ON THE GROTHENDIECK CONJECTURE FOR AFFINE HYPERBOLIC CURVES OVER KUMMER-FAITHFUL FIELDS

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(Received 10 August 2015 and revised 21 September 2016)

Abstract. In the present paper, we discuss the Grothendieck conjecture for hyperbolic curves over Kummer-faithful fields. In particular, we prove that every point-theoretic and Galois-preserving outer isomorphism between the étale/tame fundamental groups of affine hyperbolic curves over Kummer-faithful fields arises from a uniquely determined isomorphism between the original hyperbolic curves. This result generalizes results of Tamagawa and Mochizuki, i.e., our main result in the case where the basefields are either finite fields or mixed-characteristic local fields.

0. Introduction

In the present paper, we discuss the [semi-absolute version—cf. the discussion preceding Theorem A below—of the] Grothendieck conjecture for hyperbolic curves over Kummer-faithful fields. In the Introduction, let the symbol ‘□’ stand for either ‘◦’ or ‘•’. Let \( k □ \) be a perfect field and let \( X □ \) be an affine hyperbolic curve over \( k □ \). Write \( X □ _\text{cpt} \) for the smooth compactification of \( X □ \), \( D X □ \subseteq X □ _\text{cpt} \) for the divisor at infinity of \( X □ \), and
\[
\Pi_{X □} \overset{\text{def}}{=} \pi^\text{tame}_{1}(X □ _\text{cpt}, D X □ )
\]
for the tame fundamental group of \( (X □ _\text{cpt}, D X □ ) \) (for some choice of basepoint).

Then it follows immediately from the functoriality of the tame fundamental group, together with the elementary theory of algebraic curves, that an isomorphism \( X _\circ \overset{\sim}{\rightarrow} X _\bullet \) of schemes gives rise to an (continuous) outer isomorphism \( \Pi_{X _\circ} \overset{\sim}{\rightarrow} \Pi_{X _\bullet} \) of profinite groups. In this situation, one of the main questions in the anabelian geometry may be stated as follows.

Question A. When does an outer isomorphism \( \Pi_{X _\circ} \overset{\sim}{\rightarrow} \Pi_{X _\bullet} \) of profinite groups arise (in the above sense) from an isomorphism \( X _\circ \overset{\sim}{\rightarrow} X _\bullet \) of schemes?

Let
\[
\phi: \Pi_{X _\circ} \overset{\sim}{\rightarrow} \Pi_{X _\bullet}
\]
be an outer isomorphism of profinite groups. In [7], Tamagawa proved that if \( k _\circ \) (respectively, \( k _\bullet \)) is an FF (i.e., a finite field), then \( \phi \) arises from an (uniquely determined)
isomorphism $X_\circ \sim X_\bullet$ of schemes (cf. [7, Theorem 0.6]). Moreover, in [4, Section 2], Mochizuki proved the following assertion (cf. [4, Corollary 2.2]), which may be regarded as an answer to Question A in the case where $k_\circ$ (respectively, $k_\bullet$) is either an MLF (mixed-characteristic local field; i.e., a finite extension of $\mathbb{Q}_p$ for some prime number $p$) or an FF.

Suppose that $k_\circ$ (respectively, $k_\bullet$) is either an MLF or FF. Then it holds that $\phi$ arises from an isomorphism $X_\circ \sim X_\bullet$ of schemes if and only if $\phi$ is point-theoretic (i.e., satisfies the condition that

\[(*)_P \quad \phi \text{ induces a bijection between the set of decomposition subgroups of } \Pi_{X_\circ} \text{ associated with closed points of } X_\circ^{\text{cpt}} \text{ and the set of decomposition subgroups of } \Pi_{X_\bullet} \text{ associated with closed points of } X_\bullet^{\text{cpt}} \text{—cf. Definition 3.1(i); also [4, Definition 1.5(ii)]}).\]

Moreover, such an isomorphism $X_\circ \sim X_\bullet$ of schemes is uniquely determined.

In the present paper, by refining various arguments given in [4, Sections 1, 2], [6, Section 1], we generalize the above result of Mochizuki to the case of affine hyperbolic curves over arbitrary Kummer-faithful fields.

We shall say that a perfect field $k$ is Kummer-faithful if, for every finite extension $K$ of $k$ and every semi-abelian variety $A$ over $K$, the Kummer map associated with $A$ is injective, or, equivalently, it holds that

$$\bigcap N \cdot A(K) = \{0\},$$

where $N$ ranges over the positive integers (cf. Definition 1.2; also [6, Definition 1.5]). A typical example of a Kummer-faithful field of characteristic zero is a sub-$p$-adic field for some prime number $p$ (i.e., a field which is isomorphic to a subfield of a finitely generated extension of an MLF—cf. [3, Definition 15.4(i)] (cf. [6, Remark 1.5.4(i)])); a typical example of a Kummer-faithful field of positive characteristic is an FF. Moreover, there exists a Kummer-faithful field of characteristic zero which is not sub-$p$-adic (cf. Remark 1.2.3(ii)). In particular, a field that appears in the statement of the above result of Mochizuki (i.e., a field which is either an MLF or FF) is Kummer-faithful. Here, we note that, in [6, Section 1], Mochizuki also proved a result on the (semi-absolute version of the) birational Grothendieck conjecture for curves over Kummer-faithful fields (cf. [6, Corollary 1.11]).

Let us observe that one verifies easily that if the outer isomorphism $\phi$ arises from an isomorphism of schemes, then $\phi$ satisfies the above condition $(*)_P$. Next, let us introduce another necessary condition to arise from an isomorphism of schemes. We shall say that (under the assumption that $k_\circ$ and $k_\bullet$ are Kummer-faithful) the outer isomorphism $\phi$ is Galois-preserving if

\[(*)_G \quad \phi \text{ is compatible with the quotients } \Pi_{X_\circ} \to \pi_1(\text{Spec}(k_\circ)), \quad \Pi_{X_\bullet} \to \pi_1(\text{Spec}(k_\bullet))\]

(cf. Definition 3.1(ii)).

Here, we note that it follows from [5, Corollary 2.8(ii)] that if $k_\circ$ (respectively, $k_\bullet$) is either a finite extension of $\mathbb{Q}_p$, an MLF, or an FF, then every outer isomorphism $\Pi_{X_\circ} \sim \Pi_{X_\bullet}$ (hence also every outer isomorphism discussed in the above result of Mochizuki) is Galois-preserving. However, it is not clear to the author at the time of writing whether or not an arbitrary ‘$\phi$’ in the case where both $k_\circ$ and $k_\bullet$ are Kummer-faithful is Galois-preserving, i.e., satisfies the above condition $(*)_G$. (In particular, the author does not have any example of ‘$\phi$’ which does not satisfy the above condition $(*)_G$ in the case where both $k_\circ$ and $k_\bullet$ are
Kummer-faithful.) Now let us observe that it follows immediately from the elementary theory of algebraic curves that if $\phi$ arises from an isomorphism of schemes, then $\phi$ satisfies the above condition $(*)_G$.

Roughly speaking, the absolute version of the Grothendieck conjecture is a version of the Grothendieck conjecture which discusses Question A for an arbitrary outer isomorphism $\Pi_{X_0} \sim \Pi_{X_\bullet}$. On the other hand, the semi-absolute version of the Grothendieck conjecture is a version of the Grothendieck conjecture which discusses Question A for an outer isomorphism $\Pi_{X_0} \sim \Pi_{X_\bullet}$ which satisfies the above condition $(*)_G$. (Moreover, the relative version of the Grothendieck conjecture is a version of the Grothendieck conjecture which discusses Question A for an outer isomorphism $\Pi_{X_0} \sim \Pi_{X_\bullet}$ in the case where the outer isomorphism satisfies the above condition $(*)_G$, and, moreover, the resulting outer isomorphism $\pi_1(\text{Spec}(k_0)) \sim \pi_1(\text{Spec}(k_\bullet))$ arises from an isomorphism $k_\bullet \sim k_0$ of fields.)

The main result of the present paper may be summarized as follows (cf. Theorem 3.4(ii) in the case where we take ‘$\Pi_{X_\square}$’ to be ‘$\pi_1^{\text{tame}}$’).

**THEOREM A.** Let the symbol ‘$\square$’ stand for either ‘$\circ$’ or ‘$\bullet$’. Let $k_\square$ be a Kummer-faithful field (cf. Definition 1.2) and let $X_\square$ be an affine hyperbolic curve over $k_\square$. Write $X_\square^{\text{cpt}}$ for the smooth compactification of $X_\square$, $D_{X_\square} \subseteq X_\square^{\text{cpt}}$ for the divisor at infinity of $X_\square$, and $\Pi_{X_\square} \overset{\text{def}}{=} \pi_1^{\text{tame}}(X_\square^{\text{cpt}}, D_{X_\square})$ for the tame fundamental group of $(X_\square^{\text{cpt}}, D_{X_\square})$ (for some choice of basepoint). Let

$$\phi : \Pi_{X_\square} \sim \Pi_{X_\bullet}$$

be an outer isomorphism of profinite groups. Then it holds that $\phi$ arises from an isomorphism of schemes

$$X_\circ \sim X_\bullet$$

if and only if $\phi$ is point-theoretic (cf. Definition 3.1(ii)) and Galois-preserving (cf. Definition 3.1(ii)). Moreover, such an isomorphism $X_\circ \sim X_\bullet$ of schemes is uniquely determined.

Note that it is not clear to the author at the time of writing whether or not one may drop the affine hypothesis in the statement of Theorem A (cf. Remark 3.4.1).

Similar arguments to the arguments which lead us to various assertions discussed in the proof of the above result of Mochizuki (i.e., [4, Corollary 2.2])—i.e., of the case of a hyperbolic curve over either an MLF or FF—in fact still do work in the generalization to the case of a hyperbolic curve over an arbitrary Kummer-faithful field (cf. various proofs given in Section 1 of the present paper). Some of the highly non-trivial portions in a task of establishing the generalization to the case of a hyperbolic curve over an arbitrary Kummer-faithful field consist of portions related to the group-theoretic reconstruction of fundamental extensions (cf. Definition 2.7(ii); also [4, Definition 1.2(ii)]). In [4, Section 1], a fundamental extension may be reconstructed (cf. [4, Proposition 1.5(i), (ii)]) in a fashion that depends on the arithmetic of an MLF and FF, i.e., by applying the theory of Galois cohomology of an MLF and FF (cf. [4, Propositions 1.2, 1.3]). In the present paper (i.e., in Section 2 of the present paper), in order to establish a group-theoretic reconstruction of fundamental extensions for Kummer-faithful fields, we consider a certain projective system of cohomology modules (cf. Lemma 2.6).
The present paper is organized as follows. In Section 1, we discuss various objects related to the étale/tame fundamental group of a hyperbolic curve over a *Kummer-faithful* field. In particular, we consider the image, via the Kummer theory, of the multiplicative group of the function field of the hyperbolic curve in a certain injective limit of cohomology modules (cf. Lemma 1.10). In Section 2, we discuss the *maximal cuspidally abelian quotient* of the étale fundamental group of the second configuration space of a proper hyperbolic curve. In particular, in order to study the maximal cuspidally abelian quotient, we consider a *fundamental extension* of an open subgroup of the fundamental group of the second configuration space by means of a certain projective system of cohomology modules (cf. Lemma 2.6). In Section 3, we prove the main result of the present paper (cf. Theorem 3.4).

**Notation and conventions**

**Numbers**

We shall write $\mathbb{P}$rimes for the set of all prime numbers. Let $\Sigma \subseteq \mathbb{P}$rimes be a subset of $\mathbb{P}$rimes. Then we shall say that a positive integer is a $\Sigma$-*integer* if every prime divisor of the integer is contained in $\Sigma$. We shall refer to a finite extension of $\mathbb{Q}_p$ (respectively, $\mathbb{F}_p$) for some prime number $p$ as an

$$MLF \quad (\text{respectively, } FF).$$

Here,

‘MLF’ is to be understood as an abbreviation for ‘mixed-characteristic local field’; ‘FF’ is to be understood as an abbreviation for ‘finite field’.

**Profinite groups**

In the present paper, a ‘homomorphism’ (respectively, an ‘isomorphism’; an ‘automorphism’; an ‘outer isomorphism’; an ‘outer automorphism’) between profinite groups always means a continuous homomorphism (respectively, continuous isomorphism; continuous automorphism; continuous outer isomorphism; continuous outer automorphism).

We shall say that a profinite group is *slim* if every open subgroup of the profinite group is center-free. One verifies immediately that an extension of *center-free* (respectively, *slim*) profinite groups is *center-free* (respectively, *slim*).

Let $G$ be a profinite group and let $H \subseteq G$ be a closed subgroup. We shall say that $H$ is *characteristic* if every automorphism of $G$ preserves $H$. We shall write $Z_G(H)$ for the *centralizer* of $H$ in $G$, $Z(G) \overset{\text{def}}{=} Z_G(G)$ for the *center* of $G$, $G^{\text{ab}}$ for the *abelianization* of $G$ (i.e., the quotient of $G$ by the closure of the commutator subgroup of $G$), $\text{Aut}(G)$ for the group of automorphisms of $G$, $\text{Inn}(G) \subseteq \text{Aut}(G)$ for the group of inner automorphisms of $G$, and $\text{Out}(G) \overset{\text{def}}{=} \text{Aut}(G)/\text{Inn}(G)$ for the group of outer automorphisms of $G$. Note that if $G$ is *topologically finitely generated*, then it follows immediately that the topology of $G$ admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on $\text{Aut}(G)$, and hence also $\text{Out}(G)$.

Let $G$ be a *center-free* and *topologically finitely generated* profinite group, $J$ a profinite group, and $\rho : J \to \text{Out}(G)$ a homomorphism of profinite groups. Thus, we have a natural
exact sequence of profinite groups $1 \to G \to \text{Aut}(G) \to \text{Out}(G) \to 1$. Then, by pulling back this exact sequence by the homomorphism $\rho$, we obtain a profinite group $G \rtimes J$, which fits into an exact sequence of profinite groups

$$1 \to G \to G^{\text{out}} \rtimes J \to J \to 1.$$  

**Curves**

Let $S$ be a scheme and let $X$ be a scheme over $S$. Then we shall say that $X$ is a *smooth curve* over $S$ if there exist a scheme $X^{\text{cpt}}$ which is smooth, proper, geometrically connected, and of relative dimension one over $S$ and a closed subscheme $D \subseteq X^{\text{cpt}}$ which is finite and étale over $S$ such that the complement $X^{\text{cpt}} \setminus D$ of $D$ in $X^{\text{cpt}}$ is isomorphic to $X$ over $S$. Note that if $S$ is the spectrum of a field $k$, then it follows immediately from elementary algebraic geometry that the pair $(X^{\text{cpt}}, D)$ is uniquely determined up to canonical isomorphism over $k$; we shall refer to $X^{\text{cpt}}$ as the *smooth compactification* of $X$ and to $D$ as the *divisor at infinity* of $X$.

