A pro-$l$ version of the congruence subgroup problem for mapping class groups of genus one

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Let $l$ be a prime number. In the present paper, we discuss a pro-$l$ version of the congruence subgroup problem for mapping class groups of genus one. Our main result is that the pro-2 version has an affirmative answer, but the pro-$l$ version for $l \geq 11$ has a negative answer.

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Introduction

Let $l$ be a prime number. In the present paper, we discuss a pro-$l$ version of the congruence subgroup problem for mapping class groups of genus one.

Let us first recall the congruence subgroup problem for mapping class groups as follows (cf., e.g., [3], [17]): Let $(g, r)$ be a pair of nonnegative integers such that $2g - 2 + r > 0$ and $\Sigma_{g,r}$ a topological surface of type $(g, r)$, i.e., a topological space obtained by removing $r$ distinct points from a connected orientable compact topological surface of genus $g$. Write $\pi_1^{\text{top}}(\Sigma_{g,r})$ for the topological fundamental group of $\Sigma_{g,r}$ (which is well-defined up to conjugation) and $\text{MCG}_{g,r}$ for the (pure) mapping class group of $\Sigma_{g,r}$, i.e., the group of isotopy classes of orientation-preserving automorphisms of $\Sigma_{g,r}$ that fix each removed point. Then a classical result due to Dehn and Nielsen asserts that the natural homomorphism

$$\rho_{g,r}^{\text{top}} : \text{MCG}_{g,r} \to \text{Out}(\pi_1^{\text{top}}(\Sigma_{g,r}))$$

is injective. Now we shall say that a subgroup $J \subseteq \text{MCG}_{g,r}$ of $\text{MCG}_{g,r}$ is a congruence subgroup if there exists a characteristic subgroup $H \subseteq \pi_1^{\text{top}}(\Sigma_{g,r})$ of $\pi_1^{\text{top}}(\Sigma_{g,r})$ of finite index such that the inclusion

$$\ker(\text{MCG}_{g,r} \xrightarrow{\rho_{g,r}^{\text{top}}} \text{Out}(\pi_1^{\text{top}}(\Sigma_{g,r})) \to \text{Out}(\pi_1^{\text{top}}(\Sigma_{g,r})/H)) \subseteq J$$

holds. Then the congruence subgroup problem for the mapping class group of type $(g, r)$ may be stated as follows:

$$(\text{CSP})_{g,r} : \text{Is every subgroup of MCG}_{g,r} of finite index a congruence subgroup?}$$

If $g \leq 1$, then the problem $(\text{CSP})_{g,r}$ was answered affirmatively in [2, Theorems 2, 3A, 5]. If $g = 2$ and $r > 0$, then the problem $(\text{CSP})_{g,r}$ was answered affirmatively in [4, Theorem 3.5]. Moreover, the problem $(\text{CSP})_{g,r}$ in the case where $(g, r) = (2, 0)$ also has an affirmative answer by the affirmative answer to the problem $(\text{CSP})_{g,r}$ in the case where $g = 2$ and $r > 0$, together with [13, Theorem B] (cf. also Proposition 1.9 of the present paper). However, the problem $(\text{CSP})_{g,r}$ in the case where $g \geq 3$ remains unsolved.

Now let us observe that since (as is well-known) $\pi_1^{\text{top}}(\Sigma_{g,r})$ is finitely generated, if we write $\hat{\pi}_1(\Sigma_{g,r})$ for the profinite completion of the discrete group $\pi_1^{\text{top}}(\Sigma_{g,r})$, then
Moreover,\([\text{Out}(\pi^1(\Sigma_{g,r}))\) of \(\pi^1_{1}(\Sigma_{g,r})\) admits a natural structure of profinite group. In particular, if we write \(\text{MCG}^{\wedge}_{g,r}\) for the profinite completion of the discrete group \(\text{MCG}_{g,r}\), then the homomorphism \(\rho^{\text{top}}_{g,r}\) induces a continuous homomorphism

\[
\rho^{\wedge}_{g,r}: \text{MCG}^{\wedge}_{g,r} \to \text{Out}(\pi^1_{1}(\Sigma_{g,r})).
\]

Here, one verifies easily that the problem \((\text{CSP})_{g,r}\) has an affirmative answer if and only if this continuous homomorphism \(\rho^{\wedge}_{g,r}\) is injective.

Next, let us consider a pro-\(l\) version of the congruence subgroup problem for mapping class groups. Let us first recall that, for a characteristic subgroup \(H \subseteq \pi^1_{1}(\Sigma_{g,r})\) of index a power of \(l\), the group \(\text{Out}(\pi^1_{1}(\Sigma_{g,r})/H)\) is not an \(l\)-group in general; on the other hand, by [1, Corollary 7], if we write \(\Sigma^\text{cpt}_{g,r}\) for the compactification of \(\Sigma_{g,r}\) (so \(\Sigma^\text{cpt}_{g,r}\) is homeomorphic to \(\Sigma_{g,0}\)) and

\[
\text{MCG}_{g,r}^{l} := \ker(\text{MCG}_{g,r} \to \text{Aut}(H_1(\Sigma^\text{cpt}_{g,r}, \mathbb{F}_l))),
\]

then the image of the composite

\[
\text{MCG}_{g,r}^{l} \to \text{MCG}_{g,r} \xrightarrow{\rho^{\text{top}}_{g,r}} \text{Out}(\pi^1_{1}(\Sigma_{g,r})) \to \text{Out}(\pi^1_{1}(\Sigma_{g,r})/H)
\]

is always an \(l\)-group. From this observation, we shall say that a subgroup \(J \subseteq \text{MCG}_{g,r}^{l}\) of \(\text{MCG}_{g,r}^{l}\) is an \(l\)-congruence subgroup if there exists a characteristic subgroup \(H \subseteq \pi^1_{1}(\Sigma_{g,r})\) of \(\pi^1_{1}(\Sigma_{g,r})\) of index a power of \(l\) such that the inclusion

\[
\ker(\text{MCG}_{g,r} \xrightarrow{\rho^{\text{top}}_{g,r}} \text{Out}(\pi^1_{1}(\Sigma_{g,r})) \to \text{Out}(\pi^1_{1}(\Sigma_{g,r})/H)) \subseteq J
\]

holds. Then the following problem may be regarded as a pro-\(l\) version of the congruence subgroup problem for mapping class groups:

\((\text{CSP})_{g,r}^{\text{pro-}\,l}: \) Is every normal subgroup of \(\text{MCG}_{g,r}^{l}\) of index a power of \(l\) an \(l\)-congruence subgroup?

If \(g = 0\), then the problem \((\text{CSP})_{g,r}^{\text{pro-}\,l}\) was answered affirmatively in [2, Remark following the proof of Theorem 1].

Here, let us observe that, if we write \(\pi^1_{1}(\Sigma_{g,r})\), \(\text{MCG}_{g,r}^{l}(\Sigma_{g,r})\), \(\text{MCG}_{g,r}^{l}(\Sigma_{g,r})\), respectively, then the homomorphism \(\rho^{\text{top}}_{g,r}\) induces a continuous homomorphism

\[
\rho^{\text{pro-}\,l}_{g,r}: \text{MCG}_{g,r}^{l}(\Sigma_{g,r}) \to \text{Out}(\pi^1_{1}(\Sigma_{g,r})).
\]

Moreover, as in the profinite case, it holds that the problem \((\text{CSP})_{g,r}^{\text{pro-}\,l}\) has an affirmative answer if and only if this continuous homomorphism \(\rho^{\text{pro-}\,l}_{g,r}\) is injective. We note that, in
[8, Theorem 1, the discussion following Theorem 1], it was proved that if \( g \geq 2 \), then the natural continuous homomorphism from the pro-\( l \) completion of the Torelli subgroup of \( \text{MCG}_{g,r} \) (i.e., the subgroup of \( \text{MCG}_{g,r} \) obtained by forming the kernel of the natural homomorphism

\[
\text{MCG}_{g,r} \rightarrow \text{Aut}(H_1(\Sigma_{g,r}, \mathbb{Z}))
\]

to \( \text{MCG}_{g,r}[l][l] \) is not injective. In particular, the continuous homomorphism induced by \( \rho^\text{top}_{g,r} \) from the pro-\( l \) completion of (not \( \text{MCG}_{g,r}[l] \) but) the Torelli subgroup of \( \text{MCG}_{g,r} \) to \( \text{Out}(\pi_1^{\text{pro-}l}(\Sigma_{g,r})) \) is not injective.

In the present paper, we discuss the problem \((\text{CSP})^{\text{pro-}l}_{g,r}\) in the case where \( g = 1 \), i.e., a pro-\( l \) version of the congruence subgroup problem for mapping class groups of genus one. The main result of the present paper is as follows (cf. Corollaries 2.11, 3.9):

**Theorem A.** Let \( r \) be a positive integer. Then the following hold.

(i) The problem \((\text{CSP})^{\text{pro-}2}_{1,r}\) has an affirmative answer.

(ii) If \( l \geq 11 \), then the problem \((\text{CSP})^{\text{pro-}l}_{1,r}\) has a negative answer.

Theorem A, (i), is proved by a similar argument to the argument applied in [2, Theorem 5], which gives rise to an affirmative answer to the problem \((\text{CSP})^{\text{pro-}l}_{g,r}\) in the case where \( g = 1 \). Here, we note that there is an alternative proof that is purely group-theoretic of [2, Theorem 5] (cf., e.g., [7]). It seems possible that a similar purely group-theoretic proof to the proof given in [7] works in (unlike the situation of Theorem A, (ii)) the situation of Theorem A, (i).

The proof of Theorem A, (ii), in the case where \( r = 1 \) may be summarized as follows: Suppose that \( l \geq 11 \). Let us first recall from the discussion following the statement of the problem \((\text{CSP})^{\text{pro-}l}_{g,r}\) that it suffices to verify that the surjection

\[
\text{MCG}_{1,1}[l][l] \rightarrow \text{im}(\rho^{\text{pro-}l}_{1,1}),
\]

i.e., determined by the homomorphism \( \rho^{\text{pro-}l}_{1,1} \), is not an isomorphism. To this end, let us consider the modular curve \( Y(l) \) with respect to the congruence subgroup \( \Gamma(l) \) over the field \( \mathbb{Q}(\zeta_l) \), where \( \zeta_l \) is a primitive \( l \)-th root of unity. Then it follows immediately from the various definitions involved that we have a natural identification of \( \text{MCG}_{1,1}[l][l] \) with the geometric pro-\( l \) fundamental group of \( Y(l) \). Thus, one may consider the pro-\( l \) outer Galois action

\[
\rho^l_{Y(l)} : G_{\mathbb{Q}(\zeta_l)} \rightarrow \text{Out}(\text{MCG}_{1,1}[l][l])
\]

associated to \( Y(l) \). Here, one may easily verify that the kernel of the surjection \( \text{MCG}_{1,1}[l][l] \rightarrow \text{im}(\rho^{\text{pro-}l}_{1,1}) \) is preserved by this outer action \( \rho^l_{Y(l)} \), which thus implies that this outer action \( \rho^l_{Y(l)} \) determines an outer action
Now let us recall from [25, Theorem 0.5, (2)] that the kernel of \( \overline{\rho} \) contains the kernel of the pro-\( l \) outer Galois action \( \rho^l_{1}\) \( \mathbb{Q}(\zeta_l) \) \( \{0,1,\infty\} \) associated to \( \mathbb{P}^1(\mathbb{Q}(\zeta_l)) \). In particular, since the image of \( \rho^l_{1}\) \( \mathbb{Q}(\zeta_l) \) \( \{0,1,\infty\} \) is a pro-\( l \) group (cf. [14, I, §5, Proposition 7, (ii)]),

the image of \( \overline{\rho} \) is a pro-\( l \) group.

