Abstract. Let $\Sigma$ be a subset of the set of prime numbers which is either equal to the entire set of prime numbers or of cardinality one. In the present paper, we continue our study of the pro-$\Sigma$ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of $\Sigma$ are invertible. The starting point of the theory of the present paper is a combinatorial anabelian result which, unlike results obtained in previous papers, allows one to eliminate the hypothesis that cuspidal inertia subgroups are preserved by the isomorphism in question. This result allows us to [partially] generalize combinatorial cuspidalization results obtained in previous papers to the case of outer automorphisms of pro-$\Sigma$ fundamental groups of configuration spaces that do not necessarily preserve the cuspidal inertia subgroups of the various one-dimensional subquotients of such a fundamental group. Such partial combinatorial cuspidalization results allow one in effect to reduce issues concerning the anabelian geometry of configuration spaces to issues concerning the anabelian geometry of hyperbolic curves. These results also allow us, in the case of configuration spaces of sufficiently large dimension, to give purely group-theoretic characterizations of the cuspidal inertia subgroups of the various one-dimensional subquotients of the pro-$\Sigma$ fundamental group of a configuration space. We then turn to the study of tripod synchronization, i.e., roughly speaking, the phenomenon that an outer automorphism of the pro-$\Sigma$ fundamental group of a log configuration space associated to a log stable curve typically induces the same outer automorphism on the various subquotients of such a fundamental group determined by tripods [i.e., copies of the projective line minus three points]. Our study of tripod synchronization allows us to show that outer automorphisms of pro-$\Sigma$ fundamental groups of configuration spaces exhibit somewhat different behavior from the behavior that may be observed — as a consequence of the classical Dehn-Nielsen-Baer theorem — in the case of discrete fundamental groups. Other applications of the theory of tripod synchronization include a result concerning commuting profinite Dehn multi-twists that, a priori, arise from distinct semi-graphs of
anabeloids of pro-$\Sigma$ PSC-type structures [i.e., the profinite analogue of the notion of a decomposition of a hyperbolic topological surface into hyperbolic subsurfaces, such as “pants”], as well as the computation, in terms of a certain scheme-theoretic fundamental group, of the purely combinatorial/group-theoretic commensurator of the group of profinite Dehn multi-twists. Finally, we show that the condition that an outer automorphism of the pro-$\Sigma$ fundamental group of a log stable curve lift to an outer automorphism of the pro-$\Sigma$ fundamental group of the corresponding $n$-th log configuration space, where $n \geq 2$ is an integer, is compatible, in a suitable sense, with localization on the dual graph of the log stable curve. This localizability property, together with the theory of tripod synchronization, is applied to construct a purely combinatorial analogue of the natural outer surjection from the étale fundamental group of the moduli stack of hyperbolic curves over $\mathbb{Q}$ to the absolute Galois group of $\mathbb{Q}$.

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Introduction

Let $\Sigma \subseteq \text{Primes}$ be a subset of the set of prime numbers $\text{Primes}$ which is either equal to $\text{Primes}$ or of cardinality one. In the present paper, we continue our study of the pro-$\Sigma$ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of $\Sigma$ are invertible [cf. [MzTa], [CmbCsp], [NodNon], [CbTpI]]. One central theme of this study is the issue of $n$-cuspidalizability [cf. Definition 3.20], i.e., the issue of the extent to which a given isomorphism between the pro-$\Sigma$ fundamental groups of a pair of hyperbolic curves lifts [necessarily uniquely, up to a permutation of factors — cf. [NodNon], Theorem B] to a PFC-admissible [cf. [CbTpI], Definition 1.4, (iii)] isomorphism between the pro-$\Sigma$ fundamental groups of the corresponding $n$-th configuration spaces, for $n \geq 1$ a positive integer. In this context, we recall that both the algebraic and the anabelian geometry of such configuration spaces revolves around the behavior of the various diagonals that are removed from direct products of copies of the given curve in order to construct
these configuration spaces. From this point of view, it is perhaps natural to think of the issue of \( n \)-cuspidalizability as a sort of \textit{abstract profinite analogue} of the notion of \textit{\( n \)-differentiability} in the theory of differential manifolds. In particular, it is perhaps natural to think of the theory of the present paper [as well as of [MzTa], [CmbCsp], [NodNon], [CbTpI]] as a sort of \textit{abstract profinite analogue} of the classical theory constituted by the \textit{differential topology of surfaces}.

Next, we recall that, to a substantial extent, the theory of \textit{combinatorial cuspidalization} developed in [CmbCsp] may be thought of as an \textit{essentially formal consequence} of the \textit{combinatorial anabelian result} obtained in [CmbGC], Corollary 2.7, (iii). In a similar vein, the generalization of this theory of [CmbCsp] that is summarized in [NodNon], Theorem B, may be regarded as an essentially formal consequence of the combinatorial anabelian result given in [NodNon], Theorem A. The development of the theory of the present paper follows this pattern to a substantial extent. That is to say, in \( \S 1 \), we begin the development of the theory of the present paper by proving a \textit{fundamental combinatorial anabelian result} [cf. Theorem 1.9], which generalizes the combinatorial anabelian results given in [CmbGC], Corollary 2.7, (iii); [NodNon], Theorem A. A substantial portion of the main results obtained in the remainder of the present paper may be understood as consisting of various \textit{applications} of Theorem 1.9.

By comparison to the combinatorial anabelian results of [CmbGC], Corollary 2.7, (iii); [NodNon], Theorem A, the \textit{main technical feature} of the combinatorial anabelian result given in Theorem 1.9 of the present paper is that it allows one, to a substantial extent, to

\begin{center}
\textit{eliminate the group-theoretic cuspidality hypothesis}
\end{center}

— i.e., the assumption to the effect that the isomorphism between pro-\( \Sigma \) fundamental groups of log stable curves under consideration necessarily \textit{preserves cuspidal inertia subgroups} — that plays a \textit{central role} in the proofs of earlier combinatorial anabelian results. In \( \S 2 \), we apply Theorem 1.9 to obtain the following \textit{partial} \textit{combinatorial cuspidalization} result [cf. Theorem 2.3, (i), (ii); Corollary 3.22], which [partially] generalizes [NodNon], Theorem B.

\textbf{Theorem A} (Partial combinatorial cuspidalization for \( F \)-admissible automorphisms). Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(n\) a positive integer; \(\Sigma\) a set of prime numbers which is either equal to the set of all prime numbers or of cardinality one; \(X\) a \textit{hyperbolic curve} of type \((g, r)\) over an algebraically closed field of characteristic \(\not\in\Sigma\); \(X_n\) the \(n\)-th \textit{configuration space} of \(X\); \(\Pi_n\) the maximal pro-\(\Sigma\) quotient of the fundamental group of \(X_n\);

\begin{equation*}
\text{Out}_F(\Pi_n) \subseteq \text{Out}(\Pi_n)
\end{equation*}
the subgroup of $F$-admissible automorphisms [i.e., roughly speaking, outer automorphisms that preserve the fiber subgroups — cf. [CmbCsp], Definition 1.1, (ii)] of $\Pi_n$:

$$\text{Out}^F(\Pi_n) \subseteq \text{Out}^F(\Pi_n)$$

the subgroup of $FC$-admissible automorphisms [i.e., roughly speaking, outer automorphisms that preserve the fiber subgroups and the cuspidal inertia subgroups — cf. [CmbCsp], Definition 1.1, (ii)] of $\Pi_n$. Then the following hold:

(i) Write

$$n_{\text{inj}} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } r \neq 0, \\ 2 & \text{if } r = 0, \end{cases} \quad n_{\text{bij}} \overset{\text{def}}{=} \begin{cases} 3 & \text{if } r \neq 0, \\ 4 & \text{if } r = 0. \end{cases}$$

If $n \geq n_{\text{inj}}$ (respectively, $n \geq n_{\text{bij}}$), then the natural homomorphism

$$\text{Out}^F(\Pi_{n+1}) \longrightarrow \text{Out}^F(\Pi_n)$$

induced by the projections $X_{n+1} \rightarrow X_n$ obtained by forgetting any one of the $n+1$ factors of $X_{n+1}$ [cf. [CbTpI], Theorem A, (i)] is injective (respectively, bijective).

(ii) Write

$$n_{\text{FC}} \overset{\text{def}}{=} \begin{cases} 2 & \text{if } (g, r) = (0, 3), \\ 3 & \text{if } (g, r) \neq (0, 3) \text{ and } r \neq 0, \\ 4 & \text{if } r = 0. \end{cases}$$

If $n \geq n_{\text{FC}}$, then it holds that

$$\text{Out}^{FC}(\Pi_n) = \text{Out}^F(\Pi_n).$$

(iii) Suppose that $(g, r) \not\in \{(0, 3); (1, 1)\}$. Then the natural injection [cf. [NodNon], Theorem B]

$$\text{Out}^{FC}(\Pi_2) \hookrightarrow \text{Out}^{FC}(\Pi_1)$$

induced by the projections $X_2 \rightarrow X_1$ obtained by forgetting either of the two factors of $X_2$ is not surjective.

Here, we remark that the non-surjectivity discussed in Theorem A, (iii), is, in fact, obtained as a consequence of the theory of tripod synchronization developed in §3 [cf. the discussion preceding Theorem C below]. This non-surjectivity is remarkable in that it yields an important example of substantially different behavior in the theory of profinite fundamental groups of hyperbolic curves from the corresponding theory in the discrete case. That is to say, in the case of the classical discrete fundamental group of a hyperbolic topological surface, the surjectivity of the corresponding homomorphism may be derived as an essentially formal consequence of the well-known Dehn-Nielsen-Baer...
In the theory of topological surfaces [cf. the discussion of Remark 3.22.1, (i)]. In particular, it constitutes an important “counterexample” to the “line of reasoning” [i.e., for instance, of the sort which appears in the final paragraph of [Lch], §1; the discussion between [Lch], Theorem 5.1, and [Lch], Conjecture 5.2] that one should expect essentially analogous behavior in the theory of profinite fundamental groups of hyperbolic curves to the relatively well understood behavior observed classically in the theory of discrete fundamental groups of topological surfaces [cf. the discussion of Remark 3.22.1, (iii)].

Theorem A leads naturally to the following strengthening of the result obtained in [CbTpI], Theorem A, (ii), concerning the group-theoreticity of the cuspidal inertia subgroups of the various one-dimensional subquotients of a configuration space group [cf. Corollary 2.4].

**Theorem B (PFC-admissibility of automorphisms).** In the notation of Theorem A, write

\[ \text{Out}^{PF}(\Pi_n) \subseteq \text{Out}(\Pi_n) \]

for the subgroup of \textbf{PF-admissible} automorphisms [i.e., roughly speaking, outer automorphisms that preserve the fiber subgroups up to a possible permutation of the factors — cf. [CbTpI], Definition 1.4, (i)] and

\[ \text{Out}^{PFC}(\Pi_n) \subseteq \text{Out}^{PF}(\Pi_n) \]

for the subgroup of \textbf{PFC-admissible} automorphisms [i.e., roughly speaking, outer automorphisms that preserve the fiber subgroups and the cuspidal inertia subgroups up to a possible permutation of the factors — cf. [CbTpI], Definition 1.4, (iii)]. Let us regard the symmetric group on \( n \) letters \( \mathfrak{S}_n \) as a subgroup of \( \text{Out}(\Pi_n) \) via the natural inclusion \( \mathfrak{S}_n \hookrightarrow \text{Out}(\Pi_n) \) obtained by permuting the various factors of \( X_n \). Finally, suppose that \( (g, r) \notin \{(0, 3); (1, 1)\} \). Then the following hold:

(i) We have an equality

\[ \text{Out}(\Pi_n) = \text{Out}^{PF}(\Pi_n). \]

If, moreover, \( (r, n) \neq (0, 2) \), then we have equalities

\[ \text{Out}(\Pi_n) = \text{Out}^{PF}(\Pi_n) = \text{Out}^{F}(\Pi_n) \times \mathfrak{S}_n. \]

(ii) If either

\[ r > 0, \quad n \geq 3 \]

or

\[ n \geq 4, \]

then we have equalities

\[ \text{Out}(\Pi_n) = \text{Out}^{PFC}(\Pi_n) = \text{Out}^{FC}(\Pi_n) \times \mathfrak{S}_n. \]
The partial combinatorial cuspidalization of Theorem A has natural applications to the relative and semi-absolutes anabelian geometry of configuration spaces [cf. Corollaries 2.5, 2.6], which generalize the theory of [AbsTpI], §1. Roughly speaking, these results allow one, in a wide variety of cases, to reduce issues concerning the relative and semi-absolutes anabelian geometry of configuration spaces to the corresponding issues concerning the relative and semi-absolutes anabelian geometry of hyperbolic curves. Also, we remark that in this context, we obtain a purely scheme-theoretic result [cf. Lemma 2.7] that states, roughly speaking, that the theory of isomorphisms [of schemes!] between configuration spaces associated to hyperbolic curves may be reduced to the theory of isomorphisms [of schemes!] between hyperbolic curves.

In §3, we take up the study of [the group-theoretic versions of] the various tripods [i.e., copies of the projective line minus three points] that occur in the various one-dimensional fibers of the log configuration spaces associated to a log stable curve. Roughly speaking, these tripods either occur in the original log stable curve or arise as the result of blowing up various cusps or nodes that occur in the one-dimensional fibers of log configuration spaces of lower dimension [cf. Figure 1 at the end of the present Introduction]. In fact, a substantial portion of §3 is devoted precisely to the theory of classification of the various tripods that occur in the one-dimensional fibers of the log configuration spaces associated to a log stable curve [cf. Lemmas 3.6, 3.8]. This leads naturally to the study of the phenomenon of tripod synchronization, i.e., roughly speaking, the phenomenon that an automorphism [that is to say, an outer automorphism] of the pro-$\Sigma$ fundamental group of a log configuration space associated to a log stable curve typically induces the same outer automorphism on the various [group-theoretic] tripods that occur in subquotients of such a fundamental group [cf. Theorems 3.16, 3.17, 3.18]. The phenomenon of tripod synchronization, in turn, leads naturally to the definition of the tripod homomorphism [cf. Definition 3.19], which may be thought of as the homomorphism obtained by associating to an [FC-admissible] outer automorphism of the pro-$\Sigma$ fundamental group of the $n$-th log configuration space associated to a log stable curve, where $n \geq 3$ is a positive integer, the outer automorphism induced on a [group-theoretic] central tripod, i.e., roughly speaking, a tripod that arises, in the case where $n = 3$ and the given log stable curve has no nodes, by blowing up the intersection of the three diagonal divisors of the direct product of three copies of the curve.

Theorem C (Synchronization of tripods in three or more dimensions). Let $(g,r)$ be a pair of nonnegative integers such that $2g - 2 + r > 0$; $n$ a positive integer; $\Sigma$ a set of prime numbers which is
either equal to the set of all prime numbers or of cardinality one; let k be an algebraically closed field of characteristic \(\not\in \Sigma\); (Spec \(k\))^\log the log scheme obtained by equipping Spec \(k\) with the log structure determined by the fs chart \(N \rightarrow k\) that maps \(1 \mapsto 0\); \(X^\log = X_1^\log\) a stable log curve of type \((g,r)\) over (Spec \(k\))^\log. Write \(\mathcal{G}\) for the semi-graph of anabelioids of pro-\(\Sigma\) PSC-type determined by the stable log curve \(X^\log\). For each positive integer \(i\), write \(X_i^\log\) for the \(i\)-th log configuration space of the stable log curve \(X^\log\) [cf. the discussion entitled “Curves” in [CbTpI], \(\S 0\)]; \(\Pi_i\) for the maximal pro-\(\Sigma\) quotient of the kernel of the natural surjection \(\pi_1(X_i^\log) \twoheadrightarrow \pi_1((\text{Spec} \(k\))^\log)\). Let \(T \subseteq \Pi_m\) be a \(\{1, \cdots, m\}\)-tripod of \(\Pi_n\) [cf. Definitions 3.3, (i); 3.7, (ii)]. Then the following hold:

(i) The commensurator and centralizer of \(T\) in \(\Pi_m\) satisfy the equality

\[
C_{\Pi_m}(T) = T \times Z_{\Pi_m}(T).
\]

Thus, if an automorphism \(\alpha\) of \(\Pi_m\) preserves the \(\Pi_m\)-conjugacy class of \(T \subseteq \Pi_m\), then one obtains a “restriction” \(\alpha|_T \in \text{Out}(T)\).

(ii) Let \(\alpha \in \text{Out}^{\text{FC}}(\Pi_n)\) be an FC-admissible automorphism of \(\Pi_n\). Then the automorphism of \(\Pi_3\) induced by \(\alpha\) preserves the \(\Pi_3\)-conjugacy class of \(\Pi^\text{tpd} \subseteq \Pi_3\). In particular, by (i), we obtain a natural homomorphism

\[
\Sigma_{\Pi^\text{tpd}} : \text{Out}^{\text{FC}}(\Pi_n) \longrightarrow \text{Out}(\Pi^\text{tpd}).
\]

We shall refer to this homomorphism as the tripod homomorphism associated to \(\Pi_n\).

(iii) Let \(\alpha \in \text{Out}^{\text{FC}}(\Pi_n)\) be an FC-admissible automorphism of \(\Pi_n\) such that the automorphism \(\alpha_m\) of \(\Pi_m\) induced by \(\alpha\) preserves the \(\Pi_m\)-conjugacy class of \(T \subseteq \Pi_m\) and induces [cf. (i)] the identity automorphism of the set of \(T\)-conjugacy classes of cuspidal inertia subgroups of \(T\). Then there exists a geometric [cf. Definition 3.4, (ii)] outer isomorphism \(\Pi^\text{tpd} \xrightarrow{\sim} T\) with respect to which the automorphism \(\Sigma_{\Pi^\text{tpd}}(\alpha) \in \text{Out}(\Pi^\text{tpd})\) [cf. (ii)] is compatible with the automorphism \(\alpha_m|_T \in \text{Out}(T)\) [cf. (i)].

(iv) Suppose, moreover, that either \(n \geq 4\) or \(r \neq 0\). Then the homomorphism \(\Sigma_{\Pi^\text{tpd}}\) of (ii) factors through \(\text{Out}^C(\Pi^\text{tpd})^{\Delta^+} \subseteq \text{Out}(\Pi^\text{tpd})\) [cf. Definition 3.4, (i)], and, moreover, the resulting
homomorphism

\[ \mathcal{T}_{\text{tpd}} : \text{Out}^F(\Pi_n) = \text{Out}^C(\Pi_n) \to \text{Out}^C(\Pi_{\text{tpd}})^\Delta^+ \]

[cf. Theorem A, (ii)] is surjective.

Here, we remark that the surjectivity of the tripod homomorphism [cf. Theorem C, (iv)] is obtained [cf. Corollary 4.15] as a consequence of the theory of glueability of combinatorial cuspidalizations developed in §4 [cf. the discussion preceding Theorem F below]. Also, we recall that the codomain of this surjective tripod homomorphism may be identified with the \([\text{pro-}\Sigma]\) Grothendieck-Teichmüller group \(\text{GT}^\Sigma\) [cf. the discussion of [CmbCsp], Remark 1.11.1]. Since \(\text{GT}^\Sigma\) may be thought of as a sort of abstract combinatorial approximation of the absolute Galois group \(G_Q\) of the rational number field \(Q\), it is thus natural to think of the surjective tripod homomorphism

\[ \text{Out}^C(\Pi_{\text{tpd}})^\Delta^+ \]

of Theorem C, (iv), as a sort of abstract combinatorial version of the natural surjective outer homomorphism

\[ \pi_1((\mathcal{M}_{g,[r]})_Q) \to G_Q \]

induced on étale fundamental groups by the structure morphism \((\mathcal{M}_{g,[r]})_Q \to \text{Spec}(Q)\) of the moduli stack \((\mathcal{M}_{g,[r]})_Q\) of hyperbolic curves of type \((g,r)\) [cf. the discussion of Remark 3.19.1]. In particular, the kernel of the tripod homomorphism — which we denote by

\[ \text{Out}^F(\Pi_n)^\text{geo} \]

— may be thought of as a sort of abstract combinatorial analogue of the geometric étale fundamental group of \((\mathcal{M}_{g,[r]})_Q\) [i.e., the kernel of the natural outer homomorphism \(\pi_1((\mathcal{M}_{g,[r]})_Q) \to G_Q\)].

One interesting application of the theory of tripod synchronization is the following. Fix a pro-\(\Sigma\) fundamental group of a hyperbolic curve. Recall the notion of a nondegenerate profinite Dehn multi-twist [cf. [CbTpI], Definition 4.4; [CbTpI], Definition 5.8, (ii)] associated to a structure of semi-graph of anabelioids of pro-\(\Sigma\) PSC-type on such a fundamental group. Here, we recall that such a structure may be thought of as a sort of profinite analogue of the notion of a decomposition of a hyperbolic topological surface into hyperbolic subsurfaces [i.e., such as “pants”]. Then the following result asserts that, under certain technical conditions, any such nondegenerate profinite Dehn multi-twist that commutes with another nondegenerate profinite Dehn multi-twist associated to some given totally degenerate semi-graph of anabelioids of pro-\(\Sigma\) PSC-type [cf. [CbTpI], Definition 2.3, (iv)] necessarily arises from a structure of semi-graph of anabelioids of pro-\(\Sigma\) PSC-type that is “co-Dehn” to, i.e., arises by applying a deformation to, the given
totally degenerate semi-graph of anabelioids of pro-$\Sigma$ PSC-type [cf. Corollary 3.25]. This sort of result is reminiscent of topological results concerning subgroups of the mapping class group generated by pairs of positive Dehn multi-twists [cf. [Ishi], [HT]].

Theorem D (Co-Dehn-ness of degeneration structures in the totally degenerate case). In the notation of Theorem C, for $i = 1, 2$, let $Y_i^{\log}$ be a stable log curve over $(\text{Spec} \ k)^{log}$, $\mathcal{H}_i$ the “$G$” that occurs in the case where we take “$X^{log}$” to be $Y_i^{log}$, $(\mathcal{H}_i, S_i, \phi_i)$ a 3-cuspidalizable degeneration structure on $\mathcal{G}$ [cf. Definition 3.23, (iv)]; $\alpha_i \in \text{Out}(\Pi_\mathcal{G})$ a nondegenerate ($H_i, S_i, \phi_i$)-Dehn multi-twist of $\mathcal{G}$ [cf. Definition 3.23, (iv)]. Suppose that $\alpha_1$ commutes with $\alpha_2$, and that $\mathcal{H}_2$ is totally degenerate [cf. [CbtPl], Definition 2.3, (iv)]. Suppose, moreover, that one of the following conditions is satisfied:

(i) $r \neq 0$.
(ii) $\alpha_1$ and $\alpha_2$ are positive definite [cf. Definition 3.23, (iv)].

Then $(\mathcal{H}_1, S_1, \phi_1)$ is co-Dehn to $(\mathcal{H}_2, S_2, \phi_2)$ [cf. Definition 3.23, (iii)], or, equivalently [since $\mathcal{H}_2$ is totally degenerate], $(\mathcal{H}_2, S_2, \phi_2) \preceq (\mathcal{H}_1, S_1, \phi_1)$ [cf. Definition 3.23, (ii)].

Another interesting application of the theory of tripod synchronization is to the computation, in terms of a certain scheme-theoretic fundamental group, of the purely combinatorial commensurator of the subgroup of profinite Dehn multi-twists in the group of 3-cuspidalizable, FC-admissible, “geometric” outer automorphisms of the pro-$\Sigma$ fundamental group of a totally degenerate log stable curve [cf. Corollary 3.27]. Here, we remark that the scheme-theoretic [or, perhaps more precisely, “log algebraic stack-theoretic”] fundamental group that appears is, roughly speaking, the pro-$\Sigma$ geometric fundamental group of a formal neighborhood, in the corresponding logarithmic moduli stack, of the point determined by the given totally degenerate log stable curve. In particular, this computation may also be regarded as a sort of purely combinatorial algorithm for constructing this scheme-theoretic fundamental group [cf. Remark 3.27.1].

Theorem E (Commensurator of profinite Dehn multi-twists in the totally degenerate case). In the notation of Theorem C [so $n \geq 3$], suppose further that if $r = 0$, then $n \geq 4$. Also, we assume that $\mathcal{G}$ is totally degenerate [cf. [CbtPl], Definition 2.3, (iv)].

Write $s: \text{Spec} \ k \to (\mathcal{M}_{g,[r]}^\log)_k \overset{\text{def}}{=} (\mathcal{M}_{g,[r]}^\log)^{\text{Spec} \ k}$ [cf. the discussion entitled “Curves” in §6] for the underlying $(1)$-morphism of algebraic stacks of the classifying $(1)$-morphism $(\text{Spec} \ k)^{log} \to (\mathcal{M}_{g,[r]}^\log)_k \overset{\text{def}}{=} (\mathcal{M}_{g,[r]}^\log)^{\text{Spec} \ k}$ [cf. the discussion entitled “Curves” in §6] of the stable log curve
\( X^{\log} \) over \( (\text{Spec} \, k)^{\log} \); \( \tilde{\mathcal{N}}_s^{\log} \) for the log scheme obtained by equipping \( \tilde{\mathcal{N}}_s \overset{\text{def}}{=} \text{Spec} \, k \) with the log structure induced, via \( s \), by the log structure of \( (M_{g, [r]} \log)_{k} \); \( \mathcal{N}^{\log}_s \) for the log stack obtained by forming the \([\text{stack-theoretic}] \) quotient of the log scheme \( \mathcal{N}^{\log}_s \) by the natural action of the finite \( k \)-group \( s \), i.e., the fiber product over \( (M_{g, [r]} \log)_{k} \) of two copies of \( s; \mathcal{N}_s \) for the underlying stack of the log stack \( \mathcal{N}^{\log}_s \); \( I_{\mathcal{N}_s} \) for the closed subgroup of the log fundamental group \( \mathcal{N}^{\log}_s \) of \( \mathcal{N}_s \) given by the kernel of the natural surjection \( \mathcal{N}^{\log}_s \to \mathcal{N}_s \) [induced by the \((1-)\)morphism \( \mathcal{N}^{\log}_s \to \mathcal{N}_s \) obtained by forgetting the log structure]; \( \pi_1(\Sigma)(\mathcal{N}^{\log}_s) \) for the quotient of \( \pi_1(\mathcal{N}^{\log}_s) \) by the kernel of the natural surjection from \( I_{\mathcal{N}_s} \) to its maximal pro-\( \Sigma \) quotient \( I_{\mathcal{N}_s}^{\Sigma} \). Then we have an equality

\[
\mathcal{N}^{\log}_{\text{Out}}(\Pi_n)^{\text{geo}}(\text{Dehn}(\mathcal{G})) = C_{\text{Out}}^{\log}(\Pi_n)^{\text{geo}}(\text{Dehn}(\mathcal{G}))
\]

and a natural commutative diagram of profinite groups

\[
\begin{array}{ccccccc}
1 & \longrightarrow & I_{\mathcal{N}_s}^{\Sigma} & \longrightarrow & \pi_1(\mathcal{N}^{\log}_s) & \longrightarrow & \pi_1(\mathcal{N}_s) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \text{Dehn}(\mathcal{G}) & \longrightarrow & C_{\text{Out}}^{\log}(\Pi_n)^{\text{geo}}(\text{Dehn}(\mathcal{G})) & \longrightarrow & \text{Aut}(\mathcal{G}) & \longrightarrow & 1 \\
\end{array}
\]

[cf. Definition 3.1, (ii), concerning the notation "\( \mathcal{G} \)" — where the horizontal sequences are exact, and the vertical arrows are isomorphisms. Moreover, \( \text{Dehn}(\mathcal{G}) \) is open in \( C_{\text{Out}}^{\log}(\Pi_n)^{\text{geo}}(\text{Dehn}(\mathcal{G})) \).]

In §4, we show, under suitable technical conditions, that an automorphism of the pro-\( \Sigma \) fundamental group of the log configuration space associated to a log stable curve necessarily preserves the graph-theoretic structure of the various one-dimensional fibers of such a log configuration space [cf. Theorem 4.7]. This allows us to verify the glueability of combinatorial cuspidalizations, i.e., roughly speaking, that, for \( n \geq 2 \) a positive integer, the datum of an \( n \)-cuspidalizable outer automorphism of the pro-\( \Sigma \) fundamental group of a log stable curve is equivalent, up to possible composition with a profinite Dehn multi-twist, to the datum of a collection of \( n \)-cuspidalizable automorphisms of the pro-\( \Sigma \) fundamental groups of the various irreducible components of the given log stable curve that satisfy a certain gluing condition involving the induced outer actions on tripods [cf. Theorem 4.14].

**Theorem F (Glueability of combinatorial cuspidalizations).** In the notation of Theorem C, write

\[
\text{Out}^{\log}(\Pi_n)^{\text{barch}} \subseteq \text{Out}^{\log}(\Pi_n)
\]

for the closed subgroup of \( \text{Out}^{\log}(\Pi_n) \) consisting of \( \text{FC} \)-admissible outer automorphisms \( \alpha \) of \( \Pi_n \) such that the automorphism of \( \Pi_1 \) determined by
\(\alpha\) induces the identity automorphism of \(\text{Vert}(G)\), \(\text{Node}(G)\), and, moreover, fixes each of the branches of every node of \(G\) [cf. Definition 4.6, (i)];

\[
\text{Glu}(\Pi_n) \subseteq \prod_{v \in \text{Vert}(G)} \text{Out}^{\text{FC}}((\Pi_v)_n)
\]

for the closed subgroup of \(\prod_{v \in \text{Vert}(G)} \text{Out}^{\text{FC}}((\Pi_v)_n)\) consisting of “glueable” collections of automorphisms of the groups \(“(\Pi_v)_n”\) [cf. Definition 4.9, (iii)]. Then we have a natural exact sequence of profinite groups

\[
1 \rightarrow \text{Dehn}(G) \rightarrow \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}} \rightarrow \text{Glu}(\Pi_n) \rightarrow 1.
\]

This glueability result may, alternatively, be thought of as a result that asserts the localizability [i.e., relative to localization on the dual graph of the given log stable curve] of the notion of \(n\)-cuspidalizability. In this context, it is of interest to observe that this glueability result may be regarded as a natural generalization, to the case of \(n\)-cuspidalizability for \(n \geq 2\), of the glueability result obtained in \([CbTpI]\), Theorem B, (iii), in the “1-cuspidalizable” case, which is derived as a consequence of the theory of localizability [i.e., relative to localization on the dual graph of the given log stable curve] and synchronization of cyclotomes developed in \([CbTpI]\), §3, §4. From this point of view, it is also of interest to observe that the sufficiency portion of [the equivalence that constitutes] this glueability result [i.e., Theorem F] may be thought of as a sort of “converse” to the theory of tripod synchronizations developed in §3 [i.e., of which the necessity portion of this glueability result is, in essence, a formal consequence]. Indeed, the bulk of the proof given in §4 of Theorem 4.14 is devoted to the sufficiency portion of this result, which is verified by means of a detailed combinatorial analysis [cf. the proof of \([CbTpI]\), Proposition 4.10, (ii)] of the noncyclically primitive and cyclically primitive cases [cf. Lemmas 4.12, 4.13; Figures 2, 3, 4].

Finally, we apply this glueability result to derive a cuspidalization theorem — i.e., in the spirit of and generalizing the corresponding results of \([AbsCsp]\), Theorem 3.1; \([Hsh]\), Theorem 0.1; \([Wkb]\), Theorem C [cf. Remark 4.16.1] — for geometrically pro-\(l\) fundamental groups of log stable curves over finite fields [cf. Corollary 4.16]. That is to say, in the case of log stable curves over finite fields,

the condition of compatibility with the Galois action

is sufficient to imply the \(n\)-cuspidalizability of arbitrary isomorphisms between the geometric pro-\(l\) fundamental groups, for \(n \geq 1\).

In this context, it is of interest to recall that strong anabelian results [i.e., in the style of the “Grothendieck Conjecture”] for such geometrically pro-\(l\) fundamental groups of log stable curves over finite fields
are not known in general, at the time of writing. On the other hand, we observe that in the case of totally degenerate log stable curves over finite fields, such “strong anabelian results” may be obtained under certain technical conditions [cf. Corollary 4.17; Remarks 4.17.1, 4.17.2].

0. Notations and Conventions

Groups: We shall refer to an element of a group as trivial (respectively, nontrivial) if it is (respectively, is not) equal to the identity element of the group. We shall refer to a nonempty subset of a group as trivial (respectively, nontrivial) if it is (respectively, is not) equal to the set whose unique element is the identity element of the group.

Topological groups: Let $G$ be a topological group and $J, H \subseteq G$ closed subgroups. Then we shall write $Z_J(H) \overset{\text{def}}{=} \{ j \in J \mid jh = hj \text{ for any } h \in H \} = Z_G(H) \cap J$ for the centralizer of $H$ in $J$,

\[ Z(G) \overset{\text{def}}{=} Z_G(G) \]

for the center of $G$, and

\[ Z_{J}^{\text{loc}}(H) \overset{\text{def}}{=} \lim_{\rightarrow} Z_J(U) \subseteq J \]

— where the inductive limit is over all open subgroups $U \subseteq H$ of $H$ — for the “local centralizer” of $H$ in $J$. We shall write $Z_{J}^{\text{loc}}(G) \overset{\text{def}}{=} Z_{G}^{\text{loc}}(G)$ for the “local center” of $G$. Thus, a profinite group $G$ is slim [cf. the discussion entitled “Topological groups” in [CbTpI], §0] if and only if $Z_{J}^{\text{loc}}(G) = \{ 1 \}$.

Curves: Let $(g, r)$ be a pair of nonnegative integers such that $2g - 2 + r > 0$. Then we shall write $\mathcal{M}_{g,[r]}$ for the moduli stack of pointed stable curves of type $(g, r)$, where the marked points are regarded as unordered, over $\mathbb{Z}$; $\mathcal{M}_{g,[r]} \subseteq \overline{\mathcal{M}}_{g,[r]}$ for the open substack of $\overline{\mathcal{M}}_{g,[r]}$ that parametrizes smooth curves, i.e., hyperbolic curves; $\overline{\mathcal{M}}_{g,[r]}^{\log}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,[r]}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,[r]} \setminus \mathcal{M}_{g,[r]} \subseteq \mathcal{M}_{g,[r]}$; $\tilde{\mathcal{C}}_{g,[r]} \rightarrow \overline{\mathcal{M}}_{g,[r]}$ for the tautological stable curve over $\overline{\mathcal{M}}_{g,[r]}$; $\overline{\mathcal{D}}_{g,[r]} \subseteq \overline{\mathcal{C}}_{g,[r]}$ for the corresponding tautological divisor of cusps of $\overline{\mathcal{C}}_{g,[r]} \rightarrow \overline{\mathcal{M}}_{g,[r]}$. Then the divisor given by the union of $\overline{\mathcal{D}}_{g,[r]}$ with the inverse image in $\overline{\mathcal{C}}_{g,[r]}$ of the divisor $\overline{\mathcal{M}}_{g,[r]} \setminus \mathcal{M}_{g,[r]} \subseteq \overline{\mathcal{M}}_{g,[r]}$ determines a log structure on $\overline{\mathcal{C}}_{g,[r]}$; write $\overline{\mathcal{C}}_{g,[r]}^{\log}$ for the resulting log stack. In particular, we obtain a (1-)morphism of log stacks $\overline{\mathcal{C}}_{g,[r]}^{\log} \rightarrow \overline{\mathcal{M}}_{g,[r]}^{\log}$. We shall write $\mathcal{C}_{g,[r]} \subseteq \overline{\mathcal{C}}_{g,[r]}$ for the interior of $\overline{\mathcal{C}}_{g,[r]}^{\log}$ [cf. the discussion entitled “Log schemes” in
Thus, we obtain a (1-)morphism of stacks $C_{g,r} \to \mathcal{M}_{g,r}$. If $S$ is a scheme, then we shall denote by means of a subscript $S$ the result of base-changing via the structure morphism $S \to \text{Spec} \mathbb{Z}$ the various log stacks of the above discussion.
Figure 1: tripods in the various fibers of a configuration space
1. COMBINATORIAL ANABELIAN GEOMETRY IN THE ABSENCE OF GROUP-THEORETIC CUSPIDALITY

In the present §1, we discuss various combinatorial versions of the Grothendieck Conjecture for outer representations of $NN$- and $IPSC$-type [cf. Theorem 1.9 below]. These Grothendieck Conjecture-type results may be regarded as generalizations of [NodNon], Corollary 4.2; [NodNon], Remark 4.2.1, that may be applied to isomorphisms that are not necessarily group-theoretically cuspidal. For instance, we prove [cf. Theorem 1.9, (ii), below] that any isomorphism between outer representations of $IPSC$-type [cf. [NodNon], Definition 2.4, (i)] is necessarily group-theoretically vertical, i.e., roughly speaking, preserves the vertical subgroups.

A basic reference for the theory of semi-graphs of anabelioids of PSC-type is [CmbGC]. We shall use the terms “semi-graph of anabelioids of PSC-type”, “PSC-fundamental group of a semi-graph of anabelioids of PSC-type”, “finite étale covering of semi-graphs of anabelioids of PSC-type”, “vertex”, “edge”, “node”, “cusp”, “vertical subgroup”, “edge-like subgroup”, “nodal subgroup”, “cuspidal subgroup”, and “sturdy” as they are defined in [CmbGC], Definition 1.1 [cf. also Remark 1.1.2 below]. Also, we shall apply the various notational conventions established in [NodNon], Definition 1.1, and refer to the “PSC-fundamental group of a semi-graph of anabelioids of PSC-type” simply as the “fundamental group” [of the semi-graph of anabelioids of PSC-type]. That is to say, we shall refer to the maximal pro-$\Sigma$ quotient of the fundamental group of a semi-graph of anabelioids of pro-$\Sigma$ PSC-type [as a semi-graph of anabelioids!] as the “fundamental group of the semi-graph of anabelioids of PSC-type”.

In the present §1, let $\Sigma$ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of anabelioids of pro-$\Sigma$ PSC-type. Write $\mathcal{G}$ for the underlying semi-graph of $\mathcal{G}$, $\Pi_\mathcal{G}$ for the [pro-$\Sigma$] fundamental group of $\mathcal{G}$, and $\tilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of $\mathcal{G}$ corresponding to $\Pi_\mathcal{G}$. Then since the fundamental group $\Pi_\mathcal{G}$ of $\tilde{\mathcal{G}}$ is topologically finitely generated, the profinite topology of $\Pi_\mathcal{G}$ induces [profinite] topologies on $\text{Aut}(\Pi_\mathcal{G})$ and $\text{Out}(\Pi_\mathcal{G})$ [cf. the discussion entitled “Topological groups” in [CbTpi], §0]. If, moreover, we write $\text{Aut}(\mathcal{G})$ for the automorphism group of $\mathcal{G}$, then, by the discussion preceding [CmbGC], Lemma 2.1, the natural homomorphism

$$\text{Aut}(\mathcal{G}) \to \text{Out}(\Pi_\mathcal{G})$$

is an injection with closed image. [Here, we recall that an automorphism of a semi-graph of anabelioids consists of an automorphism of the underlying semi-graph, together with a compatible system of isomorphisms between the various anabelioids at each of the vertices and
edges of the underlying semi-graph which are compatible with the various morphisms of anabeloids associated to the branches of the underlying semi-graph — cf. [SemiAn], Definition 2.1; [SemiAn], Remark 2.4.2.] Thus, by equipping \( \text{Aut}(G) \) with the topology induced via this homomorphism by the topology of \( \text{Out}(\Pi_G) \), we may regard \( \text{Aut}(G) \) as being equipped with the structure of a profinite group.

**Definition 1.1.** We shall say that an element \( \gamma \in \Pi_G \) of \( \Pi_G \) is vertical (respectively, edge-like; nodal; cuspidal) if \( \gamma \) is contained in a vertical (respectively, an edge-like; a nodal; a cuspidal) subgroup of \( \Pi_G \).

**Remark 1.1.1.** Let \( \gamma \in \Pi_G \) be a nontrivial [cf. the discussion entitled “Groups” in §0] element of \( \Pi_G \). If \( \gamma \in \Pi_G \) is edge-like [cf. Definition 1.1], then it follows from [NodNon], Lemma 1.5, that there exists a unique edge \( \tilde{e} \in \text{Edge}(\tilde{G}) \) such that \( \gamma \in \Pi_{\tilde{e}} \). If \( \gamma \in \Pi_G \) is vertical, but not nodal [cf. Definition 1.1], then it follows from [NodNon], Lemma 1.9, (i), that there exists a unique vertex \( \tilde{v} \in \text{Vert}(\tilde{G}) \) such that \( \gamma \in \Pi_{\tilde{v}} \).

**Remark 1.1.2.** Here, we take the opportunity to correct an unfortunate misprint in [CmbGC]. In the final sentence of [CmbGC], Definition 1.1, (ii), the phrase “rank \( \geq 2 \)” should read “rank \( > 2 \)”.

**Lemma 1.2 (Existence of a certain connected finite étale covering).** Let \( n \) be a positive integer which is a product [possibly with multiplicities!] of primes \( e \in \Sigma \); \( \tilde{e}_1, \tilde{e}_2 \in \text{Edge}(\tilde{G}); \tilde{v} \in \text{Vert}(\tilde{G}) \). Write \( e_1 \overset{\text{def}}{=} \tilde{e}_1(G), e_2 \overset{\text{def}}{=} \tilde{e}_2(G), \) and \( v \overset{\text{def}}{=} \tilde{v}(G) \). Suppose that the following conditions are satisfied:

(i) \( G \) is untangled [cf. [NodNon], Definition 1.2].

(ii) If \( e_1 \) is a node, then the following condition holds: Let \( w, w' \in \mathcal{V}(e_1) \) be the two distinct elements of \( \mathcal{V}(e_1) \) [cf. (i)]. Then \( |\mathcal{N}(w) \cap \mathcal{N}(w')|^2 \geq 3 \).

(iii) If \( e_1 \) is a cusp, then the following condition holds: Let \( w \in \mathcal{V}(e_1) \) be the unique element of \( \mathcal{V}(e_1) \). Then \( |\mathcal{C}(w)|^2 \geq 3 \).

(iv) \( e_1 \neq e_2 \).

(v) \( v \not\in \mathcal{V}(e_1) \).

Then there exists a Galois subcovering \( G' \to G \) of \( \tilde{G} \to G \) such that \( n \) divides \( [\Pi_{\tilde{e}_1} : \Pi_{\tilde{e}_1} \cap \Pi_{G'}] \), and, moreover, \( \Pi_{\tilde{e}_2}, \Pi_{\tilde{v}} \subseteq \Pi_{G'} \).
Proof. Suppose that \( e_1 \) is a node (respectively, cusp). Write \( \mathbb{H} \) for the [uniquely determined] sub-semi-graph of \( PSC\text{-}type \) [cf. \( \text{[CbTpI]} \), Definition 2.2, (i)] of \( \mathcal{G} \) whose set of vertices is \( = \mathcal{V}(e_1) = \{ w, w' \} \) [cf. condition (ii)] (respectively, \( = \{ w \} \) [cf. condition (iii)]). Now it follows from condition (ii) (respectively, (iii)) of \( \Pi \) that there exists an \( e_3 \in \text{Node}(\mathcal{G}|_{\mathbb{H}}) = \mathcal{N}(w) \cap \mathcal{N}(w') \) (respectively, \( \in \text{Cusp}(\mathcal{G}|_{\mathbb{H}}) \cap \text{Cusp}(\mathcal{G}) = \mathcal{C}(w) \)) [cf. \( \text{[CbTpI]} \), Definition 2.2, (ii)] such that \( e_3 \neq e_2 \). Moreover, applying condition (ii) (respectively, (iii)), together with the well-known structure of the abelianization of the fundamental group of a smooth curve over an algebraically closed field of characteristic 0, we conclude that there exists a Galois covering \( \mathcal{G}'_{|\mathbb{H}} \to \mathcal{G}|_{\mathbb{H}} \) that arises from a normal open subgroup of \( \Pi \) which is unramified at every element of \( \text{Edge}(\mathcal{G}|_{\mathbb{H}}) \setminus \{ e_1, e_3 \} \) and totally ramified at \( e_1, e_3 \) with ramification indices divisible by \( n \). Now since \( \mathcal{G}'_{|\mathbb{H}} \to \mathcal{G}|_{\mathbb{H}} \) is unramified at every element of \( \text{Cusp}(\mathcal{G}|_{\mathbb{H}}) \cap \text{Node}(\mathcal{G}) \), one may extend this covering to a Galois subcovering \( \mathcal{G}' \to \mathcal{G} \to \mathcal{G} \) which restricts to the trivial covering over every vertex \( u \) of \( \mathcal{G} \) such that \( u \neq w, w' \) (respectively, \( u \neq w \)). Moreover, it follows immediately from the construction of \( \mathcal{G}' \to \mathcal{G} \) that \( n \) divides \( [\Pi_{\bar{e}_1} : \Pi_{\bar{e}_1} \cap \Pi_{\bar{e}_2}] \), and \( \Pi_{\bar{e}_2}, \Pi_{\bar{e}_2} \subseteq \Pi_{\bar{e}_3} \). This completes the proof of Lemma 1.2. \( \square \)

Lemma 1.3 (Product of edge-like elements). Let \( \gamma_1, \gamma_2 \in \Pi \) be two nontrivial edge-like elements of \( \Pi \) [cf. Definition 1.1]. Write \( \bar{\bar{e}}_1, \bar{\bar{e}}_2 \in \text{Edge}(\bar{\mathcal{G}}) \) for the unique elements of \( \text{Edge}(\bar{\mathcal{G}}) \) such that \( \gamma_1 \in \Pi_{\bar{\bar{e}}_1}, \gamma_2 \in \Pi_{\bar{\bar{e}}_2} \) [cf. Remark 1.1.1]. Suppose that the following conditions are satisfied:

(i) For every positive integer \( n \), it holds that \( \gamma_1^n \gamma_2^n \) is vertical.

(ii) \( \bar{\bar{e}}_1 \neq \bar{\bar{e}}_2 \).

Then there exists a [necessarily unique — cf. \( \text{[NodNon]} \), Remark 1.8.1, (iii)] \( \bar{\bar{v}} \in \text{Vert}(\bar{\mathcal{G}}) \) such that \( \{ \bar{\bar{e}}_1, \bar{\bar{e}}_2 \} \subseteq \mathcal{E}(\bar{\bar{v}}) \); in particular, it holds that \( \gamma_1 \gamma_2 \in \Pi_{\bar{\bar{v}}} \).

Proof. Since \( \bar{\bar{e}}_1 \neq \bar{\bar{e}}_2 \) [cf. condition (ii)], one verifies easily that there exists a Galois subcovering \( \mathcal{H} \to \mathcal{G} \) of \( \mathcal{G} \to \mathcal{G} \) that satisfies the following conditions:

1. \( \bar{\bar{e}}_1(\mathcal{H}) \neq \bar{\bar{e}}_2(\mathcal{H}) \).

2. \( \mathcal{H} \) is untangled [cf. \( \text{[NodNon]} \), Definition 1.2; \( \text{[NodNon]} \), Remark 1.2.1, (i), (ii)].

3. For \( i \in \{ 1, 2 \} \), if \( \bar{\bar{e}}_i \in \text{Node}(\bar{\mathcal{G}}) \), then the following holds: Let \( w, w' \in \mathcal{V}(\bar{\bar{e}}_i(\mathcal{H})) \) be the two distinct elements of \( \mathcal{V}(\bar{\bar{e}}_i(\mathcal{H})) \). Then \( (\mathcal{N}(w) \cap \mathcal{N}(w'))^2 \geq 3 \).
(4) For \( i \in \{1, 2\} \), if \( \overline{e}_i \in \text{Cusp}(\tilde{G}) \), then the following holds:
Let \( w \in \mathcal{V}(\overline{e}_i(\mathcal{H})) \) be the unique element of \( \mathcal{V}(\overline{e}_i(\mathcal{H})) \). Then
\( C(w)^{\sharp} \geq 3. \)

Now it is immediate that there exists a positive integer \( m \) such that
\( \gamma_1^m \in \Pi_{\overline{e}_1} \cap \Pi_\mathcal{H}, \gamma_2^m \in \Pi_{\overline{e}_2} \cap \Pi_\mathcal{H}. \) Let \( \overline{v} \in \text{Vert}(\tilde{G}) \) be such that \( \gamma_1^m \gamma_2^m \in \Pi_{\overline{v}} \) [cf. condition (i)].

Suppose that \( \overline{v}(\mathcal{H}) \not\subseteq \mathcal{V}(\overline{e}_1(\mathcal{H})) \). Then it follows from Lemma 1.2 that there exists a Galois subcovering \( \mathcal{H}' \to \mathcal{H} \) of \( \tilde{G} \to \mathcal{H} \) such that
\( \gamma_1^m \not\in \Pi_{\mathcal{H}'} \), and, moreover, \( \Pi_{\overline{e}_2} \cap \Pi_\mathcal{H}, \Pi_{\overline{v}} \cap \Pi_\mathcal{H} \subseteq \Pi_{\mathcal{H}'} \). But this implies that \( \gamma_2^m, \gamma_1^m \gamma_2^m \in \Pi_{\mathcal{H}'} \), hence that \( \gamma_1^m \in \Pi_{\mathcal{H}'}, \) a contradiction. In particular, it holds that \( \gamma_1^m \gamma_2^m \in \Pi_{\mathcal{H}'} \), hence that \( \gamma_1^m \in \Pi_{\mathcal{H}'} \), a contradiction. Thus, by applying this argument to a suitable system of connected finite étale coverings of \( \mathcal{H} \), we conclude that
\( \mathcal{V}(\overline{e}_1(\mathcal{H})) \cap \mathcal{V}(\overline{e}_2(\mathcal{H})) \neq \emptyset. \) Then since \( \Pi_{\overline{e}_1}, \Pi_{\overline{e}_2} \subseteq \Pi_{\overline{v}}, \) it follows immediately that \( \gamma_1 \gamma_2 \in \Pi_{\overline{v}} \). This completes the proof of Lemma 1.3.

**Proposition 1.4 (Group-theoretic characterization of closed subgroups of edge-like subgroups).** Let \( H \subseteq \Pi_G \) be a closed subgroup of \( \Pi_G \). Then the following conditions are equivalent:

(i) \( H \) is contained in an **edge-like subgroup**.