Let $S$ be a scheme. Then we shall say that a smooth curve $X$ over $S$ is *hyperbolic* if there exist a pair $(X^{\text{cpt}}, D)$ satisfying the condition in the above definition of the term ‘smooth curve’ and a pair $(g, r)$ of non-negative integers such that $2g - 2 + r > 0$, the (necessarily locally free) $O_S$-module $(X^{\text{cpt}} \to S)_*(\Omega^1_{X^{\text{cpt}}/S})$ is of rank $g$, and the finite étale covering $D \hookrightarrow X^{\text{cpt}} \to S$ is of degree $r$.

1. **Fundamental groups of hyperbolic curves over Kummer-faithful fields**

In this section, we discuss various objects related to the étale/tame fundamental group of a hyperbolic curve over a *Kummer-faithful* field (cf. Definition 1.2 below). In this section, let $k$ be a *perfect* field, let $\overline{k}$ be an algebraic closure of $k$, and let $X$ be a *hyperbolic curve* over $k$. Write $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$. Note that one verifies immediately from the various definitions involved that since $k$ is *perfect*, every connected finite étale covering of $X$ is a *hyperbolic curve* over the finite extension of $k$ obtained by forming the algebraic closure of $k$ in the function field of the covering.

**Definition 1.1.** We shall write

- $\mathcal{P} \text{rimes}_{x/k} \subseteq \mathcal{P} \text{rimes}$ for the set of prime numbers which are invertible in $k$;
- $\mathcal{E}_{x/k}$ for the maximal pro-$\mathcal{P} \text{rimes}_{x/k}$ quotient of $\hat{\mathbb{Z}}$;
- $X^{\text{cpt}}$ for the *smooth compactification* of $X$;
- $D_X \subseteq X^{\text{cpt}}$ for the *divisor at infinity* of $X$;
- $g_X$ for the genus of $X^{\text{cpt}}$;
- $r_X \overset{\text{def}}{=} \sharp D_X(\overline{k})$;
- $K_X$ for the function field of $X$;
- $X^{\text{cl}+}$ for the set of closed points of $X^{\text{cpt}}$; and
- $\text{Div}(X)$ for the group of divisors on $X^{\text{cpt}}$. 
If \( x \in X^{\text{cpt}}(k) \) is a \( k \)-rational point of \( X^{\text{cpt}} \), then we shall write

- \( \text{ord}_x : K_X^\times \to \mathbb{Z} \) for the (uniquely determined) surjective valuation associated to \( x \in X^{\text{cpt}}(k) \).

In the following, let

\[ \Pi_X \]

be either the étale fundamental group \( \pi_1(X) \) of \( X \) or the tame fundamental group \( \pi_1^{\text{tame}}(X^{\text{cpt}}, D_X) \) of \( (X^{\text{cpt}}, D_X) \) (for some choice of basepoint). Write

\[ \Delta_X \subseteq \Pi_X \]

for the quotient of the étale fundamental group \( \pi_1(X \otimes_k \overline{k}) \subseteq \pi_1(X) \) of \( X \otimes_k \overline{k} \) determined by \( \Pi_X \). Thus, we have an exact sequence of profinite groups

\[ 1 \to \Delta_X \to \Pi_X \to G_k \to 1. \]

Now let us recall (cf., e.g., \([7, \text{Corollary 1.4 and Proposition 1.11}]\)) that \( \Delta_X \) is slim.

**Definition 1.2.** We shall say that \( k \) is *Kummer-faithful* if, for every finite extension \( K \) of \( k \) and every semi-abelian variety \( A \) over \( K \), it holds that

\[ \bigcap_N N \cdot A(K) = \{0\}, \]

where \( N \) ranges over the positive integers (cf. Remark 1.2.1 below).

**Remark 1.2.1.** If \( k \) is of characteristic zero, then it is immediate that \( k \) is *Kummer-faithful* in the sense of Definition 1.2 if and only if \( k \) is *Kummer-faithful* in the sense of \([6, \text{Definition 1.5}]\).

**Remark 1.2.2.** If \( k \) is *Kummer-faithful*, then the following assertion holds.

If \( K \) is a finite extension of \( k \), then it holds that

\[ \bigcap_N (K^\times)^N = \{1\}, \]

where \( N \) ranges over the \( \mathfrak{P} \)rime\( \times/k \)-integers.

Indeed, this follows immediately, by considering the semi-abelian variety \( \mathbb{G}_m \), from the definition of the term *Kummer-faithful*, together with our assumption that \( k \), and hence also \( K \), is *perfect*.

**Remark 1.2.3.** (i) A typical example of a *Kummer-faithful* field of characteristic zero is a sub-\( p \)-adic field for some prime number \( p \) (i.e., a field which is isomorphic to a subfield of a finitely generated extension of an MLF—cf. \([3, \text{Definition 15.4(i)}]\)) (cf. \([6, \text{Remark 1.5.4(i)}]\)). A typical example of a *Kummer-faithful* field of positive characteristic is an FF.

(ii) There exists a *Kummer-faithful* field of characteristic zero which is *not* sub-\( p \)-adic. For instance, if \( I \) is an infinite set, then the field \( \mathbb{Q}_p(x_i)_{i \in I} \), where \( x_i \) is an indeterminate, is an example of such a field (cf. \([6, \text{Remark 1.5.4(iii)}]\)).
Definition 1.3. (i) Let $G$ be a profinite group. Then we shall write $\mathfrak{C}(G)$ for the set of closed subgroups of $G$. Note that $G$ acts on $\mathfrak{C}(G)$ by conjugation.

(ii) We shall write $$\mathfrak{S}_X: X^{\text{cl} +} \rightarrow \mathfrak{C}(\Pi_X)/\Pi_X$$ for the map given by mapping a closed point of $X^{\text{cl} +}$ to the $\Pi_X$-conjugacy class of decomposition subgroups associated with the closed point.

If $x \in X(k)$ is a $k$-rational point of $X$, then we shall often identify $\mathfrak{S}_X(x)$ with the $\Delta_X$-conjugacy class of splittings of the natural surjection $\Pi_X \twoheadrightarrow G_k$ induced by $x$.

Definition 1.4. We shall write
$$\Lambda_X$$
for the cyclotome associated with the semi-graph of anabelioids of pro-$\mathfrak{P}$-type (with no nodes) arising from the hyperbolic curve $X \otimes_k \mathfrak{K}$ (cf. [2, Definition 3.8(i)]).

Remark 1.4.1. In the notation of Definition 1.4:

(i) the cyclotome $\Lambda_X$ is isomorphic, as a $G_k$-module, to $\lim_N^{N} \mu_N(\mathfrak{K})$, where the projective limit is taken over the $\mathfrak{P}$-integers $N$, and we write $\mu_N(\mathfrak{K})$ for the group of $N$th roots of unity in $\mathfrak{K}$;

(ii) if $X$ is proper over $k$ (i.e., $r_X = 0$), then one verifies easily from the definition of $\Lambda_X$ that the cyclotome $\Lambda_X$ may be identified with $\text{Hom}_{\mathfrak{P}_X/k}(\hat{H}^2(\Delta_X, \mathcal{Z}_{X/k}), \mathcal{Z}_{X/k})$.

Proposition 1.5. Suppose that $k$ is Kummer-faithful. Then the following assertions hold.

(i) Write
$$\chi_{\text{cyc}}^{X/k}: G_k \rightarrow \text{Aut}\left(\lim_N^{N} \mu_N(\mathfrak{K})\right) = (\mathcal{Z}_{X/k})^{\times}$$
—where the projective limit is taken over the $\mathfrak{P}$-integers $N$—for the $\mathfrak{P}$-adic cyclotomic character of $G_k$. Then it holds that $Z(G_k) \cap \text{Ker}(\chi_{\text{cyc}}^{X/k}) = \{1\}$.

(ii) The profinite group $\Pi_X$ is slim.

Proof. First, we verify assertion (i). Assume that there exists a non-trivial element $\gamma \in Z(G_k) \cap \text{Ker}(\chi_{\text{cyc}}^{X/k})$. Let $K$ be a finite Galois extension of $k$ contained in $\mathfrak{K}$ such that the corresponding normal open subgroup $G_K \subseteq G_k$ does not contain $\gamma \in G_k$. Then, since $\gamma \in Z(G_k) \cap \text{Ker}(\chi_{\text{cyc}}^{X/k})$, the natural action of $\gamma$ on $H^1(G_K, \lim_N^{N} \mu_N(\mathfrak{K}))$ is trivial. On the other hand, it follows from Remark 1.2.2, together with the Kummer theory, that this triviality implies the triviality of the action of $\gamma$ on $K$. Thus, since $\gamma \notin G_K$, we obtain a contradiction. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that since a connected finite étale covering of $X$ is a hyperbolic curve over a Kummer-faithful field (cf. our assumption that $k$ is perfect), to verify assertion (ii), it suffices to verify that $\Pi_X$ is center-free. Next, let us observe that since $\Delta_X$ is center-free, the composite $Z(\Pi_X) \hookrightarrow \Pi_X \twoheadrightarrow G_k$ is an injection, whose image is contained in the center $Z(G_k)$ of $G_k$. On the other hand, it follows immediately from the various definitions involved that the natural action of $Z(\Pi_X)$ on $\Delta_X$ is trivial, i.e., that the image of the composite $Z(\Pi_X) \hookrightarrow \Pi_X \twoheadrightarrow G_k$ is contained in $\text{Ker}(\chi_{\text{cyc}}^{X/k})$ (cf. Remark 1.4.1(i)). Thus, it follows from assertion (i) that $Z(\Pi_X) = \{1\}$. This completes the proof of assertion (ii), and hence also of Proposition 1.5.
Remark 1.5.1. In the situation of Proposition 1.5(i), in general, it does not hold that \( Z(G_k) = \{1\} \). Indeed, although (one verifies easily that) an FF is Kummer-faithful, the absolute Galois group of an FF is abelian and nontrivial.

Lemma 1.6. The following assertions hold.

(i) The quotient of \( \pi_X \) by the normal closed subgroup generated by the intersections \( \Delta_X \cap H \)—where \( H \) ranges over the closed subgroups of \( \pi_X \) whose images in \( C(\pi_X)/\Pi_X \) are contained in the image of the map \( \Psi_X \) of Definition 1.3(ii)—coincides with the quotient \( \pi_X \rightarrow \pi_1(X^{\text{cpl}}) \).

(ii) The subset \( \text{Primes}^{\times/k} \subseteq \text{Primes} \) is the (uniquely determined) maximal infinite subset on which the map \( \text{Primes} \rightarrow \mathbb{Z} \) given by mapping \( p \in \text{Primes} \) to the integer \( \dim_{\mathbb{Q}_p}(\pi_1(X^{\text{cpl}} \otimes_k \mathbb{F}_p)^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}_p) \) \((< \infty)—\text{cf. [7, Proposition 1.1]} \) is constant.

(iii) For every \( p \in \text{Primes}^{\times/k} \), it holds that \( \dim_{\mathbb{Q}_p}(\Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_p) = 2g_X + \max\{0, r_X - 1\} \) and \( \dim_{\mathbb{Q}_p}(\pi_1(X^{\text{cpl}} \otimes_k \mathbb{F}_p)^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}_p) = 2g_X \).

(iv) It holds that \( X \) is proper over \( k \) (i.e., \( r_X = 0 \)) if and only if, for every \( p \in \text{Primes}^{\times/k} \), the maximal pro-\( p \) quotient of \( \Delta_X \) is not free pro-\( p \).

(v) It holds that \( \Delta_X \) is not topologically finitely generated if and only if \( \text{char}(k) \neq 0 \), \( r_X \neq 0 \), and \( \pi_X = \pi_1(X) \). In other words, it holds that \( \Delta_X \) is topologically finitely generated if and only if the natural surjection \( \pi_X \rightarrow \pi_1^{\text{tame}}(X^{\text{cpl}}, D_X) \) is an isomorphism.

Proof. Assertion (i) follows immediately from the various definitions involved. Assertions (ii) and (iii) follow immediately from [7, Corollary 1.2]. Assertions (iv) and (v) follow immediately from [7, Proposition 1.1]. This completes the proof of Lemma 1.6.

Lemma 1.7. Suppose that \( k \) is Kummer-faithful. Then the following assertions hold.

(i) The map \( \Psi_X : X^{[1]} \rightarrow C(\pi_X)/\Pi_X \) of Definition 1.3(ii) is injective.

(ii) Suppose that \( X \) is proper over \( k \) (i.e., \( r_X = 0 \)). Let \( m \) be a positive integer and let \( x_1, \ldots, x_m \in X(k) \) be distinct \( k \)-rational points. Thus, for each \( i \in \{1, \ldots, m\} \), the \( k \)-rational point \( x_i \in X(k) \) determines a splitting \( s_i \) (well-defined up to \( \Delta_X^{ab} \)-conjugation) of the exact sequence of profinite groups

\[
1 \rightarrow \Delta_X^{ab} \rightarrow \Pi_X / \text{Ker}(\Delta_X \rightarrow \Delta_X^{ab}) \rightarrow G_k \rightarrow 1.
\]

Let \( (n_1, \ldots, n_m) \in \mathbb{Z}^{\oplus m} \) be such that \( \sum_{i=1}^m n_i = 0 \). Thus, by considering the linear combination \( \sum_{i=1}^m n_i \cdot s_i \) of the splittings \( s_i \), we obtain a cohomology class \( \left[ \sum_{i=1}^m n_i \cdot s_i \right] \in H^1(G_k, \Delta_X^{ab}) \). Then the divisor \( \sum_{i=1}^m n_i \cdot x_i \) (of degree zero) on \( X \) is principal if and only if \( \left[ \sum_{i=1}^m n_i \cdot s_i \right] = 0 \) in \( H^1(G_k, \Delta_X^{ab}) \).