On the other hand, the authors of the present paper proved, in [12, Theorem A], that the assumption \( l \geq 11 \) implies that

the image of \( \rho^l_{1}\) is not a pro-\( l \) group.

Therefore, we conclude that the natural surjection \( \text{MCG}_{1,1} \mathbb{I}^l \rightarrow \text{im}(\rho^l_{1}) \) is not an isomorphism, as desired.

Here, it is of interest to observe that the problem \( \text{(CSP)}^\text{pro-}l \) (as well as the problem \( \text{(CSP)}_{g,r} \)) can be stated and formulated by a purely topological and combinatorial group-theoretic setting. Nevertheless, as discussed in the above outline of the proof, our approach to the problem \( \text{(CSP)}^\text{pro-}l \), i.e., in the case where \( g = 1 \), is based on a highly arithmetic phenomenon concerning the outer Galois actions associated to modular curves.

The present paper is organized as follows: In §1, we recall generalities on the relative pro-\( l \) completions of mapping class groups. In §2, we consider the pro-2 outer geometric monodromy action to prove Theorem A, (i). In §3, we prove Theorem A, (ii), by means of the results obtained in the previous sections. In §4, we discuss some remarks concerning the outer Galois actions on the relative pro-\( l \) completions of mapping class groups.

**Notations and conventions**

**Profinite groups:**

For a profinite group \( G \) and a closed subgroup \( H \subseteq G \) of \( G \), we shall write \( G^{\text{ab}} \) for the abelianization of \( G \) (i.e., the quotient of \( G \) by the closure of the commutator subgroup of \( G \)), \(|G : H|\) for the index of \( H \) in \( G \), \( Z(G) \) for the center of \( G \), and \( Z_G(H) \) for the centralizer of \( H \) in \( G \). We shall say that a profinite group \( G \) is torsion-free if \( G \) has no nontrivial element of finite order. We shall say that a profinite group \( G \) is center-free if \( Z(G) = \{1\} \). We shall say that a profinite group \( G \) is slim if for every open subgroup \( H \subseteq G \), it holds that \( Z_G(H) = \{1\} \).

For a profinite group \( G \), we shall denote by \( \text{Aut}(G) \) the group of (continuous) automorphisms of the topological group \( G \), by \( \text{Inn}(G) \) the group of inner automorphisms of \( G \), and by \( \text{Out}(G) \) the quotient of \( \text{Aut}(G) \) with respect to the normal subgroup \( \text{Inn}(G) \subseteq \text{Aut}(G) \). If, moreover, \( G \) is topologically finitely generated, then one verifies
that the topology of $G$ admits a basis of characteristic open subgroups, which thus induces a profinite topology on the group $\operatorname{Aut}(G)$, hence also a profinite topology on the group $\operatorname{Out}(G)$.

Let $G$ be a profinite group, $N \subseteq G$ a normal open subgroup of $G$, $l$ a prime number, and $N^l$ the maximal pro-$l$ quotient of $N$. Observe that the kernel $K$ of the natural surjection $N \to N^l$ is characteristic in $N$ and therefore normal in $G$. We shall define the relative pro-$l$ completion of $G$ with respect to $G \to G/N$ to be the quotient $G/K$. It is an extension

$$1 \longrightarrow N^l \longrightarrow G/K \longrightarrow G/N \longrightarrow 1$$

of $G/N$ by $N^l$.

Curves:

Let $k$ be a field and $(g, r)$ a pair of nonnegative integers. Then we shall say that a scheme $X$ over $k$ is a curve of type $(g, r)$ over $k$ if there exist a scheme $X^{\text{cpt}}$ which is of dimension 1, smooth, proper, geometrically connected over $k$ of genus $g$ and a closed subscheme $D \subseteq X^{\text{cpt}}$ which is finite and étale over $k$ of degree $r$ such that $X$ is isomorphic to the complement of $D$ in $X^{\text{cpt}}$ over $k$. In this case, it follows from elementary algebraic geometry that these $X^{\text{cpt}}$ and $D$ are uniquely determined by $X$ up to unique canonical isomorphism. We shall refer to $X^{\text{cpt}}$ as the smooth compactification of $X$ and $D$ as the divisor at infinity of $X$. We shall say that a scheme $X$ over $k$ is a hyperbolic curve over $k$ if there exists a pair $(g, r)$ of nonnegative integers such that $2g - 2 + r > 0$, and, moreover, $X$ is a curve of type $(g, r)$ over $k$. For a pair $(g, r)$ of nonnegative integers such that $2g - 2 + r > 0$, the notation $(\mathcal{M}_{g,r})_k$ will be used to denote the moduli stack of $r$-pointed smooth proper curves of genus $g$ over $k$ whose $r$ marked points are equipped with an ordering.

Let $n$ be a positive integer, $(g, r)$ a pair of nonnegative integers such that $2g - 2 + r > 0$, and $X$ a curve of type $(g, r)$ over $k$. Suppose that the divisor at infinity $D$ of $X$ consists of $r$ distinct $k$-rational points. Then we shall refer to the scheme obtained by pulling back the morphism $(\mathcal{M}_{g,r+n})_k \to (\mathcal{M}_{g,r})_k$ given by forgetting the last $n$ marked points via the classifying morphism $\text{Spec}(k) \to (\mathcal{M}_{g,r})_k$ of the $r$-pointed smooth proper curve of genus $g$ over $k$ obtained by equipping the $r$ marked points of $X$ with an ordering as the $n$-th configuration space of $X$. Note that one verifies immediately that the isomorphism class of this pull-back does not depend on the choice of the ordering of the $r$ marked points of $X$. Also, note that the $n$-th configuration space of $X$ is naturally isomorphic to the open subscheme of the $n$-th product $X^{\times n}$ of $X$ over $k$ whose complement consists of the diagonal divisors $\{(x_1, \ldots, x_n) \in X^{\times n} \mid x_i = x_j\}$, where $1 \leq i < j \leq n$.

1. The relative pro-$l$ completions of mapping class groups

Throughout the present paper, let $l$ be a prime number, $k$ a field of characteristic zero, and $\overline{k}$ an algebraic closure of $k$. Write $G_k := \operatorname{Gal}(\overline{k}/k)$. In the present §1, we
recall generalities on the relative pro-l completions of mapping class groups. Much of the content of the present §1 is contained in [8].

**Definition 1.1.** Let $X$ be a scheme which is geometrically connected and of finite type over $k$.

(i) We shall write $\Delta^l_X$ for the pro-$l$ geometric fundamental group of $X$, i.e., the maximal pro-$l$ quotient of the étale fundamental group $\pi_1(X \otimes_k \overline{k})$ of $X \otimes_k \overline{k}$. (In fact, $\pi_1(\cdot)$ is defined for the pair of “−” and a base point of “−”. However, since the $\pi_1(\cdot)$ is independent, up to inner automorphisms, of the choice of the base point, we shall omit the base point.)

(ii) We shall write $\Pi^l_X$ for the geometrically pro-$l$ fundamental group of $X$, i.e., the quotient of the étale fundamental group $\pi_1(X)$ of $X$ by the kernel of the natural surjection $\pi_1(X \otimes_k \overline{k}) \rightarrow \Delta^l_X$.

(iii) We shall write

$$\rho^l_X : G_k \longrightarrow \text{Out}(\Delta^l_X)$$

for the outer action determined by the natural exact sequence

$$1 \longrightarrow \Delta^l_X \longrightarrow \Pi^l_X \longrightarrow G_k \longrightarrow 1.$$  

We shall refer to $\rho^l_X$ as the pro-$l$ outer Galois action associated to $X$.

**Definition 1.2.** Let $(g, r)$ be a pair of nonnegative integers such that $2g - 2 + r > 0$.

(i) We shall write

$$\Pi_{(M_{g,r})_k}$$

for the étale fundamental group of $(M_{g,r})_k$ (for the étale fundamental group of $(M_{g,r})_k$, see [22]) and

$$\Gamma_{g,r}$$

for the kernel of the outer surjection $\Pi_{(M_{g,r})_k} \rightarrow G_k$ that arises from the structure morphism $(M_{g,r})_k \rightarrow \text{Spec}(k)$ (cf. Remark 1.3 below). Thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Gamma_{g,r} \longrightarrow \Pi_{(M_{g,r})_k} \longrightarrow G_k \longrightarrow 1.$$  

(1)
(ii) We shall write
\[ \Delta_{g,r} \]
for the kernel of the outer surjection \( \Pi_{(\mathcal{M}_{g,r+1})_k} \to \Pi_{(\mathcal{M}_{g,r})_k} \) induced by the morphism \( (\mathcal{M}_{g,r+1})_k \to (\mathcal{M}_{g,r})_k \) given by forgetting the last marked point and
\[ \Delta_{l}^{l} \]
for the maximal pro-\( l \) quotient of \( \Delta_{g,r} \). Thus, we have a natural exact sequence of profinite groups
\[
1 \longrightarrow \Delta_{g,r} \longrightarrow \Pi_{(\mathcal{M}_{g,r+1})_k} \longrightarrow \Pi_{(\mathcal{M}_{g,r})_k} \longrightarrow 1.
\]

(2)

Remark 1.3. It is well-known that the isomorphism class of the kernel of the outer surjection \( \Pi_{(\mathcal{M}_{g,r})_k} \to G_k \) that arises from the structure morphism \( (\mathcal{M}_{g,r})_k \to \text{Spec}(k) \) does not depend on the choice of the field \( k \) of characteristic zero, and that this kernel (i.e., \( \Gamma_{g,r} \)) is isomorphic to the étale fundamental group of \( (\mathcal{M}_{g,r})_{\bar{\mathbb{F}}} \).

Definition 1.4 ([8, §3]). Let \((g, r)\) be a pair of nonnegative integers such that \(2g-2+r > 0\), and \(X\) a curve of type \((g, r)\) over \(k\).

