\[
(\mathcal{M}_{g_r})_s \overset{\text{pr}^{\log}_s}{\to} N^{\log} \overset{\text{def}}{=} (\mathcal{M}_{g_r})_s
\]

to a suitable étale neighborhood of the underlying morphism of stacks of \( \sigma_{N_s}^{\log} \in \mathcal{N}^{\log}(s^{\log}) \) induce the following morphisms between the charts of \( (\mathcal{M}_{g_r})_s \), \( N^{\log} \), \( \mathcal{M}^{\log} \), and \( (\mathcal{M}_{g_r})_s \) determined by the chart of \( \mathcal{G}_{X^{\log}} \) given in Lemma 5.4, (i):

\[
\bigoplus_{e \in \text{Node}(H)} \mathbb{N}_e \xrightarrow{\sim} \left( \bigoplus_{e \in \text{Node}(H)} \mathbb{N}_e \right) \oplus \{0\}
\]

\[
\bigoplus_{e \in \text{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e \xrightarrow{\sim} \bigoplus_{e \in \text{Node}(\mathcal{G}_{X^{\log}})} \mathbb{N}_e
\]

where we use the notation \( \mathbb{N}_e \) to denote a copy of the monoid \( \mathbb{N} \) indexed by \( e \), and the “\( \sim \)” is the natural inclusion determined by the
natural inclusion \( \text{Node}(\mathcal{H}) \hookrightarrow \text{Node}(\mathcal{G}) \). Thus, by applying the functor \( \text{Hom}_{\mathcal{E}}(\mathcal{S}^p, \mathcal{S}(1)) \) to the homomorphism \( \bigoplus_{e \in \text{Node}(\mathcal{H})} \mathbb{N}_e \hookrightarrow \bigoplus_{e \in \text{Node}(\mathcal{G})} \mathbb{N}_e \) obtained by composing the morphisms of the above display and considering the [relevant] isomorphism of Lemma 5.4, (ii), we obtain a homomorphism \( I_{T_X} \to I_{T_Y} \), which makes the left-hand square of the diagram in the statement of assertion (v) commute.

On the other hand, to verify the commutativity of the right-hand square of the diagram in the statement of assertion (v), let us observe that by Theorem 4.8, (iv), it suffices to verify that for any node \( e \in \text{Node}(\mathcal{G})_Y \) of \( \mathcal{G} \), the two composites

\[
I_{T_X} \xrightarrow{\rho_{X,\log}^{\text{univ}}} \text{Dehn}(\mathcal{G}_X) \xrightarrow{\mathcal{D}_X} \mathcal{L}_{X,\log} \xrightarrow{\mathcal{L}_X} \mathcal{G}_Y \xrightarrow{\mathcal{D}_Y} \text{Dehn}(\mathcal{G}_Y);
\]

where we write \( e_X \) for the node of \( \mathcal{G}_X \) corresponding to the node \( e \in \text{Node}(\mathcal{G}_Y) \) via the natural inclusion \( \text{Node}(\mathcal{G}_Y) \hookrightarrow \text{Node}(\mathcal{G}_X) \) — coincide. But this follows immediately by comparing the natural action of \( I_{T_X} \) on the portion of \( \mathcal{G}_X \) corresponding to \( f_{e_X} \mathcal{V}(e_X) \) with the natural action of \( I_{T_Y} \) on the portion of \( \mathcal{G}_Y \) corresponding to \( \{e\} \cup \mathcal{V}(e) \). This completes the proof of assertion (v).

Finally, we verify assertion (vi). First, we observe that one may verify easily that if \( K \) is the underlying semi-graph of \( \mathcal{H} \), and \( Q = \emptyset \), then the stable log curve \( \mathcal{Y}^\log \) over \( S^\log \) obtained by equipping \( \mathcal{X}^\log \) with suitable cusps satisfies, for a suitable choice of isomorphism \( \mathcal{H} \sim \mathcal{G}_Y \), the conditions given in the statement of assertion (vi). Thus, one verifies immediately that to verify assertion (vi), we may assume without loss of generality that \( U = \emptyset \).

Write \( V \overset{\text{def}}{=} \text{Vert}(\mathcal{H}) \setminus \text{Vert}(\mathcal{H})_K \subseteq \text{Vert}(\mathcal{H}) \). Denote by \((g_{\mathcal{H}, r_{\mathcal{H}}})\) the type of \( \mathcal{H} \), and, for each \( v \in V \), by \((g_v, r_v)\) the type of \( v \). Then it follows immediately from the general theory of stable log curves that there exists a clutching “\((-\rvert_K\rangle\) morphism” corresponding to the operations “\((-\rvert_K\rangle\) and “\((-\rvert_{\mathcal{G}}\rangle\)” i.e., obtained by forming appropriate composites of the clutching morphisms discussed in [Knud], Definition 3.6

\[
\mathcal{N} \overset{\text{def}}{=} (\mathcal{M}_{g, r})_s \times_s \left( \prod_{v \in V} (\mathcal{M}_{g_v, r_v})_s \right) \longrightarrow (\mathcal{M}_{g_{\mathcal{H}, r_{\mathcal{H}}}})_s
\]

— where the fiber product “\( \prod_{v \in V} \)” is taken over \( s \) — that satisfies the following condition: write \( \mathcal{N}^\log \) for the log stack obtained by equipping the stack \( \mathcal{N} \) with the log structure induced by the log structure
of \( (\mathcal{M}_{g_n,r_n})_s \) via the above clutching morphism; then there exists an \( s^{\log} \)-valued point \( \sigma_N^{\log} \in \mathcal{N}_s^{\log}(s^{\log}) \) of \( \mathcal{N}_s^{\log} \) such that the image of \( \sigma_N^{\log} \in \mathcal{N}_s^{\log}(s^{\log}) \) via the composite of (1-)morphisms

\[
\mathcal{N}_s^{\log} \rightarrow \mathcal{N}_s^{\log} \stackrel{\text{def}}{=} (\mathcal{M}_{g,r})_s \times_s \left( \prod_{v \in V'} (\mathcal{M}_{g_v,r_v})_s \right) \frac{pr_{1,log}}{pr_{2,log}} (\mathcal{M}_{g,r})_s
\]

— where the first arrow is the (1-)morphism of log stacks obtained by “forgetting” the portion of the log structure of \( \mathcal{N}_s^{\log} \) that arises from \( \mathcal{M}_{g_n,r_n} \), determined by the irreducible components of the divisor \( (\mathcal{M}_{g_n,r_n})_s \) which contain the image of \( N \rightarrow (\mathcal{M}_{g_n,r_n})_s \) — coincides with the \( s^{\log} \)-valued point of \( (\mathcal{M}_{g,r})_s \) obtained by restricting the classifying morphism \( \sigma^{\log} \in (\mathcal{M}_{g,r})_S(S^{\log}) \) of \( X^{\log} \) to \( s^{\log} \). If, moreover, we write \( Y^{\log} \) for the stable log curve over \( s^{\log} \) corresponding to the image of \( \sigma_N^{\log} \in \mathcal{N}_s^{\log}(s^{\log}) \) via the natural strict (1-)morphism \( \mathcal{N}_s^{\log} \rightarrow (\mathcal{M}_{g,r})_s \), then one verifies immediately that, for any stable log curve \( Y^{\log} \) over \( s^{\log} \) that lifts \( Y^{\log} \), there exist a sub-semi-graph of PSC-type \( K' \) of the underlying semi-graph of \( \mathcal{G}_{Y^{\log}} \), a subset \( Q' \subseteq \text{Node}((\mathcal{G}_{Y^{\log}})|_{K'}) \), and an isomorphism of semi-graphs of anabelioids \( H \rightarrow \mathcal{G}_{Y^{\log}} \) that satisfy the following conditions:

(a) \( ((\mathcal{G}_{Y^{\log}})|_{K'})_{Q'} \) may be naturally identified with \( \mathcal{G}_{X^{\log}} \).

(b) The isomorphism \( H \rightarrow \mathcal{G}_{Y^{\log}} \) induces an isomorphism \( K' \rightarrow K' \) and a bijection \( Q \rightarrow Q' \), hence also an isomorphism \( (H|_{K'})_{Q'} \rightarrow ((\mathcal{G}_{Y^{\log}})|_{K'})_{Q'} \).

(c) The automorphism of \( \mathcal{G}_{X^{\log}} \) determined by the composite

\[
\mathcal{G}_{X^{\log}} \leftarrow (H|_{K'})_{Q'} \rightarrow (\mathcal{G}_{Y^{\log}})|_{K'} \rightarrow Q' \rightarrow \mathcal{G}_{X^{\log}}
\]

— where the first arrow is the isomorphism given in the statement of assertion (vi); the second arrow is the isomorphism of (b); the third arrow is the natural isomorphism arising from the natural identification of (a) — is contained in \( \text{Aut}^{\text{graph}}(\mathcal{G}_{X^{\log}}) \), and, moreover, the automorphism of \( \Lambda_{\mathcal{G}_{X^{\log}}} \) induced by this automorphism of \( \mathcal{G}_{X^{\log}} \) is the identity automorphism [cf. Proposition 4.10, (iii)].

Thus, by applying a similar argument to the argument used in the proof of assertion (v), one verifies easily that the stable log curve \( Y^{\log} \) and the
isomorphism \( H \cong G_{X,\log} \) satisfy the conditions given in the statement of assertion (vi). This completes the proof of assertion (vi). Q.E.D.

**Remark 5.6.1.** Here, we take the opportunity to correct a minor misprint in the discussion entitled “The Étale Fundamental Group of a Log Scheme” in \( CmbCsp \), §0. In the third paragraph of this discussion, the field \( K \) should be defined as a maximal algebraic extension of \( K_0 \) among those extensions which are unramified over \( U_{S_0} \) [i.e., but not necessarily over \( R_0 \)].

**Theorem 5.7 (Compatibility of scheme-theoretic and abstract combinatorial cyclotomic synchronizations).** Let \((g,r)\) be a pair of nonnegative integers such that \(2g-2+r>0\); \(\Sigma\) a nonempty set of prime numbers; \(R\) a complete discrete valuation ring whose residue field is separably closed of characteristic \(\not\equiv 2\); \(S_{\log}\) the log scheme obtained by equipping \(S \cong \text{Spec } R\) with the log structure defined by its closed point; \(X_{\log}\) a stable log curve of type \((g,r)\) over \(S_{\log}\); \(G_{X,\log}\) the semi-graph of anabeloids of pro-\(\Sigma\) PSC-type determined by the special fiber of the stable log curve \(X_{\log}\) [cf. \( CmbGC \), Example 2.5]; \(\hat{\mathcal{O}}\) the completion of the local ring of \((\mathcal{M}_{g,r})_S [c.f. \text{the discussion entitled \textbf{Curves} in } x_0]\) at the image of the closed point of \(S\) via the underlying \((1,1)\)-morphism of stacks \(\sigma : S \to (\mathcal{M}_{g,r})_S\) of the classifying morphism of \(X_{\log}\); \(\hat{T}_{\log}\) for the log scheme obtained by equipping \(T \defeq \text{Spec } \hat{\mathcal{O}}\) with the log structure induced by the log structure of \((\mathcal{M}_{g,r})_S \) [cf. \text{the discussion entitled \textbf{Curves} in } \S 0]; \(I_{\log} \) the maximal pro-\(\Sigma\) quotient of the log fundamental group \(\pi_1(T_{\log})\) of \(T_{\log}\). Then there exists an isomorphism

\[
\text{syn}_{X,\log} : \Lambda_{X,\log} \defeq \text{Hom}(N^{\text{gp}}, \hat{\mathbb{Z}}(1)) \cong \Lambda_{G_{X,\log}}
\]

[cf. Definition 3.8, (i)] such that the composite

\[
I_{\log} \xrightarrow{\sim} \bigoplus_{e \in \text{Node}(G_{X,\log})} \Lambda_{X,\log}[e]
\]

\[
\xrightarrow{\bigoplus \text{syn}_{X,\log}} \bigoplus_{e \in \text{Node}(G_{X,\log})} \Lambda_{G_{X,\log}} \xrightarrow{\Delta \log} \text{Dehn}(G_{X,\log})
\]

[cf. Definitions 4.4; 4.7] — where we use the notation \(\Lambda_{\Sigma}[e]\) to denote a copy of \(\Lambda_{\Sigma}\) indexed by \(e \in \text{Node}(G_{X,\log})\), and the first arrow is the [relevant] isomorphism of Lemma 5.4, (ii) — coincides with the outer representation \(\rho_{X,log}^{\text{univ}} : I_{\log} \to \text{Dehn}(G_{X,\log})\) [cf. Definition 5.5] associated
to the stable log curve over \( T^{\log} \) corresponding to the tautological strict \((1-)\)-morphism \( T^{\log} \to (\mathcal{M}^\log_{g,r})_S \).

**Proof.** In light of Theorem 4.8, (ii), (iv); Proposition 5.6, (ii), by applying Proposition 5.6, (iii), to the various generalizations of the form \( "(G_{X^{\log}})^{\sim \text{Node}(G_{X^{\log}})\setminus\{e\}}", \) it follows immediately that for each node \( e \in \text{Node}(G_{X^{\log}}) \), there exists a (necessarily unique) isomorphism

\[
\operatorname{syn}_{X^{\log}[e]} : \Lambda^\Sigma[e] \xrightarrow{\sim} \Lambda_{G_{X^{\log}}}
\]

— where \( \Lambda^\Sigma[e] \) is a copy of \( \Lambda^\Sigma \) indexed by \( e \in \text{Node}(G_{X^{log}}) \) — such that the composite

\[
I_{T^{\log}} \xrightarrow{\sim} \bigoplus_{e \in \text{Node}(G_{X^{\log}})} \Lambda^\Sigma[e] \xrightarrow{\sim} \bigoplus_{e \in \text{Node}(G_{X^{\log}})} \Lambda_{G_{X^{\log}}} \xleftarrow{\text{Dehn}(G_{X^{\log}})}
\]

— where the first \( \xrightarrow{\sim} \) is the [relevant] isomorphism of Lemma 5.4, (ii) — coincides with \( \rho_{\text{univ}}^{X^{\log}} \).

Thus, to complete the proof of Theorem 5.7, it suffices to verify that this isomorphism \( \operatorname{syn}_{X^{\log}[e]} \) is independent of the choice of \( e \). Now if \( \text{Node}(G_{X^{\log}})^2 \leq 1 \), then this independence is immediate. Thus, suppose that \( \text{Node}(G_{X^{\log}})^2 > 1 \) and fix two distinct nodes \( e_1, e_2 \in \text{Node}(G_{X^{\log}}) \) of \( G_{X^{\log}} \). The rest of the proof of Theorem 5.7 is devoted to verifying that

\[
(\dagger): \text{the two isomorphisms}
\]

\[
\Lambda^\Sigma[e_1] \xrightarrow{\sim} \Lambda_{G_{X^{\log}}}, \quad \Lambda^\Sigma[e_2] \xrightarrow{\sim} \Lambda_{G_{X^{\log}}}
\]

coincide.

Next, let us observe that one may verify easily that there exist

- a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type \( \mathcal{H}^* \),
- a sub-semi-graph of PSC-type \( \mathbb{K}^* \) of the underlying semi-graph of \( \mathcal{H}^* \),
- an omittable subset \( Q^* \subseteq \text{Cusp}(\mathcal{H}^*)|_{\mathbb{K}^*} \), and
- an isomorphism

\[
((\mathcal{H}^*)|_{\mathbb{K}^*})_{Q^*} \xrightarrow{\sim} G_{X^{\log}}
\]
such that the subset $U^* \subseteq \text{Node}(\mathcal{H}^*)$ corresponding, relative to the isomorphism $((\mathcal{H}^*)|_{\mathbb{K}^*}) \circlearrowleft \mathcal{G}_{X^\log} \rightarrow \mathcal{G}_{X^\log}$, to the subset $\{e_1, e_2\} \subseteq \text{Node}(\mathcal{G}_{X^\log})$ is not of separating type. Thus, it follows immediately from Proposition 5.6, (vi) — i.e., by replacing $X^\log$ (respectively, $e_1$, $e_2$) by the stable log curve $\mathcal{G}_{X^\log}$ obtained by applying Proposition 5.6, (vi), to the isomorphism $((\mathcal{H}^*)|_{\mathbb{K}^*}) \circlearrowleft \mathcal{G}_{X^\log}$ (respectively, by the two nodes $e_1, e_2 \in \text{Node}(\mathcal{G}_{X^\log})$ corresponding to the two nodes $\in U^*$) — that to verify the above (i), we may assume without loss of generality that the subset $\{e_1, e_2\} \subseteq \text{Node}(\mathcal{G}_{X^\log})$ is not of separating type. Thus, it follows immediately from Proposition 5.6, (iii) — i.e., by replacing $X^\log$ (respectively, $e_1$, $e_2$) by the stable log curve $\mathcal{G}_{X^\log}$ obtained by applying Proposition 5.6, (iii), to $(\mathcal{G}_{X^\log}) \rightarrow \text{Node}(\mathcal{G}_{X^\log}) \backslash \{e_1, e_2\}$ (respectively, by the two nodes $\in \text{Node}(\mathcal{G}_{X^\log})$) — that to verify the above (i), we may assume without loss of generality that $\text{Node}(\mathcal{G}_{X^\log}) = \{e_1, e_2\}$, and that $\text{Node}(\mathcal{G}_{X^\log}) = \{e_1, e_2\}$ is not of separating type. One verifies easily that these hypotheses imply that $\text{Vert}(\mathcal{G}_{X^\log})^2 = 1$.

Next, let us observe that one may verify easily that there exist [cf. Fig. 6 below]

- a semi-graph of anabelioids of pro-$\Sigma$ PSC-type $\mathcal{H}^i$,
- two distinct cusps $c_1^i, c_2^i \in \text{Cusp}(\mathcal{H}^i)$ of $\mathcal{H}^i$,
- three distinct nodes $f_1^i, f_2^i, f_3^i \in \text{Node}(\mathcal{H}^i)$ of $\mathcal{H}^i$, and
- an isomorphism

\[
(\mathcal{H}^i \rightarrow \{f_1^i, f_2^i, f_3^i\}) \circlearrowleft \mathcal{G}_{X^\log}
\]

such that

- $\text{Vert}(\mathcal{H}^i) = \{v_1^i, v_2^i, v_3^i, v_4^i\}$;
- for $i \in \{1, 2\}$, if we write $e_i^i \in \text{Node}(\mathcal{H}^i)$ for the node corresponding, relative to the isomorphism $((\mathcal{H}^i)|_{\mathbb{K}^*}) \circlearrowleft \mathcal{G}_{X^\log}$, to $e_i \in \text{Node}(\mathcal{G}_{X^\log})$, then it holds that $\mathcal{V}(e_i^i) = \{v_1^i\}$;
- $\mathcal{V}(f_1^i) = \{v_1^i, v_3^i\}$, $\mathcal{V}(f_2^i) = \{v_2^i, v_4^i\}$, $\mathcal{V}(f_3^i) = \{v_3^i, v_4^i\}$;
- $\mathcal{V}(e_1^i) = \mathcal{V}(e_2^i) = \{v_1^i\}$;
- for $i \in \{1, 2, 3\}$, $v_1^i$ is of type $(0, 3)$ [cf. Definition 2.3, (iii)].
Figure 6: The underlying semi-graph of $\mathcal{H}^\dagger$

One veriﬁes easily that these hypotheses imply that $(\mathcal{N}(v_1^\dagger) \cap \mathcal{N}(v_2^\dagger))^\sharp = (\mathcal{N}(v_2^\dagger) \cap \mathcal{N}(v_3^\dagger))^\sharp = 1$. Thus, it follows immediately from Proposition 5.6, (iv), (vi) — i.e., by replacing $X_{\log}$ (respectively, $e_1$, $e_2$) by the stable log curve $Y_{\log}$ obtained by applying Proposition 5.6, (iv), (vi), to the isomorphism $(\mathcal{H}^\dagger_{[f_1^\dagger,f_2^\dagger,f_3^\dagger]} \cup \{e_1^\dagger,e_2^\dagger\}) \cong \mathcal{G}_{X_{\log}}$ (respectively, by the two nodes $\in \text{Node}(\mathcal{G}_{Y_{\log}})$ corresponding to the two nodes $e_1^\dagger, e_2^\dagger$) — that to verify the above $(\mathcal{H}^\dagger)$, we may assume without loss of generality that there exist vertices $v_1, v_2, v_3$ of $\mathcal{G}_{X_{\log}}$ such that

- for $i \in \{1,2\}$, $\mathcal{V}(e_i) = \{v_i\}$;
- for $i \in \{1,2,3\}$, $v_i$ is of type $(0,3)$;
- $(\mathcal{N}(v_1) \cap \mathcal{N}(v_3))^\sharp = (\mathcal{N}(v_2) \cap \mathcal{N}(v_3))^\sharp = 1$.

Write $\mathcal{H}$ for the sub-semi-graph of PSC-type of the underlying semi-graph of $\mathcal{G}_{X_{\log}}$ whose set of vertices $= \{v_1, v_2, v_3\}$. Then one veriﬁes easily that these hypotheses imply that $\text{Node}(\mathcal{G}_{X_{\log}}|_{\mathcal{H}}) = \{e_1, e_2, f_1, f_2\}$, where we write $\{f_1\} = \mathcal{N}(v_1) \cap \mathcal{N}(v_3)$, $\{f_2\} = \mathcal{N}(v_2) \cap \mathcal{N}(v_3)$.