Proof. Assertion (i) follows immediately from a similar argument to the argument applied in [7, Proof of Proposition 2.8(i)]. Assertion (ii) follows immediately from a similar argument to the argument applied in [4, Proof of Proposition 2.2(i)]. This completes the proof of Lemma 1.7.

Lemma 1.8. Suppose that \( k \) is Kummer-faithful, and that \( D_X(\overline{k}) = D_X(k) \). For \( x \in D_X(k) \), let \( \mathcal{J}_x \subseteq \Delta_X \) be an inertia subgroup of \( \Delta_X \) associated to \( x \). Then the following assertions hold.
(i) The inclusions $\mathfrak{S}_x \hookrightarrow \Pi_X$—where $x$ ranges over the elements of $D_X(k)$—and the surjection $\Pi_X \to G_k$ determine an exact sequence
\[
0 \to H^1(G_k, \Lambda_X) \to H^1(\Pi_X, \Lambda_X) \to \bigoplus_{x \in D_X(k)} \text{Hom}_{\mathbb{Z}}(\mathfrak{S}_x, \Lambda_X).
\]
Thus, by considering the isomorphism (well-defined up to a $(\hat{\mathbb{Z}}_{\times}/k)^\times$-multiple)
\[
(k^\times)^{\times/k} \overset{\text{def}}{=} \underset{N}{\lim} k^\times/(k^\times)^N \sim H^1(G_k, \Lambda_X)
\]
where the projective limit is taken over the $\mathfrak{P}_{\text{Primes}}^{\times/k}$-integers $N$—obtained by the Kummer theory (cf. Remark 1.4.1(i)) and the identification
\[
\hat{\mathbb{Z}}_{\times/k} = \text{Hom}_{\hat{\mathbb{Z}}_{\times/k}}(\Lambda_X, \Lambda_X) \sim \text{Hom}_{\mathbb{Z}}(\mathfrak{S}_x, \Lambda_X)
\]
where we write $\mathfrak{S}_x^{\times/k}$ for the maximal pro-$\mathfrak{P}_{\text{Primes}}^{\times/k}$ quotient of $\mathfrak{S}_x$—obtained by the synchronization of cyclotomes $\mathfrak{S}_x^{\times/k} \sim \Lambda_X$ discussed in [2, Corollary 3.9(v)], we obtain an exact sequence
\[
0 \to (k^\times)^{\times/k} \to H^1(\Pi_X, \Lambda_X) \to \bigoplus_{x \in D_X(k)} \hat{\mathbb{Z}}_{\times/k}.
\]

(ii) The exact sequence of the final display of (i) fits into the commutative diagram
\[
\begin{array}{ccc}
0 & \to & k^\times \\
\downarrow & & \downarrow \\
0 & \to & (k^\times)^{\times/k} \\
\end{array}
\begin{array}{ccc}
\oplus_{x \in D_X(k)} \mathcal{O}_X^\times(X) & \to & \bigoplus_{x \in D_X(k)} \mathbb{Z} \\
\downarrow & & \downarrow \\
\oplus_{x \in D_X(k)} \mathbb{Z} & \to & \bigoplus_{x \in D_X(k)} \hat{\mathbb{Z}}_{\times/k} \\
\end{array}
\begin{array}{ccc}
\oplus_{x \in D_X(k)} \mathbb{Z} & \to & \bigoplus_{x \in D_X(k)} \hat{\mathbb{Z}}_{\times/k} \\
\downarrow & & \downarrow \\
0 & \to & H^1(\Pi_X, \Lambda_X) \\
\end{array}
\]
where the horizontal sequences are exact, the vertical arrows are injective, the left-hand vertical arrow is the natural homomorphism into the completion, the middle vertical arrow is the homomorphism obtained by the Kummer theory, and the right-hand vertical arrow is the homomorphism determined by the natural inclusion $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}_{\times/k}$.

(iii) Let $y \in X(k)$ be a $k$-rational point. Then the composite
\[
\mathcal{O}_X^\times(X) \to H^1(\Pi_X, \Lambda_X) \to H^1(G_k, \Lambda_X) \sim (k^\times)^{\times/k}
\]
where the first arrow is the middle vertical arrow of the diagram of (ii), and the second arrow is the homomorphism determined by the splitting (well-defined up to $\Delta_X$-conjugation) of $\Pi_X \to G_k$ induced by $y \in X(k)$, i.e., ‘$\mathcal{O}_X(y)$’—coincides (up to a $(\hat{\mathbb{Z}}_{\times/k})^\times$-multiple—cf. the isomorphism $(k^\times)^{\times/k} \sim H^1(G_k, \Lambda_X)$ discussed in (i)) with the composite
\[
\mathcal{O}_X^\times(X) \to k^\times \to (k^\times)^{\times/k}
\]
\[
f \mapsto f(y),
\]
where the second arrow is the natural homomorphism.
Proof. Since the submodule consisting of $G_k$-invariants of the $\mathfrak{P}_{\mathrm{Primes}}^{X/k}$-adic Tate module of the Jacobian variety of $X^{\mathrm{cpt}}$ is trivial (by our assumption that $k$ is Kummer-faithful), assertion (i) follows immediately from a similar argument to the argument applied in [4, Proof of Proposition 2.1(ii)]. Assertions (ii) and (iii) follow immediately—in light of Remark 1.2.2—from the functoriality of the Kummer class, together with the various definitions involved. This completes the proof of Lemma 1.8. □

Definition 1.9. Suppose that $k$ is Kummer-faithful, and that $X$ is proper (i.e., $r_X = 0$). Let $S \subseteq X(k)$ be a finite subset and $x \in X(k) \setminus S$. Thus, since $X \setminus S$ is a hyperbolic curve over $k$, it follows from Lemma 1.8(i) that we have an exact sequence

$$0 \longrightarrow (k^\times)^{X/k} \longrightarrow H^1(\Pi_{X\setminus S}, \Lambda_X) \longrightarrow \bigoplus_{s \in S} \mathbb{Z}/\mathfrak{p}_{s,k},$$

where we write $\Pi_{X \setminus S} \overset{\text{def}}{=} \pi_{1,\mathrm{tame}}(X, S)$. We shall write

$${\mathcal{P}}(X, S) \overset{\text{def}}{=} \left\{ (n_s)_{s \in S} \in \bigoplus_{s \in S} \mathbb{Z} \mid \text{The divisor } \sum_{s \in S} n_s \cdot s \text{ is principal} \right\} \subseteq \bigoplus_{s \in S} \mathbb{Z}/\mathfrak{p}_{s,k};$$

$${\mathcal{O}}^\times(\Pi_X, S) \subseteq H^1(\Pi_{X\setminus S}, \Lambda_X)$$

for the submodule obtained by forming the inverse image, via the third arrow of the above exact sequence, of the submodule $${\mathcal{P}}(X, S) \subseteq \bigoplus_{s \in S} \mathbb{Z}/\mathfrak{p}_{s,k};$$

$${\text{ev}}_x(\Pi_X, S): {\mathcal{O}}^\times(\Pi_X, S) \longrightarrow H^1(G_k, \Lambda_X)$$

for the restriction to the submodule $${\mathcal{O}}^\times(\Pi_X, S) \subseteq H^1(\Pi_{X\setminus S}, \Lambda_X)$$ of the homomorphism $H^1(\Pi_{X\setminus S}, \Lambda_X) \to H^1(G_k, \Lambda_X)$ determined by the splitting of $\Pi_{X\setminus S} \twoheadrightarrow G_k$ (well-defined up to $\Delta_{X \setminus S}$-conjugation) induced by $x \in X(k) \setminus S$, i.e., $\mathfrak{g}_{X \setminus S}(x);$

$${\mathcal{K}}^\times(\Pi_X) \overset{\text{def}}{=} \lim_{K \to \frac{k}{\mathfrak{p}_K}} {\mathcal{O}}^\times(\Pi_{X \otimes_k K}, T),$$

where the injective limit is taken over the finite extensions $K$ of $k$ contained in $\mathfrak{p}_K$ and the finite subsets $T \subseteq (X \otimes_k K)(K)$. Here, we note that the natural injection $\Pi_{X \otimes_k K} \hookrightarrow \Pi_X$ (well-defined up to $\Delta_X$-conjugation) and the natural surjection $\Pi_{X \otimes_k K} \twoheadrightarrow G_K$ determine an isomorphism $\Pi_{X \otimes_k K} \overset{\sim}{\longrightarrow} \Pi_X \times_{G_k} G_K$ (well-defined up to $(\Delta_X \times \{1\})$-conjugation).

Remark 1.9.1. The module $${\mathcal{O}}^\times(\Pi_X, S)$$ defined in Definition 1.9 is a module that may be

• regarded as an ‘approximation’ of the (scheme-theoretic) module $${\mathcal{O}}^\times_X(X \setminus S)$$ (cf., e.g., Lemma 1.10(ii) below) and

• reconstructed group-theoretically (cf. Lemma 3.3(iv) below).

Moreover, the module $${\mathcal{K}}^\times(\Pi_X)$$ defined in Definition 1.9 is a module that may be

• regarded as an ‘approximation’ of the (scheme-theoretic) module $${\mathcal{K}}^\times_{X \otimes_k K}$$ (cf., e.g., Lemma 1.10(i) below) and

• reconstructed group-theoretically (cf. Lemma 3.3(v) below).
**Lemma 1.10.** Suppose that $k$ is Kummer-faithful, and that $X$ is proper (i.e., $r_X = 0$). Then the following assertions hold.

(i) The middle vertical arrows of the diagram of Lemma 1.8(ii) in the case where we take the ‘$X$’ of Lemma 1.8(ii) to be $(X \otimes_k K) \setminus T$—where $K$ ranges over the finite extensions of $k$ contained in $\overline{k}$ and $T$ ranges over the finite subsets of $(X \otimes_k K)(K)$—determine an injective homomorphism

$$K^\times_{X \otimes_k k} \hookrightarrow K^\times(\Pi_X)$$

(cf. Definition 1.1).

(ii) Let $S \subseteq X(k)$ be a finite subset. Then the diagram of Lemma 1.8(ii) determines a commutative diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{} & k^\times \\
\downarrow & & \downarrow \\
0 & \xrightarrow{} & H^1(G_k, \Lambda_X) & \xrightarrow{} & O^\times(\Pi_X, S) & \xrightarrow{} & 0 \\
\downarrow & & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & O^\times(X \setminus S) & \xrightarrow{} & \bigoplus_{x \in S} \mathrm{ord}_x & \xrightarrow{} & P(X, S) & \xrightarrow{} & 0
\end{array}
$$

where the horizontal sequences are exact, and the vertical arrows are injective.

(iii) Let $S \subseteq X(k)$ be a finite subset and $x \in X(k) \setminus S$. Then the kernel of the homomorphism

$$\mathrm{ev}_x(\Pi_X, S): O^\times(\Pi_X, S) \longrightarrow H^1(G_k, \Lambda_X)$$

coincides, relative to the middle vertical injection of the diagram of (ii), with the subgroup

$$\{ f \in O^\times_X(X \setminus S) \mid f(x) = 1 \}$$

of $O^\times_X(X \setminus S)$; i.e.,

$$\ker(\mathrm{ev}_x(\Pi_X, S)) = \{ f \in O^\times_X(X \setminus S) \mid f(x) = 1 \} \subseteq O^\times_X(X \setminus S).$$

In particular, for every $y \in X(k) \setminus S$, relative to the left-hand vertical injection of the diagram of (ii), it holds that

$$\mathrm{ev}_y(\Pi_X, S)(\ker(\mathrm{ev}_x(\Pi_X, S))) \subseteq k^\times.$$

(iv) Let $x_1, x_2 \in X(k)$ be such that $x_1 \neq x_2$. Then

(a) the subgroups

$$\ker(\mathrm{ev}_{(x_i)K}(\Pi_{X \otimes_k K}, S_i)) \subseteq O^\times(\Pi_{X \otimes_k K}, S_i) \subseteq \mathcal{K}^\times(\Pi_X)$$

—where $i$ ranges over the elements of $\{1, 2\}$, $K$ ranges over the finite extensions of $k$ contained in $\overline{k}$, $S_i$ ranges over the finite subsets of $(X \otimes_k K)(K)$ which do not contain $(x_i)_K$, and we write $(x_i)_K \in (X \otimes_k K)(K)$ for the $K$-rational point determined by $x_i$—

and

(b) the subgroups

$$\mathrm{ev}_{(x_2)_K}(\Pi_{X \otimes_k K}, S)(\ker(\mathrm{ev}_{(x_1)_K}(\Pi_{X \otimes_k K}, S))) \subseteq H^1(G_K, \Lambda_X) \subseteq \mathcal{K}^\times(\Pi_X)$$

—where $K$ ranges over the finite extensions of $k$ contained in $\overline{k}$, $S$ ranges over the finite subsets of $(X \otimes_k K)(K)$ which do not contain $(x_1)_K$ and $(x_2)_K$, and we write $(x_i)_K \in (X \otimes_k K)(K)$ for the $K$-rational point determined by $x_i$—

generate the image of the injection of (i).
Proof. Assertions (i) and (ii) follow immediately from the various definitions involved, together with our assumption that \( k \) is Kummer-faithful. Next, we verify assertion (iii). Let us first observe that one verifies immediately from Lemma 1.8(iii), together with the various definitions involved, that, to complete the verification of assertion (iii), it suffices to verify that \( \text{Ker}(ev_x(\Pi_X, S)) \subseteq O_X^\times(X \setminus S) \). Let \( f \in \text{Ker}(ev_x(\Pi_X, S)) \). Next, let us observe that it follows immediately from assertion (ii) that there exist \( g \in O_X^\times(X \setminus S) \) and \( a \in H^1(G_k, \Lambda_X) \), such that \( f = a \cdot g \). Thus, it holds that

\[
1 = ev_x(\Pi_X, S)(f) = a \cdot ev_x(\Pi_X, S)(g),
\]

which thus implies that \( a = ev_x(\Pi_X, S)(g)^{-1} \in k^\times \) (cf. Lemma 1.8(iii)). In particular, we conclude that \( f = a \cdot g \in O_X^\times(X \setminus S) \). This completes the proof of assertion (iii).