(i) We shall write
\[ \text{Out}^C(\Delta^I_X) \subseteq \text{Out}(\Delta^I_X) \]
for the subgroup of \(C\)-admissible outer automorphisms of \( \Delta^I_X \) (i.e., outer automorphisms that determine self-bijections of the set of conjugacy classes of cuspidal inertia subgroups of \( \Delta^I_X \) — cf. [20, Definition 1.1 (ii)]),
\[ \text{Out}^C(\Delta^I_{g, r}) \subseteq \text{Out}(\Delta^I_{g, r}) \]
for the image of the above subgroup \( \text{Out}^C(\Delta^I_X) \subseteq \text{Out}(\Delta^I_X) \) via the isomorphism \( \text{Out}(\Delta^I_X) \to \text{Out}(\Delta^I_{g, r}) \) determined by the outer isomorphism \( i_X : \Delta^I_X \to \Delta^I_{g, r} \) induced by the classifying morphism \( \text{Spec}(\bar{k}) \to (\mathcal{M}_{g,r})_{\bar{\mathbb{F}}} \) of the \( r \)-pointed smooth proper curve of genus \( g \) over \( \bar{k} \) obtained by equipping the \( r \) marked points of \( X \) with an ordering (cf. Remark 1.5, (i), below), and
\[ \varphi^I_{g,r} : \text{Out}^C(\Delta^I_{g, r}) \to \text{Aut}((\Delta^I_{g, 0})^\text{ab} \otimes_{\mathbb{Z}_l} (\mathbb{Z}/l^n)), \]
for the action obtained by considering the outer surjection \( \Delta^I_{g, r} \to \Delta^I_{g, 0} \) induced by the natural morphism from \( X \) to the smooth compactification of \( X \) over \( \bar{k} \), which is a curve of type \((g, 0)\) over \( \bar{k} \) (cf. also the definition of \( \text{Out}^C(\Delta^I_{g, r}) \)).
(ii) We shall write

$$(\rho_{g,r}^{\text{puni-l}})_k : \Pi_{(\mathcal{M}_{g,r})_k} \longrightarrow \text{Out}(\Delta_{g,r}^l)$$

for the outer action determined by the exact sequence (2) of Definition 1.2, (ii), and the natural surjection $\Delta_{g,r} \twoheadrightarrow \Delta_{g,r}^l$,

$$(\Gamma_{g,r}[l])^l$$

for the maximal pro-$l$ quotient of $\Gamma_{g,r}[l] := \ker(\varphi_{g,r}^l \circ (\rho_{g,r}^{\text{puni-l}})_k) \cap \Gamma_{g,r}$ (cf. Remark 1.5, (ii), below), and

$$\Gamma_{g,r}^{\text{rel-l}}$$

for the relative pro-$l$ completion of $\Gamma_{g,r}$ with respect to $\Gamma_{g,r} \twoheadrightarrow \Gamma_{g,r}/\Gamma_{g,r}[l]$, i.e., the quotient of $\Gamma_{g,r}$ with respect to the kernel of $\Gamma_{g,r}[l] \twoheadrightarrow (\Gamma_{g,r}[l])^l$. We shall refer to $\Gamma_{g,r}^{\text{rel-l}}$ as the relative pro-$l$ completion of the mapping class group of type $(g,r)$.

(iii) We shall write

$$\Pi_{(\mathcal{M}_{g,r})_k}^{\text{rel-l}}$$

for the quotient of $\Pi_{(\mathcal{M}_{g,r})_k}$ with respect to the kernel of $\Gamma_{g,r}[l] \twoheadrightarrow (\Gamma_{g,r}[l])^l$ (cf. Remark 1.5, (iii), below) and

$$(\rho_{g,r}^{\text{rel-l}})_k : G_k \longrightarrow \text{Out}(\Gamma_{g,r}^{\text{rel-l}})$$

for the outer Galois action determined by the exact sequence

$$1 \longrightarrow \Gamma_{g,r}^{\text{rel-l}} \longrightarrow \Pi_{(\mathcal{M}_{g,r})_k}^{\text{rel-l}} \longrightarrow G_k \longrightarrow 1 \quad (3)$$

that arises from the exact sequence (1) of Definition 1.2, (i).

Remark 1.5.

(i) It is immediate that the subgroup $\text{Out}^C(\Delta_{g,r}^l)$ of $\text{Out}(\Delta_{g,r}^l)$ does not depend on the choice of a curve $X$ of type $(g,r)$ over $\kbar$.

(ii) By the construction, the image of $(\rho_{g,r}^{\text{puni-l}})_k$ is contained in the subgroup $\text{Out}^C(\Delta_{g,r}^l)$. Moreover, the homomorphism $\varphi_{g,r}^l$ (respectively, the kernel $\ker(\varphi_{g,r}^l \circ (\rho_{g,r}^{\text{puni-l}})_k) \cap \Gamma_{g,r}$) does not depend on the choice of $X$ (respectively, $k$).

(iii) Since $\Gamma_{g,r}[l]$ is normal in $\Pi_{(\mathcal{M}_{g,r})_k}$, and the kernel of $\Gamma_{g,r}[l] \twoheadrightarrow (\Gamma_{g,r}[l])^l$ is characteristic in $\Gamma_{g,r}[l]$, the subgroup $\ker(\Gamma_{g,r}[l] \twoheadrightarrow (\Gamma_{g,r}[l])^l)$ is normal in $\Pi_{(\mathcal{M}_{g,r})_k}$. 
Definition 1.6. Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\). We shall write

\[
\Gamma_{g,r}^{\text{geo-l}} \cong (\rho_{g,r}^{\text{uni-l}})_k(\Gamma_{g,r})
\]

for the quotient of \(\Gamma_{g,r}\) with respect to the kernel of the restriction to \(\Gamma_{g,r} \subseteq \Pi(M_{g,r})_k\)

of the homomorphism \((\rho_{g,r}^{\text{uni-l}})_k\),

\[
\Pi_{(M_{g,r})_k}^{\text{geo-l}}
\]

for the quotient of \(\Pi(M_{g,r})_k\) with respect to the kernel of \(\Gamma_{g,r} \twoheadrightarrow (\rho_{g,r}^{\text{uni-l}})_k(\Gamma_{g,r})\), and

\[
(\rho_{g,r}^{\text{geo-l}})_k: G_k \rightarrow \text{Out}(\Gamma_{g,r}^{\text{geo-l}})
\]

for the outer Galois action determined by the exact sequence

\[
1 \rightarrow \Gamma_{g,r}^{\text{geo-l}} \rightarrow \Pi_{(M_{g,r})_k}^{\text{geo-l}} \rightarrow G_k \rightarrow 1
\]

that arises from the exact sequence (1) of Definition 1.2, (i).

Proposition 1.7 (cf. [8, Proposition 3.1, (2)]). Let \(n\) be a positive integer, \((g, r)\) a pair of nonnegative integers such that \(2g - 2 + r > 0\), \(X\) a curve of type \((g, r)\) over \(\overline{k}\), and \(X_n\) the \(n\)-th configuration space of the curve \(X\). Then the morphism \((M_{g,r+n})_k \rightarrow (M_{g,r})_k\)

given by forgetting the last \(n\) points and the classifying morphism \(\text{Spec}(\overline{k}) \rightarrow (M_{g,r})_{\overline{k}}\) of \(X\) determine the following commutative diagram

\[
\begin{array}{ccc}
1 & \rightarrow & \Delta^l_{X_n} \rightarrow \Pi_{(M_{g,r+n})_k}^{\text{rel-l}} \rightarrow \Pi_{(M_{g,r})_k}^{\text{rel-l}} \rightarrow 1 \\
1 & \rightarrow & \Delta^l_{X_n} \rightarrow \Gamma_{g,r+n}^{\text{rel-l}} \rightarrow \Gamma_{g,r}^{\text{rel-l}} \rightarrow 1
\end{array}
\]

where the horizontal sequences are exact, the vertical arrows are injective, and the left-hand vertical arrow is the identity morphism of \(\Delta^l_{X_n}\).

In particular, by considering the case where \(n = 1\), we conclude that the homomorphism \((\rho_{g,r}^{\text{uni-l}})_k\) factors through \(\Pi_{(M_{g,r})_k}^{\text{rel-l}}\).

Proof. Let us first observe that it follows immediately from the exact sequence (3) of Definition 1.4, (iii), that, to verify Proposition 1.7, it suffices to verify the exactness of the lower sequence of the commutative diagram in the statement of Proposition 1.7. Thus, we may assume without loss of generality, by replacing \(k\) by \(\overline{k}\), that \(k\) is an algebraically closed field. Let \(Y\) be the curve of type \((g, r+1)\) over \(k\) obtained by removing a \(k\)-rational
point from \( X \) and \( Y_{n-1} \) the \( (n - 1) \)-st configuration space of \( Y \). Then it follows from the (easily verified) *right exactness* of the functor of taking maximal pro-\( l \) quotient and [11, Lemma 15, (iv)] that we have the following commutative diagram of profinite groups

\[
1 \longrightarrow \Delta_{Y_{n-1}}^l \longrightarrow \Delta_{X_n}^l \longrightarrow \Delta_X^l \longrightarrow 1 \\
\Delta_{Y_{n-1}}^l \longrightarrow \Gamma^\text{rel-l}_{g,r+n} \longrightarrow \Gamma^\text{rel-l}_{g,r+1} \longrightarrow 1 \\
\Gamma^\text{rel-l}_{g,r} \longrightarrow \Gamma^\text{rel-l}_{g,r} \\
1 \longrightarrow 1
\]

where the vertical and horizontal sequences are exact, the lower horizontal arrow is the identity morphism of \( \Gamma^\text{rel-l}_{g,r} \), and the left-hand vertical arrow is the identity morphism of \( \Delta_{Y_{n-1}}^l \). Thus, to verify Proposition 1.7, by *induction on \( n \)*, we may assume without loss of generality that \( n = 1 \). On the other hand, if \( n = 1 \), then the desired exactness follows from the proof of [8, Proposition 3.1, (2)].

**Definition 1.8.** We shall write

\[
(\rho^\text{univ-l}_{g,r})_k : \Pi^\text{rel-l}_{(M_{g,r})_k} \longrightarrow \text{Out}(\Delta_{g,r}^l)
\]

for the resulting (cf. Proposition 1.7) homomorphism, and

\[
\rho^\text{univ-l}_{g,r} : \Gamma^\text{rel-l}_{g,r} \longrightarrow \text{Out}(\Delta_{g,r}^l)
\]

for the restriction to \( \Gamma^\text{rel-l}_{g,r} \subseteq \Pi^\text{rel-l}_{(M_{g,r})_k} \) of \( (\rho^\text{univ-l}_{g,r})_k \).