Thus, it follows immediately from Proposition 5.6, (v) — i.e., by replacing $X_{\log}$ (respectively, $e_1$, $e_2$) by the stable log curve $Y_{\log}$ obtained by applying Proposition 5.6, (v), to $(\mathcal{G}_{X_{\log}}|_{\mathcal{H}})$ (respectively, by the two nodes $\in \text{Node}(\mathcal{G}_{Y_{\log}})$ corresponding to $e_1$, $e_2$) — that to verify the above $(\mathcal{H})$, we may assume without loss of generality that there exist three distinct vertices $v_1, v_2, v_3$ of $\mathcal{G}_{X_{\log}}$ such that

- for $i \in \{1,2\}$, $\mathcal{V}(e_i) = \{v_i\}$;
for } i \in \{1, 2, 3\}, v_i \text{ is of type } (0, 3);  \\
• Node(\mathcal{G}_{X^{\log}}) = \{e_1, e_2, f_1, f_2\}, \text{ where we write } \{f_1\} = (N(v_1) \cap N(v_3)), \{f_2\} = (N(v_2) \cap N(v_3)).

One verifies easily that these hypotheses imply that there exists a cusp } c \text{ of } \mathcal{G}_{X^{\log}} \text{ such that } Cusp(\mathcal{G}_{X^{\log}}) = c = C(v_3).

Then it follows immediately from the explicit structure of } \mathcal{G}_{X^{\log}} \text{ that there exists an automorphism of } X^{\log} \text{ such that the automorphism of } Node(\mathcal{G}_{X^{\log}}) = f_1 \leftrightarrow e_1, e_2 \leftrightarrow e_2, f_1 \leftrightarrow f_2, f_2 \leftrightarrow f_1, \text{ (respectively, by the corresponding permutation of factors of } N(e_1) \oplus N(e_2) \oplus N(f_1) \oplus N(f_2)\text{)}, \text{ and, moreover, } \tau \text{ preserves the cusp corresponding to } c. \text{ Now it follows immediately from Corollary 3.9, (v), together with the fact that the automorphism of the anabelioid } (\mathcal{G}_{X^{\log}}) \text{ corresponding to the cusp } c \text{ induced by } \tau \text{ is the identity automorphism [cf. the argument used in the final portion of the proof of Corollary 3.9, (vi)]}, \text{ that the automorphism of } \Lambda_{\mathcal{G}} \text{ induced by } \tau \text{ is the identity automorphism. Thus, by applying the evident functoriality of the homomorphism } \rho^{\min}_{X^{\log}} \text{ with respect to the automorphism of } \mathcal{G}_{X^{\log}} \text{ induced by } \tau, \text{ one concludes immediately from the above description of } \tau, \text{ together with Theorem 4.8, (v), that the assertion (i) holds. This completes the proof of Theorem 5.7. Q.E.D.}

\textbf{Definition 5.8.} Let } \alpha \in \text{Dehn}(\mathcal{G}) \text{ be a profinite Dehn multi-twist of } \mathcal{G} \text{ and } u \in \Lambda_{\mathcal{G}} \text{ a topological generator of } \Lambda_{\mathcal{G}}.

(i) \text{ Let } e \in \text{Node}(\mathcal{G}) \text{ be a node of } \mathcal{G}. \text{ Then since } \Lambda_{\mathcal{G}} \text{ is a free } \hat{\mathbb{Z}}\Sigma\text{-module of rank } 1 \text{ [cf. Definition 3.8, (i)]}, \text{ there exists a unique element } a_e \in \hat{\mathbb{Z}}\Sigma \text{ of } \hat{\mathbb{Z}}\Sigma \text{ such that } \hat{\mathbb{Z}}\Sigma(\alpha) = a_e u. \text{ We shall refer to } a_e \in \hat{\mathbb{Z}}\Sigma \text{ as the Dehn coordinate of } \alpha \text{ indexed by } e \text{ with respect to } u.

(ii) We shall say that a profinite Dehn multi-twist } \alpha \in \text{Dehn}(\mathcal{G}) \text{ is nondegenerate if, for each node } e \in \text{Node}(\mathcal{G}) \text{ of } \mathcal{G}, \text{ the Dehn coordinate of } \alpha \text{ indexed by } e \text{ with respect to } u \text{ [cf. (i)] topologically generates an open subgroup of } \hat{\mathbb{Z}}\Sigma. \text{ Note that it is immediate that if } \alpha \text{ is nondegenerate, then the Dehn coordinate } (e \in \hat{\mathbb{Z}}\Sigma \hookrightarrow \prod_{\ell \in \Sigma} \mathbb{Z}_\ell \subseteq \prod_{\ell \in \Sigma} \mathbb{Q}_\ell) \text{ of } \alpha \text{ indexed by } e \text{ with respect to } u \text{ is contained in } \prod_{\ell \in \Sigma} \mathbb{Q}_\ell.
We shall say that a profinite Dehn multi-twist \( \alpha \in \text{Dehn}(G) \) is positive definite if \( \alpha \) is nondegenerate [cf. (ii)], and, moreover, the following condition is satisfied: For each node \( e \in \text{Node}(G) \) of \( G \), denote by \( a_e \in \mathbb{Z} \) the Dehn coordinate of \( \alpha \) indexed by \( e \) with respect to \( u \) [cf. (i)]. [Thus, \( a_e \in \prod_{l \in \Sigma} \mathbb{Q}_l^* \) — cf. (ii).]

Then for any \( e, e' \in \text{Node}(G) \), \( a_e/a_{e'} \) is contained in the image of the diagonal map \( \mathbb{Q}_{\geq 0} \overset{\text{def}}{=} \{ a \in \mathbb{Q} | a > 0 \} \rightarrow \prod_{l \in \Sigma} \mathbb{Q}_l^* \).

**Remark 5.8.1.** One may verify easily that the notions defined in Definition 5.8, (ii), (iii), are independent of the choice of the topological generator \( u \) of \( \Lambda_G \).

**Corollary 5.9 (Properties of outer representations of PSC-type and profinite Dehn multi-twists).** Let \( \Sigma \) be a nonempty set of prime numbers and \( \rho: I \rightarrow \text{Aut}(G) \) an outer representation of pro-\( \Sigma \) PSC-type [cf. [NodNon], Definition 2.1, (i)]. Suppose that \( I \) is isomorphic to \( b \mathbb{Z} \). Then the following hold:

(i) (Outer representations of SVA-type and profinite Dehn multi-twists) The following three conditions are equivalent:

(i-1) \( \rho \) is of SVA-type [cf. [NodNon], Definition 2.4, (ii)].

(i-2) The image of any topological generator of \( I \) is a profinite Dehn multi-twist [cf. Definition 4.4].

(i-3) There exists a topological generator of \( I \) whose image via \( \rho \) is a profinite Dehn multi-twist.

(ii) (Outer representations of SNN-type and nondegenerate profinite Dehn multi-twists) The following three conditions are equivalent [cf. the related discussion of [NodNon], Remark 2.14.1]:

(ii-1) \( \rho \) is of SNN-type [cf. [NodNon], Definition 2.4, (iii)].

(ii-2) The image of any topological generator of \( I \) is a nondegenerate [cf. Definition 5.8, (ii)] profinite Dehn multi-twist.

(ii-3) There exists a topological generator of \( I \) whose image via \( \rho \) is a nondegenerate profinite Dehn multi-twist.
(iii) (Outer representations of IPSC-type and positive definite profinite Dehn multi-twists) The following three conditions are equivalent [cf. Remark 5.10.1 below; the related discussion of [NodNon], Remark 2.14.1]:

(iii-1) $\rho$ is of IPSC-type [cf. [NodNon], Definition 2.4, (i)].

(iii-2) The image of any topological generator of $I$ is a positive definite [cf. Definition 5.8, (iii)] profinite Dehn multi-twist.

(iii-3) There exists a topological generator of $I$ whose image via $\rho$ is a positive definite profinite Dehn multi-twist.

(iv) (Synchronization associated to outer representations of IPSC-type) Suppose that $\rho$ is of IPSC-type. Write

$$(\hat{Z}^\Sigma)^+ \subseteq (\hat{Z}^\Sigma)^*$$

for the intersection of the images of the diagonal map $Q_{>0} \overset{\text{def}}{=} \{ a \in \mathbb{Q} \mid a > 0 \} \hookrightarrow \prod_{\ell \in \Sigma} \mathbb{Q}_\ell$ and the composite of natural morphisms $(\hat{Z}^\Sigma)^* \hookrightarrow \hat{Z}^\Sigma \cong \prod_{\ell \in \Sigma} \mathbb{Z}_\ell \subseteq \prod_{\ell \in \Sigma} \mathbb{Q}_\ell$. [Thus, when $\Sigma = \text{Primes}$, it holds that $(\hat{Z}^\Sigma)^+ = \{1\}$.] Then there exists a natural $(\hat{Z}^\Sigma)^+$-orbit of isomorphisms of $\hat{Z}^\Sigma$-modules

$$\text{syn}_\rho : I \simto \Lambda_G$$

that is functorial, in $\rho$, with respect to isomorphisms of outer representations of PSC-type [cf. [NodNon], Definition 2.1, (ii)].

(v) (Compatibility of synchronizations with finite étale coverings) In the situation of (iv), let $\Pi \subseteq \Pi_I \overset{\text{def}}{=} \Pi_G \rtimes I$ [cf. the discussion entitled “Topological groups” in §70] be an open subgroup of $\Pi_I$ such that if we write $H \to G$ for the connected finite étale covering of $G$ corresponding to $\Pi \cap \Pi_G$ [so $\Pi_H = \Pi \cap \Pi_G$], then the outer representation $\rho_1 : J \overset{\text{def}}{=} \Pi/I \to \text{Out}(\Pi_H)$ is of IPSC-type. Then the diagram of $\hat{Z}^\Sigma$-modules

\[
\begin{array}{ccc}
J & \xrightarrow{\text{syn}_{\rho_1}} & \Lambda_H \\
\downarrow & & \downarrow \\
I & \xrightarrow{\text{syn}_\rho} & \Lambda_G
\end{array}
\]
where the left-hand vertical arrow is the natural inclusion; the right-hand vertical arrow is the isomorphism of Corollary 3.9, (iii) — commutes up to multiplication by an element $\in \mathbb{Q}_{>0}$.

Proof. Assertion (i) follows immediately from condition (2$'$) of [NodNon], Definition 2.4. Next, we verify assertions (ii) and (iii). The implication

$$(ii-1) \implies (ii-2), \quad (respectively, \quad (iii-1) \implies (iii-2))$$

follows immediately from the final portion of Lemma 5.2, (ii), concerning $\rho$ of SNN-type (respectively, Lemma 5.4, (ii); Theorem 5.7). The implications

$$(ii-2) \implies (ii-3), \quad (iii-2) \implies (iii-3)$$

are immediate.

Next, we verify the implication

$$(ii-3) \implies (ii-1).$$

It follows from the implication (i-3) $\implies$ (i-1) that $\rho$ is of SVA-type. Thus, to show the implication in question, it suffices to verify that $\rho$ satisfies condition (3) of [NodNon], Definition 2.4. Let $\tilde{\mathcal{V}} \in \text{Node}(\tilde{G})$ be an element of Node(\tilde{G}); $\Pi_I \overset{\text{def}}{=} \Pi_{\tilde{G}} \rtimes I$ [cf. the discussion entitled "Topological groups" in §0]; $\tilde{v}, \tilde{w} \in \text{Vert}(\tilde{G})$ the two distinct elements of Vert(\tilde{G}) such that $\mathcal{V}(\tilde{r}) = \{\tilde{v}, \tilde{w}\}$ [cf. [NodNon], Remark 1.2.1, (iii)]; $I_{\tilde{v}}, I_{\tilde{w}}, I_{\tilde{w}} \subseteq \Pi_I$ the inertia subgroups of $\Pi_I$ associated to $\tilde{v}, \tilde{v}, \tilde{w}$, respectively. Then since the homomorphisms of the final two displays of Lemma 5.2, (ii), coincide, and $\Lambda_{\tilde{G}_{\text{top}}}^{\text{rel}}$ and $I_{\tilde{v}}$ are free $\mathbb{Z}^\Sigma$-modules of rank 1 [cf. Definition 3.8, (i); [NodNon], Lemma 2.5, (i)], it follows immediately from the definition of nondegeneracy that the composite of the second display of Lemma 5.2, (ii), is an open injection. Thus, it follows immediately that the natural homomorphism $I_{\tilde{v}} \times I_{\tilde{w}} \to I_{\tilde{v}}$ has open image, and that $I_{\tilde{v}} \cap I_{\tilde{w}} = \{1\}$, i.e., that $I_{\tilde{v}} \times I_{\tilde{w}} \to I_{\tilde{v}}$ is injective. That is to say, $\rho$ satisfies condition (3) of [NodNon], Definition 2.4. This completes the proof of the implication in question.

Next, we verify the implication

$$(iii-3) \implies (iii-1).$$

Let $u \in \Lambda_{\tilde{G}}$ be a topological generator of $\Lambda_{\tilde{G}}$. Then it follows immediately from Lemma 5.4, (i), (ii), and Theorem 5.7 — by considering the stable
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log curve over $S_{log}$ corresponding to a suitable homomorphism of $R$-algebras $\hat{O} \simeq R[[t_1, \ldots, t_{3g-2g+1}]] \to R$ [cf. Lemma 5.4, (i)] — that to complete the proof of the implication in question, it suffices to verify that there exists a topological generator $\alpha \in I$ of $I$ which satisfies the following condition (*):

(*): The Dehn coordinates of $\rho(\alpha)$ with respect to $u$

[cf. Definition 5.8, (i)] $\in \mathbb{N} \setminus \{0\}$.

To this end, let $\alpha \in I$ be a topological generator of $I$ such that $\rho(\alpha)$ is a positive definite profinite Dehn multi-twist of $G$ [cf. condition (iii-3)]. For each node $e \in \text{Node}(G)$ of $G$, denote by $a_e \in \hat{\mathbb{Z}}^\Sigma$ the Dehn coordinate of $\rho(\alpha)$ indexed by $e$ with respect to $u$. Now since $\rho(\alpha)$ is nondegenerate, it follows immediately from the definition of nondegeneracy that for each node $e \in \text{Node}(G)$ of $G$, it holds that $a_e \in \mathbb{N} \setminus (\mathbb{Q}^\times)^\Sigma$. Thus, it follows immediately that for a given node $f \in \text{Node}(G)$ of $G$, by replacing $\alpha$ by a suitable topological generator of $I$, we may assume without loss of generality that $a_f \in \mathbb{N} \setminus \{0\}$. In particular, it follows immediately from the definition of positive definiteness that there exists an element $a \in \mathbb{N} \setminus \{0\}$ such that for each node $e \in \text{Node}(G)$ of $G$, it holds that $a \cdot a_e \in \mathbb{N} \setminus \{0\}$.

Moreover, by replacing $\alpha$ by a suitable topological generator of $I$, we may assume that every prime number dividing $a$ belongs to $\mathfrak{P}$. But then it follows from the fact that $a_e \in \hat{\mathbb{Z}}^\Sigma \cap (\frac{1}{n} \cdot \mathbb{N})$ that $a_e$ is a positive rational number that is integral at every element of $\mathfrak{P}$, i.e., that $a_e \in \mathbb{N} \setminus \{0\}$, as desired. In particular, the topological generator $\alpha \in I$ satisfies the above condition (*). This completes the proof of the implication in question, hence also of assertions (ii) and (iii).

Next, we verify assertion (iv). It follows immediately from the final portion of Lemma 5.2, (ii), concerning $\rho$ of $SNN$-type that for each $e \in \text{Node}(G)$, the homomorphism $\syn_{\rho}: I \to \Delta_G$ obtained by dividing the composite $I \xrightarrow{\Delta} \text{Dehn}(G) \xrightarrow{\rho} \Delta_G$ by $\text{img}_{\Sigma}(e, \rho)$ is an isomorphism. Moreover, by “translating into group theory” the scheme-theoretic content of Lemma 5.4, (ii), by means of the correspondence between group-theoretic and scheme-theoretic notions given in Proposition 5.6, (i); Theorem 5.7, one concludes that $\syn_{\rho}$ is independent — up to multiplication by an element of $(\hat{\mathbb{Z}}^\Sigma)^+$ — of the choice of the node $e \in \text{Node}(G)$. Now the functoriality of $\syn_{\rho}$ follows immediately from the functoriality of the homomorphism $D_{e}$ [cf. Theorem 4.8, (iv)], together with the group-theoreticity of $\text{img}_{\Sigma}(e, \rho)$. This completes the proof of assertion (iv).

Finally, assertion (v) follows immediately, in light of the group-theoretic construction of “$\syn_{\rho}$” given in the proof of assertion (iv), from the various definitions involved. Q.E.D.
Remark 5.9.1.

(i) Corollary 5.9, (iv), may be regarded as a sort of abstract combinatorial analogue of the cyclotomic synchronization given in [GalSct], Theorem 4.3 [cf. also [AbsHyp], Lemma 2.5, (ii)].

(ii) It follows from Theorem 5.7 that one may think of the isomorphisms of Corollary 5.9, (iv), as a sort of abstract combinatorial construction of the various identification isomorphisms between the various copies of \( \hat{\mathbb{Z}}^\Sigma(1) \) that appear in Lemma 5.4, (ii). Such identification isomorphisms are typically “taken for granted” in conventional discussions of scheme theory.

Remark 5.9.2.

(i) Consider the exact sequence of free \( \hat{\mathbb{Z}}^\Sigma \)-modules

\[
0 \longrightarrow M_G^{\text{vert}} \longrightarrow M_G \overset{\Pi_G^{ab}}{\longrightarrow} M_G^{\text{comb}} \overset{\text{def}}{=} M_G / M_G^{\text{vert}} \longrightarrow 0
\]

where we write \( M_G^{\text{vert}} \subseteq M_G \) for the \( \hat{\mathbb{Z}}^\Sigma \)-submodule of \( M_G \) topologically generated by the images of the vertical subgroups of \( \Pi_G \) [cf. [CmbGC], Remark 1.1.4]. Then one verifies easily that any profinite Dehn multi-twist \( \alpha \in \text{Dehn}(\hat{\mathcal{G}}) \) preserves and induces the identity automorphism on \( M_G^{\text{vert}}, M_G^{\text{comb}} \). In particular, the homomorphism \( M_G \to M_G \) obtained by considering the difference of the automorphism of \( M_G \) induced by \( \alpha \) and the identity automorphism on \( M_G \) naturally determines [and is determined by!] a homomorphism

\[\alpha_{\text{comb, vert}}: M_G^{\text{comb}} \longrightarrow M_G^{\text{vert}}.\]

Write \( M_G^{\text{edge}} \subseteq M_G^{\text{vert}} \) for the \( \hat{\mathbb{Z}}^\Sigma \)-submodule topologically generated by the image of the edge-like subgroups of \( \Pi_G \). Then the following two facts are well-known:

- If \( \text{Cusp}(\mathcal{G}) = \emptyset \), then Poincaré duality \( M_G \cong \text{Hom}_{\hat{\mathbb{Z}}^\Sigma}(M_G, \Lambda_G) \) determines an isomorphism \( M_G^{\text{edge}} \cong \text{Hom}_{\hat{\mathbb{Z}}^\Sigma}(M_G^{\text{comb}}, \Lambda_G) \) [cf. [CmbGC], Proposition 1.3].

- The natural homomorphism

\[\text{Dehn}(\hat{\mathcal{G}}) \longrightarrow \text{Hom}_{\hat{\mathbb{Z}}^\Sigma}(M_G^{\text{comb}}, M_G^{\text{vert}})\]

given by mapping \( \alpha \mapsto \alpha_{\text{comb, vert}} \) factors through the submodule \( \text{Hom}_{\hat{\mathbb{Z}}^\Sigma}(M_G^{\text{comb}}, M_G^{\text{edge}}) \subseteq \text{Hom}_{\hat{\mathbb{Z}}^\Sigma}(M_G^{\text{comb}}, M_G^{\text{vert}}) \).
Indeed, this may be verified, for instance, by applying a similar argument to the argument used in the proof of [CmbGC], Proposition 1.3, involving weights.

Thus, if Cusp(\(G\)) = \(\emptyset\), then we obtain a homomorphism

\[
\Omega_G : \text{Dehn}(G) \rightarrow M_G^\text{edge} \otimes_{\mathbb{Z}_\Sigma} M_G^\text{edge} \otimes_{\mathbb{Z}_\Sigma} \text{Hom}_{\mathbb{Z}_\Sigma}(\Lambda_G, \mathbb{Z}_\Sigma)
\]

that is manifestly functorial, in \(G\), with respect to isomorphisms of semi-graphs of anabelioids of pro-\(\Sigma\) PSC-type. The matrices that appear in the image of this homomorphism \(\Omega_G\) are often referred to as period matrices.

(ii) Now let us recall that [CmbGC], Proposition 2.6, plays a key role in the proof of the combinatorial version of the Grothendieck conjecture given in [CmbGC], Corollary 2.7, (iii). Moreover, the proof of [CmbGC], Proposition 2.6, is essentially a formal consequence of the nondegeneracy of the period matrix associated to a positive definite profinite Dehn multi-twist — i.e., of the injectivity of the homomorphism

\[
\alpha_{\text{comb,vert}} : M_G^\text{comb} \rightarrow M_G^\text{vert}
\]

of (i) in the case where \(\alpha \in \text{Dehn}(G)\) is positive definite [cf. Corollary 5.9, (iii)].