(ii) An open subgroup of \( H \) is contained in an **edge-like subgroup**.

(iii) Every element of \( H \) is **edge-like** [cf. Definition 1.1].

(iv) There exists a connected finite étale subcovering \( \mathcal{G}' \to \mathcal{G} \) of \( \tilde{G} \to \mathcal{G} \) such that for any connected finite étale subcovering \( \mathcal{G}' \to \mathcal{G} \) of \( \tilde{G} \to \mathcal{G} \) that factors through \( \mathcal{G}' \to \mathcal{G} \), the image of the composite
\[
H \cap \Pi_{\mathcal{G}'} \hookrightarrow \Pi_{\mathcal{G}'} \to \Pi_{\mathcal{G}'}^{ab/edge}
\]
— where we write \( \Pi_{\mathcal{G}'}^{ab/edge} \) for the torsion-free [cf. [CmbGC], Remark 1.1.4] quotient of the abelianization \( \Pi_{\mathcal{G}'}^{ab} \) by the closed subgroup topologically generated by the images in \( \Pi_{\mathcal{G}'}^{ab} \) of the edge-like subgroups of \( \Pi_{\mathcal{G}'} \) — is **trivial**.

**Proof.** The implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iv) are immediate. The equivalence (iii) \( \iff \) (iv) follows immediately from [NodNon], Lemma 1.6. Thus, to complete the verification of Proposition 1.4, it suffices to verify the implication (iii) \( \Rightarrow \) (i). To this end, suppose that condition (iii) holds. First, we observe that, to verify the implication (iii) \( \Rightarrow \) (i), it suffices to verify the following assertion:
Claim 1.4.A: Let $\gamma_1, \gamma_2 \in H$ be nontrivial elements. Write $e_1, e_2 \in \text{Edge}(\tilde{\mathcal{G}})$ for the unique elements of $\text{Edge}(\tilde{\mathcal{G}})$ such that $\gamma_1 \in \Pi_{e_1}, \gamma_2 \in \Pi_{e_2}$ [cf. Remark 1.1.1]. Then $e_1 = e_2$.

To verify Claim 1.4.A, let us observe that it follows from condition (iii) that, for every positive integer $n$, it holds that $\gamma_1^n \gamma_2^n$ is edge-like, hence vertical. Thus, it follows immediately from Lemma 1.3 that there exists an element $\tilde{v} \in \text{Vert}(\tilde{\mathcal{G}})$ such that $\{\tilde{e}_1, \tilde{e}_2\} \subseteq \mathcal{E}(\tilde{v})$; in particular, it holds that $\gamma_1, \gamma_2 \in \Pi_{\tilde{v}}$. Thus, to complete the verification of Claim 1.4.A, we may assume without loss of generality — by replacing $\Pi_G, H$ by $\Pi_{\tilde{v}}, \Pi_{\tilde{v}} \cap H$, respectively — that $\text{Node}(\tilde{G}) = \emptyset$ [so $\tilde{e}_1, \tilde{e}_2 \in \text{Cusp}(\tilde{G})$]. Moreover, we may assume without loss of generality — by replacing $\Pi_G$ (respectively, $\gamma_1, \gamma_2$) by a suitable open subgroup of $\Pi_G$ (respectively, suitable powers of $\gamma_1, \gamma_2$) — that $\text{Cusp}(\tilde{G})^\sharp \geq 4$. Thus, it follows immediately from the well-known structure of the abelianization of the fundamental group of a smooth curve over an algebraically closed field of characteristic $\not\equiv \Sigma$ that the direct product of any 3 cuspidal inertia subgroups of $\Pi_G$ associated to distinct cusps of $\tilde{G}$ maps injectively to the abelianization $\Pi_{\tilde{G}}^\text{ab}$ of $\Pi_{\tilde{G}}$. In particular, since $\gamma_1 \gamma_2$ is edge-like, hence cuspidal, we conclude, by considering the cuspidal inertia subgroups that contain $\gamma_1, \gamma_2$, and $\gamma_1 \gamma_2$, that $\tilde{e}_1 = \tilde{e}_2$. This completes the proof of Claim 1.4.A, hence also of the implication (iii) $\Rightarrow$ (i). This completes the proof of Proposition 1.4.

Proposition 1.5 (Group-theoretic characterization of closed subgroups of vertical subgroups). Let $H \subseteq \Pi_{\tilde{G}}$ be a closed subgroup of $\Pi_{\tilde{G}}$. Then the following conditions are equivalent:

(i) $H$ is contained in a vertical subgroup.

(ii) An open subgroup of $H$ is contained in a vertical subgroup.

(iii) Every element of $H$ is vertical [cf. Definition 1.1].

(iv) There exists a connected finite étale subcovering $\mathcal{G}^\dagger \to \mathcal{G}$ of $\tilde{\mathcal{G}} \to \mathcal{G}$ such that for any connected finite étale subcovering $\mathcal{G}' \to \mathcal{G}$ of $\tilde{\mathcal{G}} \to \mathcal{G}$ that factors through $\mathcal{G}^\dagger \to \mathcal{G}$, the image of the composite

$$H \cap \Pi_{\mathcal{G}'} \hookrightarrow \Pi_{\mathcal{G}'} \to \Pi_{\mathcal{G}'}^\text{ab-comb}$$

— where we write $\Pi_{\mathcal{G}'}^\text{ab-comb}$ for the torsion-free [cf. [CmbGC], Remark 1.1.4] quotient of the abelianization $\Pi_{\mathcal{G}'}^\text{ab}$ by the closed subgroup topologically generated by the images in $\Pi_{\mathcal{G}'}^\text{ab}$ of the vertical subgroups of $\Pi_{\mathcal{G}'}$ — is trivial.
Proof. The implications (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iv) are immediate. Next, we verify the implication (iv) \(\Rightarrow\) (iii). Suppose that condition (iv) holds. Let \(\gamma \in H\). Then to verify that \(\gamma\) is vertical, we may assume without loss of generality — by replacing \(H\) by the procyclic subgroup of \(H\) topologically generated by \(\gamma\) — that \(H\) is procyclic. Now the implication (iv) \(\Rightarrow\) (iii) follows immediately from a similar argument to the argument applied in the proof of the implication (ii) \(\Rightarrow\) (i) of [NodNon], Lemma 1.6, in the edge-like case. Here, we note that unlike the edge-like case, there is a slight complication arising from the fact [cf. [NodNon], Lemma 1.9, (i)] that an element \(e\) of \(Vert(eG)\) is not necessarily uniquely determined by the condition that \(\Pi = \Pi_{\tilde{v}} \cap \Pi_{\tilde{w}}\). On the other hand, this phenomenon is, in fact, irrelevant to the argument in question, since \(\Pi_G\) does not contain any elements that fix, but permute the branches of \(\tilde{e}\). This completes the proof of the implication (iv) \(\Rightarrow\) (iii).

Finally, we verify the implication (iii) \(\Rightarrow\) (i). Suppose that condition (iii) holds. Now if every element of \(H\) is edge-like, then the implication (iii) \(\Rightarrow\) (i) follows from the implication (iii) \(\Rightarrow\) (i) of Proposition 1.4, together with the fact that every edge-like subgroup is contained in a vertical subgroup. Thus, to verify the implication (iii) \(\Rightarrow\) (i), we may assume without loss of generality that there exists an element \(1 \neq 2 \in H\) of \(H\) that is not edge-like. Write \(\tilde{v}_1 \in Vert(\tilde{G})\) for the unique element of \(Vert(eG)\) such that \(\Pi_{\tilde{v}_1} = \Pi_{\tilde{w}}\) [cf. Remark 1.1.1].

Now we claim the following assertion:

Claim 1.5.A: \(H \subseteq \Pi_{\tilde{v}_1}\).

Indeed, let \(\gamma_2 \in H\) be a nontrivial element of \(H\). If \(\gamma_2 = \gamma_1\), then \(\gamma_2 \in \Pi_{\tilde{v}_1}\). Thus, we may assume without loss of generality that \(\gamma_1 \neq \gamma_2\). Write \(\gamma \overset{\text{def}}{=} \gamma_1\gamma_2^{-1}\).

Next, suppose that \(\gamma_2\) is not edge-like. Write \(\tilde{v}_2 \in Vert(\tilde{G})\) for the unique element of \(Vert(eG)\) such that \(\gamma_2 \in \Pi_{\tilde{v}_2}\) [cf. Remark 1.1.1]. Let \(H' \to G\) be a connected finite étale subcovering of \(G\). Then since neither \(\gamma_1\) nor \(\gamma_2\) is edge-like, one verifies easily — by applying the implication (iv) \(\Rightarrow\) (i) of Proposition 1.4 to the closed subgroups of \(\Pi_G\) topologically generated by \(\gamma_1\), \(\gamma_2\), respectively — that there exist a connected finite étale subcovering \(H' \to H\) of \(\tilde{G}\) and a positive integer \(n\) such that \(\gamma_1^n, \gamma_2^n \in \Pi_{H'} \subseteq \Pi_H\), and, moreover, the images of \(\gamma_1^n, \gamma_2^n \in \Pi_{H'}\) via the natural surjection \(\Pi_{H'} \to \Pi_{H'}^{ab/edge}\) [cf. the notation of Proposition 1.4, (iv)] are nontrivial. Thus, it follows from the existence of the natural split injection

\[
\bigoplus_{v \in Vert(H')} \Pi_{v}^{ab/edge} \longrightarrow \Pi_{H'}^{ab/edge}
\]
of [NodNon], Lemma 1.4, together with the fact that \( \gamma_1^n \gamma_2^n \in \Pi_{\mathcal{H}'} \) is vertical [cf. condition (iii)], that \( \tilde{v}_1(\mathcal{H}') = \tilde{v}_2(\mathcal{H}') \), hence that \( \tilde{v}_1(\mathcal{H}) = \tilde{v}_2(\mathcal{H}) \). Therefore, by allowing the subcovering \( \mathcal{H} \to \mathcal{G} \) of \( \tilde{\mathcal{G}} \to \mathcal{G} \) to vary, we conclude that \( \tilde{v}_1 = \tilde{v}_2 \); in particular, it holds that \( \gamma_2 \in \Pi_{\tilde{v}_1} \).

Next, suppose that \( \gamma_2 \) is edge-like, but that \( \gamma \) is not edge-like. Then, by applying the argument of the preceding paragraph concerning \( \gamma_2 \) to \( \gamma \), we conclude that \( \tilde{v}_1 = \tilde{v}_2 \), hence also \( \gamma_2 \), is contained in \( \Pi_{\tilde{v}_1} \).

Next, suppose that both \( \gamma_2 \) and \( \gamma \) are edge-like. Write \( e_{\gamma_2}, e_{\gamma} \in \text{Edge}(\mathcal{G}) \) for the unique elements of \( \text{Edge}(\mathcal{G}) \) such that \( \tilde{v}_1 = e_{\gamma_2} \), \( \tilde{v}_2 = e_{\gamma} \). Moreover, it follows from condition (iii) that for any positive integer \( n \), the element \( \gamma_2^n \) is vertical. Thus, it follows immediately from Lemma 1.3 that there exists a unique \( s_{\tilde{v}_1} \in \text{Sect}(\Pi_{\tilde{v}_1}) \) such that \( \text{Vert}(\mathcal{G}) \to \text{Sect}(\Pi_{\tilde{v}_1}) \).

Finally, the following equalities concerning centralizers of subgroups of \( \Pi_I \) in \( \Pi_{\mathcal{G}} \) hold: \( Z_{\Pi_{\mathcal{G}}}(s_{\tilde{v}}(I)) = Z_{\Pi_G}(I_{\tilde{v}}) = \Pi_{\tilde{v}} \).

(iii) The map of (ii) is injective.

Theorem 1.6 (Section conjecture-type result for outer representations of SNN-, IPSC-type). Let \( \Sigma \) be a nonempty set of prime numbers, \( \mathcal{G} \) a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type, and \( I \to \text{Aut}(\mathcal{G}) \) an outer representation of \( \text{SNN-type} \) [cf. [NodNon], Definition 2.4, (iii)]. Write \( \Pi_{\mathcal{G}} \) for the [pro-\( \Sigma \)] fundamental group of \( \mathcal{G} \) and \( \Pi_I \overset{\text{out}}{=} \Pi_{\mathcal{G}} \rtimes I \) [cf. the discussion entitled “Topological groups” in [CbTpI], §0]; thus, we have a natural exact sequence of profinite groups

\[
1 \to \Pi_{\mathcal{G}} \to \Pi_I \to I \to 1.
\]

Write \( \text{Sect}(\Pi_I/I) \) for the set of sections of the natural surjection \( \Pi_I \to I \). Then the following hold:

(i) For any \( \tilde{v} \in \text{Vert}(\tilde{\mathcal{G}}) \), the composite \( I_{\tilde{v}} \hookrightarrow \Pi_I \to I \) [cf. [NodNon], Definition 2.2, (i)] is an isomorphism. In particular, \( I_{\tilde{v}} \subseteq \Pi_I \) determines an element \( s_{\tilde{v}} \in \text{Sect}(\Pi_I/I) \); thus, we have a map

\[
\text{Vert}(\tilde{\mathcal{G}}) \to \text{Sect}(\Pi_I/I) \\
\tilde{v} \mapsto s_{\tilde{v}}.
\]

(ii) The map of (i) is injective.
(iii) If, moreover, \( I \to \text{Aut}(\mathcal{G}) \) is of IPSC-type [cf. [NodNon], Definition 2.4, (i)], then, for any \( s \in \text{Sect}(\Pi_I/I) \), the centralizer \( \Pi_I(s(I)) \) is contained in a vertical subgroup.

(iv) Let \( s \in \text{Sect}(\Pi_I/I) \). Consider the following two conditions:

1. The section \( s \) is contained in the image of the map of (i), i.e., \( s = s_{\bar{v}} \) for some \( \bar{v} \in \text{Vert}(\widehat{\mathcal{G}}) \).

2. \( \Pi_I(s(I)) = \{1\} \).

Then we have an implication

\[(1) \implies (2) .\]

If, moreover, \( I \to \text{Aut}(\mathcal{G}) \) is of IPSC-type, then we have an equivalence

\[(1) \iff (2) .\]

Proof. First, we verify assertion (i). The fact that the composite \( I_{\bar{v}} \hookrightarrow \Pi_I \to I \) is an isomorphism follows from condition (2') of [NodNon], Definition 2.4, (ii). On the other hand, the equalities \( \Pi_I(s_{\bar{v}}(I)) = Z_{\Pi_I}(I_{\bar{v}}) = \Pi_{\bar{v}} \) follow from [NodNon], Lemma 3.6, (i). This completes the proof of assertion (i). Assertion (ii) follows immediately from the final equalities of assertion (i), together with [NodNon], Lemma 1.9, (ii). Next, we verify assertion (iii). Write \( H \overset{\text{def}}{=} Z_{\Pi_I}(s(I)) \). Then it follows immediately from [CmbGC], Proposition 2.6, together with the definition of \( H = Z_{\Pi_I}(s(I)) \), that for any connected finite étale subcovering \( \mathcal{G}' \to \mathcal{G} \) of \( \widehat{\mathcal{G}} \to \mathcal{G} \), the image of the composite

\[
H \cap \Pi_{\mathcal{G}'} \to \Pi_{\mathcal{G}'} \to \Pi_{\mathcal{G}'}^{\text{ab-comb}}
\]

[cf. the notation of Proposition 1.5, (iv)] is trivial. Thus, it follows from the implication (iv) \( \implies (i) \) of Proposition 1.5 that \( H \) is contained in a vertical subgroup. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). To verify the implication (1) \( \implies (2) \), suppose that condition (1) holds. Then since \( Z_{\Pi_I}(s_{\bar{v}}(I)) = Z_{\Pi_I}(I_{\bar{v}}) = \Pi_{\bar{v}} \) [cf. assertion (i)] is commensurably terminal in \( \Pi_I \) [cf. [CmbGC], Proposition 1.2, (ii) and center-free [cf. [CmbGC], Remark 1.1.3], we conclude that \( Z_{\Pi_I}(Z_{\Pi_I}(s_{\bar{v}}(I))) = Z_{\Pi_I}(\Pi_{\bar{v}}) = \{1\} \). This completes the proof of the implication (1) \( \implies (2) \). Next, suppose that \( I \to \text{Aut}(\mathcal{G}) \) is of IPSC-type, and that condition (2) holds. Then it follows from assertion (iii) that there exists a \( \bar{v} \in \text{Vert}(\widehat{\mathcal{G}}) \) such that \( H \overset{\text{def}}{=} Z_{\Pi_I}(s(I)) \subseteq \Pi_{\bar{v}} \), so \( I_{\bar{v}} \subseteq Z_{\Pi_I}(H) \). On the other hand, since \( s(I) \subseteq Z_{\Pi_I}(H) \), and \( Z_{\Pi_I}(H) = Z_{\Pi_I}(Z_{\Pi_I}(s(I))) = \{1\} \) [cf. condition (2)], i.e., the composite of natural homomorphisms \( Z_{\Pi_I}(H) \hookrightarrow \Pi_I \to I \) is injective, it follows that \( s(I) = Z_{\Pi_I}(H) \supseteq I_{\bar{v}} \). Since \( I_{\bar{v}} \) and \( s(I) \) may be obtained as the images of sections, we thus conclude that \( I_{\bar{v}} = s(I) \), i.e., \( s = s_{\bar{v}} \). This completes the proof of the implication (2) \( \implies (1) \), hence also of assertion (iv). \( \square \)
Remark 1.6.1. Recall that in the case of outer representations of NN-type, the period matrix is not necessarily nondegenerate [cf. [CbTpI], Remark 5.9.2]. In particular, the argument applied in the proof of Theorem 1.6, (iii) — which depends, in an essential way, on the fact that, in the case of outer representations of IPSC-type, the period matrix is nondegenerate [cf. the proof of [CmbGC], Proposition 2.6] — cannot be applied in the case of outer representations of NN-type. Nevertheless, the question of whether or not Theorem 1.6, (iii), as well as the application of Theorem 1.6, (iii), given in Corollary 1.7, (ii), below, may be generalized to the case of outer representations of NN-type remains a topic of interest to the authors.

Corollary 1.7 (Group-theoretic characterization of vertical subgroups for outer representations of IPSC-type). In the notation of Theorem 1.6, let us refer to a closed subgroup of $\Pi_G$ as a section-centralizer if it may be written in the form $Z_{\Pi_G}(s(I))$ for some $s \in \text{Sect}(\Pi_I/I)$. Let $H \subseteq \Pi_G$ be a closed subgroup of $\Pi_G$. Then the following hold:

(i) Suppose that $H$ is a section-centralizer such that $Z_{\Pi_G}(H) = \{1\}$. Then the following conditions on a section $s \in \text{Sect}(\Pi_I/I)$ are equivalent:

(i-1) $H = Z_{\Pi_I}(s(I))$.

(i-2) $s(I) \subseteq Z_{\Pi_I}(H)$.

(i-3) $s(I) = Z_{\Pi_I}(H)$.

(ii) Consider the following three conditions:

(ii-1) $H$ is a vertical subgroup.

(ii-2) $H$ is a section-centralizer such that $Z_{\Pi_G}(H) = \{1\}$.

(ii-3) $H$ is a maximal section-centralizer.

Then we have implications

$(\text{ii-1}) \implies (\text{ii-2}) \implies (\text{ii-3})$.

If, moreover, $I \to \text{Aut}(G)$ is of IPSC-type [cf. [NodNon], Definition 2.4, (i)], then we have equivalences

$(\text{ii-1}) \iff (\text{ii-2}) \iff (\text{ii-3})$.

Proof. First, we verify assertion (i). The implication (i-1) $\implies$ (i-2) is immediate. To verify the implication (i-2) $\implies$ (i-3), suppose that condition (i-2) holds. Then since $Z_{\Pi_I}(H) \cap \Pi_G = Z_{\Pi_G}(H) = \{1\}$, the composite $Z_{\Pi_I}(H) \to \Pi_I \to I$ is injective. Thus, since the composite $s(I) \to Z_{\Pi_I}(H) \to \Pi_I \to I$ is an isomorphism, it follows immediately
that condition (i-3) holds. This completes the proof of the implication (i-2) \(\Rightarrow\) (i-3). Finally, to verify the implication (i-3) \(\Rightarrow\) (i-1), suppose that condition (i-3) holds. Then since \(H\) is a section-centralizer, there exists a \(t \in \text{Sect}(\Pi_I/I)\) such that \(H = Z_{\Pi_I}(t(I))\). In particular, \(t(I) \subseteq Z_{\Pi_I}(H) = s(I)\) [cf. condition (i-3)]. We thus conclude that \(t = s\), i.e., that condition (i-1) holds. This completes the proof of assertion (i).

Next, we verify assertion (ii). The implication (ii-1) \(\Rightarrow\) (ii-2) follows immediately from Theorem 1.6, (i), (iv). To verify the implication (ii-2) \(\Rightarrow\) (ii-3), suppose that \(H\) satisfies condition (ii-2); let \(s \in \text{Sect}(\Pi_I/I)\) be such that \(H \subseteq Z_{\Pi_G}(s(I))\). Then it follows immediately that \(s(I) \subseteq Z_{\Pi_I}(H)\). Thus, it follows immediately from the equivalence (i-1) \(\Leftrightarrow\) (i-2) of assertion (i) that \(H = Z_{\Pi_G}(s(I))\). This completes the proof of the implication (ii-2) \(\Rightarrow\) (ii-3). Finally, observe that the implication (ii-3) \(\Rightarrow\) (ii-1) in the case where \(I \neq \text{Aut}(G)\) is of IPSC-type follows immediately from Theorem 1.6, (iii), together with the fact that every vertical subgroup is a section-centralizer [cf. the implication (ii-1) \(\Rightarrow\) (ii-2) verified above]. This completes the proof of Corollary 1.7.

**Lemma 1.8 (Group-theoretic characterization of vertical subgroups for outer representations of SNN-type).** Let \(H \subseteq \Pi_G\) be a closed subgroup of \(\Pi_G\) and \(I \to \text{Aut}(G)\) an outer representation of SNN-type [cf. [NodNon], Definition 2.4, (iii)]. Write \(\Pi_I \defeq \Pi_G \rtimes I\) [cf. the discussion entitled “Topological groups” in [CbTpI], §9]; thus, we have a natural exact sequence of profinite groups

\[
1 \rightarrow \Pi_G \rightarrow \Pi_I \rightarrow I \rightarrow 1.
\]

Suppose that \(G\) is untangled [cf. [NodNon], Definition 1.2]. Then \(H\) is a vertical subgroup if and only if \(H\) satisfies the following four conditions:

1. The composite \(I_H \defeq Z_{\Pi_I}(H) \hookrightarrow \Pi_I \rightarrow I\) is an isomorphism.
2. It holds that \(H = Z_{\Pi_G}(I_H)\).
3. For any \(\gamma \in \Pi_G\), it holds that \(\gamma \in \Pi_I\) if and only if \(H \cap (\gamma \cdot H \cdot \gamma^{-1}) \neq \{1\}\).
4. \(H\) contains a nontrivial vertical element of \(\Pi_G\) [cf. Definition 1.1].

**Proof.** If \(H\) is a vertical subgroup, then it is immediate that condition (iv) is satisfied; moreover, it follows from condition (2’) of [NodNon], Definition 2.4, (ii) (respectively, [NodNon], Lemma 3.6, (i); [NodNon], Remark 1.10.1), that \(H\) satisfies condition (i) (respectively, (ii); (iii)). This completes the proof of necessity.
To verify sufficiency, suppose that $H$ satisfies conditions (i), (ii), (iii), and (iv). It follows from condition (iv) that there exists a $\tilde{v} \in \text{Vert}(G)$ such that $\tilde{v} \in H \cap \Pi_{\tilde{v}} \neq \{1\}$. If either $J = \Pi_{\tilde{v}}$ or $J = H$, i.e., either $\Pi_{\tilde{v}} \subseteq H$ or $H \subseteq \Pi_{\tilde{v}}$, then it is immediate that either $I_H \subseteq I_{\tilde{v}}$ or $I_{\tilde{v}} \subseteq I_H$ [cf. [NodNon], Definition 2.2, (i)]. Thus, it follows from condition (i) for $H$ and $\Pi_{\tilde{v}}$ that $I_H = I_{\tilde{v}}$. But then it follows from condition (ii) for $H$ and $\Pi_{\tilde{v}}$ that $H = Z_{\Pi_{\tilde{v}}}(I_H) = Z_{\Pi_{\tilde{v}}}(I_{\tilde{v}}) = \Pi_{\tilde{v}}$; in particular, $H$ is a vertical subgroup.

Thus, we may assume without loss of generality that $J \neq H$, $\Pi_{\tilde{v}}$. Let $\gamma \in H \setminus J$. Write $J' \overset{\text{def}}{=} \gamma \cdot J \cdot \gamma^{-1}$. Then we have inclusions

$$\Pi_{\tilde{v}} \supseteq J \subseteq H \supseteq J' \subseteq \Pi_{\tilde{v}^\gamma} (= \gamma \cdot \Pi_{\tilde{v}} \cdot \gamma^{-1}).$$

Now we claim the following assertion:


Indeed, let $\sigma \in N_{\Pi_{\tilde{v}}}(J)$. Then since $\{1\} \neq J = J \cap (\sigma \cdot J \cdot \sigma^{-1}) \subseteq \Pi_{\tilde{v}} \cap \Pi_{\tilde{v}^\sigma}$, it follows from condition (iii) [for $\Pi_{\tilde{v}}$] that $\sigma \in \Pi_{\tilde{v}}$. Similarly, since $\{1\} \neq J = J \cap (\sigma \cdot J \cdot \sigma^{-1}) \subseteq H \cap (\sigma \cdot H \cdot \sigma^{-1})$, it follows from condition (ii) [for $H$] that $\sigma \in H$. Thus, $\sigma \in \Pi_{\tilde{v}} \cap H = J$. In particular, we obtain that $N_{\Pi_{\tilde{v}}}(J) = J$. A similar argument implies that $N_{\Pi_{\tilde{v}}}(J') = J'$. This completes the proof of Claim 1.8.A.

Now the composites $N_{\Pi_{\tilde{v}}}(J)$, $N_{\Pi_{\tilde{v}}}(J') \hookrightarrow \Pi_{\tilde{v}} \twoheadrightarrow I$ fit into exact sequences of profinite groups

$$1 \rightarrow N_{\Pi_{\tilde{v}}}(J) \rightarrow N_{\Pi_{\tilde{v}}}(J') \rightarrow I,$$

$$1 \rightarrow N_{\Pi_{\tilde{v}}}(J') \rightarrow N_{\Pi_{\tilde{v}}}(J) \rightarrow I.$$ Thus, since we have inclusions

$$I_H = Z_{\Pi_{\tilde{v}}}(H) \subseteq Z_{\Pi_{\tilde{v}}}(J) \subseteq N_{\Pi_{\tilde{v}}}(J),$$

$$I_H = Z_{\Pi_{\tilde{v}}}(H) \subseteq Z_{\Pi_{\tilde{v}}}(J') \subseteq N_{\Pi_{\tilde{v}}}(J'),$$

$$I_{\tilde{v}} = Z_{\Pi_{\tilde{v}}}(\Pi_{\tilde{v}}) \subseteq Z_{\Pi_{\tilde{v}}}(I_{\tilde{v}}) \subseteq N_{\Pi_{\tilde{v}}}(I_{\tilde{v}}),$$

$$I_{\tilde{v}^\gamma} = Z_{\Pi_{\tilde{v}}}(\Pi_{\tilde{v}^\gamma}) \subseteq Z_{\Pi_{\tilde{v}}}(I_{\tilde{v}^\gamma}) \subseteq N_{\Pi_{\tilde{v}}}(I_{\tilde{v}^\gamma}),$$

it follows immediately from Claim 1.8.A, together with condition (i) for $H$ and $\Pi_{\tilde{v}}$, that

$$N_{\Pi_{\tilde{v}}}(J) = J \cdot I_H = J \cdot I_{\tilde{v}}, \quad N_{\Pi_{\tilde{v}}}(J') = J' \cdot I_H = J' \cdot I_{\tilde{v}^\gamma}.$$ In particular, we obtain that

$$I_H \subseteq N_{\Pi_{\tilde{v}}}(J) = J \cdot I_{\tilde{v}} \subseteq \Pi_{\tilde{v}} \cdot D_{\tilde{v}} = D_{\tilde{v}},$$

$$I_H \subseteq N_{\Pi_{\tilde{v}}}(J') = J' \cdot I_{\tilde{v}^\gamma} \subseteq \Pi_{\tilde{v}^\gamma} \cdot D_{\tilde{v}^\gamma} = D_{\tilde{v}^\gamma}.$$ [cf. [NodNon], Definition 2.2, (i)], i.e., $I_H \subseteq D_{\tilde{v}} \cap D_{\tilde{v}^\gamma}$. On the other hand, since $H \ni \gamma \notin J = H \cap \Pi_{\tilde{v}}$, it follows from condition (iii) [for $\Pi_{\tilde{v}}$] that $\Pi_{\tilde{v}} \cap \Pi_{\tilde{v}} = \{1\}$; thus, it follows immediately from the fact that $D_{\tilde{v}} \cap D_{\tilde{v}^\gamma} \cap \Pi_{\tilde{v}} = \Pi_{\tilde{v}} \cap \Pi_{\tilde{v}^\gamma} = \{1\}$ [cf. [CmbGC], Proposition 1.2, (ii)], together with condition (i), that $I_H = D_{\tilde{v}} \cap D_{\tilde{v}^\gamma}$, which implies,
by [NodNon], Proposition 3.9, (iii), that there exists a \( \tilde{w} \in \text{Vert}(\tilde{G}) \) such that \( I_H = I_{\tilde{w}} \). In particular, it follows from condition (ii) [for \( H \) and \( \Pi_{\tilde{w}} \)] that \( H = \Pi_{\tilde{w}}(I_H) = \Pi_{\tilde{w}}(I_{\tilde{w}}) = \Pi_{\tilde{w}} \). Thus, \( H \) is a vertical subgroup. This completes the proof of Lemma 1.8. \( \square \)

**Theorem 1.9** (Group-theoretic verticiality/nodality of isomorphisms of outer representations of NN-, IPSC-type). Let \( \Sigma \) be a nonempty set of prime numbers, \( G \) (respectively, \( H \)) a semi-graph of anabelioids of pro-\( \Sigma \) PSC-type, \( \Pi_G \) (respectively, \( \Pi_H \)) the [pro-\( \Sigma \)] fundamental group of \( G \) (respectively, \( H \)), \( \alpha : \Pi_G \to \Pi_H \) an isomorphism of profinite groups, \( I \) (respectively, \( J \)) a profinite group, \( \rho_I : I \to \text{Aut}(G) \) (respectively, \( \rho_J : J \to \text{Aut}(H) \)) a continuous homomorphism, and \( \beta : I \to J \) an isomorphism of profinite groups. Suppose that the diagram

\[
\begin{array}{c}
I & \longrightarrow & \text{Out}(\Pi_G) \\
\downarrow \beta & & \downarrow \text{Out}(\alpha) \\
J & \longrightarrow & \text{Out}(\Pi_H)
\end{array}
\]

— where the right-hand vertical arrow is the isomorphism induced by \( \alpha \); the upper and lower horizontal arrows are the homomorphisms determined by \( \rho_I \) and \( \rho_J \), respectively — commutes. Then the following hold:

(i) Suppose, moreover, that \( \rho_I, \rho_J \) are of NN-type [cf. [NodNon], Definition 2.4, (iii)]. Then the following three conditions are equivalent:

1. The isomorphism \( \alpha \) is group-theoretically verticial [i.e., roughly speaking, preserves vertical subgroups — cf. [CmbGC], Definition 1.4, (iv)].

2. The isomorphism \( \alpha \) is group-theoretically nodal [i.e., roughly speaking, preserves nodal subgroups — cf. [NodNon], Definition 1.12].

3. There exists a nontrivial vertical element \( \gamma \in \Pi_G \) such that \( \alpha(\gamma) \in \Pi_H \) is vertical [cf. Definition 1.1].

(ii) Suppose, moreover, that \( \rho_I \) is of NN-type, and that \( \rho_J \) is of IPSC-type [cf. [NodNon], Definition 2.4, (i)]. [For example, this will be the case if both \( \rho_I \) and \( \rho_J \) are of IPSC-type — cf. [NodNon], Remark 2.4.2.] Then \( \alpha \) is group-theoretically verticial, hence also [cf. (i)] group-theoretically nodal.

**Proof.** First, we verify assertion (i). The implication (1) \( \Rightarrow \) (2) follows from [NodNon], Proposition 1.13. The implication (2) \( \Rightarrow \) (3)
follows from the fact that any nodal subgroup is contained in a vertical subgroup. [Note that if $\text{Node}(\mathcal{H}) = \emptyset$, then every element of $\Pi_\mathcal{H}$ is vertical.] Finally, we verify the implication $(3) \Rightarrow (1)$. Suppose that condition $(3)$ holds. Since vertical subgroups are commensurably terminal [cf. [CmbGC], Proposition 1.2, (ii)], to verify the implication $(3) \Rightarrow (1)$, by replacing $\Pi_I, \Pi_J$ by open subgroups of $\Pi_I, \Pi_J$, we may assume without loss of generality that $\rho_I, \rho_J$ are of SNN-type [cf. [NodNon], Definition 2.4, (iii)], and, moreover, that $\mathcal{G}$ and $\mathcal{H}$ are untangled [cf. [NodNon], Definition 1.2; [NodNon], Remark 1.2.1, (i), (ii)]. Let $\bar{v} \in \text{Vert}(\mathcal{G})$ be such that $\gamma \in \Pi_\mathcal{G}$. Then it is immediate that $\alpha(\Pi_\mathcal{G})$ satisfies conditions (i), (ii), and (iii) in the statement of Lemma 1.8. On the other hand, it follows from condition (3) that $\alpha(\Pi_\mathcal{G})$ satisfies condition (iv) in the statement of Lemma 1.8. Thus, it follows from Lemma 1.8 that $\alpha(\Pi_\mathcal{G}) \subseteq \Pi_\mathcal{H}$ is a vertical subgroup. Now it follows from [NodNon], Theorem 4.1, that $\alpha$ is group-theoretically vertical. This completes the proof of the implication $(3) \Rightarrow (1)$.

Finally, we verify assertion (ii). It is immediate that, to verify assertion (ii) — by replacing $I, J$ by open subgroups of $I, J$ — we may assume without loss of generality that $\rho_I$ is of SNN-type. Let $H \subseteq \Pi_\mathcal{G}$ be a vertical subgroup of $\Pi_\mathcal{G}$. Then it follows from Corollary 1.7, (ii), that $H$, hence also $\alpha(H)$, is a maximal section-centralizer [cf. the statement of Corollary 1.7]. Thus, since $\rho_J$ is of IPSC-type, again by Corollary 1.7, (ii), we conclude that $\alpha(H) \subseteq \Pi_\mathcal{H}$ is a vertical subgroup of $\Pi_\mathcal{H}$. In particular, it follows from [NodNon], Theorem 4.1, together with [NodNon], Remark 2.4.2, that $\alpha$ is group-theoretically vertical and group-theoretically nodal. This completes the proof of assertion (ii).

Remark 1.9.1. Thus, Theorem 1.9, (i), may be regarded as a generalization of [NodNon], Corollary 4.2. Of course, ideally, one would like to be able to prove that conditions (1) and (2) of Theorem 1.9, (i), hold automatically [i.e., as in the case of outer representations of IPSC-type treated in Theorem 1.9, (ii)], without assuming condition (3). Although this topic lies beyond the scope of the present paper, perhaps progress could be made in this direction if, say, in the case where $\Sigma$ is either equal to the set of all prime numbers or of cardinality one, one starts with an isomorphism $\alpha$ that arises from a PF-admissible [cf. [CbTpI], Definition 1.4, (i)] isomorphism between configuration space groups corresponding to $m$-dimensional configuration spaces [where $m \geq 2$] associated to stable curves that give rise to $\mathcal{G}$ and $\mathcal{H}$, respectively [i.e., one assumes the condition of “$m$-cuspidalizability” discussed in Definition 3.20, below, where we replace the condition of “PFC-admissibility” by the condition of “PF-admissibility”]. For instance, if $\text{Cusp}(\mathcal{G}) \neq \emptyset$, then it follows from [CbTpI], Theorem 1.8, (iv); [NodNon], Corollary
4.2, that this condition on $\alpha$ is sufficient to imply that conditions (1) and (2) of Theorem 1.9, (i), hold.
2. Partial combinatorial cuspidalization for $F$-admissible automorphisms

In the present §2, we apply the results obtained in the preceding §1, together with the theory developed by the authors in earlier papers, to prove combinatorial cuspidalization-type results for $F$-admissible automorphisms [cf. Theorem 2.3, (i), below]. We also show that any $F$-admissible automorphism of a configuration space group [arising from a configuration space] of sufficiently high dimension [i.e., $\geq 3$ in the affine case; $\geq 4$ in the proper case] is necessarily $C$-admissible, i.e., preserves the cuspidal inertia subgroups of the various subquotients corresponding to surface groups [cf. Theorem 2.3, (ii), below]. Finally, we discuss applications of these combinatorial anabelian results to the anabelian geometry of configuration spaces associated to hyperbolic curves over arithmetic fields [cf. Corollaries 2.5, 2.6, below].

In the present §2, let $\Sigma$ be a set of prime numbers which is either equal to the set of all prime numbers or of cardinality one; $n$ a positive integer; $k$ an algebraically closed field of characteristic $\not\in \Sigma$; $X$ a hyperbolic curve of type $(g, r)$ over $k$. For each positive integer $i$, write $X_i$ for the $i$-th configuration space of $X$; $\Pi_i$ for the maximal pro-$\Sigma$ quotient of the fundamental group of $X_i$.

Definition 2.1. Let $\alpha \in \text{Aut}(\Pi_n)$ be an automorphism of $\Pi_n$.

(i) Write

$$\{1\} = K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_2 \subseteq K_1 \subseteq K_0 = \Pi_n$$

for the standard fiber filtration on $\Pi_n$ [cf. [CmbCsp], Definition 1.1, (i)]. For each $m \in \{1, 2, \cdots, n\}$, write $C_m$ for the [finite] set of $K_{m-1}/K_m$-conjugacy classes of cuspidal inertia subgroups of $K_{m-1}/K_m$ [where we recall that $K_{m-1}/K_m$ is equipped with a natural structure of pro-$\Sigma$ surface group — cf. [MzTa], Definition 1.2]. Then we shall say that $\alpha$ is $wC$-admissible [i.e., “weakly $C$-admissible”] if $\alpha$ preserves the standard fiber filtration on $\Pi_n$ and, moreover, satisfies the following conditions:

- If $m \in \{1, 2, \cdots, n-1\}$, then the automorphism of $K_{m-1}/K_m$ determined by $\alpha$ induces an automorphism of $C_m$.
- It follows immediately from the various definitions involved that we have a natural injection $C_{n-1} \hookrightarrow C_n$. That is to say, if one thinks of $K_{n-2}$ as the two-dimensional configuration space group associated to some hyperbolic curve, then the image of $C_{n-1} \hookrightarrow C_n$ corresponds to the set of cusps of a fiber [of the two-dimensional configuration space over the hyperbolic curve] that arise from the
cusps of the hyperbolic curve. Then the automorphism of $K_{n-1}$ determined by $\alpha$ induces an automorphism of the image of the natural injection $C_{n-1} \hookrightarrow C_n$.

Write

$$\text{Aut}^{wC}(\Pi_n) \subseteq \text{Aut}(\Pi_n)$$

for the subgroup of $wC$-admissible automorphisms and

$$\text{Out}^{wC}(\Pi_n) \overset{\text{def}}{=} \text{Aut}^{wC}(\Pi_n)/\text{Inn}(\Pi_n) \subseteq \text{Out}(\Pi_n).$$

We shall refer to an element of $\text{Out}^{wC}(\Pi_n)$ as a $wC$-admissible automorphism.

(ii) We shall say that $\alpha$ is $FwC$-admissible if $\alpha$ is $F$-admissible [cf. [CmbCsp], Definition 1.1, (ii)] and $wC$-admissible [cf. (i)]. Write

$$\text{Aut}^{FwC}(\Pi_n) \subseteq \text{Aut}^F(\Pi_n)$$

for the subgroup of $FwC$-admissible automorphisms and

$$\text{Out}^{FwC}(\Pi_n) \overset{\text{def}}{=} \text{Aut}^{FwC}(\Pi_n)/\text{Inn}(\Pi_n) \subseteq \text{Out}^F(\Pi_n).$$

We shall refer to an element of $\text{Out}^{FwC}(\Pi_n)$ as an $FwC$-admissible automorphism.

(iii) We shall say that $\alpha$ is $DF$-admissible [i.e., “diagonal-fiber-admissible”] if $\alpha$ is $F$-admissible, and, moreover, $\alpha$ induces the same automorphism of $\Pi_1$ relative to the various quotients $\Pi_n \rightarrow \Pi_1$ by fiber subgroups of co-length 1 [cf. [MzTa], Definition 2.3, (iii)]. Write

$$\text{Aut}^{DF}(\Pi_n) \subseteq \text{Aut}^F(\Pi_n)$$

for the subgroup of $DF$-admissible automorphisms.

Remark 2.1.1. Thus, it follows immediately from the definitions that

$$C\text{-admissible} \implies wC\text{-admissible}.$$

In particular, we have inclusions

$$\text{Aut}^{FC}(\Pi_n) \subseteq \text{Aut}^{FwC}(\Pi_n) \cap \text{Out}^{FC}(\Pi_n) \subseteq \text{Out}^{FwC}(\Pi_n)$$

$$\text{Aut}^{C}(\Pi_n) \subseteq \text{Aut}^{wC}(\Pi_n) \cap \text{Out}^{C}(\Pi_n) \subseteq \text{Out}^{wC}(\Pi_n)$$

[cf. Definition 2.1, (i), (ii)].
Lemma 2.2 (F-admissible automorphisms and inertia subgroups).
Let \( \alpha \in \text{Aut}^F(\Pi_n) \) be an F-admissible automorphism of \( \Pi_n \). Then the following hold:

(i) There exist \( \beta \in \text{Aut}^{DF}(\Pi_n) \) [cf. Definition 2.1, (iii)] and \( \iota \in \text{Inn}(\Pi_n) \) such that \( \alpha = \beta \circ \iota \).

(ii) For each positive integer \( i \), write \( Z_i^{\log} \) for the \( i \)-th log configuration space of \( X \) [cf. the discussion entitled “Curves” in [CbTpI], x]

\[ U_{Z_i} \subseteq Z_i \] for the interior of \( Z_i^{\log} \) [cf. the discussion entitled “Log schemes” in [CbTpI], §0], which may be identified with \( X_i \). Let \( \epsilon \) be an irreducible component of the complement \( Z_{n-1} \setminus U_{Z_{n-1}} \) [cf. [CmbCsp], Proposition 1.3]; \( \mathbb{I}_\epsilon \subseteq \Pi_{n-1} \) an inertia subgroup of \( \Pi_{n-1} \) associated to the divisor \( \epsilon \) of \( Z_{n-1} \); \( \text{pr}: U_{Z_n} \to U_{Z_{n-1}} \) the projection obtained by forgetting the factor labeled \( n \); \( \text{pr}^\Pi: \Pi_n \to \Pi_{n-1} \) the surjection induced by \( \text{pr}; \Pi_{n/n-1} \overset{\text{def}}{=} \text{Ker}(\text{pr}^\Pi) \); \( \theta \) an irreducible component of the fiber of the [uniquely determined] extension \( Z_n \to Z_{n-1} \) of \( \text{pr} \) over the generic point of \( \epsilon \) [so \( \theta \) naturally determines an irreducible component of the complement \( Z_n \setminus U_{Z_n} \); \( \mathbb{D}_\theta \subseteq \Pi_n \times_{\Pi_{n-1}} \mathbb{I}_\epsilon \) (\( \subseteq \Pi_n \)) — where the homomorphism \( \Pi_n \to \Pi_{n-1} \) implicit in the fiber product is the surjection \( \text{pr}^\Pi: \Pi_n \to \Pi_{n-1} \) — a decomposition subgroup of \( \Pi_n \times_{\Pi_{n-1}} \mathbb{I}_\epsilon \) (\( \subseteq \Pi_n \)) associated to the divisor [naturally determined by] \( \theta \) of \( Z_n \); \( \Pi_\theta \overset{\text{def}}{=} \mathbb{D}_\theta \cap \Pi_{n/n-1} \) [cf. [CmbCsp], Proposition 1.3, (iv)]. Suppose that the automorphism of \( \Pi_{n-1} \) induced by \( \alpha \in \text{Aut}^F(\Pi_n) \) relative to \( \text{pr}^\Pi \) stabilizes \( \mathbb{I}_\epsilon \subseteq \Pi_{n-1} \). Then \( \alpha \) preserves the \( \Pi_{n/n-1} \)-conjugacy class of \( \Pi_\theta \).

Proof. Assertion (i) follows immediately from [CbTpI], Theorem A, (i). Assertion (ii) follows immediately from Theorem 1.9, (ii) [cf. also the proof of [CmbCsp], Proposition 1.3, (iv)]. \( \Box \)

Theorem 2.3 (Partial combinatorial cuspidalization for F-admissible automorphisms). Let \( \Sigma \) be a set of prime numbers which is either equal to the set of all prime numbers or of cardinality one; \( n \) a positive integer; \( X \) a hyperbolic curve of type \((g,r)\) over an algebraically closed field of characteristic \( \not\in \Sigma \); \( X_n \) the \( n \)-th configuration space of \( X \); \( \Pi_n \) the maximal pro-\( \Sigma \) quotient of the fundamental group of \( X_n \);

\[ \text{Out}^F(\Pi_n) \subseteq \text{Out}(\Pi_n) \]

the subgroup of F-admissible automorphisms [i.e., roughly speaking, automorphisms that preserve the fiber subgroups — cf. [CmbCsp], Definition 1.1, (ii)] of \( \Pi_n \);

\[ \text{Out}^{FC}(\Pi_n) \subseteq \text{Out}^F(\Pi_n) \]
the subgroup of \textbf{FC-admissible} automorphisms [i.e., roughly speaking, automorphisms that preserve the fiber subgroups and the cuspidal inertia subgroups — cf. [CmbCsp], Definition 1.1, (ii)] of $\Pi_n$;

\[(\text{Out}^{\text{FC}}(\Pi_n) \subseteq \text{Out}^{\text{FwC}}(\Pi_n) \subseteq \text{Out}^{\text{F}}(\Pi_n))\]

the subgroup of \textbf{FwC-admissible} automorphisms [cf. Definition 2.1, (ii); Remark 2.1.1] of $\Pi_n$. Then the following hold:

(i) Write

\[n_{\text{inj}} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } r \neq 0, \\ 2 & \text{if } r = 0, \end{cases} \quad n_{\text{bij}} \overset{\text{def}}{=} \begin{cases} 3 & \text{if } r \neq 0, \\ 4 & \text{if } r = 0. \end{cases}\]

If $n \geq n_{\text{inj}}$ (respectively, $n \geq n_{\text{bij}}$), then the natural homomorphism

\[\text{Out}^{\text{F}}(\Pi_{n+1}) \longrightarrow \text{Out}^{\text{F}}(\Pi_n)\]

induced by the projections $X_{n+1} \rightarrow X_n$ obtained by forgetting any one of the $n+1$ factors of $X_{n+1}$ [cf. [CbTpI], Theorem A, (i)] is injective (respectively, bijective).

(ii) Write

\[n_{\text{FC}} \overset{\text{def}}{=} \begin{cases} 2 & \text{if } (g, r) = (0, 3), \\ 3 & \text{if } (g, r) \neq (0, 3) \text{ and } r \neq 0, \\ 4 & \text{if } r = 0. \end{cases}\]

If $n \geq n_{\text{FC}}$, then it holds that

\[\text{Out}^{\text{FC}}(\Pi_n) = \text{Out}^{\text{F}}(\Pi_n).\]

(iii) Write

\[n_{\text{FwC}} \overset{\text{def}}{=} \begin{cases} 2 & \text{if } r \geq 2, \\ 3 & \text{if } r = 1, \\ 4 & \text{if } r = 0. \end{cases}\]

If $n \geq n_{\text{FwC}}$, then it holds that

\[\text{Out}^{\text{FwC}}(\Pi_n) = \text{Out}^{\text{F}}(\Pi_n).\]

(iv) Consider the natural inclusion

\[\mathfrak{S}_n \hookrightarrow \text{Out}(\Pi_n)\]

— where we write $\mathfrak{S}_n$ for the symmetric group on $n$ letters — obtained by permuting the various factors of $X_n$. If $(r, n) \neq (0, 2)$, then the image of this inclusion is contained in the centralizer $Z_{\text{Out}(\Pi_n)}(\text{Out}^{\text{F}}(\Pi_n))$.

Proof. First, we verify assertion (iii) in the case where $n = 2$, which implies that $r \geq 2$ [cf. the statement of assertion (iii)]. To verify
assertion (iii) in the case where \( n = 2 \), it is immediate that it suffices to verify that
\[
\text{Aut}^{\text{FwC}}(\Pi_2) = \text{Aut}^F(\Pi_2).
\]

Let \( \alpha \in \text{Aut}^F(\Pi_2) \). Let us assign the cusps of \( X \) the labels \( a_1, \cdots, a_r \).

Now, for each \( i \in \{1, \cdots, r\} \), recall that there is a uniquely determined cusp of the geometric generic fiber \( X_{2/1} \) of the projection \( X_2 \to X \) to the factor labeled 1 that corresponds naturally to the cusp of \( X \) labeled \( a_i \); we assign to this uniquely determined cusp the label \( b_i \). Thus, there is precisely one cusp of \( X_{2/1} \) that has not been assigned a label in \( \{b_1, \cdots, b_r\} \); we assign to this uniquely determined cusp the label \( b_{r+1} \). Then since the automorphism of \( \Pi_1 \) induced by \( \alpha \) relative to either \( p_1 \) or \( p_2 \) — where we write \( p_1, p_2 \) for the surjections \( \Pi_2 \to \Pi_1 \) induced by the projections \( X_2 \to X \) to the factors labeled 1, 2, respectively — is \( \text{FC-admissible} \) [cf. [CbTpI], Theorem A, (ii)], it follows from the various definitions involved that, to verify that \( \alpha \in \text{Aut}^{\text{FwC}}(\Pi_2) \), it suffices to verify the following assertion:

**Claim 2.3.A:** For any \( b \in \{b_1, \cdots, b_r\} \), if \( I_b \subseteq \Pi_{2/1} \) is a cuspidal inertia subgroup associated to the cusp labeled \( b \), then \( \alpha(I_b) \) is a cuspidal inertia subgroup.