**Proposition 1.9.** Let \((g,r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\). Then the homomorphism

\[
\rho^\text{univ-l}_{g,r} : \Gamma^\text{rel-l}_{g,r} \longrightarrow \text{Out}(\Delta_{g,r}^l)
\]

is injective if and only if the homomorphism

\[
\rho^\text{univ-l}_{g,r+1} : \Gamma^\text{rel-l}_{g,r+1} \longrightarrow \text{Out}(\Delta_{g,r+1}^l)
\]

is injective.
Proof. Let us first observe that it follows immediately from the definition of the homomorphisms under consideration that, to verify Proposition 1.9, we may assume without loss of generality, by replacing \( k \) by \( \overline{k} \), that \( k \) is an algebraically closed field. Let \( X \) be a curve of type \((g, r)\) over \( k \), \( X_2 \) the 2-nd configuration space of \( X \), and \( Y \) the curve of type \((g, r + 1)\) over \( k \) obtained by removing a \( k \)-rational point from \( X \). Then it follows from Proposition 1.7 and [11, Lemma 15, (iv)] that we have the following commutative diagram of profinite groups

\[
\begin{array}{ccccccc}
1 & 1 \\
\downarrow & \downarrow \\
\Delta_Y & \Delta_Y \\
\downarrow & \downarrow \\
1 & \rightarrow & \Delta_{X_2} & \rightarrow & \Gamma_{g,r+2}^{rel-l} & \rightarrow & \Gamma_{g,r}^{rel-l} & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \rightarrow & \Delta_X & \rightarrow & \Gamma_{g,r+1}^{rel-l} & \rightarrow & \Gamma_{g,r}^{rel-l} & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

where the horizontal and vertical sequences are exact, the upper horizontal arrow is the identity morphism of \( \Delta_Y \), and the right-hand vertical arrow is the identity morphism of \( \Gamma_{g,r}^{rel-l} \). Now let us observe that one verifies easily that the outer action \( \Gamma_{g,r}^{rel-l} \rightarrow \text{Out}(\Delta_{X_2}^l) \) determined by the middle horizontal sequence of the above diagram factors through the subgroup

\[
\text{Out}^{\text{FC}}(\Delta_{X_2}^l) \subseteq \text{Out}(\Delta_{X_2}^l)
\]

of FC-admissible outer automorphisms (i.e., roughly speaking, outer automorphisms that preserve the fiber subgroups of \( \Delta_{X_2}^l \) and the cuspidal inertia subgroups of these fiber subgroups — cf. [20, Definition 1.1 (ii)]) of \( \Delta_{X_2}^l \). Therefore, it follows from [11, Lemma 17, (ii)] and [2, Remark following the proof of Theorem 1] that we obtain the following commutative diagram of profinite groups

\[
\begin{array}{cccc}
1 & \rightarrow & \Delta_X & \rightarrow & \Gamma_{g,r+1}^{rel-l} & \rightarrow & \Gamma_{g,r}^{rel-l} & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \rightarrow & \Delta_X & \rightarrow & \Gamma_{g,r+1}^{geo-l} & \rightarrow & \Gamma_{g,r}^{geo-l} & \rightarrow & 1
\end{array}
\]
where the horizontal sequences are exact, and the left-hand vertical arrow is the identity morphism of $\Delta^l_X$. In particular, $\rho_{g,r}^{\text{univ}-l}$ is injective (i.e., the right-hand vertical arrow of this diagram is injective) if and only if $\rho_{g,r+1}^{\text{univ}-l}$ is injective (i.e., the middle vertical arrow of this diagram is injective). This completes the proof of Proposition 1.9. □

**Remark 1.10.** A similar result to Proposition 1.9 for the profinite case can be found in [4, Lemma 3.6].

### 2. A pro-$2$ version of the congruence subgroup problem for mapping class groups of genus one

In the present §2, we maintain the notation of the preceding §1. In the present §2, we consider the congruence subgroup problem for the relative pro-$2$ completions of mapping class groups. In particular, we prove that the quotient of the profinite completion of the mapping class group of genus one determined by the pro-$2$ outer geometric monodromy representation coincides with the relative pro-$2$ completion of the mapping class group of genus one.

**Definition 2.1.** We shall write

$$(\mathcal{M}_{\text{Lgd}})_k$$

for the affine algebraic surface over $k$ defined by the equation $y^2 = x(x - 1)(x - \lambda)$ in $\text{Spec}(k[x, y, \lambda, \lambda^{-1}, (\lambda-1)^{-1}])$, where $x$, $y$, and $\lambda$ are indeterminates. Thus, the projection

$$(\mathcal{M}_{\text{Lgd}})_k \to \mathbb{P}^1_k \setminus \{0, 1, \infty\} \simeq (\mathcal{M}_{0,4})_k, \quad (x, y, \lambda) \mapsto \lambda$$

is a family of curves of type $(1,1)$, which we shall refer to as the Legendre family of elliptic curves. We shall write

$$\Pi^l_{\text{Lgd}}$$

for the geometrically pro-$l$ fundamental group of $(\mathcal{M}_{\text{Lgd}})_k$ and

$$\Delta^l_{\text{Lgd}}$$

for the pro-$l$ geometric fundamental group of $(\mathcal{M}_{\text{Lgd}})_k$.

**Remark 2.2.** It is well-known that the classifying morphism $(\mathcal{M}_{0,4})_k \to (\mathcal{M}_{1,1})_k$ of the family $(\mathcal{M}_{\text{Lgd}})_k \to (\mathcal{M}_{0,4})_k$ of curves of type $(1,1)$ is a finite étale covering of $(\mathcal{M}_{1,1})_k$. In particular, $\Pi^{\text{univ}}_{(\mathcal{M}_{0,4})_k}$ may be regarded as an open subgroup of $\Pi^{\text{univ}}_{(\mathcal{M}_{1,1})_k}$. 
Lemma 2.3. The following hold:

(i) The open subgroup $\Pi_{(\mathcal{M}_{0,4})} \rightarrow \Pi_{(\mathcal{M}_{1,1})}$ (cf. Remark 2.2) is contained in the kernel of the composite

$$\Pi_{(\mathcal{M}_{1,1})} \xrightarrow{(\rho_{\text{uni}}^{-2})_{k}} \text{Out}^C(\Delta_{1,1}) \xrightarrow{\varphi_{1,1}} \text{Aut}((\Delta_{1,0}^2)^{ab} \otimes \mathbb{Z}/2)).$$

(ii) The family $(\mathcal{M}_{Lgd})_{k} \rightarrow (\mathcal{M}_{0,4})_{k}$ of curves of type $(1,1)$ gives rise to an exact sequence

$$1 \longrightarrow \Delta_{1,1}^2 \longrightarrow \Pi_{Lgd_{k}}^{2} \longrightarrow \Pi_{(\mathcal{M}_{0,4})_{k}}^{\text{rel-2}} \longrightarrow 1.$$ 

(iii) The natural open injection $\Pi_{(\mathcal{M}_{0,4})_{k}} \rightarrow \Pi_{(\mathcal{M}_{1,1})_{k}}$ determines an open injection $\Gamma_{0,4}^{\text{rel-2}} \rightarrow \Gamma_{1,1}^{\text{rel-2}}$.

**Proof.** Assertion (i) follows from the fact that every “2-torsion point” of the family of elliptic curves over $(\mathcal{M}_{0,4})_{k}$ determined by $(\mathcal{M}_{Lgd})_{k} \rightarrow (\mathcal{M}_{0,4})_{k}$ is rational over $(\mathcal{M}_{0,4})_{k}$. Assertion (ii) follows from assertion (i), together with a similar argument to the argument applied in the proof of [9, Proposition 1.2]. Assertion (iii) follows formally from assertion (ii). \Box

**Definition 2.4.** We shall write

$$(\rho_{\text{Lgd}}^{2})_{k} : \Pi_{(\mathcal{M}_{0,4})_{k}}^{\text{rel-2}} \longrightarrow \text{Out}(\Delta_{1,1}^2)$$

for the outer action determined by the exact sequence of Lemma 2.3, (ii), and

$$\rho_{\text{Lgd}}^{2} : \Gamma_{0,4}^{\text{rel-2}} \longrightarrow \text{Out}(\Delta_{1,1}^2)$$

for the restriction of $(\rho_{\text{Lgd}}^{2})_{k}$ to $\Gamma_{0,4}^{\text{rel-2}} \subseteq \Pi_{(\mathcal{M}_{0,4})_{k}}^{\text{rel-2}}$.

**Definition 2.5.** We shall write

$$[2] : (\mathcal{M}_{Lgd\setminus Lgd[2]})_{k} \longrightarrow (\mathcal{M}_{Lgd})_{k}$$

for the finite étale covering over $(\mathcal{M}_{0,4})_{k}$ given by multiplication by 2 (i.e., relative to the operation on the family of elliptic curves given by the canonical relative compactification of $(\mathcal{M}_{Lgd})_{k}$ over $(\mathcal{M}_{0,4})_{k}$),

$$\Pi_{Lgd\setminus Lgd[2]}^{l}$$

for the geometrically pro-$l$ fundamental group of the covering $(\mathcal{M}_{Lgd\setminus Lgd[2]})_{k}$.
\[ \Delta^t_{\text{Lgd}\setminus \text{Lgd}[2]} \]

for the pro-\(t\) geometric fundamental group of the covering \((\mathcal{M}_{\text{Lgd}\setminus \text{Lgd}[2]})_k\), and

\[ \pi_1([2]): \Pi^2_{\text{Lgd}\setminus \text{Lgd}[2]}_k \rightarrow \Pi^2_{\text{Lgd}_k} \]

for the outer injection induced by the finite étale covering \((\mathcal{M}_{\text{Lgd}\setminus \text{Lgd}[2]})_k \to (\mathcal{M}_{\text{Lgd}})_k\).

**Lemma 2.6.** The following hold:

(i) The composite \((\mathcal{M}_{\text{Lgd}\setminus \text{Lgd}[2]})_k \to (\mathcal{M}_{\text{Lgd}})_k \to (\mathcal{M}_{0,4})_k\) is a family of curves of type \((1,4)\).

(ii) The exact sequence of Lemma 2.3, (ii), determines an exact sequence

\[ 1 \longrightarrow \Delta^2_{1,4} \longrightarrow \Pi^2_{\text{Lgd}\setminus \text{Lgd}[2]}_k \longrightarrow \Pi^{\text{rel}-2}_{(\mathcal{M}_{0,4})_k} \longrightarrow 1. \]

**Proof.** These assertions follow from the fact that the morphism \([2]\) is a finite étale abelian covering of degree 4. \(\square\)

**Definition 2.7.** We shall write

\[ (\rho^2_{\text{Lgd}\setminus \text{Lgd}[2]})_k: \Pi^{\text{rel}-2}_{(\mathcal{M}_{0,4})_k} \rightarrow \text{Out}(\Delta^2_{1,4}) \]

for the outer action determined by the exact sequence of Lemma 2.6, (ii), and

\[ \rho^2_{\text{Lgd}\setminus \text{Lgd}[2]}: \Gamma^{\text{rel}-2}_{0,4} \rightarrow \text{Out}(\Delta^2_{1,4}) \]

for the restriction of \((\rho^2_{\text{Lgd}\setminus \text{Lgd}[2]})_k\) to \(\Gamma^{\text{rel}-2}_{0,4} \subseteq \Pi^{\text{rel}-2}_{(\mathcal{M}_{0,4})_k}\).