Finally, we verify assertion (iv). Write \( \mathcal{F} \subseteq K^\times(\Pi_X) \) for the subgroup generated by the various subgroups (a), (b) appearing in the statement of assertion (iv) and regard \( K_{X \otimes_k \overline{k}}^\times \) as a subgroup of \( K^\times(\Pi_X) \) by means of the injection of assertion (i). Then let us observe that it follows from assertion (iii) that \( \mathcal{F} \subseteq K_{X \otimes_k \overline{k}}^\times \). Moreover, by considering the subgroups (b), one verifies immediately—in light of Lemma 1.11(i) below—from assertion (iii), together with Lemma 1.8(iii), that

\[
\overline{k}^\times \subseteq \mathcal{F} \subseteq K_{X \otimes_k \overline{k}}^\times.
\]

In particular, by considering the subgroups (a), we conclude from assertion (iii), together with Lemma 1.8(iii), that, for a rational function \( f \in K_{X \otimes_k \overline{k}}^\times \), if \( f((x_i)_{\overline{\tau}}) \notin \{0, \infty\} \) for some \( i \in \{1, 2\} \) (where we write \( (x_i)_{\overline{\tau}} \in (X \otimes_k \overline{k})^\tau(\overline{k}) \) for the \( \overline{k} \)-valued point determined by \( x_i \)), then \( f \in \mathcal{F} \). Thus, the equality \( \mathcal{F} = K_{X \otimes_k \overline{k}}^\times \) follows immediately from Lemma 1.11(ii) below. This completes the proof of assertion (iv).

\[
\text{LEMMA 1.11.} \quad \text{Let} \ \Omega \ \text{be an algebraically closed field,} \ C \ \text{a proper hyperbolic curve over} \ \Omega, \ \text{and} \ x, y \in C(\Omega) \ \text{distinct} \ \Omega-\text{valued points of} \ C. \ \text{Then the following assertions hold.}
\]

\begin{enumerate}
\item \quad \text{For every} \ \lambda \in \Omega \setminus \{0, 1\} \ \text{there exists a rational function} \ f \in K_C^\times \ \text{(cf. Definition 1.1)} \ \text{such that} \ f(x) = 1 \ \text{and} \ f(y) = \lambda.
\item \quad \text{The multiplicative group} \ K_C^\times \ \text{is generated by rational functions} \ f \in K_C^\times \ \text{such that} \ f((x, y)) \notin \{0, \infty\}.
\end{enumerate}

\[
\text{Proof.} \ \text{Assertion (i) follows immediately by considering, for instance, a suitable linear fractional transformation} \ (ag + b)/(cg + d) \ \text{(where} \ a, b, c, d \in \Omega) \ \text{of a rational function} \ g \in K_C^\times \ \text{such that} \ g(x) \neq g(y). \ \text{Next, we verify assertion (ii). Write} \ \mathcal{F} \subseteq K_C^\times \ \text{for the subgroup of} \ K_C^\times \ \text{generated by rational functions} \ f \in K_C^\times \ \text{such that} \ f((x, y)) \notin \{0, \infty\}. \ \text{To complete the verification of the equality} \ \mathcal{F} = K_C^\times, \ \text{let us take a rational function} \ g \in K_C^\times \ \text{such that} \ g((x, y)) \subseteq \{0, \infty\}. \ \text{Now, to verify} \ g \in \mathcal{F}, \ \text{we may assume without loss of generality, by replacing} \ g \ \text{by} \ g^{-1} \ \text{if necessary, that} \ g(x) = \infty, \ \text{i.e., that} \ \text{ord}_x(g) < 0. \ \text{Then one verifies immediately from the Riemann–Roch theorem that there exists a rational function} \ h \in K_C^\times \ \text{such that} \ \text{ord}_x(g) = \text{ord}_x(h)(= \text{ord}_x(h + 1)) \ \text{and} \ h(y) = 0 \ \text{(i.e.,} \ (h + 1)(y) = 1). \ \text{Thus, since} \ g/(h + 1), \ h + 1 \in \mathcal{F}, \ \text{we conclude that} \ g \in \mathcal{F}. \ \text{This completes the proof of assertion (ii).} \ \square
\]
Definition 1.12. We shall write
\[ \Delta_X^{c-ab} \] (respectively, \( \Delta_X^{c-cn} \))
for the maximal quotient of \( \Delta_X \) such that the natural surjection \( \Delta_X \to \pi_1(X^{cpt} \otimes_k \overline{k}) \) factors through the surjection \( \Delta_X \to \Delta_X^{c-ab} \) (respectively, \( \Delta_X^{c-cn} \)), and, moreover, the kernel of the resulting surjection \( \Delta_X^{c-ab} \) (respectively, \( \Delta_X^{c-cn} \)) is pro-\( \mathbf{Primes} \times /k \) and abelian (respectively, pro-\( \mathbf{Primes} \times /k \) and contained in the center of \( \Delta_X^{c-cn} \)). We shall write
\[ \Pi_X^{c-ab} \] (respectively, \( \Pi_X^{c-cn} \))
for the quotient of \( \Pi_X \) by the kernel of \( \Delta_X \to \Delta_X^{c-ab} \) (respectively, \( \Delta_X^{c-cn} \)). Thus, we have a commutative diagram of profinite groups
\[
\begin{array}{ccc}
1 & \rightarrow & \Delta_X \\
\downarrow & & \downarrow \\
\Delta_X^{c-ab} & \rightarrow & \Pi_X^{c-ab} \\
\downarrow & & \downarrow \\
\Delta_X^{c-cn} & \rightarrow & \Pi_X^{c-cn} \\
\downarrow & & \downarrow \\
1 & \rightarrow & G_k \\
\end{array}
\]
where the horizontal sequences are exact and the vertical arrows are surjective.

Remark 1.12.1. The extension \( \Pi_X^{c-cn} \), as well as \( \Pi_X^{c-ab} \), of \( \pi_1(X^{cpt}) \) defined in Definition 1.12 is an extension that may be
- regarded as an ‘approximation’ of the fundamental group \( \Pi_X \) (cf., e.g., Lemma 1.13(i) below) and
- reconstructed group-theoretically (cf. Lemma 3.3(iii) below).

Lemma 1.13. The following assertions hold.
(i) The natural surjections
\[ \Pi_X \to \Pi_X^{c-ab} \to \Pi_X^{c-cn} \]
determine isomorphisms
\[ H^1(\Pi_X^{c-cn}, \Delta_X) \xrightarrow{\sim} H^1(\Pi_X^{c-ab}, \Delta_X) \xrightarrow{\sim} H^1(\Pi_X, \Delta_X). \]
(ii) Suppose that \( X \) is proper (i.e., \( r_X = 0 \)). Let \( x_1, \ldots, x_n \in X(k) \) be distinct \( k \)-rational points. For \( i \in \{1, \ldots, n\} \), write \( U_i \equiv X \setminus \{x_i\} \) and \( U \equiv X \setminus \{x_1, \ldots, x_n\} \). Then the natural open immersions \( U \hookrightarrow U_i \) (where \( i \) ranges over the elements of \( \{1, \ldots, n\} \))—
determine an isomorphism of profinite groups (well-defined up to \( \Delta_{U_n}^{c-cn} \times \Delta_X \cdots \times \Delta_X \Delta_{U_n}^{c-cn} \) conjugation)
\[ \Pi_{U_n}^{c-cn} \xrightarrow{\sim} \Pi_{U_1}^{c-cn} \times \Pi_X \cdots \times \Pi_X \Pi_{U_n}^{c-cn}. \]

Proof. Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from a similar argument to the argument applied in the proof of the final portion of [4, Proposition 1.6(iii)]. \( \square \)
2. Maximal cuspidally abelian quotients

In this section, we discuss the maximal cuspidally abelian quotients $\Pi_{X \times_k X}^{\text{c-ab}}$ (cf. Definition 2.1(ii) below) of the étale fundamental groups of the second configuration spaces of proper hyperbolic curves. In this section, we maintain the notation of the preceding Section 1. Suppose, moreover, that $X$ is proper over $k$ (i.e., $r_X = 0$). Write

$$\Pi_{X \times_k X}^\text{def} = \pi_1(X \times_k X) \geq \Delta_{X \times_k X}^\text{def} = \pi_1((X \times_k X) \otimes_k \overline{k})$$

for the respective étale fundamental groups of $X \times_k X$, $(X \times_k X) \otimes_k \overline{k}$ (for some choice of basepoints). Then let us recall that the two projections $X \times_k X \to X$ determine an isomorphism of profinite groups

$$\Pi_{X \times_k X} \sim \to \Pi_X \times G_k \Pi_X,$$

which restricts to an isomorphism of profinite groups

$$\Delta_{X \times_k X} \sim \to \Delta_X \times \Delta_X.$$ 

Let $N$ be a $\mathfrak{Primes}x/k$-integer. Write

$$\Lambda_{X,N}^\text{def} = \Lambda_X / N \Lambda_X = \text{Hom}_{\widehat{\mathbb{Z}}}^\wedge(\Delta_X, \widehat{\mathbb{Z}}/N\mathbb{Z})$$

(cf. Definition 1.4; Remark 1.4.1(ii)).

Definition 2.1. (i) We shall write $U_{X \times_k X} \subseteq X \times_k X$ for the second configuration space of $X$, i.e., the open subscheme of $X \times_k X$ obtained by forming the complement of the diagonal divisor $X \subseteq X \times_k X$. Thus, the natural inclusion $U_{X \times_k X} \hookrightarrow X \times_k X$ determines a commutative diagram of profinite groups

$$\begin{array}{cccccc}
1 & \longrightarrow & \pi_1((U_{X \times_k X}) \otimes_k \overline{k}) & \longrightarrow & \pi_1(U_{X \times_k X}) & \longrightarrow & G_k & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta_{X \times_k X} & \longrightarrow & \Pi_{X \times_k X} & \longrightarrow & G_k & \longrightarrow & 1,
\end{array}$$

where the horizontal sequences are exact, and the vertical arrows are surjective.

(ii) We shall write $\Delta_{U_{X \times_k X}}^{c,(\times/k)}$ (respectively, $\Delta_{U_{X \times_k X}}^{c-ab}$, $\Delta_{U_{X \times_k X}}^{c-cn}$) for the maximal quotient of $\pi_1((U_{X \times_k X}) \otimes_k \overline{k})$ such that the left-hand vertical arrow of the diagram of (i) factors through the surjection $\pi_1((U_{X \times_k X}) \otimes_k \overline{k}) \to \Delta_{U_{X \times_k X}}^{c,(\times/k)}$ (respectively, $\Delta_{U_{X \times_k X}}^{c-ab}$, $\Delta_{U_{X \times_k X}}^{c-cn}$), and, moreover, the kernel of the resulting surjection $\Delta_{U_{X \times_k X}}^{c,(\times/k)}$ (respectively, $\Delta_{U_{X \times_k X}}^{c-ab}$, $\Delta_{U_{X \times_k X}}^{c-cn}$) $\to \Delta_{X \times_k X}$ is pro-$\mathfrak{Primes}x/k$ (respectively, pro-$\mathfrak{Primes}x/k$ and abelian; pro-$\mathfrak{Primes}x/k$ and contained in the center of $\Delta_{U_{X \times_k X}}^{c-cn}$). We shall write $\Pi_{U_{X \times_k X}}^{c,(\times/k)}$ (respectively, $\Pi_{U_{X \times_k X}}^{c-ab}$, $\Pi_{U_{X \times_k X}}^{c-cn}$).
for the quotient of $\pi_1(U_{X \times k})$ by the kernel of $\pi_1((U_{X \times k}) \otimes_k \mathbb{T}) \to \Delta^{c,(x/k)}_{U_{X \times k}}$ (respectively, $\Delta^{c-ab}_{U_{X \times k}}, \Delta^{c-cn}_{U_{X \times k}}$). Thus, the diagram of (i) determines a commutative diagram of profinite groups

\begin{align*}
1 & \longrightarrow \Delta^{c,(x/k)}_{U_{X \times k}} \longrightarrow \Pi^{c,(x/k)}_{U_{X \times k}} \longrightarrow G_k \longrightarrow 1 \\
1 & \longrightarrow \Delta^{c-ab}_{U_{X \times k}} \longrightarrow \Pi^{c-ab}_{U_{X \times k}} \longrightarrow G_k \longrightarrow 1 \\
1 & \longrightarrow \Delta^{c-cn}_{U_{X \times k}} \longrightarrow \Pi^{c-cn}_{U_{X \times k}} \longrightarrow G_k \longrightarrow 1 \\
1 & \longrightarrow \Delta_{X \times k} \longrightarrow \Pi_{X \times k} \longrightarrow G_k \longrightarrow 1,
\end{align*}

where the horizontal sequences are exact and the vertical arrows are surjective.

**Lemma 2.2.** Let $x \in X(k)$ be a $k$-rational point. Write $U \overset{\text{def}}{=} X \setminus \{x\}$. Then the splitting (well-defined up to $\Delta_X$-conjugation) $s: G_k \to \Pi_X$ induced by $x$, i.e., $\langle \Sigma_X(x) \rangle$, determines an isomorphism of profinite groups over $G_k$:

$$\Pi^{c-cn}_{U} \cong \Pi^{c-cn}_{U_{X \times k}} \times \Pi_{X \times k} (s(G_k) \times G_k \Pi_X).$$

**Proof.** This follows immediately from a similar argument to the argument applied in [4, Proof of Proposition 1.6(iii), (iv)]. \qed

**Lemma 2.3.** Suppose that $\Pi_X$ is slim and that $k$ is $p$-cyclotomically full for every $p \in \mathbb{P}_{\text{Primes}}^{\times/k}$ (i.e., the image of the $p$-adic cyclotomic character of $G_k$ is open in $\mathbb{Z}_p^\times$ for every $p \in \mathbb{P}_{\text{Primes}}^{\times/k}$). Then an extension of an open subgroup of either $\Pi_X$ or $\Pi_{X \times k}$ by a (possibly empty) finite product of copies of $\Delta_X$ is slim.