Now observe that to verify Claim 2.3.A, by replacing \( \alpha \) by the composite of \( \alpha \) with a suitable element of \( \text{Aut}^{\text{FC}}(\Pi_2) \) [cf. [CmbCsp], Lemma 2.4], we may assume without loss of generality that the [necessarily FC-admissible] automorphism of \( \Pi_1 \) induced by \( \alpha \) relative to \( p_1 \), hence also relative to \( p_2 \) [cf. [CbTpI], Theorem A, (i)], induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of \( \Pi_1 \).

To verify Claim 2.3.A, let us fix \( b \in \{b_1, \cdots, b_r\} \), together with a cuspidal inertia subgroup \( I_b \subseteq \Pi_{2/1} \) associated to the cusp labeled \( b \) of \( \Pi_{2/1} \). Also, let us fix

- \( a \in \{a_1, \cdots, a_r\} \) such that if \( b = b_i \) and \( a = a_j \), then \( i \neq j \) [cf. the assumption that \( r \geq 2 \)];
- a cuspidal inertia subgroup \( I_a \subseteq \Pi_1 \) associated to the cusp labeled \( a \) of \( \Pi_1 \).

Now observe that since the [necessarily FC-admissible] automorphism of \( \Pi_1 \) induced by \( \alpha \) relative to \( p_1 \) induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of \( \Pi_1 \), to verify the fact that \( \alpha(I_b) \) is a cuspidal inertia subgroup, we may assume without loss of generality [by replacing \( \alpha \) by a suitable \( \Pi_2 \)-conjugate of \( \alpha \)] that the automorphism of \( \Pi_1 \) induced by \( \alpha \) relative to \( p_1 \) fixes \( I_a \). Let \( \Pi_{F_a} \subseteq \Pi_{2/1} \) be a major vertical subgroup at \( a \) [cf. [CmbCsp], Definition 1.4, (ii)] such that \( I_b \subseteq \Pi_{F_a} \). Then it follows from Lemma 2.2, (ii),...
that $\alpha$ fixes the $\Pi_{2/1}$-conjugacy class of $\Pi_{F_a}$, i.e., that $\Pi_{F_a}^\dagger \overset{\text{def}}{=} \alpha(\Pi_{F_a})$ is a $\Pi_{2/1}$-conjugate of $\Pi_{F_a}$. Thus, one verifies easily that, to verify that $\alpha(I_b)$ is a cuspidal inertia subgroup, it suffices to verify that the isomorphism $\Pi_{F_a} \sim \Pi_{F_a}^\dagger$ induced by $\alpha$ is group-theoretically cuspidal — cf. [CmbGC], Definition 1.4, (iv). [Note that it follows immediately from the various definitions involved that $\Pi_{F_a}$ and $\Pi_{F_a}^\dagger$ may be regarded as pro-$\Sigma$ fundamental groups of semi-graphs of anabelioids of pro-$\Sigma$ PSC-type.] On the other hand, it follows immediately from the various definitions involved that this isomorphism factors as the composite

$$\Pi_{F_a} \sim \Pi_1 \sim \Pi_1 \leftarrow \Pi_{F_a}$$

— where the first and third arrows are the isomorphisms induced by $p_2: \Pi_2 \to \Pi_1$ [cf. [CmbCsp], Definition 1.4, (ii)], and the second arrow is the automorphism induced by $\alpha$ relative to $p_2$ — and that the three arrows appearing in this composite are group-theoretically cuspidal. Thus, we conclude that $\alpha(I_b)$ is a cuspidal inertia subgroup. This completes the proof of Claim 2.3.A, hence also of assertion (iii) in the case where $n = 2$.

Next, we verify assertion (ii) in the case where $(g, r, n) = (0, 3, 2)$. In the following, we shall use the notation “$a_i$” [for $i = 1, 2, 3$] and “$b_j$” [for $j = 1, 2, 3, 4$] introduced in the proof of assertion (iii) in the case where $n = 2$. Now, to verify assertion (ii) in the case where $(g, r, n) = (0, 3, 2)$, it is immediate that it suffices to verify that

$$\text{Aut}^F(\Pi_2) = \text{Aut}^F(\Pi_2).$$

Let $\alpha \in \text{Aut}^F(\Pi_2)$. Then let us observe that to verify that $\alpha \in \text{Aut}^F(\Pi_2)$, by replacing $\alpha$ by the composite of $\alpha$ with a suitable element of $\text{Aut}^F(\Pi_2)$ [cf. [CmbCsp], Lemma 2.4], we may assume without loss of generality that the [necessarily FC-admissible — cf. [CbTpI], Theorem A, (ii)] automorphism of $\Pi_1$ induced by $\alpha$ relative to $p_1$, hence also relative to $p_2$ [cf. [CbTpI], Theorem A, (i)] — where we write $p_1$, $p_2$ for the surjections $\Pi_2 \to \Pi_1$ induced by the projections $X_2 \to X$ to the factors labeled 1, 2, respectively — induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of $\Pi_1$. Now it follows from assertion (iii) in the case where $n = 2$ that $\alpha$ is $\text{FwC-admissible}$; thus, to verify the fact that $\alpha$ is FC-admissible, it suffices to verify the following assertion:

Claim 2.3.B: If $I_{b_4} \subseteq \Pi_{2/1} \overset{\text{def}}{=} \text{Ker}(p_1) \subseteq \Pi_2$ is a cuspidal inertia subgroup associated to the cusp labeled $b_4$, then $\alpha(I_{b_4})$ is a cuspidal inertia subgroup.

On the other hand, as is well-known [cf. e.g., [CbTpI], Lemma 6.10, (ii)], there exists an automorphism of $X_2$ over $X$ relative to the projection to the factor labeled 1 which switches the cusps on the geometric
generic fiber $X_{2/1}$ labeled $b_1$ and $b_4$. In particular, there exists an automorphism $\iota$ of $\Pi_2$ over $\Pi_1$ relative to $p_1$ which switches the respective $\Pi_{2/1}$-conjugacy classes of cuspidal inertia subgroups associated to $b_1$ and $b_4$. Write $\beta = \iota^{-1} \circ \alpha \circ \iota$.

Now let us verify that Claim 2.3.B follows from the following assertion:

Claim 2.3.C: $\beta \in \text{Aut}^F(\Pi_2)$.

Indeed, if Claim 2.3.C holds, then it follows from assertion (iii) in the case where $n = 2$ that, for any cuspidal inertia subgroup $I_{b_1} \subseteq \Pi_{2/1}$ associated to the cusp labeled $b_1$, $\beta(I_{b_1})$ is a cuspidal inertia subgroup. Thus, it follows immediately from our choice of $\iota$ that, for any cuspidal inertia subgroup $I_{b_4} \subseteq \Pi_{2/1}$ associated to the cusp labeled $b_4$, $\alpha(I_{b_4})$ is a cuspidal inertia subgroup. This completes the proof of the assertion that Claim 2.3.C implies Claim 2.3.B.

Finally, we verify Claim 2.3.C. Since $\alpha$ and $\iota$, hence also $\beta$, preserve $\Pi_{2/1} \subseteq \Pi_2$, it follows immediately from [CmbCsp], Proposition 1.2, (i), that, to verify Claim 2.3.C, it suffices to verify that $\beta$ preserves $\Xi_2 \subseteq \Pi_2$ [cf. [CmbCsp], Definition 1.1, (iii)], i.e., the normal closed subgroup of $\Pi_2$ topologically normally generated by a cuspidal inertia subgroup associated to $b_1$. On the other hand, this follows immediately from the fact that $\alpha$ preserves the $\Pi_{2/1}$-conjugacy class of cuspidal inertia subgroups associated to $b_1$ [cf. assertion (iii) in the case where $n = 2$], together with our choice of $\iota$. This completes the proof of Claim 2.3.C, hence also of assertion (ii) in the case where $(g, r, n) = (0, 3, 2)$.

Next, we verify assertion (ii) in the case where $(g, r, n) \neq (0, 3, 2)$. Thus, $n \geq 3$. Write $\Pi_3^f$ (respectively, $\Pi_2^f$, $\Pi_1^f$) for the kernel of the surjection $\Pi_n \to \Pi_{n-3}$ (respectively, $\Pi_{n-1} \to \Pi_{n-3}$; $\Pi_{n-2} \to \Pi_{n-3}$) induced by the projection obtained by forgetting the factor(s) labeled $n, n-1, n-2$ (respectively, $n-1, n-2; n-2$). Here, if $n = 3$, then we set $\Pi_{n-3} = \Pi_0 \overset{\text{def}}{=} \{1\}$. Then recall [cf., e.g., the proof of [CmbCsp], Theorem 4.1, (i)] that we have natural isomorphisms

$$\Pi_n \simeq \Pi_3^f \times \Pi_{n-3}; \quad \Pi_{n-1} \simeq \Pi_2^f \times \Pi_{n-3}; \quad \Pi_{n-2} \simeq \Pi_1^f \times \Pi_{n-3}$$

[cf. the discussion entitled “Topological groups” in [CbTpI], §0]. Also, we recall [cf. [MzTa], Proposition 2.4, (i)] that one may interpret the surjections $\Pi_3^f \to \Pi_2^f \to \Pi_1^f$ induced by the surjections $\Pi_n \to \Pi_{n-1} \to \Pi_{n-2}$ as the surjections “$\Pi_3 \to \Pi_2 \to \Pi_1$” that arise from the projections $X_3 \to X_2 \to X$ in the case of an “$X$” of type $(g, r + n - 3)$. Moreover, one verifies easily that this interpretation is compatible with the definition of the various “$\text{Out}(-)$’s” involved. Thus, since $\nu_{\text{FC}} = 4$ if $r = 0$, the above natural isomorphisms, together with [CbTpI], Theorem A, (ii), allow one to reduce the equality in question to the case where $n = 3$ and $r \neq 0$. 
Now one verifies easily that, to verify the equality in question in the case where \( n = 3 \) and \( r \neq 0 \), it is immediate that it suffices to verify that

\[
\text{Aut}^{FC}(\Pi_3) = \text{Aut}^F(\Pi_3).
\]

Let \( \alpha \in \text{Aut}^F(\Pi_3) \). Then let us observe that to verify \( \alpha \in \text{Aut}^{FC}(\Pi_3) \), by replacing \( \alpha \) by the composite of \( \alpha \) with a suitable element of \( \text{Aut}^{FC}(\Pi_3) \) [cf. \( \text{CmbCsp} \), Lemma 2.4], we may assume without loss of generality that the [necessarily FC-admissible — cf. \( \text{CbTpI} \), Theorem A, (i)] automorphism of \( \Pi_1 \) induced by \( \alpha \) relative to \( q_1 \), hence also relative to either \( q_2 \) or \( q_3 \) [cf. \( \text{CbTpI} \), Theorem A, (i)] — where we write \( q_1, q_2, q_3 \) for the surjections \( \Pi_3 \to \Pi_1 \) induced by the projections \( X_3 \to X \) to the factors labeled 1, 2, 3, respectively — induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of \( \Pi_1 \); in particular, one verifies easily that the [necessarily FC-admissible — cf. \( \text{CbTpI} \), Theorem A, (ii)] automorphism of \( \Pi_{2/1} \) — where we write \( p_1 : \Pi_2 \to \Pi_1 \) for the surjection induced by the projection \( X_2 \to X \) to the factor labeled 1 and \( \Pi_{2/1}^{\text{def}} = \text{Ker}(p_1) \subseteq \Pi_2 \) — induced by \( \alpha \) induces the identity automorphism on the set of conjugacy classes of cuspidal inertia subgroups of \( \Pi_{2/1} \). Write \( X_{2/1} \) (respectively, \( X_{3/2} \); \( X_{3/1} \)) for the geometric generic fiber of the projection \( X_2 \to X \) (respectively, \( X_3 \to X_2 \); \( X_3 \to X \)) to the factor(s) labeled 1 (respectively, 1, 2, 1). Let us assign the cusps of \( X \) the labels \( a_1, \cdots, a_r \). For each \( i \in \{1, \cdots, r\} \), we assign to the cusp of \( X_{2/1} \) that corresponds naturally to the cusp of \( X \) labeled \( a_i \) the label \( b_i \). Thus, there is precisely one cusp of \( X_{2/1} \) that has not been assigned a label \( \in \{b_1, \cdots, b_r\} \); we assign to this uniquely determined cusp the label \( b_{r+1} \). For each \( i \in \{1, \cdots, r+1\} \), we assign to the cusp of \( X_{3/2} \) that corresponds naturally to the cusp of \( X_{2/1} \) labeled \( b_i \) the label \( c_i \). Thus, there is precisely one cusp of \( X_{3/2} \) that has not been assigned a label \( \in \{c_1, \cdots, c_{r+1}\} \); we assign to this uniquely determined cusp the label \( c_{r+2} \). Now it follows from assertion (iii) in the case where \( n = 2 \), applied to the restriction of \( \alpha \) to \( \Pi_{3/1}^{\text{def}} = \text{Ker}(q_1) \), together with \( \text{CbTpI} \), Theorem A, (ii), that \( \alpha \) is \( FwC\)-admissible. Write \( q_{12} : \Pi_3 \to \Pi_2 \) for the surjection induced by the projection \( X_3 \to X_2 \) to the factors labeled 1, 2; \( \Pi_{3/2}^{\text{def}} = \text{Ker}(q_{12}) \subseteq \Pi_3 \). Thus, to verify the fact that \( \alpha \) is \( FC\)-admissible, it suffices to verify the following assertion:

Claim 2.3.D: If \( I_{c_{r+2}} \subseteq \Pi_{3/2} \) is a cuspidal inertia subgroup associated to the cusp labeled \( c_{r+2} \), then \( \alpha(I_{c_{r+2}}) \) is a cuspidal inertia subgroup.

To verify Claim 2.3.D, let us fix a cusp labeled \( b \in \{b_1, \cdots, b_r\} \) [where we recall that \( r \neq 0 \)], a cuspidal inertia subgroup \( I_b \subseteq \Pi_{2/1} \) associated to the cusp labeled \( b \) of \( X_{2/1} \), and a cuspidal inertia subgroup \( I_{c_{r+2}} \subseteq \Pi_{3/2} \) associated to the cusp labeled \( c_{r+2} \) of \( \Pi_{3/2} \). Now observe
that since the [necessarily FC-admissible] automorphism of $\Pi_{2/1}$
induced by $\alpha$ induces the identity automorphism on the set of conjugacy
classes of cuspidal inertia subgroups of $\Pi_{2/1}$, to verify the assertion
that $\alpha(I_{e+2})$ is a cuspidal inertia subgroup, we may assume without
loss of generality [by replacing $\alpha$ by a suitable $\Pi_3$-conjugate of $\alpha$] that
the automorphism of $\Pi_{2/1}$ induced by $\alpha$ fixes $I_b$. Let $\Pi_{E_b} \subseteq \Pi_{3/2}$
be a minor vertical subgroup, relative to the two-dimensional configuration
space $X_{3/1}$ associated to the hyperbolic curve $X_{2/1}$, at the cusp labeled
$b$ [cf. [CmbCsp], Definition 1.4, (ii)] such that $I_{e+2} \subseteq \Pi_{E_b}$. Then it fol-

— where the second arrow is the surjection determined by the surjection
$q_{13}: \Pi_3 \rightarrow \Pi_2$ induced by the projection $X_3 \rightarrow X_2$ to the factors
labeled $1, 3$ — are injective, and that the $\Pi_{2/1}$-conjugacy class of the
image in $\Pi_{2/1}$ of either of these composite injections coincides with the
$\Pi_{2/1}$-conjugacy class of a minor vertical subgroup at the cusp labeled
$a_i$ [where we write $b = b_i$ — cf. [CmbCsp], Definition 1.4, (ii)]. In
particular, since the automorphism of $\Pi_2$ induced by $\alpha$ relative to $q_{13}$
is FC-admissible [cf. [CbTpI], Theorem A, (ii)], it follows immediately
that $\Pi_{E_b} \rightarrow \Pi_{E_b}^\dagger$ induced by $\alpha$ is group-theoretically cuspidal — cf. [CmbGC],
Definition 1.4, (iv).

Now assertion (iii) in the case where $n \neq 2$ follows immediately
from assertion (ii), together with the natural inclusions $\text{Out}^{\text{FC}}(\Pi_n) \subseteq
\text{Out}^{\text{FC}}(\Pi_n) \subseteq \text{Out}^{\text{FwC}}(\Pi_n) \subseteq \text{Out}^{\text{F}}(\Pi_n)$ [cf. Remark 2.1.1]. This completes the proof
of assertion (iii).

Next, we verify assertion (i). The bijectivity portion of assertion
(i) follows from assertion (ii), together with the bijectivity portion of
[NodNon], Theorem B. Thus, it suffices to verify the injectivity portion
of assertion (i). First, we observe that injectivity in the case where
$(g, r) = (0, 3)$ follows from assertion (ii), together with the injectivi-
ty portion of [NodNon], Theorem B. Write $\Pi_2^\dagger$ (respectively, $\Pi_1^\dagger$) for
the kernel of the surjection $\Pi_{n+1} \rightarrow \Pi_{n-1}$ (respectively, $\Pi_n \rightarrow \Pi_{n-1}$)
induced by the projection obtained by forgetting the factor(s) labeled $n + 1$, $n$ (respectively, $n$). Here, if $n = 1$, then we set $\Pi_{n-1} = \Pi_0 \overset{\text{def}}{=} \{1\}$. Then recall [cf. e.g., the proof of [CmbCsp], Theorem 4.1, (i)] that we have natural isomorphisms

$$\Pi_{n+1} \simeq \Pi_2^{\text{out}} \times \Pi_{n-1} ; \quad \Pi_n \simeq \Pi_1^{\text{out}} \times \Pi_{n-1}$$

[cf. the discussion entitled “Topological groups” in [CbTpI], §0]. Also, we recall [cf. [MzTa], Proposition 2.4, (i)] that one may interpret the surjection $\Pi_2 \rightarrow \Pi_1$ induced by the surjection $\Pi_{n+1} \rightarrow \Pi_n$ in question as the surjection “$\Pi_2 \rightarrow \Pi_1$” that arises from the projection $X_2 \rightarrow X$ in the case of an “$X$” of type $(g, r + n - 1)$. Moreover, one verifies easily that this interpretation is compatible with the definition of the various “Out($-$)’s” involved. Thus, since $n_{\text{inj}} = 2$ if $r = 0$, the above natural isomorphisms allow one to reduce the injectivity in question to the case where $n = 1$ and $r \neq 0$. On the other hand, this injectivity follows immediately from a similar argument to the argument used in the proof of [CmbCsp], Corollary 2.3, (ii), by replacing [CmbCsp], Proposition 1.2, (iii) (respectively, the non-resp’d portion of [CmbCsp], Proposition 1.3, (iv); [CmbCsp], Corollary 1.12, (i)), in the proof of [CmbCsp], Corollary 2.3, (ii), by Lemma 2.2, (i) (respectively, Lemma 2.2, (ii); the injectivity in question in the case where $(g, r) = (0, 3)$, which was verified above). This completes the proof of the injectivity portion of assertion (i), hence also of assertion (i).

Finally, assertion (iv) follows immediately from assertion (i), together with a similar argument to the argument applied in the proof of [CmbCsp], Theorem 4.1, (iv). This completes the proof of Theorem 2.3. $\Box$

**Corollary 2.4 (PFC-admissibility of automorphisms).** In the notation of Theorem 2.3, write

$$\text{Out}^{\text{PF}}(\Pi_n) \subseteq \text{Out}(\Pi_n)$$

for the subgroup of **PF-admissible** automorphisms [i.e., roughly speaking, automorphisms that preserve the fiber subgroups up to a possible permutation of the factors — cf. [CbTpI], Definition 1.4, (i)] and

$$\text{Out}^{\text{PFC}}(\Pi_n) \subseteq \text{Out}^{\text{PF}}(\Pi_n)$$

for the subgroup of **PFC-admissible** automorphisms [i.e., roughly speaking, automorphisms that preserve the fiber subgroups and the cuspidal inertia subgroups up to a possible permutation of the factors — cf. [CbTpI], Definition 1.4, (iii)]. Let us regard the symmetric group on $n$ letters $\mathfrak{S}_n$ as a subgroup of Out$(\Pi_n)$ via the natural inclusion of Theorem 2.3, (iv). Finally, suppose that $(g, r) \notin \{(0, 3); (1, 1)\}$. Then the following hold:
We have an equality
\[ \text{Out}(\Pi_n) = \text{Out}^{PF}(\Pi_n). \]

If, moreover, \((r, n) \neq (0, 2)\), then we have equalities
\[ \text{Out}(\Pi_n) = \text{Out}^{PF}(\Pi_n) = \text{Out}^F(\Pi_n) \times \mathfrak{S}_n \]
[cf. the notational conventions introduced in Theorem 2.3].

(ii) If either
\[ r > 0, \quad n \geq 3 \]
or
\[ n \geq 4, \]
then we have equalities
\[ \text{Out}(\Pi_n) = \text{Out}^{PFC}(\Pi_n) = \text{Out}^{FC}(\Pi_n) \times \mathfrak{S}_n \]
[cf. the notational conventions introduced in Theorem 2.3].

Proof. First, we verify assertion (i). The equality in the first display of assertion (i) follows from [MzTa], Corollary 6.3, together with the assumption that \((g, r) \notin \{(0, 3); (1, 1)\}\). The second equality in the second display of assertion (i) follows from Theorem 2.3, (iv). This completes the proof of assertion (i). Next, we verify assertion (ii). The first equality of assertion (ii) follows immediately from Theorem 2.3, (ii), together with the first equality of assertion (i). The second equality of assertion (ii) follows from [NodNon], Theorem B. This completes the proof of assertion (ii).

\[ \square \]

Corollary 2.5 (Anabelian properties of hyperbolic curves and associated configuration spaces I). Let \(\Sigma\) be a set of prime numbers which is either equal to the set of all prime numbers or of cardinality one; \(m \leq n\) positive integers; \((g, r)\) a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(k\) a field of characteristic \(\notin \Sigma\); \(\overline{k}\) a separable closure of \(k\); \(X\) a hyperbolic curve of type \((g, r)\) over \(k\). Write \(G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)\). For each positive integer \(i\), write \(X_i\) for the \(i\)-th configuration space of \(X\); \((X_i)_\overline{k} = X_i \times_k \overline{k}\); \(\Delta_{X_i}\) for the maximal pro-\(\Sigma\) quotient of the étale fundamental group of \((X_i)_\overline{k}\);
\[ \rho^\Sigma_{X_i}: G_k \rightarrow \text{Out}(\Delta_{X_i}) \]
for the pro-\(\Sigma\) outer Galois representation associated to \(X_i\); \(\mathfrak{S}_i\) for the symmetric group on \(i\) letters;
\[ \Phi_i: \mathfrak{S}_i \rightarrow \text{Out}(\Delta_{X_i}) \]
for the outer representation arising from the permutations of the factors of \(X_i\). Suppose that the following conditions are satisfied:
(1) \((g, r) \notin \{(0, 3); (1, 1)\} \).

(2) If \((r, n, m) \in \{(0, 2, 1); (0, 2, 2); (0, 3, 1)\}\), then there exists an \(l \in \Sigma\) such that \(k\) is \(l\)-cyclo-
tomically full, i.e., the \(l\)-adic cyclotomic character of \(G_k\) has open image.

Then the following hold:

(i) Let \(\alpha \in \text{Out}(\Delta_{X_n})\). Then there exists a unique element \(\sigma_\alpha \in \mathfrak{S}_n\) such that \(\alpha \circ \Phi_n(\sigma_\alpha) \in \text{Out}^F(\Delta_{X_n})\) [cf. the notational conventions introduced in Theorem 2.3]. Write

\[ \alpha_m \in \text{Out}^F(\Delta_{X_m}) \]

for the automorphism of \(\Delta_{X_m}\) induced by \(\alpha \circ \Phi_n(\sigma_\alpha)\), relative to the quotient \(\Delta_{X_n} \rightarrow \Delta_{X_m}\) by a fiber subgroup of co-length \(m\) of \(\Delta_{X_*}\). [Note that it follows from \(\text{[CbTpi]}\), Theorem A, (i), that \(\alpha_m\) does not depend on the choice of fiber subgroup of co-length \(m\) of \(\Delta_{X_*}\).

(ii) If \((r, n, m) \in \{(0, 2, 1); (0, 2, 2); (0, 3, 1)\}\), then

\[ C_{\text{Out}(\Delta_{X_n})}(\text{Im}(\rho_{\Delta_{X_n}}^\Sigma)) \subseteq \text{Out}^{\text{PFC}}(\Delta_{X_n}) \]

[cf. the notational conventions introduced in Corollary 2.4].

(iii) The map

\[ \text{Out}(\Delta_{X_n}) \xrightarrow{\alpha} \text{Out}(\Delta_{X_m}) \]

[cf. (i)] determines an exact sequence of homomorphisms of profinite groups

\[ 1 \rightarrow \mathfrak{S}_n \xrightarrow{\Phi_n} \text{Out}^{\text{PFC}}(\Delta_{X_n}) \rightarrow \text{Out}(\Delta_{X_m}) \]

— where the second arrow is a split injection whose image commutes with \(\text{Out}^{\text{PFC}}(\Delta_{X_n})\) and has trivial intersection with \(\text{Im}(\rho_{\Delta_{X_n}}^\Sigma)\). If \((r, n) \neq (0, 2)\), then the map \(\alpha \mapsto \alpha_m\) determines a sequence of homomorphisms of profinite groups

\[ 1 \rightarrow \mathfrak{S}_n \xrightarrow{\Phi_n} \text{Out}(\Delta_{X_n}) \rightarrow \text{Out}(\Delta_{X_m}) \]

— where the second arrow is a split injection whose image commutes with \(\text{Out}^F(\Delta_{X_n})\) and has trivial intersection with \(\text{Im}(\rho_{\Delta_{X_n}}^\Sigma)\) — which is exact if, moreover, \((r, n, m) \neq (0, 3, 1)\).

(iv) Let \(\alpha \in \text{Out}(\Delta_{X_n})\). If \((r, n, m) \in \{(0, 2, 1); (0, 3, 1)\}\), then we suppose further that \(\alpha \in \text{Out}^{\text{PFC}}(\Delta_{X_n})\), which is the case if, for instance, \(\alpha \in C_{\text{Out}(\Delta_{X_n})}(\text{Im}(\rho_{\Delta_{X_n}}^\Sigma))\) [cf. (ii)]. Then it holds that

\[ \alpha \in Z_{\text{Out}(\Delta_{X_n})}(\text{Im}(\rho_{\Delta_{X_n}}^\Sigma)) \]

(respectively, \(N_{\text{Out}(\Delta_{X_n})}(\text{Im}(\rho_{\Delta_{X_n}}^\Sigma)); C_{\text{Out}(\Delta_{X_n})}(\text{Im}(\rho_{\Delta_{X_n}}^\Sigma)))\).
if and only if
\[ \alpha_m \in Z_{\text{Out}}(\Delta_X^m)(\text{Im}(\rho_X^m)) \]
(respectively, \( N_{\text{Out}}(\Delta_X^m)(\text{Im}(\rho_X^m)); C_{\text{Out}}(\Delta_X^m)(\text{Im}(\rho_X^m)) \)).

(v) For each positive integer \( i \), write \( \text{Aut}_k(X_i) \) for the group of automorphisms of \( X_i \) over \( k \). Then if the natural homomorphism
\[ \text{Aut}_k(X_m) \rightarrow Z_{\text{Out}}(\Delta_X^m)(\text{Im}(\rho_X^m)) \]
is bijective, then the natural homomorphism
\[ \text{Aut}_k(X_n) \rightarrow Z_{\text{Out}}(\Delta_X^m)(\text{Im}(\rho_X^m)) \]
is bijective.

(vi) For each positive integer \( i \), write \( \text{Aut}((X_i)_k/k) \) for the group of automorphisms of \( (X_i)_k \) that are compatible with some automorphism of \( k; \) \( \text{Aut}^G(G_k) \) for the group of automorphisms of \( G_k \) that preserve \( \text{Ker}(\rho_X^G) \subseteq G_k \) [where we note that, by [NodNon], Corollary 6.2, (i), for any positive integer \( i \), it holds that \( \text{Ker}(\rho_X^G) = \text{Ker}(\rho_X^G) \)]. Then if the natural homomorphism
\[ \text{Aut}((X_m)_k/k) \rightarrow \text{Aut}^G(G_k) \times_{\text{Aut}(\text{Im}(\rho_X^G))} N_{\text{Out}}(\Delta_X^m)(\text{Im}(\rho_X^m)) \]
is bijective, then the natural homomorphism
\[ \text{Aut}((X_n)_k/k) \rightarrow \text{Aut}^G(G_k) \times_{\text{Aut}(\text{Im}(\rho_X^G))} N_{\text{Out}}(\Delta_X^m)(\text{Im}(\rho_X^m)) \]
is bijective.

Proof. First, we verify assertion (i). The existence of such a \( \sigma_n \) follows from the fact that \( \text{Out}(\Delta_X^n) = \text{Out}^\text{PF}(\Delta_X^n) \) [cf. Corollary 2.4, (i), together with assumption (1)]. The uniqueness of such a \( \sigma_n \) follows immediately from the easily verified faithfulness of the action of \( \mathfrak{S}_n \), via \( \Phi_n \), on the set of fiber subgroups of \( \Delta_X^n \). This completes the proof of assertion (i). Next, we verify assertion (ii). Since \( \text{Out}(\Delta_X^n) = \text{Out}^\text{PF}(\Delta_X^n) \) [cf. Corollary 2.4, (i), together with assumption (1)], assertion (ii) follows immediately from [CmbGC], Corollary 2.7, (i), together with condition (2). This completes the proof of assertion (ii).

Next, we verify assertion (iii). First, let us observe that it follows immediately from the various definitions involved that \( \text{Im}(\Phi_n) \subseteq \text{Out}^\text{PF}(\Delta_X^n) \). Now since \( \text{Out}(\Delta_X^n) = \text{Out}^\text{PF}(\Delta_X^n) \) [cf. Corollary 2.4, (i), together with assumption (1)], and \( \text{Out}^\text{PF}(\Delta_X^n) \) is normalized by \( \text{Out}^\text{PF}(\Delta_X^n) \), one verifies easily [i.e., by considering the action of elements of \( \text{Out}^\text{PF}(\Delta_X^n) \) on the set of fiber subgroups of \( \Delta_X^n \)] that the second arrow in either of the two displayed sequences is a split injection. Moreover, since [as is easily verified] the outer action of \( G_k \) via \( \rho_X^G \), on \( \Delta_X^n \), fixes every fiber subgroup of \( \Delta_X^n \), it follows immediately from the faithfulness of the action of \( \mathfrak{S}_n \), via \( \Phi_n \), on the set of fiber subgroups of \( \Delta_X^n \), that the image of the second arrow in either of the
two displayed sequences has \textit{trivial intersection} with \(\text{Im}(\rho_X^n)\). Now it follows from [NodNon], Theorem B, that the image of the second arrow of the first displayed sequence \textit{commutes} with \(\text{Out}^\text{FC}(\Delta X_n)\); in particular, one verifies easily from the various definitions involved [cf. also Corollary 2.4, (i), together with assumption (1)] that the third arrow of the first displayed sequence is a \textit{homomorphism}. If \((r, n) \neq (0, 2)\), then it follows from Corollary 2.4, (i), together with assumption (1), that the image of the second arrow of the second displayed sequence \textit{commutes} with \(\text{Out}^{\text{F}}(\Delta X_n)\); in particular, one verifies easily from the various definitions involved [cf. also Corollary 2.4, (i), together with assumption (1)] that the third arrow of the second displayed sequence is a \textit{homomorphism}. Now if \((r, m) \neq (0, 1)\), then it follows immediately from the injectivity portion of Theorem 2.3, (i), together with the equality \(\text{Out}(\Delta X_n) = \text{Out}^{\text{PF}}(\Delta X_n)\) [cf. Corollary 2.4, (i), together with assumption (1)], that the kernel of the third arrow in the first displayed sequence is \(\text{Im}(\Phi_n)\). On the other hand, if \((r, m) = (0, 1)\) and \(n \not\in \{2, 3\}\), then it follows immediately from the injectivity portion of [NodNon], Theorem B, together with Corollary 2.4, (ii), together with assumption (1), that the kernel of the third arrow in either of the two displayed sequences is \(\text{Im}(\Phi_n)\). This completes the proof of assertion (iii).

Next, we verify assertion (iv). Now since the permutations of the factors of \(X_n\) give rise to \textit{automorphisms} of \(X_n\) over \(k\), it follows immediately that \(\text{Im}(\Phi_n) \subseteq \text{Z}_{\text{Out}(\Delta X_n)}(\text{Im}(\rho_X^n))\). In particular, to verify assertion (iv), we may assume without loss of generality — by replacing \(\alpha\) by \(\alpha_n\) [cf. assertion (i)] — that \(\alpha \in \text{Out}^{\text{F}}(\Delta X_n)\), and that \(m < n\). Then \textit{necessity} follows immediately. On the other hand, \textit{sufficiency} follows immediately from the exact sequences of assertion (iii). This completes the proof of assertion (iv). Assertion (v) (respectively, (vi)) follows immediately from assertions (i), (ii), (iii), (iv), together with Lemma 2.7, (iii), below (respectively, Lemma 2.7, (iv), below). This completes the proof of Corollary 2.5. 

\begin{corollary} \textbf{(Anabelian properties of hyperbolic curves and associated configuration spaces II).} \label{cor2.6} \textit{Let} \(\Sigma\) \textit{be a set of prime numbers which is either equal to the set of all prime numbers or of cardinality one;} \(m \leq n\) \textit{positive integers;} \((g_X, r_X), (g_Y, r_Y)\) \textit{pairs of nonnegative integers such that} \(2g_X - 2 + r_X, 2g_Y - 2 + r_Y > 0\); \(k_X, k_Y\) \textit{fields;} \(\overline{k}_X, \overline{k}_Y\) \textit{separable closures of} \(k_X, k_Y\), \textit{respectively;} \(X, Y\) \textit{hyperbolic curves of type} \((g_X, r_X), (g_Y, r_Y)\) \textit{over} \(k_X, k_Y\), \textit{respectively.} \textit{Write} \(G_{k_X} \overset{\text{def}}{=} \text{Gal}(\overline{k}_X/k_X)\); \(G_{k_Y} \overset{\text{def}}{=} \text{Gal}(\overline{k}_Y/k_Y)\). \textit{For each positive integer} \(i\), \textit{write} \(X_i, Y_i\) \textit{for the} \(i\)-\textit{th configuration spaces of} \(X, Y\),
respectively; $(X_i)_{\bar{\mathbb{F}}_X} \overset{def}{=} X_i \times_{k_X} \bar{k}_X$; $(Y_i)_{\bar{\mathbb{F}}_Y} \overset{def}{=} Y_i \times_{k_Y} \bar{k}_Y$; $\pi_1^{\Sigma}((X_i)_{\bar{\mathbb{F}}_X})$, $\pi_1^{\Sigma}((Y_i)_{\bar{\mathbb{F}}_Y})$ for the maximal pro-$\Sigma$ quotients of the étale fundamental groups $\pi_1((X_i)_{\bar{\mathbb{F}}_X})$, $\pi_1((Y_i)_{\bar{\mathbb{F}}_Y})$ of $(X_i)_{\bar{\mathbb{F}}_X}$, $(Y_i)_{\bar{\mathbb{F}}_Y}$, respectively; $\pi_1^{\Sigma}(X_i)$, $\pi_1^{\Sigma}(Y_i)$ for the geometrically pro-$\Sigma$ étale fundamental groups of $X_i$, $Y_i$, respectively, i.e., the quotients of the étale fundamental groups $\pi_1(X_i)$, $\pi_1(Y_i)$ of $X_i$, $Y_i$ by the respective kernels of the natural surjections $\pi_1((X_i)_{\bar{\mathbb{F}}_X}) \twoheadrightarrow \pi_1^{\Sigma}((X_i)_{\bar{\mathbb{F}}_X})$, $\pi_1((Y_i)_{\bar{\mathbb{F}}_Y}) \twoheadrightarrow \pi_1^{\Sigma}((Y_i)_{\bar{\mathbb{F}}_Y})$. Suppose that the following conditions are satisfied:

1. $\{(g_X, r_X); (g_Y, r_Y)\} \cap \{(0, 3); (1, 1)\} = \emptyset$.

2. If $(r_X, n, m)$ (respectively, $(r_Y, n, m)$) is contained in the set $\{(0, 2, 1); (0, 2, 2); (0, 3, 1)\}$, then there exists an $l \in \Sigma$ such that $k_Y \overset{l}{\twoheadrightarrow} k_Y$ (respectively, $k_Y$) is $l$-cyclicotomically full, i.e., the $l$-adic cyclotomic character of $G_{k_X}$ (respectively, $G_{k_Y}$) has open image.

Then the following hold:

(i) Let $\theta: \bar{k}_X \overset{\sim}{\rightarrow} \bar{k}_Y$ be an isomorphism of fields that determines an isomorphism $k_X \overset{\sim}{\rightarrow} k_Y$. For each positive integer $i$, write $\text{Isom}_\theta(X_i, Y_i)$ for the set of isomorphisms of $X_i$ with $Y_i$ that are compatible with the isomorphism $k_X \overset{\sim}{\rightarrow} k_Y$ determined by $\theta$; $\text{Isom}_\theta(\pi_1^{\Sigma}(X_i), \pi_1^{\Sigma}(Y_i))$ for the set of isomorphisms of $\pi_1^{\Sigma}(X_i)$ with $\pi_1^{\Sigma}(Y_i)$ that are compatible with the isomorphism $G_{k_X} \overset{\sim}{\rightarrow} G_{k_Y}$ determined by $\theta$. Then if the natural map $\text{Isom}_\theta(X_m, Y_m) \rightarrow \text{Isom}_\theta(\pi_1^{\Sigma}(X_m), \pi_1^{\Sigma}(Y_m))/\text{Inn}(\pi_1^{\Sigma}((Y_m)_{\bar{\mathbb{F}}_Y}))$ is bijective, then the natural map $\text{Isom}_\theta(X_n, Y_n) \rightarrow \text{Isom}_\theta(\pi_1^{\Sigma}(X_n), \pi_1^{\Sigma}(Y_n))/\text{Inn}(\pi_1^{\Sigma}((Y_n)_{\bar{\mathbb{F}}_Y}))$ is bijective.

(ii) For each positive integer $i$, write $\text{Isom}((X_i)_{\bar{\mathbb{F}}_X}/k_X, (Y_i)_{\bar{\mathbb{F}}_Y}/k_Y)$ for the set of isomorphisms of $(X_i)_{\bar{\mathbb{F}}_X}$ with $(Y_i)_{\bar{\mathbb{F}}_Y}$ that are compatible with some field isomorphism of $k_X$ with $k_Y$; $\text{Isom}(\pi_1^{\Sigma}(X_i)/G_{k_X}, \pi_1^{\Sigma}(Y_i)/G_{k_Y})$ for the set of isomorphisms of $\pi_1^{\Sigma}(X_i)$ with $\pi_1^{\Sigma}(Y_i)$ that are compatible with some isomorphism of $G_{k_X}$ with $G_{k_Y}$. Then if the natural map $\text{Isom}((X_m)_{\bar{\mathbb{F}}_X}/k_X, (Y_m)_{\bar{\mathbb{F}}_Y}/k_Y) \rightarrow \text{Isom}(\pi_1^{\Sigma}(X_m)/G_{k_X}, \pi_1^{\Sigma}(Y_m)/G_{k_Y})/\text{Inn}(\pi_1^{\Sigma}((Y_m)_{\bar{\mathbb{F}}_Y}))$ is bijective, then the natural map $\text{Isom}((X_n)_{\bar{\mathbb{F}}_X}/k_X, (Y_n)_{\bar{\mathbb{F}}_Y}/k_Y)$ is bijective.
\[ \rightarrow \text{Isom}(\pi_1^{(\Sigma)}(X_n)/G_{k_X}, \pi_1^{(\Sigma)}(Y_n)/G_{k_Y})/\text{Inn}(\pi_1^{(\Sigma)}((Y_n)_{\overline{k}_Y})) \]

is bijective.

Proof. Consider assertion (i) (respectively, (ii)). If the set 
\[ \text{Isom}_\theta(\pi_1^{(\Sigma)}(X_n), \pi_1^{(\Sigma)}(Y_n))/\text{Inn}(\pi_1^{(\Sigma)}((Y_n)_{\overline{k}_Y})) \]
(respectively, 
\[ \text{Isom}(\pi_1^{(\Sigma)}(X_n)/G_{k_X}, \pi_1^{(\Sigma)}(Y_n)/G_{k_Y})/\text{Inn}(\pi_1^{(\Sigma)}((Y_n)_{\overline{k}_Y})) \]
is empty, then assertion (i) (respectively, (ii)) is immediate. Thus, we may suppose without loss of generality that this set is nonempty. Then one verifies easily from [MzTa], Corollary 6.3, together with condition (1), that the set 
\[ \text{Isom}_\theta(\pi_1^{(\Sigma)}(X_m), \pi_1^{(\Sigma)}(Y_m))/\text{Inn}(\pi_1^{(\Sigma)}((Y_m)_{\overline{k}_Y})) \]
(respectively, 
\[ \text{Isom}(\pi_1^{(\Sigma)}(X_m)/G_{k_X}, \pi_1^{(\Sigma)}(Y_m)/G_{k_Y})/\text{Inn}(\pi_1^{(\Sigma)}((Y_m)_{\overline{k}_Y})) \]
is nonempty. Thus, it follows immediately from the bijectivity assumed in assertion (i) (respectively, (ii)) that there exists an isomorphism \( X_m \sim Y_m \) that is compatible with the isomorphism \( k_X \sim k_Y \) determined by \( \theta \) (respectively, an isomorphism \( (X_m)_{\overline{k}_X} \sim (Y_m)_{\overline{k}_Y} \) that is compatible with some isomorphism \( k_X \sim k_Y \)). In particular, it follows immediately from Lemma 2.7, (iii), below (respectively, Lemma 2.7, (iv), below) that there exists an isomorphism \( X \sim Y \) that is compatible with the isomorphism \( k_X \sim k_Y \) determined by \( \theta \) (respectively, an isomorphism \( X \times_{k_X} \overline{k}_X \sim Y \times_{k_Y} \overline{k}_Y \) that is compatible with some isomorphism \( k_X \sim k_Y \)). Thus, by pulling back the various objects involved via this isomorphism, to verify assertion (i) (respectively, (ii)), we may assume without loss of generality that \( (X, k_X, \overline{k}_X, \theta) = (Y, k_Y, \overline{k}_Y, \text{id}_{\overline{k}_X}) \) (respectively, \( (X, k_X, \overline{k}_X) = (Y, k_Y, \overline{k}_Y) \)). Then assertion (i) (respectively, (ii)) follows from Corollary 2.5, (v) (respectively, Corollary 2.5, (vi)). This completes the proof of Corollary 2.6.

Lemma 2.7 (Isomorphisms between configuration spaces of hyperbolic curves). Let \( n \) be a positive integer; \((g_X, r_X), (g_Y, r_Y)\) pairs of nonnegative integers such that \( 2g_X - 2 + r_X, 2g_Y - 2 + r_Y > 0 \); \( k_X, k_Y \) fields; \( \overline{k}_X, \overline{k}_Y \) separable closures of \( k_X, k_Y \), respectively; \( X, Y \) hyperbolic curves of type \((g_X, r_X), (g_Y, r_Y)\) over \( k_X, k_Y \), respectively. Write \( X_n, Y_n \) for the \( n \)-th configuration spaces of \( X, Y \), respectively; 
\[
X_{\overline{k}_X} \overset{\text{def}}{=} X \times_{k_X} \overline{k}_X; \ Y_{\overline{k}_Y} \overset{\text{def}}{=} Y \times_{k_Y} \overline{k}_Y; \ (X_n)_{\overline{k}_X} \overset{\text{def}}{=} X_n \times_{k_X} \overline{k}_X; \ (Y_n)_{\overline{k}_Y} \overset{\text{def}}{=}
\]
$Y_n \times_{k_Y} \Phi_Y; \mathcal{G}_n$ for the symmetric group on $n$ letters; $\text{Aut}_{k_X}(X_n)$ for the group of automorphisms of $X_n$ over $k_X$:

$$\Psi_n: \mathcal{G}_n \rightarrow \text{Aut}_{k_X}(X_n)$$

for the action of $\mathcal{G}_n$ on $X_n$ over $k_X$ obtained by permuting the factors of $X_n$. Suppose that $(g_X, r_X), (g_Y, r_Y) \not\in \{(0, 3); (1, 1)\}$. Then the following hold:

(i) Let $\alpha: X_n \rightarrow Y_n$ be an isomorphism. Then there exists a unique isomorphism $\alpha_0: k_Y \rightarrow k_X$ that is compatible with $\alpha$ relative to the structure morphisms of $X_n, Y_n$.

(ii) Let $\alpha: X_n \rightarrow Y_n$ be an isomorphism. Then there exist a unique permutation $\sigma \in \Psi_n(\mathcal{G}_n) \subseteq \text{Aut}_{k_X}(X_n)$ and a unique isomorphism $\alpha_1: X \rightarrow Y$ that is compatible with $\alpha \circ \sigma$ relative to the projections $X_n \rightarrow X, Y_n \rightarrow Y$ to each of the $n$ factors.

(iii) Write $\text{Isom}(X_n, Y_n)$ for the set of isomorphisms of $X_n$ with $Y_n$; $\text{Isom}(X, Y) \overset{\text{def}}{=} \text{Isom}(X_1, Y_1)$. Then the natural map

$$\text{Isom}(X, Y) \times \Psi_n(\mathcal{G}_n) \rightarrow \text{Isom}(X_n, Y_n)$$

is bijective.

(iv) Write $\text{Isom}((X_n)_{\Phi_X}/k_X, (Y_n)_{\Phi_Y}/k_Y)$ for the set of isomorphisms $(X_n)_{\Phi_X} \rightarrow (Y_n)_{\Phi_Y}$ that are compatible with some isomorphism $k_Y \rightarrow k_X$; $\text{Isom}((X_n)_{\Phi_X}/k_X, (Y_n)_{\Phi_Y}/k_Y) \overset{\text{def}}{=} \text{Isom}((X_1)_{\Phi_X}/k_X, (Y_1)_{\Phi_Y}/k_Y)$. Then the natural map

$$\text{Isom}((X_n)_{\Phi_X}/k_X, (Y_n)_{\Phi_Y}/k_Y) \times \Psi_n(\mathcal{G}_n) \rightarrow \text{Isom}((X_n)_{\Phi_X}/k_X, (Y_n)_{\Phi_Y}/k_Y)$$

is bijective.

Proof. First, we verify assertion (i). Write $(C_n^X)_{\log}, (C_n^Y)_{\log}$ for the $n$-th log configuration spaces [cf. the discussion entitled “Curves” in [CbTpI], §80] of the smooth log curves over $k_X, k_Y$ determined by $X, Y$, respectively. Then recall [cf. the discussion at the beginning of [MzTa], §2] that $(C_n^X)_{\log}, (C_n^Y)_{\log}$ are log regular log schemes whose interiors are naturally isomorphic to $X_n, Y_n$, respectively, and that the underlying schemes $C_n^X, C_n^Y$ of $(C_n^X)_{\log}, (C_n^Y)_{\log}$ are proper over $k_X, k_Y$, respectively. Thus, by applying [ExtFam], Theorem A, (1), to the composite

$$X_n \overset{\alpha}{\rightarrow} Y_n \hookrightarrow C_n^Y \hookrightarrow \mathcal{M}_{g_Y,r_Y+n}$$

— where we refer to the discussion entitled “Curves” in [CbTpI], §80, concerning the notation “$\mathcal{M}_{g_Y,r_Y+n}$”; the third arrow is the natural $(1)$-morphism arising from the definition of $C_n^Y$ — we conclude that the composite

$$X_n \overset{\alpha}{\rightarrow} Y_n \hookrightarrow C_n^Y \hookrightarrow \mathcal{M}_{g_Y,r_Y+n} \rightarrow (\mathcal{M}_{g_Y,r_Y+n})^e$$
Next, let us observe that if \( Z_{\overline{k}} \) is the \( \overline{k} \)-points of a \( k \)-scheme, then any \( \overline{k} \)-morphism from an \( \overline{k} \)-scheme to \( Z_{\overline{k}} \) in \( k \)-algebraic geometry induces an \( \overline{k} \)-algebraic geometry over \( k \)-morphism between the respective structure sheaves, that there exists a unique homomorphism \( \alpha_0 : k_Y \to k_X \) that is compatible with \( \alpha \). Moreover, by applying a similar argument to \( \alpha^{-1} \), it follows that \( \alpha_0 \) is an isomorphism. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, let us observe that, by replacing \( Y \) by the result of base-changing \( Y \) via \( \alpha_0 : k_Y \to k_X \) [cf. assertion (i)], we may assume without loss of generality that \( k_Y = k_X \), \( \overline{k}_Y = \overline{k}_X \), and that \( \alpha \) is an isomorphism over \( k_X \). Next, let us observe that it is immediate that \( \sigma \) and \( \alpha_1 \) as in the statement of assertion (ii) are unique; thus, it remains to verify the existence of such \( \sigma \) and \( \alpha_1 \). Next, let us observe that it follows immediately from [MzTa], Corollary 6.3, that there exists a permutation \( \sigma \in \Psi_n(\mathfrak{S}_n) \) such that if we identify the respective sets of fiber subgroups of \( \Delta_{X_n} \), \( \Delta_{Y_n} \) — where we write \( \Delta_{X_n} \), \( \Delta_{Y_n} \) for the maximal \( \text{pro-}l \) quotients of the \( \text{étale} \) fundamental groups of \( (X_n)_{\overline{k}_X} \), \( (Y_n)_{\overline{k}_X} \), respectively, for some prime number \( l \) that is invertible in \( k_X \) — with the set \( 2^{\{1, \ldots, n\}} \) [cf. the discussion entitled “Sets” in [CbTpi], §0] in the evident way, then the automorphism of the set \( 2^{\{1, \ldots, n\}} \) induced by the composite \( \beta \overset{\text{def}}{=} \alpha \circ \sigma \) is the identity automorphism. Write \( \text{pr}_X : X_n \to X \), \( \text{pr}_Y : Y_n \to Y \) for the projections to the factor labeled \( n \), respectively. Then we claim that the following assertion holds:

**Claim 2.7.A:** There exists an isomorphism \( \alpha_1 : X \xrightarrow{\sim} Y \) that is compatible with \( \beta \) relative to \( \text{pr}_X \), \( \text{pr}_Y \).