**Lemma 2.8.** The quotient of \((\mathcal{M}_{\text{Lgd}\setminus \text{Lgd}[2]})_k\) by the automorphism over \((\mathcal{M}_{0,4})_k\) given by multiplication by \(-1\) (i.e., relative to the operation on the family of elliptic curves given by the canonical relative compactification of \((\mathcal{M}_{\text{Lgd}\setminus \text{Lgd}[2]})_k\) over \((\mathcal{M}_{0,4})_k\)) is isomorphic to \((\mathcal{M}_{0,5})_k\) over \((\mathcal{M}_{0,4})_k\).

**Proof.** Let us recall that \((\mathcal{M}_{\text{Lgd}\setminus \text{Lgd}[2]})_k\) is isomorphic to the affine algebraic surface over \(k\) defined by the equation \(y^2 = x(x - 1)(x - \lambda)\) in the affine scheme \(\text{Spec}(k[x, y, \lambda, y^{-1}, \lambda^{-1}, (\lambda - 1)^{-1}])\), where \(x\), \(y\), and \(\lambda\) are indeterminates. Moreover, let us also recall that the automorphism of \((\mathcal{M}_{\text{Lgd}\setminus \text{Lgd}[2]})_k\) under consideration is, relative to the above isomorphism, given by mapping \((x, y, \lambda)\) to \((x, -y, \lambda)\). Thus, Lemma 2.8 is immediate. \(\square\)
Definition 2.9. We shall write
\[ q: (\mathcal{M}_{\text{Lgd}\setminus\text{Lgd}[2]})_k \to (\mathcal{M}_{0,5})_k \]
for the resulting (cf. Lemma 2.8) finite étale covering over \((\mathcal{M}_{0,4})_k\) and
\[ \pi_1(q): \Pi^2_{\text{Lgd}\setminus\text{Lgd}[2]}_k \to \Pi^{\text{rel}-2}(\mathcal{M}_{0,5})_k \]
for the outer injection induced by \(q\).

Theorem 2.10. The homomorphism
\[ \rho^2_{\text{Lgd}}: \Gamma^{\text{rel}-2}_0 \to \text{Out}(\Delta^2_{1,1}) \]
is injective.

Proof. Let us first observe that we have the following commutative diagram of profinite groups

\[
\begin{array}{cccccc}
1 & \to & \Delta^2_{1,1} & \to & \Delta^2_{\text{Lgd}} & \to & \Gamma^{\text{rel}-2}_{0,4} & \to & 1 \\
\pi_1([2]) & & & & \pi_1([2]) & & & & \\
1 & \to & \Delta^2_{1,4} & \to & \Delta^2_{\text{Lgd}\setminus\text{Lgd}[2]} & \to & \Gamma^{\text{rel}-2}_{0,4} & \to & 1 \\
& & \pi_1(q) & & \pi_1(q) & & & & \\
1 & \to & \Delta^2_{0,4} & \to & \Gamma^{\text{rel}-2}_{0,5} & \to & \Gamma^{\text{rel}-2}_{0,4} & \to & 1 \\
\end{array}
\]

where the horizontal sequences are exact, the vertical arrows are injective, and the right-hand vertical arrows are the identity morphisms of \(\Gamma^{\text{rel}-2}_{0,4}\). By [11, Lemma 23, (i), (iii)], \(\ker(\rho^2_{\text{Lgd}\setminus\text{Lgd}[2]})\) is an open subgroup of \(\ker(\rho^2_{\text{Lgd}})\) and a subgroup of \(\ker(\rho^{\text{univ}-2}_{0,4})\). Thus, since \(\ker(\rho^{\text{univ}-2}_{0,4})\) is trivial (cf. [2, Remark following the proof of Theorem 1]), \(\ker(\rho^2_{\text{Lgd}\setminus\text{Lgd}[2]})\) is trivial. In particular, \(\ker(\rho^2_{\text{Lgd}})\) is a finite group. On the other hand, since \(\Gamma^{\text{rel}-2}_{0,4} \cong \Delta^2_{0,2}\) is torsion-free (cf., e.g., [21, Remark 1.2.2]), \(\ker(\rho^2_{\text{Lgd}})\) is trivial. This completes the proof of Theorem 2.10. \(\square\)

Corollary 2.11. Let \(r\) be a positive integer. Then the homomorphism
\[ \rho^{\text{univ}-2}_{1,r}: \Gamma^{\text{rel}-2}_{1,r} \to \text{Out}(\Delta^2_{1,r}) \]
is injective.

In particular, the problem \((\text{CSP})^{\text{pro}-2}_{1,r}\) in the Introduction has an affirmative answer.
Proof. First, we consider the first portion of Corollary 2.11. Let us first observe that, by Proposition 1.9, to verify the first portion of Corollary 2.11, it suffices to verify the first portion of Corollary 2.11 in the case where \( r = 1 \). Suppose that \( r = 1 \). Next, recall from Lemma 2.3, (iii), that the natural open injection \( \Pi(\mathcal{M}_{0,1}) \hookrightarrow \Pi(\mathcal{M}_{1,1}) \) determines an open injection \( \Gamma_{0,1} \hookrightarrow \Gamma_{1,1} \). Thus, since the kernel of the homomorphism

\[
\varphi_{1,1} \circ \rho_{1,1}^{\text{univ}} : \Gamma_{1,1} \longrightarrow \text{Aut}(\Delta_{1,0}^2 \otimes \mathbb{Z}_4 (\mathbb{Z}/4))
\]

is torsion-free (cf., e.g., [18, §1.4], [21, Remark 1.2.2]), we conclude from Theorem 2.10 that \( \ker(\rho_{1,1}^{\text{univ}}) \) is trivial, as desired. This completes the proof of the first portion of Corollary 2.11.

The final portion of Corollary 2.11 follows immediately from the first portion of Corollary 2.11, together with the discussion following the statement of the problem (CSP)$_{g,r}^{\text{pro-l}}$ in the Introduction. This completes the proof of Corollary 2.11. \( \Box \)

Remark 2.12. The argument given in the proof of Corollary 2.11 is essentially the same as the argument applied in [2] to prove [2, Theorem 5].

3. A pro-l version of the congruence subgroup problem for mapping class groups of genus one: the general case

In the present §3, we maintain the notation of §1 and the preceding §2. In the present §3, we continue our study of the congruence subgroup problem for the relative pro-l completions of mapping class groups. In particular, we prove that, if \( l \neq 2, 3, 5, 7 \), then the quotient of the profinite completion of the mapping class group of genus one determined by the pro-l outer geometric monodromy representation does not coincide with the relative pro-l completion of the mapping class group of genus one.

In the present §3, by means of an injection \( \overline{\mathbb{Q}} \hookrightarrow \overline{k} \), let us regard \( \overline{\mathbb{Q}} \) as a subfield of \( \overline{k} \). Write \( G_{\overline{Q}} := \text{Gal}(\overline{Q}/\mathbb{Q}) \), and \( \zeta_l \in \overline{\mathbb{Q}} \) for a primitive \( l \)-th root of unity. For a subfield \( F \) of \( \overline{k} \) which contains \( \zeta_l \), write \( Y(l)_F \) for the modular curve over \( F \) with respect to the congruence subgroup \( \Gamma(l) \), i.e., the coarse moduli scheme of elliptic curves with \( \Gamma(l) \)-structures (cf., e.g., [19, Chapter 3]), and \( X(l)_F \) for the smooth compactification of \( Y(l)_F \) over \( F \).
Lemma 3.1. Let

\[
\begin{array}{ccccccc}
1 & \to & \Delta & \to & \Pi_1 & \to & G_1 & \to & 1 \\
\downarrow & & \downarrow i_{\Pi_1} & & \downarrow & & \downarrow & & \\
1 & \to & \Pi_2 & \to & \Pi_3 & \to & G_1 & \to & 1 \\
\downarrow i_{\Pi_2} & & \downarrow i_{\Pi_2} & & \downarrow & & \downarrow & & \\
G_2 & & G_2 & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & & & & & 
\end{array}
\]

be a commutative diagram of profinite groups, where the horizontal and vertical sequences are exact, the right-hand vertical arrow is the identity morphism of \(G_1\), and the lower horizontal arrow is the identity morphism of \(G_2\). Write

\[\rho_1 : G_1 \to \text{Out}(\Delta) \quad \text{and} \quad \rho_2 : G_2 \to \text{Out}(\Delta)\]

for the outer actions associated to the top horizontal and left-hand vertical exact sequences, respectively. Then any element of \(\text{im}(\rho_2)\) commutes with any element of \(\text{im}(\rho_1)\).

Proof. One verifies easily that the left-hand upper square in the diagram in the statement of Lemma 3.1 is cartesian, i.e., the equality \(i_{\Pi_1} \circ i^{\Pi_1}(\Delta) = i_{\Pi_1}(\Pi_1) \cap i_{\Pi_2}(\Pi_2)\) holds. Thus, it follows that the commutator subgroup \([i_{\Pi_1}(\Pi_1), i_{\Pi_2}(\Pi_2)] \subseteq \Pi_3\) is contained in \(i_{\Pi_1} \circ i^{\Pi_1}(\Delta)\). In particular, one verifies immediately from the various definitions involved that any element of \(\text{im}(\rho_2)\) commutes with any element of \(\text{im}(\rho_1)\). This completes the proof of Lemma 3.1. \(\square\)

Lemma 3.2. Let

\[
\begin{array}{ccccccc}
1 & \to & \Delta_1 & \to & \Pi_1 & \to & G & \to & 1 \\
\downarrow \alpha & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \Delta_2 & \to & \Pi_2 & \to & G & \to & 1 
\end{array}
\]

be a commutative diagram of profinite groups, where the horizontal sequences are exact and the right-hand vertical arrow is the identity morphism of \(G\). Write

\[\rho_1 : G \to \text{Out}(\Delta_1), \quad \rho_2 : G \to \text{Out}(\Delta_2)\]
for the outer actions of $G$ on $\Delta_1$, $\Delta_2$ determined by the upper, lower horizontal sequences of the above diagram, respectively. Suppose that the profinite group $\Delta_1$ is center-free, that the homomorphism $\alpha$ is injective, and that the image of the middle vertical arrow $\Pi_1 \to \Pi_2$ of the above diagram is normal. Then it holds that $\ker(\rho_1) \subseteq \ker(\rho_2)$.