(iii) In general, the period matrix associated to a profinite Dehn multi-twist may fail to be nondegenerate even if the profinite Dehn multi-twist is nondegenerate. Indeed, suppose that \(\Sigma^2 = 1\), that \(G\) is the double [cf. [CmbGC], Proposition 2.2, (i)] of a semi-graph of anabelioids of pro-\(\Sigma\) PSC-type \(H\) such that

\[
(\text{Vert}(H)^2, \text{Node}(H)^2, \text{Cusp}(H)^2) = (1, 0, 2).
\]

Suppose, moreover, that \(H\) admits an automorphism which permutes the two cusps of \(H\) and extends to an automorphism \(\phi\) of \(G\). [One verifies easily that such data exist.] Then one may verify easily that Node(\(G\))^2 = 2, that Cusp(\(G\))^2 = 0, and that the free \(\mathbb{Z}_\Sigma\)-module \(M_G^\text{comb}\), hence also \(M_G^\text{edge} \otimes_{\mathbb{Z}_\Sigma} M_G^\text{edge} \otimes_{\mathbb{Z}_\Sigma} \text{Hom}_{\mathbb{Z}_\Sigma}(\Lambda_G, \mathbb{Z}_\Sigma)\) [cf. (i)], is of rank 1. Now let us recall that the period matrix associated to a positive definite profinite Dehn multi-twist is necessarily nondegenerate [cf. Corollary 5.9, (iii); the proof of [CmbGC], Proposition 2.6]. Thus, since \(\Sigma^2 = 1\), it follows immediately from the functoriality of \(\Omega_G\) [cf. (i)] and
\[ \mathcal{D}_G \text{ [cf. Theorem 4.8, (iv)] with respect to } \phi \text{ that the kernel of} \]
\[ \bigoplus_{\text{Node}(G)} \mathcal{D}_G \xleftarrow{\text{Dehn}(G)} M^\text{edge}_G \otimes_{\mathbb{Z}_2} M^\text{edge}_G \otimes_{\mathbb{Z}_2} \text{Hom}_{\mathbb{Z}_2}(A_G, \mathbb{F}_2^\Sigma) \]

is a free \( \mathbb{F}_2^2 \)-submodule of \( \bigoplus_{\text{Node}(G)} A_G \) of rank 1 that is stabilized by \( \phi \). On the other hand, since profinite Dehn multi-twists of the form \((u, u)^2 \oplus \text{Node}(G)\) of \( u \in A_G \), where \( u \neq 0 \), are manifestly positive definite, we thus conclude that the kernel in question is equal to

\[ \{ (u, -u) \in \bigoplus_{\text{Node}(G)} A_G \mid u \in A_G \}. \]

In particular, any nonzero element of this kernel yields an example of a nondegenerate profinite Dehn multi-twist whose associated period matrix fails to be nondegenerate.

**Corollary 5.10 (Combinatorial/group-theoretic nature of scheme-theoreticity)** Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(\Sigma\) a nonempty set of prime numbers; \(R\) a complete discrete valuation ring whose residue field \(k\) is separably closed of characteristic \(\not \equiv 2\); \(S\) a complete discrete valuation ring whose completion \(\hat{S}\) has residue field \(\overline{k}\) of characteristic \(\not \equiv 2\); \(x_0\) the completion of the local ring of \((\mathcal{M}_{g, r})_S\) at the image of \(x\); \(T_{\log}\) the log scheme obtained by equipping \(T \equiv \text{Spec } \hat{S}\) with the log structure determined by the maximal ideal of \(R\); \(x \in (\mathcal{M}_{g, r})_S(k)\) a \(k\)-valued point of the moduli stack of curves \((\mathcal{M}_{g, r})_S\) of type \((g, r)\) over \(S\) [cf. the discussion entitled “Curves” in §0.]; \(\mathcal{O}\) the completion of the local ring of \((\mathcal{M}_{g, r})_S\) at the image of \(x\); \(T_{\log}\) the log scheme obtained by equipping \(T \equiv \text{Spec } \hat{S}\) with the log structure induced by the log structure of \((\mathcal{M}_{g, r})_S\) [cf. the discussion entitled “Curves” in §0]; \(t_{\log}\) the log scheme obtained by equipping the closed point of \(T\) with the log structure induced by the log structure of \(T_{\log}\); \(X^\text{log}_{t_{\log}}\) the stable log curve over \(t_{\log}\) corresponding to the natural strict (1-)morphism \(t_{\log} \to (\mathcal{M}_{g, r})_S\); \(I_{t_{\log}}\) the maximal pro-\(\Sigma\) quotient of the log fundamental group \(\pi_1(T_{\log})\) of \(T_{\log}\); \(I_{S_{\log}}\) the maximal pro-\(\Sigma\) quotient of the log fundamental group \(\pi_1(S_{\log})\) of \(S_{\log}\); \(G^\text{univ}_{X^\text{log}_{t_{\log}}}\) the semi-graph of anabeloids of pro-\(\Sigma\) PSC-type determined by the stable log curve \(X^\text{log}_{t_{\log}}\) [cf. [CmbGC], Example 2.5]; \(\rho^\text{univ}_{X^\text{log}_{t_{\log}}}: I_{t_{\log}} \to \text{Aut}(G^\text{univ}_{X^\text{log}_{t_{\log}}})\) the natural outer representation associated to \(X^\text{log}_{t_{\log}}\) [cf. Definition 5.5]; \(\rho: I \to \text{Aut}(G^\text{univ}_{X^\text{log}_{t_{\log}}})\) an
outer representation of pro-$\Sigma$ PSC-type [cf. [NodNon], Definition 2.1, (i)]. Then the following conditions are equivalent:

(i) $\rho$ is of $\text{IPSC-type}$.

(ii) There exist a morphism of log schemes $\phi_{\text{log}}: S_{\text{log}} \to T_{\text{log}}$ over $S$ and an isomorphism of outer representations of pro-$\Sigma$ PSC-type $\rho \Rightarrow \rho_{\text{Xlog}}^\text{univ} \circ I_{\phi_{\text{log}}}$ [cf. [NodNon], Definition 2.1, (i)] — where we write $I_{\phi_{\text{log}}} : I_{S_{\text{log}}} \to I_{T_{\text{log}}}$ for the homomorphism induced by $\phi_{\text{log}}$ — i.e., there exist an automorphism $\beta$ of $G_{X_{\text{log}}}$ and an isomorphism $\alpha : I \Rightarrow I_S^\text{log}$ such that the diagram

\[
\begin{array}{ccc}
I & \xrightarrow{\rho} & \text{Aut}(G_{X_{\text{log}}}) \\
\alpha \downarrow & & \downarrow \alpha \\
I_{S_{\text{log}}} & \xrightarrow{\rho_{X_{\text{log}}}^\text{log} \circ I_{\phi_{\text{log}}}} & \text{Aut}(G_{X_{\text{log}}})
\end{array}
\]

— where the right-hand vertical arrow is the automorphism of $\text{Aut}(G_{X_{\text{log}}})$ induced by $\beta$ — commutes.

(iii) There exist a morphism of log schemes $\phi_{\text{log}}: S_{\text{log}} \to T_{\text{log}}$ over $S$ and an isomorphism $\alpha : I \Rightarrow I_S^\text{log}$ such that $\rho = \rho_{X_{\text{log}}}^\text{univ} \circ I_{\phi_{\text{log}}} \circ \alpha$ — where we write $I_{\phi_{\text{log}}} : I_{S_{\text{log}}} \to I_{T_{\text{log}}}$ for the homomorphism induced by $\phi_{\text{log}}$ — i.e., the automorphism “$\beta$” of (ii) may be taken to be the identity.

Proof. The equivalence (i) $\iff$ (ii) follows from the definition of the term “IPSC-type” [cf. [NodNon], Definition 2.4, (i)]. The implication (iii) $\Rightarrow$ (ii) is immediate. The implication (ii) $\Rightarrow$ (iii) follows immediately, in light of the functoriality asserted in Theorem 4.8, (iv), from Lemma 5.4, (i), (ii), and Theorem 5.7. Q.E.D.

Remark 5.10.1.

(i) The equivalence of Corollary 5.10 essentially amounts to the equivalence

“$\text{IPSC-type} \iff \text{positive definite}$”

which was discussed in [HM], Remark 2.14.1, without proof.
(ii) One way to understand the equivalence of Corollary 5.10 is as
the statement that the property that an outer representation
of PSC-type be of scheme-theoretic origin may be formulated
purely in terms of combinatorics/group theory.

In the final portion of the present §5, we apply the theory developed so far [i.e., in particular, the equivalences of Corollary 5.9, (ii), (iii)] to derive results [cf. Theorem 5.14] concerning normalizers and commensurators of groups of profinite Dehn multi-twists.

**Definition 5.11.** Let $M \subseteq H \subseteq \text{Out}(\Pi_G)$ be closed subgroups of $\text{Out}(\Pi_G)$. Suppose further that $M$ is an abelian pro-$\Sigma$ group [such as $\text{Dehn}(G)$ — cf. Theorem 4.8, (iv)].

(i) We shall write

$$N^\text{scal}_H(M) \subseteq N_H(M) \subseteq H$$

for the [closed] subgroup of $H$ consisting of $\alpha \in H$ satisfying the following condition: $\alpha \in N_H(M)$, and, moreover, the action of $\alpha$ on $M$ by conjugation coincides with the automorphism of $M$ given by multiplication by an element of $(\hat{Z}^\Sigma)^*$. We shall refer to $N^\text{scal}_H(M)$ as the scalar-normalizer of $M$ in $H$.

(ii) We shall write

$$C^\text{scal}_H(M) \subseteq C_H(M) \subseteq H$$

for the subgroup of $H$ consisting of $\alpha \in H$ satisfying the following condition: there exists an open $\hat{Z}^\Sigma$-submodule $M'_\alpha \subseteq M$ of $M$ [possibly depending on $\alpha$] such that the action of $\alpha$ on $H$ by conjugation determines an automorphism of $M'_\alpha$ given by multiplication by an element of $(\hat{Z}^\Sigma)^*$. We shall refer to $C^\text{scal}_H(M)$ as the scalar-commensurator of $M$ in $H$.

**Lemma 5.12 (Scalar-normalizers and scalar-commensurators).** Let $M \subseteq H \subseteq \text{Out}(\Pi_G)$ be closed subgroups of $\text{Out}(\Pi_G)$. Suppose further that $M$ is an abelian pro-$\Sigma$ group. Then:
(i) It holds that
\[ M \subseteq Z_H(M) \subseteq N_H^{scal}(M) \subseteq C_H^{scal}(M). \]

(ii) If \( M' \subseteq M \) is a \( \hat{\mathbb{Z}}^\Sigma \)-submodule of \( M \), then
\[ N_H^{scal}(M) \subseteq N_H^{scal}(M') ; \quad C_H^{scal}(M) \subseteq C_H^{scal}(M'). \]

If, moreover, \( M' \subseteq M \) is open in \( M \), then
\[ C_H^{scal}(M) = C_H^{scal}(M'). \]

Proof. These assertions follow immediately from the various definitions involved. \( \quad \) Q.E.D.

**Definition 5.13.** Let \( H \subseteq \mathrm{Out}(\Pi_G) \) be a closed subgroup of \( \mathrm{Out}(\Pi_G) \). Then we shall say that \( H \) is IPSC-ample (respectively, NN-ample) if \( H \) contains a positive definite (respectively, nondegenerate) \[ \text{cf. Definition 5.8} \] profinite Dehn multi-twist \( \in \mathrm{Dehn}(\mathcal{G}) \).

**Remark 5.13.1.** It follows immediately from Theorem 4.8, (iv), that any open subgroup of \( \mathrm{Dehn}(\mathcal{G}) \) is IPSC-ample, hence also NN-ample \[ \text{cf. Definition 5.13} \].

**Theorem 5.14 (Normalizers and commensurators of groups of profinite Dehn multi-twists).** Let \( \Sigma \) be a nonempty set of prime numbers, \( \mathcal{G} \) a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type, \( \mathrm{Out}_G^C(\Pi_G) \) the group of group-theoretically cuspidal \[ \text{cf. [CmbGC], Definition 1.4, (iv)] \} automorphisms of \( \Pi_G \), and \( M \subseteq \mathrm{Out}_G^C(\Pi_G) \) a closed subgroup of \( \mathrm{Out}_G^C(\Pi_G) \) which is abelian pro-\( \Sigma \). Then the following hold:

(i) Suppose that one of the following two conditions is satisfied:

1. \( M \) is IPSC-ample \[ \text{cf. Definition 5.13} \].
2. \( M \) is NN-ample \[ \text{cf. Definition 5.13} \], and \( \mathrm{Cusp}(\mathcal{G}) \neq \emptyset \).
Then it holds that
\[ N^{scal}(\Pi_G)(M) \subseteq C^{scal}(\Pi_G)(M) \subseteq \text{Aut}(G) \]
[cf. Definition 5.11]. If, moreover, \( M \subseteq \text{Dehn}(G) \) [cf. Definition 4.4], then
\[ \text{Aut}^{\text{Node}(G)}(G) \subseteq N^{scal}(\Pi_G)(M) \subseteq C^{scal}(\Pi_G)(M) \subseteq \text{Aut}(G) \]
[cf. Definition 2.6, (i)]. In particular,
\[ N^{scal}(\Pi_G)(M), C^{scal}(\Pi_G)(M) \subseteq \text{Aut}(G) \]
are open subgroups of \( \text{Aut}(G) \).

(ii) If \( M \) is an open subgroup of \( \text{Dehn}(G) \), then it holds that
\[ \text{Aut}(G) = C^{\text{Out}^c}(\Pi_G)(M) \cdot \]
If, moreover, \( \text{Node}(G) \neq \emptyset \), then
\[ \text{Aut}^{\text{Node}(G)}(G) \cap \text{Ker}(\chi_G) = Z^{\text{Out}^c}(\Pi_G)(M) \]
[cf. Definition 3.8, (ii)].

(iii) It holds that
\[ \text{Aut}(G) = N^{\text{Out}^c}(\Pi_G)(\text{Dehn}(G)) = C^{\text{Out}^c}(\Pi_G)(\text{Dehn}(G)) \cdot \]

Proof. First, we verify the inclusion \( C^{\text{scal}}(\Pi_G)(M) \subseteq \text{Aut}(G) \) asserted in assertion (i). Suppose that condition (1) (respectively, (2)) is satisfied. Let \( \alpha \in C^{\text{scal}}(\Pi_G)(M) \). Then since \( \alpha \in C^{\text{scal}}(\Pi_G)(M) \), and \( M \) is IPSC-ample (respectively, NN-ample), it follows immediately that there exists an element \( \beta \in M \) of \( M \) such that both \( \beta \) and \( \alpha \beta \alpha^{-1} = \beta^{\lambda} \), where \( \lambda \in (\mathbb{Z}^2)^* \), are positive definite (respectively, nondegenerate) profinite Dehn multi-twists. Thus, the graphicity of \( \alpha \) follows immediately from [NodNon], Remark 4.2.1, together with Corollary 5.9, (iii) (respectively, from [NodNon], Theorem A, together with Corollary 5.9, (ii)). This completes the proof of the inclusion \( C^{\text{scal}}(\Pi_G)(M) \subseteq \text{Aut}(G) \), hence also, by Lemma 5.12, (i), of the two inclusions in the first display of assertion (i).

If, moreover, \( M \subseteq \text{Dehn}(G) \), then the inclusion \( \text{Aut}^{\text{Node}(G)}(G) \subseteq N^{\text{scal}}(\Pi_G)(M) \) follows immediately from Theorem 4.8, (v). Thus, since
In the notation of Theorem 5.14, (i) (respectively, Theorem 5.14, (ii)), in general, the inclusion
\[ \text{Aut}^{\text{Node}(\mathcal{G})}(\mathcal{G}) \] is an open subgroup of \( \text{Aut}(\mathcal{G}) \) [cf. Proposition 2.7, (iii)], it follows immediately that \( N^{\text{scal}}_{\text{Out}^c(\Pi_\mathcal{G})}(M) \), hence also \( C^{\text{scal}}_{\text{Out}^c(\Pi_\mathcal{G})}(M) \), is an open subgroup of \( \text{Aut}(\mathcal{G}) \). This completes the proof of assertion (i).

Next, we verify the equality \( \text{Aut}(\mathcal{G}) = C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \) in the first display of assertion (ii). It follows immediately from Theorem 4.8, (i), that \( \text{Aut}(\mathcal{G}) \subseteq N_{\text{Out}^c(\Pi_\mathcal{G})}(\text{Dehn}(\mathcal{G})) \subseteq C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \). Thus, to verify the equality \( \text{Aut}(\mathcal{G}) = C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \), it suffices to verify the inclusion \( C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \subseteq \text{Aut}(\mathcal{G}) \). To this end, let \( \alpha \in C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \). Then it follows from Lemma 5.12, (ii), that
\[ C_{\text{Out}^c(\Pi_\mathcal{G})}(M) = C_{\text{Out}^c(\Pi_\mathcal{G})}(\alpha \cdot M \cdot \alpha^{-1}) = \alpha \cdot C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \cdot \alpha^{-1}, \]
i.e., \( \alpha \in N_{\text{Out}^c(\Pi_\mathcal{G})}(C_{\text{Out}^c(\Pi_\mathcal{G})}(M)) \). Thus, since \( C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \) is an open subgroup of \( \text{Aut}(\mathcal{G}) \) [cf. assertion (i); Remark 5.13.1], we conclude that \( \alpha \in C_{\text{Out}^c(\Pi_\mathcal{G})}(\text{Aut}(\mathcal{G})) \). Thus, the fact that \( \alpha \in \text{Aut}(\mathcal{G}) \) follows from the commensurable terminality of \( \text{Aut}(\mathcal{G}) \) in \( \text{Out}(\Pi_\mathcal{G}) \), i.e., the equality \( \text{Aut}(\mathcal{G}) = C_{\text{Out}(\Pi_\mathcal{G})}(\text{Aut}(\mathcal{G})) \) [cf. [CmbGC], Corollary 2.7, (iv)]. This completes the proof of the equality \( \text{Aut}(\mathcal{G}) = C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \).

Next, we verify the equality
\[ \text{Aut}^{\text{Node}(\mathcal{G})}(\mathcal{G}) \cap \text{Ker}(\chi_\mathcal{G}) = Z_{\text{Out}^c(\Pi_\mathcal{G})}(M) \]
in the second display of assertion (ii). Now it follows immediately from Theorem 4.8, (v), that \( \text{Aut}^{\text{Node}(\mathcal{G})}(\mathcal{G}) \cap \text{Ker}(\chi_\mathcal{G}) \subseteq Z_{\text{Out}^c(\Pi_\mathcal{G})}(M) \). Thus, to show the equality in question, it suffices to verify the inclusion
\[ Z_{\text{Out}^c(\Pi_\mathcal{G})}(M) \subseteq \text{Aut}^{\text{Node}(\mathcal{G})}(\mathcal{G}) \cap \text{Ker}(\chi_\mathcal{G}). \]
To this end, let us observe that since \( \text{Aut}(\mathcal{G}) = C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \) [cf. the preceding paragraph], it holds that \( Z_{\text{Out}^c(\Pi_\mathcal{G})}(M) \subseteq \text{Aut}(\mathcal{G}) \). Thus, since the action of \( Z_{\text{Out}^c(\Pi_\mathcal{G})}(M) \) on \( M \) by conjugation preserves and induces the identity automorphism on the intersection of \( M \) with each direct summand
\[ \bigoplus_{\mathcal{G} \in \text{Node}(\mathcal{G})} \text{Dehn}(\mathcal{G}) \] [i.e., each “\( \mathcal{G}_a \)"], it follows immediately from Theorem 4.8, (v), in light of our assumption that \( \text{Node}(\mathcal{G}) \neq \emptyset \), that
\[ Z_{\text{Out}^c(\Pi_\mathcal{G})}(M) \subseteq \text{Aut}^{\text{Node}(\mathcal{G})}(\mathcal{G}) \cap \text{Ker}(\chi_\mathcal{G}). \] This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (ii), together with Theorem 4.8, (i). This completes the proof of Theorem 5.14. Q.E.D.

**Remark 5.14.1.** In the notation of Theorem 5.14, (i) (respectively, Theorem 5.14, (ii)), in general, the inclusion
\[ C_{\text{Out}^c(\Pi_\mathcal{G})}(M) \subseteq \text{Aut}(\mathcal{G}) \]

[hence, a fortiori, by the inclusions of the first display of Theorem 5.14, (i), the inclusion $N_{Out^C(PG)}^{scal}(M) \subseteq Aut(G)$] (respectively, in general, the inclusion $N_{Out^C(PG)}(M) \subseteq Aut(G)$)

is strict. Indeed, suppose that there exist a node $e \in \text{Node}(G)$ and an automorphism $\alpha \in \text{Aut}(G)$ of $G$ such that $\alpha$ does not stabilize $e$, and $\chi_G(\alpha) = 1$. [For example, in the notation of the final paragraph of the proof of Theorem 5.7, the node $e_1$ and the automorphism induced by $X_\log$ satisfy these conditions.] Now fix a prime number $l \in \Sigma$; write

$$M \overset{\text{def}}{=} l \cdot (\Lambda_G)_e \oplus \left( \bigoplus_{f \neq e} \Lambda_G \right) \subseteq \bigoplus_{f \in \text{Node}(G)} \Lambda_G \hookrightarrow \text{Dehn}(G)$$

— where we use the notation $(\Lambda_G)_e$ to denote a copy of $\Lambda_G$ indexed by $e \in \text{Node}(G)$. Then $M$ is an open subgroup of $\text{Dehn}(G)$, hence also IPSCample [cf. Remark 5.13.1], but it follows immediately from Theorem 4.8, (v), that $\alpha \notin C_{Out^C(PG)}^{scal}(M)$ (respectively, $\alpha \notin N_{Out^C(PG)}(M)$).

### §6. Centralizers of geometric monodromy

In the present §, we study the centralizer of the image of certain geometric monodromy groups. As an application, we prove a “geometric version of the Grothendieck conjecture” for the universal curve over the moduli stack of pointed smooth curves [cf. Theorem 6.13 below].