Indeed, write \( \Gamma \subseteq X \times_{k_X} Y \) for the scheme-theoretic image via \( X_n \times_{k_X} Y \) \( (\text{pr}_X,\text{id}_Y) \) \( X \times_{k_X} Y \) of the graph of the composite \( X_n \xrightarrow{\beta} Y_n \xrightarrow{\text{pr}_Y} Y \). Next, let us observe that if \( Z \) is an irreducible scheme of finite type over \( \overline{k}_X \), then any nonconstant [i.e., dominant] \( \overline{k}_X \)-morphism \( Z \to Y_{\overline{k}_X} \) induces an open homomorphism between the respective fundamental groups. Thus, since the automorphism of the set \( 2^{\{1, \ldots, n\}} \) induced by \( \beta \) is the identity automorphism, it follows immediately that, for any \( \overline{k}_X \)-valued geometric point \( \overline{x} \) of \( X \), if we write \( F \) for the geometric
fiber of $\text{pr}_X : X_n \to X$ at $\pi$, then the composite $F \to (X_n)_{\tilde{F}_X}^{\beta_X}$ $(Y_n)_{\tilde{F}_X} \to \tilde{F}_X$ is constant. In particular, one verifies immediately that $\tilde{F}$ is an integral, separated scheme of dimension 1. Thus, since $\text{pr}_X$ is surjective, geometrically connected, smooth, and factors through the composite $\Gamma \hookrightarrow X \times_{k_X} Y \rightarrow X$, it follows immediately that this composite morphism $\Gamma \to X$ is surjective and induces an isomorphism between the respective function fields. Therefore, one concludes easily, by applying Zariski’s main theorem, that the composite $\Gamma \to X \times_{k_X} Y \rightarrow X$ is an isomorphism, hence that there exists a unique morphism $\alpha_1 : X \to Y$ such that $\text{pr}_Y \circ \beta = \alpha_1 \circ \text{pr}_X$. Moreover, by applying a similar argument to $\beta^{-1}$, it follows that $\alpha_1$ is an isomorphism. This completes the proof of Claim 2.7.A.

Write $\gamma$ for the composite of $\beta$ with the isomorphism $Y_n \sim \to X_n$ determined by $\alpha_1^{-1}$. Then it is immediate that $\gamma$ is an automorphism of $X_n$ over $X$ relative to $\text{pr}_X$; in particular, the automorphism of $\Delta_{X_n}$ induced by $\gamma$ is contained in the kernel of the homomorphism $\text{Out}^F(\Delta_{X_n}) \to \text{Out}^F(\Delta_X)$ — where we write $\Delta_X$ for the maximal pro-$l$ quotient of the étale fundamental group of $X_{\tilde{F}_X}$ induced by $\text{pr}_X$. Now, by applying a similar argument to the argument of the proof of Claim 2.7.A, one verifies easily that, for each $i \in \{1, \ldots, n\}$, there exists an automorphism $\gamma_{1,i}$ of $X$ that is compatible with $\gamma$ relative to the projection $X_n \to X$ to the factor labeled $i$. [Thus, $\gamma_{1,n} = \text{id}_X$.] Moreover, since, by applying induction on $n$, we may assume that assertion (ii) has already been verified for $n - 1$, it follows immediately that the automorphism of $\Delta_{X_n}$ induced by $\gamma$ is contained in $\text{Out}^{\text{FC}}(\Delta_{X_n})$, hence in the kernel of the homomorphism $\text{Out}^{\text{FC}}(\Delta_{X_n}) \to \text{Out}^{\text{FC}}(\Delta_X)$ induced by the projections $X_n \to X$ to each of the $n$ factors [cf. [CmbCsp], Proposition 1.2, (iii)]. Therefore, it follows immediately from the argument of the first paragraph of the proof of [LocAn], Theorem 14.1, that, for each $i \in \{1, \ldots, n\}$, $\gamma_{1,i}$ is the identity automorphism of $X$, hence also that $\gamma$ is the identity automorphism of $X_n$. This completes the proof of assertion (ii).

Assertions (iii), (iv) follow immediately from assertion (ii), together with the various definitions involved. This completes the proof of Lemma 2.7. \[ \square \]
3. Synchronization of tripods

In the present §3, we introduce and study the notion of a tripod of the log fundamental group of the log configuration space of a stable log curve [cf. Definition 3.3, (i), below]. In particular, we discuss the phenomenon of synchronization among the various tripods of the log fundamental group [cf. Theorems 3.17; 3.18, below]. One interesting consequence of this phenomenon of tripod synchronization is a certain non-surjectivity result [cf. Corollary 3.22 below]. Finally, we apply the theory of synchronization of tripods to show that, under certain conditions, commuting profinite Dehn multi-twists are “co-Dehn” [cf. Corollary 3.25 below] and to compute the commensurator of certain purely combinatorial/group-theoretic groups of profinite Dehn multi-twists in terms of scheme theory [cf. Corollary 3.27 below].

In the present §3, let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(n\) a positive integer; \(\Sigma\) a set of prime numbers which is either the set of all prime numbers or of cardinality one; \(k\) an algebraically closed field of characteristic \(\not\in \Sigma\); \((\Spec k)_{\log}\) the log scheme obtained by equipping \(\Spec k\) with the log structure determined by the fs chart \(N \to k\) that maps \(1 \mapsto 0\); \(X^\log = X^\log_1\) a stable log curve of type \((g, r)\) over \((\Spec k)_{\log}\). For each [possibly empty] subset \(E \subseteq \{1, \ldots, n\}\), write

\[
X^\log_E \quad \text{for the } E^2\text{-th log configuration space of the stable log curve } X^\log \quad \text{[cf. the discussion entitled “Curves” in [CbTpI], §0], where we think of the factors as being labeled by the elements of } E \subseteq \{1, \ldots, n\};
\]

\[
\Pi_E \quad \text{for the maximal pro-}\Sigma\text{ quotient of the kernel of the natural surjection } \pi_1(X^\log_E) \to \pi_1((\Spec k)_{\log}). \]

Thus, by applying a suitable specialization isomorphism — cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1 — one verifies easily that \(\Pi_E\) is equipped with a natural structure of pro-\(\Sigma\) configuration space group — cf. [MzTa], Definition 2.3, (i). For each \(1 \leq m \leq n\), write

\[
X^\log_m \quad \text{def} \quad X^\log_{\{1, \ldots, m\}}; \quad \Pi_m \quad \text{def} \quad \Pi_{\{1, \ldots, m\}}.
\]

Thus, for subsets \(E' \subseteq E \subseteq \{1, \ldots, n\}\), we have a projection

\[
p^\log_{E/E'} : X^\log_E \to X^\log_{E'}
\]

obtained by forgetting the factors that belong to \(E \setminus E'\). For \(E' \subseteq E \subseteq \{1, \ldots, n\}\) and \(1 \leq m' \leq m \leq n\), we shall write

\[
p^\Pi_{E/E'} : \Pi_E \to \Pi_{E'}
\]

for the surjection induced by \(p^\log_{E/E'}\);

\[
\Pi_{E/E'} \quad \text{def} \quad \Ker(p^\Pi_{E/E'}) \subseteq \Pi_E;
\]
Finally, recall [cf. the statement of Theorem 2.3, (iv)] the natural inclusion

$$G_n \hookrightarrow \text{Out}(\Pi_n)$$

— where we write $G_n$ for the symmetric group on $n$ letters — obtained by permuting the various factors of $X_n$. We shall regard $G_n$ as a subgroup of Out($\Pi_n$) by means of this natural inclusion.

**Definition 3.1.** Let $i \in E \subseteq \{1, \cdots, n\}; x \in X_n(k)$ a $k$-valued geometric point of the underlying scheme $X_n$ of $X_n^\log$.

(i) Let $E' \subseteq \{1, \cdots, n\}$ be a subset. Then we shall write $x_{E'} \in X_{E'}(k)$ for the $k$-valued geometric point of $X_{E'}$ obtained by forming the image of $x \in X_n(k)$ via $p_{\{1, \cdots, n\}/E'}: X_n \to X_{E'}$;

$$x_{E'}^\log = x_{E'} \times_{X_{E'}} X_{E'}^\log.$$

(ii) We shall write

$$\mathcal{G}$$

for the semi-graph of anabelioids of pro-$\Sigma$ PSC-type determined by the stable log curve $X^\log$ over $(\text{Spec } k)^\log$ [cf. [CmbGC], Example 2.5];

$$\mathcal{G}$$

for the underlying semi-graph of $\mathcal{G}$;

$$\Pi_\mathcal{G}$$

for the [pro-$\Sigma$] fundamental group of $\mathcal{G}$. Thus, we have a natural outer isomorphism

$$\Pi_1 \sim \to \Pi_\mathcal{G}.$$

(iii) We shall write

$$\mathcal{G}_{i \in E, x}$$

for the semi-graph of anabelioids of pro-$\Sigma$ PSC-type determined by the geometric fiber of the projection $p_{E/(E \setminus \{i\})}^\log: X_E^\log \to X_{E \setminus \{i\}}^\log$ over $x_{E \setminus \{i\}}^\log \to X_{E \setminus \{i\}}^\log$ [cf. (i)];

$$\Pi_{\mathcal{G}_{i \in E, x}}$$

for the [pro-$\Sigma$] fundamental group of $\mathcal{G}_{i \in E, x}$. Thus, we have a natural identification

$$\mathcal{G} = \mathcal{G}_{i \in \{i\}, x}.$$
and a natural $\Pi_E$-orbit [i.e., relative to composition with automorphisms induced by conjugation by elements of $\Pi_E$] of isomorphisms

$$(\Pi_E \supseteq) \Pi_{E/(E\setminus\{i\})} \simto \Pi_{G_i \in E,x}.$$  

For the remainder of the present §3, let us fix an outer isomorphism

$$\Pi_{E/(E\setminus\{i\})} \simto \Pi_{G_i \in E,x}$$

whose constituent isomorphisms belong to the $\Pi_E$-orbit of isomorphisms just discussed.

(iv) Let $v \in \text{Vert}(G_{i \in E,x})$ (respectively, $e \in \text{Cusp}(G_{i \in E,x}); e \in \text{Node}(G_{i \in E,x}); e \in \text{Edge}(G_{i \in E,x}); z \in \text{VCN}(G_{i \in E,x}))$. Then we shall refer to the image [in $\Pi_E$] of a vertical (respectively, a cuspidal; a nodal; an edge-like; a VCN-) subgroup of $\Pi_{G_{i \in E,x}}$ associated to $v \in \text{Vert}(G_{i \in E,x})$ (respectively, $e \in \text{Cusp}(G_{i \in E,x}); e \in \text{Node}(G_{i \in E,x}); e \in \text{Edge}(G_{i \in E,x}); z \in \text{VCN}(G_{i \in E,x}))$ via the inverse $\Pi_{G_{i \in E,x}} \simto \Pi_{E/(E\setminus\{i\})} \subseteq \Pi_E$ of any isomorphism that lifts the fixed outer isomorphism discussed in (iii) as a vertical (respectively, a cuspidal; a nodal; an edge-like; a VCN-) subgroup of $\Pi_E$ associated to $v \in \text{Vert}(G_{i \in E,x})$ (respectively, $e \in \text{Cusp}(G_{i \in E,x}); e \in \text{Node}(G_{i \in E,x}); e \in \text{Edge}(G_{i \in E,x}); z \in \text{VCN}(G_{i \in E,x}))$. Thus, the notion of a vertical (respectively, a cuspidal; a nodal; an edge-like; a VCN-) subgroup of $\Pi_E$ associated to $v \in \text{Vert}(G_{i \in E,x})$ (respectively, $e \in \text{Cusp}(G_{i \in E,x}); e \in \text{Node}(G_{i \in E,x}); e \in \text{Edge}(G_{i \in E,x}); z \in \text{VCN}(G_{i \in E,x}))$ depends on the choice of the fixed outer isomorphism of (iii) [but cf. Lemma 3.2, (i), below, in the case of cusps!].

(v) We shall say that a vertex $v \in \text{Vert}(G_{i \in E,x})$ of $G_{i \in E,x}$ is a(n) [E]-tripod of $X^\log_n$ if $v$ is of type $(0, 3)$ [cf. [CbTPL], Definition 2.3, (iii)]. If, in this situation, $C(v) \neq \emptyset$, then we shall say that the tripod $v$ is cuspidal-supporting.

(vi) We shall say that a cusp $c \in \text{Cusp}(G_{i \in E,x})$ of $G_{i \in E,x}$ is diagonal if $c$ does not arise from a cusp of the copy of $X^\log_n$ given by the factor of $X^\log_E$ labeled $i \in E$.

Lemma 3.2 (Cusps of various fibers). Let $i \in E \subseteq \{1, \ldots, n\}; x \in X_n(k)$. Then the following hold:

(i) Let $c \in \text{Cusp}(G_{i \in E,x})$ and $\Pi_c \subseteq \Pi_{G_{i \in E,x}} \simto \Pi_{E/(E\setminus\{i\})}$ a cuspidal subgroup of $\Pi_{G_{i \in E,x}} \simto \Pi_{E/(E\setminus\{i\})}$ associated to $c \in \text{Cusp}(G_{i \in E,x})$. Then any $\Pi_E$-conjugate of $\Pi_c$ is, in fact, a $\Pi_{E/(E\setminus\{i\})}$-conjugate of $\Pi_c$. 


(ii) Each diagonal cusp of $G_{i \in E,x}$ [cf. Definition 3.1, (vi)] admits a natural label $\in E \setminus \{i\}$. More precisely, for each $j \in E \setminus \{i\}$, there exists a unique diagonal cusp of $G_{i \in E,x}$ that arises from the divisor of the fiber product over $k$ of $E^2$ copies of $X$ consisting of the points whose $i$-th and $j$-th factors coincide.

(iii) Let $\alpha \in \text{Aut}^F(\Pi_n) \ [\text{cf. [CmbCsp], Definition 1.1, (ii)}]. Suppose that either $E^6 = f_1; \cdots ; n$ or $n \geq n_{FC} \ [\text{cf. Theorem 2.3, (ii)}]$. Then the automorphism of $G_{i \in E,x} \simeq \Pi_{E/(E\setminus \{i\})}$ determined by $\alpha$ is group-theoretically cuspidal [cf. [CmbGC], Definition 1.4, (iv)].

(iv) Let $\alpha \in \text{Aut}^F(\Pi_n)$ and $c \in \text{Cusp}(G_{i \in E,x})$ a diagonal cusp of $G_{i \in E,x}$. Suppose that the automorphism of $G_{i \in E,x} \simeq \Pi_{E/(E\setminus \{i\})}$ determined by $\alpha$ is group-theoretically cuspidal. Then this automorphism preserves the $\Pi_{G_{i \in E,x}}$-conjugacy class of cuspidal subgroups of $\Pi_{G_{i \in E,x}} \simeq \Pi_{E/(E\setminus \{i\})}$ associated to $c \in \text{Cusp}(G_{i \in E,x})$.

Proof. Assertion (i) follows immediately from the [easily verified] fact that the restriction of $p_{\Pi_{E/(E\setminus \{i\})}}: \Pi_E \twoheadrightarrow \Pi_{E\setminus \{i\}}$ to the normalizer of $\Pi_c$ in $\Pi_E$ is surjective. Assertion (ii) follows immediately from the various definitions involved. Next, we verify assertion (iii). If $E \neq \{1, \cdots , n\}$ (respectively, $n \geq n_{FC}$), then assertion (iii) follows immediately from [CbTpl], Theorem A, (ii) (respectively, Theorem 2.3, (ii), of the present paper), together with assertion (i). This completes the proof of assertion (iii). Finally, assertion (iv) follows immediately from the definition of $F$-admissibility [cf. also assertion (ii)]. This completes the proof of Lemma 3.2. □

**Definition 3.3.** Let $E \subseteq \{1, \cdots , n\}$.

(i) We shall say that a closed subgroup $H \subseteq \Pi_E$ of $\Pi_E$ is a(n) $[E\text{-}]$tripod of $\Pi_n$ if $H$ is a vertical subgroup of $\Pi_E$ [cf. Definition 3.1, (iv)] associated to a(n) $[E\text{-}]$tripod $v$ of $X^\log_n$ [cf. Definition 3.1, (v)]. If, in this situation, the tripod $v$ is cuspsupporting [cf. Definition 3.1, (v)], then we shall say that the tripod $H$ is cuspsupporting.

(ii) We shall say that an $E$-tripod of $\Pi_n$ [cf. (i)] is trigonal if, for every $j \in E$, the image of the tripod via $p_{\Pi_{E/(\{j\})}}: \Pi_E \twoheadrightarrow \Pi_{\{j\}}$ is trivial.

(iii) Let $T \subseteq \Pi_E$ be an $E$-tripod of $\Pi_n$ [cf. (i)] and $E' \subseteq E$. Then we shall say that $T$ is $E'$-strict if the image $p_{\Pi_E/E'}(T) \subseteq \Pi_{E'}$ of $T$ via $p_{\Pi_E/E'}: \Pi_E \twoheadrightarrow \Pi_{E'}$ is an $E'$-tripod of $\Pi_n$, and, moreover,
for every $E'' \subset E'$, the image of the $E'$- tripod $p^E_{E/E'}(T)$ via $p^E_{E/E'}: \Pi_E \rightarrow \Pi_{E''}$ is not a tripod of $\Pi_n$.

(iv) Let $h$ be a positive integer. Then we shall say that an $E$- tripod $T$ of $\Pi_n$ [cf. (i)] is $h$-descendable if there exists a subset $E' \subset E$ such that the image of $T$ via $p^E_{E/E'}: \Pi_E \rightarrow \Pi_{E'}$ is an $E'$- tripod of $\Pi_n$, and, moreover, $(E')^2 \leq n - h$. [Thus, one verifies immediately that an $E$- tripod $T \subset \Pi_E$ of $\Pi_n$ is 1-descendable if and only if either $E \neq \{1, \cdots , n\}$ or $T$ fails to be $E$- strict — cf. (iii).]

Remark 3.3.1. In the notation of Definition 3.1, let $v \in \text{Vert}(G_{i \in E,x})$ be an $E$- tripod of $X^\log_n$ [cf. Definition 3.1, (v)] and $T \subset \Pi_E$ an $E$- tripod of $\Pi_n$ associated to $v$ [cf. Definition 3.3, (i)]. Write $F_v$ for the irreducible component of the geometric fiber of $p_{E/(E \setminus \{i\})}: X_E \rightarrow X_{E \setminus \{i\}}$ at $x_{E \setminus \{i\}}$ corresponding to $v$; $F^\log_v$ for the log scheme obtained by equipping $F_v$ with the log structure induced by the log structure of $X^\log_E$, $n_v$ for the rank of the group-characteristic of $F^\log_v$ [cf. [MzTa], Definition 5.1, (i)] at the generic point of $F_v$. Then it is immediate that the $n_v$-interior $U_v \subset F_v$ of $F^\log_v$ [cf. [MzTa], Definition 5.1, (i)] is a nonempty open subset of $F_v$ which is isomorphic to $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over $k$. Moreover, one verifies easily that if we write $U^\log_v$ for the log scheme obtained by equipping $U_v$ with the log structure induced by the log structure of $X^\log_E$, then the natural morphism $U^\log_v \rightarrow U_v$ [obtained by forgetting the log structure of $U^\log_v$] determines a natural outer isomorphism $T \stackrel{\sim}{\rightarrow} \pi^\Sigma_1(U_v)$ — where we write “$\pi^\Sigma_1(\cdot)$” for the maximal pro-$\Sigma$ quotient of the étale fundamental group of “$(\cdot)$”. In particular, we obtain a natural outer isomorphism

$$T \stackrel{\sim}{\rightarrow} \pi^\Sigma_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\})$$

that is well-defined up to composition with an automorphism of $\pi^\Sigma_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\})$ that arises from an automorphism of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over $k$.

Definition 3.4. Let $E \subset \{1, \cdots , n\}$.

(i) Let $T \subset \Pi_E$ be an $E$- tripod of $\Pi_n$ [cf. Definition 3.3, (i)]. Then $T$ may be regarded as the “$\Pi_i$” that occurs in the case where we take “$X^\log$” to be the smooth log curve associated to $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ [cf. Remark 3.3.1]. We shall write

$$\text{Out}^C(T) \subset \text{Out}(T)$$

for the [closed] subgroup of $\text{Out}(T)$ consisting of $C$-admissible automorphisms of $T$ [cf. [CmbCsp], Definition 1.1, (ii)];

$$\text{Out}^C(T)_{\text{cusp}} \subset \text{Out}^C(T)$$
for the [closed] subgroup of Out($T$) consisting of $C$-admissible automorphisms of $T$ that induce the identity automorphism of the set of $T$-conjugacy classes of cuspidal inertia subgroups;

$$\text{Out}(T)^\Delta \subseteq \text{Out}(T)$$

for the centralizer of the subgroup $[\simeq \mathfrak{S}_3$, where we write $\mathfrak{S}_3$ for the symmetric group on 3 letters] of Out($T$) consisting of the outer modular symmetries [cf. [CmbCsp], Definition 1.1, (vi)];

$$\text{Out}(T)^+ \subseteq \text{Out}(T)$$

for the [closed] subgroup of Out($T$) given by the image of the natural homomorphism $\text{Out}^F(T_2) = \text{Out}^{FC}(T_2) \to \text{Out}(T)$ [cf. Theorem 2.3, (ii); [CmbCsp], Proposition 1.2, (iii)] — where we write $T_2$ for the “$\Pi_2$” that occurs in the case where we take “$X^{\log}$” to be the smooth log curve associated to $\mathbb{P}_k^1 \setminus \{0, 1, \infty\}$;

$$\text{Out}^C(T)^\Delta \overset{\text{def}}{=} \text{Out}^C(T) \cap \text{Out}(T)^\Delta;$$

$$\text{Out}^C(T)^\Delta^+ \overset{\text{def}}{=} \text{Out}^C(T)^\Delta \cap \text{Out}(T)^+$$

[cf. [CmbCsp], Definition 1.11, (i)].

(ii) Let $E' \subseteq \{1, \cdots, n\}$; let $T \subseteq \Pi_E, T' \subseteq \Pi_{E'}$ be $E, E'$-tripods of $\Pi_n$ [cf. Definition 3.3, (i)], respectively. Then we shall say that an outer isomorphism $\alpha: T \overset{\sim}{\rightarrow} T'$ is geometric if the composite

$$\pi^\Sigma_1(\mathbb{P}_k^1 \setminus \{0, 1, \infty\}) \overset{\alpha}{\sim} T \overset{\sim}{\rightarrow} T' \overset{\sim}{\rightarrow} \pi^\Sigma_1(\mathbb{P}_k^1 \setminus \{0, 1, \infty\})$$

— where the first and third arrows are natural outer isomorphisms of the sort discussed in Remark 3.3.1 — arises from an automorphism of $\mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ over $k$.

**Remark 3.4.1.** In the notation of Definition 3.4, (ii), one verifies easily that every geometric outer isomorphism $\alpha: T \overset{\sim}{\rightarrow} T'$ preserves cuspidal inertia subgroups and outer modular symmetries [cf. [CmbCsp], Definition 1.1, (vi)], and, moreover, lifts to an outer isomorphism $T_2 \overset{\sim}{\rightarrow} T'_2$ [i.e., of the corresponding “$\Pi_2$’s”] that arises from an isomorphism of two-dimensional configuration spaces. In particular, the isomorphism $\text{Out}(T) \overset{\sim}{\rightarrow} \text{Out}(T')$ induced by $\alpha$ determines isomorphisms

$$\text{Out}^C(T) \overset{\sim}{\rightarrow} \text{Out}^C(T'), \quad \text{Out}^C(T)^\text{cusp} \overset{\sim}{\rightarrow} \text{Out}^C(T')^\text{cusp},$$

$$\text{Out}(T)^\Delta \overset{\sim}{\rightarrow} \text{Out}(T')^\Delta, \quad \text{Out}(T)^+ \overset{\sim}{\rightarrow} \text{Out}(T')^+$$

[cf. Definition 3.4, (i)].

**Lemma 3.5** (Triviality of the action on the set of cusps). *In the notation of Definition 3.4, it holds that $\text{Out}^C(T)^\Delta \subseteq \text{Out}^C(T)^\text{cusp}$.***
Lemma 3.6 (Vertices, cusps, and nodes of various fibers). Let \( i, j \in E \) be two distinct elements of a subset \( E \subseteq \{1, \ldots, n\} \); \( x \in X_n(k) \). Write \( z_{i,j,x} \in \text{VCN}(G_{j \in E \setminus \{i\},x}) \) for the element of \( \text{VCN}(G_{j \in E \setminus \{i\},x}) \) on which \( x_{E \setminus \{i\}} \) lies, that is to say: If \( x_{E \setminus \{i\}} \) is a cusp or node of the geometric fiber of the projection \( p_{E \setminus \{i\}/E \setminus \{i,j\}}^\log : X_{E \setminus \{i\}}^\log \to X_{E \setminus \{i,j\}}^\log \) over \( x_{E \setminus \{i,j\}}^\log \) corresponding to an edge \( e \in \text{Edge}(G_{j \in E \setminus \{i\},x}) \), then \( z_{i,j,x} \overset{\text{def}}{=} e \); if \( x_{E \setminus \{i\}} \) is neither a cusp nor a node of the geometric fiber of the projection \( p_{E \setminus \{i\}/E \setminus \{i,j\}}^\log : X_{E \setminus \{i\}}^\log \to X_{E \setminus \{i,j\}}^\log \) over \( x_{E \setminus \{i,j\}}^\log \), but lies on the irreducible component of the geometric fiber corresponding to a vertex \( v \in \text{Vert}(G_{j \in E \setminus \{i\},x}) \), then \( z_{i,j,x} \overset{\text{def}}{=} v \). Then the following hold:

(i) The automorphism of \( X_E^\log \) determined by permuting the factors labeled \( i, j \) induces natural bijections

\[
\begin{align*}
\text{Vert}(G_{j \in E \setminus \{i\},x}) & \sim \text{Vert}(G_{i \in E \setminus \{j\},x}) ; \\
\text{Cusp}(G_{j \in E \setminus \{i\},x}) & \sim \text{Cusp}(G_{i \in E \setminus \{j\},x}) ; \\
\text{Node}(G_{j \in E \setminus \{i\},x}) & \sim \text{Node}(G_{i \in E \setminus \{j\},x}) .
\end{align*}
\]

(ii) Let us write

\[
c_{i,j,x}^{\text{diag}} \in \text{Cusp}(G_{i \in E,x})
\]

for the diagonal cusp of \( G_{i \in E,x} \) [cf. Definition 3.1, (vi)] labeled \( j \in E \setminus \{i\} \) [cf. Lemma 3.2, (ii)]. Then \( p_{E/(E \setminus \{j\})}^\log : X_E^\log \to X_{E \setminus \{j\}}^\log \) induces a bijection

\[
\begin{align*}
\text{Cusp}(G_{i \in E,x}) \setminus \{c_{i,j,x}^{\text{diag}}\} & \sim \text{Cusp}(G_{i \in E \setminus \{j\},x}) .
\end{align*}
\]

(iii) Suppose that \( z_{i,j,x} \in \text{Vert}(G_{j \in E \setminus \{i\},x}) \). Then \( p_{E/(E \setminus \{j\})}^\log : X_E^\log \to X_{E \setminus \{j\}}^\log \) induces a bijection

\[
\text{Vert}(G_{j \in E \setminus \{i\},x}) \sim \text{Vert}(G_{i \in E \setminus \{j\},x}) .
\]

(iv) Suppose that \( z_{i,j,x} \in \text{Edge}(G_{j \in E \setminus \{i\},x}) \). Then there exists a unique vertex

\[
v_{i,j,x}^{\text{new}} \in \text{Vert}(G_{i \in E,x})
\]

such that \( p_{E/(E \setminus \{j\})}^\log : X_E^\log \to X_{E \setminus \{j\}}^\log \) induces a bijection

\[
\text{Vert}(G_{i \in E,x}) \setminus \{v_{i,j,x}^{\text{new}}\} \sim \text{Vert}(G_{i \in E \setminus \{j\},x}) .
\]

Moreover, \( v_{i,j,x}^{\text{new}} \) is of type \((0,3)\) [i.e., \( v_{i,j,x}^{\text{new}} \) is an \( E \)-tripod of \( X_n^\log \) — cf. Definition 3.1, (v)] ; and \( c_{i,j,x}^{\text{diag}} \in C(v_{i,j,x}^{\text{new}}) \) [cf.
(ii). Finally, any vertical subgroup of $\Pi_E$ associated to $v_{i,j,x}^{\text{new}}$ surjects, via $p_{E/(E\setminus\{j\})}$, onto an edge-like subgroup of $\Pi_{E\setminus\{j\}}$ associated to the edge $e \in \text{Edge}(G_{i,j,x})$ determined by $z_{i,j,x} \in \text{Edge}(G_{j \in E \setminus \{i\}, x})$ via the bijections of (i).

(v) Suppose that $E^x = 3$. Write $h_{i;j;g}$ for the unique element of $E_{\mathfrak{nf} i;j;g}$. Suppose, moreover, that $z_{i,j,x} = c_{j,h,x}^{\text{diag}}$. Then the $\Pi_E$-conjugacy class of a vertical subgroup of $\Pi_E$ associated to the edge $v_{i,j,x}^{\text{new}} \in \text{Vert}(G_{i \in E, x})$ [cf. (ii)] depends only on $i$ and not on the choice of the pair $(j, x)$. Moreover, these three $\Pi_E$-conjugacy classes [cf. the dependence on the choice of $i \in E$] may also be characterized uniquely as the $\Pi_E$-conjugacy classes of subgroups of $\Pi_E$ associated to some trigonal $E$-tripod of $\Pi_n$ [cf. Definition 3.3, (ii)].

Proof. First, we verify assertions (i), (ii), (iii), and (iv). To verify assertions (i), (ii), (iii), and (iv) — by replacing $X^E_{\log}$ by the base-change of $p_{E/(E\setminus\{i,j\})}^\log: X^E_{\log} \to X^{E\setminus\{i,j\}}_{\log}$ via a suitable morphism of log schemes $(\text{Spec } k)^{\log} \to X^{E\setminus\{i,j\}}_{\log}$ whose image lies on $x_{E\setminus\{i,j\}} \in X_{\log}(k)$ [cf. Definition 3.1, (i)] — we may assume without loss of generality that $E^x = 2$. Then one verifies easily from the various definitions involved that assertions (i), (ii), (iii), and (iv) hold. This completes the proof of assertions (i), (ii), (iii), and (iv). Finally, we consider assertion (v). First, we observe the easily verified fact [cf. assertions (iii), (iv)] that the irreducible component corresponding to an $E$-tripod of $X^E_{\log}$ [cf. Definition 3.1, (v)] that gives rise to a trigonal $E$-tripod of $\Pi_n$ necessarily collapses to a point upon projection to $X^{E'}_{\log}$ for any $E' \subseteq E$ of cardinality $\leq 2$. In light of this observation, it follows immediately [cf. assertions (i), (ii), (iii), (iv)] that any $E$-tripod of $X^E_{\log}$ that gives rise to a trigonal $E$-tripod of $\Pi_n$ arises as a vertex “$v_{i,j,x}^{\text{new}}$” as described in the statement of assertion (v). Now the remainder of assertion (v) follows immediately from the various definitions involved [cf. also the situation discussed in [CmbCsp], Definition 1.8, Proposition 1.9, Corollary 1.10, as well as the discussion, concerning specialization isomorphisms, preceding [CmbCsp], Definition 2.1; [CbTpI], Remark 5.6.1]. This completes the proof of Lemma 3.6.

\[\square\]

**Definition 3.7.** Let $E \subseteq \{1, \cdots, n\}$.

(i) Let $v$ be an $E$-tripod of $X^{E}_{\log}$ [cf. Definition 3.1, (v)]; thus, $v$ belongs to $\text{Vert}(G_{i \in E, x})$ for some choice of $i \in E$ and $x \in X_n(k)$. Let $j \in E \setminus \{i\}$ and $e \in \text{Edge}(G_{j \in E \setminus \{i\}, x})$. Then we shall say that $v$, or equivalently, an $E$-tripod of $\Pi_n$ associated to $v$ [cf.
Definition 3.3, (i)], arises from $e$ if $e = z_{i,j,x}$ [cf. the statement of Lemma 3.6], and $v = v_{i,j,x}^{\text{new}}$ [cf. Lemma 3.6, (iv)].

(ii) Let $i \in E$. Then we shall say that an $E$-tripod of $\Pi_n$ is $i$-central if $E^i = 3$, and, moreover, the tripod is a vertical subgroup of the sort discussed in Lemma 3.6, (v), i.e., the unique, up to $\Pi_E$-conjugacy, trigonal $E$-tripod of $\Pi_n$ contained in $\Pi_{E/(E \setminus \{i\})}$ [cf. the final portion of Lemma 3.6, (iv)]. We shall say that an $E$-tripod of $\Pi_n$ is central if it is $j$-central for some $j \in E$.

Remark 3.7.1. Let $E \subseteq \{1, \cdots, n\}$; $T \subseteq \Pi_E$ an $E$-tripod of $\Pi_n$ [cf. Definition 3.3, (i)]; $\sigma \in \mathcal{S}_n \subseteq \text{Out}(\Pi_n)$ [cf. the discussion at the beginning of the present §3]; $\tilde{\sigma} \in \text{Aut}(\Pi_n)$ a lifting of $\sigma \in \mathcal{S}_n \subseteq \text{Out}(\Pi_n)$. Write

$$T^{\tilde{\sigma}} \subseteq \Pi_{\sigma(E)}$$

for the image of $T \subseteq \Pi_E$ by the isomorphism $\Pi_E \xrightarrow{\sim} \Pi_{\sigma(E)}$ determined by $\tilde{\sigma} \in \text{Aut}(\Pi_n)$.

(i) One verifies easily that $T^{\tilde{\sigma}} \subseteq \Pi_{\sigma(E)}$ is a $\sigma(E)$-tripod of $\Pi_n$.

(ii) If, moreover, the equality $E^i = 3$ holds, and $T$ is $i$-central [cf. Definition 3.7, (ii)] for some $i \in E$, then one verifies easily from Lemma 3.6, (v), that $T^{\tilde{\sigma}} \subseteq \Pi_{\sigma(E)}$ is $\sigma(i)$-central.

(iii) In the situation of (ii), let $T' \subseteq \Pi_E$ be a central $E$-tripod of $\Pi_n$. Then it follows from Lemma 3.6, (v), that there exist an element $\tau \in \mathcal{S}_n \subseteq \text{Out}(\Pi_n)$ and a lifting $\tilde{\tau} \in \text{Aut}(\Pi_n)$ of $\tau$ such that $\tau$ preserves the subset $E \subseteq \{1, \cdots, n\}$, and, moreover, the image of $T \subseteq \Pi_E$ by the automorphism of $\Pi_E$ determined by $\tilde{\tau} \in \text{Aut}(\Pi_n)$ coincides with $T' \subseteq \Pi_E$.

Lemma 3.8 (Strict tripods). Let $E \subseteq \{1, \cdots, n\}$ and $T \subseteq \Pi_E$ an $E$-tripod of $\Pi_n$ [cf. Definition 3.3, (i)] that arises as a vertical subgroup associated to a vertex $v \in \text{Vert}(G_{i \in E,x})$ for some $i \in \{1, \cdots, n\}$ [which thus implies that $T \subseteq \Pi_{E/(E \setminus \{i\})} \subseteq \Pi_E$]. Then the following hold:

(i) There exists a [not necessarily unique!] subset $E' \subseteq E$ such that $T$ is $E'$-strict [cf. Definition 3.3, (iii)]. In this situation, $i \in E'$, and, moreover, $p^n_{E/E'}: \Pi_E \twoheadrightarrow \Pi_{E'}$ induces an isomorphism $T \xrightarrow{\sim} T_{E'}$ onto an $E'$-tripod $T_{E'}$ of $\Pi_n$.

(ii) $T$ is $E$-strict if and only if one of the following conditions is satisfied:

(1) $E^i = 1$. 


(2c) \( E^2 = 2; T \subseteq \Pi_E \) is a vertical subgroup of \( \Pi_E \) associated to the vertex \( v_{i,j,x}^{\text{new}} \in \text{Vert}(G_{i \in E,x}) \) of Lemma 3.6, (iv), for some choice of \((i, j, x)\) such that \( z_{i,j,x} \in \text{Cusp}(G_{j \in E \setminus \{i\}, x}) \).

[In particular, \( T \) arises from \( z_{i,j,x} \in \text{Cusp}(G_{j \in E \setminus \{i\}, x}) \) — cf. Definition 3.7, (i).]

(2N) \( E^2 = 2; T \subseteq \Pi_E \) is a vertical subgroup of \( \Pi_E \) associated to the vertex \( v_{i,j,x}^{\text{new}} \in \text{Vert}(G_{i \in E,x}) \) of Lemma 3.6, (iv), for some choice of \((i, j, x)\) such that \( z_{i,j,x} \in \text{Node}(G_{j \in E \setminus \{i\}, x}) \).

[In particular, \( T \) arises from \( z_{i,j,x} \in \text{Node}(G_{j \in E \setminus \{i\}, x}) \) — cf. Definition 3.7, (i).]

(3) \( E^3 = 3, \) and \( T \) is central [cf. Definition 3.7, (ii)].

(iii) Suppose that \( T \) is trigonal [cf. Definition 3.3, (ii)]. Then there exists a [not necessarily unique!] subset \( E' \subseteq E \) such that \( (E')^3 = 3, \) and, moreover, the image of \( T \subseteq \Pi_E \) via \( p^E_{E/E'}: \Pi_E \rightarrow \Pi_{E'} \) is a central tripod.

Proof. Assertion (i) follows immediately from the various definitions involved by induction on \( E^3 \), together with the well-known elementary fact that any surjective endomorphism of a topologically finitely generated profinite group is necessarily bijective. Next, we verify assertion (ii). First, let us observe that sufficiency is immediate. Thus, it remains to verify necessity. Suppose that \( T \) is \( E\)-strict. Now one verifies easily that if there exists an element \( j \in E \setminus \{i\} \) such that \( c_{i,j,x}^{\text{diag}} \not\in \mathcal{C}(v) \) [cf. Lemma 3.6, (ii)], then it follows immediately that the image of \( T \subseteq \Pi_E \) via \( p^E_{E/(E \setminus \{j\})}: \Pi_E \rightarrow \Pi_{E \setminus \{j\}} \) is an \((E \setminus \{j\})\)-tripod [cf. also Lemma 3.6, (iii), (iv)]. Thus, since \( T \) is \( E\)-strict, we conclude that every cusp of \( G_{i \in E,x} \) that is \( \not\in \mathcal{C}(v) \) is non-diagonal. In particular, since \( v \) is of type \((0, 3)\), it follows immediately from Lemma 3.2, (ii), that \( 0 \leq E^3 - 1 \leq \mathcal{C}(v)^3 \leq 3 \). If \( \mathcal{C}(v)^3 = 0 \), then it follows from the inequality \( E^3 - 1 \leq \mathcal{C}(v)^2 \) that \( E^3 = 1 \), i.e., condition (1) is satisfied. If \( \mathcal{C}(v)^3 = 3 \), then one verifies easily that \( E^3 = 1 \), i.e., condition (1) is satisfied. Thus, it remains to verify assertion (ii) in the case where \( \mathcal{C}(v)^3 \in \{1, 2\} \).

Suppose that \( \mathcal{C}(v)^3 = 1 \) and \( E^3 \neq 1 \). Then it follows immediately from the inequality \( E^3 - 1 \leq \mathcal{C}(v)^2 \) that \( E^3 = 2 \). Now let us recall [cf. Lemma 3.2, (ii)] that the number of diagonal cusps of \( G_{i \in E,x} \) is \( = E^3 - 1 = 1 \). Moreover, the unique cusp on \( v \) is the unique diagonal cusp of \( G_{i \in E,x} \) [cf. the argument of the preceding paragraph]. Thus, one verifies easily that \( T \) satisfies condition (2N). Next, suppose that \( \mathcal{C}(v)^3 = 2 \) and \( E^3 \neq 1 \). Then it follows immediately from the inequality \( E^3 - 1 \leq \mathcal{C}(v)^2 \) that \( E^3 \in \{2, 3\} \). Now let us recall [cf. Lemma 3.2, (ii)] that if \( E^3 = 2 \) (respectively, \( E^3 = 3 \)), then the number of diagonal cusps of \( G_{i \in E,x} \) is \( = E^3 - 1 \), i.e., 1 (respectively, 2). Moreover, the set of diagonal cusp(s) of \( G_{i \in E,x} \) is contained in (respectively, is equal to)
\[c(v)\text{ [cf. the argument of the preceding paragraph]. Thus, one verifies easily that } T \text{ satisfies condition (2C) (respectively, (3)). This completes the proof of assertion (ii).}

Finally, we verify assertion (iii). It follows from assertion (i) that there exists a subset \( E' \subseteq E \) such that \( T \) is \( E' \)-strict. Moreover, it follows immediately from the definition of a trigonal tripod that the \( E' \)-tripod \( T \) satisfies any of conditions (1), (2C), (2N) of assertion (ii), then one verifies easily that \( p_{E/E'}^H(T) \) is not trigonal [cf. the final portion of Lemma 3.6, (iv)]. Thus, \( p_{E/E'}^H(T) \) satisfies condition (3) of assertion (ii); in particular, \( p_{E/E'}^H(T) \) is central. This completes the proof of assertion (iii). \( \Box \)

Lemma 3.9 (Generalities on normalizers and commensurators). Let \( G \) be a profinite group, \( N \subseteq G \) a normal closed subgroup of \( G \), and \( H \subseteq G \) a closed subgroup of \( G \). Then the following hold:

(i) It holds that \( C_G(H) \subseteq C_G(H \cap N) \).

(ii) It holds that \( C_G(H) \subseteq N_G(Z_{G}^{loc}(H)) \) [cf. the discussion entitled "Topological groups" in \( \S 0 \)].

(iii) Suppose that \( H \subseteq N \). Then it holds that \( C_G(H) \subseteq N_G(C_N(H)) \). In particular, if, moreover, \( H \) is commensurably terminal in \( N \), then it holds that \( C_G(H) = N_G(H) \).

(iv) Write \( \overline{H} \overset{\text{def}}{=} H/(H \cap N) \subseteq \overline{G} \overset{\text{def}}{=} G/N \). If \( H \cap N \) is commensurably terminal in \( N \), and the image of \( C_G(H) \subseteq G \) in \( \overline{G} \) is contained in \( N_{\overline{G}}(\overline{H}) \), then \( C_G(H) = N_G(H) \).

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). Let \( g \in C_G(H) \) and \( a \in Z_{G}^{loc}(H) \). Since \( Z_{G}^{loc}(H) = Z_{G}^{loc}(H \cap (g^{-1} \cdot H \cdot g)) = Z_{G}^{loc}(g^{-1} \cdot H \cdot g) \), there exists an open subgroup \( U \subseteq H \) of \( H \) such that \( a \in Z_G(g^{-1} \cdot U \cdot g) \). But this implies that \( gag^{-1} \in Z_G(U) \subseteq Z_{G}^{loc}(H) \). This completes the proof of assertion (ii). Next, we verify assertion (iii). Let \( g \in C_G(H) \) and \( a \in C_N(H) \). Since \( C_N(H) \subseteq C_G(H) \), we have \( C_N(H) = C_G(H \cap (g^{-1} \cdot H \cdot g)) \) = \( C_G(g^{-1} \cdot H \cdot g) \), we conclude that \( ag^{-1} \cdot H \cdot ga^{-1} \) is commensurate with \( g^{-1} \cdot H \cdot g \). In particular, \( gag^{-1} \cdot H \cdot ga^{-1}g^{-1} \) is commensurate with \( H \), i.e., \( gag^{-1} \in C_G(H) \cap N = C_N(H) \). This completes the proof of assertion (iii). Finally, we verify assertion (iv). First, we observe that since \( H \cap N \) is commensurably terminal in \( N \), one verifies easily that \( H = N_{H \cap N}(H \cap N) \). Let \( g \in C_G(H) \). Then since the image of \( C_G(H) \subseteq G \) in \( \overline{G} \) is contained in \( N_{\overline{G}}(\overline{H}) \), it is immediate that \( g \cdot H \cdot g^{-1} \subseteq H \cdot N \). On the other hand, again by applying the fact that \( H \cap N \) is commensurably terminal in \( N \), we conclude immediately from
assertions (i), (iii), that $C_G(H) \subseteq C_G(H \cap N) = N_G(H \cap N)$. Thus, we obtain that $(g \cdot H \cdot g^{-1}) \cap N = H \cap N$; in particular, $g \cdot H \cdot g^{-1} \subseteq N_{H \cap N}((g \cdot H \cdot g^{-1}) \cap N) = N_{H \cap N}(H \cap N) = H$, i.e., $g \in N_G(H)$. This completes the proof of assertion (iv).

□

Lemma 3.10 (Restrictions of outomorphisms). Let $G$ be a profinite group and $H \subseteq G$ a closed subgroup of $G$. Write $\text{Out}^H(G) \subseteq \text{Out}(G)$ for the group of outomorphisms of $G$ that preserve the $G$-conjugacy class of $H$. Suppose that the homomorphism $N_G(H) \to \text{Aut}(H)$ determined by conjugation factors through $\text{Inn}(H) \subseteq \text{Aut}(H)$. Then the following hold:

(i) For $\alpha \in \text{Out}^H(G)$, let us write $\alpha|_H$ for the outomorphism of $H$ determined by the restriction to $H \subseteq G$ of a lifting $\widetilde{\alpha} \in \text{Aut}(G)$ of $\alpha$ such that $\widetilde{\alpha}(H) = H$. Then $\alpha|_H$ does not depend on the choice of the lifting “$\widetilde{\alpha}$”, and the map

$$\text{Out}^H(G) \longrightarrow \text{Out}(H)$$

given by assigning $\alpha \mapsto \alpha|_H$ is a group homomorphism.

(ii) The homomorphism

$$\text{Out}^H(G) \longrightarrow \text{Out}(H)$$

of (i) depends only on the $G$-conjugacy class of the closed subgroup $H \subseteq G$, i.e., if we write $H^\gamma \overset{\text{def}}{=} \gamma \cdot H \cdot \gamma^{-1}$ for $\gamma \in G$, then the diagram

$$\begin{array}{ccc}
\text{Out}^H(G) & \longrightarrow & \text{Out}(H) \\
\| & & \downarrow \\
\text{Out}^{H^\gamma}(G) & \longrightarrow & \text{Out}(H^\gamma)
\end{array}$$

— where the upper (respectively, lower) horizontal arrow is the homomorphism given by mapping $\alpha \mapsto \alpha|_H$ (respectively, $\alpha \mapsto \alpha|_{H^\gamma}$), and the right-hand vertical arrow is the isomorphism obtained by conjugation via the isomorphism $H \overset{\sim}{\to} H^\gamma$ determined by conjugation by $\gamma \in G$ — commutes.

Proof. Assertion (i) follows immediately from our assumption that the homomorphism $N_G(H) \to \text{Aut}(H)$ determined by conjugation factors through $\text{Inn}(H) \subseteq \text{Aut}(H)$, together with the various definitions involved. Assertion (ii) follows immediately from the various definitions involved. This completes the proof of Lemma 3.10. □
Lemma 3.11 (Commensurator of a tripod arising from an edge). In the notation of Lemma 3.6, suppose that \((j, i) = (1, 2); E = \{i, j\} ; z_{i,j,x} \in \text{Edge}(\mathcal{G}_{i \in E \setminus \{i\}})\). Thus, \(\mathcal{G}_j \in E \setminus \{i\} = \mathcal{G}_i \in E \setminus \{j\} = \mathcal{G}\); \(\Pi_2 = \Pi_E; \Pi_1 = \Pi(\langle j \rangle) \twoheadrightarrow \Pi_{j \in E \setminus \{i, j\}} = \Pi_{G}; \Pi_{2/1} = \Pi_E / (E \setminus \{i\}) \twoheadrightarrow \Pi_{G \setminus E,x}\).