**Proof.** Let us observe that, to verify Lemma 3.2, it follows immediately from the various definitions involved that it suffices to verify the inclusion

$$Z_{\Pi_1}(\Delta_1) \subseteq Z_{\Pi_2}(\Delta_2).$$

Now since $\Delta_1 = \Pi_1 \cap \Delta_2$ is normal in $\Pi_2$, hence also in $\Delta_2$, we have the outer action $\Delta_2/\Delta_1 \to \text{Out}(\Delta_1)$ induced by the action $\Delta_2 \to \text{Aut}(\Delta_1)$ by conjugation. Write $\text{Aut}(\Delta_1) \times_{\text{Out}(\Delta_1)} (\Delta_2/\Delta_1)$ for the fiber product of the natural surjection $\text{Aut}(\Delta_1) \to \text{Out}(\Delta_1)$ and the above outer action $\Delta_2/\Delta_1 \to \text{Out}(\Delta_1)$ over $\text{Out}(\Delta_1)$. Since $\Delta_1$ is center-free, the homomorphism

$$\Delta_2 \ni \delta \mapsto ((\Delta_1 \ni \gamma \mapsto \delta \cdot \gamma \cdot \delta^{-1} \in \Delta_1), \overline{\delta}) \in \text{Aut}(\Delta_1) \times_{\text{Out}(\Delta_1)} (\Delta_2/\Delta_1)$$

where $\overline{\delta}$ is the image of $\delta$ in $\Delta_2/\Delta_1$ — is an isomorphism. Thus, by considering the action $\Pi_2 \to \text{Aut}(\Delta_1) \times_{\text{Out}(\Delta_1)} (\Delta_2/\Delta_1)$ induced by the action $\Pi_2 \to \text{Aut}(\Delta_2)$ by conjugation and the above isomorphism $\Delta_2 \cong \text{Aut}(\Delta_1) \times_{\text{Out}(\Delta_1)} (\Delta_2/\Delta_1)$, to verify the above inclusion $Z_{\Pi_1}(\Delta_1) \subseteq Z_{\Pi_2}(\Delta_2)$, it suffices to verify the following assertion:

If $\alpha \in Z_{\Pi_1}(\Delta_1)$, then the automorphism of $\Delta_2/\Delta_1$ obtained by conjugation by $\alpha \in \Pi_1 \subseteq \Pi_2$ is the identity automorphism.

On the other hand, the commutative diagram of the statement of Lemma 3.2 determines an isomorphism $\Delta_2/\Delta_1 \cong \Pi_2/\Pi_1$. Thus, since $\alpha \in Z_{\Pi_1}(\Delta_1) \subseteq \Pi_1$, it follows that the automorphism of $\Delta_2/\Delta_1$ obtained by conjugation by $\alpha$ is the identity automorphism. This completes the proof of Lemma 3.2. \(\square\)

**Proposition 3.3.** Suppose that $k$ contains $\zeta_l$, and that $l > 2$. Then the equality

$$\ker(\rho^l_{1 (l)_k}) = \ker((\rho^1_{1, 1})_k)$$

(cf. Definitions 1.1, (iii); 1.4, (iii)) holds.

**Proof.** Let us first observe that, by the various definitions involved, we have the following commutative diagram of profinite groups
where the vertical and horizontal sequences are exact, the lower horizontal arrow is the identity morphism of $G_k$, and the right-hand vertical arrow is the identity morphism of the special linear group $SL_2(\mathbb{F}_l)$ of degree 2 over $\mathbb{F}_l$. In particular, since $\Delta_{Y(l)k}^l$ is center-free (cf., e.g., [18, §1.4], [21, Proposition 1.4]), it follows from Lemma 3.2 that, to verify Proposition 3.3, it suffices to verify that

$$\ker(\rho_{Y(l)k}^l) \supseteq \ker((\rho_{1,1}^l)_k).$$

Write $\varphi_{SL}^l: SL_2(\mathbb{F}_l) \to \text{Out}(\Delta_{Y(l)k}^l)$ for the homomorphism determined by the upper horizontal sequence of the above commutative diagram. Let $\sigma$ be an element of $\ker((\rho_{1,1}^l)_k)$. Note that it follows immediately from the various definitions involved that $\rho_{Y(l)k}^l(\sigma)$ is contained in $\im(\varphi_{SL}^l)$.

First, suppose that $l > 3$. Since (one verifies easily that)

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{F}_l)$$

is contained in the image of the restriction to $Z(\Gamma_{1,1}^{\text{rel}-l}) \subseteq \Gamma_{1,1}^{\text{rel}-l}$ of $\Gamma_{1,1}^{\text{rel}-l} \to SL_2(\mathbb{F}_l)$, it follows immediately that the homomorphism $\varphi_{SL}^l: SL_2(\mathbb{F}_l) \to \text{Out}(\Delta_{Y(l)k}^l)$ factors through the projective special linear group $PSL_2(\mathbb{F}_l)$ of degree 2 over $\mathbb{F}_l$. Hence, it follows from the simplicity of the group $PSL_2(\mathbb{F}_l)$ (cf., e.g., [5, Chapter II, §10, Exercise 14]) that the image of $\varphi_{SL}^l$ is isomorphic to either $\{1\}$ or $PSL_2(\mathbb{F}_l)$, which thus implies that $\im(\varphi_{SL}^l)$ is center-free. In particular, by Lemma 3.1, together with the fact that $\rho_{Y(l)k}^l(\sigma) \in \im(\varphi_{SL}^l)$ (already verified above), we conclude that $\sigma$ is contained in $\ker(\rho_{Y(l)k}^l)$. This completes the proof of the case where $l > 3$.

Next, suppose that $l = 3$. Write $\text{Cusp}(\Delta_{Y(3)k}^3)$ for the set of conjugacy classes of cuspidal inertia subgroups of $\Delta_{Y(3)k}^3$. Let us recall that $Y(3)_k$ is a curve of type $(0, 4)$ over $k$,
and every cusp of $Y(3)_k$ is $k$-rational (cf. [18, (A1.5.1)], [18, §1.4]). Also, \( \varphi^3_{SL}: SL_2(\mathbb{F}_3) \to \text{Out}(\Delta^3_{Y(3)_k}) \) factors through the natural homomorphism \( \text{Aut}_k(Y(3)_k) \to \text{Out}(\Delta^3_{Y(3)_k}) \) by the various definitions involved. Thus, the natural action of \( \text{im}(\varphi^3_{SL}) \) on \( \text{Cusp}(\Delta^3_{Y(3)_k}) \) is faithful. On the other hand, by means of the fact that every cusp of \( Y(3)_k \) is defined over \( k \), we may verify that the natural action of \( \text{im}(\varphi^3_{SL}) \) on \( \text{Cusp}(\Delta^3_{Y(3)_k}) \) is trivial. Therefore, by the fact that \( \rho^3_{Y(3)_k}(\sigma) \in \text{im}(\varphi^3_{SL}) \) (already verified above), we conclude that \( \sigma \) is contained in \( \ker(\rho^3_{Y(3)_k}) \). This completes the proof of the case where \( l = 3 \), hence also of Proposition 3.3. \( \square \)

**Theorem 3.4** (cf. [16, Corollary 3.8]). Let \( (g, r) \) be a pair of nonnegative integers such that \( 3g - 3 + r > 0 \). Then the kernel of the homomorphism \( (\rho^l_{g,r})_k \) is contained in the kernel of the homomorphism

\[
\rho^l_{E^1_k \setminus \{0,1,\infty\}} : G_k \to \text{Out}(\Delta^l_{E^1_k \setminus \{0,1,\infty\}}).
\]

**Proof.** If either \( (g, r) \neq (1, 1) \) or \( l = 2 \), then Theorem 3.4 follows from [16, Corollary 3.8]. Thus, to verify Theorem 3.4, we may assume that \( (g, r) = (1, 1) \) and \( l > 2 \). Next, let us observe that one verifies immediately that, to complete the verification of Theorem 3.4, it suffices to verify Theorem 3.4 in the case where we take “\( k \)” to be \( \mathbb{Q} \). Moreover, we claim that

Theorem 3.4 in the case where we take “\( k \)” to be \( \mathbb{Q}(\xi_l) \) implies Theorem 3.4 (i.e., Theorem 3.4 in the case where we take “\( k \)” to be \( \mathbb{Q} \)).

Indeed, suppose that \( \ker((\rho^l_{1,1})_{\mathbb{Q}(\xi_l)}) \) is contained in \( \ker(\rho^l_{E^1_{\mathbb{Q}(\xi_l)} \setminus \{0,1,\infty\}}) \). Then we have that

\[
\ker((\rho^l_{1,1})_{\mathbb{Q}(\xi_l)}) = \ker((\rho^l_{1,1})_{\mathbb{Q}}) \cap G_{\mathbb{Q}(\xi_l)} \subseteq \ker(\rho^l_{E^1_{\mathbb{Q}(\xi_l)} \setminus \{0,1,\infty\}}) \subseteq \ker(\rho^l_{E^1_k \setminus \{0,1,\infty\}}).
\]

In particular, the image of \( \ker((\rho^l_{1,1})_{\mathbb{Q}}) \) by the natural homomorphism \( G_{\mathbb{Q}} \to G_{\mathbb{Q}}/\ker(\rho^l_{E^1_{\mathbb{Q}} \setminus \{0,1,\infty\}}) \) is a finite normal subgroup of \( G_{\mathbb{Q}}/\ker(\rho^l_{E^1_k \setminus \{0,1,\infty\}}) \). On the other hand, it follows from [10, Lemma 4.3, (ii)] that \( G_{\mathbb{Q}}/\ker(\rho^l_{E^1_{\mathbb{Q}} \setminus \{0,1,\infty\}}) \) is slim. Thus, the above claim follows from Lemma 3.5 below. This completes the proof of the claim. It follows from the above claim that, to complete the verification of Theorem 3.4, we may assume without loss of generality that \( k = \mathbb{Q}(\xi_l) \).

Since \( Y(l)_k \) is a hyperbolic curve over \( k \), it follows from [13, Theorem C] that

\[
\ker(\rho^l_{Y(l)_k}) \subseteq \ker(\rho^l_{E^1_k \setminus \{0,1,\infty\}}).
\]

Thus, since (we have assumed that) \( l > 2 \), Theorem 3.4 follows immediately from Proposition 3.3. This completes the proof of Theorem 3.4. \( \square \)
Lemma 3.5. Let $G$ be a profinite group. If $G$ is slim, then any finite normal closed subgroup of $G$ is trivial.

**Proof.** Let $N \subseteq G$ be a finite normal closed subgroup of $G$. Since $G$ is a profinite group, there exists a normal open subgroup $H \subseteq G$ of $G$ which satisfies the equality $N \cap H = \{1\}$. Thus, since both $N$ and $H$ are normal in $G$, it follows that $N$ is contained in $Z_G(H)$. On the other hand, it follows from the slimness of $G$ that $Z_G(H)$, hence also $N$, is trivial. This completes the proof of Lemma 3.5. □

The following Theorem says that if $l$ is odd, and regular, then Ihara’s problem concerning the pro-$l$ outer Galois action associated to a tripod (cf., e.g., [15, Lecture I, §2], [23, Introduction]) has an affirmative answer.