**Proof.** Let us first observe that since the profinite group $\Pi_{X \times k} \twoheadrightarrow \Pi_X \times G_k$ $\Pi_X$ has a structure of extension of $\Pi_X$ by $\Delta_X$, and $\Delta_X$ is slim, it follows that $\Pi_{X \times k}$ is slim. In particular, since $k$ is $p$-cyclotomically full for every $p \in \mathbb{P}_{\text{Primes}}^{\times/k}$, it follows immediately from Remark 1.4.1(i) that an extension of an open subgroup of either $\Pi_X$ or $\Pi_{X \times k}$ by a finite product of copies of $\Delta_X$ is slim. This completes the proof of Lemma 2.3. \qed

**Remark 2.3.1.** It follows immediately from Proposition 1.5(ii), together with a similar argument to the argument given in [6, Remark 1.5.1], that if $k$ is Kummer-faithful, then the two assumptions in the statement of Lemma 2.3 are satisfied. In particular, in this situation, it follows from Lemma 2.3 that an extension of an open subgroup of either $\Pi_X$ or $\Pi_{X \times k}$ by a (possibly empty) finite product of copies of $\Delta_X$ is slim.

**Definition 2.4.** Let $\Delta \subseteq \Delta_X$ be a characteristic open subgroup of $\Delta_X$ and let $\Pi_Y \subseteq \Pi_X$ be an open subgroup of $\Pi_X$ such that $\Delta_Y = \Pi_Y \cap \Delta_X = \Delta$. Write $G_{kY} \subseteq G_k$ for the image of the composite $\Pi_Y \hookrightarrow \Pi_X \twoheadrightarrow G_k$. (Thus, the connected finite étale covering $Y \rightarrow X$
(corresponding to \( \Pi_Y \subseteq \Pi_X \)) is a hyperbolic curve over the finite extension \( k_Y \) of \( k \) (corresponding to \( G_{k_Y} \subseteq G_k \)).

(i) By conjugation, we obtain an action \( \Pi_X \to \text{Aut}(\Delta) \), and hence also a semi-direct product \( \Delta \rtimes \Pi_X \), which fits into an exact sequence of profinite groups

\[
1 \longrightarrow \Delta \rtimes \Pi_X \longrightarrow G_k \longrightarrow 1.
\]

Observe that since \( \Delta_X \) is slim, it follows that \( \Delta \rtimes \Delta_X \) is slim.

(ii) By restricting the action \( \Pi_X \to \text{Aut}(\Delta) \) of (i) to the closed subgroups \( \Delta_Y \subseteq \Pi_Y \subseteq \Pi_X \), we obtain semi-direct products \( \Delta \rtimes \Delta_Y \subseteq \Delta \rtimes \Pi_Y \subseteq \Delta \rtimes \Pi_X \). Now observe that the resulting action \( \Delta_Y \to \text{Aut}(\Delta) \) is, under the natural identification \( \Delta = \Delta_Y \), just the conjugation action of \( \Delta \) on itself. Thus, one verifies easily from an explicit computation (i.e., by considering the subgroup of \( \Delta \rtimes \Delta_Y \) consisting of elements of the form \( (g^{-1}, g) \)) that the composite \( Z_{\Delta \rtimes \Delta_Y} \to \Delta \rtimes \Delta_Y \to \Delta_Y \) is surjective. In particular, it follows from the fact that \( \Delta \) is center-free that the centralizer \( Z_{\Delta \rtimes \Delta_Y} \subseteq \Delta \rtimes \Delta_Y \) determines a splitting of the exact sequence of profinite groups

\[
1 \longrightarrow \Delta \longrightarrow \Delta \rtimes \Delta_Y \longrightarrow \Delta_Y \longrightarrow 1.
\]

By this observation, we conclude that the two natural surjections \( \Delta \rtimes \Delta_Y \to (\Delta \rtimes \Delta_Y)/Z_{\Delta \rtimes \Delta_Y} (\Delta) \) and \( \Delta \rtimes \Delta_Y \to \Delta_Y \) determine isomorphisms of profinite groups

\[
\Delta \rtimes \Delta_Y \sim \to \left((\Delta \rtimes \Delta_Y)/Z_{\Delta \rtimes \Delta_Y} (\Delta)\right) \times \Delta_Y \sim \to \Delta \rtimes \Delta_Y.
\]

Note that one verifies easily from an explicit computation that the composite of these isomorphisms maps bijectively the image of the natural splitting of the surjection \( \Delta \rtimes \Delta_Y \to \Delta_Y \) to the ‘diagonal’ subgroup of \( \Delta \rtimes \Delta_Y \).

(iii) Next, let us observe that one verifies easily from the construction that the isomorphisms of the final display of (ii) are compatible with the natural outer actions of \( G_{k_Y} \).

In particular, by considering \( (-) \rtimes G_{k_Y} \) (cf. the slimness of \( \Delta \rtimes \Delta_X \) discussed in (i))—i.e., applying the final discussion of Section 0 to the natural outer actions \( G_{k_Y} \to \text{Out}(\Delta \rtimes \Delta_Y) \) and \( G_{k_Y} \to \text{Out}(\Delta \rtimes \Delta_Y) \)—we obtain an isomorphism of profinite groups

\[
\Delta \rtimes \Pi_Y = (\Delta \rtimes \Delta_Y) \rtimes G_{k_Y} \sim \to (\Delta \rtimes \Delta_Y) \rtimes G_{k_Y} = \Pi_Y \rtimes G_{k_Y} \Pi_Y \sim \to (\Delta \rtimes \Delta_Y) \rtimes G_{k_Y} \Pi_Y.
\]

Observe that one verifies immediately from the final discussion of (ii) that the natural splitting of the surjection \( \Pi_Y \times_k Y \to \Delta \rtimes \Pi_Y \to \Pi_Y \) arises from the diagonal morphism \( Y \to Y \times_{k_Y} Y \).

(iv) We shall write \( Z_{\Delta} \to X \times_k X \) for the connected finite étale covering corresponding to the open subgroup

\[
\Pi_{Z_{\Delta}} \overset{\text{def}}{=} \Delta \rtimes \Pi_X \subseteq \Delta_X \rtimes \Pi_X \to \Pi_X \times_{k_X} X,
\]

where the ‘\( \sim \)’ is the isomorphism obtained in (iii) in the case where we take ‘\( \Delta \)’ to be \( \Delta_X \).

Thus, the exact sequence of (i) determines an exact sequence of profinite groups

\[
1 \longrightarrow \Delta_{Z_{\Delta}} \overset{\text{def}}{=} \text{Ker}(\Pi_{Z_{\Delta}} \to G_k) \longrightarrow \Pi_{Z_{\Delta}} \longrightarrow G_k \longrightarrow 1.
\]
Observe that one verifies immediately from the various definitions involved that the surjection \( \Pi_{Z_\Delta} = \Delta \times \Pi_X \to \Pi_X \) arises from an ‘isotrivial’ (cf. (iii)) hyperbolic curve \( Z_\Delta \to X \) over \( X \), and the natural splitting of the surjection \( \Pi_{Z_\Delta} = \Delta \times \Pi_X \to \Pi_X \) arises from a section

\[ \iota_\Delta : X \hookrightarrow Z_\Delta \]

—that lies over the diagonal morphism \( X \hookrightarrow X \times_k X \) (cf. (iii))—of this hyperbolic curve \( Z_\Delta \to X \).

**Remark 2.4.1.** One verifies easily from the various definitions involved that, in the notation of Definition 2.4, if \( k \) is either an MLF or FF, then the finite étale covering \( Z_\Delta \to X \) of Definition 2.4(iv) is the diagonal covering associated to the covering \( Y \to X \) in the sense of [4, Definition 1.2(i)].

**Lemma 2.5.** In the notation of Definition 2.4, the following assertions hold.

(i) Let \( i \neq 0 \) be an integer, \( G \in \{ \Delta, \Pi_{Z_\Delta} \} \), and let \( A \) be a finite \( G \)-module annihilated by a \( \mathfrak{P} \times \mathbb{Z}/k \)-integer. Then

\[
\lim_{\rightarrow H} H^i(H, A) = \{0\},
\]

where the injective limit is taken over the open subgroups \( H \subseteq G \), and the transition morphisms in the limit are given by the restriction maps.

(ii) Let \( i \neq 2 \) be an integer and let \( A \) be a finite module equipped with the trivial action of \( \Delta \) that is annihilated by a \( \mathfrak{P} \times \mathbb{Z}/k \)-integer. Then

\[
\lim_{\rightarrow H} H^i(H, A) = \{0\},
\]

where the projective limit is taken over the open subgroups \( H \subseteq \Delta \), and the transition morphisms in the limit are given by the corestriction maps.

(iii) Let \( i \) be an integer and let \( A \) be a finite \( \Pi_{Z_\Delta} \)-module annihilated by a \( \mathfrak{P} \times \mathbb{Z}/k \)-integer. Then the natural homomorphism

\[
H^i(\Pi_{Z_\Delta}, A) \to H^i(Z_\Delta, A)
\]

is an isomorphism.

**Proof.** Assertion (i) follows immediately from a similar argument to the argument applied in [1, Proof of Lemma 4.2(iii)]. Next, we verify assertion (ii). Let us recall (cf., e.g., [4, Proposition 1.3(ii)]) that the homomorphism

\[
H^i(H, A) \to \text{Hom}_{\mathbb{Z}/k}(H^2-i(H, \Lambda_X), A)
\]

determined by the cup product in group cohomology and the natural isomorphism of \( \Lambda_X \) with ‘\( \Lambda_X \)’ with respect to \( H \) (cf., e.g., [4, Remark 1]) is an isomorphism. Thus, assertion (ii) follows immediately from assertion (i). This completes the proof of assertion (ii). Assertion (iii) is a formal consequence of assertion (i) (cf., e.g., [1, Proof of Lemma 4.2(iii)]). This completes the proof of Lemma 2.5.

**Lemma 2.6.** In the notation of Definition 2.4, write

\[
E^{i,j}_2(\Delta) = H^i(\Pi_X, H^j(\Delta, \Lambda_{X,N})) \Rightarrow E^{i+j}(\Delta) = H^{i+j}(\Pi_{Z_\Delta}, \Lambda_{X,N})
\]
for the spectral sequence associated to the exact sequence of profinite groups
\[
1 \rightarrow \Delta \rightarrow \Pi_{Z_{\Delta}} (= \Delta \times \Pi_X) \rightarrow \Pi_X \rightarrow 1.
\]

Then the following assertions hold.

(i) The natural homomorphism (i.e., the homomorphism which arises from the definition of \(\lim\)
\[
\lim_{\Delta^t} E_0^{0,2}(\Delta^t) \rightarrow E_2^{0,2}(\Delta) \quad (= H^0(\Pi_X, H^2(\Delta, \Lambda_{X,N})) = \mathbb{Z}/N\mathbb{Z})
\]
—where the projective limit is taken over the characteristic open subgroups \(\Delta^t \subseteq \Delta_X\) contained in \(\Delta\), and the transition morphisms in the limit are given by the corestriction maps—is an isomorphism.

(ii) The natural homomorphism (i.e., the homomorphism determined by the various natural inclusions \(\Delta^t \hookrightarrow \Pi_{Z_{\Delta^t}}\)
\[
\lim_{\Delta^t} E^2(\Delta^t) \rightarrow \lim_{\Delta^t} E_0^{0,2}(\Delta^t)
\]
—where the projective limits are taken over the characteristic open subgroups \(\Delta^t \subseteq \Delta_X\) contained in \(\Delta\), and the transition morphisms in the limits are given by the corestriction maps—is an isomorphism.

(iii) The image of \(1 \in \mathbb{Z}/N\mathbb{Z}\) via the composite
\[
\mathbb{Z}/N\mathbb{Z} = H^0(\Pi_X, H^2(\Delta, \Lambda_{X,N})) = E_2^{0,2}(\Delta) \xrightarrow{\sim} \lim_{\Delta^t} E_2^{0,2}(\Delta^t)
\]
\[
\xleftarrow{\sim} \lim_{\Delta^t} E^2(\Delta^t) \rightarrow E^2(\Delta) = H^2(\Pi_{Z_{\Delta}}, \Lambda_{X,N}) \xrightarrow{\sim} H^2(Z_{\Delta}, \Lambda_{X,N})
\]
—where the first \(\xrightarrow{\sim}\) is the isomorphism of (i), the second \(\xleftarrow{\sim}\) is the isomorphism of (ii), and the \(\xrightarrow{\sim}\) is the isomorphism of Lemma 2.5(iii)—coincides with the first Chern class \(c_1(\iota_{\Delta}/\Delta^t)\) of the divisor \(\iota_{\Delta}/\Delta^t \subseteq Z_{\Delta}\) obtained by forming the scheme-theoretic image of the section \(\iota_{\Delta} : X \hookrightarrow Z_{\Delta}\) of the hyperbolic curve \(Z_{\Delta} \rightarrow X\) of Definition 2.4(iv).

Proof. First, we verify assertion (i). Let us recall (cf., e.g., [4, Proposition 1.3(ii)]) that the homomorphism
\[
(E_2^{0,2}(\Delta^t)) = H^2(\Delta^t, \Lambda_{X,N}) \rightarrow \text{Hom}_{\mathbb{Z}_{/k}}(H^0(\Delta^t, \Lambda_X), \Lambda_{X,N}) \quad (= \mathbb{Z}/N\mathbb{Z})
\]
determined by the cup product in group cohomology and the natural isomorphism of \(\Lambda_X\) with \(\Lambda_X^t\) with respect to \(\Delta^t\) (cf., e.g., [4, Remark 1]) is an isomorphism. Thus, assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from Lemma 2.5(ii). Assertion (iii) follows immediately from the (easily verified, i.e., by the consideration of the degrees) fact that the image of the compatible system
\[
(c_1(\iota_{\Delta^t}(X)))_{\Delta^t} \in \lim_{\Delta^t} E^2(\Delta^t)
\]
(cf. Lemma 2.5(iii)) via the composite of natural homomorphisms
\[
\lim_{\Delta^t} E^2(\Delta^t) \rightarrow \lim_{\Delta^t} E_0^{0,2}(\Delta^t) \rightarrow E_0^{0,2}(\Delta) = \mathbb{Z}/N\mathbb{Z}
\]
coincides with \(1 \in \mathbb{Z}/N\mathbb{Z}\). This completes the proof of Lemma 2.6. \(\square\)
**Definition 2.7.** We use the notation of Lemma 2.6.