Write \(\mathcal{G}_{2/1} \triangleq \mathcal{G}_i \in E \setminus \{x\}; \mathcal{G}_{1/2} \triangleq \mathcal{G}_j \in E \setminus \{x\}; p_{2/1}^\Pi \triangleq p_{E / (E \setminus \{i\})}^\Pi; \Pi_2 \twoheadrightarrow \Pi(\{2\}); \Pi_{1/2} \triangleq \text{Ker}(p_{1/2}^\Pi) = \Pi_E / (\{2\}) \twoheadrightarrow \Pi_{G \setminus E,x}; z_x \triangleq z_{i,j,x} \in \text{Edge}(\mathcal{G}); c_{i,j,x}^{\text{diag}} \triangleq c_{i,j,x}^{\text{diag}} \in \text{Cusp}(\mathcal{G}_{2/1})\), \(\text{cf. Lemma 3.6, (ii)}\);

\(v_{\text{new}}^\text{def} \triangleq v_{2/1}^\text{new} = v_{i,j,x}^\text{new} \in \text{Vert}(\mathcal{G}_{2/1})\), \(\text{cf. Lemma 3.6, (iv)}\);

\(v_{1/2}^\text{new} \in \text{Vert}(\mathcal{G}_{1/2})\) for the vertex that corresponds to \(v_{\text{new}}^\text{def} \in \text{Vert}(\mathcal{G}_{2/1})\) via the natural bijection \(\text{Vert}(\mathcal{G}_{2/1}) \sim \text{Vert}(\mathcal{G}_{1/2})\) induced by the automorphism of \(X_E^\log\) determined by permuting the factors labeled \(i, j; Y \twoheadrightarrow X_E\) for the base-change — by the morphism \(X_E \to X_1 \times_k X_2 = X \times_k X\) determined by \(p_{E / (\{1\})}^\log\) and \(p_{E / (\{2\})}^\log\) — of the geometric point of \(X \times_k X\) determined by the geometric points \(x, x\) of \(X\) and \(x, x\) of \(X\) of Definition 3.1, (i) \(\text{i.e., as opposed to the geometric point of \(X_\{1\} \times_k X_\{2\} = X \times_k X\) determined by the geometric points \(x, x\) of \(X\) and \(x, x\) of \(X\); \(Y^\log\) for the log scheme obtained by equipping \(Y\) with the log structure induced by the log structure of \(X_E^\log; U \subseteq Y\) for the 2-interior of \(Y^\log\); \(U\) for the maximal pro-\(\Sigma\) quotient of the kernel of the natural surjection \(\pi_1(U^\log) \twoheadrightarrow \pi_1(\text{Spec }k^\log)\). Thus, one verifies easily that \(Y\) is isomorphic to \(\mathbb{P}^1_k\); that the complement \(Y \setminus U\) consists of three closed points of \(Y\); that the vertices \(v_{2/1}^\text{new}\) and \(v_{1/2}^\text{new}\) correspond to the closed irreducible subscheme \(Y \subseteq X_E\); and that the point corresponding to the cusp \(c_{i,j,x}^{\text{diag}}\) is contained in \(Y\) — \(\text{cf. Lemma 3.6, (iv)}\).

Let \(\Pi_{z_x} \subseteq \Pi_1\) be an edge-like subgroup associated to \(z_x \in \text{Edge}(\mathcal{G}); \Pi^{\text{diag}}_{z_x} \subseteq \Pi_{1 \setminus \{2\}} \cap \Pi_{1/2}^{\text{new}}\) a cuspidal subgroup associated to \(c_{i,j,x}^{\text{diag}}\); \(\Pi_{z_x}^{\text{new}} \subseteq \Pi_{2/1}\) a vertical subgroup associated to \(v_{\text{new}}^\text{new}\) that contains \(\Pi_{z_x}^{\text{diag}} \subseteq \Pi_2; \Pi_{z_x}^{\text{new}} \triangleq \Pi_{z_x}^{\text{new}}; \Pi_{z_x}^{\text{new}} \subseteq \Pi_{1 \setminus \{2\}} \twoheadrightarrow \text{a vertical subgroup associated to } v_{\text{new}}\) that contains \(\Pi_{z_x}^{\text{diag}} \subseteq \Pi_2\). Write \(\Pi_{1/2} z_x = \Pi_{2 \times_k \Pi_1 z_x} \subseteq \Pi_2; D_{c_{i,j,x}^{\text{diag}}} = \Pi_{z_x}^{\text{new}} \subseteq D_{\text{new}} z_x = \Pi_{z_x}^{\text{new}} \subseteq D_{\text{new}} z_x = \Pi_{z_x}^{\text{new}}\).

Then the following hold:

(i) It holds that \(D_{c_{i,j,x}^{\text{diag}}} \cap \Pi_{1 \setminus \{2\}} = D_{c_{i,j,x}^{\text{diag}}} \cap \Pi_{1 \setminus \{2\}} = \Pi_{\Pi z_x}^{\text{diag}} \cap \Pi_{1 \setminus \{2\}} = C_{\Pi_{z_x}^{\text{new}}} = D_{c_{i,j,x}^{\text{diag}}} \cap \Pi_{1 \setminus \{2\}} = D_{c_{i,j,x}^{\text{diag}}} \cap \Pi_{1 \setminus \{2\}} = D_{c_{i,j,x}^{\text{diag}}} \cap \Pi_{1 \setminus \{2\}}

(ii) It holds that \(C_{\Pi_{z_x}^{\text{new}}} = D_{c_{i,j,x}^{\text{diag}}} \cap \Pi_{1 \setminus \{2\}} = D_{c_{i,j,x}^{\text{diag}}} \cap \Pi_{1 \setminus \{2\}}

(iii) The surjections \(p_{2/1}^\Pi : \Pi_2 \twoheadrightarrow \Pi_1, p_{1/2}^\Pi : \Pi_2 \twoheadrightarrow \Pi_{1 \setminus \{2\}}\) determine isomorphisms \(D_{c_{i,j,x}^{\text{diag}}} / D_{c_{i,j,x}^{\text{diag}}} \sim \Pi_1, D_{c_{i,j,x}^{\text{diag}}} / D_{c_{i,j,x}^{\text{diag}}} \sim \Pi_{1 \setminus \{2\}}\), respectively, such that the resulting composite outer isomorphism \(\Pi_1 \sim \Pi_{1 \setminus \{2\}}\) is the identity outer isomorphism.
(iv) The natural inclusions $\Pi_{v,\text{new}}^i$, $I_{v,\text{new}}|_{z_x} \hookrightarrow D_{v,\text{new}}|_{z_x}$ determine an isomorphism $\Pi_{v,\text{new}} \times I_{v,\text{new}}|_{z_x} \xrightarrow{\sim} D_{v,\text{new}}|_{z_x} = C_{\Pi_2/1}(\Pi_{v,\text{new}})$. Moreover, the composite $I_{v,\text{new}}|_{z_x} \hookrightarrow D_{v,\text{new}}|_{z_x} \rightarrow \Pi_{z_x}$ is an isomorphism.

(v) It holds that $C_{\Pi_2}(D_{v,\text{new}}|_{z_x}) \subseteq C_{\Pi_2}(\Pi_{v,\text{new}})$.

(vi) $D_{v,\text{new}}|_{z_x}$ is commensurably terminal in $\Pi_2$, i.e., it holds that $D_{v,\text{new}}|_{z_x} = C_{\Pi_2}(D_{v,\text{new}}|_{z_x})$.

(vii) It holds that $Z_{\Pi_2}(\Pi_{v,\text{new}}) = Z_{\Pi_2}^{\text{log}}(\Pi_{v,\text{new}}) = I_{v,\text{new}}|_{z_x}$. Moreover, these profinite groups are isomorphic to $\hat{\mathbb{Z}}^2$ [cf. the discussion entitled “Numbers” in [CbTpI], §8].

(viii) It holds that $C_{\Pi_2}(\Pi_{v,\text{new}}) = D_{v,\text{new}}|_{z_x} = \Pi_{v,\text{new}} \times Z_{\Pi_2}(\Pi_{v,\text{new}})$. In particular, the equality $C_{\Pi_2}(\Pi_{v,\text{new}}) = N_{\Pi_2}(\Pi_{v,\text{new}})$ holds.

(ix) It holds that $Z(C_{\Pi_2}(\Pi_{v,\text{new}})) = Z_{\Pi_2}(\Pi_{v,\text{new}})$.

(x) It holds that

$$C_{\Pi_2}(\Pi_{v,\text{new}})_{1/2} \cap \Pi_{2/1} = \Pi_{v,\text{new}}_{1/2}, \quad C_{\Pi_2}(\Pi_{v,\text{new}}) \cap \Pi_{1/2} = \Pi_{v,\text{new}}_{1/2},$$

$$C_{\Pi_2}(\Pi_{v,\text{new}})_{1/2} = C_{\Pi_2}(\Pi_{v,\text{new}}_{1/2}).$$

Moreover, for suitable choices of basepoints of the log schemes $U^{\log}$ and $X_E^{\log}$, the natural morphism $U^{\log} \rightarrow X_E^{\log}$ induces an isomorphism $\Pi_{U} \xrightarrow{\sim} C_{\Pi_2}(\Pi_{v,\text{new}}) = C_{\Pi_2}(\Pi_{v,\text{new}}_{1/2})$.

**Proof.** First, we verify assertion (i). Now it is immediate that we have inclusions $\Pi_{\text{diag}} \subseteq D_{\text{diag}} \subseteq C_{\Pi_2}(\Pi_{v,\text{diag}})$. In particular, since $\Pi_{\text{diag}}$ is commensurably terminal in $\Pi_{2/1}$ and $\Pi_{1/2}$ [cf. [CmbGC] Proposition 1.2, (ii)], we obtain that $\Pi_{\text{diag}} \subseteq D_{\text{diag}} \cap \Pi_{2/1} \subseteq C_{\Pi_2}(\Pi_{v,\text{diag}}) \cap \Pi_{2/1} = C_{\Pi_{2/1}}(\Pi_{v,\text{diag}}) = \Pi_{\text{diag}}$; $\Pi_{\text{diag}} \subseteq D_{\text{diag}} \cap \Pi_{1/2} \subseteq C_{\Pi_2}(\Pi_{v,\text{diag}}) \cap \Pi_{1/2} = C_{\Pi_{1/2}}(\Pi_{v,\text{diag}}) = \Pi_{\text{diag}}$. This completes the proof of assertion (i). Assertions (ii), (iii) follow immediately from assertion (i), together with the [easily verified] fact that the composites $D_{\text{diag}} \hookrightarrow \Pi_2 \twoheadrightarrow \Pi_1$ and $D_{\text{diag}} \hookrightarrow \Pi_2 \twoheadrightarrow \Pi_{\{2\}}$ are surjective.

Next, we verify assertion (iv). It follows immediately from the various definitions involved — by considering a suitable stable log curve of type $(g, r)$ over $(\text{Spec} k)^{\text{log}}$ and applying a suitable specialization isomorphism [cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1] — that, to verify assertion (iv), we may assume without loss of generality that $\text{Cusp}(\mathcal{G}) \cup \{z_x\} = \text{Edge}(\mathcal{G})$. Then, in light of the well-known local structure of $X^{\log}$ in a neighborhood of the node or cusp corresponding to $z_x$, one verifies easily that the outer action $\Pi_{z_x} \rightarrow \text{Out}(\Pi_{2/1}) \xrightarrow{\sim} \text{Out}(\Pi_{\{2\}})$ arising from the
natural exact sequence

\[ 1 \rightarrow \Pi_{2/1} \rightarrow \Pi_{2}|_{z_x} \rightarrow \Pi_{z_x} \rightarrow 1 \]

is of \textit{SNN-type} [cf. [NodNon], Definition 2.4, (iii)], hence, in particular, that the composite \( I_{\nu_{\text{new}}}|_{z_x} \hookrightarrow D_{\nu_{\text{new}}}|_{z_x} \rightarrow \Pi_{z_x} \) is an \textit{isomorphism}. Thus, assertion (iv) follows immediately from Lemma 3.6, (ii), together with the \textit{commensurable terminality} of \( \Pi_{\nu_{\text{new}}} \) in \( \Pi_{2/1} \) [cf. [CmbGC], Proposition 1.2, (ii)] and the fact that the composite \( D_{\nu_{\text{new}}}|_{z_x} \hookrightarrow \Pi_{2}|_{z_x} \rightarrow \Pi_{z_x} \) is \textit{surjective}. This completes the proof of assertion (iv).

Next, we verify assertion (v). It follows immediately from assertion (iv), together with the \textit{commensurable terminality} of \( \Pi_{\nu_{\text{new}}} \) in \( \Pi_{2/1} \) [cf. [CmbGC], Proposition 1.2, (ii)], that \( D_{\nu_{\text{new}}}|_{z_x} \cap \Pi_{2/1} = \Pi_{\nu_{\text{new}}} \). Thus, since \( \Pi_{2/1} \) is \textit{normal} in \( \Pi_{2} \), assertion (v) follows immediately from Lemma 3.9, (i). This completes the proof of assertion (v).

Next, we verify assertion (vi). Since the image of the composite

\[ D_{\nu_{\text{new}}}|_{z_x} \hookrightarrow \Pi_{2} \rightarrow \Pi_{1} \] coincides with \( \Pi_{z_x} \subseteq \Pi_{1} \) [cf. assertion (iv)], and \( \Pi_{z_x} \subseteq \Pi_{1} \) is \textit{commensurably terminal} in \( \Pi_{1} \) [cf. [CmbGC], Proposition 1.2, (ii)], it follows immediately that \( C_{\Pi_{2}}(D_{\nu_{\text{new}}}|_{z_x}) \subseteq \Pi_{2}|_{z_x} \). In particular, it follows immediately from assertions (iv), (v) that \( D_{\nu_{\text{new}}}|_{z_x} \subseteq C_{\Pi_{2}}(D_{\nu_{\text{new}}}|_{z_x}) \subseteq C_{\Pi_{2}}(\Pi_{\nu_{\text{new}}}) \cap \Pi_{2}|_{z_x} = C_{\Pi_{2}|_{z_x}}(\Pi_{\nu_{\text{new}}}) = D_{\nu_{\text{new}}}|_{z_x} \). This completes the proof of assertion (vi).

Next, we verify assertion (vii). It follows from assertion (iv) and [CmbGC], Remark 1.1.3, that \( I_{\nu_{\text{new}}}|_{z_x} \) is \textit{isomorphic} to \( \hat{\mathbb{Z}}^{\Sigma} \). Moreover, it follows from the various definitions involved that we have inclusions \( I_{\nu_{\text{new}}}|_{z_x} \subseteq Z_{\Pi_{2}}(\Pi_{\nu_{\text{new}}}) \subseteq Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\nu_{\text{new}}}) \). Thus, to verify assertion (vii), it suffices to verify that \( Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\nu_{\text{new}}}) \subseteq I_{\nu_{\text{new}}}|_{z_x} \). To this end, let us observe that it follows immediately from the final portion of Lemma 3.6, (iv), that the image \( p_{1/2}^{\Pi}(\Pi_{\nu_{\text{new}}}) \subseteq \Pi_{2} \cong \Pi_{G} \) is an edge-like subgroup of \( \Pi_{2} \cong \Pi_{G} \) associated to \( z_{x} \in \text{Edge}(G) \). Thus, since every edge-like subgroup is \textit{commensurably terminal} [cf. [CmbGC], Proposition 1.2, (ii)], it follows that the image \( p_{1/2}^{\Pi}(Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\nu_{\text{new}}})) \subseteq \Pi_{2} \cong \Pi_{G} \) is \textit{contained} in an edge-like subgroup of \( \Pi_{2} \cong \Pi_{G} \) associated to \( z_{x} \in \text{Edge}(G) \). On the other hand, since \( \Pi_{\text{diag}} \subseteq \Pi_{\nu_{\text{new}}} \), we have \( Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\nu_{\text{new}}}) \subseteq Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\text{diag}}) \subseteq C_{\Pi_{2}}(\Pi_{\text{diag}}) = D_{\text{diag}} \) [cf. assertion (ii)]. In particular, it follows immediately from assertion (iii), together with the fact [cf. assertion (iv)] that \( I_{\nu_{\text{new}}}|_{z_x} \subseteq Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\nu_{\text{new}}}) \) \textit{surjects} onto \( \Pi_{z_x} \) [cf. also [NodNon], Lemma 1.5], that \( p_{2/1}^{\Pi}(Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\nu_{\text{new}}})) \subseteq \Pi_{1} \) is \textit{contained} in \( \Pi_{z_x} \subseteq \Pi_{1} \), i.e., \( Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\nu_{\text{new}}}) \subseteq \Pi_{2}|_{z_x} \). Thus, it follows immediately from assertion (iv), together with the \textit{slimness} of \( \Pi_{\nu_{\text{new}}} \) [cf. [CmbGC], Remark 1.1.3], that \( Z_{\Pi_{2}}^{\text{loc}}(\Pi_{\nu_{\text{new}}}) \subseteq I_{\nu_{\text{new}}}|_{z_x} \). This completes the proof of assertion (vii).
Next, we verify assertion (viii). It follows from assertion (vii), together with Lemma 3.9, (ii), that
\[ C_{\Pi_2} \left( \Pi_{v_{\text{new}}} \right) \subseteq N_{\Pi_2} \left( I_{v_{\text{new}}^2} \right). \]
In particular, since \( D_{v_{\text{new}}} \) is generated by \( \Pi_{v_{\text{new}}} \), \( I_{v_{\text{new}}^2} \) [cf. assertion (iv)], it follows immediately that \( \left( D_{v_{\text{new}}} \right) \subseteq C_{\Pi_2} \left( \Pi_{v_{\text{new}}} \right) \subseteq C_{\Pi_2} \left( D_{v_{\text{new}}} \right) \).
Thus, the first equality of assertion (viii) follows from assertion (vi); the second equality of assertion (viii) follows immediately from assertions (iv), (vii). This completes the proof of assertion (viii).

Next, we verify assertion (ix). Let us recall from [CmbGC], Remark 1.1.3, that \( \Pi_{v_{\text{new}}} \) is slim. Thus, assertion (ix) follows from assertion (viii), together with the final portion of assertion (vii). This completes the proof of assertion (ix).

Finally, we verify assertion (x). The first two equalities follow from [CmbGC], Proposition 1.2, (ii). Next, let us observe that since it is immediate that the automorphism of \( X_E^\log \) determined by permuting the factors labeled \( i, j \) stabilizes \( U \), but permutes \( v_{1/2}^{\text{new}} \) and \( v_{1/2}^{\text{new}} \), one verifies immediately that, to verify assertion (x), it suffices to verify that, for suitable choices of basepoints of the log schemes \( U^\log \) and \( X_E^\log \), the natural morphism \( U^\log \to X_E^\log \) induces an isomorphism \( \Pi_U \sim C_{\Pi_2} \left( \Pi_{v_{\text{new}}} \right) \) (\( = C_{\Pi_2} \left( \Pi_{v_{\text{new}}} \right) \)). To this end, let us observe that since the vertex \( v_{\text{new}} \) corresponds to the closed irreducible subscheme \( Y \subseteq X_E \) [cf. the discussion following the definition of \( \Pi_U \) in the statement of Lemma 3.11], it follows immediately from the various definitions involved that, for suitable choices of basepoints of the log schemes \( U^\log \) and \( X_E^\log \), the natural morphism \( U^\log \to X_E^\log \) gives rise to a commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \Pi_{U/z} & \longrightarrow & \Pi_U & \longrightarrow & \Pi_{zx} & \longrightarrow & 1 \\
| & \downarrow & \downarrow & \downarrow & | & | & | & | \\
1 & \longrightarrow & \Pi_{v_{\text{new}}} & \longrightarrow & D_{v_{\text{new}}} \mid_{z_x} & \longrightarrow & \Pi_{z_x} & \longrightarrow & 1
\end{array}
\]

— where we write \( \Pi_{U/z} \) for the kernel of the natural surjection \( \Pi_U \to \Pi_{z_x} \); the horizontal sequences are exact; the exactness of the lower horizontal sequence follows from assertion (iv); the left-hand vertical arrow is an isomorphism. Thus, it follows from assertion (viii) that, for suitable choices of basepoints of the log schemes \( U^\log \) and \( X_E^\log \), the natural morphism \( U^\log \to X_E^\log \) induces an isomorphism \( \Pi_U \sim D_{v_{\text{new}}} \mid_{z_x} = C_{\Pi_2} \left( \Pi_{v_{\text{new}}} \right) \), as desired. This completes the proof of assertion (x), hence also of Lemma 3.11. \( \square \)

The first item of the following result [i.e., Lemma 3.12, (i)] is, along with its proof, a routine generalization of [CmbCsp], Corollary 1.10, (ii).
Lemma 3.12 (Commensurator of a tripod). Let $E \subseteq \{1, \cdots, n\}$ and $T \subseteq \Pi_E$ an $E$-tripod of $\Pi_n$ [cf. Definition 3.3, (i)]. Then the following hold:

(i) It holds that $C_{\Pi_E}(T) = T \times Z_{\Pi_E}(T)$. Thus, if an automorphism $\alpha$ of $\Pi_E$ preserves the $\Pi_E$-conjugacy class of $T$, then one may define $\alpha|_T \in \text{Out}(T)$ [cf. Lemma 3.10, (i)].

(ii) Suppose that $n = \sharp E = 3$, and that $T$ is central [cf. Definition 3.7, (ii)]. Let $T' \subseteq \Pi_E = \Pi_n$ be a central $E$-tripod of $\Pi_n$. Then $C_{\Pi_n}(T)$ (respectively, $N_{\Pi_n}(T); Z_{\Pi_n}(T)$) is a $\Pi_n$-conjugate of $C_{\Pi_n}(T')$ (respectively, $N_{\Pi_n}(T'); Z_{\Pi_n}(T')$).

Proof. Let $i \in E; x \in X_n(k); v \in \text{Vert}(G_{i \in E,x})$ be such that $v$ is of type $(0, 3)$, and, moreover, $T$ is a vertical subgroup of $\Pi_E$ associated to $v \in \text{Vert}(G_{i \in E,x})$. Thus, we have an inclusion $T \subseteq \Pi_E((E \setminus \{i\}) \subseteq \Pi_E$ — cf. Definition 3.1, (iv).

First, we verify assertion (i). Since $T \subseteq \Pi_E((E \setminus \{i\}) \subseteq \Pi_E$, and $T$ is commensurably terminal in $\Pi_E((E \setminus \{i\})$ [cf. [CmbGC], Proposition 1.2, (ii)], it follows from Lemma 3.9, (iii), that $C_{\Pi_E}(T) = N_{\Pi_E}(T)$. Thus, in light of the slimness of $T$ [cf. [CmbGC], Remark 1.1.3], to verify assertion (i), it suffices to verify that the natural outer action of $N_{\Pi_E}(T)$ on $T$ is trivial. To this end, let $E' \subseteq E$ be such that $T$ is $E'$-strict [cf. Lemma 3.8, (i)]; write $T_{E'} \subseteq \Pi_{E'}$ for the image of $T$ via $p_{E/E'}^{\Pi}: \Pi_E \rightarrow \Pi_{E'}$. Then it is immediate that the image of $N_{\Pi_E}(T)$ via $p_{E/E'}^{\Pi}: \Pi_E \rightarrow \Pi_{E'}$ is contained in $N_{\Pi_{E'}}(T_{E'})$, and that the natural surjection $T \rightarrow T_{E'}$ is an isomorphism [cf. Lemma 3.8, (i)]. Thus, one verifies easily — by replacing $E, T$ by $E', T_{E'},$ respectively — that, to verify that the natural outer action of $N_{\Pi_E}(T)$ on $T$ is trivial, we may assume without loss of generality that $T$ is $E'$-strict. If $T$ satisfies condition (1) of Lemma 3.8, (ii), then assertion (i) follows from the commensurable terminality of $T$ in $\Pi_E$ [cf. [CmbGC], Proposition 1.2, (ii)]. If $T$ satisfies either condition (2c) or condition (2N) of Lemma 3.8, (ii), then assertion (i) follows immediately from Lemma 3.11, (viii). If $T$ satisfies condition (3) of Lemma 3.8, (ii), then one verifies easily from the various definitions involved — by considering a suitable stable log curve of type $(g, r)$ over $(\text{Spec } k)^{\text{log}}$ and applying a suitable specialization isomorphism [cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbT], Remark 5.6.1] — that, to verify assertion (i), we may assume without loss of generality that $\text{Node}(G) = \emptyset$. Thus, assertion (i) follows immediately from [CmbCsp], Corollary 1.10, (ii). This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us recall from Remark 3.7, (iii), that there exist an element $\tau \in \mathfrak{S}_3 \subseteq \text{Out}(\Pi_3)$ [cf. the discussion at the beginning of the present §3] and a lifting $\tilde{\tau} \in \text{Aut}(\Pi_3)$ of $\tau$ such that the image of $T \subseteq \Pi_3$ by the automorphism $\tilde{\tau} \in \text{Aut}(\Pi_3)$
coincides with $T' \subseteq \Pi_3$. Next, let us observe that one verifies easily that $\tau \in \mathcal{S}_3$ may be written as a product of transpositions in $\mathcal{S}_3$. Thus, in the remainder of the proof of assertion (ii), we may assume without loss of generality that $\tau$ is a transposition in $\mathcal{S}_3$. Moreover, in the remainder of the proof of assertion (ii), we may assume without loss of generality, by conjugating by a suitable element of $\mathcal{S}_3$, that $\tau$ is the transposition \(\langle 1, 2 \rangle\) in $\mathcal{S}_3$. Thus, if, moreover, $i = 3$ [i.e., the $E$-tripod $T$ is 3-central], then it follows from Lemma 3.6, (v), that $T$ is a $\Pi_3$-conjugate of $T'$, hence that $C_{\Pi_3}(T)$ (respectively, $N_{\Pi_3}(T)$; $Z_{\Pi_3}(T)$) is a $\Pi_3$-conjugate of $C_{\Pi_3}(T')$ (respectively, $N_{\Pi_3}(T')$; $Z_{\Pi_3}(T')$). In particular, in the remainder of the proof of assertion (ii), we may assume without loss of generality, by conjugating by $\tau \in \mathcal{S}_3$ if necessary, that $i = 2$, i.e., that the $E$-tripods $T$, $T'$ are 2-central, 1-central, respectively.

Next, let us observe that, in this situation, one verifies immediately from the various definitions involved that there exists a natural identification between $\Pi_{\langle 1, 2, 3 \rangle}/\{3\}$ and the “$\Pi_2$” that arises in the case where we take \(X^{\log}\) to be the base-change of \(p^{\log}_{\{3\}}: X^{\log}_{\langle 2, 3 \rangle} \to X^{\log}_{\{3\}}\) via a suitable morphism of log schemes \((\text{Spec } k)^{\log} \to X^{\log}_{\{3\}}\). Moreover, one also verifies immediately from the various definitions involved [cf. also Lemma 3.6, (v)] that this natural identification maps suitable $\Pi_3$-conjugates of $T$, $T'$, respectively, bijectively onto the closed subgroups “$\Pi_{\langle 1, 2, 3 \rangle}/\{3\}$” “$\Pi_{\langle 1, 2 \rangle}/\{2, 3\}$” of the “$\Pi_2$” that appears in the statement of Lemma 3.11. In particular, it follows from Lemma 3.11, (viii), (ix), (x), that the following assertions hold:

(a) The following equalities hold:

$$C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T) = T \times Z_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T),$$
$$C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T') = T' \times Z_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T').$$

(b) The following equalities hold:

$$C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T) \cap \Pi_{\{1, 2, 3 \rangle}/\{1, 3\}} = T,$$
$$C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T') \cap \Pi_{\{1, 2, 3 \rangle}/\{2, 3\}} = T'.$$

(c) The subgroup $C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T)$ (respectively, $Z_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T)$) is a $\Pi_{\{1, 2, 3 \rangle}/\{3\}$-conjugate of the subgroup $C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T')$ (respectively, $Z_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T')$).

In particular, it follows from (c) that, to verify assertion (ii), it suffices to verify the following assertion:

Claim 3.12.A: The following equalities hold:

$$C_{\Pi_3}(T) = C_{\Pi_3}(C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T)),$$
$$C_{\Pi_3}(T') = C_{\Pi_3}(C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T')),$$
$$N_{\Pi_3}(T) = N_{\Pi_3}(C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T)),$$
$$N_{\Pi_3}(T') = N_{\Pi_3}(C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T')),$$
$$Z_{\Pi_3}(T) = Z_{\Pi_3}(C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T)),$$
$$Z_{\Pi_3}(T') = Z_{\Pi_3}(C_{\Pi_{\langle 1, 2, 3 \rangle}/\{3\}}(T')).$$
First, we verify the first four equalities of Claim 3.12.A. Observe that since $\Pi_{(1,2,3)/3}$ is a normal closed subgroup of $\Pi_3$ and contains both $T$ and $T'$, it follows from Lemma 3.9, (iii), that the inclusions

$$N_{\Pi_3}(T) \subseteq C_{\Pi_3}(T) \subseteq N_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T)) \subseteq C_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T)),$$

$$N_{\Pi_3}(T') \subseteq C_{\Pi_3}(T') \subseteq N_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T')) \subseteq C_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T'))$$

hold. Moreover, by the normality of $\Pi_{(1,2,3)/1,3}$ and $\Pi_{(1,2,3)/2,3}$ in $\Pi_3$, one verifies easily, by applying (b), that the inclusions

$$N_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T)) \subseteq N_{\Pi_3}(T), \quad C_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T)) \subseteq C_{\Pi_3}(T),$$

$$N_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T')) \subseteq N_{\Pi_3}(T'), \quad C_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T')) \subseteq C_{\Pi_3}(T')$$

hold. This completes the proof of the first four equalities of Claim 3.12.A.

Finally, we verify the final two equalities of Claim 3.12.A. Let us first observe that the inclusions $T \subseteq C_{\Pi_{(1,2,3)/3}}(T), \quad T' \subseteq C_{\Pi_{(1,2,3)/3}}(T')$ imply that

$$Z_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T)) \subseteq Z_{\Pi_3}(T), \quad Z_{\Pi_3}(C_{\Pi_{(1,2,3)/3}}(T')) \subseteq Z_{\Pi_3}(T').$$

Thus, it follows immediately from (a) that, to verify the final two equalities of Claim 3.12.A, it suffices to verify the following assertion:

Claim 3.12.B: The following inclusions hold:

$$Z_{\Pi_3}(T) \subseteq Z_{\Pi_3}(Z_{\Pi_{(1,2,3)/3}}(T)), \quad Z_{\Pi_3}(T') \subseteq Z_{\Pi_3}(Z_{\Pi_{(1,2,3)/3}}(T')).$$

First, let us observe that one verifies immediately from the various definitions involved that the natural identification that appears in the discussion preceding assertion (a) in the present proof of Lemma 3.12, (ii), determines a natural identification between $\Pi_{(2,3)/3}$ and the “$\Pi_1 = \Pi_{(2)}$” that arises in the case where we take “$X^{\log}$” to be as in the discussion preceding assertion (a) in the present proof of Lemma 3.12, (ii). Thus, it follows immediately from the final portion of Lemma 3.6, (iv), that the image $J_T \subseteq \Pi_{(2,3)/3}$ of $T \subseteq \Pi_{(1,2,3)/3}$ in $\Pi_{(2,3)/3}$ corresponds, via the natural identification just discussed, to an edge-like subgroup of “$\Pi_1 = \Pi_{(2)}$” associated to the edge $z_x \in \text{Edge}(G)$ that appears in the statement of Lemma 3.11. Moreover, it follows immediately from (c) and Lemma 3.11, (iv), (vii), that the surjection $\Pi_{(1,2,3)/3} \rightarrow \Pi_{(2,3)/3}$ induces an isomorphism

$$\Pi_{(1,2,3)/3} \cong Z_{\Pi_{(1,2,3)/3}}(T) \rightarrow J_Z \subseteq \Pi_{(2,3)/3}$$

— where the closed subgroup $J_Z \subseteq \Pi_{(2,3)/3}$ corresponds, via the natural identification just discussed, to an edge-like subgroup of “$\Pi_1 = \Pi_{(2)}$” associated to the edge $z_x \in \text{Edge}(G)$ that appears in the statement of Lemma 3.11. Thus, we conclude immediately from [CmbGC], Proposition 1.2, (ii), together with the various definitions involved, that $J_T = J_Z$ ($\rightarrow Z_{\Pi_{(1,2,3)/3}}(T)$). In particular, since $Z_{\Pi_3}(T) \subseteq$
Lemma 3.13 (Preservation of vertical subgroups). In the notation of Lemma 3.11, let $\tilde{\alpha}$ be an $F$-admissible automorphism of $\Pi_E = \Pi_2$, $v \in \text{Vert}(\mathcal{G})$. Write $v^o \in \text{Vert}(\mathcal{G}_{2/1})$ for the vertex of $\mathcal{G}_{2/1}$ that corresponds to $v \in \text{Vert}(\mathcal{G})$ via the bijection of Lemma 3.6, (iv); $\tilde{\alpha}_1$, $\tilde{\alpha}_{2/1}$ for the automorphisms of $\Pi_1$, $\Pi_{2/1}$ determined by $\tilde{\alpha}$; $\alpha_1$, $\alpha_{2/1}$ for the automorphisms of $\tilde{\Pi}_2$, $\tilde{\Pi}_1$, $\tilde{\Pi}_{2/1}$ determined by $\tilde{\alpha}$, $\tilde{\alpha}_1$, $\tilde{\alpha}_{2/1}$, respectively. Then the following hold:

(i) Recall the edge-like subgroup $\Pi_{z_x} \subseteq \Pi_1 \rightarrow \Pi_G$ associated to the edge $z_x \in \text{Edge}(\mathcal{G})$. Suppose that

$$\tilde{\alpha}_1(\Pi_{z_x}) = \Pi_{z_x}.$$ 

Suppose, moreover, either that

(a) the automorphism $\alpha_{2/1}$ of $\Pi_{G_{2/1}} \overset{\sim}{\rightarrow} \Pi_{2/1}$ maps some cuspidal inertia subgroup of $\Pi_{G_{2/1}} \overset{\sim}{\rightarrow} \Pi_{2/1}$ to a cuspidal inertia subgroup of $\Pi_{G_{2/1}} \overset{\sim}{\rightarrow} \Pi_{2/1}$, or that

(b) $z_x \in \text{Cusp}(\mathcal{G})$.

[For example, condition (a) holds if the automorphism $\alpha_{2/1}$ of $\Pi_{G_{2/1}} \overset{\sim}{\rightarrow} \Pi_{2/1}$ is group-theoretically cuspidal — cf. [CmbGC], Definition 1.4, (iv).] Then $\alpha_{2/1}$ preserves the $\Pi_{2/1}$-conjugacy class of the vertical subgroup $\Pi_{v^{\text{new}}} \subseteq \Pi_{2/1} \overset{\sim}{\rightarrow} \Pi_{G_{2/1}}$ associated to the vertex $v^{\text{new}} \in \text{Vert}(\mathcal{G}_{2/1})$. If, moreover, $\alpha_{2/1}$ is group-theoretically cuspidal, then the induced automorphism of $\Pi_{v^{\text{new}}}$ [cf. Lemma 3.12, (i)] is itself group-theoretically cuspidal.

(ii) In the situation of (i), suppose, moreover, that there exists a vertical subgroup $\Pi_v \subseteq \Pi_G \overset{\sim}{\rightarrow} \Pi_1$ of $\Pi_G \overset{\sim}{\rightarrow} \Pi_1$ associated to $v \in \text{Vert}(\mathcal{G})$ such that $\tilde{\alpha}_1$ preserves the $\Pi_1$-conjugacy class of $\Pi_v$. Then $\alpha_{2/1}$ preserves the $\Pi_{2/1}$-conjugacy class of a vertical subgroup of $\Pi_{G_{2/1}} \overset{\sim}{\rightarrow} \Pi_{2/1}$ associated to the vertex $v^o \in \text{Vert}(\mathcal{G}_{2/1})$.

(iii) In the situation of (i), suppose, moreover, that $X^{\log}$ is of type $(0, 3)$ [which implies that $\Pi_v \overset{\text{def}}{=} \Pi_G \overset{\sim}{\rightarrow} \Pi_1$ is the unique vertical subgroup of $\Pi_G$ associated to $v$], and that $\alpha_1 \in \text{Out}^{\log}(\Pi_v)^{\text{cusp}}$.
sequence of profinite groups where the second arrow is the outer action determined by the exact
condition (b)) from Theorem 1.9, (i) (respectively, Theorem 1.9, (ii)),
statement of assertion (i) in the case of condition (a) (respectively,
it follows immediately [in light of the various assumptions made in the
SNN-type [CbTpI], Definition 2.10] is of
(iii) (respectively,
from [CbTpI], Proposition 2.9,
theoretically vertical
the following easily verified fact: a vertex of \((G_{2/1})\) is a vertical subgroup of \(\Pi_{2/1}\), to an abelian
subgroup of \(\Pi_{2/1}\).]
If, moreover, \(\alpha_{2/1}\) is group-theoretically cuspidal, then the group-theoretic cuspidality of the resulting automorphism of

Proof. First, we verify assertions (i), (ii). Write \(S \overset{\text{def}}{=} \text{Node}(G_{2/1}) \setminus \mathcal{N}(v_{\text{new}})\). Then it follows immediately from the well-known local structure of \(X^\log\) in a neighborhood of the edge corresponding to \(z_x\) that if \(z_x \in \text{Node}(G)\) (respectively, \(z_x \in \text{Cusp}(G)\)), then the outer action of \(\Pi_{z_x}\) on \(\Pi_{(G_{2/1})}\) [cf. [CbTpI], Definition 2.8] obtained by conjugating
the natural outer action \(\Pi_{z_x} \hookrightarrow \Pi_1 \rightarrow \text{Out}(\Pi_{2/1}) \sim \text{Out}(\Pi_{(G_{2/1})})\) —
where the second arrow is the outer action determined by the exact sequence of profinite groups

\[1 \longrightarrow \Pi_{2/1} \longrightarrow \Pi_{2} \overset{p_{2/1}}{\longrightarrow} \Pi_1 \longrightarrow 1\]

— by the natural outer isomorphism \(\Phi_{(G_{2/1})\rightarrow S}: \Pi_{(G_{2/1})\rightarrow S} \sim \Pi_{G_{2/1}}\) [cf. [CbTpI], Definition 2.10] is of SNN-type [cf. [NodNon], Definition 2.4, (iii)] (respectively, IPSC-type [cf. [NodNon], Definition 2.4, (i)]). Thus, it follows immediately [in light of the various assumptions made in the statement of assertion (i)!!] in the case of condition (a) (respectively, condition (b)) from Theorem 1.9, (i) (respectively, Theorem 1.9, (ii)), that the automorphism \(\alpha_{(G_{2/1})\rightarrow S}\) of \(\Pi_{(G_{2/1})\rightarrow S}\) obtained by conjugating \(\alpha_{2/1}\) by the composite \(\Pi_{2/1} \sim \Pi_{G_{2/1}} \overset{\sim}{\leftarrow} \Pi_{(G_{2/1})\rightarrow S}\) is group-theoretically vertical [cf. [CmbGC], Definition 1.4, (iv)] and group-theoretically nodal [cf. [NodNon], Definition 1.12]. On the other hand, it follows immediately from condition (3) of [CbTpI], Proposition 2.9, (i), that the image via \(\Phi_{(G_{2/1})\rightarrow S}: \Pi_{(G_{2/1})\rightarrow S} \sim \Pi_{G_{2/1}}\) of any vertical subgroup of \(\Pi_{(G_{2/1})\rightarrow S}\) associated to the vertex of \((G_{2/1})\) corresponding to \(v_{\text{new}}\) is a vertical subgroup of \(\Pi_{G_{2/1}}\) associated to \(v_{\text{new}}\). Thus, since \(\alpha_{(G_{2/1})\rightarrow S}\) is group-theoretically vertical, it follows immediately that \(\alpha_{2/1}\) preserves the \(\Pi_{2/1}\)-conjugacy class of the vertical subgroup \(\Pi_{v_{\text{new}}} \subseteq \Pi_{2/1} \sim \Pi_{G_{2/1}}\) associated to \(v_{\text{new}}\). [Here, we observe in passing the following easily verified fact: a vertex of \((G_{2/1})\rightarrow S\) corresponds to \(v_{\text{new}}\) if and only if the vertical subgroup of \(\Pi_{(G_{2/1})\rightarrow S}\) associated to this vertex maps, via the composite \(\Pi_{(G_{2/1})\rightarrow S} \sim \Pi_{G_{2/1}} \overset{p_{2/1}\Pi}{\longrightarrow} \Pi_{2},\) to an abelian subgroup of \(\Pi_{2}\).]
follows immediately from the group-theoretic cuspidality of $\alpha_{2/1}$ and the \textit{group-theoretic nodality} of $\alpha_{(G_{2/1})\to S}$. This completes the proof of assertion (i).

To verify assertion (ii), let us first observe that it follows immediately from \[\text{[CbTpI]}, \text{Theorem A, (i)}, \text{that — after possibly replacing $\tilde{\alpha}$ by the composite of $\tilde{\alpha}$ with an inner automorphism of $\Pi_2$ determined by conjugation by an element of $\Pi_{2/1}$ — we may assume without loss of generality that if we write $\tilde{\alpha}_{(2)}$ for the automorphism of $\Pi_{(2)}$ determined by $\tilde{\alpha}$, then}

$$\tilde{\alpha}_{(2)}(\Pi_v) = \Pi_v$$

— where, by abuse of notation, we write $\Pi_v$ for some \textit{fixed} subgroup of $\Pi_{(2)}$ whose image in $\Pi_2 \overset{\sim}{\to} \Pi_{(2)}$ is a vertical subgroup associated to $v$.

Next, let us fix a vertical subgroup $\Pi_v \subseteq \Pi_{2/1} \overset{\sim}{\to} \Pi_{G_{2/1}}$ of $\Pi_{G_{2/1}}$ associated to the vertex $v^0 \in \text{Vert}(\mathcal{G}_{2/1})$ such that the composite $\Pi_{G_{2/1}} \overset{\Phi_{G_{2/1}}}\to \Pi_{(2)}$ via the composite $\Pi_{(G_{2/1})\to S} \overset{\sim}{\to} \Pi_{G_{2/1}} \overset{\sim}{\to} \Pi_{2/1}$ of some vertical subgroup of $\Pi_{(G_{2/1})\to S}$ associated to $w^0$ contains the vertical subgroup $\Pi_{w^0} \subseteq \Pi_{2/1} \overset{\sim}{\to} \Pi_{G_{2/1}}$. Thus, it follows immediately from the various definitions involved that the composite $\Pi_{w^0} \overset{\Phi_{G_{2/1}}}\to \Pi_{2/1} \overset{\Phi_{G_{2/1}}}\to \Pi_{(2)}$ is an \textit{injective} homomorphism whose image $\Pi_w \subseteq \Pi_{(2)}$ maps via the composite $\Pi_{(2)} \overset{\Phi_{G_{2/1}}}\to \Pi_2 \overset{\alpha_{(G_{2/1})\to S}}\to \Pi_{(Z)}$ — where we write $\alpha_{(G_{2/1})\to S} \overset{\text{def}}{=} \text{Node}(\mathcal{G}) \setminus (\text{Node}(\mathcal{G}) \cap \{z_v\})$ — to a vertical subgroup of $\Pi_{G_{w^0}}$ associated to a vertex $w \in \text{Vert}(\mathcal{G}_{w^0})$. Here, we note that the vertex $w$ may also be characterized as the \textit{unique} vertex of $\mathcal{G}_{w^0}$ such that the image via the natural outer isomorphism $\Phi_{G_{w^0}} : \Pi_{G_{w^0}} \overset{\sim}{\to} \Pi_G$ of some vertical subgroup associated to $w$ contains a vertical subgroup associated to $v \in \text{Vert}(\mathcal{G})$. Thus, we obtain an isomorphism $\Pi_{w^0} \overset{\sim}{\to} \Pi_w$, hence also an isomorphism $\tilde{\alpha}_{2/1}(\Pi_{w^0}) \overset{\sim}{\to} \tilde{\alpha}_{(2)}(\Pi_w)$.

Next, let us observe that since $\alpha_{(G_{2/1})\to S}$ is \textit{group-theoretically vertical} [cf. the argument given in the proof of assertion (i)], it follows immediately that $\tilde{\alpha}_{2/1}(\Pi_{w^0}) \subseteq \Pi_{2/1} \overset{\sim}{\to} \Pi_{(G_{2/1})\to S}$ is a vertical subgroup of $\Pi_{(G_{2/1})\to S}$ that maps isomorphically to a vertical subgroup $\tilde{\alpha}_{(2)}(\Pi_w) \subseteq \Pi_{(2)} \overset{\sim}{\to} \Pi_{G_{w^0}}$ of $\Pi_{G_{w^0}}$ that contains $\tilde{\alpha}_{(2)}(\Pi_w) = \Pi_v$. On the other hand, in light of the \textit{unique} characterization of $w$ given above, this implies that $\tilde{\alpha}_{(2)}(\Pi_w) \subseteq \Pi_{(2)} \overset{\sim}{\to} \Pi_{G_{w^0}}$ is a vertical subgroup associated to $w$, and hence [as is easily verified] that
\(\tilde{\alpha}_{2/1}(\Pi_{w^0}) \subseteq \Pi_{2/1} \sim \Pi_{(G_{2/1})_{\sim \mathcal{S}}}\) is a vertical subgroup associated to \(w^0\). In particular, one may apply the natural outer isomorphisms \(\Pi((G_{2/1})_{\sim \mathcal{S}})_{\sim \mathcal{T}_{w^0}} \sim \tilde{\alpha}_{2/1}(\Pi_{w^0})\); \(\Pi((G||_{w^0})_{\sim \mathcal{T}_{w^0}} \sim \tilde{\alpha}_{(2)}(\Pi_{w^0})\) [cf. [CbTpI], Definitions 2.2, (ii); 2.5, (ii)] arising from condition (3) of [CbTpI], Proposition 2.9, (i); moreover, one verifies easily that the resulting outer isomorphism \(\Pi((G_{2/1})_{\sim \mathcal{S}})_{\sim \mathcal{T}_{w^0}} \sim \tilde{\alpha}_{(2)}(\Pi_{w^0})\) [induced by the above isomorphism \(\tilde{\alpha}_{2/1}(\Pi_{w^0}) \sim \tilde{\alpha}_{(2)}(\Pi_{w^0})\)] arises from scheme theory, hence is graphic [cf. [CmbGC], Definition 1.4, (i)]. Therefore, we conclude that the closed subgroup \(\tilde{\alpha}_{2/1}(\Pi_{w^0}) \subseteq (\tilde{\alpha}_{2/1}(\Pi_{w^0}) \subseteq) \Pi_{2/1} \sim \Pi_{(G_{2/1})_{\sim \mathcal{S}}}\) is a vertical subgroup of \(\Pi_{(G_{2/1})_{\sim \mathcal{S}}}\) associated to \(v^0\). This completes the proof of assertion (ii).

Finally, we verify assertion (iii). First, we recall from [CmbCsp], Corollary 1.14, (ii), that there exists an outer modular symmetry \(\sigma \in (\mathcal{S}_1 \subseteq) \text{Out}(\Pi_2)\) such that the composite \(\Pi_{v_{\text{new}}} \leftarrow \Pi_2 \sim \Pi_2 \stackrel{\sigma}{\rightarrow} \Pi_1 = \Pi_v\) determines \(a(n)\) [necessarily geometric] outer isomorphism \(\Pi_{v_{\text{new}}} \sim \Pi_v\). The remainder of the proof of assertion (iii) is devoted to verifying that this outer isomorphism \(\Pi_{v_{\text{new}}} \sim \Pi_v\) satisfies the condition of assertion (iii). First, suppose that \(\alpha_1 \in \text{Out}(\Pi_1)^\Delta\). Then since \(\text{Out}^F(\Pi_2) = \text{Out}^F(C_{\Pi_2}) = \text{Out}^F(C_{\Pi_2}) [cf. [CmbCsp], Definition 1.1, (iv); Theorem 2.3, (ii), (iv), of the present paper; our assumption that \(X^{\log}\) is of type (0,3)], it follows from [CmbCsp], Corollary 1.14, (i), together with the injectivity portion of [CmbCsp], Theorem A, (i), that \(\alpha\) commutes with every modular outer symmetry on \(\Pi_2\); in particular, \(\alpha\) commutes with \(\sigma\). Thus, it follows immediately from [CmbCsp], Corollary 1.14, (iii), that the above outer isomorphism \(\Pi_{v_{\text{new}}} \sim \Pi_v\) satisfies the condition of assertion (iii).