**Theorem 3.6** (Sharifi, Brown). Suppose that $l$ is odd, and regular. Then the extension $\mathbb{Q}^{ker(\rho_{1,l}^{\text{rel}})}_{\mathbb{Q}(\{0,1,\infty\})}$ of $\mathbb{Q}$ is equal to the maximal Galois extension $\mathbb{Q}^{\text{un-l}}$ of $\mathbb{Q}$ that satisfies the following conditions:

1. The extension $\mathbb{Q}^{\text{un-l}}/\mathbb{Q}$ is unramified at every prime number $\neq l$.
2. The field $\mathbb{Q}(\zeta_l)$ is a subfield of $\mathbb{Q}^{\text{un-l}}$, and the extension $\mathbb{Q}^{\text{un-l}}/\mathbb{Q}(\zeta_l)$ is pro-$l$.

**Proof.** See [24, Theorem 1.1], and the main result of [6]. □

**Theorem 3.7.** The equality

$$\ker((\rho_{1,1}^{\text{rel-l}})_\mathbb{Q}) = \ker(\rho_{\mathbb{Q}[\zeta_1 \setminus \{0,1,\infty\}]})$$

holds if and only if

$$l \in \{2,3,5,7\}.$$ 

In particular, if $r$ is a positive integer, and

$$l \notin \{2,3,5,7\},$$

then

$$\ker((\rho_{1,r}^{\text{rel-l}})_\mathbb{Q}) \subsetneq \ker(\rho_{\mathbb{Q}[\zeta_1 \setminus \{0,1,\infty\}])$$

**Proof.** Let us first observe that, for any positive integer $r$, we have the following commutative diagram of profinite groups
where the horizontal sequences are exact, the vertical arrows are surjective, and the right-hand vertical arrow is the identity morphism of $G_Q$. Thus, one verifies easily that \( \ker((\rho^l_{1,r+1})_Q) \subseteq \ker((\rho^l_{1,r})_Q) \). In particular, it follows from Theorem 3.7 that, to verify Theorem 3.7, it suffices to verify the case \( r = 1 \) of Theorem 3.7.

First, suppose that \( l = 2 \). Then it follows from Corollary 2.11 and [16, Theorem 3.4] that the equalities

\[
\ker((\rho^l_{1,1})_Q) = \ker((\rho^{geo-2}_{1,1})_Q) = \ker(\rho^2_{\mathcal{E}_l} \setminus \{0,1,\infty\})
\]

hold. This completes the proof of the case where \( l = 2 \).

Next, suppose that \( l \in \{3,5,7\} \). Thus, since \( l \) is regular and odd, and \( (\rho^l_{1,1})_Q(G_Q(\mathcal{G})) \) is pro-\( l \) (cf. [12, Theorem A], Proposition 3.3), it follows immediately from Theorem 3.6, together with [8, Theorem 3], that the equality

\[
\ker((\rho^l_{1,1})_Q) = \ker(\rho^l_{\mathcal{E}_l} \setminus \{0,1,\infty\})
\]

holds. This completes the proof of the case where \( l \in \{3,5,7\} \).

Finally, suppose that \( l \notin \{2,3,5,7\} \). Then it follows from Proposition 3.3 that, to complete the verification of this case, it suffices to prove that

\[
\ker(\rho^l_{Y(l)_Q(\mathcal{G})}) \neq \ker(\rho^l_{\mathcal{E}_l(\mathcal{G})} \setminus \{0,1,\infty\}).
\]

On the other hand, if

\[
\ker(\rho^l_{Y(l)_Q(\mathcal{G})}) = \ker(\rho^l_{\mathcal{E}_l(\mathcal{G})} \setminus \{0,1,\infty\}),
\]

then it follows from [14, I, §5, Proposition 7, (ii)] that the image of \( \rho^l_{Y(l)_Q(\mathcal{G})} \) is pro-\( l \), which contradicts [12, Theorem A]. This completes the proof of the case where \( l \notin \{2,3,5,7\} \), hence also of Theorem 3.7. \( \square \)

The following Theorem says that Oda’s problem concerning the universal pro-\( l \) outer monodromy representation (cf., e.g., [25, §0]) has an affirmative answer.

**Theorem 3.8** (Ihara, Oda, Nakamura, Takao). The kernel of the composite

\[
G_Q \xrightarrow{\sim} \Pi^rel_{(\mathcal{M}_{g,r})_Q} / \Gamma^{rel-l} \rightarrow \text{im}((\rho^l_{\text{univ-l}})_Q) / (\rho^l_{\text{univ-l}})_Q(\Gamma^{rel-l})
\]
where the first homomorphism is the isomorphism determined by the exact sequence (3) of Definition 1.4, (iii), and the second homomorphism is the surjection determined by \((\rho_{g,r}^{\text{univ-l}})_Q\) — is equal to the kernel of \(\rho_{\mathbb{Z}_q \setminus \{0,1,\infty\}}^l\).

**Proof.** See [25, Theorem 0.5, (2)]. \(\square\)

**Corollary 3.9.** Let \(r\) be a positive integer. Suppose that

\[ l \notin \{2, 3, 5, 7\}. \]

Then the homomorphism

\[ \rho_{1,r}^{\text{univ-l}}: \Gamma_{1,r}^{\text{rel-l}} \to \text{Out}(\Delta_{1,r}^l) \]

is not injective.

In particular, the problem \((\text{CSP})_{1,r}^{\text{pro-l}}\) in the Introduction has a negative answer.

**Proof.** Let us first observe that it follows from Theorem 3.7 that

\[ \ker((\rho_{1,r}^{\text{rel-l}})_Q) \subseteq \ker(\rho_{\mathbb{Z}_q \setminus \{0,1,\infty\}}^l). \]

On the other hand, by Theorem 3.8, we have that

\[ \ker(\rho_{\mathbb{Z}_q \setminus \{0,1,\infty\}}^l) \subseteq \ker((\rho_{1,r}^{\text{geo-l}})_Q). \]

Thus, since the injectivity of \(\rho_{1,r}^{\text{univ-l}}\) implies, by definition, the equality

\[ \ker((\rho_{1,r}^{\text{rel-l}})_Q) = \ker((\rho_{1,r}^{\text{geo-l}})_Q), \]

we conclude that \(\rho_{1,r}^{\text{univ-l}}\) is not injective. This completes the proof of the first portion of Corollary 3.9. Moreover, the final portion of Corollary 3.9 follows immediately from the discussion following the statement of the problem \((\text{CSP})_{1,r}^{\text{pro-l}}\) in the Introduction. This completes the proof of Corollary 3.9. \(\square\)

**Corollary 3.10.** Let \(r\) be a positive integer, \(N \subseteq \Gamma_{1,r}\) a normal open subgroup of \(\Gamma_{1,r}\), and \((\Gamma_{1,r})_N^l\) the relative pro-l completion of \(\Gamma_{1,r}\) with respect to \(\Gamma_{1,r} \to \Gamma_{1,r}/N\) (cf. “Profinite groups” in “Notations and Conventions”). Suppose that

\[ l \notin \{2, 3, 5, 7\}, \]

and that the homomorphism \((\rho_{1,r}^{\text{univ-l}})_{k|\Gamma_{1,r}}\) factors through \((\Gamma_{1,r})_N^l\). Then the resulting homomorphism
\[(\rho_{1,r}^{\text{univ-}l})_N : (\Gamma_{1,r})_N \rightarrow \text{Out}(\Delta_{1,r}^l)\]

is not injective.

**Proof.** Corollary 3.10 follows immediately (in light of Lemma 3.12 below) from Corollary 3.9, together with Lemma 3.11 below, in the case where we take “\((G, H, N_1, N_2)\)” in the statement of Lemma 3.11 below, to be \((\Gamma_{1,r}, \text{Out}(\Delta_{1,r}^l), \Gamma_{1,r}[l], N)\). □

**Lemma 3.11.** Let \(f : G \rightarrow H\) be a homomorphism of profinite groups. For \(i = 1, 2\), let \(N_i \subseteq G\) be a normal open subgroup of \(G\), \(N_i^l\) the maximal pro-\(l\) quotient of \(N_i\), and \(G_{N_i}^l\) the relative pro-\(l\) completion of \(G\) with respect to \(G \rightarrow G/N_i\) (cf. “Profinite groups” in “Notations and Conventions”). Suppose that \(N_1^l\) is torsion-free, and that \(f : G \rightarrow H\) factors through \(G_{N_1}^l\) and \(G_{N_2}^l\). Write \(f_{N_i} : G_{N_i}^l \rightarrow H\) for the resulting homomorphism for \(i = 1, 2\). Suppose, moreover, that the kernel of \(f : G \rightarrow H\) is contained in \(N_1\), and that \(f_{N_2} : G_{N_2}^l \rightarrow H\) is injective. Then \(f_{N_1} : G_{N_1}^l \rightarrow H\) is injective.

**Proof.** Write \(N_3 := N_1 \cap N_2\) and \(G_{N_3}^l\) for the relative pro-\(l\) completion of \(G\) with respect to \(G \rightarrow G/N_3\). Let us observe that since \(f_{N_2} : G_{N_2}^l \rightarrow H\) is injective, one verifies easily that \(\ker(f) = \ker(N_2 \rightarrow N_2^l)\), which thus implies that \(N_2/\ker(f)\) is pro-\(l\). Thus, it follows that \(N_3\) is a normal open subgroup of \(G\) which contains the kernel of \(f : G \rightarrow H\). In particular, the quotient \(N_2/N_3\), hence also \(N_2/\ker(N_3 \rightarrow N_3^l)\), is pro-\(l\). Therefore, by considering the natural exact sequence of profinite groups

\[
1 \longrightarrow \ker(N_3 \rightarrow N_3^l) \longrightarrow \ker(N_2 \rightarrow N_2^l) \longrightarrow N_2/\ker(N_3 \rightarrow N_3^l),
\]

we conclude that \(\ker(N_3 \rightarrow N_3^l) = \ker(N_2 \rightarrow N_2^l)\), i.e., \(f : G \rightarrow H\) determines an injection \(G_{N_3}^l \rightarrow H\). In particular, by replacing \(N_2\) by \(N_3\), we may assume that \(N_2 \subseteq N_1\). Then since \(f_{N_2} : G_{N_2}^l \rightarrow H\) factors through \(G_{N_1}^l\), and \(f_{N_2} : G_{N_2}^l \rightarrow H\) is injective, we have the following commutative diagram of profinite groups:

\[
\begin{array}{ccc}
N_2^l & \longrightarrow & N_1^l \\
\downarrow f_{N_2|N_2^l} & & \downarrow f_{N_1|N_1^l} \\
H. & &
\end{array}
\]

Thus, since the top arrow \(N_2^l \rightarrow N_1^l\) is an open injection, and \(N_1^l\) is torsion-free, it holds that \(f_{N_1|N_1^l}\), hence also \(f_{N_1}\), is injective. This completes the proof of Lemma 3.11. □

**Lemma 3.12.** Let \(r\) be a positive integer. Suppose that \(l > 2\). Then \((\Gamma_{1,r}[l])^l\) (cf. Definition 1.4, (ii)) is slim and torsion-free.