**Definition 6.1.** Let $\Sigma$ be a nonempty set of prime numbers and $\Pi$ a pro-$\Sigma$ surface group [cf. [MT], Definition 1.2]. Then we shall write

$$\text{Out}^C(\Pi) = \text{Out}^{FC}(\Pi) = \text{Out}^{PFC}(\Pi)$$

for the group of automorphisms of $\Pi$ which induce bijections on the set of cuspidal inertia subgroups of $\Pi$. We shall refer to an element of $\text{Out}^C(\Pi) = \text{Out}^{FC}(\Pi) = \text{Out}^{PFC}(\Pi)$ as a $C$-, $FC$-, or $PFC$-admissible automorphism of $\Pi$.

**Remark 6.1.1.** In the notation of Definition 6.1, suppose that either $\Sigma^3 = 1$ or $\Sigma = \mathbb{P}rim$. Then it follows from the various definitions involved that $\Pi$ is equipped with a natural structure of pro-$\Sigma$ configuration space group [cf. [MT], Definition 2.3, (i)]. Thus, the
terms “$C$-/FC-/PFC-admissible automorphism of $\Pi$” and the notation “$\text{Out}^C(\Pi) = \text{Out}^{\text{FC}}(\Pi)$” have already been defined in [CmbCsp], Definition 1.1, (ii), and Definition 1.4, (iii), of the present paper. In this case, however, one may verify easily that these definitions coincide.

**Lemma 6.2 (Extensions arising from log configuration spaces).** Let $(g, r)$ be a pair of nonnegative integers such that $2g+2-r > 0$; $0 < m < n$ positive integers; $\Sigma_F $ and $\Sigma_B $ nonempty sets of prime numbers; $k$ an algebraically closed field of characteristic zero; $(\text{Spec } k)^{\log}$ the log scheme obtained by equipping $\text{Spec } k$ with the log structure given by the fs chart $N \to k$ that maps $1 \mapsto 0$; $X^{\log} = X_1^{\log}$ a stable log curve of type $(g, r)$ over $(\text{Spec } k)^{\log}$. Suppose that $\Sigma_F \subseteq \Sigma_B$ satisfy one of the following two conditions:

1. $\Sigma_F$ and $\Sigma_B$ determine PT-formations i.e., are either of cardinality one or equal to $\mathfrak{P} \text{rimes}$ — cf. [MT], Definition 1.1, (ii)].

2. $n - m = 1$ and $\Sigma_B = \mathfrak{P} \text{rimes}$. Write

$$X_n^{\log}, X_m^{\log}$$

for the $n$-th, $m$-th log configuration spaces of the stable log curve $X^{\log}$ [cf. the discussion entitled “Curves” in §0], respectively; $\Pi_n , \Pi_B \overset{\text{def}}{=} \Pi_m$ for the respective maximal pro-$\Sigma_B$ quotients of the kernels of the natural surjections $\pi_1( X_n^{\log}) \twoheadrightarrow \pi_1((\text{Spec } k)^{\log})$, $\pi_1( X_m^{\log}) \twoheadrightarrow \pi_1((\text{Spec } k)^{\log})$; $\Pi_{n/m} \subseteq \Pi_n$ for the kernel of the surjection $\Pi_n \twoheadrightarrow \Pi_B = \Pi_m$ induced by the projection $X_{n/m}^{\log} \twoheadrightarrow X_m^{\log}$ obtained by forgetting the last $(n - m)$ factors; $\Pi_F$ for the maximal pro-$\Sigma_F$ quotient of $\Pi_{n/m} , \Pi_T$ for the quotient of $\Pi_n$ by the kernel of the natural surjection $\Pi_{n/m} \twoheadrightarrow \Pi_F$. Thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Pi_F \longrightarrow \Pi_T \longrightarrow \Pi_B \longrightarrow 1 ,$$

which determines an outer representation

$$\rho_{n/m} : \Pi_B \longrightarrow \text{Out}(\Pi_F) .$$

Then the following hold:

(i) The isomorphism class of the exact sequence of profinite groups

$$1 \longrightarrow \Pi_F \longrightarrow \Pi_T \longrightarrow \Pi_B \longrightarrow 1$$
depends only on \((g,r)\) and the pair \((\Sigma_F, \Sigma_B)\), i.e., if \(1 \rightarrow \Pi_F^* \rightarrow \Pi_T^* \rightarrow \Pi_B^* \rightarrow 1\) is the exact sequence “1 \(\rightarrow \Pi_F \rightarrow \Pi_T \rightarrow \Pi_B \rightarrow 1\)” associated, with respect to the same \((\Sigma_F, \Sigma_B)\), to another stable log curve of type \((g,r)\) over \((\text{Spec} k)_{\text{log}}\), then there exists a commutative diagram of profinite groups

\[
\begin{array}{c}
1 \longrightarrow \Pi_F^* \longrightarrow \Pi_T^* \longrightarrow \Pi_B^* \longrightarrow 1 \\
| \quad | \quad | \\
1 \longrightarrow \Pi_F \longrightarrow \Pi_T \longrightarrow \Pi_B \longrightarrow 1
\end{array}
\]

— where the vertical arrows are isomorphisms which may be chosen to arise scheme-theoretically.

(ii) The profinite group \(\Pi_B\) is equipped with a natural structure of \textit{pro-\(\Sigma_B\) configuration space group} [cf. [MT], Definition 2.3, (i)]. If, moreover, \(\Sigma_F \subseteq \Sigma_B\) satisfies condition (1) (respectively, (2)), then the profinite group \(\Pi_F\) is equipped with a natural structure of \textit{pro-\(\Sigma_F\) configuration space group} (respectively, \textit{surface group} [cf. [MT], Definition 1.2]).

(iii) The outer representation \(\rho_{n/m} : \Pi_B \rightarrow \text{Out}(\Pi_F)\) factors through the closed subgroup \(\text{Out}^c(\Pi_F) \subseteq \text{Out}(\Pi_F)\) [cf. Definition 6.1; [CmbCsp], Definition 1.1, (ii)].

Proof. Assertion (i) follows immediately by considering a suitable specialization isomorphism [cf. the discussion preceding [CmbCsp], Definition 2.1, as well as Remark 5.6.1 of the present paper]. Assertion (ii) follows immediately from assertion (i), together with the various definitions involved. Assertion (iii) follows immediately from the various definitions involved. This completes the proof of Lemma 6.2. Q.E.D.

Definition 6.3. In the notation of Lemma 6.2 in the case where \((m,n,\Sigma_B) = (1, 2, \mathbb{P}\text{rimes})\), let \(x \in X(k)\) be a \(k\)-valued point of the underlying scheme \(X\) of \(X_{\text{log}}\).

(i) We shall denote by

\[\mathcal{G}\]

the semi-graph of anabelioids of \textit{pro-\(\mathbb{P}\text{rimes}\) PSC-type} determined by the stable log curve \(X_{\text{log}}\); by

\[\mathcal{G}_x\]
the semi-graph of anabelioids of pro-$\Sigma_F$ PSC-type determined by the geometric fiber of $X_2^{\log} \to X^{\log}$ over $x^{\log} \overset{\text{def}}{=} x \times_X X^{\log}$; by $\Pi_{Gx}$, $\Pi_{Gx'}$, the [pro-$\mathfrak{Primes}$, pro-$\Sigma_F$] fundamental groups of $G$, $G_{x'}$, respectively. Thus, we have a natural outer isomorphism
\[
\Pi_B \sim \Pi_{Gx}
\]
and a natural $\text{Im}(\rho_{2/1}) (\subseteq \text{Out}(\Pi_F))$-torsor of outer isomorphisms
\[
\Pi_F \sim \Pi_{Gx}.
\]
Let us fix isomorphisms $\Pi_B \sim \Pi_{Gx}$, $\Pi_F \sim \Pi_{Gx}$ that belong to these collections of isomorphisms.

(ii) Denote by
\[
e^F_{\text{diag},x} \in \text{Cusp}(G_x)
\]
the cusp of $G_x$ [i.e., the cusp of the geometric fiber of $X_2^{\log} \to X^{\log}$ over $x^{\log}$] determined by the diagonal divisor of $X_2^{\log}$. For $v \in \text{Vert}(G)$ (respectively, $e \in \text{Cusp}(G)$) [i.e., which corresponds to an irreducible component (respectively, a cusp) of $X^{\log}$], denote by
\[
e^F_x \in \text{Vert}(G_x) \quad \text{(respectively, } e^F_x \in \text{Cusp}(G_x))
\]
the vertex (respectively, cusp) of $G_x$ that corresponds naturally to $v \in \text{Vert}(G)$ (respectively, $e \in \text{Cusp}(G)$).

(iii) Let $e \in \text{Edge}(G)$, $v \in \text{Vert}(G)$, $S \subseteq \text{VCN}(G)$, and $z \in \text{VCN}(G)$. Then we shall say that $x$ lies on $e$ if the image of $x$ is the cusp or node corresponding to $e \in \text{Edge}(G)$. We shall say that $x$ lies on $v$ if $x$ does not lie on any edge of $G$, and, moreover, the image of $x$ is contained in the irreducible component corresponding to $v \in \text{Vert}(G)$. We shall write $x \cap S$ if $x$ lies on some $s \in S$. We shall write $x \cap \{z\}$.

Lemma 6.4 (Cusps and vertices of fibers). In the notation of Definition 6.3, let $x, x' \in X(k)$ be $k$-valued points of $X$. Then the following hold:

(i) The isomorphism $\Pi_{Gx} \sim \Pi_{G_{x'}}$ obtained by forming the composite of the isomorphisms $\Pi_{Gx} \sim \Pi_F \sim \Pi_{G_{x'}}$ [cf. Definition 6.3,
(i) is group-theoretically cuspidal [cf. [CmbGC], Definition 1.4, (iv)].

(ii) The injection $\text{Cusp}(G) \hookrightarrow \text{Cusp}(G_x)$ given by mapping $c \mapsto c_x^F$ determines a bijection

$$\text{Cusp}(G) \simto \text{Cusp}(G_x) \setminus \{c_x^F\}$$

[cf. Definition 6.3, (ii)]. Moreover, if we regard $\text{Cusp}(G)$ as a subset of each of the sets $\text{Cusp}(G_x)$, $\text{Cusp}(G_{x'})$ by means of the above injections, then the bijection $\text{Cusp}(G_x) \simto \text{Cusp}(G_{x'})$ determined by the group-theoretically cuspidal isomorphism $\Pi_{G_x} \simto \Pi_{G_{x'}}$ of (i) maps $c_{\text{diag},x} \mapsto c_{\text{diag},x'}$ and induces the identity automorphism on $\text{Cusp}(G)$. Thus, in the remainder of the present $\xi$, we shall omit the subscript “$x$” from the notation $c_x^F$ and $c_{\text{diag},x}^F$.

(iii) The injection $\text{Vert}(G) \hookrightarrow \text{Vert}(G_x)$ given by mapping $v \mapsto v_x^F$ [cf. Definition 6.3, (ii)] is bijective if and only if $x \rtarrow \text{Vert}(G)$ [cf. Definition 6.3, (iii)]. If $x \rtarrow \text{Edge}(G)$, then the complement of the image of $\text{Vert}(G)$ in $\text{Vert}(G_x)$ is of cardinality one; in this case, we shall write

$$v_{\text{new},x}^F \in \text{Vert}(G_x) \setminus \text{Vert}(G)$$

for the unique element of $\text{Vert}(G_x) \setminus \text{Vert}(G)$.

(iv) Suppose that $x \rtarrow \text{Cusp}(G)$ (respectively, $\text{Node}(G)$). Then it holds that $c_{\text{diag}}^F \in \mathcal{C}(v_{\text{new},x}^F)$ [cf. (iii)], and $(\mathcal{C}(v_{\text{new},x}^F)^2, \mathcal{N}(v_{\text{new},x}^F)^2) = (2,1)$ (respectively, $= (1,2)$). Moreover, for any element $e^F \in \mathcal{N}(v_{\text{new},x}^F)$, it holds that $\mathcal{V}(e^F)^2 = 2$.

Proof. These assertions follow immediately from the various definitions involved. Q.E.D.

**Definition 6.5.** In the notation of Definition 6.3:

(i) Write

$$\text{Cusp}^F(G) \overset{\text{def}}{=} \text{Cusp}(G) \cup \{c_{\text{diag}}^F\}$$

[cf. Definition 6.3, (ii); Lemma 6.4, (ii)].
(ii) Let \( \alpha \in \text{Out}^C(\Pi_F) \) be an \( C\)-admissible automorphism of \( \Pi_F \) [cf. Definition 6.1; Lemma 6.2, (ii)]. Then it follows immediately from Lemma 6.4, (ii), that for any \( k \)-valued point \( x \in X(k) \) of \( X \), the automorphism of \( \text{Cusp}^F(\mathcal{G}) \) [cf. (i)] obtained by conjugating the natural action of \( \alpha \) on \( \text{Cusp}(\mathcal{G}_x) \) by the natural bijection \( \text{Cusp}^F(\mathcal{G}) \cong \text{Cusp}(\mathcal{G}_x) \) implicit in Lemma 6.4, (ii), does not depend on the choice of \( x \). We shall refer to this automorphism of \( \text{Cusp}^F(\mathcal{G}) \) as the automorphism of \( \text{Cusp}^F(\mathcal{G}) \) determined by \( \alpha \). Thus, we have a natural homomorphism \( \text{Out}^C(\Pi_F) \to \text{Aut}(\text{Cusp}^F(\mathcal{G})) \).

(iii) For \( c \in \text{Cusp}^F(\mathcal{G}) \) [cf. (i)], we shall refer to a closed subgroup of \( F \) obtained as the image \( \cong \) \( \Pi_F \) [cf. Definition 6.3, (i)] for some \( k \)-valued point \( x \in X(k) \) — of a cuspidal subgroup of \( \Pi_{G_x} \) associated to the cuspid of \( \mathcal{G}_x \) corresponding to \( c \in \text{Cusp}^F(\mathcal{G}) \) as a cuspidal subgroup of \( \Pi_F \) associated to \( c \in \text{Cusp}^F(\mathcal{G}) \). Note that it follows immediately from Lemma 6.4, (ii), that the \( F \)-conjugacy class of a cuspidal subgroup of \( F \) associated to \( c \in \text{Cusp}^F(\mathcal{G}) \) depends only on \( c \in \text{Cusp}^F(\mathcal{G}) \), i.e., does not depend on the choice of \( x \) or on the choices of isomorphisms made in Definition 6.3, (i).

Lemma 6.6 (Images of VCN-subgroups of fibers). In the notation of Definition 6.3, let \( \Pi_{x_{\text{diag}}} \subseteq \Pi_F \) be a cuspidal subgroup of \( \Pi_F \) associated to \( c_{\text{diag}}^F \in \text{Cusp}^F(\mathcal{G}) \) [cf. Definition 6.5, (i), (iii)], \( x \in X(k) \) a \( k \)-valued point of \( X \), \( z^F \in \text{VCN}(\mathcal{G}_x) \backslash \{c_{\text{diag}}^F\} \), and \( \Pi_{z^F} \subseteq \Pi_{G_x} \) a VCN-subgroup of \( \Pi_{G_x} \) associated to \( z^F \). Write \( N_{\text{diag}} \subseteq \Pi_F \) for the normal closed subgroup of \( \Pi_F \) topologically normally generated by \( \Pi_{x_{\text{diag}}} \) [Note that it follows immediately from Lemma 6.4, (i), that \( N_{\text{diag}} \) is normal in \( \Pi_T \).] Then the following hold:

(i) Write \( \mathcal{G}^{z^F} \) for the semi-graph of anabelioids of pro-\( \Sigma_F \) PSC-type obtained by forming the pro-\( \Sigma_F \) completion of \( \mathcal{G} \) [cf. [SemiAn], Definition 2.9, (ii)]. Then there exists a natural outer isomorphism \( \Pi_F/N_{\text{diag}} \cong \Pi_{G^{z^F}} \) that satisfies the following conditions:

- Suppose that \( x \sim \text{Vert}(\mathcal{G}) \) [cf. Definition 6.3, (iii)]. Then the \( \Pi_{G^{z^F}} \)-conjugacy class of the image of the composite

\[
\Pi_{z^F} \hookrightarrow \Pi_{G_x} \cong \Pi_F \rightarrow \Pi_F/N_{\text{diag}} \cong \Pi_{G^{z^F}}
\]
coincides with the $\Pi_{G_z^\Sigma_g}$-conjugacy class of any VCN-subgroup of $\Pi_{G_z^\Sigma_g}$ associated to the element of $\text{VCN}(G_z^\Sigma_g) = \text{VCN}(G_z)$ naturally determined by $z^F$.

- Suppose that $x \curvearrowright e \in \text{Edge}(G)$, and that $z^F \notin \{v_{\text{new},x}^F\} \cup E(v_{\text{new},x}^F)$ (respectively, $z^F \in \{v_{\text{new},x}^F\} \cup E(v_{\text{new},x}^F)$) [cf. Lemma 6.4, (iii)]. Then the $\Pi_{G_z^\Sigma_g}$-conjugacy class of the image of the composite

\[ \Pi_{z^F} \hookrightarrow \Pi_{G_z^\Sigma_g} \xrightarrow{\sim} \Pi_F \twoheadrightarrow \Pi_F/\Pi_{N_{\text{diag}}} \xrightarrow{\sim} \Pi_{G_z^\Sigma_g} \]

coincides with the $\Pi_{G_z^\Sigma_g}$-conjugacy class of any VCN-subgroup of $\Pi_{G_z^\Sigma_g}$ associated to the element of $\text{VCN}(G_z^\Sigma_g) = \text{VCN}(G_z)$ natural determined by $z^F$ (respectively, associated to $e \in \text{Edge}(G_z^\Sigma_g) = \text{Edge}(G_z)$).

(ii) The image of the composite

\[ \Pi_{z^F} \hookrightarrow \Pi_{G_z^\Sigma_g} \xrightarrow{\sim} \Pi_F \twoheadrightarrow \Pi_F/\Pi_{N_{\text{diag}}} \]

is commensurably terminal.

(iii) Suppose that either

- $z^F \in \text{Edge}(G_z)$,

or

- $z^F = v_x^F$ for $v \in \text{Vert}(G)$ such that $x$ does not lie on $v$.

Then the composite

\[ \Pi_{z^F} \hookrightarrow \Pi_{G_z^\Sigma_g} \xrightarrow{\sim} \Pi_F \twoheadrightarrow \Pi_F/\Pi_{N_{\text{diag}}} \]

is injective.

(iv) Let $\Pi_{(z')^F} \subseteq \Pi_{G_z^\Sigma_g}$ be a VCN-subgroup of $\Pi_{G_z^\Sigma_g}$ associated to an element $(z')^F \in \text{VCN}(G_z) \setminus \{e_{\text{diag}}^F\}$. Suppose that either

- $x \curvearrowright \text{Vert}(G)$,

or

- $x \curvearrowright \text{Edge}(G)$, and $z^F, (z')^F \notin \{v_{\text{new},x}^F\} \cup E(v_{\text{new},x}^F)$.

Then if the $\Pi_F/\Pi_{N_{\text{diag}}}$-conjugacy classes of the images of $\Pi_{z^F}$, $\Pi_{(z')^F} \subseteq \Pi_{G_z^\Sigma_g}$ via the composite

\[ \Pi_{G_z^\Sigma_g} \xrightarrow{\sim} \Pi_F \twoheadrightarrow \Pi_F/\Pi_{N_{\text{diag}}} \]
coincide, then \( z^F = (z')^F \).

**Proof.** Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from [CmbGC], Proposition 1.2, (ii), and assertion (i), together with the various definitions involved. Assertion (iii) follows immediately from assertion (i), together with the various definitions involved. Assertion (iv) follows immediately from [CmbGC], Proposition 1.2, (i), and assertion (i), together with the various definitions involved. Q.E.D.

**Lemma 6.7 (Outomorphisms preserving the diagonal).** In the notation of Definition 6.3, let \( H \subseteq \Pi_B \) be an open subgroup of \( \Pi_B \), \( \tilde{\alpha} \) an automorphism of \( \Pi_H^{[H]} \) defined \( \Pi_T \times_H H \) over \( H \), \( \alpha_F \in \text{Out}(\Pi_F) \) the automorphism of \( \Pi_F \) determined by the restriction \( \tilde{\alpha}|_{\Pi_F} \) of \( \tilde{\alpha} \) to \( \Pi_F \subseteq \Pi_T|_H \), and \( \Pi_{\text{diag}} \subseteq \Pi_F \) a cuspidal subgroup of \( \Pi_F \) associated to \( \text{Cusp}^F(\mathcal{G}) \) [cf. Definition 6.5, (i), (iii)]. Then the following hold:

(i) Suppose that \( e \) preserves \( \Pi_{\text{diag}}^x \subseteq \Pi_F \). Then the automorphism of \( \Pi_{\text{diag}}^x/\text{N}_{\text{diag}} \) [where we refer to the statement of Lemma 6.6 concerning \( \text{N}_{\text{diag}} \)] induced by \( \tilde{\alpha} \) is the identity automorphism. If, moreover, \( \alpha_F \) is C-admissible [cf. Definition 6.1; Lemma 6.2, (ii)], then the automorphism of \( \text{Cusp}^F(\mathcal{G}) \) induced by \( \alpha_F \) [cf. Definition 6.5, (ii)] is the identity automorphism.

(ii) Let \( e \in \text{Edge}(\mathcal{G}), x \in X(k) \) be such that \( x \triangleright e \). Suppose that \( \alpha_F \) is C-admissible, and that \( \text{Edge}(\mathcal{G}) = \{e\} \cup \text{Cusp}(\mathcal{G}) \). Then it holds that \( \alpha_F \in \text{Aut}(G_x) \subseteq \text{Out}(\Pi_G) \subseteq \text{Out}(\Pi_F) \).