Next, suppose that \(\alpha|_{\Pi_{v_{\text{new}}} \subseteq} \text{Out}(\Pi_{v_{\text{new}}})^\Delta\). If we write \(\alpha^\sigma \overset{\text{def}}{=} \sigma \circ \alpha \circ \sigma^{-1}\) (\(\in \text{Out}^F(\Pi_2)_{\text{cusp}}\) — cf. [CmbCsp], Corollary 1.14, (i); Theorem 2.3, (ii), and Lemma 3.5 of the present paper) and \((\alpha^{\sigma})_1 \in \Pi(v)\) for the outer isomorphism of \(\Pi_v\) determined by \(\alpha^\sigma\), then it follows immediately from [CmbCsp], Corollary 1.14, (iii), that the outer isomorphisms \(\alpha|_{\Pi_{v_{\text{new}}} \subseteq}, (\alpha^{\sigma})_1 \Pi_{v_{\text{new}}} \Pi_v\) are compatible relative to the outer isomorphism \(\Pi_{v_{\text{new}}} \sim \Pi_v\) discussed above. Thus, since \(\alpha|_{\Pi_{v_{\text{new}}} \subseteq} \text{Out}(\Pi_{v_{\text{new}}})^\Delta\), we conclude that \((\alpha^{\sigma})_1 \in \Pi(v)^\Delta\). In particular, \(\text{Out}(\Pi_2) = \text{Out}^F(\Pi_2) = \text{Out}^F(C_{\Pi_2}) [cf. [CmbCsp], Definition 1.1, (iv); Theorem 2.3, (ii), (iv), of the present paper; our assumption that \(X^{\log}\) is of type (0,3)] it follows from [CmbCsp], Corollary 1.14, (i), together with the injectivity portion of [CmbCsp], Theorem A, (i), that \(\alpha^\sigma\) commutes with every modular outer symmetry on \(\Pi_2\). Thus, we conclude that \(\alpha^\sigma\) commutes with \(\sigma^{-1}\), which implies that \(\alpha = \alpha^\sigma\). This completes the proof of assertion (iii). □
Lemma 3.14 (Commensurator of the closed subgroup arising from a certain second log configuration space). Let \( i \in E, j \in E, x, \) and \( z_{i,j,x} \) be as in Lemma 3.6; let \( v \in \text{Vert}(G_{j \in E \setminus \{i\}, x}) \). Then, by applying a similar argument to the argument used in [CmbCsp], Definition 2.1, (iii), (vi), or [NodNon], Definition 5.1, (ix), (x) [i.e., by considering the portion of the underlying scheme \( X_{E} \) of \( X_{E}^{\log} \) corresponding to the underlying scheme \( (X_{E})_{2} \) of the 2-nd log configuration space \( (X_{E})_{2}^{\log} \) of the stable log curve \( X_{E}^{\log} \) determined by \( G_{j \in E \setminus \{i\}, x} \) v — cf. [CbTpI], Definition 2.1, (iii)], one obtains a closed subgroup

\[
(\Pi_{v})_{2} \subseteq \Pi_{E/\{E \setminus \{i,j\}\}}
\]

[which is well-defined up to \( \Pi_{E} \)-conjugation]. Write

\[
(\Pi_{v})_{2/1} \triangleq (\Pi_{v})_{2} \cap \Pi_{E/\{E \setminus \{i\}\}} \subseteq (\Pi_{v})_{2}.
\]

[Thus, one verifies easily that there exists a natural commutative diagram

\[
\begin{array}{c}
1 \longrightarrow (\Pi_{v})_{2/1} \longrightarrow (\Pi_{v})_{2} \longrightarrow \Pi_{v} \longrightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
1 \longrightarrow \Pi_{E/\{E \setminus \{i\}\}} \longrightarrow \Pi_{E/\{E \setminus \{i,j\}\}} \longrightarrow \Pi_{E/\{E \setminus \{i,j\}\})/\Pi_{E/\{E \setminus \{i\}\}} \longrightarrow 1
\end{array}
\]

where we use the notation \( \Pi_{v} \) to denote a vertical subgroup of \( \Pi_{G_{j \in E \setminus \{i\}, x}} \), \( \Pi_{E/\{E \setminus \{i\}\}} \) associated to \( v \in \text{Vert}(G_{j \in E \setminus \{i\}, x}) \), the horizontal sequences are exact, and the vertical arrows are injective.] Then the following hold:

(i) Suppose that \( z_{i,j,x} \in \text{VCN}(G_{j \in E \setminus \{i\}, x}) \) is contained in \( \mathcal{E}(v) \). Write \( v^{o} \in \text{Vert}(G_{j \in E, x}) \) for the vertex of \( G_{j \in E, x} \) that corresponds to \( v \in \text{Vert}(G_{j \in E \setminus \{i\}, x}) \) via the bijections of Lemma 3.6, (i), (iv). Let \( \Pi_{e}, \Pi_{e_{i,j,x}}^{new} \subseteq \Pi_{G_{j \in E, x}} \trianglerighteq \Pi_{E/\{E \setminus \{i\}\}} \) be vertical subgroups of \( \Pi_{G_{j \in E, x}} \) associated to the vertices \( v^{o}, v_{i,j,x}^{new} \in \text{Vert}(G_{j \in E, x}) \), respectively, such that \( \Pi_{e_{i,j,x}}^{new} \subseteq (\Pi_{v})_{2/1} \), and, moreover, \( \Pi_{e} \cap \Pi_{e_{i,j,x}}^{new} \neq \{1\} \). Let us say that two \( \Pi_{E/\{E \setminus \{i\}\}} \)-conjugates \( \Pi_{e}, \Pi_{e_{i,j,x}}^{new} \) [i.e., where \( \gamma, \delta \in \Pi_{E/\{E \setminus \{i\}\}} \)] of \( \Pi_{e}, \Pi_{e_{i,j,x}}^{new} \) are conjugate-adjacent if \( \Pi_{e} \cap \Pi_{e_{i,j,x}}^{new} \neq \{1\} \). Let us say that a finite sequence of \( \Pi_{E/\{E \setminus \{i\}\}} \)-conjugates of \( \Pi_{e}, \Pi_{e_{i,j,x}}^{new} \) is a conjugate-chain if any two adjacent members of the finite sequence are conjugate-adjacent. Let us say that a subgroup of \( \Pi_{E/\{E \setminus \{i\}\}} \) is conjugate-tempered if it appears as the first member of a conjugate-chain whose final member is equal to \( \Pi_{e_{i,j,x}}^{new} \). Then \( (\Pi_{v})_{2/1} \) is equal to the subgroup of \( \Pi_{E/\{E \setminus \{i\}\}} \) topologically generated by the conjugate-tempered subgroups and the elements \( \delta \in \Pi_{E/\{E \setminus \{i\}\}} \) such that \( \Pi_{e_{i,j,x}}^{new} \) is conjugate-tempered.
(ii) If $N_{\Pi_{v,1}}(\Pi_v) = C_{\Pi_{v,1}}(\Pi_v)$, then $N_{\Pi_v}(\Pi_v) = C_{\Pi_v}(\Pi_v)$.

(iii) If $C_{\Pi_{v,1}}(\Pi_v) = \Pi_v \times Z_{\Pi_{v,1}}(\Pi_v)$, then $C_{\Pi_v}(\Pi_v) = (\Pi_v) \times Z_{\Pi_v}(\Pi_v)$.

(iv) Suppose that $v$ is of type $(0, 3)$, i.e., that $\Pi_v$ is an $(E \setminus \{i\})$-tripod of $\Pi_n$ [cf. Definition 3.3, (i)]. Then it holds that $C_{\Pi_v}(\Pi_v) = (\Pi_v) \times Z_{\Pi_v}(\Pi_v)$. Thus, if an automorphism $\alpha$ of $\Pi_v$ preserves the $\Pi_v$-conjugacy class of $(\Pi_v)$, then one may define $\alpha : (\Pi_v) \in \text{Out}((\Pi_v))$ [cf. Lemma 3.10, (i)].

Proof. First, we verify assertion (i). We begin by observing that the subgroup described in the final portion of the statement of assertion (i) is contained in $(\Pi_v)_2$. If $(v^o) \cap N(v^o)$ then assertion (i) follows immediately from a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.5, (iii), together with the various definitions involved [cf. also [NodNon], Lemma 1.9, (ii)]. Thus, we may assume without loss of generality that $(v^o) \cap N(v^o)^2 = 2$.

Write

- $e_1 \in N(v^o) \cap N(v^\text{new}_{i,j,x})$ for the [uniquely determined] — cf. [NodNon], Lemma 1.5] node such that $\Pi_v \cap \Pi_{v,i,j,x}^\text{new} (\neq \{i\})$ is a nodal subgroup associated to $e_1$ [cf. [NodNon], Lemma 1.9, (i)];
- $e_2$ for the unique element of $N(v^o) \cap N(v^\text{new}_{i,j,x})$ such that $e_2 \neq e_1$ [so $N(v^o) \cap N(v^\text{new}_{i,j,x}) = \{e_1, e_2\}$];
- $\mathbb{H}$ for the sub-semi-graph of $\text{PSC-type}$ [cf. [CbTpI], Definition 2.2, (i)] of the underlying semi-graph of $G_{i \in E,x}$ whose set of vertices $\{v^o, v^\text{new}_{i,j,x}\}$;
- $S \overset{\text{def}}{=} \text{Node}(G_{i \in E,x}) \setminus \{e_1, e_2\}$ [cf. [CbTpI], Definition 2.2, (ii)];
- $\mathcal{H} \overset{\text{def}}{=} (G_{i \in E,x})_{|H} \bullet S$ [which is well-defined since, as is easily verified, $S$ is not of separating type as a subset of Node($G_{i \in E,x}$) — cf. [CbTpI], Definition 2.5, (i), (ii)].

Then it follows immediately from the construction of $\mathcal{H}$ that $\mathcal{H}_{\sim\{e_1\}}$ [cf. [CbTpI], Definition 2.8], where we observe that one verifies easily that the node $e_1$ of $G_{i \in E,x}$ may be regarded as a node of $\mathcal{H}$, is cyclically primitive [cf. [CbTpI], Definition 4.1]. Moreover, it follows immediately from [NodNon], Lemma 1.9, (ii), together with the various definitions involved, that $(\Pi_v)_2 \subseteq \Pi_{E/(E \setminus \{i\})} \nu_{g_i \in E,x}$ may be characterized uniquely as the closed subgroup of $(\Pi_{v,i,j,x})$ that contains $\Pi_{v,i,j,x}^\text{new} \subseteq \Pi_{g_i \in E,x}$ and, moreover, belongs to the $\Pi_{g_i \in E,x}$-conjugacy class.
of closed subgroups of $\prod_{i \in E,x} \Phi$ obtained by forming the image of the composite of outer homomorphisms

$$\Phi_{H \rightarrow \{e\}} \sim \prod H \leftrightarrow \prod_{i \in E,x} \Phi$$

[cf. [CbTpI], Definition 2.10] — where the second arrow is the outer injection discussed in [CbTpI], Proposition 2.11. In particular, it follows from the commensurable terminality of $(\Pi_v)_{2/1}$ in $\prod_{i \in E,x} \Phi$ [cf. [CmbGC], Proposition 1.2, (ii)] that this characterization of $(\Pi_v)_{2/1}$ determines an outer isomorphism $\prod H \rightarrow \{e\}$.

On the other hand, it follows immediately from a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.5, (iii), together with the various definitions involved [cf. also [NodNon], Lemma 1.9, (ii)], that the image of the closed subgroup of $(\Pi_v)_{2/1}$ topologically generated by $\Pi_v$ and $\Pi_v_{x,x}$ via the inverse $(\Pi_v)_{2/1} \rightarrow \prod H_{\rightarrow \{e\}}$ of this outer isomorphism is a vertical subgroup of $\prod H \rightarrow \{e\}$ associated to the unique vertex of $H \rightarrow \{e\}$. Thus, since $H \rightarrow \{e\}$ is cyclically primitive, assertion (i) follows immediately from [CmbGC], Proposition 1.2, (ii); [NodNon], Lemma 1.9, (ii), together with the description of the structure of a certain tempered covering of $H \rightarrow \{e\}$ given in [CbTpI], Lemma 4.3. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since $(\Pi_v)_{2/1} = (\Pi_v)_{2} \cap \Pi_{E/(E \setminus \{i\})}$ is commensurable terminal in $\Pi_{E/(E \setminus \{i\})}$ [cf. [CmbGC], Proposition 1.2, (ii)], assertion (ii) follows immediately from Lemma 3.9, (iv). This completes the proof of assertion (ii). Next, we verify assertion (iii). First, let us observe that if $\mathcal{E}(v) = \emptyset$, then one verifies immediately that the vertical arrows of the commutative diagram in the statement of Lemma 3.14 are isomorphisms, and hence that assertion (iii) holds. Thus, we may assume that $\mathcal{E}(v) \neq \emptyset$. Next, let us observe that it follows from assertion (ii) that $N_{\Pi_E}((\Pi_v)_{2}) = C_{\Pi_E}((\Pi_v)_{2})$. Thus, in light of the slimness of $(\Pi_v)_{2}$ [cf. [MzTa], Proposition 2.2, (ii)], to verify assertion (iii), it suffices to verify that the natural outer action of $N_{\Pi_E}((\Pi_v)_{2})$ on $(\Pi_v)_{2}$ is trivial. On the other hand, since [one verifies easily that] the natural outer action $N_{\Pi_E}((\Pi_v)_{2}) \rightarrow \text{Out}((\Pi_v)_{2})$ factors through $\text{Out}^F((\Pi_v)_{2}) \subseteq \text{Out}((\Pi_v)_{2})$, it follows from the injectivity portion of Theorem 2.3, (i) [cf. our assumption that $\mathcal{E}(v) \neq \emptyset$], that to verify the triviality in question, it suffices to verify that the natural outer action of $N_{\Pi_E}((\Pi_v)_{2})$ on $\Pi_v$ is trivial. But this follows from the equality $C_{\Pi_{E/(i)}}(\Pi_v) = \Pi_v \times Z_{\Pi_{E/(i)}}(\Pi_v)$. This completes the proof of assertion (iii). Assertion (iv) follows immediately from assertion (iii), together with Lemma 3.12, (i). This completes the proof of Lemma 3.14. □

Lemma 3.15 (Preservation of various subgroups of geometric origin). In the notation of Lemma 3.14, let $\tilde{\alpha}$ be an $F$-admissible
automorphism of $\Pi_E$. Write $\tilde{\alpha}_{E\setminus\{i\}}, \tilde{\alpha}_{E/(E\setminus\{i\})}$ for the automorphisms of $\Pi_{E\setminus\{i\}}, \Pi_E/(E\setminus\{i\})$ determined by $\tilde{\alpha}; \alpha, \alpha_{E\setminus\{i\}}, \alpha_{E/(E\setminus\{i\})}$ for the automorphisms of $\Pi_E, \Pi_{E\setminus\{i\}}, \Pi_{E/(E\setminus\{i\})}$ determined by $\tilde{\alpha}, \tilde{\alpha}_{E\setminus\{i\}}, \tilde{\alpha}_{E/(E\setminus\{i\})}$, respectively. Suppose that there exist an edge $e \in \text{Edge}(G_{j\in E\setminus\{i\}}, E)$ of $G_{j\in E\setminus\{i\},x}$ that belongs to $E(v) \subseteq \text{Edge}(G_{j\in E\setminus\{i\}}, x)$ and a pair $\Pi_v \subseteq \Pi_{E\setminus\{i\}} \supseteq \Pi_v \subseteq \Pi_{E\setminus\{i\},x} \cong \Pi_{(E\setminus\{i\})/(E\setminus\{i,j\})}$ of VCN-subgroups associated to $e \in \text{Edge}(G_{j\in E\setminus\{i\}}, x), v \in \text{Vert}(G_{j\in E\setminus\{i\}, x})$, respectively, such that

$$\tilde{\alpha}_{E\setminus\{i\}}(\Pi_v) = \Pi_v \subseteq \tilde{\alpha}_{E\setminus\{i\}}(\Pi_v) = \Pi_v.$$ 

Suppose, moreover, either that

(a) the automorphism $\alpha_{E/(E\setminus\{i\})}$ of $\Pi_{\tilde{\alpha}_E\setminus\{i\}} \cong \Pi_{E/(E\setminus\{i\})}$ maps some cuspidal inertia subgroup of $\Pi_{\tilde{\alpha}_E\setminus\{i\}} \cong \Pi_{E/(E\setminus\{i\})}$ to a cuspidal inertia subgroup of $\Pi_{\tilde{\alpha}_E\setminus\{i\}} \cong \Pi_{E/(E\setminus\{i\})}$, or that

(b) $e \in \text{Cusp}(G_{j\in E\setminus\{i\}, x}).$

[For example, condition (a) holds if the automorphism $\alpha_{E/(E\setminus\{i\})}$ of $\Pi_{\tilde{\alpha}_E\setminus\{i\}} \cong \Pi_{E/(E\setminus\{i\})}$ is group-theoretically cuspidal — cf. [CmbGC], Definition 1.4, (iv,)]. Write $T \subseteq \Pi_E$ for the $E$-tripod of $\Pi_n$ [cf. Definition 3.3, (i)] arising from $e \in \text{Edge}(G_{j\in E\setminus\{i\}, x})$ [cf. Definition 3.7, (i)]. Then the following hold:

(i) The automorphism $\alpha$ preserves the $\Pi_E$-conjugacy classes of $T, (\Pi_v)_2 \subseteq \Pi_E$. If, moreover, the automorphism $\alpha_{E/(E\setminus\{i\})}$ of $\Pi_{\tilde{\alpha}_E\setminus\{i\}} \cong \Pi_{E/(E\setminus\{i\})}$ is group-theoretically cuspidal [cf. [CmbGC], Definition 1.4, (iv)], then the automorphism $\alpha |_T$ [cf. Lemma 3.12, (i)] of $T$ is contained in $\text{Out}^C(T)^{\text{cusp}}$ [cf. Definition 3.4, (i)].

(ii) Suppose, moreover, that $v$ is of type $(0, 3)$ — i.e., that $\Pi_v$ is an $(E\setminus\{i\})$-tripod of $\Pi_n$ — and that $\alpha_{E\setminus\{i\}}|\Pi_v \in \text{Out}^C(\Pi_v)^{\text{cusp}}$ [cf. Lemma 3.12, (i)]. Then there exists a geometric [cf. Definition 3.4, (ii)] outer isomorphism $T \tilde{\rightarrow} \Pi_v$ which satisfies the following condition:

If either $\alpha |_T \in \text{Out}(T)^{\Delta}$ [cf. (i)] or $\alpha_{E\setminus\{i\}}|\Pi_v \in \text{Out}(\Pi_v)^{\Delta}$, then the automorphisms $\alpha |_T, \alpha_{E\setminus\{i\}}|\Pi_v$ of $T, \Pi_v$ are compatible relative to the outer isomorphism in question $T \tilde{\rightarrow} \Pi_v$.

If, moreover, $\Pi_v$ is $(E \setminus \{i\})$-strict [cf. Definition 3.3, (iii)], then the following hold:

(1) If $(E \setminus \{i\})^2 = 1$ [i.e., $\Pi_v$ satisfies condition (1) of Lemma 3.8, (ii)], then $T$ is $E$-strict [i.e., $T$ satisfies one of the two conditions $(2C), (2N)$ of Lemma 3.8, (ii)].

(2) If $(E \setminus \{i\})^4 = 2$ [i.e., $\Pi_v$ satisfies one of the two conditions $(2C), (2N)$ of Lemma 3.8, (ii)], and the edge $e \in$
Edge($\mathcal{G}_{j \in E \setminus \{i\}, x}$) is the unique diagonal cusp of $\mathcal{G}_{j \in E \setminus \{i\}, x}$ [cf. Lemma 3.2, (ii)], then $T$ is $E$-strict [i.e., $T$ satisfies condition (3) of Lemma 3.8, (ii)], hence also central [cf. Definition 3.7, (ii)].

Proof. First, let us observe that one verifies easily — by replacing $x$ by a suitable $k$-valued geometric point of $X_n(k)$ that lifts $x_{E \setminus \{i,j\}} \in X_{E \setminus \{i,j\}}(k)$ [note that this does not affect “$\mathcal{G}_{j \in E \setminus \{i\}, x}$”!] — that, to verify Lemma 3.15, we may assume without loss of generality that $z_{i,j,x} = e \in \text{Edge}(\mathcal{G}_{j \in E \setminus \{i\}, x})$.

Now we verify assertion (i). First, let us observe that one verifies easily — by replacing $X_{E \setminus \{i,j\}}^x$ by the base-change of $p_{E \setminus \{i,j\}}^x : X_{E \setminus \{i,j\}}^\log \to X_{E \setminus \{i,j\}}^\log$ by a suitable morphism of log schemes $(\text{Spec } k)^\log \to X_{E \setminus \{i,j\}}^\log$ that lies over $x_{E \setminus \{i,j\}} \in X_{E \setminus \{i,j\}}(k)$ [cf. Definition 3.1, (i)] — that, to verify assertion (i), we may assume without loss of generality that $E^\sharp_2 = 2$. Then it follows immediately from Lemma 3.13, (i), that $\alpha_{E/(E \setminus \{i\})}$ preserves the $\Pi_{E/(E \setminus \{i\})}$-conjugacy class of $T (= \Pi_{v_{\text{new}}}^\circ) \subseteq \Pi_{E/(E \setminus \{i\})}$. Moreover, it follows immediately from Lemma 3.13, (i), (ii), together with Lemma 3.14, (i), that $\alpha_{E/(E \setminus \{i\})}$ preserves the $\Pi_{E/(E \setminus \{i\})}$-conjugacy classes of the normally terminal closed subgroups $\Pi_{v_{\circ}} \subseteq (\Pi_v)_{/2/1} \subseteq \Pi_{E/(E \setminus \{i\})}$ [cf. [CmbGC], Proposition 1.2, (ii)]. In particular, since $\widetilde{\alpha}_{E \setminus \{i\}}(\Pi_{v}) = \Pi_v$, by considering the natural isomorphism $(\Pi_v)_{/2} \cong (\Pi_v)_{/2/1} \rtimes \Pi_v$ [cf. the upper exact sequence of the commutative diagram in the statement of Lemma 3.14; the discussion entitled “Topological groups” in [CbtP]], §0], we conclude that $\alpha_E$ preserves the $\Pi_E$-conjugacy class of $(\Pi_v)_{/2} \subseteq \Pi_E$.

Next, suppose that the automorphism $\alpha_{E/(E \setminus \{i\})}$ of $\Pi_{\mathcal{G}_{i \in E \setminus \{i\}, x}} \sim \Pi_{E/(E \setminus \{i\})}$ is group-theoretically cuspidal. Then it follows from Lemma 3.13, (i), that $\alpha_{T} \in \text{Out}^C(T)$. Moreover, since $\alpha_{E/(E \setminus \{i\})}$ is group-theoretically cuspidal, it follows immediately from Lemma 3.2, (iv), that $\alpha_{E/(E \setminus \{i\})}$ fixes the $\Pi_{E/(E \setminus \{i\})}$-conjugacy class of cuspidal inertia subgroups associated to each element $\in \mathcal{N}(u_{i,j,x}^\text{new}) (\geq e_{i,j,x}^\text{diag})$. Thus, to verify that $\alpha_{T} \in \text{Out}^C(T)$ cuspid, it suffices to verify that $\alpha_{E/(E \setminus \{i\})}$ fixes the $\Pi_{E/(E \setminus \{i\})}$-conjugacy class of nodal subgroups of $\Pi_{\mathcal{G}_{i \in E \setminus \{i\}, x}} \sim \Pi_{E/(E \setminus \{i\})}$ associated to each element of $\mathcal{N}(u_{i,j,x}^\text{new}) \cap N(v_{\circ})$. To this end, let $e_{\circ} \in \mathcal{N}(u_{i,j,x}^\text{new}) \cap N(v_{\circ})$ and $\Pi_{\circ} \subseteq \Pi_{\mathcal{G}_{i \in E \setminus \{i\}, x}} \sim \Pi_{E/(E \setminus \{i\})}$ a nodal subgroup associated to the node $e_{\circ}$ such that $\Pi_{\circ} \subseteq \Pi_{e_{\circ}}$. Now let us observe that one verifies easily that the closed subgroups $\Pi_{e_{\circ}} \subseteq \Pi_{e_{\circ}} \subseteq \Pi_{\mathcal{G}_{i \in E \setminus \{i\}, x}} \sim \Pi_{E/(E \setminus \{i\})}$ map bijectively onto VCN-subgroups of $\Pi_{\mathcal{G}_{i \in E \setminus \{i\}, x}} \sim \Pi_{E/(E \setminus \{i\})}$ associated, respectively, to the edge and vertex of $\mathcal{G}_{i \in E \setminus \{i\}, x}$ that correspond, via the bijections of Lemma 3.6, (i), to $e, v \in \text{VCN}(\mathcal{G}_{i \in E \setminus \{i\}, x})$. In particular, if $\widetilde{\beta}$ is the composite of $\widetilde{\alpha}$ with some $\Pi_{E/(E \setminus \{i\})}$-inner automorphism such that $\widetilde{\beta}(\Pi_{e_{\circ}}) = \Pi_{e_{\circ}}$ [cf. the preceding paragraph],
then it follows immediately from our assumption that $\tilde{\alpha}_{E \setminus \{i\}}(\Pi_e) = \Pi_e \subseteq \tilde{\alpha}_{E \setminus \{i\}}(\Pi_e) = \Pi_e$, together with [CbTpi], Theorem A, (i), and [CmbGC], Proposition 1.2, (ii), that the automorphism of $\Pi_e$ determined by $\beta$ preserves the $\Pi_e$-conjugacy class of $\Pi_e$. Thus, $\alpha_{E/(E\setminus\{i\})}$ fixes the $\Pi_{E/(E\setminus\{i\})}$-conjugacy class of $\Pi_e$, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since $v$ is of type $(0, 3)$, it follows from assertion (i), together with Lemma 3.14, (iv), that one may define $j(\Pi_v)$ into $\Out((\Pi_v)^2)$. Thus, by applying Lemma 3.13, (iii), to $\alpha|_{(\Pi_v)^2} \in \Out((\Pi_v)^2)$, one veriﬁes easily that the first portion of assertion (ii) holds. The final portion of assertion (ii) follows immediately from the descriptions given in the four conditions of Lemma 3.8, (ii), together with the various deﬁnitions involved. This completes the proof of assertion (ii).

\begin{flushright}
$\square$
\end{flushright}

\textbf{Theorem 3.16 (Outomorphisms preserving tripods).} In the notation of the beginning of the present §3, let $E \subseteq \{1, \cdots, n\}$ and $T \subseteq \Pi_E$ an $E$-tripod of $\Pi_n$ [cf. Definition 3.3, (i)]. Let us write

$$\Out^F(\Pi_n)[T] \subseteq \Out^F(\Pi_n)$$

for the [closed] subgroup of $\Out^F(\Pi_n)$ [cf. [CmbCsp], Definition 1.1, (ii)] consisting of $F$-admissible outomorphisms $\alpha$ of $\Pi_n$ such that the outomorphism of $\Pi_E$ determined by $\alpha$ preserves the $\Pi_E$-conjugacy class of $T \subseteq \Pi_E$. Then the following hold:

(i) It holds that

$$C_{\Pi_E}(T) = T \times Z_{\Pi_E}(T).$$

Thus, by applying Lemma 3.10, (i), to outomorphisms of $\Pi_E$ determined by elements of $\Out^F(\Pi_n)[T]$, one obtains a natural homomorphism

$$\Sigma_T : \Out^F(\Pi_n)[T] \longrightarrow \Out(T).$$

Let us write

$$\Out^F(\Pi_n)[T : \{C\}], \Out^F(\Pi_n)[T : \{|C|\}], \Out^F(\Pi_n)[T : \{\Delta\}], \Out^F(\Pi_n)[T : \{+\}] \subseteq \Out^F(\Pi_n)[T]$$

for the [closed] subgroups of $\Out^F(\Pi_n)[T]$ obtained by forming the respective inverse images via $\Sigma_T$ of the closed subgroups $\Out^C(T), \Out^C(T)^\text{cusp}, \Out(T)^\Delta, \Out(T)^+ \subseteq \Out(T)$ [cf. Definition 3.4, (i)]. For each subset $S \subseteq \{C, |C|, \Delta, +\}$, let us write

$$\Out^F(\Pi_n)[T : S] \overset{\text{def}}{=} \bigcap_{\square \in S} \Out^F(\Pi_n)[T : \{\square\}] \subseteq \Out^F(\Pi_n)[T];$$
Out^{FC}(\Pi_n)[T : S] \overset{\text{def}}{=} \text{Out}^F(\Pi_n)[T : S] \cap \text{Out}^{FC}(\Pi_n) \subseteq \text{Out}^{FC}(\Pi_n)

[cf. [CmbCsp], Definition 1.1, (ii)]. Suppose, moreover, that we are given an element \sigma \in \mathcal{G}_n \subseteq \text{Out}(\Pi_n) [cf. the discussion at the beginning of the present \S 3] and a lifting \bar{\sigma} \in \text{Aut}(\Pi_n) of \sigma \in \mathcal{G}_n \subseteq \text{Out}(\Pi_n). Write

\[ T^{\bar{\sigma}} \subseteq \Pi_{\sigma(E)} \]

for the image of \( T \subseteq \Pi_E \) by the isomorphism \( \Pi_E \overset{\sim}{\rightarrow} \Pi_{\sigma(E)} \) determined by \( \bar{\sigma} \in \text{Aut}(\Pi_n) \) [which thus implies that \( T^{\bar{\sigma}} \subseteq \Pi_{\sigma(E)} \) is a \( \sigma(E) \)-tripod of \( \Pi_n \) — cf. Remark 3.7.1] and

\[ \text{Out}^F(\Pi_n)[T, \bar{\sigma}] \overset{\text{def}}{=} \text{Out}^F(\Pi_n)[T] \cap \text{Out}^F(\Pi_n)[T^{\bar{\sigma}}] \subseteq \text{Out}^F(\Pi_n), \]

\[ \text{Out}^{FC}(\Pi_n)[T, \bar{\sigma}] \overset{\text{def}}{=} \text{Out}^F(\Pi_n)[T, \bar{\sigma}] \cap \text{Out}^{FC}(\Pi_n) \subseteq \text{Out}^{FC}(\Pi_n). \]

Then the resulting isomorphism \( T \overset{\sim}{\rightarrow} T^{\bar{\sigma}} \) is geometric [cf. Definition 3.4, (ii)]. Moreover, we have a commutative diagram

\[
\begin{array}{ccc}
\text{Out}^F(\Pi_n)[T, \bar{\sigma}] & \cong & \text{Out}^F(\Pi_n)[T, \bar{\sigma}] \\
\downarrow \xi_T & & \downarrow \xi_{T^{\bar{\sigma}}} \\
\text{Out}(T) & \overset{\sim}{\rightarrow} & \text{Out}(T^{\bar{\sigma}})
\end{array}
\]

— where the upper horizontal equality is an equality of subgroups of the group \text{Out}^F(\Pi_n), and the lower horizontal arrow is the isomorphism obtained by conjugating by the above geometric isomorphism \( T \overset{\sim}{\rightarrow} T^{\bar{\sigma}} \) [i.e., induced by \( \bar{\sigma} \in \text{Aut}(\Pi_n) \)].

Finally, the equalities

\[ \text{Out}^{FC}(\Pi_n)[T, \bar{\sigma}] = \text{Out}^{FC}(\Pi_n)[T] = \text{Out}^{FC}(\Pi_n)[T^{\bar{\sigma}}] \]

hold; if, moreover, one of the following conditions is satisfied, then the equalities

\[ \text{Out}^F(\Pi_n)[T, \bar{\sigma}] = \text{Out}^F(\Pi_n)[T] = \text{Out}^F(\Pi_n)[T^{\bar{\sigma}}] \]

hold:

(i-1) \((r, n) \neq (0, 2),\)

(i-2) \( T \) is \textbf{E}-strict [cf. Definition 3.3, (iii)].

(ii) It holds that

\[ \text{Out}^F(\Pi_n)[T : \{C, \Delta\}] = \text{Out}^F(\Pi_n)[T : \{|C|, \Delta\}] . \]

(iii) Suppose that \( T \) is \textbf{1}-descendable [cf. Definition 3.3, (iv)]. Then it holds that

\[ \text{Out}^{FC}(\Pi_n)[T : \{|C|\}] = \text{Out}^{FC}(\Pi_n)[T : \{|C|, +\}] . \]
If, moreover, one of the following conditions is satisfied, then it holds that

\[ \text{Out}^F(\Pi_n)[T : \{ |C| \}] = \text{Out}^F(\Pi_n)[T : \{ |C|, + \}] : \]

(iii-1) \( T \) is 2-descendable [cf. Definition 3.3, (iv)].

(iii-2) There exists a subset \( E' \subseteq E \) such that:

(iii-2-a) \( E' \neq \{1, \ldots, n\} \);

(iii-2-b) the image \( p^\Pi_{E/E'}(T) \subseteq \Pi_{E'} \) is a cusp-supporting \( E' \)-tripod of \( \Pi_n \) [cf. Definition 3.3, (i)].

(iv) Let \( i, j \in E \) be two distinct elements of \( E \); \( e \in \text{Edge}(G_{j \in E \setminus \{i\}, x}) \) [cf. Definition 3.1, (iii)]; \( \alpha \in \text{Out}^F(\Pi_n) \). Suppose that \( T \) arises from \( e \in \text{Edge}(G_{j \in E \setminus \{i\}, x}) \) [cf. Definition 3.7, (i)], and that the automorphism of \( \Pi_{E \setminus \{i\}} \) determined by \( \alpha \) preserves the \( \Pi_{E \setminus \{i\}} \)-conjugacy class of an edge-like subgroup of \( \Pi_{E \setminus \{i\}} \) associated to \( e \in \text{Edge}(G_{j \in E \setminus \{i\}, x}) \) [cf. Definition 3.1, (iv)]. Suppose, moreover, that one of the following conditions is satisfied:

(iv-1) \( \alpha \in \text{Out}^{FC}(\Pi_n) \).

(iv-2) \( E^2 \leq n - 1 \).

(iv-3) \( e \in \text{Cusp}(G_{j \in E \setminus \{i\}, x}) \).

Then \( \alpha \in \text{Out}^F(\Pi_n)[T] \). Suppose, further, that either condition (iv-1) or condition (iv-2) is satisfied. Then \( \alpha \in \text{Out}^F(\Pi_n)[T : \{ |C| \}] \); if, in addition, condition (iv-3) is satisfied, then \( \alpha \in \text{Out}^F(\Pi_n)[T : \{ |C| \}] \).

(v) Suppose that \( T \) is central [cf. Definition 3.7, (ii)]. If \( n \geq 4 \) [i.e., \( T \) is 1-descendable], then it holds that

\[ \text{Out}^F(\Pi_n) = \text{Out}^{FC}(\Pi_n)[T : \{ |C|, \Delta, + \}] . \]

If \( n = 3 \) [i.e., \( T \) is not 1-descendable], then it holds that

\[ \text{Out}^{FC}(\Pi_n) = \text{Out}^{FC}(\Pi_n)[T : \{ |C|, \Delta \}] \]

\[ \subseteq \text{Out}^F(\Pi_n) = \text{Out}^F(\Pi_n)[T : \{ \Delta \}] ; \]

if, moreover, \( r \neq 0 \), then

\[ \text{Out}^F(\Pi_n) = \text{Out}^{FC}(\Pi_n)[T : \{ |C|, \Delta, + \}] . \]

**Proof.** We begin the proof of Theorem 3.16 with the following claim:

Claim 3.16.1: Let \( E' \subseteq E \) be a subset such that the image \( T_{E'} \) of \( T \) via \( p^\Pi_{E/E'} : \Pi_E \to \Pi_{E'} \) is an \( E' \)-tripod. Thus, one verifies easily that one obtains a(n) [necessarily geometric] outer isomorphism \( T \sim T_{E'} \) [induced
by $p_{E/E'}^\Pi$. Then we have an inclusion $\Out^F(\Pi_n)[T] \subseteq \Out^F(\Pi_n)[T_{E'}]$, and, moreover, the diagram

$$
\begin{array}{ccc}
\Out^F(\Pi_n)[T] & \subseteq & \Out^F(\Pi_n)[T_{E'}] \\
\downarrow & & \downarrow \\
\Out(T) & \sim & \Out(T_{E'})
\end{array}
$$

— where the lower horizontal arrow is the isomorphism determined by the isomorphism $T \sim T_{E'}$ induced by $p_{E/E'}^\Pi$ — commutes.

Indeed, this follows immediately from the various definitions involved. This completes the proof of Claim 3.16.A.

Next, we verify assertion (i). The equality $C_{\Pi_n}(T) = T \times \Z_{\Pi_n}(T)$ of the first display in assertion (i) follows from Lemma 3.12, (i). Moreover, the geometricity of the isomorphism $T \sim T^\sigma$ follows immediately from the various definitions involved. Next, let us observe that if $(r, n) \neq (0, 2)$, then the commutativity of the displayed diagram in assertion (i) and the equalities

$$
\Out^F(\Pi_n)[T, \tilde{\sigma}] = \Out^F(\Pi_n)[T] = \Out^F(\Pi_n)[T^\sigma]
$$

in assertion (i) may be easily derived from the fact that the closed subgroup $\Out^F(\Pi_n) \subseteq \Out(\Pi_n)$ centralizes the closed subgroup $\mathfrak{S}_n \subseteq \Out^F(\Pi_n)$ [cf. Theorem 2.3, (iv)]. Moreover, the equalities

$$
\Out^{FC}(\Pi_n)[T, \tilde{\sigma}] = \Out^{FC}(\Pi_n)[T] = \Out^{FC}(\Pi_n)[T^\sigma]
$$

in assertion (i) may be easily derived from the fact that the closed subgroup $\Out^{FC}(\Pi_n) \subseteq \Out(\Pi_n)$ centralizes the closed subgroup $\mathfrak{S}_n \subseteq \Out^F(\Pi_n)$ [cf. [NodNon], Theorem B].

Next, let us observe that if $T$ is $E'$-strict for some subset $E' \subseteq E$ of cardinality one, then the commutativity of the displayed diagram in assertion (i) follows immediately from Claim 3.16.A and [CbTpI], Theorem A, (i). Thus, it follows from Lemma 3.8, (ii), that, to complete the verification of assertion (i), it suffices to verify, under the assumption that $\sigma \neq \text{id}$,

(a) the commutativity of the displayed diagram in assertion (i) in the case where $(r, n) = (0, 2)$, and $T$ is $\{1, 2\}$-strict, and

(b) the equalities

$$
\Out^F(\Pi_n)[T, \tilde{\sigma}] = \Out^F(\Pi_n)[T] = \Out^F(\Pi_n)[T^\sigma]
$$

in assertion (i) in the case where $(r, n) = (0, 2)$, and $T$ is $\{1, 2\}$-strict.

In particular, to verify assertion (i), we may assume without loss of generality [cf. conditions $(2_C)$ and $(2_N)$ of Lemma 3.8, (ii)] that we
are in the situation of Lemma 3.11 in the case where we take the \("n\)”, “\(E\)” of Lemma 3.11 to be 2, \(\{1, 2\}\), respectively. Moreover, it follows immediately from Lemma 3.8, (ii), that the \(\Pi_n\)-conjugacy classes of \(T, T^\sigma\) coincide with the \(\Pi_n\)-conjugacy classes of the closed subgroups \(\Pi_{E_1}^{\text{new}}, \Pi_{E_2}^{\text{new}}\) of \(\Pi_n\) that appear in the statement of Lemma 3.11, respectively. Then the above equalities in (b) follows immediately from Lemma 3.11, (x). Moreover, it follows from Lemma 3.11, (viii), (ix), that the composites

\[
T \hookrightarrow C_{\Pi_n}(T) \rightarrow C_{\Pi_n}(T)/Z(C_{\Pi_n}(T)),
\]

\[
T^\sigma \hookrightarrow C_{\Pi_n}(T^\sigma) \rightarrow C_{\Pi_n}(T^\sigma)/Z(C_{\Pi_n}(T^\sigma))
\]

are isomorphisms. Thus, the commutativity in (a) follows immediately from Lemma 3.11, (x). This completes the proof of assertion (i). Assertion (ii) follows from Lemma 3.5.

Next, we verify assertion (iii). First, to verify the first displayed equality of assertion (iii), let us observe that since \(T\) is 1-descendable, there exists a subset \(E' \subseteq E\) such that the image of \(T \subseteq \Pi_E\) via \(p_{E/E'}: \Pi_E \rightarrow \Pi_{E'}\) is an \(E'\)-tripod, and, moreover, \((E')^2 \leq n - 1\). Thus, it follows immediately from Claim 3.16.A, together with Remark 3.4.1 — by replacing \(T, E\) by \(p_{E/E'}(T), E'\), respectively — that, to verify the first displayed equality of assertion (iii), we may assume without loss of generality that \(E \neq \{1, \cdots, n\}\). Then the first displayed equality of assertion (iii) follows immediately from Lemma 3.14, (iv); the portion of Lemma 3.15, (i) [where we observe that the “\(T\)” of Lemma 3.15 differs from the \(T\) of the present discussion!], concerning \(\Pi_n\) [cf. condition (a) of Lemma 3.15]. This completes the proof of the first displayed equality of assertion (iii).

Next, suppose that condition (iii-1) is satisfied; thus, there exists a subset \(E' \subseteq E\) such that the image \(p_{E/E'}^\Pi(T) \subseteq \Pi_E\) is an \(E'\)-tripod, and, moreover, \((E')^2 \leq n - 2\). Then — by replacing \(T, E\) by \(p_{E/E'}^\Pi(T), E'\), respectively [and applying Claim 3.16.A] — we may assume without loss of generality that \(E^2 \leq n - 2\). Thus, by applying [CbTP], Theorem A, (ii), we conclude that the second displayed equality of assertion (iii) follows immediately from the first displayed equality of assertion (iii).

Next, suppose that condition (iii-2) is satisfied. Then — by replacing \(T, E\) by the \(p_{E/E'}^\Pi(T), E'\) in condition (iii-2) [and applying Claim 3.16.A] — we may assume without loss of generality that \(E \neq \{1, \cdots, n\}\), and, moreover, that \(T\) is a cusp-supporting \(E\)-tripod. Then it follows immediately from Lemma 3.14, (iv); the portion of Lemma 3.15, (i), concerning \(\Pi_n\) [cf. condition (b) of Lemma 3.15], that the second displayed equality of assertion (iii) holds. This completes the proof of assertion (iii).

Next, we verify assertion (iv). If either condition (iv-1) or condition (iv-3) is satisfied, then one reduces immediately to the case where
\( n = 2 \), in which case it follows immediately from Lemma 3.13, (i), that \( \alpha \in \text{Out}^F(\Pi_n)[T] \). If condition (iv-1) is satisfied, then one reduces immediately to the case where \( n = 2 \), in which case it follows immediately from Lemma 3.13, (i), that \( \alpha \in \text{Out}^F(\Pi_n)[T : \{C\}] \). If both condition (iv-1) and condition (iv-3) are satisfied, then — by applying a suitable specialization isomorphism [cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1] — one reduces immediately to the case where \( n = 2 \) and \( \text{Node}(\mathcal{G}) = \emptyset \), in which case it follows immediately from Lemma 3.15, (i), that \( \alpha \in \text{Out}^F(\Pi_n)[T : \{C\}] \). Finally, if condition (iv-2) is satisfied, then, by applying [CbTpI], Theorem A, (ii), one reduces immediately to the case where \( n \) is taken to be \( n - 1 \), and condition (iv-1) is satisfied. This completes the proof of assertion (iv).

Finally, we verify assertion (v). First, we claim that the following assertion holds:

**Claim 3.16.B:** \( \text{Out}^F(\Pi_n) = \text{Out}^F(\Pi_n)[T] \).

Indeed, to verify Claim 3.16.B, by reordering the factors of \( X_n \), we may assume without loss of generality that \( E = \{1, 2, 3\} \). Let \( \bar{\alpha} \in \text{Aut}^F(\Pi_n) \). Then since \( n \geq 3 \), it follows immediately from [CbTpI], Theorem A, (ii), together with Lemma 3.2, (iv), that the automorphism of \( \Pi_{2/1} \) determined by \( \bar{\alpha} \) preserves the \( \Pi_{2/1} \)-conjugacy class of cuspidal subgroups of \( \Pi_{2/1} \) associated to the [unique — cf. Lemma 3.2, (ii)] diagonal cusp. Thus, it follows immediately from assertion (iv) in the case where condition (iv-3) is satisfied that the automorphism of \( \Pi_3 \) determined by \( \bar{\alpha} \) preserves the \( \Pi_3 \)-conjugacy class of \( T \subseteq \Pi_3 \). This completes the proof of Claim 3.16.B.

Next, we claim that the following assertion holds:

**Claim 3.16.C:** \( \text{Out}^F(\Pi_n)[T] = \text{Out}^F(\Pi_n)[T : \{\Delta\}] \).

Indeed, since \( n \geq 3 \), this follows immediately from Theorem 2.3, (iv), together with a similar argument to the argument used in the proof of [CmbCsp], Corollary 3.4, (i). This completes the proof of Claim 3.16.C.

Now it follows immediately from Claims 3.16.B, 3.16.C that we have an equality \( \text{Out}^F(\Pi_n) = \text{Out}^F(\Pi_n)[T : \{\Delta\}] \). Thus, it follows from assertion (ii) and the first displayed equality of assertion (iii), together with Theorem 2.3, (ii), that, to complete the proof of the content of the first two displays of assertion (v), it suffices to verify the equality \( \text{Out}^{FC}(\Pi_n) = \text{Out}^{FC}(\Pi_n)[T : \{C\}] \). On the other hand, this follows immediately from the portion of Lemma 3.15, (i), concerning \( \alpha|_T \). [Note that one verifies easily that every central tripod arises from a cusp.]

Thus, it remains to verify the equality of the final display of assertion (v). In light of what has already been verified [cf. also Theorem 2.3, (ii)], to verify the final equality of assertion (v), it suffices to verify the condition "+" on the right-hand side of this equality. On the other hand, it follows immediately — by replacing an element of the
left-hand side of the equality under consideration by a composite of the element with a suitable automorphism arising from an element of \( \text{Out}^{\text{FC}}(\Pi_4) \) [cf. the equality of the first display of assertion (v)] — from [CmbCsp], Lemma 2.4, that it suffices to verify the condition “+” on an element of the left-hand side of the equality under consideration that induces the \text{identity automorphism} on \text{Cusp}(\mathcal{G}). Then the equality under consideration follows immediately, in light of the assumption that \( r \neq 0 \), by first applying Lemma 3.15, (i) [in the case where we take the “\( E \) of loc. cit.” to be a subset of \( E \) of cardinality two, and we apply the argument involving \text{specialization isomorphisms} applied in the proof of assertion (iv)], and then applying Lemma 3.15, (i), (ii) [in the case where we take the “\( E \) of loc. cit.” to be \( E \)]. This completes the proof of assertion (v). \( \square \)

Remark 3.16.1. Theorem 3.16, (i), may be regarded as a \text{generalization} of [CmbCsp], Corollary 1.10, (ii). On the other hand, Theorem 3.16, (v), may be regarded as a \text{more precise version} of [CmbCsp], Corollary 3.4.

Theorem 3.17 (Synchronization of tripods in two dimensions).
In the notation of Theorem 3.16, suppose that \( n = 2 \), and that \( E^2 = 1 \); thus, one may regard the \( E \)-tripod \( T \) of \( \Pi_2 \) as a vertical subgroup of \( \Pi_E \rightarrow \Pi_G \) associated to a vertex \( v_T \in \text{Vert}(\mathcal{G}) \) of type \((0,3)\) [cf. Definition 3.1, (ii)]. Let \( E' \subseteq \{1, \ldots, n\} \) and \( T' \subseteq \Pi_{E'} \) an \( E' \)-tripod of \( \Pi_n \). Then the following hold:

(i) Suppose that there exists an edge \( e \in \mathcal{E}(v_T) \) from which \( T' \) arises [cf. Definition 3.7, (i)]. [Thus, it holds that \( E' = \{1, 2\} \).] Then it holds that
\[
\text{Out}^{\text{FC}}(\Pi_n)[T: \{\{C\}, \Delta\}] \subseteq \text{Out}^{\text{FC}}(\Pi_n)[T': \{\{C\}, \Delta, +\}]
\]
[cf. the notational conventions of Theorem 3.16, (i)]. Moreover, there exists a \text{geometric} [cf. Definition 3.4, (ii)] outer isomorphism \( T \rightarrow T' \) such that the diagram
\[
\text{Out}^{\text{FC}}(\Pi_n)[T: \{\{C\}, \Delta\}] \subseteq \text{Out}^{\text{FC}}(\Pi_n)[T': \{\{C\}, \Delta, +\}]
\]
\[
\mathfrak{T}_T \downarrow \quad \downarrow \mathfrak{T}_{T'}
\]
\[
\text{Out}(T) \rightarrow \text{Out}(T')
\]
[cf. the notation of Theorem 3.16, (i)] — where the lower horizontal arrow is the isomorphism induced by the outer isomorphism in question \( T \rightarrow T' \) — commutes.
(ii) Suppose that \((E')^4 = 1\). Thus, one may regard the \(E'\)-tripod \(T'\) of \(\Pi_n \cong \Pi_G\) associated to a vertex \(v_{E'} \in \text{Vert}(G)\) of type \((0, 3)\). Suppose, moreover, that \(\mathcal{N}(v_T) \cap \mathcal{N}(v_{E'}) \neq \emptyset\). Then there exists a geometric [cf. Definition 3.4, (ii)] outer isomorphism \(T \sim T'\) such that if we write

\[
\text{Out}^{FC}(\Pi_n)[T, T': \{\mid C\mid, \Delta\}] \]

then the diagram

\[
\begin{array}{ccc}
\text{Out}^{FC}(\Pi_n)[T, T': \{\mid C\mid, \Delta\}] & \xrightarrow{\circlearrowright} & \text{Out}^{FC}(\Pi_n)[T, T': \{\mid C\mid, \Delta\}] \\
\downarrow^{\pi_T} & & \downarrow^{\pi_{T'}} \\
\text{Out}(T) & \sim & \text{Out}(T')
\end{array}
\]

— where the lower horizontal arrow is the isomorphism induced by the outer isomorphism in question \(T \sim T'\) — commutes.

**Proof.** First, we verify assertion (i). Let us observe that the inclusion \(\text{Out}^{FC}(\Pi_n)[T : \{\mid C\mid\}] \subseteq \text{Out}^{FC}(\Pi_n)[T']\), hence also the inclusion \(\text{Out}^{FC}(\Pi_n)[T : \{\mid C\mid, \Delta\}] \subseteq \text{Out}^{FC}(\Pi_n)[T']\), follows immediately from Theorem 3.16, (iv), in the case where condition (iv-1) is satisfied. Thus, one verifies easily from Lemma 3.15, (i), (ii) [cf. also Lemma 3.14, (iv)], that the remainder of assertion (i) holds. This completes the proof of assertion (i). Next, we verify assertion (ii). It follows immediately from [CmbCsp], Proposition 1.2, (iii), that we may assume without loss of generality that \(E' = E\). Write \(T'' \subseteq \Pi_n\) for the \(\{1, 2\}\)-tripod of \(\Pi_n\) arising from \(e \in \mathcal{N}(v_T) \cap \mathcal{N}(v_{E'})\). Then it follows from assertion (i) that there exist geometric outer isomorphisms \(T \sim T'', T' \sim T''\) that satisfy the condition of assertion (i) [i.e., for the pairs \((T, T'')\) and \((T', T'')\)]. Thus, one verifies easily that the [necessarily geometric] outer isomorphism \(T \sim T'' \sim T'\) obtained by forming the composite of these two outer isomorphisms satisfies the condition of assertion (ii). This completes the proof of assertion (ii). \(\square\)

**Theorem 3.18 (Synchronization of tripods in three or more dimensions).** In the notation of Theorem 3.16, suppose that \(n \geq 3\). Then the following hold:

(i) It holds that

\[
\text{Out}^{FC}(\Pi_n)[T : \{\mid C\mid\}] = \text{Out}^{FC}(\Pi_n)[T : \{\mid C\mid, \Delta\}]
\]

[cf. the notational conventions of Theorem 3.16, (i)]. If, moreover, \(n \geq 4\) or \(r \neq 0\), then it holds that

\[
\text{Out}^{FC}(\Pi_n)[T : \{\mid C\mid\}] = \text{Out}^{FC}(\Pi_n)[T : \{\mid C\mid, \Delta, +\}]
\]
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(cf. the notational conventions of Theorem 3.16, (i)).