(i) We shall refer to an extension of $\Pi_{Z_{\Delta}}$ by $\Lambda_{X,N}$

$$1 \to \Lambda_{X,N} \to E_{\Delta,N} \to \Pi_{Z_{\Delta}} \to 1$$

whose associated extension class is given by the image of $1 \in \mathbb{Z}/N\mathbb{Z}$ via the composite

$$\mathbb{Z}/N\mathbb{Z} = H^0(\Pi_X, H^2(\Delta, \Lambda_{X,N})) = E_2^{0,2}(\Delta) \sim \lim_{\Delta^1} E_2^{0,2}(\Delta^1)$$

$$\sim \lim_{\Delta^1} E^2(\Delta^1) \to E^2(\Delta) = H^2(\Pi_{Z_{\Delta}}, \Lambda_{X,N})$$

—where the first `$\sim$' is the isomorphism of Lemma 2.6(i), and the second `$\sim$' is the isomorphism of Lemma 2.6(ii)—as a mod $N$ fundamental extension of $\Pi_{Z_{\Delta}}$.

(ii) We shall refer to the extension of $\Pi_{Z_{\Delta}}$ by $\Lambda_{X}$

$$1 \to \Lambda_{X} \to E_{\Delta} \overset{def}{=} \lim_{N} E_{\Delta,N} \to \Pi_{Z_{\Delta}} \to 1$$

—where the projective limit is taken over the $\mathcal{P}\mathfrak{rimes}^{x/k}$-integers $N$—obtained by forming the projective limit of a compatible system of mod $N$ fundamental extensions $\{E_{\Delta,N}\}_N$ (cf. (i)) as a fundamental extension of $\Pi_{Z_{\Delta}}$. (Here, let us observe that the natural surjection $\Pi_{Z_{\Delta}} \to G_k$, together with the Kummer theory, determines an exact sequence

$$1 \to k^x/(k^x)^N \to H^1(\Pi_{Z_{\Delta}}, \Lambda_{X,N}) \to H^1(\Delta_{Z_{\Delta}}, \Lambda_{X,N});$$

moreover, one verifies easily that $H^1(\Delta_{Z_{\Delta}}, \Lambda_{X,N})$ is finite—cf. Lemma 1.6(v). In particular, for every $N$, there exists a $\mathcal{P}\mathfrak{rimes}^{x/k}$-integer $N_0$ which is divisible by $N$ such that, if $N'$ is a $\mathcal{P}\mathfrak{rimes}^{x/k}$-integer which is divisible by $N_0$, then the image of the homomorphism $H^1(\Pi_{Z_{\Delta}}, \Lambda_{X,N'}) \to H^1(\Pi_{Z_{\Delta}}, \Lambda_{X,N})$ coincides with the image of the homomorphism $H^1(\Pi_{Z_{\Delta}}, \Lambda_{X,N_0}) \to H^1(\Pi_{Z_{\Delta}}, \Lambda_{X,N})$.

**Remark 2.7.1.** One verifies easily from Lemma 2.6(iii), together with the various definitions involved, that, in the notation of Definition 2.7, if $k$ is either an MLF or FF, then the notion of a fundamental extension of $\Pi_{Z_{\Delta}}$ defined in Definition 2.7(ii) coincides with the notion of a fundamental extension of $\Pi_{Z_{\Delta}}$ defined in [4, Definition 1.2(i)] (cf. also Remark 2.4.1).

**Remark 2.7.2.** As already observed in Section 1, similar arguments to the arguments which lead us to various assertions discussed in order to prove [4, Corollary 2.2]—i.e., a result of the case of a hyperbolic curve over either an MLF or FF—in fact still do work in the generalization to the case of a hyperbolic curve over an arbitrary Kummer-faithful field. Some of the highly non-trivial portions in a task of establishing the generalization to the case of a hyperbolic curve over an arbitrary Kummer-faithful field consist of portions related to the group-theoretic reconstruction of fundamental extensions. In [4, Section 1], a fundamental extension may be reconstructed (cf. [4, Proposition 1.5(i), (ii)]) in a fashion that depends on the arithmetic of an MLF and FF, i.e., by applying the theory of Galois cohomology of an MLF and FF (cf. [4, Propositions 1.2, 1.3]). In the present paper, in order to establish a group-theoretic reconstruction of fundamental extensions for Kummer-faithful fields, we have considered a certain projective system of cohomology modules (cf. Lemma 2.6).
Definition 2.8. Suppose that $k$ is Kummer-faithful. Let

$$\Delta^\dagger \subseteq \Delta^\ddagger \subseteq \Delta \subseteq \Delta_X$$

be characteristic open subgroups of $\Delta_X$; let

$$1 \rightarrow \Lambda_X \rightarrow E_{\Delta^\dagger} \rightarrow \Pi_{Z_{\Delta^\dagger}} \rightarrow 1,$$
$$1 \rightarrow \Lambda_X \rightarrow E_{\Delta^\ddagger} \rightarrow \Pi_{Z_{\Delta^\ddagger}} \rightarrow 1$$

be respective fundamental extensions of $\Pi_{Z_{\Delta^\dagger}}$, $\Pi_{Z_{\Delta^\ddagger}}$; and let $s : G_k \rightarrow \Pi_X$ be a splitting of the natural surjection $\Pi_X \rightarrow G_k$. Write

$$Y^\dagger \rightarrow Y^\ddagger \rightarrow Y \rightarrow X$$

for the connected finite étale coverings corresponding to the open subgroups

$$\Pi_{Y^\dagger} \overset{\text{def}}{=} \Delta^\dagger \cdot \text{Im}(s) \subseteq \Pi_{Y^\ddagger} \overset{\text{def}}{=} \Delta^\ddagger \cdot \text{Im}(s) \subseteq \Pi_Y \overset{\text{def}}{=} \Delta \cdot \text{Im}(s) \subseteq \Pi_X.$$

(i) By similar procedures to the procedures given in the discussion following [4, Proposition 1.6] in the case where we take $(X'' \rightarrow X' \rightarrow X^* \rightarrow X, D')$ in the discussion following [4, Proposition 1.6] to be $(Y^\dagger \rightarrow Y^\ddagger \rightarrow Y \rightarrow X, E_{\Delta^\dagger})$ (cf. Remarks 2.4.1, 2.7.1), together with Remark 2.3.1, one may define extensions

$$1 \rightarrow \prod_{\Delta^\dagger} \Lambda_X \rightarrow S_{Y^\dagger/Y}(E_{\Delta^\dagger}) \rightarrow \Pi_{Z_{\Delta^\dagger}} \rightarrow 1,$$
$$1 \rightarrow \prod_{\Delta^\ddagger} \Lambda_X \rightarrow \text{Tr}_{Y^\dagger/Y^\ddagger}(E_{\Delta^\ddagger}) \rightarrow \Pi_{Z_{\Delta^\ddagger}} \rightarrow 1$$

in which $\Pi_{Z_{\Delta^\dagger}}$ is only determined up to $\Delta^\dagger \times \{1\}$-inner automorphisms. We shall refer to $S_{Y^\dagger/Y}(E_{\Delta^\dagger})$ as the $[Y^\dagger/Y]$-symmetrized fundamental extension (cf. [4, Definition 1.3(i)]) and to $\text{Tr}_{Y^\dagger/Y^\ddagger}(E_{\Delta^\ddagger})$ as the $[Y^\dagger/Y^\ddagger]$-trace-symmetrized fundamental extension (cf. [4, Definition 1.3(ii)]).

(ii) By a similar argument to the argument given in [4, Definition 1.3(ii)], together with similar results to the results given in [4, Proposition 1.7(i), (ii)], one may define the notion of a morphism of trace type

$$S_{Y^\dagger/X}(E_{\Delta^\dagger}) \rightarrow (S_{Y^\dagger/X}(\text{Tr}_{Y^\dagger/Y^\ddagger}(E_{\Delta^\ddagger})) \rightarrow) \ S_{Y^\dagger/X}(E_{\Delta^\ddagger}).$$

Definition 2.9. Suppose that $k$ is Kummer-faithful. Let $s : G_k \rightarrow \Pi_X$ be a splitting of the natural surjection $\Pi_X \rightarrow G_k$ and let

$$\cdots \subseteq \Delta_i \subseteq \cdots \subseteq \Delta_j \subseteq \cdots \subseteq \Delta_0 = \Delta_X$$

be a system of characteristic open subgroups of $\Delta_X$ indexed by the non-negative integers such that

$$\bigcap_{i \geq 0} \Delta_i = \{1\}.$$  

(Note that since $\Delta_X$ is topologically finitely generated—by Lemma 1.6(v)—such a system always exists.) Write

$$\cdots \rightarrow Y_i \rightarrow \cdots \rightarrow Y_j \rightarrow \cdots \rightarrow Y_0 = X$$
for the connected finite étale coverings corresponding to the open subgroups
\[ \cdots \subseteq \Pi Y_i \overset{\text{def}}{=} \Delta_i \cdot \text{Im}(s) \subseteq \cdots \subseteq \Pi Y_j \overset{\text{def}}{=} \Delta_j \cdot \text{Im}(s) \subseteq \cdots \subseteq \Pi X. \]

(i) We shall refer to a compatible system of morphisms of trace type, up to inner automorphisms of the appropriate type, between symmetrized fundamental extensions
\[ \cdots \rightarrow S_i \rightarrow \cdots \rightarrow S_j \rightarrow \cdots \rightarrow S_0 \rightarrow \Pi_{X \times_k X} \]
—where \( S_i \) is the \( Y_i/X \)-symmetrized fundamental extension—as a pro-symmetrized fundamental extension of \( \Pi_{X \times_k X} \) (cf. [4, Definition 1.3(iii)]). In this situation, we shall refer to the profinite group
\[ S_\infty \overset{\text{def}}{=} \lim_{i \geq 0} S_i \rightarrow \Pi_{X \times_k X} \]
as a pro-fundamental extension of \( \Pi_{X \times_k X} \) (cf. [4, Definition 1.3(iii)]).

(ii) Let \( S_\infty \) be a pro-fundamental extension of \( \Pi_{X \times_k X} \) (cf. (i)). Then, by a similar argument to the argument given in [4, Definition 1.4(ii)], together with a similar result to the result given in [4, Proposition 1.8(i)], one may define the notion of a pro-fundamental section
\[ \Pi_{U_{X \times_k X}} \rightarrow S_\infty. \]

**Proposition 2.10.** Suppose that \( k \) is Kummer-faithful, and that the natural surjection \( \Pi_X \rightarrow G_k \) has a splitting. Then the following assertions hold.

(i) Let \( S_\infty \) be a pro-fundamental extension of \( \Pi_{X \times_k X} \). Then a pro-fundamental section \( \Pi_{U_{X \times_k X}} \rightarrow S_\infty \) determines isomorphisms of profinite groups
\[ \Pi_{c-ab}^{U_{X \times_k X}} \sim \rightarrow S_\infty, \quad \Pi_{c-cn}^{U_{X \times_k X}} \sim \rightarrow S_0, \]
where we write \( S_0 \) for the \( X/X \)-symmetrized fundamental extension (i.e., a fundamental extension of \( \Pi_{Z_{\Delta X}} = \Pi_{X \times_k X} \)) appearing in the pro-symmetrized fundamental extension of \( \Pi_{X \times_k X} \) that determines \( S_\infty \).

(ii) Let \( \mathcal{D}_X \subseteq \Pi_{c-ab}^{U_{X \times_k X}} \) be a decomposition subgroup associated with the diagonal divisor \( X \subseteq X \times_k X \) such that the image of the composite \( \mathcal{D}_X \hookrightarrow \Pi_{c-ab}^{U_{X \times_k X}} \rightarrow \Pi_{X \times_k X} \) coincides with the image of the diagonal homomorphism \( \Pi_X \hookrightarrow \Pi_X \times G_k, \Pi_X \twoheadrightarrow \Pi_{X \times_k X} \). Then if an automorphism \( \phi \) of \( \Pi_{c-ab}^{U_{X \times_k X}} \) satisfies the following two conditions, then \( \phi \) is Ker(\( \Delta_{c-ab}^{U_{X \times_k X}} \rightarrow \Delta_{X \times_k X} \))-inner:
(a) \( \phi \) preserves the quotient \( \Pi_{c-ab}^{U_{X \times_k X}} \rightarrow \Pi_{X \times_k X} \), and, moreover, the resulting automorphism of \( \Pi_{X \times_k X} \) is the identity automorphism;
(b) \( \phi \) preserves the inertia subgroup of \( \mathcal{D}_X \).

(iii) Let \( \{ S_i \} \) be a pro-symmetrized fundamental extension of \( \Pi_{X \times_k X} \) and let \( x \in X(k) \) be a \( k \)-rational point. Write \( U \overset{\text{def}}{=} X \setminus \{ x \} \) and \( s : G_k \rightarrow \Pi_X \) for the splitting (well-defined up to \( \Delta_X \)-conjugation) induced by \( x \), i.e., ‘\( G_X(x) \)’. Then the first isomorphism of (i) determines an isomorphism of profinite groups
\[ \Pi_{c-ab}^{U} \overset{\sim}{\rightarrow} \lim_{i \geq 0} (S_i \times \Pi_{X \times_k X} (s(G_k) \times G_k \Pi_X)). \]
(iv) In the notation of (iii), let $D_x \subseteq \Pi_U^{c,ab}$ be a decomposition subgroup associated with $x \in X(k)$. Then if an automorphism $\phi$ of $\Pi_U^{c,ab}$ satisfies the following two conditions, then $\phi$ is $\text{Ker}(\Delta_U^{c,ab} \to \Delta_X)$-inner:

(a) $\phi$ preserves the quotient $\Pi_U^{c,ab} \to \Pi_X$, and, moreover, the resulting automorphism of $\Pi_X$ is the identity automorphism;
(b) $\phi$ preserves the inertia subgroup of $D_x$.

Proof. First, let us observe that if we regard $\Pi_X$ as a closed subgroup of $\Pi_{X \times \kappa} \times \Pi_X$ by means of the diagonal homomorphism $\Pi_X \hookrightarrow \Pi_{X \times \kappa} \times \Pi_X$, then one verifies immediately that the set of $(\Pi_X \times \kappa)\text{-conjugates}$ of $\Pi_X$ coincides with the set of $(\Delta_{X \times \kappa})\text{-conjugates}$ of $\Pi_X$. Thus, assertion (i) (respectively, (iii)) follows immediately from a similar argument to the argument applied in [4, Proof of Proposition 1.9(ii)] (respectively, [4, Proposition 1.9(i)]). In particular, assertion (ii) (respectively, (iv)) follows immediately—in light of Lemma 2.11 below and a similar result to the result given as [4, Lemma 1.1]—from a similar argument to the argument applied in [4, Proof of Proposition 1.10(i)] (respectively, [4, Proposition 2.3(i)], together with Lemma 1.7(i)). This completes the proof of Proposition 2.10.