If, moreover, \( \tilde{\alpha} \) preserves \( \Pi_{\text{diag}}^x \subseteq \Pi_F \), then \( \alpha_F \in \text{Aut}^{[\text{graph}]}(G_x) \subseteq \text{Aut}(G_x) \).

**Proof.** First, we verify assertion (i). Now let us observe that it follows immediately from a similar argument to the argument used in the proof of [CmbCsp], Proposition 1.2, (iii) — i.e., by considering the action of \( \tilde{\alpha} \) on the decomposition subgroup \( D \subseteq \Pi_T|_H \) of \( \Pi_T|_H \) associated to the diagonal divisor of \( X_2^\log \) such that \( \Pi_{\text{diag}}^x \subseteq D \), and applying the fact that \( D = \text{N}_{\Pi_T|_H}(\Pi_{\text{diag}}^x) \subseteq \Pi_T|_H \) — that \( \tilde{\alpha} \) induces the identity automorphism on some normal open subgroup \( J \subseteq \Pi_F/\text{N}_{\text{diag}} \) of \( \Pi_F/\text{N}_{\text{diag}} \). Thus, it follows immediately from the slimness [cf. [CmbGC], Remark
1.1.3] of $\Pi_{G^e} \sim \Pi_F/N_{\text{diag}} \hookrightarrow \text{Aut}(J)$ that $\tilde{\alpha}$ induces the identity automorphism on $\Pi_F/N_{\text{diag}}$. This completes the proof of the fact that $\tilde{\alpha}$ induces the identity automorphism of $\Pi_F/N_{\text{diag}}$. On the other hand, if, moreover, $\alpha_F$ is $C$-admissible, then since $\tilde{\alpha}$ induces the identity automorphism of $\Pi_F/N_{\text{diag}}$, it follows immediately from [CmbGC], Proposition 1.2, (i), applied to the cuspidal inertia subgroups of $\Pi_{G^e} \sim \Pi_G$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $\Pi_e \subseteq \Pi_G \sim \Pi_B$ be an edge-like subgroup associated to the edge $e \in \text{Edge}(G)$. By abuse of notation, we shall write $H \cap \Pi_e \subseteq \Pi_B$ for the intersection of $H$ with the image of $\Pi_e$ in $\Pi_B$. Now since $\alpha_F$ is $C$-admissible, and $\tilde{\alpha}$ is an automorphism of $\Pi_{G^e}\big|_H$ over $H$, it holds that $\alpha_F \in Z_{\text{Out}^C(\Pi_F)}(\rho_{2/1}(H))$ [cf. the discussion entitled “Topological groups” in §0], hence also that $\alpha_F \in Z_{\text{Out}^C(\Pi_F)}(\rho_{2/1}(H \cap \Pi_e))$. On the other hand, in light of the well-known structure of $X^{\log}$ in a neighborhood of the cusp or node corresponding to $e$, one verifies easily — by applying [NodNon], Proposition 2.14, together with our assumption that $\text{Edge}(G) = \{e\} \cup \text{Cusp}(G)$ — that the image of the composite

$$\Pi_e \hookrightarrow \Pi_G \sim \Pi_B \xrightarrow{\rho_{2/1}} \text{Out}(\Pi_F) \sim \text{Out}(\Pi_{G^e}),$$

hence also the image $\rho_{2/1}(H \cap \Pi_e) \subseteq \text{Out}(\Pi_F) \sim \text{Out}(\Pi_{G^e})$, is $NN$-ample [cf. Definition 6.3; Theorem 5.9, (ii)]. Thus, since $v_{\text{diag}}^{F} \in \text{Cusp}(G_x) \neq \emptyset$, it follows immediately from Theorem 5.14, (i), that $\alpha_F \in \text{Aut}(G_x)$. This completes the proof of the fact that $\alpha_F \in \text{Aut}(G_x)$. Now suppose, moreover, that $\tilde{\alpha}$ preserves $\Pi_{G^e} \subseteq \Pi_F$. Then it follows from assertion (i) that $\alpha_F$ fixes the cusps of $G_x$, hence that it fixes $v_{\text{new}}^{G_x}$. On the other hand, since $\tilde{\alpha}$ induces the identity automorphism of $\Pi_F/N_{\text{diag}}$ [cf. assertion (i)], it follows from Lemma 6.6, (iii), (iv), that $\alpha_F$ fixes the vertices of $G_x$ that are $\neq v_{\text{new}}^{G_x}$, as well as [cf. [CmbGC], Proposition 1.2, (i)] the branches of nodes of $G_x$ that abut to such vertices. Thus, $\alpha_F \in \text{Aut}^{|\text{graph}|}(G_x)$, as desired. This completes the proof of assertion (ii).

Q.E.D.

Lemma 6.8 (Triviality of certain automorphisms). In the notation of Definition 6.3, let $\Pi_{G^e} \subseteq \Pi_F$ be a cuspidal subgroup of $\Pi_F$ associated to $v_{\text{diag}}^{F} \in \text{Cusp}(G)$ [cf. Definition 6.5, (i), (iii)], $H \subseteq \Pi_B$ an open subgroup of $\Pi_B$, and $\alpha \in Z_{\text{Out}^C(\Pi_F)}(\rho_{2/1}(H))$ [cf. Definition 6.1;
Lemma 6.2, (ii)]. Suppose that \( \alpha \) preserves the \( \Pi_F \)-conjugacy class of \( \Pi_{F, \text{diag}} \subseteq \Pi_F \). Then \( \alpha \) is the identity automorphism.

Proof. The following argument is essentially the same as the argument applied in [CmbCsp], [NodNon] to prove [CmbCsp], Corollary 2.3, (ii); [NodNon], Corollary 5.3.

Let \( \Pi_T|_H \equiv \Pi_T \times_{\Pi_F} H \) and \( \tilde{\alpha} \in \text{Aut}_H(\Pi_T|_H) \) a lifting of \( \alpha \in Z_{\text{Out}}(\Pi_T)(\rho_2/1(H)) \subseteq Z_{\text{Out}}(\Pi_F)(\rho_2/1(H)) \overset{\sim}{\to} \text{Aut}_H(\Pi_T|_H)/\text{Inn}(\Pi_F) \) [cf. the discussion entitled “Topological groups” in §6]. Since we have assumed that \( \alpha \) preserves the \( \Pi_F \)-conjugacy class of \( \Pi_{F, \text{diag}} \subseteq \Pi_F \), it follows from Lemma 6.7, (i), (ii), that by replacing \( \tilde{\alpha} \) by a suitable \( \Pi_F \)-conjugate of \( \tilde{\alpha} \), we may assume without loss of generality that \( \tilde{\alpha} \) preserves \( \Pi_{F, \text{diag}} \subseteq \Pi_F \), and, moreover, that

(a) the automorphism of \( \Pi_F/N_{\text{diag}} \) induced by \( \tilde{\alpha} \) is the identity automorphism;

(b) for \( e \in \text{Edge}(\mathcal{G}) \), \( x \in X(k) \) such that \( x \sim e \), if \( \text{Edge}(\mathcal{G}) = \{ e \} \) \( \cup \) \( \text{Cusp}(\mathcal{G}) \), then \( \alpha \in \text{Aut}^{\text{graph}}(\mathcal{G}_x) \) \( \subseteq \text{Out}(\Pi_{G_x}) \overset{\sim}{\to} \text{Out}(\Pi_F) \).

Next, we claim that

\((\ast)_1): \) if \((g, r) = (0, 3)\), then \( \alpha \) is the identity automorphism.

Indeed, write \( c_1, c_2, c_3 \in \text{Cusp}(\mathcal{G}) \) for the three distinct cusps of \( \mathcal{G} \); \( v \in \text{Vert}(\mathcal{G}) \) for the unique vertex of \( \mathcal{G} \). For \( i \in \{1, 2, 3\} \), let \( x_i \in X(k) \) be such that \( x_i \sim c_i \). Next, let us observe that since our assumption that \((g, r) = (0, 3)\) implies that \( \text{Node}(\mathcal{G}) = \emptyset \), it follows immediately from (b) that for \( i \in \{1, 2, 3\} \), the automorphism \( \alpha \) of \( \Pi_{G_{x_i}} \overset{\sim}{\to} \Pi_F \) is \( \in \text{Aut}^{\text{graph}}(\mathcal{G}_{x_i}) \) \( \subseteq \text{Out}(\Pi_{G_{x_i}}) \overset{\sim}{\to} \text{Out}(\Pi_F) \). Next, let us fix a vertical subgroup \( \Pi_{F, x_2} \subseteq \Pi_{G_{x_2}} \overset{\sim}{\to} \Pi_F \) associated to \( v_{x_2} \in \text{Vert}(\mathcal{G}_{x_2}) \) [cf. Definition 6.3, (ii)]. Then since \( \alpha \in \text{Aut}^{\text{graph}}(\mathcal{G}_{x_2}) \), it follows immediately from the commensurable terminality of the image of the composite \( \Pi_{F, x_2} \overset{\sim}{\to} \Pi_{G_{x_2}} \overset{\sim}{\to} \Pi_F \) that there exists an \( N_{\text{diag}} \)-conjugate \( \tilde{\beta} \) of \( \tilde{\alpha} \) such that \( \tilde{\beta}(\Pi_{x_2}) = \Pi_{x_2} \). Thus, since the composite \( \Pi_{F, x_2} \overset{\sim}{\to} \Pi_{G_{x_2}} \overset{\sim}{\to} \Pi_F \) is injective [cf. Lemma 6.6, (ii)], together with (a), that \( \beta \) induces the identity automorphism on \( \Pi_{x_2} \) \( \subseteq \Pi_{G_{x_2}} \overset{\sim}{\to} \Pi_F \). Next, let \( \Pi_F \subseteq \Pi_F \) be a cuspidal subgroup of \( \Pi_F \) associated to \( c_1 \in \text{Cusp}(\mathcal{G}) \) [cf. Definition 6.5, (iii)] which is contained in \( \Pi_{F, x_2} \subseteq \Pi_{G_{x_2}} \overset{\sim}{\to} \Pi_F ; \Pi_{F, x_3} \subseteq \Pi_{G_{x_3}} \overset{\sim}{\to} \Pi_F \) a
vertical subgroup associated to \( v_{x_3}^F \in \text{Vert}(G_{x_3}) \) that contains \( \Pi_{x_3}^F \subseteq \Pi_F \). Then since \( \tilde{\beta} \) induces the identity automorphism on \( \Pi_{x_3}^F \subseteq \Pi_{x_3}^F \sim \), it follows from the inclusion \( \Pi_{x_3}^F \subseteq \Pi_{x_3}^F \) that \( \tilde{\beta}(\Pi_{x_3}^F) = \Pi_{x_3}^F \). Thus, since the vertical subgroup \( \Pi_{x_3}^F \subseteq \Pi_{x_3}^F \) is the unique vertical subgroup of \( \Pi_{x_3}^F \) associated to \( v_{x_3}^F \in \text{Vert}(G_{x_3}) \) which contains \( \Pi_{x_3}^F \) [cf. Lemma 6.4, (iv)], it follows immediately from the fact that \( \alpha \in \text{Aut}[\text{vert}](G_{x_3}) \) that \( \tilde{\beta}(\Pi_{x_3}^F) = \Pi_{x_3}^F \). In particular, since the composite \( \Pi_{x_3}^F \sim \Pi_F \to \Pi_F/\text{N}_{\text{dias}} \) is injective [cf. Lemma 6.6, (iii)], it follows immediately from (a) that \( \tilde{\beta} \) induces the identity automorphism on \( \Pi_{x_3}^F \subseteq \Pi_{x_3}^F \sim \Pi_F \). On the other hand, since \( \Pi_F \) is topologically generated by \( \Pi_{x_3}^F \subseteq \Pi_{G_{x_3}} \sim \Pi_F \) and \( \Pi_{x_3}^F \subseteq \Pi_{G_{x_3}} \sim \Pi_F \) [cf. [CmbCsp], Lemma 1.13], this implies that \( \tilde{\beta} \) induces the identity automorphism on \( \Pi_F \). This completes the proof of the claim \((\ast_1)\).

Next, we claim that

\((\ast_2)\): for arbitrary \((g, r)\), \(\alpha\) is the identity automorphism.

Indeed, we verify the claim \((\ast_2)\) by induction on \(3g - 3 + r\). If \(3g - 3 + r = 0\), i.e., \((g, r) = (0, 3)\), then the claim \((\ast_2)\) amounts to the claim \((\ast_1)\). Now suppose that \(3g - 3 + r > 1\), and that the induction hypothesis is in force. Since \(3g - 3 + r > 1\), one verifies easily that there exists a stable log curve \(Y^{\log}\) of type \((g, r)\) over \((\text{Spec} k)^{\log}\) such that \(Y^{\log}\) has precisely one node. Thus, it follows immediately from Lemma 6.2, (i), that to verify the claim \((\ast_2)\), by replacing \(X^{\log}\) by \(Y^{\log}\), we may assume without loss of generality that \(\text{Node}(G)^2 = 1\). Let \(e\) be the unique node of \(G\) and \(x \in X(k)\) such that \(x \sim e\). Now let us observe that since \(\text{Node}(G)^2 = 1\), and \(e \in \text{Node}(G)\), it follows from (b) that \(\alpha \in \text{Aut}[\text{vert}](G_x) (\subseteq \text{Out}(\Pi_{G_{x}}) \sim \text{Out}(\Pi_F))\). Write \(\{e_1^F, e_1^F\} = N(v_{\text{new}, x})\) [cf. Lemma 6.4, (iv)]. Also, for \(i \in \{1, 2\}\), denote by \(v_i \in \text{Vert}(G)\) the vertex of \(G\) such that \((v_i)_{x_3}^F \in \text{Vert}(G_{x_3})\) is the unique element of \(\mathcal{V}(e_i^F) \setminus \{v_{\text{new}, x}\}\) [cf. Lemma 6.4, (iv)]; by \(\mathbb{H}_i\) the sub-semi-graph of PSC-type of the underlying semi-graph \(G_{x_3}\) of \(G_x\) whose set of vertices \(= \{v_{\text{new}, x}, (v_i)_{x_3}^F\}\); and by \(S_i \overset{\text{def}}{=} \text{Node}(G_x)_{[\mathbb{H}_i]} \setminus \{e_i^F\} \subseteq \text{Node}(G_x)_{[\mathbb{H}_i]}\) the complement of \(\{e_i^F\}\). Thus, if \(G\) is noncyclically primitive (respectively, cyclically primitive) [cf. Definition 4.1], then \(\mathbb{H}_i \neq G_x\) and \(S_i = \emptyset\) (respectively, \(\mathbb{H}_i = G_x\) and \(S_i = \{e_{3-i}^F\}\)). In particular, \(S_i \subseteq \text{Node}(G_x)_{[\mathbb{H}_i]}\) is not of separating type.

Next, let us observe that to complete the proof of the above claim \((\ast_2)\), it suffices to verify that
\(\alpha \in \text{Dehn}(G_x)\), and, moreover, for \(i \in \{1, 2\}\), \(\alpha\) is contained in the kernel of the natural surjection \(\text{Dehn}(G_x) \twoheadrightarrow \text{Dehn}((G_x)_{\|_{i} \rightarrow S_i})\) [cf. Theorem 4.8, (iii), (iv)].

Indeed, since \([\text{as is easily verified}]\) \(\text{Node}(G_x) = N(v_{\text{new}, x}^F) = \{ e_1^F, e_2^F \}\), it follows immediately from Theorem 4.8, (iii), (iv), that

\[
\bigcap_{i=1}^{2} \text{Ker} \left( \text{Dehn}(G_x) \twoheadrightarrow \text{Dehn}((G_x)_{\|_{i} \rightarrow S_i}) \right) = \{ 1 \}.
\]

In particular, the implication \(\left(\dagger\right) \Rightarrow (\ast_2)\) holds. The remainder of the proof of the claim \((\ast_2)\) is devoted to verifying the above \(\left(\dagger\right)\).

For \(i \in \{1, 2\}\), let \(\Pi_{(v_i)^F_x} \subseteq \Pi_{G_x} \subseteq \Pi_F\) be a vertical subgroup of \(\Pi_{G_x} \xrightarrow{\sim} \Pi_F\) associated to the vertex \((v_i)^F_x \in V(e_i^F) \setminus \{ v_{\text{new}, x}^F \}\). Then since \(\alpha \in \text{Aut}^{[\text{grph}]}(G_x)\), it follows that \(\tilde{\alpha}\) preserves the \(\Pi_F\)-conjugacy class of \(\Pi_{(v_i)^F_x} \subseteq \Pi_{G_x} \subseteq \Pi_F\). Thus, since the image of the composite \(\Pi_{(v_i)^F_x} \hookrightarrow \Pi_F \rightarrow \Pi_F/\text{N}_{\text{diag}}\) is commensurably terminal [cf. Lemma 6.6, (ii)], it follows immediately from (a) that there exists an \(\text{N}_{\text{diag}}\)-conjugate \(\tilde{\beta}_i\) [which may depend on \(i \in \{1, 2\}\)] of \(\tilde{\alpha}\) such that \(\tilde{\beta}_i(\Pi_{(v_i)^F_x}) = \Pi_{(v_i)^F_x}\). Therefore, since the composite \(\Pi_{(v_i)^F_x} \hookrightarrow \Pi_F \rightarrow \Pi_F/\text{N}_{\text{diag}}\) is injective [cf. Lemma 6.6, (iii)], it follows from (a) that \(\tilde{\beta}_i\) induces the identity automorphism of \(\Pi_{(v_i)^F_x}\).

Next, let \(\Pi_{e_i^F} \subseteq \Pi_{(v_i)^F_x}\) be a nodal subgroup of \(\Pi_{G_x} \xrightarrow{\sim} \Pi_F\) associated to \(e_i^F \in \text{Node}(G_x)\) that is contained in \(\Pi_{(v_i)^F_x}\); \(\Pi_{e_{\text{new}, x}, i} \subseteq \Pi_{G_x} \xrightarrow{\sim} \Pi_F\) a vertical subgroup [which may depend on \(i \in \{1, 2\}\)] associated to \(v_{\text{new}, x}^F \in \text{Vert}(G_x)\) which contains \(\Pi_{e_i^F}\):

\[
\Pi_{e_{\text{new}, x}, i} \supseteq \Pi_{e_i^F} \subseteq \Pi_{(v_i)^F_x} \subseteq \Pi_{G_x} \xrightarrow{\sim} \Pi_F.
\]

Then since \(\tilde{\beta}_i\) preserves and induces the identity automorphism on \(\Pi_{(v_i)^F_x}\), it follows from the inclusion \(\Pi_{e_i^F} \subseteq \Pi_{(v_i)^F_x}\) that \(\tilde{\beta}_i(\Pi_{e_i^F}) = \Pi_{e_i^F}\). Moreover, since \(\Pi_{e_{\text{new}, x}, i}\) is the unique vertical subgroup of \(\Pi_{G_x} \xrightarrow{\sim} \Pi_F\) associated to \(v_{\text{new}, x}^F\) which contains \(\Pi_{e_i^F}\) [cf. [CmbGC], Proposition 1.5, (i)], it follows immediately from the fact that \(\alpha \in \text{Aut}^{[\text{grph}]}(G_x)\) that \(\beta_i(\Pi_{e_{\text{new}, x}, i}) = \Pi_{e_{\text{new}, x}, i}\). Thus, \(\tilde{\beta}_i\) preserves the closed subgroup \(\Pi_{F_i} \subseteq \Pi_F\) of \(\Pi_F\) obtained by forming the image of the natural homomorphism

\[
\lim_{\rightarrow} \left( \Pi_{e_{\text{new}, x}, i} \hookrightarrow \Pi_{e_i^F} \hookrightarrow \Pi_{(v_i)^F_x} \twoheadrightarrow \Pi_F \right)
\]
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where the inductive limit is taken in the category of pro-$\Sigma_{F}$ groups. Now one may verify easily that the $F$-conjugacy class of $F_{i}$ coincides with the $F$-conjugacy class of the image of the natural outer injection $\Pi_{((G_{x})_{[v]})_{[v]}} \rightarrow \Pi_{G_{x}} \sim \Pi_{F}$ discussed in Proposition 2.11; in particular, $\Pi_{F_{i}}$ is commensurably terminal in $\Pi_{F}$ [cf. Proposition 2.11].

Moreover, by applying a similar argument to the argument used in [CmbCsp], Definition 2.1, (iii), (vi), or [NodNon], Definition 5.1, (ix), (x) [i.e., by considering the portion of the underlying scheme $X_{2}$ of $X_{2}^{\log}$ corresponding to the underlying scheme $(X_{v})_{2}$ of the 2nd log configuration space $(X_{v})_{2}$ determined by $G_{x}$], one concludes that there exists a vertical subgroup $\Pi_{v} \subseteq \Pi_{G} \sim \Pi_{B}$ associated to $v_{i} \in \text{Vert}(G)$ such that the outer representation of $\Pi_{v}$ on $F_{i}$ determined by the composite $v_{i} \circ \text{Out}(F_{i})$ preserves the $F_{i}$-conjugacy class of $F_{i}$ [so we obtain a natural outer representation $v_{i} \text{Out}(F_{i}) \mid \text{cf. Lemma 2.12, (iii)}$, and, moreover, that if we write $T_{i} \overset{\text{def}}{=} F_{i} \langle V_{v} \rangle \otimes \Pi_{v} (\langle V_{v} \rangle)$ [cf. the discussion entitled “Topological groups” in §6], then $T_{i}$ is naturally isomorphic to the $T_{i}$ obtained by taking “$G$” to be $G_{v}$.