(ii) Let \( E' \subseteq \{1, \ldots, n\} \) and \( T' \subseteq \Pi_{E'} \) an \( E' \)-tripod of \( \Pi_n \). Then there exists a geometric [cf. Definition 3.4, (ii)] outer isomorphism \( T \sim T' \) such that if we write
\[
\text{Out}^{FC}(\Pi_n)[T,T' : \{|C|\}] \defeq \text{Out}^{FC}(\Pi_n)[T : \{|C|\}] \cap \text{Out}^{FC}(\Pi_n)[T' : \{|C'|\}],
\]
then the diagram
\[
\begin{array}{ccc}
\text{Out}^{FC}(\Pi_n)[T,T' : \{|C|\}] & \overset{\sim_T}{\longrightarrow} & \text{Out}^{FC}(\Pi_n)[T,T' : \{|C'|\}] \\
\downarrow \sim_T & & \downarrow \sim_{T'} \\
\text{Out}(T) & \overset{\sim}{\longrightarrow} & \text{Out}(T')
\end{array}
\]
(cf. the notation of Theorem 3.16, (i)) — where the lower horizontal arrow is the isomorphism induced by the outer isomorphism in question \( T \sim T' \) — commutes.

Proof. First, we verify the first displayed equality of assertion (i). Observe that it follows immediately from Lemma 3.8, (i), together with a similar argument to the argument applied in the proof of the first displayed equality of Theorem 3.16, (iii), that we may assume without loss of generality that \( T \) is \( E \)-strict [cf. Lemma 3.8, (ii)]. Now we apply induction on \( 3 - E^\sharp \in \{0, 1, 2\} \). If \( 3 - E^\sharp = 0 \), i.e., \( T \) is central [cf. Lemma 3.8, (ii)], then the first displayed equality of assertion (i) follows immediately from Theorem 3.16, (v). Now suppose that \( 3 - E^\sharp > 0 \), and that the induction hypothesis is in force. Let \( \alpha \in \text{Out}^{FC}(\Pi_n)[T : \{|C|\}] \). Then it follows immediately from Lemma 3.15, (i), (ii) [cf. also conditions (1), (2) of Lemma 3.15, (ii), where we note that the \( E, E', T, T' \) of the present discussion correspond, respectively, to the \( "E \setminus \{i\}" , "E" , "\Pi_n" , "T" \) of Lemma 3.15], that there exist a subset \( E \subseteq E' \subseteq \{1, \ldots, n\} \) and an \( E' \)-tripod \( T' \subseteq \Pi_{E'} \) such that \( 3 - (E')^\sharp < 3 - E^\sharp \), \( T' \subseteq \Pi_{E'} \) is \( E' \)-strict, and \( \alpha \in \text{Out}^{FC}(\Pi_n)[T' : \{|C'|\}] \) [cf. Lemma 3.15, (i)]. Thus, it follows immediately from the induction hypothesis that \( \alpha \in \text{Out}^{FC}(\Pi_n)[T' : \{|C'|, \Delta\}] \). In particular, it follows immediately from Lemma 3.15, (ii), that — for a suitable choice of the pair \( (E', T') \) [cf. the statement of Lemma 3.15, (ii)] — the actions of \( \alpha \) on \( T \) and \( T' \) may be related by means of a geometric outer isomorphism, which thus implies that \( \alpha \in \text{Out}^{FC}(\Pi_n)[T : \{|C|, \Delta\}] \) [cf. Remark 3.4.1]. This completes the proof of the first displayed equality of assertion (i).

Next, we verify assertion (ii). First, we claim that the following assertion holds:

Claim 3.18.A: If both \( T \) and \( T' \) are central, then the pair \( (T,T') \) satisfies the property stated in assertion (ii).
Indeed, this assertion follows immediately from the commutativity of the displayed diagram of Theorem 3.16, (i).

Next, we claim that the following assertion holds:

Claim 3.18.B: Suppose that $T$ is $E$-strict, and that $E^2 \neq 3$ [i.e., $E^2 \in \{1, 2\}$ — cf. Lemma 3.8, (ii)]. Then there exist a subset $E \subseteq E'' \subseteq \{1, \ldots, n\}$ and an $E''$-tripod $T'' \subseteq \Pi_{E''}$ such that $T''$ is $E''$-strict, $\text{Out}^{\text{FC}}(\Pi_n)[T : \{\{C\}\}] \subseteq \text{Out}^{\text{FC}}(\Pi_n)[T'' : \{\{C\}\}]$, and, moreover, the pair $(T, T'')$ satisfies the property stated in assertion (ii) [i.e., where one takes $"T'"$ to be $T'\prime$].

Indeed, this follows immediately from Lemma 3.15, (i), (ii) [cf. also conditions (1), (2) of Lemma 3.15, (ii), where we note that the $E$, $E'$, $T$, $T'$ of the present discussion correspond, respectively, to the "$E \setminus \{i\}$", "$E'$", "$\Pi_n'$", "$T'$" of Lemma 3.15], together with the first displayed equality of assertion (i). This completes the proof of Claim 3.18.B.

To verify assertion (ii), let us observe that it follows immediately from Lemma 3.8, (i), (ii) [cf. also conditions (1), (2) of Lemma 3.15, (ii), where we note that the $E$, $E''$, $T$, $T''$ of the present discussion correspond, respectively, to the "$E \setminus \{i\}$", "$E''$, "$\Pi_n'$", "$T'$" of Lemma 3.15], together with the first displayed equality of Theorem 3.16, (iii), that we may assume without loss of generality that $T$ is $E$-strict; in particular, $E^2 \in \{1, 2, 3\}$ [cf. Lemma 3.8, (ii)]. Next, let us observe that, by comparing two arbitrary tripods of $\Pi_n$ to a fixed central tripod of $\Pi_n$ [and applying Theorem 3.16, (v)], one may reduce immediately to the case where $T'$ is central. Moreover, by successive application of Claim 3.18.B, one reduces immediately to the case where both $T$ and $T'$ are central, which was verified in Claim 3.18.A. This completes the proof of assertion (ii). Finally, the second displayed equality of assertion (i) follows immediately from assertion (ii), together with Theorem 3.16, (v). This completes the proof of Theorem 3.18. □

**Definition 3.19.** Suppose that $n \geq 3$. Let us write

$$\Pi_{\text{ipd}}$$

for the $i$-central $E$-tripod of $\Pi_n$ [cf. Definitions 3.3, (i); 3.7, (ii)], where $E \subseteq \{1, \ldots, n\}$ is a subset of cardinality 3, and $i \in E$. Then it follows from Theorem 3.16, (i), (v), that one has a natural homomorphism

$$\mathfrak{T}_{\Pi_{\text{ipd}}}: \text{Out}^{\text{FC}}(\Pi_n) = \text{Out}^{\text{FC}}(\Pi_n)[\Pi_{\text{ipd}} : \{\{C\}, \Delta\}] \longrightarrow \text{Out}^{\text{C}}(\Pi_{\text{ipd}})^\Delta$$

[cf. Definition 3.4, (i)], which is in fact independent of $E$ and $i$ [cf. Theorem 3.16, (i)]. We shall refer to this homomorphism as the tripod homomorphism associated to $\Pi_n$ and write

$$\text{Out}^{\text{FC}}(\Pi_n)^\geq \subseteq \text{Out}^{\text{FC}}(\Pi_n)$$

for the kernel of this homomorphism [cf. Remark 3.19.1 below]. Note that it follows from Theorem 3.16, (v), that if $n \geq 4$ or $r \neq 0$, then the
image of the tripod homomorphism is contained in \( \text{Out}^C(\Pi_{\text{tpd}})\Delta^+ \subseteq \text{Out}^C(\Pi_{\text{tpd}})\Delta \) [cf. Definition 3.4, (i)]. If \( n \geq 4 \) or \( r \neq 0 \), then \( \Sigma_{\Pi_{\text{tpd}}} \) may also be regarded as a homomorphism defined on \( \text{Out}^F(\Pi_n) (= \text{Out}^F(\Pi_n)) \) — cf. Theorem 2.3, (ii)); in this case, we shall write \( \text{Out}^F(\Pi_n)_{\text{geo}} \stackrel{\text{def}}{=} \text{Out}^F(\Pi_n)^{\text{geo}} \).

Remark 3.19.1. Let us recall that if we write \( \pi_1((\mathcal{M}_{g,[r]})_\mathbb{Q}) \) for the \( \acute{\text{e}} \text{tale fundamental group of the moduli stack } (\mathcal{M}_{g,[r]})_\mathbb{Q} \text{ of hyperbolic curves of type } (g,r) \text{ over } \mathbb{Q} \text{ [cf. the discussion entitled "Curves" in } \S 0], then we have a natural outer homomorphism

\[
\pi_1((\mathcal{M}_{g,[r]})_\mathbb{Q}) \longrightarrow \text{Out}^F(\Pi_n).
\]

Suppose that \( n \geq 4 \). Then \( \text{Out}^F(\Pi_n) = \text{Out}^F(\Pi_n) \) does not depend on \( n \) [cf. Theorem 2.3, (ii); [NodNon], Theorem B]. Moreover, one verifies easily that the image of the geometric fundamental group \( \pi_1((\mathcal{M}_{g,[r]})_\mathbb{Q}) \subseteq \pi_1((\mathcal{M}_{g,[r]})_\mathbb{Q}) \) — where we use the notation \( \overline{\mathbb{Q}} \) to denote an algebraic closure of \( \mathbb{Q} \) — via the above displayed outer homomorphism is contained in the kernel \( \text{Out}^F(\Pi_n)^{\text{geo}} \subseteq \text{Out}^F(\Pi_n) \) of the tripod homomorphism associated to \( \Pi_n \) [cf. Definition 3.19]. Thus, the outer homomorphism of the above display fits into a commutative diagram of profinite groups

\[
\begin{array}{cccc}
1 & \longrightarrow & \pi_1((\mathcal{M}_{g,[r]})_\mathbb{Q}) & \longrightarrow & \pi_1((\mathcal{M}_{g,[r]})_\mathbb{Q}) & \longrightarrow & \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & \\
1 & \longrightarrow & \text{Out}^F(\Pi_n)^{\text{geo}} & \longrightarrow & \text{Out}^F(\Pi_n) & \xrightarrow{\Sigma_{\Pi_{\text{tpd}}}} & \text{Out}^C(\Pi_{\text{tpd}})\Delta^+ & \longrightarrow & 1
\end{array}
\]

— where the horizontal sequences are exact. In \( \S 4 \) below, we shall verify that the lower right-hand horizontal arrow \( \Sigma_{\Pi_{\text{tpd}}} \) is surjective [cf. Corollary 4.15 below]. On the other hand, if \( \Sigma \) is the set of all prime numbers, then it follows from Beli’s Theorem that the right-hand vertical arrow is injective; moreover, the surjectivity of the right-hand vertical arrow has been conjectured in the theory of the Grothendieck-Teichmüller group. From this point of view, one may regard the quotient \( \text{Out}^F(\Pi_n) / \Sigma_{\Pi_{\text{tpd}}} \longrightarrow \text{Out}^C(\Pi_{\text{tpd}})\Delta^+ \) as a sort of arithmetic quotient of \( \text{Out}^F(\Pi_n) \) and the subgroup \( \text{Out}^F(\Pi_n)^{\text{geo}} \subseteq \text{Out}^F(\Pi_n) \) as a sort of geometric portion of \( \text{Out}^F(\Pi_n) \).

Definition 3.20. Let \( m \) be a positive integer and \( Y^\log \) a stable log curve over \( (\text{Spec } k)^\log \). For each nonnegative integer \( i \), write \( Y^i \Pi_i \) for the “\( \Pi_i \)” that occurs in the case where we take “\( X^\log \)” to be \( Y^\log \). Then we shall say that an isomorphism (respectively, outer isomorphism)

\( \Pi_1 \overset{\sim}{\longrightarrow} Y^i \Pi_1 \) is \( m \)-cuspidalizable if it arises from a [necessarily unique,
Definition 3.20. Then the following hold:

(i) The outer isomorphism \( Y_{\alpha} \) of \( Y_{\Pi_{1}} \) determined by \( \alpha \) is \( n \)-cuspidalizable [cf. Definition 3.20].

(ii) Suppose that \( n \geq 3 \). Let \( \Pi^{\text{pd}} \subseteq \Pi_{3} \), \( Y_{\Pi^{\text{pd}}} \subseteq Y_{\Pi_{3}} \) be 1-central \( \{1,2,3\}\)-tripods [cf. Definitions 3.3, (i); 3.7, (ii)] of \( \Pi_{n} \), \( Y_{\Pi_{n}} \), respectively. Write \( \alpha_{n}, Y_{\alpha_{n}} \) for the respective FC-admissible outer isomorphisms of \( \Pi_{n} \), \( Y_{\Pi_{n}} \) determined by the \( n \)-cuspidalizable outer isomorphisms \( \alpha, Y_{\alpha} \) [cf. (i)]. Then there exists a geometric [cf. Definition 3.4, (ii)] outer isomorphism \( \phi^{\text{pd}}: \Pi^{\text{pd}} \to Y_{\Pi^{\text{pd}}} \) such that the outer isomorphism \( \xi_{\Pi^{\text{pd}}(\alpha_{n})} \) [cf. Definition 3.19] of \( \Pi^{\text{pd}} \) is compatible with the outer isomorphism \( \xi_{Y_{\Pi^{\text{pd}}}(Y_{\alpha_{n}})} \) [cf. Definition 3.19] of \( Y_{\Pi^{\text{pd}}} \) relative to \( \phi^{\text{pd}} \).

Proof. First, let us observe that, to verify Proposition 3.21 — by applying a suitable specialization isomorphism [cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1] — we may assume without loss of generality that \( X_{n} \) and \( Y_{n} \) are smooth log curves over \( (\text{Spec} \, k)^{\log} \). Write \( (U_{X})_{n}, (U_{Y})_{n} \) for the open subschemes of \( X_{n}, Y_{n} \) determined by the \( 1 \)-interiors [cf. [MzTa], Definition 5.1, (i)] of \( X_{n}, Y_{n} \), respectively. [Here, we note that in the present situation, the \( 0 \)-interior of \( (\text{Spec} \, k)^{\log} \), hence also of \( X_{n}^{\log}, Y_{n}^{\log} \), is empty!]

Thus, one verifies easily that \( U_{X} \subseteq (U_{X})_{1}, U_{Y} \subseteq (U_{Y})_{1} \) are hyperbolic curves over \( k \), and that \( (U_{X})_{n}, (U_{Y})_{n} \) are naturally isomorphic to the \( n \)-th configuration spaces of \( U_{X}, U_{Y} \), respectively. Write \( U_{X}^{\times n}, U_{Y}^{\times n} \) for the respective fiber products of \( n \) copies of \( U_{X}, U_{Y} \) over \( k; \Pi_{1}^{\times n}, Y_{\Pi_{1}^{\times n}} \) for the respective direct products of \( n \) copies of \( \Pi_{1}, Y_{\Pi_{1}}; V_{n} \) for the fiber product of the natural open immersion \( (U_{X})_{n} \to U_{X}^{\times n} \) and the natural finite étale covering \( U_{Y}^{\times n} \to U_{X}^{\times n} \). Then one verifies easily that the resulting open immersion \( V_{n} \to U_{Y}^{\times n} \) factors through the natural open immersion \( (U_{Y})_{n} \to U_{Y}^{\times n} \), i.e., we obtain an open immersion
\[ V_n \hookrightarrow (U_Y)_n. \] That is to say, whereas \((U_Y)_n\) is the open subscheme of \(U^\times Y\) obtained by removing the various diagonals of \(U^\times Y\), the scheme \(V_n\) may be thought of as the open subscheme of \(U^\times Y\) obtained by removing the various Galois conjugates of these diagonals, relative to the action of the Galois group \(\text{Gal}(U^\times Y/U^\times X) = \text{Gal}(U_Y/U_X)^\times\). In particular, we obtain a natural outer isomorphism and outer surjection
\[
\Pi_n \times_{\Pi_1^\times} \Pi_1^\times \overset{\sim}{\longrightarrow} \Pi V_n \twoheadrightarrow \Pi_1
\]
— where we write \(\Pi V_n\) for the maximal pro-\(\Sigma\) quotient of the étale fundamental group of \(V_n\).

Now we verify assertion (i). Let \(\tilde{\alpha}_n\) be an FC-admissible automorphism of \(\Pi_n\) that lies over the automorphism \(\tilde{\alpha}\) of \(\Pi_1\) with respect to each of the \(n\) natural projections \(\Pi_n \twoheadrightarrow \Pi_1\). Then since \(\tilde{\alpha}_n\) is FC-admissible and commutes with the image of the natural inclusion \(\mathcal{G}_n \hookrightarrow \text{Out}(\Pi_n)\) [cf. [NodNon], Theorem B], one verifies easily, in light of the description given above of \(V_n\), that the isomorphism of \(\Pi_n \times_{\Pi_1^\times} \Pi_1^\times \) induced by \(\tilde{\alpha}_n\) and \(\gamma \alpha\) preserve the inertia subgroups associated to each irreducible component of the complement \(U^\times Y \setminus V_n\).

Thus, since [by the Zariski-Nagata purity theorem] the inertia subgroups of the irreducible components of the complement \((U_Y)_n \setminus V_n\) normally topologically generate the kernel of the above outer surjection \(\Pi V_n \twoheadrightarrow \Pi_1\), we conclude, by applying the morphisms of the above display, that the isomorphism of \(\Pi_n \times_{\Pi_1^\times} \Pi_1^\times \) induced by \(\tilde{\alpha}_n\) and \(\gamma \alpha\) determines an FC-admissible isomorphism of \(\Pi_1\). Moreover, one verifies easily that the resulting isomorphism of \(\Pi_1\) lies over the automorphism \(\gamma \alpha\) of \(\Pi_1\). This completes the proof of assertion (i).

Next, we verify assertion (ii). First, let us observe that the natural inclusion \(\Pi^{\text{tpd}} \hookrightarrow \Pi_3\), together with the trivial homomorphism \(\Pi^{\text{tpd}} \to (\{1\} \hookrightarrow \Pi_1^3)\) [cf. Definition 3.3, (ii); Lemma 3.6, (v); Definition 3.7, (ii)], determines an injection \(\Pi^{\text{tpd}} \hookrightarrow \Pi_3 \times_{\Pi_1^\times 3} \Pi_1^\times 3 \hookrightarrow \Pi V_3\). Moreover, it follows immediately from the fact that the blow-up operation that gives rise to a central tripod is compatible with étale localization [cf. the discussion of [CmbCsp], Definition 1.8] that — after possibly replacing \(\gamma \Pi^{\text{tpd}} \subseteq \gamma \Pi_3\) by a suitable \(\gamma \Pi_3\)-conjugate of \(\gamma \Pi^{\text{tpd}}\) — the composite of this injection \(\Pi^{\text{tpd}} \hookrightarrow \Pi V_3\) with the natural outer surjection \(\Pi V_3 \twoheadrightarrow \gamma \Pi_3\) of the above display determines a geometric outer [cf. Lemma 3.12, (i)] isomorphism \(\phi^{\text{tpd}}: \Pi^{\text{tpd}} \overset{\sim}{\twoheadrightarrow} \gamma \Pi^{\text{tpd}} \subseteq \gamma \Pi_3\). On the other hand, one verifies easily [cf. the construction of \(\gamma \alpha_n\) given in the proof of assertion (i)] that this outer isomorphism \(\phi^{\text{tpd}}\) satisfies the property stated in assertion (ii). This completes the proof of assertion (ii). \(\square\)
Corollary 3.22 (Non-surjectivity result). In the notation of Theorem 3.16, suppose that \((g, r) \not\in \{(0, 3); (1, 1)\}\). Then the natural injection
\[
\text{Out}^{\text{FC}}(\Pi_2) \hookrightarrow \text{Out}^{\text{FC}}(\Pi_1)
\]
of [NodNon], Theorem B, is not surjective.

Proof. First, let us observe — by considering a suitable stable log curve of type \((g, r)\) over \((\text{Spec} \, k)^{\log}\) and applying a suitable specialization isomorphism [cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1] — that, to verify Corollary 3.22, we may assume without loss of generality that \(\mathcal{G}\) is totally degenerate [cf. [CbTpI], Definition 2.3, (iv)], i.e., that every vertex of \(\mathcal{G}\) is a tripod of \(X^\log\) [cf. Definition 3.1, (v)]. Note that [since \((g, r) \not\in \{(0, 3); (1, 1)\}\)] this implies that \(\text{Vert}(\mathcal{G})^2 \geq 2\). Let us fix a vertex \(v_0 \in \text{Vert}(\mathcal{G})\) and write \(\alpha_{v_0} \overset{\text{def}}{=} \text{id}_{\mathcal{G}|v_0} \in \text{Aut}^{[\text{grph}]}(\mathcal{G}|v_0)\) [cf. [CbTpI], Definitions 2.1, (iii), and 2.6, (i); Remark 4.1.2 of the present paper]. For each \(v \in \text{Vert}(\mathcal{G}) \setminus \{v_0\}\), let \(\alpha_v \in \text{Aut}^{[\text{grph}]}(\mathcal{G}|v)\) be a nontrivial automorphism of \(\mathcal{G}|v\) such that \(\alpha_v \in \text{Out}^C(\Pi|v)_\Delta\), and, moreover, \(\chi_{\mathcal{G}|v}(\alpha_v) = 1\) [cf. [CbTpI], Definition 3.8, (ii)]. Here, we note that since the image of the natural outer Galois representation of the absolute Galois group of \(\mathbb{Q}\) associated to \(\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\}\) is contained in \(\text{“Out}^C(-)_\Delta\)”, by considering a nontrivial element of this image whose image via the cyclotomic character is trivial, one verifies immediately [e.g., by applying [LocAn], Theorem A] that such an automorphism \(\alpha_v \in \text{Aut}^{[\text{grph}]}(\mathcal{G}|v)\) always exists. Then it follows immediately from [CbTpI], Theorem B, (iii), that there exists an automorphism \(\alpha \in \text{Aut}^{[\text{grph}]}(\mathcal{G})\) such that \(\rho_{\mathcal{G}}^{\text{Vert}}(\alpha) = (\alpha_v)_{v \in \text{Vert}(\mathcal{G})}\). Now assume that there exists an automorphism \(\alpha_2 \in \text{Out}^{\text{FC}}(\Pi_2)\) such that \(\alpha \in \text{Aut}^{[\text{grph}]}(\mathcal{G})\) \(\subseteq \text{Out}(\Pi|v)^\Delta\) is equal to the image of \(\alpha_2\) via the injection in question \(\text{Out}^{\text{FC}}(\Pi_2) \hookrightarrow \text{Out}^{\text{FC}}(\Pi_1)\). Then, for each \(v \in \text{Vert}(\mathcal{G})\), since \(\alpha_v \in \text{Out}^C(\Pi|v)_\Delta\), and \(\alpha \in \text{Aut}^{[\text{grph}]}(\mathcal{G})\), it follows immediately from the various definitions involved that \(\alpha_2 \in \text{Out}^{\text{FC}}(\Pi_2)|\Pi_v : \{|C|, \Delta\}\) — where we use the notation \(\Pi_v\) to denote a vertical subgroup of \(\Pi|v\) associated to \(v \in \text{Vert}(\mathcal{G})\). Thus, since \(\alpha_{v_0} \overset{\text{def}}{=} \text{id}_{\mathcal{G}|v_0}\), it follows from Theorem 3.17, (ii), that \(\alpha_v = \text{id}_{\mathcal{G}|v}\) for every \(v \in \text{Vert}(\mathcal{G})\), in contradiction to the fact that for \(v \in \text{Vert}(\mathcal{G}) \setminus \{v_0\}\), the automorphism \(\alpha_v \in \text{Aut}^{[\text{grph}]}(\mathcal{G}|v)\) is nontrivial. This completes the proof of Corollary 3.22. \qed

Remark 3.22.1.

(i) Let us recall from [NodNon], Corollary 6.6, that, in the discrete case, the homomorphism that corresponds to the homomorphism discussed in Corollary 3.22 is, in fact, surjective;
moreover, this surjectivity may be regarded as an immediate consequence of the Dehn-Nielsen-Baer theorem — cf. the proof of [CmbCsp], Theorem 5.1, (ii). This phenomenon illustrates that, in general, analogous constructions in the discrete and profinite cases may in fact exhibit quite different behavior.

(ii) In the context of (i), we recall another famous example of substantially different behavior in the discrete and profinite cases. As is well-known, in classical algebraic topology, singular cohomology with coefficients in $\mathbb{Z}$ yields a “good” cohomology theory with coefficients in $\mathbb{Z}$. On the other hand, in the 1960’s, Serre gave an argument involving supersingular elliptic curves in characteristic $p > 0$ which shows that such a “good” cohomology theory with coefficients in $\mathbb{Z}$ [or even in $\mathbb{Z}_p$] cannot exist for smooth varieties of positive characteristic.

(iii) In [Lch], various conjectures concerning [in the notation of the present paper] the profinite group “Out($\Pi_1$)” were introduced. However, at the time of writing, the authors of the present paper were unable to find any justification for the validity of these conjectures that goes beyond the observation that the discrete analogues of these conjectures are indeed valid. That is to say, there does not appear to exist any justification for excluding the possibility that — just as in the case of the examples discussed in (i), (ii), i.e., the Dehn-Nielsen-Baer theorem and singular cohomology with coefficients in $\mathbb{Z}$ — the discrete and profinite cases exhibit substantially different behavior. In particular, it appears to the authors that it is desirable that this issue be addressed in a satisfactory fashion in the context of these conjectures.

Remark 3.22.2. As discussed in Remark 3.22.1, (i), in the discrete case, the homomorphism that corresponds to the homomorphism discussed in Corollary 3.22 is, in fact, bijective. The proof of Corollary 3.22 fails in the discrete case for the following reason: The pro-$\Sigma$ “$\Pi_1$” of a tripod admits nontrivial $C$-admissible automorphisms that commute with the outer modular symmetries and, moreover, lie in the kernel of the cyclotomic character [cf. the proof of Corollary 3.22]. By contrast, the discrete “$\Pi_1$” of a tripod does not admit such automorphisms. Indeed, it follows from a classical result of Nielsen [cf. [CmbCsp], Remark 5.3.1] that the discrete “Out$^C(\Pi_1)^{\text{cusp}}$” in the case of a tripod is a finite group of order 2 whose unique nontrivial element arises from complex conjugation.
Remark 3.22.3. It follows from [NodNon], Theorem B, together with Corollary 3.22, that if \((g, r) \notin \{(0, 3); (1, 1)\}\), then the homomorphism \(\text{Out}^{FC}(\Pi_{n+1}) \to \text{Out}^{FC}(\Pi_n)\) of [NodNon], Theorem B, fits into the following sequences of homomorphisms of profinite groups: If \(r \neq 0\), then for any \(n \geq 3\),
\[
\text{Out}^{FC}(\Pi_n) \xrightarrow{\sim} \text{Out}^{FC}(\Pi_3) \xrightarrow{\phi_2} \text{Out}^{FC}(\Pi_2) \xrightarrow{\phi_1} \text{Out}^{FC}(\Pi_1).
\]
If \(r = 0\), then for any \(n \geq 4\),
\[
\text{Out}^{FC}(\Pi_n) \xrightarrow{\sim} \text{Out}^{FC}(\Pi_4) \xrightarrow{\phi_2} \text{Out}^{FC}(\Pi_3) \xrightarrow{\phi_1} \text{Out}^{FC}(\Pi_2) \xrightarrow{\phi_1} \text{Out}^{FC}(\Pi_1).
\]

Definition 3.23. Let \(\Sigma_0\) be a nonempty set of prime numbers and \(\mathcal{G}_0\) a semi-graph of anabelioids of pro-\(\Sigma_0\) PSC-type. Write \(\Pi_{\mathcal{G}_0}\) for the \([\text{pro-}\Sigma_0]\) fundamental group of \(\mathcal{G}_0\).

(i) Let \(\mathcal{H}\) be a semi-graph of anabelioids of pro-\(\Sigma_0\) PSC-type, \(S \subseteq \text{Node}(\mathcal{H})\), and \(\phi: \mathcal{H} \to \mathcal{G}_0\) [cf. [CbTpI], Definition 2.8, for more on this notation] an isomorphism \([\text{of semi-graphs of anabelioids of PSC-type}]\). Then we shall refer to the triple \((\mathcal{H}, S, \phi)\) as a \textit{degeneration structure} on \(\mathcal{G}_0\).

(ii) Let \((\mathcal{H}_1, S_1, \phi_1), (\mathcal{H}_2, S_2, \phi_2)\) be two degeneration structures on \(\mathcal{G}_0\) [cf. (i)]. Then we shall write
\[
(\mathcal{H}_2, S_2, \phi_2) \preceq (\mathcal{H}_1, S_1, \phi_1)
\]
if there exist a subset \(S_{2,1} \subseteq S_2\) of \(S_2\) and a(n) \textit{uniquely determined, by \(\phi_1\) and \(\phi_2\)} — cf. [CmbGC], Proposition 1.5, (ii) — isomorphism \(\phi_{2,1}: (\mathcal{H}_2)_{\to S_{2,1}} \to (\mathcal{H}_1)_{\to S_1}\) [i.e., a degeneration structure \((\mathcal{H}_2, S_{2,1}, \phi_{2,1})\) on \(\mathcal{H}_1\)] such that \(\phi_{2,1}\) maps \(S_2 \setminus S_{2,1}\) bijectively onto \(S_1\), and the diagram
\[
\begin{array}{ccc}
((\mathcal{H}_2)_{\to S_{2,1}})_{\to S_2 \setminus S_{2,1}} & \xrightarrow{\sim} & (\mathcal{H}_1)_{\to S_1} \\
\downarrow \phi_{2,1} & & \downarrow \phi_1 \\
(\mathcal{H}_2)_{\to S_2} & \xrightarrow{\phi_2} & \mathcal{G}_0
\end{array}
\]
— where the upper horizontal arrow is the isomorphism induced by \(\phi_{2,1}\), and the left-hand vertical arrow is the natural isomorphism — \textit{commutes}. [Here, we note that the subset \(S_{2,1}\) is also \textit{uniquely determined} by \(\phi_1\) and \(\phi_2\) — cf. [CmbGC], Proposition 1.2, (i).]

(iii) Let \((\mathcal{H}_1, S_1, \phi_1), (\mathcal{H}_2, S_2, \phi_2)\) be two degeneration structures on \(\mathcal{G}_0\) [cf. (i)]. Then we shall say that \((\mathcal{H}_1, S_1, \phi_1)\) is \textit{co-Dehn} to \((\mathcal{H}_2, S_2, \phi_2)\) if there exists a degeneration structure \((\mathcal{H}_3, S_3, \phi_3)\) on \(\mathcal{G}_0\) such that
\[
(\mathcal{H}_3, S_3, \phi_3) \preceq (\mathcal{H}_1, S_1, \phi_1); \quad (\mathcal{H}_3, S_3, \phi_3) \preceq (\mathcal{H}_2, S_2, \phi_2)
\]
(iv) Let \((\mathcal{H}, S, \phi)\) be a degeneration structure on \(G_0\) [cf. (i)] and \(\alpha \in \text{Out}(\Pi_{G_0})\). Then we shall say that \(\alpha\) is an \((\mathcal{H}, S, \phi)\)-Dehn multi-twist of \(G_0\) if \(\alpha\) is contained in the image of the composite
\[
\text{Dehn}(\mathcal{H}) \hookrightarrow \text{Out}(\Pi_{\mathcal{H}}) \overset{\sim}{\twoheadrightarrow} \text{Out}(\Pi_{\mathcal{H}_{\rightarrow S}}) \overset{\sim}{\twoheadrightarrow} \text{Out}(\Pi_{G_0})
\]
— where the first arrow is the natural inclusion [cf. [CbTpI], Definition 4.4], the second arrow is the isomorphism determined by \(\Phi_{\mathcal{H}_{\rightarrow S}}\) [cf. [CbTpI], Definition 2.10], and the third arrow is the isomorphism determined by \(\phi\). We shall say that \(\alpha\) is a nondegenerate (respectively, positive definite) \((\mathcal{H}, S, \phi)\)-Dehn multi-twist of \(G_0\) if \(\alpha\) is the image of a nondegenerate [cf. [CbTpI], Definition 5.8, (ii)] (respectively, positive definite [cf. [CbTpI], Definition 5.8, (iii)]) profinite Dehn multi-twist of \(\mathcal{H}\) via the above composite.

(v) Let \(m\) be a positive integer and \(Y^{\log}\) a stable log curve over \((\text{Spec} \, k)^{\log}\). If \(m \geq 2\), then suppose that \(\Sigma_0\) is either equal to \(\mathfrak{Primes}\) or of cardinality one. For each nonnegative integer \(i\), write \(\gamma\Pi_i\) (respectively, \(\mathcal{H}\)) for the “\(\Pi_i\)” (respectively, “\(G\)”) that occurs in the case where we take “\(X^{\log}\)” to be \(Y^{\log}\). Then we shall say that a degeneration structure \((\mathcal{H}, S, \phi)\) on \(G\) [cf. (i)] is \(m\)-cuspidalizable if the composite
\[
\gamma\Pi_1 \overset{\Phi_{\mathcal{H}_{\rightarrow S}}}{\longrightarrow} \Pi_{\mathcal{H}} \overset{\phi}{\leftarrow} \Pi_{\mathcal{H}_{\rightarrow S}} \overset{\sim}{\longrightarrow} \Pi_{G_0} \overset{\sim}{\longleftarrow} \Pi_1
\]
— where the first and fourth arrows are the natural outer isomorphisms [cf. Definition 3.1, (ii)], and the second arrow \(\Phi_{\mathcal{H}_{\rightarrow S}}\) is the natural outer isomorphism of [CbTpI], Definition 2.10 — is \(m\)-cuspidalizable [cf. Definition 3.20].

Remark 3.23.1. One interesting open problem in the theory of profinite Dehn multi-twists developed in [CbTpI], §4, is the following: In the notation of Definition 3.23, for \(i = 1, 2\), let \((\mathcal{H}_i, S_i, \phi_i)\) be a degeneration structure on \(G_0\) [cf. Definition 3.23, (i)]; \(\alpha_i \in \text{Out}(\Pi_{G_0})\) a nondegenerate \((\mathcal{H}_i, S_i, \phi_i)\)-Dehn multi-twist [cf. Definition 3.23, (iv)]. Then:

Suppose that \(\alpha_1\) commutes with \(\alpha_2\). Then is \((\mathcal{H}_1, S_1, \phi_1)\) co-Dehn to \((\mathcal{H}_2, S_2, \phi_2)\) [cf. Definition 3.23, (iii)]?

It is not clear to the authors at the time of writing whether or not this question may be answered in the affirmative. Nevertheless, we are able to obtain a partial result in this direction [cf. Corollary 3.25 below].
Proposition 3.24 (Compatibility of tripod homomorphisms). Suppose that $n \geq 3$. Then the following hold:

(i) Let $Y^\log$ be a stable log curve over $(\text{Spec } k)^\log$. For each non-negative integer $i$, write $\hat{\Pi}_i$ (respectively, $\hat{\mathcal{H}}$) for the “$\Pi_i$” (respectively, “$\mathcal{G}$”) that occurs in the case where we take “$X^\log$” to be $Y^\log$. Let $(\hat{\mathcal{H}}, S, \phi)$ be an $n$-cuspidalizable degeneration structure on $\mathcal{G}$ [cf. Definition 3.23, (i), (v)]; $\phi_n: \hat{\Pi}_n \xrightarrow{\sim} \Pi_n$ a PFC-admissible outer isomorphism [cf. [CbTpI], Definition 1.4, (iii)] that lies over the displayed composite isomorphism of Definition 3.23, (v); $\Pi^{\text{tpd}} \subseteq \Pi_3$, $\hat{\Pi}^{\text{tpd}} \subseteq \hat{\Pi}_3$ 1-central $\{1,2,3\}$-tripods [cf. Definitions 3.3, (i); 3.7, (ii)] of $\Pi_n$, $\hat{\Pi}_n$, respectively. Then there exists an outer isomorphism $\phi^{\text{tpd}}: \hat{\Pi}^{\text{tpd}} \xrightarrow{\sim} \Pi^{\text{tpd}}$ such that the diagram

\[
\begin{array}{ccc}
\text{Out}^{\text{FC}}(\hat{\Pi}_n) & \xrightarrow{\sim} & \text{Out}^{\text{FC}}(\Pi_n) \\
\xi_{\Pi^{\text{tpd}}} \downarrow & & \downarrow \xi_{\Pi^{\text{tpd}}} \\
\text{Out}(\hat{\Pi}^{\text{tpd}}) & \xrightarrow{\sim} & \text{Out}(\Pi^{\text{tpd}})
\end{array}
\]

[cf. Definition 3.19] — where the upper and lower horizontal arrows are the isomorphisms induced by $\phi_n$, $\phi^{\text{tpd}}$, respectively — commutes, up to inner automorphisms of Out$(\Pi^{\text{tpd}})$. In particular, $\phi_n$ determines an isomorphism

\[
\text{Out}^{\text{FC}}(\hat{\Pi}_n)^{\text{geo}} \xrightarrow{\sim} \text{Out}^{\text{FC}}(\Pi_n)^{\text{geo}}
\]

[cf. Definition 3.19].

(ii) If we regard $\text{Out}^{\text{FC}}(\Pi_n)$ as a closed subgroup of $\text{Out}^{\text{FC}}(\Pi_1)$ by means of the natural injection $\text{Out}^{\text{FC}}(\Pi_n) \hookrightarrow \text{Out}^{\text{FC}}(\Pi_1)$ of [NodNon], Theorem B, then the closed subgroup $\text{Dehn}(\mathcal{G}) \subseteq (\text{Aut}(\mathcal{G}) \subseteq \text{Out}(\Pi_1)) \xrightarrow{\sim} \text{Out}(\Pi_1)$ [cf. [CbTpI], Definition 4.4] is contained in $\text{Out}^{\text{FC}}(\Pi_n)^{\text{geo}} \subseteq \text{Out}^{\text{FC}}(\Pi_n)$, i.e.,

\[
\text{Dehn}(\mathcal{G}) \subseteq \text{Out}^{\text{FC}}(\Pi_n)^{\text{geo}}.
\]

Proof. First, we verify assertion (i). Let us observe that if the outer isomorphism $\phi_n$ arises scheme-theoretically as a specialization isomorphism — cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1 — then the commutativity in question follows immediately from the various definitions involved [cf. also the discussion preceding [CmbCsp], Definition 2.1]. Now the general case follows from the observation that the scheme-theoretic case treated above allows one to reduce to the case where $Y^\log = X^\log$, and $\phi_n$ is an FC-admissible outeromorphism, in which case the commutativity in question is a tautological consequence of the fact that $\xi_{\Pi^{\text{tpd}}}$ is a group homomorphism. This completes the proof of assertion (i).
Next, we verify assertion (ii). The inclusion \( \text{Dehn}(\mathcal{G}) \subseteq \text{Out}^F(\Pi_n) \) follows immediately from the fact that every profinite Dehn multi-twist arises \( \mathcal{T} \)-\textit{scheme-theoretically}, i.e., from scheme theory over \( \mathcal{T} \) [cf. the commutative diagram of Remark 3.19.1], or as a consequence of the following argument: Observe that it follows immediately from assertion (i), together with [CbTpI], Theorem 4.8, (ii), (iv), that, by applying a suitable specialization isomorphism — cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1 — we may assume without loss of generality that \( G \) is totally degenerate. Then the inclusion \( \text{Dehn}(\mathcal{G}) \subseteq \text{Out}^F(\Pi_n)^{\text{geo}} \) follows immediately from Theorem 3.18, (ii) [cf. also Theorem 3.16, (v); [CbTpI], Definition 4.4]. This completes the proof of assertion (ii). \( \square \)

Corollary 3.25 (Co-Dehn-ness of degeneration structures in the totally degenerate case). In the notation of Theorem 3.16, for \( i = 1, 2 \), let \( Y_i^{\log} \) be a stable log curve over \((\text{Spec } k)^{\log})\); \( H_i \) the \( \mathcal{G} \) that occurs in the case where we take \( X^{\log} \) to be \( Y_i^{\log}; \) \((H_i, S_i, \phi_i)\) a 3-cuspidalizable degeneration structure on \( G \) [cf. Definition 3.23, (i), (iv)]; \( \alpha_i \in \text{Out}(\Pi_G) \) a nondegenerate \((H_i, S_i, \phi_i)\)-Dehn multi-twist of \( G \) [cf. Definition 3.23, (iv)]. Suppose that \( \alpha_1 \) commutes with \( \alpha_2 \), and that \( H_2 \) is totally degenerate [cf. [CbTpI], Definition 2.3, (iv)]. Suppose, moreover, that one of the following conditions is satisfied:

(a) \( r \neq 0 \).

(b) \( \alpha_1 \) and \( \alpha_2 \) are positive definite [cf. Definition 3.23, (iv)].

Then \((H_1, S_1, \phi_1)\) is co-Dehn to \((H_2, S_2, \phi_2)\) [cf. Definition 3.23, (iii)], or, equivalently [since \( H_2 \) is totally degenerate], \((H_2, S_2, \phi_2) \preceq (H_1, S_1, \phi_1)\) [cf. Definition 3.23, (ii)].

Proof. For \( i = 1, 2 \), write \( \psi_i : \Pi_G \xrightarrow{\sim} \Pi_{H_i} \) for the composite outer isomorphism

\[
\psi_i : \Pi_G \xleftarrow{\phi_i^{-1}} \Pi_{(H_i) \rightarrow S_i} \xrightarrow{\Phi_{(H_i) \rightarrow S_i}} \Pi_{H_i}
\]

and \( \psi = \psi_1 \circ \psi_2^{-1} \). Write \( \alpha_1[H_2] \in \text{Out}(\Pi_{H_2}) \) for the automorphism obtained by conjugating \( \alpha_1 \) by \( \psi_2 \). First, we claim that the following assertion holds:

Claim 3.25.A: There exists a positive integer \( a \) such that \( \beta \overset{\text{def}}{=} \alpha_1[H_2]^a \in \text{Dehn}(H_2) \).

Indeed, since \( \alpha_1 \) is an \((H_1, S_1, \phi_1)\)-Dehn multi-twist of \( \mathcal{G} \), the automorphism \( \alpha_1[H_2] \) of \( \Pi_{H_2} \) is \textit{group-theoretically cuspidal}. Thus, since \( \alpha_1 \)
morphism associated to $\Pi$ Proposition 3.24, (i), (ii), that the image of an integer $H$ since the underlying semi-graph of $G$ commutes with $\mathcal{C}bT\mathcal{P}I$, Corollary 5.9, (iii)), that may be applied in light of $\mathcal{C}bT\mathcal{P}I$, Corollary 5.9, (ii) (respectively, (b)), from Theorem 1.9, (i) (respectively Theorem 1.9, (ii)), which may be applied in light of $\mathcal{C}bT\mathcal{P}I$, Corollary 5.9, (iii)), that $\alpha_1[\mathcal{H}_2]^a \in \text{Aut}[^{\text{grph}}(\mathcal{H}_2)]$ [cf. $\mathcal{C}bT\mathcal{P}I$, Definition 2.6, (i); Remark 4.1.2 of the present paper]. On the other hand, since $\alpha_1$ is an $(\mathcal{H}_1, S_1, \phi_1)$-Dehn multi-twist of $\mathcal{G}$, it follows immediately from Proposition 3.24, (i), (ii), that the image of $\alpha_1$ via the tripod homomorphism associated to $\Pi_3$ [cf. Definition 3.19] is trivial. Thus, since $\mathcal{H}_2$ is totally degenerate, and $\alpha_1[\mathcal{H}_2]^a \in \text{Aut}[^{\text{grph}}(\mathcal{H}_2)]$, by applying Theorem 3.18, (ii), together with Proposition 3.24, (i), we conclude that $\beta = \alpha_1[\mathcal{H}_2]^a \in \text{Dehn}(\mathcal{H}_2)$. This completes the proof of Claim 3.25.A.

Next, let us fix an element $l \in \Sigma$. For $i \in \{1, 2\}$, write $\mathcal{H}_i^{[l]}$ for the semi-graph of anabelioids of pro-$l$ PSC-type obtained by forming the pro-$l$ completion of $\mathcal{H}_i$ [cf. $\text{SemiAn}$, Definition 2.9, (ii)]. Then it follows immediately from Claim 3.25.A, together with $\mathcal{C}bT\mathcal{P}I$, Theorem 4.8, (ii), (iv), that there exists a subset $S \subseteq \text{Node}(\mathcal{H}_2)$ [which may depend on $l$!] such that the automorphism $\beta^{[l]} \in \text{Aut}(\mathcal{H}_2^{[l]})$ induced by $\beta$ is contained in $\text{Dehn}(\mathcal{H}_2^{[l]}) \subseteq \text{Dehn}(\mathcal{H}_2^{[l]}) \subseteq \text{Aut}(\mathcal{H}_2^{[l]})$ [i.e., $\beta^{[l]}$ is a profinite Dehn multi-twist of $(\mathcal{H}_2^{[l]})_{\sim s}$], and, moreover, $\beta^{[l]}$ is nondegenerate as a profinite Dehn multi-twist of $(\mathcal{H}_2^{[l]})_{\sim s}$. Write $\alpha^{[l]}$ for the automorphism of the pro-$l$ group $\Pi_{\mathcal{H}_1^{[l]}}$ [which is naturally isomorphic to the maximal pro-$l$ quotient of $\Pi_{\mathcal{H}_1}$] obtained by conjugating $\alpha_1$ by $\psi_1$ and $\psi^{[l]}$: $\Pi_{\mathcal{H}_1^{[l]}} \xrightarrow{\sim} \Pi_{\mathcal{H}_1}$ for the outer isomorphism induced by $\psi$ [cf. the discussion preceding Claim 3.25.A].

Next, we claim that the following assertion holds:

Claim 3.25.B: The composite outer isomorphism

$$\psi_S: \Pi_{(\mathcal{H}_2)_{\sim s}} \xrightarrow{\Phi_{(\mathcal{H}_2)_{\sim s}}} \Pi_{\mathcal{H}_2} \xrightarrow{\psi} \Pi_{\mathcal{H}_1}$$

is graphic, i.e., arises from an isomorphism $(\mathcal{H}_2)_{\sim s} \xrightarrow{\sim} \mathcal{H}_1$.

Indeed, let $\widetilde{\psi}_S: \Pi_{(\mathcal{H}_2)_{\sim s}} \xrightarrow{\sim} \Pi_{\mathcal{H}_1}$ be an isomorphism that lifts $\psi_S$. Then it follows immediately from $\text{CmbGC}$, Proposition 1.5, (ii) — by considering the functorial bijections between the sets “VCN” [cf. $\text{NodNon}$, Definition 1.1, (iii)] of various connected finite étale coverings of $\mathcal{H}_1$, $(\mathcal{H}_2)_{\sim s}$ — that, to verify Claim 3.25.B, it suffices to verify the following:

Let $\mathcal{I}_2 \rightarrow (\mathcal{H}_2)_{\sim s}$ be a connected finite étale covering of $(\mathcal{H}_2)_{\sim s}$ that corresponds to a characteristic open subgroup $\Pi_{\mathcal{I}_2} \subseteq \Pi_{(\mathcal{H}_2)_{\sim s}}$. Write $\mathcal{I}_1 \rightarrow \mathcal{H}_1$ for
the connected finite étale covering of $\mathcal{H}_1$ that corresponds to the [necessarily characteristic] open subgroup $\Pi_{\mathcal{I}_1} \subseteq \Pi_{\mathcal{H}_1}$ and $\mathcal{I}_1^{(l)}$, $\mathcal{I}_2^{(l)}$ for the semi-graphs of anabelioids of pro-$l$ PSC-type obtained by forming the pro-$l$ completions of $\mathcal{I}_1$, $\mathcal{I}_2$, respectively. Then the outer isomorphism $\Pi_{\mathcal{I}_1^{(l)}} \sim \Pi_{\mathcal{I}_1}$ determined by $\sim_S$ is graphic.

To verify this graphicity, let us first recall that the automorphisms $\beta^{(l)} \in \text{Aut}((\mathcal{H}_2^{(l)})_{-S})$ and $\alpha_1 \in \text{Aut}(\mathcal{H}_1)$ are nondegenerate profinite Dehn multi-twists. Thus, it follows immediately from Lemma 3.26, (i), (ii), below [cf. also Claim 3.25.A], that there exist liftings $\tilde{\beta} \in \text{Aut}(\Pi_{\mathcal{H}_2^{(l)-S}})$, $\tilde{\alpha}_1 \in \text{Aut}(\Pi_{\mathcal{H}_1})$ of $\beta$, $\alpha_1$, respectively, and a positive integer $b$ such that the automorphisms $\gamma_2$, $\gamma_1$ of $\Pi_{\mathcal{I}_2^{(l)}}$, $\Pi_{\mathcal{I}_1^{(l)}}$ determined by $\tilde{\beta}^b$, $\tilde{\alpha}_1^b$ are nondegenerate profinite Dehn multi-twists of $\mathcal{I}_2^{(l)}$, $\mathcal{I}_1^{(l)}$, respectively, and, moreover, $\gamma_2$ and $\gamma_1^b$ are compatible relative to the outer isomorphism in question $\Pi_{\mathcal{I}_2^{(l)}} \sim \Pi_{\mathcal{I}_1^{(l)}}$. Moreover, if condition (b) is satisfied, then $\gamma_1$ is a positive definite profinite Dehn multi-twist of $\mathcal{I}_1^{(l)}$ [cf. Lemma 3.26, (ii), below]. Thus, it follows, in the case of condition (a) (respectively, (b)), from Theorem 1.9, (i) (respectively Theorem 1.9, (ii)), which may be applied in light of [CbTpI], Corollary 5.9, (ii) (respectively, [CbTpI], Corollary 5.9, (iii)), that the outer isomorphism in question $\Pi_{\mathcal{I}_2^{(l)}} \sim \Pi_{\mathcal{I}_1^{(l)}}$ is graphic. This completes the proof of Claim 3.25.B. On the other hand, one verifies easily from the various definitions involved that Claim 3.25.B implies that $(\mathcal{H}_2, S_2, \phi_2) \preceq (\mathcal{H}_1, S_1, \phi_1)$. This completes the proof of Corollary 3.25. □

Lemma 3.26 (Profinite Dehn multi-twists and pro-$\Sigma$ completions of finite étale coverings). Let $\Sigma_1 \subseteq \Sigma_0$ be nonempty sets of prime numbers, $G_0$ a semi-graph of anabelioids of pro-$\Sigma_0$ PSC-type, $\mathcal{H}_0 \to \mathcal{G}_0$ a connected finite étale Galois covering that arises from a normal open subgroup $\Pi_{\mathcal{H}_0} \subseteq \Pi_{\mathcal{G}_0}$ of $\Pi_{\mathcal{G}_0}$, and $\tilde{\alpha} \in \text{Aut}(\Pi_{\mathcal{G}_0})$. Write $\mathcal{G}_1$, $\mathcal{H}_1$ for the semi-graphs of anabelioids of pro-$\Sigma_1$ PSC-type obtained by forming the pro-$\Sigma_1$ completions of $\mathcal{G}_0$, $\mathcal{H}_0$, respectively [cf. [SemiAn], Definition 2.9, (ii)]. Suppose that $\tilde{\alpha} \in \text{Aut}(\Pi_{\mathcal{G}_0})$ preserves the normal open subgroup $\Pi_{\mathcal{H}_0} \subseteq \Pi_{\mathcal{G}_0}$ corresponding to $\mathcal{H}_0 \to \mathcal{G}_0$. Write $\alpha_{\mathcal{G}_0}$, $\alpha_{\mathcal{H}_0}$, $\alpha_{\mathcal{G}_1}$, $\alpha_{\mathcal{H}_1}$ for the respective automorphisms of $\Pi_{\mathcal{G}_0}$, $\Pi_{\mathcal{H}_0}$, $\Pi_{\mathcal{G}_1}$, $\Pi_{\mathcal{H}_1}$ induced by $\tilde{\alpha}$. Suppose, moreover, that $\alpha_{\mathcal{G}_0} \in \text{Dehn}(\mathcal{G}_0)$ [cf. [CbTpI], Definition 4.4]. Then the following hold:
exists a positive integer \( a \) such that \( \alpha_{\mathcal{H}_0}^a \in \text{Dehn}(\mathcal{H}_0) \), \( \alpha_{\mathcal{H}_1}^a \in \text{Dehn}(\mathcal{H}_1) \).