2.11. Suppose that $k$ is Kummer-faithful. Let $G$ be an open subgroup of either $\Pi_X \times \kappa \Pi_X$ which surjects onto $G_k$. Then the natural surjection $G \to G_k$ determines an isomorphism $H^1(G_k, \Lambda_X) \cong H^1(G, \Lambda_X)$.

In particular, it holds that $\bigcap_n n \cdot H^1(G, \Lambda_X) = \{0\}$, where $n$ ranges over the $\mathfrak{P} \times /k$-integers.

Proof. The first portion of the statement follows immediately from a similar argument to the argument applied in the proof of Lemma 1.8(i). The final portion of the statement follows immediately from the first portion of the statement, together with the Kummer theory (cf. Remark 1.4.1(i)). This completes the proof of Lemma 2.11.

3. The Grothendieck conjecture over Kummer-faithful fields

In this section, we discuss the (semi-absolute version of the) Grothendieck conjecture for affine hyperbolic curves over Kummer-faithful fields. In this section, let the symbol ‘□’ stand for either ‘◦’ or ‘•’. Let $k_□$ be a Kummer-faithful field, let $\kappa_□$ be an algebraic closure of $k_□$, let $X_□$ be a hyperbolic curve over $k_□$, and let

$$\Pi_{X_□}$$

be either the étale fundamental group $\pi_1(X_□)$ of $X_□$ or the tame fundamental group $\pi_1^{\text{tame}}(X_□^{\text{cpt}}, D_{X_□})$ of $(X_□^{\text{cpt}}, D_{X_□})$ (for some choice of basepoint). Write $G_{k_□}$ def $\text{Gal}(\kappa_□/k_□)$ and

$$\Delta_X \subseteq \Pi_{X_□}$$

for the quotient of the étale fundamental group $\pi_1(X_□ \otimes_{k_□} \kappa_□) \subseteq \pi_1(X_□)$ of $X_□ \otimes_{k_□} \kappa_□$ determined by $\Pi_{X_□}$.
Definition 3.1. Let \( \phi : \pi_{X, o} \sim \pi_{X, o} \) be an isomorphism of profinite groups.

(i) We shall say that \( \phi \) is point-theoretic if \( \phi \) determines a bijection between the set of decomposition subgroups of \( \pi_{X, o} \) and the set of decomposition subgroups of \( \pi_{X, o} \) associated with closed points of \( X^\text{pt} \). In particular, we obtain an isomorphism of profinite groups \( \phi_G : G_o \sim G_o \).

(ii) We shall say that \( \phi \) is Galois-preserving if \( \phi \) determines an isomorphism of profinite groups \( \phi : \pi_{X, o} \sim \pi_{X, o} \). In particular, we obtain an isomorphism of profinite groups \( \phi : G_{k, o} \sim G_{k, o} \).

We shall say that an outer isomorphism \( \pi_{X, o} \sim \pi_{X, o} \) is point-theoretic (respectively, Galois-preserving) if it arises from a point-theoretic (respectively, Galois-preserving) isomorphism.

Remark 3.1.1. One verifies easily from [4, Proposition 2.2(ii)] that, in the notation of Definition 3.1, if \( k \) is either an MLF or FF, then \( \phi \) is point-theoretic in the sense of Definition 3.1(i) if and only if \( \phi \) is point-theoretic in the sense of [4, Definition 1.5(ii)].

Lemma 3.2. Let \( \phi : \pi_{X, o} \sim \pi_{X, o} \) be a point-theoretic and Galois-preserving isomorphism of profinite groups. Then the following assertions hold.

(i) The isomorphism \( \phi \) determines a bijection \( \phi^{cl+} : X^{cl+}_o \sim X^{cl+}_o \) such that, for every \( x_o \in X^{cl+}_o \), if we write \( x_o \) \( \text{def} = \phi^{cl+}(x_o) \in X^{cl+}_o \), then the following conditions are satisfied.

(a) The diagram

\[
\begin{array}{ccc}
X^{cl+}_o & \xrightarrow{\phi^{cl+}} & X^{cl+}_o \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{C}(\pi_{X, o})/\pi_{X, o} & \longrightarrow & \mathcal{C}(\pi_{X, o})/\pi_{X, o}
\end{array}
\]

—where the lower horizontal arrow is the bijection induced by \( \phi \)—commutes.

(b) It holds that \( x_o \) lies on \( X_o \) if and only if \( x_o \) lies on \( X_o \).

(c) If we write \( \kappa(x_o) \), \( \kappa(x_o) \) for the residue fields at \( x_o \), \( x_o \), respectively, then it holds that \( [\kappa(x_o) : k_o] = [\kappa(x_o) : k_o] \).

We shall write \( \text{Div}(\phi) : \text{Div}(X_o) \sim \text{Div}(X_o) \)

for the isomorphism of groups determined by \( \phi^{cl+} \).

(ii) The isomorphism \( \phi \) determines an isomorphism of profinite groups

\[ \phi^{cl+} : \pi_1(X^{cl+}_o) \sim \pi_1(X^{cl+}_o), \]

which restricts to an isomorphism of profinite groups

\[ \phi^{cl+} : \pi_1(X^{cl+}_o \otimes_{k_o} k_o) \sim \pi_1(X^{cl+}_o \otimes_{k_o} k_o). \]

(iii) It holds that \( (g_{X, o}, r_{X, o}, \text{char}(k_o)) = (g_{X, o}, r_{X, o}, \text{char}(k_o)). \)

(iv) The isomorphism \( \phi \) determines an isomorphism \( \Lambda_{\phi} : \Lambda_{X, o} \sim \Lambda_{X, o} \).

Proof. First, we verify assertion (i). One verifies immediately from Lemma 1.7(i) that there exists a (uniquely determined) bijection \( \phi^{cl+} : X^{cl+}_o \sim X^{cl+}_o \) that satisfies condition (a).
The assertion that \( \phi^{cl+} \) satisfies condition (b) follows from the easily verified fact that, for \( x_0 \in X \), it holds that \( x_0 \) lies on \( X \) if and only if \( \Delta_{X_o} \cap \mathcal{G}_{X_o}(x_0) = \{1\} \), together with condition (a). The assertion that \( \phi^{cl+} \) satisfies condition (c) follows from the easily verified fact that, for \( x_0 \in X \), the index of the image of \( \mathcal{G}_{X_o}(x_0) \) in \( G_{k_o} \) coincides with \( [\kappa(x_0) : k_o] \), together with condition (a). This completes the proof of assertion (i).

Assertion (ii) follows immediately from Lemma 1.6(i). Assertion (iii) follows—in light of assertion (ii)—from Lemma 1.6(ii), (iii), (iv). Assertion (iv) follows immediately—in light of the equality \( \text{char}(\kappa(x_0)) = \text{char}(k_o) \) for assertion (ii)—from Lemma 3.2(ii), (iii), (iv). Assertion (v) follows immediately—in light of the fact that, for \( x_0 \in X \), the assertion that \( \phi^{cl+} \) satisfies condition (c) follows—in light of the equality \( \text{char}(\kappa(x_0)) = \text{char}(k_o) \), together with condition (a). This completes the proof of assertion (i).

**Lemma 3.3.** In the situation of Lemma 3.2, suppose, moreover, that the natural surjection \( \Pi_{X_o} \to G_{k_o} \) has a splitting, and that \( X_o \) is proper over \( k_o \). (Thus, it follows from Lemma 3.2(iii) that \( X_o \) is proper over \( k_o \).) Let \( S_o \subseteq X_o(k_o) \) be a finite subset. Write \( S_o \overset{\text{def}}{=} \phi^{cl+}(S_o) \subseteq X_o(k_o) \) (cf. condition (c) of Lemma 3.2(i)), \( U_o \overset{\text{def}}{=} X_o \setminus S_o \), and \( U_o \overset{\text{def}}{=} X_o \setminus S_o \). Then the following assertions hold.

(i) Let \( m \) be a positive integer; \( x_1^0, \ldots, x_m^0 \in X_o(k_o) \); \( n_1, \ldots, n_m \in \mathbb{Z} \). Then the divisor \( D_o \overset{\text{def}}{=} \sum_{i=1}^m n_i \cdot x_i^0 \in \text{Div}(X_o) \) on \( X_o \) is principal if and only if the divisor \( \text{Div}(\phi)(D_o) \in \text{Div}(X_o) \) on \( X_o \) is principal.

(ii) The isomorphism \( \phi \) determines an isomorphism \( \phi^{c-ab} : \Pi_{U_o \times k_o}^{c-ab} \overset{\sim}{\to} \Pi_{U_o \times k_o}^{c-ab} \) (well-defined up to \( \text{Ker}(\Pi_{U_o \times k_o}^{c-ab}) \to \Pi_{X_o \times k_o}^{c-ab} \))—inner automorphisms such that the diagram

\[
\begin{array}{ccc}
\Pi_{U_o \times k_o}^{c-ab} & \xrightarrow{\phi^{c-ab}} & \Pi_{U_o \times k_o}^{c-ab} \\
\downarrow & & \downarrow \\
\Pi_{X_o \times k_o} & \xrightarrow{\phi^{c-ab}} & \Pi_{X_o \times k_o}
\end{array}
\]

—where the lower horizontal arrow is an isomorphism induced by \( \phi \) and the natural isomorphism \( \Pi_{X_o \times k_o} \to \Pi_{X_o \times k_o} \to \Pi_{X_o \times k_o} \) commutes, and, moreover, \( \phi^{c-ab} \) maps a decomposition subgroup of \( \Pi_{U_o \times k_o}^{c-ab} \) associated with the diagonal divisor of \( X_o \times k_o \) to a decomposition subgroup of \( \Pi_{U_o \times k_o}^{c-ab} \) associated with the diagonal divisor of \( X_o \times k_o \).

(iii) The isomorphism \( \phi \) determines an isomorphism \( \phi^{c-cn} : \Pi_{U_o}^{c-cn} \overset{\sim}{\to} \Pi_{U_o}^{c-cn} \) (well-defined up to \( \text{Ker}(\Pi_{U_o}^{c-cn}) \to \Pi_{X_o}^{c-cn} \))—inner automorphisms such that the diagram

\[
\begin{array}{ccc}
\Pi_{U_o}^{c-cn} & \xrightarrow{\phi^{c-cn}} & \Pi_{U_o}^{c-cn} \\
\downarrow & & \downarrow \\
\Pi_{X_o} & \xrightarrow{\phi} & \Pi_{X_o}
\end{array}
\]

commutes, and, moreover, \( \phi^{c-cn} \) maps the (uniquely determined) inertia subgroup of \( \Pi_{U_o}^{c-cn} \) associated with \( x_o \in S_o \) to the (uniquely determined) inertia subgroup of \( \Pi_{U_o}^{c-cn} \) associated with \( \phi^{cl+}(x_o) \in S_o \).
(iv) The isomorphism $H^1(\Pi^\circ_0, \Lambda_{X_0}) \sim H^1(\Pi^\circ_* , \Lambda_{X_*})$ determined by the isomorphisms $\phi^\circ_S$ of (iii) and $\Delta_{\phi}$ of Lemma 3.2(iv) determines—relative to the isomorphism of Lemma 1.13(i)—an isomorphism

$$O^\times (\phi, S_0) : O^\times (\Pi^\circ_{X_0}, S_0) \sim O^\times (\Pi^\circ_{X_*}, S_*)$$

(cf. Definition 1.9) such that the diagram

$$
\begin{array}{c}
0 \rightarrow H^1(G_{k^\circ}, \Lambda_{X_0}) \rightarrow O^\times (\Pi^\circ_{X_0}, S_0) \rightarrow P(X_0, S_0) \rightarrow 0 \\
\downarrow \quad \downarrow O^\times (\phi, S_0) \quad \downarrow \\
0 \rightarrow H^1(G_{k_*}, \Lambda_{X_*}) \rightarrow O^\times (\Pi^\circ_{X_*}, S_*) \rightarrow P(X_*, S_*) \rightarrow 0
\end{array}
$$

(where the horizontal sequences are the lower horizontal exact sequence of the diagram of Lemma 1.10(ii) (in the case where we take '$X$' in Lemma 1.10 to be $X_0, X_*$); the left-hand vertical arrow is the isomorphism induced by the isomorphisms $\phi_G$ of Definition 3.1(ii) and $\Delta_{\phi}$ of Lemma 3.2(iv); the right-hand vertical arrow is the isomorphism determined by the isomorphism $\text{Div}(\phi)$ of Lemma 3.2(i)—commutes.

(v) The isomorphisms '$O^\times (\phi, S_0)' of (iv) for various finite extensions of $k_0$ contained in $\overline{k_0}$ and '$S_0'$ (cf. the definition of '$K^\times (\Pi X)$' given in Definition 1.9) determine an isomorphism of abelian groups

$$K^\times (\phi) : K^\times (\Pi^\circ_{X_0}) \sim K^\times (\Pi^\circ_{X_*}).$$

Proof. Assertion (i) follows immediately from Lemma 1.7(ii), together with conditions (a) and (c) of Lemma 3.2(i). Assertion (ii) follows immediately—in light of Proposition 2.10(i); Lemma 2.11—from a similar argument to the argument applied in [4, Proof of Theorem 1.1(iii)], together with similar results to the results given as [4, Lemma 1.1 and Proposition 1.8]. Assertion (iii) follows immediately—in light of condition (a) of Lemma 3.2(i)—from assertion (ii), together with Lemma 1.13(ii); Lemma 2.2. Assertion (iv) follows immediately from assertion (i), together with the various definitions involved. Assertion (v) follows immediately from the various definitions involved. This completes the proof of Lemma 3.3.