Now since $\tilde{\beta}_{v}(\Pi_{F_{i}}) = \Pi_{F_{i}}$, and $\alpha \in Z_{\text{Out}^{c}(\Pi_{F_{i}})}(\rho_{2/1}(H))$, one may verify easily that the automorphism of $\Pi_{F_{i}}$, determined by $\tilde{\beta}_{v}(\Pi_{F_{i}})$ [cf. Lemma 2.12, (iii)] is $\alpha \in Z_{\text{Out}^{c}(\Pi_{F_{i}})}(\rho_{2/1}(H \cap \Pi_{v}))$ — where, by abuse of notation, we write $H \cap \Pi_{v} \subseteq \Pi_{B}$ for the intersection of $H$ with the image of $\Pi_{v}$ in $\Pi_{B}$. Therefore, since the quantity “$3g - 3 + r$” associated to $G_{v}$ is $< 3g - 3 + r$, by considering a similar diagram to the diagram in [CmbCsp], Definition 2.1, (vi), or [NodNon], Definition 5.1, (x), and applying the induction hypothesis, we conclude that $\tilde{\beta}_{v}(\Pi_{F_{i}})$ is a $\Pi_{F_{i}}$-inner automorphism. In particular, it follows immediately [by allowing $i \in \{1, 2\}$ to vary] that the automorphism $\alpha$ is $\in \text{Dehn}(G_{x})$, and, moreover — by considering the natural identification outer isomorphism $\Pi_{F_{i}} \sim \Pi_{((G_{x})_{[v]})_{[v]}}$ — that $\alpha$ is contained in the kernel of the natural surjection $\text{Dehn}(G_{x}) \twoheadrightarrow \text{Dehn}((((G_{x})_{[v]})_{[v]})_{[v]})$, as desired. This completes the proof of (1), hence also of Lemma 6.8. Q.E.D.

**Definition 6.9.** In the notation of Definition 6.3:

(i) Suppose that $2g - 2 + r > 1$, i.e., $(g, r) \notin \{(0, 3), (1, 1)\}$. Then we shall write

$$A_{g, r} \overset{\text{def}}{=} \{1\} \subseteq \text{Aut}(\text{Cusp}^{F}(G))$$
(ii) Suppose that $(g, r) = (1, 1)$. Then we shall write
\[
\left(\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}\right) A_{g,r} \overset{\text{def}}{=} \text{Aut}(\text{Cusp}^F(G)) .
\]

(iii) Suppose that $(g, r) = (0, 3)$. Then we shall write
\[
\left(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}\right) A_{g,r} \subseteq \text{Aut}(\text{Cusp}^F(G))
\]
for the subgroup of $\text{Aut}(\text{Cusp}^F(G))$ obtained as the image of the subgroup of the symmetric group on 4 letters
\[
\{\text{id}, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)\} \subseteq \mathfrak{S}_4
\]
via the isomorphism $\mathfrak{S}_4 \overset{\sim}{\rightarrow} \text{Aut}(\text{Cusp}^F(G))$ arising from a bijection $\{1, 2, 3, 4\} \rightarrow \text{Cusp}^F(G)$. [Note that since the above subgroup of $\mathfrak{S}_4$ is normal, the subgroup $A_{g,r} \subseteq \text{Aut}(\text{Cusp}^F(G))$ does not depend on the choice of the bijection $\{1, 2, 3, 4\} \rightarrow \text{Cusp}^F(G)$.

Lemma 6.10 (Permutations of cusps arising from certain $C$-admissible outomorphisms). In the notation of Definition 6.3, let $H \subseteq \Pi_B$ be an open subgroup of $\Pi_B$. Then the following hold:

(i) The composite
\[
Z_{\text{Out}^C(\Pi_B)}(\rho_{2/1}(H)) \hookrightarrow \text{Out}^C(\Pi_F) \rightarrow \text{Aut}(\text{Cusp}^F(G))
\]
[cf. Definition 6.5, (ii)] factors through the subgroup $A_{g,r} \subseteq \text{Aut}(\text{Cusp}^F(G))$ [cf. Definition 6.9], hence determines a homomorphism
\[
Z_{\text{Out}^C(\Pi_B)}(\text{Im}(\rho_{2/1})) \rightarrow A_{g,r} .
\]

(ii) The composite
\[
\text{Aut}_{X_{\log}^c}(X_{2,\log}^c) \rightarrow Z_{\text{Out}^C(\Pi_B)}(\text{Im}(\rho_{2/1})) \rightarrow A_{g,r}
\]
of the natural homomorphism
\[
\text{Aut}_{X_{\log}^c}(X_{2,\log}^c) \rightarrow Z_{\text{Out}^C(\Pi_B)}(\text{Im}(\rho_{2/1}))
\]
with the homomorphism of (i) is an isomorphism. In particular, the homomorphism \( Z_{\text{Out}}(\Pi_3)(\text{Im}(\rho_2/1)) \rightarrow A_{g,r} \) of (i) is a split surjection [cf. the discussion entitled “Topological groups” in §9].

Proof. First, we verify assertion (i). If \((g,r) = (1,1)\), then since \(A_{g,r} = \text{Aut}(\text{Cusp}^F(\mathcal{G}))\), assertion (i) is immediate. On the other hand, if \(r = 0\), then since \(\text{Cusp}^F(\mathcal{G})^F = 1\), assertion (i) is immediate. Thus, in the remainder of the proof of assertion (i), we suppose that \((g,r) \neq (1,1), r \geq 1\).

Now we verify assertion (i) in the case where \(r = 1\). Let us observe that it follows immediately from Lemma 6.2, (i), that by replacing \(X^\log\) by a suitable stable log curve of type \((g,r)\) over \((\text{Spec} \ k)^{\log}\), we may assume without loss of generality [cf. our assumption that \(r = 1\), which implies that \((g,r) \neq (0,3)\)] that \(\mathcal{G}\) is cyclically primitive [cf. Definition 4.1]. Let \(c \in \text{Cusp}(\mathcal{G})\) be the unique cusp of \(\mathcal{G}\), \(e \in \text{Node}(\mathcal{G})\) the unique node of \(\mathcal{G}\), \(x \in X(k)\) such that \(x \sim c\), and \(\alpha \in Z_{\text{Out}}(\Pi_3)(\rho_2/1(\mathcal{H}))\). Then let us observe that it follows immediately from our assumption that \(\mathcal{G}\) is cyclically primitive of type \((g,r) \neq (1,1)\) (respectively, the various definitions involved) that the vertex of \(\mathcal{G}_x\) to which \(c^\mathcal{F}\) (respectively, \(c^\mathcal{F}_{\text{diag}}\)) abuts is not of type \((0,3)\) (respectively, is of type \((0,3)\)). Moreover, it follows immediately from Lemma 6.7, (ii), that the automorphism \(\alpha\) of \(\Pi_{\mathcal{G}_x} \simeq \Pi_F\) is \(\in \text{Aut}(\mathcal{G}_x)\). Thus, we conclude that the automorphism of \(\text{Cusp}^F(\mathcal{G})\) induced by \(\alpha\) is the identity automorphism. This completes the proof of assertion (i) in the case where \(r = 1\).

Next, we verify assertion (i) in the case where \(r > 1\). Let us observe that it follows immediately from Lemma 6.2, (i), that by replacing \(X^\log\) by a suitable stable log curve of type \((g,r)\) over \((\text{Spec} \ k)^{\log}\), we may assume without loss of generality that \(\text{Node}(\mathcal{G}) = \emptyset\). Let \(v \in \text{Vert}(\mathcal{G})\) be the unique vertex of \(\mathcal{G}\) [cf. our assumption that \(\text{Node}(\mathcal{G}) = \emptyset\)] and \(\alpha \in Z_{\text{Out}}(\Pi_3)(\rho_2/1(\mathcal{H}))\). Now let us observe that for any \(e \in \text{Cusp}(\mathcal{G})\), \(x \in X(k)\) such that \(x \sim e\), it follows immediately from the various definitions involved that \(\text{Vert}(\mathcal{G}_x) = \{v^F_x, v^F_{\text{new},x}\}; C(v^F_{\text{new},x}) = \{c^F, c^F_{\text{diag}}\}; C(v^F_x) = \text{Cusp}(\mathcal{G}_x) \setminus \{c^F, c^F_{\text{diag}}\}\); \(v^F_x\) is of type \((g,r)\); \(v^F_{\text{new},x}\) is of type \((0,3)\). Moreover, it follows immediately from Lemma 6.7, (ii), that the automorphism \(\alpha\) of \(\Pi_{\mathcal{G}_x} \simeq \Pi_F\) is \(\in \text{Aut}(\mathcal{G}_x)\). Thus, if \((g,r) \neq (0,3)\), then since \(v^F_x\) is of type \((g,r)\), and \(v^F_{\text{new},x}\) is of type \((0,3)\), it follows immediately that \(\alpha\) induces the identity automorphism on \(\text{Vert}(\mathcal{G}_x)\), hence that \(\alpha\) preserves the subset \(\{c, c^F_{\text{diag}}\} \subseteq \text{Cusp}^F(\mathcal{G})\) corresponding to \(C(v^F_{\text{new},x}) = \{c^F, c^F_{\text{diag}}\}\). In particular, if \((g,r) \neq (0,3)\), (respectively, \((g,r) = (0,3)\)), then — by allowing “\(c\) to vary among the elements of
Cusp(\mathcal{G}) — one may verify easily that the automorphism of Cusp^F(\mathcal{G}) induced by \alpha is the identity automorphism (respectively, satisfies the condition that

for any subset \( S \in \text{Cusp}^F(\mathcal{G}) \) of cardinality 2, the automorphism of Cusp^F(\mathcal{G}) induced by \alpha determines an automorphism of the set \( \{ S, \text{Cusp}^F(\mathcal{G}) \setminus S \} \subseteq 2^{\text{Cusp}^F(\mathcal{G})} \),

hence, by Lemma 6.11 below, is contained in \( A_{\alpha,r} \subseteq \text{Aut}(\text{Cusp}^F(\mathcal{G})) \). This completes the proof of assertion (i) in the case where \( r > 1 \), hence also of assertion (i).

Next, we verify assertion (ii). One verifies easily that the composite

\[
\text{Aut}_{X^\log}(X_2^{\log}) \to \text{Aut}_{\Pi_T}(\Pi_T)/\text{Inn}(\Pi_F) \xrightarrow{\sim} Z_{\text{Out}(\Pi_F)}(\text{Im}(\rho_{p/2}^1))
\]

[cf. the discussion entitled “Topological groups” in \[\text{[0]}\] factors through \( Z_{\text{Out}^C(\Pi_F)}(\text{Im}(\rho_{p/2}^1)) \subseteq Z_{\text{Out}(\Pi_F)}(\text{Im}(\rho_{p/2}^1)) \). In particular, we obtain a natural homomorphism \( \text{Aut}_{X^\log}(X_2^{\log}) \to Z_{\text{Out}^C(\Pi_F)}(\text{Im}(\rho_{p/2}^1)) \). Now the fact that the composite

\[
\text{Aut}_{X^\log}(X_2^{\log}) \to Z_{\text{Out}^C(\Pi_F)}(\text{Im}(\rho_{p/2}^1)) \hookrightarrow \text{Out}^C(\Pi_F) \to \text{Aut}(\text{Cusp}^F(\mathcal{G}))
\]

determines a surjection \( \text{Aut}_{X^\log}(X_2^{\log}) \to A_{\alpha,r} \) is well-known and easily verified. To verify that this surjection is injective, observe that an element of the kernel of this surjection determines an automorphism of the trivial family \( X^\log \times_{(\text{Spec} \mathcal{O})^\log} X^\log \to X^\log \) over \( X^\log \) that preserves the image of the diagonal. On the other hand, since the relative tangent bundle of this trivial family has no nonzero global sections, one concludes immediately that such an automorphism is constant, i.e., arises from a single automorphism of the fiber \( X^\log \) over \( (\text{Spec} \mathcal{O})^\log \) that is compatible with the diagonal, hence [as is easily verified] equal to the identity automorphism, as desired. This completes the proof of assertion (ii). Q.E.D.

\textbf{Lemma 6.11 (A subgroup of the symmetric group on 4 letters).} Write \( G \subseteq \mathfrak{S}_4 \) for the subgroup of the symmetric group on 4 letters \( \mathfrak{S}_4 \) consisting of \( g \in \mathfrak{S}_4 \) such that

\((\dagger)\): for any subset \( S \subseteq \{ 1, 2, 3, 4 \} \) of cardinality 2, the automorphism \( g \) of \( \{ 1, 2, 3, 4 \} \) determines an automorphism of the set \( \{ S, \{ 1, 2, 3, 4 \} \setminus S \} \subseteq 2^{\{1,2,3,4\}} \),
Then
\[ G = \{ \text{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3) \}. \]

**Proof.** First, let us observe that one may verify easily that
\[ \{ \text{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3) \} \subseteq G. \]

Thus, to verify Lemma 6.11, it suffices to verify that \( G^4 = 4 \). Next, let us observe that it follows immediately from the condition \((*)\) that for any element \( g \in G \), it holds that \( g^4 = \text{id} \); in particular, by the Sylow Theorem, together with the fact that \( \mathfrak{S}_4^2 = 2^3 \cdot 3 \), we conclude that \( G \) is a 2-group. Thus, to verify Lemma 6.11, it suffices to verify that \( G^2 \neq 8 \). Next, let us observe that it follows immediately from the condition \((*)\) that \( G \subseteq \mathfrak{S}_4 \) is normal. Thus, if \( G^2 = 8 \), then since \( \mathfrak{S}_4^2 = 2^3 \cdot 3 \), and \( (1\,2) \in \mathfrak{S}_4 \) is of order 2, again by the Sylow Theorem, we conclude that \( (1\,2) \in G \), in contradiction to the fact that \( (1\,2) \) does not satisfy the condition \((*)\). This completes the proof of Lemma 6.11. Q.E.D.

**Theorem 6.12** (Centralizers of geometric monodromy groups arising from configuration spaces). Let \((g, r)\) be a pair of nonnegative integers such that \( 2g - 2 + r > 0; 0 < m < n \) positive integers; \( \Sigma_F \subseteq \Sigma_B \) nonempty sets of prime numbers; \( k \) an algebraically closed field of characteristic zero; \((\text{Spec } k)^{\log}\) the log scheme obtained by equipping \( \text{Spec } k \) with the log structure given by the fs chart \( \mathbb{N} \rightarrow k \) that maps \( 1 \mapsto 0; X^{\log} = X_1^{\log} \) a **stable log curve** of type \((g, r)\) over \((\text{Spec } k)^{\log}\). Suppose that \( \Sigma_F \subseteq \Sigma_B \) satisfy one of the following two conditions:

1. \( \Sigma_F \) and \( \Sigma_B \) determine **PT-formations** i.e., are either of cardinality one or equal to \( \mathfrak{P}_{\text{rimes}} \) — cf. [MT], Definition 1.1, (ii).
2. \( n - m = 1 \) and \( \Sigma_B = \mathfrak{P}_{\text{rimes}} \).

Write
\[ X_n^{\log}, \ X_m^{\log} \]
for the \( n \)-th, \( m \)-th **log configuration spaces** of the stable log curve \( X^{\log} \) [cf. the discussion entitled “Curves” in §0], respectively; \( \Pi_n, \ \Pi_B^{\text{def}} = \Pi_m \) for the respective maximal pro-\( \Sigma_B \) quotients of the kernels of the natural surjections \( \pi_1(X_n^{\log}) \rightarrow \pi_1((\text{Spec } k)^{\log}), \pi_1(X_m^{\log}) \rightarrow \pi_1((\text{Spec } k)^{\log}); \Pi_{n/m} \subseteq \Pi_n \) for the kernel of the surjection \( \Pi_n \rightarrow \Pi_B = \Pi_m \) induced by the projection \( X_n^{\log} \rightarrow X_m^{\log} \) obtained by forgetting the last \((n - m)\)
factors; $\Pi_F$ for the maximal pro-$\Sigma_F$ quotient of $\Pi_{n/m}$; $\Pi_T$ for the quotient of $\Pi_n$ by the kernel of the natural surjection $\Pi_{n/m} \to \Pi_F$. Thus, we have a natural exact sequence of profinite groups

$$1 \to \Pi_F \to \Pi_T \to \Pi_B \to 1,$$

which determines an outer representation

$$\rho_{n/m} : \Pi_B \to \text{Out}(\Pi_F).$$

Then the following hold:

(i) Let $H \subseteq \Pi_B$ be an open subgroup of $\Pi_B$. Recall that $X_{m+1}^{\log} \to X_m^{\log}$ may be regarded as the $(n - m)$-th log configuration space of the family of stable log curves $X_{m+1}^{\log} \to X_m^{\log}$ over $X_m^{\log}$. Then the composite of natural homomorphisms

$$\text{Aut}_{X_m^{\log}}(X_{m+1}^{\log}) \to \text{Aut}_{X_m^{\log}}(X_n^{\log}) \to \text{Aut}_{\Pi_B}(\Pi_T)/\text{Inn}(\Pi_F)$$

$$\sim \to \mathbb{Z}_{\text{Out}(\Pi_F)}(\text{Im}(\rho_{n/m})) \subseteq \mathbb{Z}_{\text{Out}(\Pi_F)}(\rho_{n/m}(H))$$

— where the first arrow is the homomorphism arising from the functoriality of the construction of the log configuration space; the third arrow is the isomorphism appearing in the discussion entitled “Topological groups” in §7 — determines an isomorphism

$$\text{Aut}_{X_m^{\log}}(X_{m+1}^{\log}) \sim \to \mathbb{Z}_{\text{Out}^{\text{FC}}(\Pi_F)}(\rho_{n/m}(H))$$

— where we write $\text{Out}^{\text{FC}}(\Pi_F)$ for the group of $\text{FC}$-admissible [cf. Definition 6.1; [CmbCsp], Definition 1.1, (ii)] automorphisms of $\Pi_F$ [cf. Lemma 6.2, (ii)]. Here, we recall that the automorphism group $\text{Aut}_{X_m^{\log}}(X_{m+1}^{\log})$ is isomorphic to

$$\begin{cases} 
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } (g, r, m) = (0, 3, 1); \\
\mathbb{Z}/2\mathbb{Z} & \text{if } (g, r, m) = (1, 1, 1); \\
\{1\} & \text{if } (g, r, m) \notin \{(0, 3, 1), (1, 1, 1)\}.
\end{cases}$$

(ii) The isomorphism of (i) and the natural inclusion $\mathcal{S}_{n-m} \hookrightarrow \mathbb{Z}_{\text{Out}^{\text{FC}}(\Pi_F)}(\rho_{n/m}(H))$ — where we write $\text{Out}^{\text{FC}}(\Pi_F)$ for the group of $\text{PFC}$-admissible [cf. Definitions 1.4, (iii); 6.1] automorphisms of $\Pi_F$ [cf. Lemma 6.2, (ii)] — determine an isomorphism

$$\text{Aut}_{X_m^{\log}}(X_{m+1}^{\log}) \times \mathcal{S}_{n-m} \sim \to \mathbb{Z}_{\text{Out}^{\text{FC}}(\Pi_F)}(\rho_{n/m}(H)).$$
(iii) Let $H$ be a closed subgroup of $\text{Out}^{\text{PFC}}(\Pi_F)$ that contains an open subgroup of $\text{Im}(\rho_{n/m}) \subseteq \text{Out}(\Pi_F)$. Then $H$ is almost slim [cf. the discussion entitled “Topological groups” in §0]. If, moreover, $H \subseteq \text{Out}^{\text{PFC}}(\Pi_F)$, and $(g, r, m) \notin \{(0, 3, 1), (1, 1, 1)\}$, then $H$ is slim [cf. the discussion entitled “Topological groups” in §0].

Proof. First, we verify assertion (i). We begin by observing that the description of the automorphism group $\text{Aut}_{\log}^B(X_{m+1}^\log)$ given in the statement of assertion (i) follows immediately from Lemma 6.10, (ii). Next, let us observe that

\begin{equation}
(*)_1: \text{to verify assertion (i), it suffices to verify assertion (i) in the case where } \Sigma_B = \mathcal{P}rimes.
\end{equation}

Indeed, this follows immediately from the various definitions involved. Thus, in the remainder of the proof of assertion (i), we suppose that $\Sigma_B = \mathcal{P}rimes$.

Next, we claim that

\begin{equation}
(*)_2: \text{the composite homomorphism of assertion (i) determines an injection}
\end{equation}

$$
\text{Aut}_{X_{m+1}^\log}(X_{m+1}^\log) \hookrightarrow Z_{\text{Out}}^{\text{rc}}(\Pi_F)(\rho_{n/m}(H)).
$$

Indeed, one verifies easily that the composite as in assertion (i) factors through $Z_{\text{Out}}^{\text{rc}}(\Pi_F)(\rho_{n/m}(H))$. On the other hand, by considering the action of $\text{Aut}_{X_{m+1}^\log}(X_{m+1}^\log)$ on the set of conjugacy classes of cuspidal inertia subgroups of suitable subquotients [arising from fiber subgroups] of $\Pi_F$, it follows immediately that the composite as in assertion (i) is injective [cf. Lemma 6.10, (ii)]. This completes the proof of the claim $(*)_2$.