**Proof.** Let us first observe that it follows from Proposition 1.7 and the definition of \(\Gamma_{1,r}^{\text{univ}-l}\) that we have the following exact sequence
Thus, since $\Delta_{l_1,r}$ is slim and torsion-free (cf., e.g., [21, Proposition 1.4], [21, Remark 1.2.2]), it follows from induction on $r$ that, to verify Lemma 3.12, we may assume without loss of generality that $r = 1$. Then it follows from the various definitions involved that $(\Gamma_{1,1}[l])^l$ is isomorphic to $\Delta_{Y(\mathbb{Q}(\Omega))}$. In particular, by [18, §1.4], [21, Remark 1.2.2], and [21, Proposition 1.4], $(\Gamma_{1,1}[l])^l$ is slim and torsion-free. This completes the proof of Lemma 3.12. □

4. Some remarks

In the present §4, let $(g, r)$ be a pair of nonnegative integers such that $3g - 3 + r > 0$.

4.1. In the summer of 2011, Makoto Matsumoto gave the second author the following problem:

$$(M_{g,r,l}) : \text{Does the kernel of the homomorphism } (\rho_{g,r}^{rel-l})_k \text{ coincide with the kernel of the homomorphism}$$

$$\rho^l_{\mathbb{P}_k^1 \\setminus \{0,1,\infty\}} : G_k \to \text{Out}(\Delta^l_{\mathbb{P}_k^1 \\setminus \{0,1,\infty\}})?$$

The second author proved, in response to this problem, the following Theorem (cf. [16, Theorem 3.4]):

Suppose that either $(g, r) \neq (1,1)$ or $l = 2$. Then the kernel of the homomorphism $(\rho_{g,r}^{rel-l})_k$ coincides with the kernel of the homomorphism

$$\rho^l_{\mathbb{P}_k^1 \\setminus \{0,1,\infty\}} : G_k \to \text{Out}(\Delta^l_{\mathbb{P}_k^1 \\setminus \{0,1,\infty\}}).$$

However, the problem $(M_{g,r,l})$ itself was answered in the negative if $g = 1$ and $l \neq 2, 3, 5, 7$ (cf. Theorem 3.7).

4.2. Let us recall that, as is well-known, there exists an isomorphism of $(\mathcal{M}_{0,4})_k$ with $\mathbb{P}_k^1 \setminus \{0,1,\infty\}$ over $k$. Thus, it follows immediately from Definition 1.4, (iii), that we have an equality

$$\ker(\rho^l_{\mathbb{P}_k^1 \\setminus \{0,1,\infty\}}) = \ker((\rho_{0,4}^{rel-l})_k).$$

In particular, the problem $(M_{g,r,l})$ of §4.1 is equivalent to the following problem:

Does the equality $\ker((\rho_{g,r}^{rel-l})_k) = \ker((\rho_{0,4}^{rel-l})_k)$ hold?
That is to say, roughly speaking, the problem \((M_{g,r,l})\) of §4.1 concerns the issue of whether or not the kernel \(\ker((\rho_{g,r}^{rel-l})_k)\) is independent of the pair \((g,r)\) (i.e., depends only on \(l\)).

4.3. From the point of view of the discussion of §4.2, one can pose the following problem, which may be regarded as a weaker version of the problem \((M_{g,r,l})\) of §4.1:

\((M_{g,r,l}^w)\): Does the kernel of the homomorphism \((\rho_{g,r}^{rel-l})_k\) coincide with the kernel of the homomorphism \((\rho_{g,r+1}^{rel-l})_k\)?

That is to say, roughly speaking, this problem \((M_{g,r,l}^w)\) concerns the issue of whether or not the kernel \(\ker((\rho_{g,r}^{rel-l})_k)\) is independent of \(r\) (i.e., depends only on \(g\) and \(l\)).

Now let us observe that we have the following commutative diagram of profinite groups

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & I_{g,r+1}^{rel-l} & \longrightarrow & \Pi_{(M_{g,r+1})_k}^{rel-l} & \longrightarrow & G_k & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I_{g,r}^{rel-l} & \longrightarrow & \Pi_{(M_{g,r})_k}^{rel-l} & \longrightarrow & G_k & \longrightarrow & 1 \\
\end{array}
\]

where the horizontal sequences are exact, the vertical arrows are surjective, and the right-hand vertical arrow is the identity morphism of \(G_k\). In particular, since the left-hand vertical arrow of this diagram is surjective, it follows from [12, Lemma 2.2] that

\[
\ker((\rho_{g,r+1}^{rel-l})_k) \subseteq \ker((\rho_{g,r}^{rel-l})_k).
\]

Thus, we conclude that

the problem \((M_{g,r,l}^w)\) has an affirmative answer if and only if the inclusion

\[
\ker((\rho_{g,r}^{rel-l})_k) \subseteq \ker((\rho_{g,r+1}^{rel-l})_k)
\]

holds.

4.4. By [20, Corollary 4.2, (ii)], the problem \((M_{0,r,l})\) of §4.1, hence also the problem \((M_{0,r,l}^w)\) of §4.3, is answered in the affirmative for any integer \(r > 3\). Also, by Corollary 2.11 and [16, Theorem 3.4], the problem \((M_{1,r,2})\) of §4.1, hence also the problem \((M_{1,r,2}^w)\) of §4.3, is answered in the affirmative for any positive integer \(r\).
4.5. Moreover, we can prove the following assertion:

Suppose that

\[
    r > \begin{cases} 
        3 & \text{if } g = 0, \\
        1 & \text{if } g = 1, \\
        0 & \text{if } g > 1. 
    \end{cases}
\]

Then \((M_{g,r,l}^w)\) is answered in the affirmative.

Indeed, let \(\gamma\) be an element of \(\ker((\rho_{g,r}^{rel,l})_k)\), \(X\) a curve of type \((g, r - 1)\) over \(\overline{k}\), \(X_2\) the 2-nd configuration space of \(X\), and \(Y\) the curve of type \((g, r)\) over \(\overline{k}\) obtained by removing a \(\overline{k}\)-rational point from \(X\). Let us first observe that it follows from Proposition 1.7 that we have the following commutative diagram of profinite groups

\[
\begin{array}{ccccccccc}
1 & & 1 & & & & & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Delta_Y & & \Delta_Y & & & & \Delta_Y & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \Gamma_{g,r+1}^{rel,l} & \rightarrow & \Pi_{(M_{g,r+1})_k}^{rel,l} & \rightarrow & G_k & \rightarrow & 1 \\
& & & & & & & & & \\
& & & & & & & & & \\
1 & \rightarrow & \Gamma_{g,r}^{rel,l} & \rightarrow & \Pi_{(M_{g,r})_k}^{rel,l} & \rightarrow & G_k & \rightarrow & 1 \\
& & & & & & & & & \\
& & & & & & & & & \\
1 & & 1 & & & & & & & \\
\end{array}
\]

where the vertical and horizontal sequences are exact, the top horizontal arrow is the identity morphism of \(\Delta_Y\), and the right-hand vertical arrow is the identity morphism of \(G_k\). Since \(\gamma\) is an element of \(\ker((\rho_{g,r}^{rel,l})_k)\), there exists an element \(\gamma'\) of \(p_{g,r}^{-1}(\{\gamma\}) \cap Z_{H^{rel,l}_{(M_{g,r})_k}}(\Gamma_{g,r}^{rel,l})\). Let \(\tilde{\gamma}\) be an element of \((p_{g,r+1}^{rel,l})^{-1}(\{\gamma'\})\). Then since \(\Delta_Y\) is center-free and topologically finitely generated (cf., e.g., [21, Remark 1.2.2], [21, Proposition 1.4]), it follows from [9, Lemma 4.10] that, to verify the inclusion \(\ker((\rho_{g,r}^{rel,l})_k) \subseteq \ker((\rho_{g,r+1}^{rel,l})_k)\), it suffices to prove that, after possibly multiplying \(\tilde{\gamma}\) by a suitable element of \(\Delta_Y\), \(\tilde{\gamma}\) is contained in \(Z_{H^{rel,l}_{(M_{g,r+1})_k}}(\Delta_Y)\).

Now, by Proposition 1.7 and [21, Proposition 2.2, (i)], we have the following commutative diagram of profinite groups
where the horizontal sequences are exact, the vertical arrows are injective, and the left-hand vertical arrows are the identity morphisms of \( \Delta^l_Y \). Thus, since the image of \( \Delta^l_{X_2} \) in \( \Pi^{rel-l}_{(\mathcal{M}_{g,r+1})_k} \) is normal, \( \gamma' \in Z_{\Pi^{rel-l}_{(\mathcal{M}_{g,r})_k}}(\Gamma^{rel-l}_{g,r}) \), and \( \Delta^l_X \) is center-free, it follows from [25, Theorem 0.1] or [13, Theorem B] that, after possibly multiplying \( \tilde{\gamma} \) by a suitable element of \( \Delta^l_Y \), \( \tilde{\gamma} \) is contained in \( Z_{\Pi^{rel-l}_{(\mathcal{M}_{g,r})_k}}(\Delta^l_Y) \). This completes the proof of the inclusion \( \ker((\rho^{rel-l}_{g,r})_k) \subseteq \ker((\rho^{rel-l}_{g,r+1})_k) \), hence also (cf. the final equivalence of §4.3) of the above assertion.

4.6. Finally, we give a remark concerning the main problem \( (\text{CSP})^{pro-l}_{g,r} \) of the present paper. Let us first observe that one may think of the problem \( (\text{CSP})^{pro-l}_{g,r} \) as a sort of geometric analogue of Ihara’s problem concerning the pro-\( l \) outer Galois action associated to a tripod (cf., e.g., [15, Lecture I, §2], [23, Introduction]). The conjecture due to Rasmussen and Tamagawa given in [23, Conjecture 1] was motivated by this problem of Ihara and asserts the finiteness of abelian varieties that satisfy certain conditions, one of which is closely related to the pro-\( l \)-ness of the image of the associated \( l \)-adic Galois representation. On the other hand, to the knowledge of the authors, at least at the time of writing, it does not appear that any argument has been obtained for deriving an answer of Ihara’s problem from the conjecture of Rasmussen–Tamagawa. In this context, it is of interest to observe that the problem \( (\text{CSP})^{pro-l}_{g,r} \) — which may be thought of as a sort of geometric analogue of Ihara’s problem — directly relates, as discussed in the proof of Theorem 3.7, to the consideration of the issue of whether or not a modular curve “\( Y(l) \)” satisfies a similar condition to the condition studied in the conjecture of Rasmussen–Tamagawa, i.e., the pro-\( l \)-ness of the image of the associated pro-\( l \) outer Galois action.

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