THEOREM 3.4. Let the symbol '$\square$' stand for either '$\circ$' or '$\bullet$'. Let $k_\square$ be a Kummer-faithful field (cf. Definition 1.2) and let $X_\square$ be a hyperbolic curve over $k_\square$. Write $X_{\square \text{cpt}}$ for the smooth compactification of $X_\square$ and $D_{X_\square} \subseteq X_{\square \text{cpt}}$ for the divisor at infinity of $X_\square$. Let

$$\Pi_{X_\square}$$

be either the étale fundamental group $\pi_1(X_\square)$ of $X_\square$ or the tame fundamental group $\pi^\text{tame}_1(X^\text{cpt}_\square, D_{X_\square})$ of $(X^\text{cpt}_\square, D_{X_\square})$ (for some choice of basepoint). Write

$$\text{Isom}(\Pi_{X_0}, \Pi_{X_*})$$

for the set of isomorphisms of profinite groups $\Pi_{X_0} \sim \Pi_{X_*}$.

$$\text{Isom}_{\text{PG}}(\Pi_{X_0}, \Pi_{X_*}) \subseteq \text{Isom}(\Pi_{X_0}, \Pi_{X_*})$$
for the subset of point-theoretic (cf. Definition 3.1(i)) and Galois-preserving (cf. Definition 3.1(ii)) isomorphisms of profinite groups \( \Pi_{X_o} \sim \Pi_{X_\bullet} \), and

\[ \text{Isom}(X_o, X_\bullet) \]

for the set of isomorphisms of schemes \( X_o \sim X_\bullet \). Then the following assertions hold.

(i) Suppose that \( \text{Isom}_{PG}(\Pi_{X_o}, \Pi_{X_\bullet}) \) is non-empty. Then it holds that \( \Pi_{X_o} = \pi_1(X_o) \) if and only if \( \Pi_{X_\bullet} = \pi_1(X_\bullet) \).

(ii) Suppose, moreover, that either \( X_o \) or \( X_\bullet \) is affine. Then the natural map

\[ \text{Isom}(X_o, X_\bullet) \rightarrow \text{Isom}(\Pi_{X_o}, \Pi_{X_\bullet})/\Pi_{X_\bullet} \]

(cf. (i)) determines a bijection

\[ \text{Isom}(X_o, X_\bullet) \sim \text{Isom}_{PG}(\Pi_{X_o}, \Pi_{X_\bullet})/\Pi_{X_\bullet} \]

Proof. Assertion (i) follows immediately from Lemma 1.6(v). Next, we verify assertion (ii).

First, let us observe that it follows immediately from \([7, \text{Lemma } 4.2]\) that the natural map

\[ \text{Isom}(X_o, X_\bullet) \rightarrow \text{Isom}(\Pi_{X_o}, \Pi_{X_\bullet})/\Pi_{X_\bullet} \]

factors through the subset \( \text{Isom}_{PG}(\Pi_{X_o}, \Pi_{X_\bullet})/\Pi_{X_\bullet} \subseteq \text{Isom}(\Pi_{X_o}, \Pi_{X_\bullet})/\Pi_{X_\bullet} \). Next, let us observe that the injectivity of the map under consideration follows immediately from Lemma 3.2(i), together with Lemma 3.5 below. Thus, to complete the verification of assertion (ii), it suffices to verify the surjectivity of the map under consideration. To this end, let \( \phi: \Pi_{X_o} \rightarrow \Pi_{X_\bullet} \) be a point-theoretic and Galois-preserving isomorphism of profinite groups.

Next, let us observe that it follows immediately from the injectivity of the map under consideration (i.e., by applying the Galois descent with respect to the respective actions of \( \Pi_{X_o}, \Pi_{X_\bullet} \) by conjugation) that we may assume without loss of generality, by replacing \( \Pi_{X_o} \) by a suitable open subgroup of \( \Pi_{X_o} \) if necessary, that \( g_{X_o} \geq 2 \), and hence also \( g_{X_\bullet} \geq 2 \) (cf. Lemma 3.2(iii)), and, moreover, \( r_{X_o} \geq 3 \), and hence also \( r_{X_\bullet} \geq 3 \) (cf. Lemma 3.2(iii)). (Note that one verifies easily from the Riemann–Hurwitz formula that there exists a connected finite étale covering of \( X_o \) such that \( g_{X_o} \geq 2 \) and \( r_{X_o} \geq 3 \).)

Next, again by the injectivity of the map under consideration (i.e., by applying the Galois descent with respect to the respective actions of \( \Pi_{X_o}, \Pi_{X_\bullet} \) by conjugation), we may assume without loss of generality, by replacing \( G_{k_o} \) by a suitable open subgroup of \( G_{k_o} \) if necessary, that \( D_{X_o}(\overline{k_o}) = D_{X_o}(k_o) \), and hence also \( D_{X_\bullet}(\overline{k_\bullet}) = D_{X_\bullet}(k_\bullet) \) (cf. conditions (b), (c) of Lemma 3.2(ii)), and that the natural surjection \( \Pi_{X_o} \rightarrow G_{k_o} \), and hence also the natural surjection \( \Pi_{X_\bullet} \rightarrow G_{k_\bullet} \), has a splitting.

Let \( S_o \subseteq X_o^{\text{cpt}}(k_o) \) be a finite subset such that \( D_{X_o}(k_o) \nsubseteq S_o \) and \( x_o \in D_{X_o}(k_o) \setminus (D_{X_o}(k_o) \cap S_o) \).

Write \( S_\bullet \stackrel{\text{def}}{=} \phi^{\text{cl}+}(S_o) \subseteq X_\bullet^{\text{cpt}}(k_\bullet) \) (cf. condition (c) of Lemma 3.2(ii)),

\[ x_\bullet \stackrel{\text{def}}{=} \phi^{\text{cl}+}(x_o) \in D_{X_\bullet}(k_\bullet) \setminus (D_{X_\bullet}(k_\bullet) \cap S_\bullet) \]

(cf. conditions (b), (c) of Lemma 3.2(ii)), \( U_o \stackrel{\text{def}}{=} X_o^{\text{cpt}} \setminus S_o \), \( U_\bullet \stackrel{\text{def}}{=} X_\bullet^{\text{cpt}} \setminus S_\bullet \), and

\[ \phi^{\text{cpt}}: \Pi_{X_o^{\text{cpt}}} \rightarrow \Pi_{X_\bullet^{\text{cpt}}} \]
for the isomorphism of profinite groups obtained in Lemma 3.2(ii). Let us observe that one verifies immediately from the various definitions involved that $\phi^{\text{cpt}}$ is point-theoretic and Galois-preserving. Now I claim that the following assertion holds.

**Claim 3.4.A.** The diagram

\[
\begin{array}{ccc}
\mathcal{O}^\times(\Pi_{X_0^{\text{cpt}}, S_0}) & \xrightarrow{\mathcal{O}^\times(\phi^{\text{cpt}}, S_0)} & \mathcal{O}^\times(\Pi_{X_0^{\text{cpt}}, S_0}) \\
\xrightarrow{\text{ev}_{X_0}(\Pi_{X_0^{\text{cpt}}, S_0})} & & \xrightarrow{\text{ev}_{X_0}(\Pi_{X_0^{\text{cpt}}, S_0})} \\
H^1(G_{k_0}, \Lambda_{X_0}) & \xrightarrow{H^1(G_{k_0}, \Lambda_{X_0})} & H^1(G_{k_0}, \Lambda_{X_0})
\end{array}
\]

—where the upper horizontal arrow is the isomorphism obtained in Lemma 3.3(iv); the lower horizontal arrow is the isomorphism induced by the isomorphisms $\phi_G$ of Definition 3.1(ii) and $/\Lambda_1\phi$ of Lemma 3.2(iv)—commutes.

Indeed, since (we have assumed that) $x_0$, and hence also $x_*$, is a cusp of the given hyperbolic curve, this follows immediately—in light of Proposition 2.10(ii), (iv)—from a similar argument to the argument applied in [4, Remarks 15, 21]. This completes the proof of Claim 3.4.A.

Since (we have assumed that) $\sharp D_{X_0}(k_0) = r_{X_0}$, $\sharp D_{X_0}(k_*) = r_{X_*} \geq 3 \geq 2$, by applying Claim 3.4.A to the isomorphisms $'\mathcal{O}^\times(\phi, S_0)'$ of Lemma 3.3(iv) for various finite extensions of $k_0$ contained in $\overline{k}_0$ and $'S_0'$ (cf. the definition of $'\mathcal{K}^\times(\Pi_X)'$ given in Definition 1.9), we conclude from Lemma 1.10(iv) that the isomorphism $\mathcal{K}^\times(\phi^{\text{cpt}}): \mathcal{K}^\times(\Pi_{X_0^{\text{cpt}}}) \sim \mathcal{K}^\times(\Pi_{X_0^{\text{cpt}}})$ of Lemma 3.3(v) determines an isomorphism of abelian groups

\[
K_\phi^\times: K_{X_0^{\text{cpt}}}^\times \otimes_{k_0} \overline{k}_0 \sim K_{X_0^*}^\times \otimes_{k_*} \overline{k}_*
\]

(relative to the injections

\[
K_{X_0^{\text{cpt}}}^\times \otimes_{k_0} \overline{k}_0 \hookrightarrow \mathcal{K}^\times(\Pi_{X_0^{\text{cpt}}}) \quad \text{and} \quad K_{X_0^*}^\times \otimes_{k_*} \overline{k}_* \hookrightarrow \mathcal{K}^\times(\Pi_{X_0^{\text{cpt}}})
\]

of Lemma 1.10(i) in the case where we take $'X'$ in Lemma 1.10 to be $X_0^{\text{cpt}}$, $X_0^{\text{cpt}}$), which restricts to an isomorphism of abelian groups

\[
\overline{k}_0^\times \sim \overline{k}_*^\times.
\]

Moreover, since (we have assumed that) $\sharp D_{X_0}(k_0) = r_{X_0}$, $\sharp D_{X_0}(k_*) = r_{X_*} \geq 3$, it follows immediately—in light of Claim 3.4.A and the commutativity of the right-hand square of the diagram of Lemma 3.3(iv)—from [7, Lemma 4.7] that the bijection

\[
K_{X_0^{\text{cpt}}}^\times \otimes_{k_0} \overline{k}_0 = K_{X_0^{\text{cpt}}}^\times \otimes_{k_0} \overline{k}_0 \cup \{0\} \sim K_{X_0^{\text{cpt}}}^\times \otimes_{k_*} \overline{k}_* = K_{X_0^{\text{cpt}}}^\times \otimes_{k_*} \overline{k}_* \cup \{0\}
\]

induced by $K_\phi^\times$ is an isomorphism of fields, which restricts to an isomorphism of fields

\[
\overline{k}_0 = \overline{k}_0^\times \cup \{0\} \sim \overline{k}_* = \overline{k}_*^\times \cup \{0\}.
\]
Thus, by considering the subfields consisting of \( \Pi_{X_\circ} \), \( \Pi_{X_\bullet} \)-invariants, we obtain a commutative diagram of schemes

\[
\begin{array}{ccc}
X_\circ^{\text{cpt}} & \xrightarrow{\sim} & X_\bullet^{\text{cpt}} \\
\downarrow & & \downarrow \\
\text{Spec}(k_\circ) & \xrightarrow{\sim} & \text{Spec}(k_\bullet),
\end{array}
\]

where the horizontal arrows are isomorphisms. Now let us observe that it follows immediately from our construction of the above diagram that the bijection \( X_\circ^{\text{cl}+} \xrightarrow{\sim} X_\bullet^{\text{cl}+} \) induced by the upper horizontal arrow coincides with the bijection \( \phi^{\text{cl}+} \) of Lemma 3.2(i). Thus, it follows from condition (b) of Lemma 3.2(i) that the upper horizontal arrow of the above diagram determines an isomorphism \( X_\circ \xrightarrow{\sim} X_\bullet \). The assertion that the outer isomorphism \( \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet} \) induced by this isomorphism coincides with the outer isomorphism determined by \( \phi \) follows immediately from a similar argument to the argument given in the discussion preceding [3, Theorem 14.1]. This completes the proof of assertion (ii), and hence also of Theorem 3.4.

\[ \square \]

Remark 3.4.1. If, in the notation of Theorem 3.4, the given hyperbolic curves \( X_\circ \) and \( X_\bullet \) are proper, then it is not clear to the author at the time of writing whether or not a similar assertion to the assertion of Claim 3.4.A—i.e., the compatibility of the isomorphism \( O^\times(\phi^{\text{cpt}}, S_\circ) \) and the ‘evaluation homomorphism’—holds. Thus, it is not clear to the author at the time of writing whether or not one may drop the affine hypothesis in the statement of Theorem 3.4(ii).

It seems to the author that the following result is likely to be well-known. However, the result could not be found in literature.

**Lemma 3.5.** Let \( C \) be a proper smooth curve over a field \( F \) and let \( \psi \) be an automorphism of an abstract scheme \( C \) (i.e., not necessarily over \( F \)). Suppose that \( \psi \) induces the identity automorphism on the underlying set of \( C \). Then \( \psi \) is the identity automorphism of \( C \).

**Proof.** By abuse of notation, we shall write \( \psi \) for the automorphism of the function field \( K \) of \( C \) induced by \( \psi \). Let \( x \in C \) be a closed point of \( C \) and \( f \in K^\times \). Then it follows from our assumption that \( \psi \) preserves the local ring \( O_x \) of \( C \) at \( x \), as well as the maximal ideal \( m_x \subseteq O_x \). Thus, one verifies immediately that \( \psi \) induces the identity automorphism on the quotient \( K^\times / O^\times_x (\sim \mathbb{Z}) \) of the group \( K^\times \). In particular, it follows that there exists an element \( \delta_f \in F^\times \) such that \( \psi(f) = \delta_f \cdot f \).

Suppose that \( f - 1 \in m_x \). Thus, it holds that \( \delta_f \cdot f - 1 = \psi(f) - 1 = \psi(f - 1) \in m_x \). Moreover, since \( F \cdot m_x \subseteq m_x \), it holds that \( \delta_f \cdot (f - 1) \in m_x \). In particular, it follows that \( \delta_f - 1 = (\delta_f \cdot f - 1) - \delta_f \cdot (f - 1) \in m_x \cap F = \{0\} \), which thus implies that \( \psi \) induces the identity automorphism on \( m_x \). Thus, since (one verifies easily that) the field \( K \) is generated by the \( m_x \), where \( x \) ranges over the closed points of \( C \), we conclude that \( \psi \) is the identity automorphism of \( C \). This completes the proof of Lemma 3.5. \[ \square \]

**Acknowledgements.** The author would like to thank the referee for some comments. This research was supported by Grant-in-Aid for Scientific Research (C), No. 24540016, Japan Society for the Promotion of Science.
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