Next, we claim that

\begin{equation}
(*)_3: \text{the injection of } (*)_2 \text{ is an isomorphism.}
\end{equation}

Indeed, it follows immediately from the various definitions involved that if $N_B \subseteq \Pi_B$ is a fiber subgroup of $\Pi_B$ of length 1 [cf. Lemma 6.2, (ii); [MT], Definition 2.3, (iii)], then the natural surjection $\Pi_T \times_{\Pi_B} N_B \twoheadrightarrow N_B$ may be regarded as the “$\Pi_T \twoheadrightarrow \Pi_B$” obtained by taking “$(g, r, m, n)$” to be $(g, r + m - 1, 1, n - m + 1)$. Thus, by applying the inclusion $Z_{\text{Out}}^{\text{rc}}(\Pi_F)(\rho_{n/m}(H)) \subseteq Z_{\text{Out}}^{\text{rc}}(\Pi_F)(\rho_{n/m}(H \cap N_B))$ and replacing $\Pi_T \twoheadrightarrow \Pi_B$ by $\Pi_T \times_{\Pi_B} N_B \twoheadrightarrow N_B$, we may assume without loss of generality that $m = 1$. On the other hand, it follows immediately from the various definitions involved that if $N_F \subseteq \Pi_F$ is a fiber subgroup of $\Pi_F$ of length $n - 2$, then the natural surjection $\Pi_T/N_F \twoheadrightarrow \Pi_B$ may
be regarded as the “$\Pi_T \to \Pi_B$” obtained by taking “$(g, r, m, n)$” to be $(g, r, 1, 2)$. Thus, since the natural homomorphism $\text{Out}^{FC}(\Pi_F) \to \text{Out}^{FC}(\Pi_F/N_F)$ is injective [cf. [NodNon], Theorem B], by replacing $\Pi_T \to \Pi_B$ by $\Pi_T/N_F \to \Pi_B$, we may assume without loss of generality that $(m, n) = (1, 2)$. In particular — in light of our assumption that $\Sigma_B = \text{Primes}$ [cf. $(*)_1$] — we are in the situation of Definition 6.3.

Let $\alpha \in Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H))$. Then it follows immediately from Lemma 6.10, (ii), that there exists an element $\beta$ of the image of the injection of $(*)_2$ such that $\alpha \circ \beta \in Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H))$ induces the identity automorphism of $C_{\text{cusp}}(\mathcal{G})$ [cf. Definition 6.5, (i), (ii)]. In particular, $\alpha \circ \beta$ preserves the $\Pi_F$-conjugacy class of a cuspidal subgroup $c_{\text{diag}}$ associated to $c_{\text{diag}}$. Thus, it follows from Lemma 6.8 that $\alpha \circ \beta$ is the identity automorphism of $\Pi_F$. In particular, we conclude that the injection of $(*)_2$ is surjective. This completes the proof of the claim $(*)_3$, hence also of assertion (i).

Next, we verify assertion (ii). First, let us observe that by considering the action of $Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H))$ on the set of fiber subgroups of $\Pi_F$ of length 1, we obtain an exact sequence of profinite groups

$$1 \to Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H)) \to Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H)) \to \mathfrak{S}_{n-m}.$$ 

Now by considering the action of $\mathfrak{S}_{n-m}$ on $X_{log}^n$ over $X_{log}^{n+1}$ obtained by permuting the first $n - m$ factors of $X_{log}^{n+1}$, we obtain a section $\mathfrak{S}_{n-m} \hookrightarrow Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H))$ of the third arrow in the above exact sequence; in particular, the third arrow is surjective. On the other hand, it follows from [NodNon], Theorem B, that the image of the section $\mathfrak{S}_{n-m} \hookrightarrow Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H))$ commutes with $Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H))$. Thus, the composite of natural homomorphisms

$$\text{Aut}_{X_{log}^n}(X_{log}^{n+1}) \cong Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H)) \to Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H))$$

[cf. assertion (i)] and the section $\mathfrak{S}_{n-m} \hookrightarrow Z^{Out^{FC}(\Pi_F)}(\rho_{n/m}(H))$ determine an isomorphism as in the statement of assertion (ii). This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertions, (i), (ii). This completes the proof of Theorem 6.12.

Q.E.D.

Remark 6.12.1. By considering a suitable specialization isomorphism, one may replace the expression “$k$ an algebraically closed field of characteristic zero” in the statement of Theorem 6.12 by the expression “$k$ an algebraically closed field of characteristic $\not\in \Sigma_B$”.

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Theorem 6.13 (Centralizers of geometric monodromy groups arising from moduli stacks of pointed curves). Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(\Sigma\) a nonempty set of prime numbers; \(k\) an algebraically closed field of characteristic zero. Write \(\Pi_{M_{g,r}} \overset{\text{def}}{=} \pi_1((M_{g,r})_k)\) for the étale fundamental group of the moduli stack \((M_{g,r})_k\) [cf. the discussion entitled “Curves” in §0]; \(\Pi_{g,r}\) for the maximal pro-\(\Sigma\) quotient of the kernel \(N_{g,r}\) of the natural surjection \(\pi_1((C_{g,r})_k) \twoheadrightarrow \pi_1((M_{g,r})_k) = \Pi_{M_{g,r}}\) [cf. the discussion entitled “Curves” in §0]; \(\Pi_{C_{g,r}}\) for the quotient of the étale fundamental group \(\pi_1((C_{g,r})_k)\) of \((C_{g,r})_k\) by the kernel of the natural surjection \(N_{g,r} \twoheadrightarrow \Pi_{g,r}\). Thus, we have a natural exact sequence of profinite groups

\[
1 \rightarrow \Pi_{g,r} \rightarrow \Pi_{C_{g,r}} \rightarrow \Pi_{M_{g,r}} \rightarrow 1,
\]

which determines an outer representation

\[
\rho_{g,r}: \Pi_{M_{g,r}} \rightarrow \text{Out}(\Pi_{g,r}).
\]

Then the following hold:

(i) The profinite group \(\Pi_{g,r}\) is equipped with a natural structure of pro-\(\Sigma\) surface group [cf. [MT], Definition 1.2].

(ii) Let \(H \subseteq \Pi_{M_{g,r}}\) be an open subgroup of \(\Pi_{M_{g,r}}\). Suppose that

\[
2g - 2 + r > 1, \text{ i.e., } (g, r) \not\in \{(0, 3), (1, 1)\}.
\]

Then the composite of natural homomorphisms

\[
\text{Aut}(\Pi_{M_{g,r}}) \rightarrow \text{Aut}_{\Pi_{M_{g,r}}}((\Pi_{C_{g,r}})/\text{Inn}(\Pi_{g,r}))
\]

\[
\twoheadrightarrow \mathbb{Z}^{\text{Out}(\Pi_{g,r})}(\text{Inn}(\rho_{g,r})) \subseteq \mathbb{Z}^{\text{Out}(\Pi_{g,r})}(\rho_{g,r}(H))
\]

[cf. the discussion entitled “Topological groups” in §0] determines an isomorphism

\[
\text{Aut}_{\Pi_{M_{g,r}}}(((\Pi_{C_{g,r}})_h) \rightarrow \mathbb{Z}^{\text{Out}_{\Pi_{g,r}}}(\rho_{g,r}(H))
\]

[cf. (i); Definition 6.1]. Here, we recall that the automorphism group \(\text{Aut}_{\Pi_{M_{g,r}}}(((\Pi_{C_{g,r}})_h) is isomorphic to

\[
\begin{cases} 
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } (g, r) = (0, 4); \\
\mathbb{Z}/2\mathbb{Z} & \text{if } (g, r) \in \{(1, 2), (2, 0)\}; \\
\{1\} & \text{if } (g, r) \not\in \{(0, 4), (1, 2), (2, 0)\}.
\end{cases}
\]
(iii) Let $H \subseteq \text{Out}^C(\Pi_{g,r})$ be a closed subgroup of $\text{Out}^C(\Pi_{g,r})$ that contains an open subgroup of $\text{Im}(\rho_{g,r}) \subseteq \text{Out}(\Pi_{g,r})$. Suppose that

$$2g - 2 + r > 1, \text{ i.e., } (g, r) \notin \{(0, 3), (1, 1)\}.$$ 

Then $H$ is almost slim [cf. the discussion entitled “Topological groups” in §0]. If, moreover,

$$2g - 2 + r > 2, \text{ i.e., } (g, r) \notin \{(0, 3), (0, 4), (1, 1), (1, 2), (2, 0)\},$$ 

then $H$ is slim [cf. the discussion entitled “Topological groups” in §0].

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). First, we recall that the description of the automorphism group $\text{Aut}_{(M_{g,r})k}((C_{g,r})_k)$ given in the statement of assertion (ii) is well-known [cf., e.g., [CorHyp], Theorem B, if $2g - 2 + r > 2$, i.e., $(g, r) \notin \{(0, 4), (1, 2), (2, 0)\}$]. Next, we claim that

(*)1: the composite homomorphism of assertion (ii) determines an injection

$$\text{Aut}_{(M_{g,r})k}((C_{g,r})_k) \hookrightarrow Z_{\text{Out}^C(\Pi_{g,r})}(\rho_{g,r}(H)).$$

Indeed, one verifies easily that the composite as in assertion (ii) factors through $Z_{\text{Out}^C(\Pi_{g,r})}(\rho_{g,r}(H))$. Thus, the claim (*)1 follows immediately from the well-known fact that any nontrivial automorphism of a hyperbolic curve over an algebraically closed field of characteristic $\not\in \Sigma$ induces a nontrivial automorphism of the maximal pro-$\Sigma$ quotient of the étale fundamental group of the hyperbolic curve [cf., e.g., [LocAn], the proof of Theorem 14.1]. This completes the proof of the claim (*1).

Next, we claim that

(*)2: if $r > 0$, then the injection of (*)1 is an isomorphism.

Indeed, write $N \subseteq \Pi_{M_{g,r}}$ for the kernel of the surjection $\Pi_{M_{g,r}} \twoheadrightarrow \pi_1((M_{g,r-1})_k)$ determined by the (1-)morphism $(M_{g,r})_k \to (M_{g,r-1})_k$ obtained by forgetting the last section. Then it follows immediately from the various definitions involved that there exists a commutative diagram of profinite groups

$$
\begin{array}{ccccccc}
1 & \longrightarrow & \Pi_{g,r} & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Pi_F & \longrightarrow & \Pi_T & \longrightarrow & \Pi_B & \longrightarrow & 1
\end{array}
$$
— where the upper sequence is the exact sequence obtained by pulling back the exact sequence \( 1 \to \Pi_{g,r} \to \Pi_{C_{g,r}} \to \Pi_{M_{g,r}} \to 1 \) by the natural inclusion \( N \), the lower sequence is the exact sequence
\[
1 \to \Pi_F \to \Pi_T \to \Pi_B \to 1
\]
"obtained by applying the procedure given in the statement of Theorem 6.12 in the case where \((m,n,\Sigma_F,\Sigma_B) = (1,2,\Sigma,\Primes)\) to a stable log curve of type \((g,r-1)\) over \((\text{Spec} \, k)^{\log}\); the vertical arrows are isomorphisms. Thus, it follows immediately from Theorem 6.12, (i), that
\[
\text{Out}_{C_{g,r}}(\Pi_{M_{g,r}}) \cong \text{Out}_{C_{g,r}}(\Pi_{H \cap N}) \cong \text{Out}_{X_{g,r}}(X_{g,r}^{\log})
\]
for the stable log curve \( X_{g,r}^{\log} \) over \((\text{Spec} \, k)^{\log}\) of type \((g,r-1)\). In particular, by the claim \((\ast_1)\), we obtain that
\[
\text{Aut}_{X_{g,r}}(X_{g,r}^{\log}) \cong \text{Out}_{C_{g,r}}(\Pi_{M_{g,r}}) \cong \text{Out}_{C_{g,r}}(\Pi_{H \cap N}) \cong \text{Out}_{X_{g,r}}(X_{g,r}^{\log})
\]
Thus, by comparing \((\text{Aut}_{X_{g,r}}(X_{g,r}^{\log}))^{\#}\) with \((\text{Aut}_{X_{g,r}}(X_{g,r}^{\log}))^{\#}\) [cf. Theorem 6.12, (i)], we conclude that the injection of the claim \((\ast_2)\) is an isomorphism. This completes the proof of the claim \((\ast_2)\). Moreover, it follows immediately from the proof of the claim \((\ast_2)\) that
\[
(\ast_3): \text{if } \alpha \in \text{Out}_{C_{g,r}}(\Pi_{0,4}(H)) \text{ induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of } \Pi_{0,4}, \text{ then } \alpha \text{ is the identity automorphism of } \Pi_{0,4}.
\]
In light of the claim \((\ast_2)\), in the remainder of the proof of assertion (ii), we assume that
\[
r = 0, \text{ hence also that } g \geq 2.
\]
For \( x \in (\mathcal{M}_{g,0})_k(k) \), write
\[
\mathcal{G}_x
\]
for the semi-graph of anabeloids of pro-\(\Sigma\) PSC-type associated to the geometric fiber of \((\mathcal{C}_{g,r}^{\log})_k \to (\mathcal{M}_{g,0})_k\) over \( x^{\log} \overset{\text{def}}{=} x \times (\mathcal{M}_{g,0})_k(\mathcal{M}_{g,0}^{\log})_k \); thus, we have a natural \( \text{Im}(\rho_{0,4}) \subseteq \text{Out}(\Pi_{0,4})\)-torsor of outer isomorphisms \( \Pi_{g,0} \to \Pi_{0,4} \). Let us fix an isomorphism \( \Pi_{g,0} \to \Pi_{0,4} \) that belongs to this collection of isomorphisms. Moreover, for \( x \in (\mathcal{M}_{g,0})_k(k) \), we shall say that \( x \) satisfies the condition \((\dagger)\) if
\[
(\dagger_1) \quad \text{Vert}(\mathcal{G}_x) = \{v_1, v_2\}; \quad \text{Node}(\mathcal{G}_x) = \{e_1, e_2, \cdots, e_g+1\};
\]
\[
(\dagger_2) \quad \mathcal{N}(v_1) = \mathcal{N}(v_2) = \text{Node}(\mathcal{G}_x);
\]
\[
(\dagger_3) \quad v_1 \text{ and } v_2 \text{ are of type } (0, g+1);
\]
we shall say that \( x \) satisfies the condition \((\ddagger)\) if

\[(\ddagger_1) \quad \text{Vert}(G_x) = \{v_1^*, v_2, w^*\}; \quad \text{Node}(G_x) = \{e_1^*, e_2^*, \ldots, e_{g+1}^*, f^*\}; \]

\[(\ddagger_2) \quad \mathcal{N}(v_1^*) = \{e_1^*, e_2^*, \ldots, e_{g+1}^*\}; \quad \mathcal{N}(w^*) = \{e_g^*, e_{g+1}^*, f^*\}; \]

\[(\ddagger_3) \quad v_1^* \text{ is of type } (0, g + 1), \quad v_2^* \text{ is of type } (0, g), \quad \text{and } w^* \text{ is of type } (0, 3). \]

Let us observe that one may verify easily that there exists a \( k \)-valued point \( x_2 \in (\mathcal{M}_{g,0})_k \) that satisfies \((\ddagger)\); if, moreover, \( g > 2 \), then there exists a \( k \)-valued point \( x_2 \in (\mathcal{M}_{g,0})_k \) that satisfies \((\ddagger)\).

Let \( x \in (\mathcal{M}_{g,0})_k \) be a \( k \)-valued point. Then we claim that

\[(\ast_4) \quad \text{if } x \text{ satisfies } (\ddagger), \text{ and, relative to the isomorphism } \Pi_{g,0} \to \Pi_{G_x \text{ fixed above}}, \]

\[\alpha \in \mathbb{Z} \text{Out}^c(\Pi_{g,0})(\rho_{g,0}(H)) \]

determines an element of \( \text{Aut}^{[\text{grph}]}(G_x) \subseteq \text{Out}(\Pi_{G_x}) \)

corresponding to \( \text{Out}(\Pi_{g,0}) \), then for any \( e \in \text{Node}(G_x) \), the image \( \alpha_e \) of \( \alpha \) via the natural inclusion \( \text{Aut}^{[\text{grph}]}(G_x) \to \text{Aut}^{[\text{grph}]}((G_x)_{-\{e\}}) \) [cf. Proposition 2.9, (ii)] satisfies

\[\alpha_e \in \text{Dehn}((G_x)_{-\{e\}}).\]

Indeed, let \( e \in \text{Node}(G_x) \) and \( y \in (\mathcal{M}_{g,0})_k \) a \( k \)-valued point such that \( G_y \) corresponds to \( (G_x)_{-\{e\}} \) [cf. the special fibers of the stable log curves over “\( S_{\log}^{\text{log}} \)” that appear in Proposition 5.6, (iii)]. Write \( v \in \text{Vert}(G_y) \) for the unique vertex of \( G_y \). [Note that it follows from the definition of the condition \((\ddagger)\) that \( \text{Vert}(G_y)^2 = 1 \).] Then it follows immediately from the general theory of stable log curves that there exist a “clutching \((1-)\text{morphism}” corresponding to the operation of resolving the nodes of \( G_y \) [i.e., obtained by forming appropriate composites of the clutching morphisms discussed in [Knud], Definition 3.6]

\[(\mathcal{M}_{0,2g})_k \to (\mathcal{M}_{g,0})_k \]

and a \( k \)-valued point \( \tilde{y} \in (\mathcal{M}_{0,2g})_k \) such that the image of \( \tilde{y} \) via the above clutching morphism coincides with \( y \), and, moreover, \( G_{\tilde{y}} \) is naturally isomorphic to \( (G_y)_{\downarrow v} \). Write \((\mathcal{M}_{0,2g}^{\log})_k \) for the log stack obtained by equipping \((\mathcal{M}_{0,2g})_k \) with the log structure induced by the log structure of \((\mathcal{M}_{g,0}^{\log})_k \) via the above clutching morphism. Then one verifies easily
that the composite

\[
\Pi_{\mathcal{M}_0,2g} \overset{\text{def}}{=} \pi_1((\mathcal{M}^{\log}_{0,2g})_k) \longrightarrow \pi_1((\mathcal{M}^{\log}_{g,0})_k) \cong \Pi_{\mathcal{M}_0,0} \overset{\rho_{0,0}}{\longrightarrow} \text{Out}(\Pi_{\mathcal{M}_0,0})
\]

where the first arrow is the outer homomorphism induced by the above clutching morphism, and the second arrow is the outer isomorphism obtained by applying the "log purity theorem" to the natural (1-)morphism \((\mathcal{M}_0,0) \hookrightarrow (\mathcal{M}^{\log}_{g,0})_k\) [cf. [ExtFam], Theorem B] — factors through \(\text{Aut}([\mathcal{G}_y]) \subseteq \text{Out}(\Pi_{\mathcal{G}_y}) \overset{\sim}{\longrightarrow} \text{Out}(\Pi_{\mathcal{M}_0,0})\). Moreover, the resulting homomorphism \(\Pi_{\mathcal{M}_0,2g} \rightarrow \text{Aut}([\mathcal{G}_y])\) fits into a commutative diagram of profinite groups

\[
\begin{array}{ccc}
\Pi_{\mathcal{M}_0,2g} & \longrightarrow & \Pi_{\mathcal{M}_0,2g} \\
\downarrow & & \downarrow \\
\text{Aut}([\mathcal{G}_y]) \overset{\rho_{0,2g}}{\longrightarrow} \text{Glu}(\mathcal{G}_y) & = & \text{Aut}([\mathcal{G}_y])
\end{array}
\]

[cf. Definition 4.9; Proposition 4.10, (iii)] — where the upper horizontal arrow is the outer homomorphism induced by the (1-)morphism \((\mathcal{M}^{\log}_{2g})_k \rightarrow (\mathcal{M}_0,2g)_k\) obtained by forgetting the log structure. Moreover, one verifies easily that there exists a natural outer isomorphism \(\Pi(\mathcal{G}_y)_0 \rightarrow \Pi_{0,2g}\) such that the homomorphism \(\Pi_{\mathcal{M}_0,2g} \rightarrow \text{Out}(\Pi_{0,2g})\) obtained by conjugating the outer action implicit in the right-hand vertical arrow of the above diagram \(\Pi_{\mathcal{M}_0,2g} \rightarrow \text{Aut}([\mathcal{G}_y])_v\) by the outer isomorphism \(\Pi(\mathcal{G}_y)_0 \rightarrow \Pi_{0,2g}\) coincides with \(\rho_{0,2g}\). Thus, by considering the image in \(\Pi_{\mathcal{M}_0,2g}\) of the inverse image of \(H \subseteq \Pi_{\mathcal{M}_0,0}\) in \(\Pi_{\mathcal{M}_0,2g}\) [cf. the diagrams of the above displays], it follows immediately from the claims \(\ast_2\) [in the case where \(\ast(g,r) = (0,2g)\)] and \(\ast_3\) [in the case where \(g = 2\)], together with the various definitions involved, that if \(\alpha \in \mathcal{Z}(\Pi_{\mathcal{M}_0,0})(\rho_{0,0}(H))\) determines an element of \(\text{Aut}([\mathcal{G}_x])\) \((\subseteq \text{Out}(\Pi_{\mathcal{G}_x}) \overset{\sim}{\longrightarrow} \text{Out}(\Pi_{\mathcal{M}_0,0}))\), then the image of \(\alpha\) via

\[
\begin{array}{ccc}
\text{Aut}([\mathcal{G}_x]) & \rightarrow & \text{Aut}([\mathcal{G}_x]_{\rightarrow(\varepsilon)}) \\
\rho_{0,2g} & = & \text{Glu}(\mathcal{G}_y) \\
\alpha & = & \text{Aut}([\mathcal{G}_y])_{\rightarrow(\varepsilon)}
\end{array}
\]

[cf. Proposition 2.9, (ii)] is trivial. In particular, it follows from Proposition 4.10, (ii), that the image \(\alpha_{\varepsilon}\) of \(\alpha\) via \(\text{Aut}([\mathcal{G}_x]) \rightarrow \text{Aut}([\mathcal{G}_x]_{\rightarrow(\varepsilon)})\) satisfies \(\alpha_{\varepsilon} \in \text{Dehn}([\mathcal{G}_x]_{\rightarrow(\varepsilon)})\). This completes the proof of the claim \(\ast_4\).

Next, we claim that
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if $x$ satisfies (†), and $\alpha \in Z_{\Out^C(\Pi_g,0)}(\rho_{g,0}(H))$
determines an element of $\Aut^{\graph}(\G_x)$ ($\subseteq \Out(\Pi_{\G_x})$
$\leftarrow \Out(\Pi_{g,0})$), then $\alpha$ is the identity automorphism
of $\Pi_{g,0}$.

Indeed, it follows from the claim (∗4) that

$$\alpha \in \bigcap_{e \in \Node(\G_x)} \operatorname{Im} \left( \operatorname{Dehn}(\G_x) \to \Dehn(\G_x) \right)$$

[cf. Theorem 4.8, (ii)]. On the other hand, it follows immediately from
Theorem 4.8, (ii), (iv), that the right-hand intersection is $\{1\}$. This
completes the proof of the claim (∗5).

Next, we claim that

(∗6): we have $Z_{\Out^C(\Pi_g,0)}(\rho_{g,0}(H)) \subseteq \Aut^{\Node(\G_x)}(\G_x)$ ($\subseteq \Out(\Pi_{\G_x})$
$\leftarrow \Out(\Pi_{g,0})$);

if, moreover, $x$ satisfies (†), then

$$Z_{\Out^C(\Pi_g,0)}(\rho_{g,0}(H)) \subseteq \Aut^{\graph}(\G_x).$$

Indeed, it follows immediately from Proposition 5.6, (ii), together with
the definition of $x^{\log} = x \times (\M_{g,0})_k \to (\M_{g,0})_k$, that the composite

$$\pi_1(x^{\log}) \to \pi_1((\M_{g,0})_k) \leftarrow \Pi_{\M_{g,0}} \xrightarrow{\rho_{g,0}} \Out(\Pi_{g,0})$$

— where the second arrow is the outer isomorphism obtained by applying the “log purity theorem” to the natural (1-)morphism $(\M_{g,0})_k \to (\M_{g,0})_k$ [cf. ExtFam, Theorem B] — determines a surjection $\pi_1(x^{\log})$
$\to \Dehn(\G_x)$ ($\subseteq \Out(\Pi_{\G_x})$ $\leftarrow \Out(\Pi_{g,0})$) [i.e., which induces an isomorphism between the respective maximal pro-$\Sigma$ quotients]. Thus, it follows immediately from the various definitions involved that there exists an open subgroup $M \subseteq \Dehn(\G_x)$ such that $Z_{\Out^C(\Pi_g,0)}(\rho_{g,0}(H)) \subseteq Z_{\Out^C(\Pi_{\G_x})}(M)$ relative to the identification $\Out^C(\Pi_{g,0}) \leftarrow \Out^C(\Pi_{\G_x})$
 arising from our choice of an isomorphism $\Pi_{g,0} \leftarrow \Pi_{\G_x}$. Therefore, the inclusion $Z_{\Out^C(\Pi_g,0)}(\rho_{g,0}(H)) \subseteq \Aut^{\Node(\G_x)}(\G_x)$ follows immediately
from Theorem 5.14, (ii). This completes the proof of the inclusion
$Z_{\Out^C(\Pi_g,0)}(\rho_{g,0}(H)) \subseteq \Aut^{\graph}(\G_x)$. On the other hand, if, moreover,
$x$ satisfies (†), then it follows immediately from the definition of the condition (†) that $\Aut^{\graph}(\G_x) = \Aut^{\Node(\G_x)}(\G_x)$. In particular,
we obtain that $Z_{\text{Out}^c(\Pi_y,0)}(\rho_{g,0}(H)) \subseteq \text{Aut}_{\text{graph}}(G_x)$. This completes the proof of the claim (**6). 

Next, we claim that

(**7): if $x$ satisfies (**), then for any $\alpha \in Z_{\text{Out}^c(\Pi_y,0)}(\rho_{g,0}(H))$, there exists an element $\beta$ of the image of the injection of (**1) such that the automorphism $\alpha \circ \beta$ of $\Pi_{g,0} \rightarrow \Pi_G$ is $\in \text{Aut}^{\text{graph}}(G_x)$

$(\subseteq \text{Out}(\Pi_{G_x}) \leftarrow \text{Out}(\Pi_{g,0}))$.

Indeed, suppose that $g > 2$. Then by the definitions of (**), (**), one may verify easily that there exist $y \in (\mathcal{M}_{g,0})_k(k)$ and $f \in \text{Node}(G_y)$ such that $y$ satisfies (**), and, moreover, $G_x$ corresponds to $(\mathcal{G}_y)_{\sim}(f)$ [cf. Proposition 5.6, (iv)]. Thus, it follows immediately from the claim (**6) that $Z_{\text{Out}^c(\Pi_y,0)}(\rho_{g,0}(H)) \subseteq \text{Aut}^{\text{graph}}(G_y) \hookrightarrow \text{Aut}^{\text{graph}}(G_x)$ [cf. Proposition 2.9, (ii)], i.e., so we may take $\beta$ to be the identity automorphism. This completes the proof of the claim (**7) in the case where $g > 2$.

Next, suppose that $g = 2$. Write $G_x$ for the underlying semi-graph of $G_x$, and $\text{Aut}^{\text{Node}}(G_x)$ for the group of automorphisms of $G_x$ which induce the identity automorphism of the set of nodes of $G_x$. Then one may verify easily from the explicit structure of $G_x$ [cf. the definition of the condition (**)] that $\text{Aut}^{\text{Node}}(G_x)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus, since the automorphism group $\text{Aut}(\mathcal{M}_{2,0})_k((C_{2,0})_k)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, it follows immediately from the claim (**6), together with the various definitions involved, that — to complete the proof of the claim (**7) in the case where $g = 2$ — it suffices to verify that the composite of natural homomorphisms

$$\text{Aut}(\mathcal{M}_{2,0})_k((C_{2,0})_k) \longrightarrow \text{Aut}(G_x) \longrightarrow \text{Aut}(G_z)$$

factors through $\text{Aut}^{\text{Node}}(G_x) \subseteq \text{Aut}(G_x)$ and is injective. Now the fact that the composite in question factors through $\text{Aut}^{\text{Node}}(G_x) \subseteq \text{Aut}(G_x)$ follows immediately from the claim (**6), applied to elements of the image of the injection of (**1). On the other hand, the injectivity of the composite in question follows immediately from the injectivity of the natural homomorphism $\text{Aut}(\mathcal{M}_{2,0})_k((C_{2,0})_k) \rightarrow \text{Aut}(G_x)$ [cf. the proof of the claim (**1)] and the claim (**5). This completes the proof of the claim (**7) in the case where $g = 2$, hence also — in light of the above proof of the claim (**7) in the case where $g > 2$ — of the claim (**7). Thus, the surjectivity of the injection of (**1) follows immediately from the claims (**5) and (**7). This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii). This completes the proof of Theorem 6.13.

Q.E.D.
Remark 6.13.1. In the notation of Theorem 6.13, since $\Pi_{\mathcal{M}_0,3} = \{1\}$, it is immediate that a similar result to the results stated in Theorem 6.13, (ii), (iii), does not hold in the case where $(g,r) = (0,3)$. On the other hand, it is not clear to the authors at the time of writing whether or not a similar result to the results stated in Theorem 6.13, (ii), (iii), holds in the case where $(g,r) = (1,1)$. Nevertheless, we are able to obtain a conditional result concerning the centralizer of the geometric monodromy group in the case where $(g,r) = (1,1)$ [cf. Theorem 6.14, (iii), (iv) below].

Theorem 6.14 (Centralizers of geometric monodromy groups arising from moduli stacks of punctured semi-elliptic curves). In the notation of Theorem 6.13, write $(C_{1,1})^+_k$ for the stack-theoretic quotient of $(C_{1,1})_k$ by the natural action of $\operatorname{Aut}(\mathcal{M}_{1,1})_k((C_{1,1})_k)$ over the moduli stack $(\mathcal{M}_{1,1})_k$; $\Pi^+_1$ for the maximal pro-$\Sigma$ quotient of the kernel $N^+_1 \overset{\text{def}}{=} \ker(\pi_1((C_{1,1})^+_k) \to \pi_1((\mathcal{M}_{1,1})_k) = \Pi_{\mathcal{M}_{1,1}})$ of the natural surjection $\pi_1((C_{1,1})^+_k) \to \pi_1((\mathcal{M}_{1,1})_k) = \Pi_{\mathcal{M}_{1,1}}$; $C_{1,1}^+$ for the quotient of the étale fundamental group $\pi_1((C_{1,1})^+_k)$ of the stack $(C_{1,1})^+_k$ by the kernel of the natural surjection $N^+_1 \to \Pi^+_1$, Thus, we have a natural exact sequence of profinite groups

$$1 \to \Pi^+_1 \to C_{1,1}^+ \to \Pi_{\mathcal{M}_{1,1}} \to 1,$$

which determines an outer representation

$$\rho^+_1 : \Pi_{\mathcal{M}_{1,1}} \to \operatorname{Out}(\Pi^+_1).$$

Write $\operatorname{Out}^C(\Pi^+_1)$ for the group of automorphisms of $\Pi^+_1$ which induce bijections on the set of cuspidal inertia subgroups of $\Pi^+_1$. Suppose that $2 \in \Sigma$.

Then the following hold:

(i) The profinite group $\Pi^+_1$ is slim [cf. the discussion entitled “Topological groups” in §0].

(ii) Let $H \subseteq \Pi_{\mathcal{M}_{1,1}}$ be an open subgroup of $\Pi_{\mathcal{M}_{1,1}}$. Then the composite of natural homomorphisms

$$\operatorname{Aut}(\mathcal{M}_{1,1})_k((C_{1,1})^+_k) \to \operatorname{Aut}(\Pi^+_1) / \operatorname{Im}(\Pi^+_1)$$

$$\sim \to \operatorname{ZOut}(\Pi^+_1)(\operatorname{Im}(\rho^+_1)) \subseteq \operatorname{ZOut}(\Pi^+_1)(\rho^+_1(H))$$
[cf. (i); the discussion entitled “Topological groups” in §0] determines an isomorphism
\[ \text{Aut}_k((C^+_1)_k) \xrightarrow{\sim} Z_{\text{Out}^c}(\Pi^+_1)(\rho^{++}_{1,1}(H)) . \]

Here, we recall that \( \text{Aut}_k((C^+_1)_k) = \{1\} \).

(iii) Let \( H \subseteq \Pi_{M_{1,1}} \) be an open subgroup of \( \Pi_{M_{1,1}} \). Then the composite of natural homomorphisms
\[ \text{Aut}_k((C_1)_k) \rightarrow \text{Aut}_{\Pi_{M_{1,1}}}((\Pi_{C_1,1})/\text{Inn}(\Pi_{1,1}) \rightarrow Z_{\text{Out}^c}(\Pi_{1,1})(\text{Im}(\rho_{1,1})) \subseteq Z_{\text{Out}^c}(\Pi_{1,1})(\rho_{1,1}(H)) \]
[cf. Theorem 6.13, (i); the discussion entitled “Topological groups” in §0] determines an injection
\[ \text{Aut}_k((C_1)_k) \hookrightarrow Z_{\text{Out}^c}(\Pi_{1,1})(\rho_{1,1}(H)). \]
Moreover, the image of this injection is centrally terminal in \( Z_{\text{Out}^c}(\Pi_{1,1})(\rho_{1,1}(H)) \) [cf. the discussion entitled “Topological groups” in §0]. Here, we recall that \( \text{Aut}_k((C_1)_k) \cong \mathbb{Z}/2\mathbb{Z} \).

(iv) The composite of natural homomorphisms
\[ \text{Aut}_k((C_1)_k) \rightarrow \text{Aut}_{\Pi_{M_{1,1}}}((\Pi_{C_1,1})/\text{Inn}(\Pi_{1,1}) \rightarrow Z_{\text{Out}^c}(\Pi_{1,1})(\text{Im}(\rho_{1,1})) \]
[cf. Theorem 6.13, (i); the discussion entitled “Topological groups” in §0] determines an isomorphism
\[ \text{Aut}_k((C_1)_k) \xrightarrow{\sim} Z_{\text{Out}^c}(\Pi_{1,1})(\text{Im}(\rho_{1,1})). \]

Proof. Assertion (i) follows immediately from a similar argument to the argument used in the proof of [MT], Proposition 1.4. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, let us recall that the description of the automorphism group \( \text{Aut}_k((C^+_1)_k) \) given in the statement of assertion (ii) is well-known and easily verified. Write \( \mathcal{E} \rightarrow (M_{1,1})_k \) for the family of elliptic curves determined by the family of hyperbolic curves \( (C_1)_k \rightarrow (C_1)_k \) of type \((1,1); \mathcal{U} \rightarrow (C_1)_k \) for the restriction of the finite étale covering \( \mathcal{E} \rightarrow \mathcal{E} \) over \((M_{1,1})_k \) given by multiplication by 2 to \((C_1)_k \subseteq \mathcal{E} \). Then one verifies easily that the action of
Indeed, let us observe that one verifi es easily that the natural surjections \( \text{Aut}_k((C_{1,1})_k) \) on \((C_{1,1})_k\) lifts naturally to an action [i.e., given by “multiplication by \( \pm 1 \)’] on \( \mathcal{U} \) over \((M_{1,1})_k\). Write \( \mathcal{P} \) for the stack-theoretic quotient of \( \mathcal{U} \) by the action of \( \text{Aut}_k((C_{1,1})_k) \) on \( \mathcal{U} \); \( \Pi_{\mathcal{P}/M} \) for the maximal pro-\( \Sigma \) quotient of the kernel of the natural surjection \( \pi_1(\mathcal{P}) \to \pi_1((M_{1,1})_k) \):

\[
\rho_{\mathcal{P}/M} : \Pi_{M_{1,1}} \longrightarrow \text{Out}(\Pi_{\mathcal{P}/M})
\]

for the natural pro-\( \Sigma \) outer representation arising from the family of hyperbolic curves \( \mathcal{P} \to (M_{1,1})_k \). Thus, since \( 2 \in \Sigma \), one verifi es easily that \( \Pi_{\mathcal{P}/M} \) may be regarded as a normal open subgroup of \( \Pi_{1,1}^\pm \). Now let us observe that one verifi es easily that

\((*) : \mathcal{P} \to (M_{1,1})_k \) is a family of hyperbolic curves of type \((0,4)\). If, moreover, we denote by \( T \to (M_{1,1})_k \) the connected fi nite étale covering that trivializes the fi nite étale covering determined by the four cusps of \( \mathcal{P} \to (M_{4,1})_k \), then the classifying \((1-)\)morphism \( T \to (M_{0,4})_k \) of \( \mathcal{P} \times (M_{1,1})_k \) determines \( T \) which is well-defined up to the natural action of \( \mathfrak{S}_4 \) on \( (M_{0,4})_k \) is dominant.

Now we claim that

\((**): \) every element of \( \text{Out}^C(\Pi_{1,1}^\pm) \) preserves the normal open subgroup \( \Pi_{\mathcal{P}/M} \subseteq \Pi_{1,1}^\pm \).

Indeed, let us observe that one verifi es easily that the natural surjections \( \Pi_{1,1}^\pm \to \Pi_{1,1}^\pm/\Pi_{1,1}, \Pi_{1,1}^\pm/\Pi_{\mathcal{P}/M} \) determine an isomorphism

\[
(\Pi_{1,1}^\pm)^{ab} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} (\Pi_{1,1}^\pm/\Pi_{1,1}) \times (\Pi_{1,1}^\pm/\Pi_{\mathcal{P}/M}) .
\]

Moreover, it follows immediately from the various de nitions involved that the natural action of \( (\Pi_{1,1}^\pm)^{ab} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \) on the set of conjugacy classes of cuspidal inertia subgroups of the kernel of the natural surjection \( \Pi_{1,1}^\pm \to (\Pi_{1,1}^\pm)^{ab} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \) [which is equipped with a natural structure of pro-\( \Sigma \) surface group of type \((1,4)\)] factors through \( (\Pi_{1,1}^\pm)^{ab} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to (\Pi_{1,1}^\pm/\Pi_{1,1}) \times (\Pi_{1,1}^\pm/\Pi_{\mathcal{P}/M}) \) and that the resulting action of \( (\Pi_{1,1}^\pm)^{ab} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to (\Pi_{1,1}^\pm/\Pi_{1,1}) \times (\Pi_{1,1}^\pm/\Pi_{\mathcal{P}/M}) \) is faithful. Thus, we conclude that every element of \( \text{Out}^C(\Pi_{1,1}^\pm) \) preserves the normal open subgroup \( \Pi_{\mathcal{P}/M} \subseteq \Pi_{1,1}^\pm \). This completes the proof of the claim \((**\).

To verify assertion \((\text{ii})\), take an element \( \alpha_\mathcal{P} \in \mathbb{Z}_{\text{Out}^C(\Pi_{1,1}^\pm)}(\rho_{1,1}(H)) \). Then it follows from the claim \((**)\) that \( \alpha_\mathcal{P} \) naturally determines an element \( \alpha_\mathcal{P} \in \text{Aut}(\Pi_{\mathcal{P}/M})/\text{Inn}(\Pi_{1,1}^\pm) \). Let us fi x a lifting \( \beta \in \text{Out}^C(\Pi_{\mathcal{P}/M}) \)
of $\alpha_P$. Next, let us observe that since $\Pi_{1,1}^+/\Pi_{P/M}$ is finite, to verify assertion (ii), by replacing $H$ by an open subgroup of $\Pi_{M_{1,1}}$ contained in $H$, we may assume without loss of generality that $\beta$ commutes with $\rho_{P/M}(H) \subseteq \Out(\Pi_{P/M})$, i.e., $\beta \in Z^c_{\Out^c(\Pi_{P/M})}(\rho_{P/M}(H))$. Then it follows immediately from Theorem 6.13, (ii), in the case where $(g, r) = (0, 4)$, together with (*1), that $\beta$ is contained in the image of the natural injection $\Pi_{1,1}^+/\Pi_{P/M} \hookrightarrow \Out(\Pi_{P/M})$ obtained by conjugation. Thus, $\alpha_P$, hence also — by the manifest injectivity [cf. assertion (i)] of the homomorphism $\Out^C(\Pi_{1,1}^+) \to \Out(\Pi_{P/M})/\Inn(\Pi_{1,1}^+)$ implicit in the content of the claim (v2) — $\alpha^\pm$, is trivial. This completes the proof of assertion (ii).

Next, we verify assertion (iii). First, recall that the description of $\Aut(\Aut_{(M_{1,1})_k}((C_{1,1})_k))$ given in the statement of assertion (iii) is well-known and easily verified. Next, let us observe that the fact that the composite in the statement of assertion (iii) determines an injection

$$\Aut(\Aut_{(M_{1,1})_k}((C_{1,1})_k)) \hookrightarrow Z^c_{\Out^c(\Pi_{1,1})}(\rho_{1,1}(H))$$

follows immediately from a similar argument to the argument used in the proof of the claim (v1) in the proof of Theorem 6.13, (ii), together with the various definitions involved. Next, let us observe that by applying the natural outer isomorphism $\Pi_{1,1}^+ \xrightarrow{\text{out}} \Pi_{1,1} \rtimes \Aut(\Aut_{(M_{1,1})_k}((C_{1,1})_k))$, we obtain an exact sequence of profinite groups

$$1 \to \Aut(\Aut_{(M_{1,1})_k}((C_{1,1})_k)) \to Z^c_{\Out^c(\Pi_{1,1})}(\Aut(\Aut_{(M_{1,1})_k}((C_{1,1})_k))) \to \Out(\Pi_{1,1}^+)^\wedge$$

where we regard $\Aut(\Aut_{(M_{1,1})_k}((C_{1,1})_k))$ as a closed subgroup of $\Out(\Pi_{1,1})$ by means of the injection "$\hookrightarrow$" of the above display. Thus, the central terminality asserted in the statement of assertion (iii) follows immediately, in light of the above exact sequence, from assertion (ii). This completes the proof of assertion (iii).

Finally, we verify assertion (iv). It follows immediately from assertion (iii) that the image of the homomorphism $\Aut(\Aut_{(M_{1,1})_k}((C_{1,1})_k)) \to Z^c_{\Out^c(\Pi_{1,1})}(\Im(\rho_{1,1}))$ determined by the composite in the statement of assertion (iv) is centrally terminal. On the other hand, as is well-known, this image of $\Aut(\Aut_{(M_{1,1})_k}((C_{1,1})_k))$ in $\Out(\Pi_{1,1})$ is contained in $\Im(\rho_{1,1}) \subseteq \Out(\Pi_{1,1})$. [Indeed, recall that there exists a natural outer isomorphism $\SL_2(\Z)^\wedge \to \Pi_{M_{1,1}}$, where we write $\SL_2(\Z)^\wedge$ for the profinite completion of $\SL_2(\Z)$, such that the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \SL_2(\Z)^\wedge$
in $\text{Out}(\Pi_{1,1})$ coincides with the image of the unique nontrivial element of $\text{Aut}(\mathcal{M}_{1,1})((C_{1,1})_k) \simeq \mathbb{Z}/2\mathbb{Z}$ in $\text{Out}(\Pi_{1,1})$. Now assertion (iv) follows immediately. This completes the proof of assertion (iv). Q.E.D.

**Remark 6.14.1.** The authors hope to be able to address the issue of whether or not a similar result to the results stated in Theorem 6.13, (ii), (iii), holds for other families of pointed curves [e.g., the universal curves over moduli stacks of hyperelliptic curves or more general Hurwitz stacks] in a sequel to the present paper.

**References**


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