(ii) If, moreover, \( \alpha_{\mathcal{G}_1} \in \text{Dehn}(\mathcal{G}_1) \) [cf. (i)] is nondegenerate (respectively, positive definite) [cf. [CbTpI], Definition 5.8, (ii), (iii)], then \( \alpha_{\mathcal{H}_1}^a \in \text{Dehn}(\mathcal{H}_1) \) [cf. (i)] is nondegenerate (respectively, positive definite).

Proof. First, we verify assertion (i). One verifies easily from [NodNon], Lemma 2.6, (i), together with [CbTpI], Corollary 5.9, (i), that there exists a positive integer \( a \) such that \( \alpha_{\mathcal{H}_0}^a \in \text{Dehn}(\mathcal{H}_0) \). Now since \( \alpha_{\mathcal{G}_0} \in \text{Dehn}(\mathcal{G}_0) \), \( \alpha_{\mathcal{H}_0}^a \in \text{Dehn}(\mathcal{H}_0) \), it follows immediately from the various definitions involved that \( \alpha_{\mathcal{G}_1} \in \text{Dehn}(\mathcal{G}_1) \), \( \alpha_{\mathcal{H}_1}^a \in \text{Dehn}(\mathcal{H}_1) \). This completes the proof of assertion (i). Assertion (ii) follows immediately, in the nondegenerate (respectively, positive definite) case, from [NodNon], Lemma 2.6, (i), together with [CbTpI], Corollary 5.9, (ii) (respectively, from Corollary 5.9, (iii), (v)). This completes the proof of Lemma 3.26.

Corollary 3.27 (Commensurator of profinite Dehn multi-twists in the totally degenerate case). In the notation of Theorem 3.16, Definition 3.19 [so \( n \geq 3 \)], suppose further that \( \mathcal{G} \) is totally degenerate [cf. [CbTpI], Definition 2.3, (iv)]. Write \( s \) : Spec \( k \to (\mathcal{M}_{g,[r]}^\log)_k \) [cf. the discussion entitled “Curves” in \( \S 0 \)] for the underlying (1-)morphism of algebraic stacks of the classifying (1-)morphism \( (\text{Spec} \ k)^{\log} \to (\mathcal{M}_{g,[r]}^\log)_k \) [cf. the discussion entitled “Curves” in \( \S 0 \)] of the stable log curve \( X_{s}^{\log} \) over \( (\text{Spec} \ k)^{\log} \); \( \mathcal{N}^{\log}_{s} \) for the log scheme obtained by equipping \( \mathcal{N}^{\log}_{s} \) with the log structure induced, via \( s \), by the log structure of \( (\mathcal{M}_{g,[r]}^\log)_k \); \( \mathcal{N}^{\log}_{s} \) for the log stack obtained by forming the [stack-theoretic] quotient of the log scheme \( \mathcal{N}^{\log}_{s} \) by the natural action of the finite \( k \)-group “\( s \times (\mathcal{M}_{g,[r]}^\log)_k \)”, i.e., the fiber product over \( (\mathcal{M}_{g,[r]}^\log)_k \) of two copies of \( s \); \( \mathcal{N}^{\log}_{s} \) for the underlying stack of the log stack \( \mathcal{N}^{\log}_{s} \); \( I_{\mathcal{N}^{\log}_{s}} \subseteq \pi_1(\mathcal{N}^{\log}_{s}) \) for the closed subgroup of the log fundamental group \( \pi_1(\mathcal{N}^{\log}_{s}) \) of \( \mathcal{N}^{\log}_{s} \) given by the kernel of the natural surjection \( \pi_1(\mathcal{N}^{\log}_{s}) \to \pi_1(\mathcal{N}^{\log}_{s}) \) [induced by the (1-)morphism \( \mathcal{N}^{\log}_{s} \to \mathcal{N}^{\log}_{s} \) obtained by forgetting the log structure]; \( \pi_1(\mathcal{N}^{\log}_{s}) \) for the quotient of \( \pi_1(\mathcal{N}^{\log}_{s}) \) by the kernel of the natural surjection from \( I_{\mathcal{N}^{\log}_{s}} \) to its maximal pro-\( \Sigma \) quotient \( I_{\mathcal{N}^{\log}_{s}}^{\Sigma} \). Then the following hold:

(i) The natural homomorphism \( \pi_1(\mathcal{N}^{\log}_{s}) \to \text{Out}(\Pi_1) \) [cf. the natural outer homomorphism of the first display of Remark 3.19.1] factors through the quotient \( \pi_1(\mathcal{N}^{\log}_{s}) \to \pi_1(\mathcal{N}^{\log}_{s})^{\Sigma} \) and the
natural inclusion $N_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) \hookrightarrow \text{Out}(\Pi_1)$ [cf. Proposition 3.24, (ii)]. In particular, we obtain a homomorphism

$$\pi_1(\mathcal{N}_s^{\log}) \rightarrow N_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})),$$

hence also a homomorphism

$$\pi_1(\mathcal{N}_s^{\log}) \rightarrow C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})).$$

(ii) The second displayed homomorphism of (i) fits into a natural commutative diagram of profinite groups

\[
\begin{array}{cccccc}
1 & \rightarrow & I_{\mathcal{N}_s}^\Sigma & \rightarrow & \pi_1(\mathcal{N}_s^{\log}) & \rightarrow & \pi_1(\mathcal{N}_s) & \rightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \text{Dehn}(\mathcal{G}) & \rightarrow & C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) & \rightarrow & \text{Aut}(\mathbb{G}) & \rightarrow & 1
\end{array}
\]

[cf. Definition 3.1, (ii), concerning the notation “\(\mathbb{G}\)”] — where the horizontal sequences are exact, and the vertical arrows are isomorphisms.

(iii) \(\text{Dehn}(\mathcal{G})\) is open in \(C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G}))\).

(iv) We have an equality

$$N_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) = C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})).$$

Proof. First, we verify assertion (i). The fact that the image of the homomorphism in question is contained in \(\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}\) follows immediately from the [tautological] fact that this image arises “\(\overline{\mathbb{Q}}\)-scheme-theoretically”, i.e., from scheme theory over \(\overline{\mathbb{Q}}\) [cf. the discussion of Remark 3.19.1]. Thus, assertion (i) follows immediately from the fact that the natural homomorphism \(\pi_1(\mathcal{N}_s^{\log}) \rightarrow \text{Out}(\Pi_1)\) determines an isomorphism \(I_{\mathcal{N}_s}^\Sigma \rightarrow \text{Dehn}(\mathcal{G})\) [cf. \([\text{CbTpI}]\), Proposition 5.6, (ii)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, let us observe that it follows from \([\text{CbTpI}]\), Theorem 5.14, (iii), that \(C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) \subseteq \text{Aut}(\mathbb{G})\). Thus, we obtain a natural homomorphism \(C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) \rightarrow \text{Aut}(\mathbb{G})\), whose kernel contains \(\text{Dehn}(\mathcal{G})\) [cf. the definition of a profinite Dehn multi-twist given in \([\text{CbTpI}]\), Definition 4.4]. On the other hand, if an element \(\alpha \in C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G}))\) acts trivially on \(\mathbb{G}\), then, since \(\mathcal{G}\) is totally degenerate, it follows immediately from Theorem 3.18, (ii), that \(\alpha \in \text{Dehn}(\mathcal{G})\). This completes the proof of the existence of the lower exact sequence in the diagram of assertion (ii), except for the surjectivity of the third arrow of this sequence. Thus, it follows immediately from the proof of assertion (i) that, to complete the proof of assertion (ii), it suffices to verify that the right-hand vertical arrow \(\pi_1(\mathcal{N}_s) \rightarrow \text{Aut}(\mathbb{G})\) of the diagram is an isomorphism. Write \(X_{\mathcal{N}_s}^{\log}\) for the stable log curve over \(\overline{\mathcal{N}_s}^{\log}\) whose classifying (1-)morphism is given by
the natural (1-)morphism \( \tilde{N}_s^\log \to (\mathcal{N}_s^\log)_{s,\mathbb{C}} \) and \( \text{Aut}^\log_s(X_s^\log) \) for the group of automorphisms of \( X_s^\log \) over \( \tilde{N}_s^\log \). Then since \( X_s^\log \), hence also \( X_s^\log \), is \textit{totally degenerate}, one verifies easily that the natural homomorphism \( \text{Aut}^\log_s(X_s^\log) \to \text{Aut}(G) \) is an \textit{isomorphism}. Thus, it follows immediately from the various definitions involved that the right-hand vertical arrow \( \pi_1(\tilde{N}_s) \to \text{Aut}(G) \) of the diagram is an \textit{isomorphism}. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from the \textit{exactness} of the lower sequence of the diagram of assertion (ii), together with the \textit{finiteness} of \( G \). Assertion (iv) follows immediately from the fact that the middle vertical arrow of the diagram of assertion (ii) is an \textit{isomorphism} which \textit{factors} through \( N_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) \subseteq C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) \) [cf. assertion (i)]. This completes the proof of Corollary 3.27. \( \Box \)

\textbf{Remark 3.27.1.} One interesting consequence of Corollary 3.27 is the following: The profinite group \( \text{Out}^\text{FC}(\Pi_n)_{\text{geo}} \) [which, as discussed in Remark 3.19.1, may be regarded as the \textit{geometric portion} of the group of FC-admissible automorphisms of the configuration space group \( \Pi_n \)], hence also the commensurator \( C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) \), is defined in a \textit{purely combinatorial/group-theoretic} fashion. In particular, it follows from the commutative diagram of Corollary 3.27, (ii), that this commensurator \( C_{\text{Out}^\text{FC}(\Pi_n)_{\text{geo}}}(\text{Dehn}(\mathcal{G})) \) yields a \textit{purely combinatorial/group-theoretic} algorithm for reconstructing the profinite groups of \textit{scheme-theoretic} origin that appear in the upper sequence of this diagram.
4. Glueability of combinatorial cuspidalizations

In the present §4, we discuss the glueability of combinatorial cuspidalizations. The resulting theory may be regarded as a higher-dimensional analogue of the displayed exact sequence of [CbTpI], Theorem B, (iii) [cf. Theorem 4.14, (iii), below, of the present paper]. This theory implies a certain key surjectivity property of the tripod homomorphism [cf. Corollary 4.15 below]. Finally, we apply this result to construct cuspidalizations of the log fundamental group of a stable log curve over a finite field [cf. Corollary 4.16 below] and to compute certain commensurators of the corresponding Galois image in the totally degenerate case [cf. Corollary 4.17 below].

In the present §4, we maintain the notation of the preceding §3 [cf. also Definition 3.1]. In addition, let $\Sigma_0$ be a nonempty set of prime numbers and $G_0$ a semi-graph of anabelioids of pro-$\Sigma_0$ PSC-type. Write $G_0$ for the underlying semi-graph of $G_0$ and $\Pi_{G_0}$ for the [pro-$\Sigma_0$] fundamental group of $G_0$.

**Definition 4.1.**

(i) We shall write

$$\text{Aut}^{\text{Brch}(G_0)}(G_0) \subseteq (\text{Aut}^{\text{Vert}(G_0)}(G_0) \cap \text{Aut}^{\text{Node}(G_0)}(G_0)) \subseteq \text{Aut}(G_0)$$

[cf. [CbTpI], Definition 2.6, (i)] for the [closed] subgroup of $\text{Aut}(G_0)$ consisting of automorphisms $\alpha$ of $G_0$ that induce the identity automorphism of $\text{Vert}(G_0)$, $\text{Node}(G_0)$, and, moreover, fix each of the branches of every node of $G_0$. Thus, we have a natural exact sequence of profinite groups

$$1 \rightarrow \text{Aut}^{\text{grph}}(G_0) \rightarrow \text{Aut}^{\text{Brch}(G_0)}(G_0) \rightarrow \text{Aut}(\text{Cusp}(G_0))$$

[cf. [CbTpI], Definition 2.6, (i); Remark 4.1.2 of the present paper].

(ii) Let $v \in \text{Vert}(G_0)$. Then we shall write

$$E(G_0|_v : G_0) \subseteq \text{Edge}(G_0|_v) (= \text{Cusp}(G_0|_v))$$

[cf. [CbTpI], Definition 2.1, (iii)] for the subset of $\text{Edge}(G_0|_v)$ ($= \text{Cusp}(G_0|_v)$) consisting of cusps of $G_0|_v$ that arise from nodes of $G_0$.

(iii) We shall write

$$\text{Glu}^{\text{brch}}(G_0) \subseteq \prod_{v \in \text{Vert}(G_0)} \text{Aut}^{E(G_0|_v : G_0)}(G_0|_v)$$

[cf. (ii); [CbTpI], Definition 2.6, (i)] for the [closed] subgroup of

$$\prod_{v \in \text{Vert}(G_0)} \text{Aut}^{E(G_0|_v : G_0)}(G_0|_v)$$

consisting of “glueable” collections.
of automorphisms of the various $\mathcal{G}_0|_v$, i.e., the subgroup consisting of $(\alpha_v)_{v \in \text{Vert}(\mathcal{G}_0)}$ such that, for every $v, w \in \text{Vert}(\mathcal{G}_0)$, it holds that $\chi_v(\alpha_v) = \chi_w(\alpha_w)$ [cf. [CbTpI], Definition 3.8, (ii)].

**Remark 4.1.1.** In the notation of Definition 4.1, one verifies easily from the various definitions involved that

$$\text{Glu}(\mathcal{G}_0) = \text{Glu}^\text{brch}(\mathcal{G}_0) \cap \left( \prod_{v \in \text{Vert}(\mathcal{G}_0)} \text{Aut}^{\text{grph}}(\mathcal{G}_0|_v) \right)$$

[cf. [CbTpI], Definitions 2.6, (i), and 4.9; Remark 4.1.2 of the present paper].

**Remark 4.1.2.** Here, we take the opportunity to correct a minor error in the exposition of [CbTpI]. In [CbTpI], Definition 2.6, (i), “$\text{Aut}^{\text{grph}}(\mathcal{G})$” should be defined as the subgroup of $\text{Aut}(\mathcal{G})$ of automorphisms of $\mathcal{G}$ which induce the identity automorphism on the underlying semi-graph of $\mathcal{G}$ [cf. the definition given in [CbTpI], Theorem B]. In a similar vein, in [CbTpI], Definition 2.6, (iii), “$\text{Aut}^{\text{H}}(\mathcal{G})$” should be defined as the subgroup of $\text{Aut}(\mathcal{G})$ of automorphisms of $\mathcal{G}$ which preserve the sub-semi-graph $\mathcal{H}$ of the underlying semi-graph of $\mathcal{G}$ and, moreover, induce the identity automorphism of $\mathcal{H}$. Since the correct definitions are applied throughout the exposition of [CbTpI], these errors in the statement of the definitions have no substantive effect on the exposition of [CbTpI], except for the following two instances [which themselves do not have any substantive effect on the exposition of [CbTpI]]:

(i) In [CbTpI], Proposition 2.7, (ii), “$\text{Aut}^{\text{grph}}(\mathcal{G})$” should be replaced by “$\text{Aut}^{\text{VCN}(\mathcal{G})}(\mathcal{G})$”.

(ii) In [CbTpI], Proposition 2.7, (iii), the phrase “In particular” should be replaced by the word “Finally”.

**Theorem 4.2 (Glueability of combinatorial cuspidalizations in the one-dimensional case).** Let $\Sigma_0$ be a nonempty set of prime numbers and $\mathcal{G}_0$ a semi-graph of anabelioids of pro-$\Sigma_0$ PSC-type. Write $\Pi_{\mathcal{G}_0}$ for the [pro-$\Sigma_0$] fundamental group of $\mathcal{G}_0$. Then the following hold:

(i) The closed subgroup $\text{Dehn}(\mathcal{G}_0) \subseteq \text{Aut}(\mathcal{G}_0)$ [cf. [CbTpI], Definition 4.4] is contained in $\text{Aut}^{\text{Brch}(\mathcal{G}_0)}(\mathcal{G}_0) \subseteq \text{Aut}(\mathcal{G}_0)$ [cf. Definition 4.1, (i)], i.e., $\text{Dehn}(\mathcal{G}_0) \subseteq \text{Aut}^{\text{Brch}(\mathcal{G}_0)}(\mathcal{G}_0)$.

(ii) The natural homomorphism

$$\text{Aut}^{\text{Brch}(\mathcal{G}_0)}(\mathcal{G}_0) \twoheadrightarrow \prod_{v \in \text{Vert}(\mathcal{G}_0)} \text{Aut}(\mathcal{G}_0|_v)$$

$$\alpha \mapsto (\alpha|_{\mathcal{G}_0|_v})_{v \in \text{Vert}(\mathcal{G}_0)}$$
factors through
\[ \text{Glu}^{\text{brch}}(G_0) \subseteq \prod_{v \in \text{Vert}(G_0)} \text{Aut}(G_0|_v) \]
[cf. Definition 4.1, (iii)].

(iii) The natural inclusion \( \text{Dehn}(G_0) \hookrightarrow \text{Aut}^{[\text{Brch}(G_0)]}(G_0) \) of (i) and the natural homomorphism \( \rho_{G_0}^{\text{brch}}: \text{Aut}^{[\text{Brch}(G_0)]}(G_0) \rightarrow \text{Glu}^{\text{brch}}(G_0) \)
[cf. (ii)] fit into an exact sequence of profinite groups
\[ 1 \rightarrow \text{Dehn}(G_0) \rightarrow \text{Aut}^{[\text{Brch}(G_0)]}(G_0) \rho_{G_0}^{\text{brch}} \rightarrow \text{Glu}^{\text{brch}}(G_0) \rightarrow 1. \]

Proof. Assertions (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from [CbTpI], Corollary 3.9, (iv). Assertion (iii) follows from the exact sequence of [CbTpI], Theorem B, (iii), together with the existence of automorphisms of \( G_0 \) that induce arbitrary permutations of the cusps and, moreover, restrict to automorphisms of each \( G_0|_v \) that lie in the kernel of \( \chi_v \) [cf. the automorphisms constructed in the proof of [CmbCsp], Lemma 2.4]. □

**Definition 4.3.** Let \( \mathbb{H} \) be a sub-semi-graph of PSC-type [cf. [CbTpI], Definition 2.2, (i)] of \( G \) [cf. Definition 3.1, (ii)] and \( S \subseteq \text{Node}(G|_{\mathbb{H}}) \) [cf. [CbTpI], Definition 2.2, (ii)] a subset of \( \text{Node}(G|_{\mathbb{H}}) \) that is not of separating type [cf. [CbTpI], Definition 2.5, (i)]. Then, by applying a similar argument to the argument applied in [CmbCsp], Definition 2.1, (iii), (vi), or [NodNon], Definition 5.1, (ix), (x) [i.e., by considering the portion of the underlying scheme \( X_n \) of \( X_n^{\text{log}} \) corresponding to the underlying scheme \( (X_{\mathbb{H},S})_n \) of the \( n \)-th log configuration space \( (X_{\mathbb{H},S})_n^{\text{log}} \) of the stable log curve \( X_{\mathbb{H},S}^{\text{log}} \) determined by \( (G|_{\mathbb{H}})^{\neq} S \) — cf. [CbTpI], Definition 2.5, (ii)], one obtains a closed subgroup
\[ (\Pi_{\mathbb{H},S})_n \subseteq \Pi_n \]
[which is well-defined up to \( \Pi_n \)-conjugation]. We shall refer to \( (\Pi_{\mathbb{H},S})_n \subseteq \Pi_n \) as a configuration space subgroup [associated to \( (\mathbb{H}, S) \)]. For each \( 0 \leq i \leq j \leq n \), we shall write
\[ (\Pi_{\mathbb{H},S})_{n/i} \overset{\text{def}}{=} (\Pi_{\mathbb{H},S})_n \cap \Pi_{n/i} \subseteq \Pi_{n/i} \]
[which is well-defined up to \( \Pi_n \)-conjugation];
\[ (\Pi_{\mathbb{H},S})_{j/i} \overset{\text{def}}{=} (\Pi_{\mathbb{H},S})_{n/i}/(\Pi_{\mathbb{H},S})_{n/j} \subseteq \Pi_{j/i} \]
[which is well-defined up to \( \Pi_j \)-conjugation];
\[ (\Pi_{\mathbb{H},S})_j \overset{\text{def}}{=} (\Pi_{\mathbb{H},S})_{j/0} \subseteq \Pi_j \]
[which is well-defined up to \( \Pi_j \)-conjugation]. Thus, each \((\Pi_{H,S})_{j/i}\) is a pro-\(\Sigma\) configuration space group [cf. [MzTa], Definition 2.3, (i)], and we have natural exact sequences of profinite groups

\[
1 \to (\Pi_{H,S})_{j/i} \to (\Pi_{H,S})_j \to (\Pi_{H,S})_i \to 1.
\]

Finally, let \( v \in \text{Vert}(G) \). Then the semi-graph of anabelioids of PSC-type \( G \vert_v \) [cf. [CbTpI], Definition 2.1, (iii)] may be naturally identified with \((G|_{H_v},S_v)\) for suitable choices of \( H_v, S_v \) [cf. [CbTpI], Remark 2.5.1, (ii)]. We shall refer to \((\Pi_v)_n \) as a configuration space subgroup associated to \( v \). Thus, \((\Pi_v)_1 \subseteq \Pi_1\) is a vertical subgroup associated to \( v \in \text{Vert}(G) \), i.e., a subgroup that is typically denoted “\( \Pi_v \)”. We shall write

\[
(\Pi_{j/i})_v \overset{\text{def}}{=} (\Pi_{H_v,S_v})_{j/i} \subseteq \Pi_{j/i}; \quad (\Pi_v)_j \overset{\text{def}}{=} (\Pi_{H_v,S_v})_j \subseteq \Pi_j.
\]

**Remark 4.3.1.** In the notation of Definition 4.3, one verifies easily — by applying a suitable specialization isomorphism [cf. the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1] — that there exist a stable log curve \( Y_{\log} \) over \((\text{Spec} k)_{\log}\) and an \( n \)-cuspidalizable degeneration structure \((G,S,\phi)\) on \( Y_{\log} \) [cf. Definition 3.23, (i), (v)] — where we write \( Y_{\log} \) for the “\( G \)” that occurs in the case where we take “\( X_{\log} \)” to be \( Y_{\log} \) — which satisfy the following: Write \( Y_{\log} \) for the “\( \Pi_n \)” that occurs in the case where we take “\( X_{\log} \)” to be \( Y_{\log} \). Then:

The image of a configuration space subgroup of \( \Pi_n \) associated to \((H,S)\) [cf. Definition 4.3] via a PFC-admissible outer isomorphism \( \Pi_n \overset{\sim}{\to} Y_{\log} \) that lies over the displayed composite isomorphism of Definition 3.23, (v) [where we note that, in \textit{loc. cit.}, the roles of “\( \Pi_n \)” and “\( \Pi_n \)” are reversed!], is a configuration space subgroup of \( Y_{\log} \) associated to a vertex of \( Y_{\log} \).

**Lemma 4.4 (Commensurable terminality and slimness).** Every configuration space subgroup [cf. Definition 4.3] of \( \Pi_n \) is topologically finitely generated, slim, and commensurably terminal in \( \Pi_n \).

**Proof.** Since any configuration space subgroup is, in particular, a configuration space group, the fact that such a subgroup is topologically finitely generated and slim follows from [MzTa], Proposition 2.2, (ii). Thus, it remains to verify commensurable terminality. By applying the observation of Remark 4.3.1, we reduce immediately to the case
of a configuration space subgroup associated to a vertex. But then the desired commensurable terminality follows, in light of Lemma 4.5 below, by induction on \( n \), together with the corresponding fact for \( n = 1 \) [cf. [CmbGC], Proposition 1.2, (ii)]. This completes the proof of Lemma 4.4.

Lemma 4.5 (Extensions and commensurable terminality). Let

\[
\begin{array}{cccccc}
1 & \longrightarrow & N_H & \longrightarrow & H & \longrightarrow & Q_H & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1
\end{array}
\]

be a commutative diagram of profinite groups, where the horizontal sequences are exact, and the vertical arrows are injective. Suppose that \( N_H \subseteq N, Q_H \subseteq Q \) are commensurably terminal in \( N, Q \), respectively. Then \( H \subseteq G \) is commensurably terminal in \( G \).

Proof. This follows immediately from Lemma 3.9, (i). □

Definition 4.6.

(i) We shall write

\[ \Out^{\text{FC}}(\Pi_n)^{\text{brch}} \subseteq \Out^{\text{FC}}(\Pi_n) \]

for the closed subgroup of \( \Out^{\text{FC}}(\Pi_n) \) given by the inverse image of

\[ \Aut^{[\text{Brch}(\mathcal{G})]}(\mathcal{G}) \subseteq (\Aut(\mathcal{G}) \subseteq \Out(\Pi_G) \subseteq \Out(\Pi_1) \]

[cf. Definition 4.1, (i)] via the natural injection \( \Out^{\text{FC}}(\Pi_n) \hookrightarrow \Out(\Pi_1) \subseteq \Out(\Pi_1) \) of [NodNon], Theorem B.

(ii) Let \( v \in \text{Vert}(\mathcal{G}) \); write \( \Pi_v \overset{\text{def}}{=} (\Pi_v)_1 \) [cf. Definition 4.3]. Then we shall write

\[ \Out^{\text{FC}}((\Pi_v)_n)^{\mathcal{G} \text{-node}} \subseteq \Out^{\text{FC}}((\Pi_v)_n) \]

for the [closed] subgroup of \( \Out^{\text{FC}}((\Pi_v)_n) \) given by the inverse image of

\[ \Aut^{[\mathcal{E}(\mathcal{G} | v)]}(\mathcal{G}_v) \subseteq (\Aut(\mathcal{G}_v) \subseteq \Out(\Pi_v) \]

[cf. Definition 4.1, (ii); [CbTpI], Definition 2.6, (i)] via the natural injection \( \Out^{\text{FC}}((\Pi_v)_n) \hookrightarrow \Out^{\text{FC}}(\Pi_v) \subseteq \Out(\Pi_v) \) of [NodNon], Theorem B.
Theorem 4.7 (Graphicity of automorphisms of certain subquotients). In the notation of the preceding §3 [cf. also Definition 3.1], let \( x \in X_n(k) \). Write

\[ C_x \subseteq \text{Cusp}(\mathcal{G}) \]

for the [possibly empty] set consisting of cusps \( c \) of \( \mathcal{G} \) such that, for some \( i \in \{1, \cdots, n\} \), \( x_{(i)} \in X_{(i)}(k) = X(k) \) [cf. Definition 3.1, (i)] lies on the cusp of \( X_{\log} \) corresponding to \( c \in \text{Cusp}(\mathcal{G}) \). For each \( i \in \{1, \cdots, n\} \), write

\[ \mathcal{G}_{i/i-1,x} \overset{\text{def}}{=} \mathcal{G}_{i \in \{1, \cdots, i\}, x} \]

[cf. Definition 3.1, (iii)] and

\[ z_{i/i-1,x} \in \text{VCN}(\mathcal{G}_{i/i-1,x}) \]

for the element of \( \text{VCN}(\mathcal{G}_{i/i-1,x}) \) on which \( x_{\{1, \cdots, i\}} \) lies, that is to say: If \( x_{\{1, \cdots, i\}} \in X_i(k) \) [cf. the notation given in the discussion preceding Definition 3.1] is a cusp or node of the geometric fiber of the projection \( p_{i/i-1}^{\log}: X_{i}^{\log} \to X_{i-1}^{\log} \) over \( x_{\{1, \cdots, i-1\}} \), then \( z_{i/i-1,x} \overset{\text{def}}{=} e \); if \( x_{\{1, \cdots, i\}} \in X_i(k) \) is neither a cusp or a node of the geometric fiber of the projection \( p_{i/i-1}^{\log}: X_{i}^{\log} \to X_{i-1}^{\log} \) over \( x_{\{1, \cdots, i-1\}} \), but lies on the irreducible component of the geometric fiber corresponding to a vertex \( v \in \text{Edge}(\mathcal{G}_{i/i-1,x}) \), then \( z_{i/i-1,x} \overset{\text{def}}{=} v \).

Let

\[ \alpha \in \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}}[\text{cf. Definition 4.6, (i)}]. \]

Suppose that the element of

\[ \text{Aut}^{\text{Brch}(\mathcal{G})}/(\mathcal{G}) \subseteq (\text{Aut}(\mathcal{G}) \subseteq \text{Out}(\Pi_n) \overset{\sim}{\hookrightarrow} \text{Out}(\Pi_1)) \]

[cf. Definition 4.1, (i)] determined by \( \alpha \in \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}}[\text{cf. Definition 4.6, (i)}] \) is contained in

\[ \text{Aut}^{[C_{z}]}(\mathcal{G}) \subseteq \text{Aut}(\mathcal{G}) \]

[cf. [CbTpI], Definition 2.6, (i)]. Then there exist

- a lifting \( \tilde{\alpha} \in \text{Aut}(\Pi_n) \) of \( \alpha \), and,
- for each \( i \in \{1, \cdots, n\} \), a VCN-subgroup \( \Pi_{z_{i/i-1,x}} \subseteq \Pi_{i/i-1} \overset{\sim}{\rightsquigarrow} \Pi_{\tilde{G}_{i/i-1,x}} \) [cf. Definition 3.1, (iii)] associated to the element \( z_{i/i-1,x} \in \text{VCN}(\mathcal{G}_{i/i-1,x}) \)

such that the following properties hold:

(a) For each \( i \in \{1, \cdots, n\} \), the automorphism of \( \Pi_{i/i-1} \overset{\sim}{\rightsquigarrow} \Pi_{\tilde{G}_{i/i-1,x}} \)

determined by \( \tilde{\alpha} \) fixes the VCN-subgroup \( \Pi_{z_{i/i-1,x}} \subseteq \Pi_{i/i-1} \overset{\sim}{\rightsquigarrow} \Pi_{\tilde{G}_{i/i-1,x}} \).
Thus, since we have assumed that \( \varPi \mathcal{1} \) that the automorphism of \( \varPi_{i/i-1} \sim \varPi_{\varphi_{i/i-1},x} \)
iinduced by \( \bar{\alpha} \) is contained in
\[
\text{Aut}^{[\text{Brch}(\varphi_{i/i-1},x)]}(\varphi_{i/i-1,x}) \subseteq \text{Out}(\varPi_{\varphi_{i/i-1},x}) \prec \text{Out}(\varPi_{i/i-1}).
\]

**Proof.** We verify Theorem 4.7 by induction on \( n \). If \( n = 1 \), then Theorem 4.7 follows immediately from the various definitions involved. Now suppose that \( n \geq 2 \), and that the induction hypothesis is in force. In particular, [since the homomorphism \( p_{n/n-1}^1 : \Pi_n \to \Pi_{n-1} \) is surjective] we have a lifting \( \bar{\alpha} \in \text{Aut}(\Pi_n) \) of \( \alpha \) and, for each \( i \in \{1, \cdots, n-1\} \), a VCN-subgroup \( \varPi_{z_{i/i-1},x} \subseteq \Pi_{i/i-1} \sim \varPi_{\varphi_{i/i-1},x} \) associated to the element \( z_{i/i-1,x} \in \text{VCN}(\varphi_{i/i-1,x}) \) such that, for each \( i \in \{1, \cdots, n-1\} \), the automorphism of \( \varPi_i \) determined by \( \bar{\alpha} \) fixes \( \varPi_{z_{i/i-1},x} \subseteq \Pi_{i/i-1} \subseteq \Pi_i \) and, moreover, the automorphism of \( \Pi_{n-1} \) determined by \( \bar{\alpha} \) satisfies the property (b) in the statement of Theorem 4.7. Now we claim that the following assertion holds:

**Claim 4.7.A:** The automorphism of \( \Pi_{n/n-1} \sim \varPi_{\varphi_{n/n-1},x} \)
iinduced by the lifting \( \bar{\alpha} \) is contained in
\[
\text{Aut}^{[\text{Brch}(\varphi_{n/n-1},x)]}(\varphi_{n/n-1,x}) \subseteq \text{Out}(\varPi_{\varphi_{n/n-1},x}) \prec \text{Out}(\Pi_{n/n-1}).
\]

To this end, let us first observe that it follows immediately — by replacing \( X_n^{\log} \) by the base-change of \( p_{n/n-2}^{\log} : X_n^{\log} \to X_{n-2}^{\log} \) via a suitable morphism of log schemes (Spec \( k^{\log} \to X_{n-2}^{\log} \)) whose image lies on \( x_{\{1, \cdots, n-2\}} \in X_{n-2}(k) \) — from Lemma 3.2, (iv), that, to verify Claim 4.7.A, we may assume without loss of generality that \( n = 2 \). Also, one verifies easily, by applying Lemma 3.14, (i) \[ \text{cf. also [CbtPl], Proposition 2.9, (i)], that we may assume without loss of generality that } x_{\{1\}} \text{ is a node of } X^{\log} \text{ [i.e., } z_{1/0,x} \in \text{Edge}(\varphi_{1/0,x}) \].

Next, let us recall that the automorphism of \( \Pi_1 \sim \varPi_{\varphi_{1/0,x}} \) determined by \( \bar{\alpha} \) fixes the edge-like subgroup \( \varPi_{z_{1/0,x}} \subseteq \Pi_1 \sim \varPi_{\varphi_{1/0,x}} \) associated to the edge \( z_{1/0,x} \) of \( \varphi_{1/0,x} \) \[ \text{cf. the discussion preceding Claim 4.7.A}, \]
Thus, since [we have assumed that] \( \alpha \in \text{Out}^{FC}(\Pi_2)^{\text{brch}} \) which implies that the automorphism of \( \Pi_1 \sim \varPi_{\varphi_{1/0,x}} \) determined by \( \alpha \) preserves the \( \Pi_1 \)-conjugacy class of each vertical subgroup of \( \Pi_1 \sim \varPi_{\varphi_{1/0,x}} \), it follows immediately from Lemma 3.13, (i), (ii), that the automorphism of \( \varPi_{\varphi_{2/1},x} \sim \varPi_{2/1} \) induced by \( \bar{\alpha} \) is group-theoretically vertical, hence [cf. [NodNon], Proposition 1.13; [CmbGC], Proposition 1.5, (ii); the fact that \( \alpha \) is \( C \)-admissible] graphic, i.e., \( \in \text{Out}(\varphi_{2/1,x}) \). Moreover, since the automorphism of \( \varPi_{\varphi_{2/1},x} \sim \varPi_{2/1} \) induced by \( \bar{\alpha} \) is, by assumption, contained in \( \text{Aut}^{[\text{Brch}(\varphi)]}(\varphi) \) \[ \text{cf. [CmbCsp], Proposition 1.2, (iii)], one verifies easily, by considering the map on vertices/nodes/branches induced by the projection}
\[
p_{1,2/1}^\Pi : \Pi_{2/1} \to \Pi_{(2)}
\]
[cf. Lemma 3.6, (i), (iv)], that the automorphism of \( \Pi_{G/1, x} \) induced by \( \bar{\alpha} \) is contained in the subgroup \( \text{Aut}^{\text{Brch}}(G_{2/1,x}))(G_{2/1,x}) \). This completes the proof of Claim 4.7.A.

On the other hand, one verifies easily from Claim 4.7.A, together with the various definitions involved, that there exist a \( \Pi_{n/n-1} \)-conjugate \( \bar{e} \) of \( e \) and a VCN-subgroup \( \Pi_{z_{n/n-1}, x} \) of \( \Pi_{n/n-1} \) associated to \( z_{n/n-1,x} \). In particular, the lifting \( \bar{\beta} \) of \( \beta \) and the VCN-subgroups \( \Pi_{z_{i/i-1}, x} \) [where \( i \in \{1, \ldots, n\} \)] satisfy the properties (a), (b) in the statement of Theorem 4.7. This completes the proof of Theorem 4.7. □

**Lemma 4.8** (Preservation of configuration space subgroups).

The following hold:

(i) Let \( \alpha \in \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}} \) [cf. Definition 4.6, (i)]. Then \( \alpha \) preserves the \( \Pi_n \)-conjugacy class of each configuration space subgroup [cf. Definition 4.3] of \( \Pi_n \). Thus, by applying the portion of Lemma 4.4 concerning commensurable terminality, together with Lemma 3.10, (i), we obtain a natural homomorphism

\[
\text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}} \to \prod_{v \in \text{Vert}(G)} \text{Out}(\Pi_v). 
\]

(ii) The displayed homomorphism of (i) factors through

\[
\prod_{v \in \text{Vert}(G)} \text{Out}^{\text{FC}}((\Pi_v)_n)^{G_{-\text{node}}} \subseteq \prod_{v \in \text{Vert}(G)} \text{Out}((\Pi_v)_n)
\]

[cf. Definition 4.6, (ii)].

**Proof.** First, we verify assertion (i). We begin by observing that, in light of the observation of Remark 4.3.1 [cf. also [CbTpI], Proposition 2.9, (ii)], to complete the verification of assertion (i), it suffices to verify the following assertion:

Claim 4.8.A: For each \( v \in \text{Vert}(G) \), \( \alpha \) preserves the \( \Pi_v \)-conjugacy class of configuration space subgroups \( (\Pi_v)_n \subseteq \Pi_n \) of \( \Pi_n \) associated to \( v \).

We verify Claim 4.8.A by induction on \( n \). If \( n = 1 \), then Claim 4.8.A follows immediately from the various definitions involved. Now suppose that \( n \geq 2 \), and that the induction hypothesis is in force. Then it follows from the induction hypothesis that the automorphism of \( \Pi_n \) induced by \( \alpha \) preserves the \( \Pi_{n-1} \)-conjugacy class of configuration space subgroups \( (\Pi_v)_{n-1} \subseteq \Pi_{n-1} \) associated to each \( v \). On the other hand, it follows immediately from Theorem 4.7 that \( \alpha \) preserves the \( \Pi_n \)-conjugacy
class of \((\Pi_v)_{n-1} \subseteq \Pi_n\). In particular, by considering the natural isomorphism \((\Pi_v)_n \cong (\Pi_v)_{n-1} \times (\Pi_v)_{n-1}\) [cf. the displayed exact sequence of Definition 4.3; the discussion entitled “Topological groups” in [CbTpI], §0 for a suitable choice of the pair \((\Pi_v)_{n-1}, (\Pi_v)_{n-1}\) [whose existence is a consequence of the existence of the closed subgroup \((\Pi_v)_n\) of \(\Pi_n\)], we conclude that \(\alpha\) preserves the \(\Pi_n\)-conjugacy class of \((\Pi_v)_n\). This completes the proof of Claim 4.8.A, hence also of assertion (i).

Next, we verify assertion (ii). Let \(\alpha \in \text{Out}^{\text{FC}}((\Pi_v)_n)\), \(v \in \text{Vert}(G)\).

Write \(v\) for the automorphism of \((\Pi_v)_n\) induced by \(\alpha\), and \(\chi_v = \chi_w\). Then the F-admissibility of \(\alpha_v\) follows immediately from the F-admissibility of \(\alpha\). The C-admissibility of \(\alpha_v\) follows immediately from Theorem 4.7; [CmbGC], Proposition 1.5, (i), together with the definition of C-admissibility. Finally, the fact that \(\alpha_v \in \text{Out}^{\text{FC}}((\Pi_v)_n)^{G\text{-node}}\) follows immediately from the fact that \(\alpha \in \text{Out}^{\text{FC}}(\Pi_n)\). This completes the proof of assertion (ii).

\(\square\)

Definition 4.9. We shall write

\[ \text{Glu}(\Pi_n) = \prod_{v \in \text{Vert}(G)} \text{Out}^{\text{FC}}((\Pi_v)_n)^{G\text{-node}} \]

for the [closed] subgroup of \(\prod_{v \in \text{Vert}(G)} \text{Out}^{\text{FC}}((\Pi_v)_n)^{G\text{-node}}\) consisting of “glueable” collections of automorphisms of \((\Pi_v)_n\), i.e., the subgroup defined as follows:

(i) Suppose that \(n = 1\). Then \(\text{Glu}(\Pi_n)\) consists of those collections \((\alpha_v)_{v \in \text{Vert}(G)}\) such that, for every \(v, w \in \text{Vert}(G)\), it holds that \(\chi_v(\alpha_v) = \chi_w(\alpha_w)\) [cf. [CbTpI], Definition 3.8, (ii)] — where we note that one verifies easily that \(\alpha_v\) may be regarded as an element of \(\text{Aut}(\mathcal{G}|_v)\).

(ii) Suppose that \(n = 2\). Then \(\text{Glu}(\Pi_n)\) consists of those collections \((\alpha_v)_{v \in \text{Vert}(G)}\) that satisfy the following condition: Let \(v, w \in \text{Vert}(G)\); \(e \in \mathcal{N}(v) \cap \mathcal{N}(w)\); \(T \subseteq \Pi_2/1 \subseteq \Pi_2 = \Pi_n\) a \(\{1, 2\}\)-tripod of \(\Pi_n\) arising from \(e \in \mathcal{N}(v) \cap \mathcal{N}(w)\) [cf. Definitions 3.3, (i); 3.7, (i)]. Then one verifies easily from the various definitions involved that there exist \(\Pi_n\)-conjugates \(T_v, T_w\) of \(T\) such that \(T_v, T_w\) are contained in \((\Pi_v)_n\), \((\Pi_w)_n\), respectively, and, moreover,

\[
T_v \subseteq (\Pi_v)_2/1 \subseteq (\Pi_v)_2 = (\Pi_v)_n,
\]

\[
T_w \subseteq (\Pi_w)_2/1 \subseteq (\Pi_w)_2 = (\Pi_w)_n
\]

are the tripods of \((\Pi_v)_n\), \((\Pi_w)_n\) arising from [the cusps of \(\mathcal{G}|_v\), \(\mathcal{G}|_w\) corresponding to] the node \(e\), respectively. Moreover, since \(\alpha_v \in \text{Out}^{\text{FC}}((\Pi_v)_n)^{G\text{-node}}, \alpha_w \in \text{Out}^{\text{FC}}((\Pi_w)_n)^{G\text{-node}},\) it follows
from Theorem 3.16, (iv), that $\alpha_v \in \Out^{\FC}((\Pi_v)_n)[T_v]$, $\alpha_w \in \Out^{\FC}((\Pi_w)_n)[T_w]$; thus, we obtain that $\mathcal{T}_{T_v}(\alpha_v) \in \Out(T_v) \sim \Out(T)$; $\mathcal{T}_{T_w}(\alpha_w) \in \Out(T_w) \sim \Out(T)$ [cf. Theorem 3.16, (i)]. Then we require that $\mathcal{T}_{T_v}(\alpha_v) = \mathcal{T}_{T_w}(\alpha_w)$.

(iii) Suppose that $n \geq 3$. Then $\Glu(\Pi_n)$ consists of those collections $(\alpha_v)_{v \in \Vert(G)}$ that satisfy the following condition: Write $\Pi^\tpd \subseteq \Pi_3$ for the central $\{1, 2, 3\}$-tripod of $\Pi_n$ [cf. Definitions 3.3, (i); 3.7, (ii)]. Then one verifies easily from the various definitions involved that, for every $v \in \Vert(G)$, there exists a $\Pi^\tpd$-conjugate $\Pi^\tpd_v$ of $\Pi^\tpd$ such that $\Pi^\tpd_v \subseteq (\Pi_v)_3$ is the central tripod of $(\Pi_v)_3$. Thus, since $\alpha_v \in \Out^{\FC}((\Pi_v)_n)^\brch$, we obtain $\mathcal{T}_{\Pi^\tpd_v}(\alpha_v) \in \Out(\Pi^\tpd_v) \sim \Glu^\brch(G)$ [cf. Theorem 3.16, (i), (v)]. Then, for every $v, w \in \Vert(G)$, we require that $\mathcal{T}_{\Pi^\tpd_v}(\alpha_v) = \mathcal{T}_{\Pi^\tpd_w}(\alpha_w)$.

Remark 4.9.1. In the notation of Definition 4.9, one verifies easily from the various definitions involved that the natural outer isomorphism $\Pi_1 \sim \Pi_G$ determines a natural isomorphism $\Glu(\Pi_1) \sim \Glu^\brch(G)$ [cf. Definition 4.1, (iii)].

Lemma 4.10 (Basic properties concerning groups of glueable collections). For $n \geq 1$, the following hold:

(i) The natural injections

$$\Out^{\FC}((\Pi_v)_{n+1}) \hookrightarrow \Out^{\FC}((\Pi_v)_n)$$

of [NodNon], Theorem B — where $v$ ranges over the vertices of $G$ — determine an injection

$$\Glu(\Pi_{n+1}) \hookrightarrow \Glu(\Pi_n).$$

(ii) The displayed homomorphism of Lemma 4.8, (i),

$$\Out^{\FC}(\Pi_n)^\brch \longrightarrow \prod_{v \in \Vert(G)} \Out((\Pi_v)_n)$$

factors through

$$\Glu(\Pi_n) \subseteq \prod_{v \in \Vert(G)} \Out((\Pi_v)_n).$$

Proof. First, we verify assertion (i). The fact that the image of the composite

$$\Glu(\Pi_{n+1}) \hookrightarrow \prod_{v \in \Vert(G)} \Out^{\FC}((\Pi_v)_{n+1}) \hookrightarrow \prod_{v \in \Vert(G)} \Out^{\FC}((\Pi_v)_n)$$

...
is contained in
\[ \prod_{v \in \text{Vert}(G)} \text{Out}^{FC}((\Pi_v)_n)^{G\text{-node}} \leq \prod_{v \in \text{Vert}(G)} \text{Out}^{FC}((\Pi_v)_n) \]
follows immediately from the various definitions involved. The fact that the image of the composite
\[ \text{Glu}(\Pi_{n+1}) \ni \prod_{v \in \text{Vert}(G)} \text{Out}^{FC}((\Pi_v)_{n+1}) \ni \prod_{v \in \text{Vert}(G)} \text{Out}^{FC}((\Pi_v)_n) \]
is contained in
\[ \text{Glu}(\Pi_n) \leq \prod_{v \in \text{Vert}(G)} \text{Out}^{FC}((\Pi_v)_n) \]
follows immediately from the various definitions involved when \( n \geq 3 \) and from Theorems 3.16, (iv), (v); 3.18, (ii) [applied to each \( (\Pi_v)_{n+1} \)], when \( n = 2 \). Thus, it remains to verify assertion (i) in the case where \( n = 1 \). Suppose that \( n = 1 \). Let \( (\alpha_{v_1})_{v \in \text{Vert}(G)} \in \text{Glu}(\Pi_2) \).
Write \( ((\alpha_{v_1})_{v \in \text{Vert}(G)}) \ni \prod_{v \in \text{Vert}(G)} \text{Out}^{FC}((\Pi_v)_1)^{G\text{-node}} \) for the image of \( (\alpha_{v_1})_{v \in \text{Vert}(G)} \). Since \( G \) is connected, to verify assertion (i) in the case where \( n = 1 \), it suffices to verify that, for any two vertices \( v, w \) of \( G \) such that \( \mathcal{N}(v) \cap \mathcal{N}(w) \neq \emptyset \), it holds that \( \chi_v((\alpha_{v_1})_1) = \chi_w((\alpha_{v_1})_1) \). Let \( x \in X_2(k) \) be a \( k \)-valued geometric point of \( X_2 \) such that \( x_{\{1\}} \in X(k) \) [cf. Definition 3.1, (i)] is a node of \( X^{\log} \) corresponding to an element of \( \mathcal{N}(v) \cap \mathcal{N}(w) \neq \emptyset \). Then by applying Theorem 4.7 to a suitable lifting \( \tilde{\alpha}_v \in \text{Aut}^{FC}((\Pi_v)_2) \) of the automorphism \( \alpha_v \) of \( (\Pi_v)_2 \) where we take the “\( \Pi_n \)” in the statement of Theorem 4.7 to be \( (\Pi_v)_2 \), we conclude that the automorphism \( (\alpha_{v_1})_{2/1} \) of \( (\Pi_{[v]})_{2 \in \{1,2\}, x} \) is \( (\Pi_{[v]})_{2/1} \) [cf. Definition 3.1, (iii)] determined by \( \tilde{\alpha}_v \), is graphic and fixes each of the vertices of \( (\Pi_{[v]})_{2 \in \{1,2\}, x} \). Thus, if we write \( (\alpha_{v_1})_{2} \) for the automorphism of the “\( \Pi_{[v]} \)” that occurs in the case where we take “\( \Pi_2 \)” to be \( (\Pi_v)_2 \), and \( T_v \subseteq (\Pi_v)_2 \) for the \( \{1,2\} \)-tripod arising from the cusp \( x_{\{1\}} \) of \( G_v \) [cf. Definitions 3.3, (i); 3.7, (i)], then it follows from \( [\text{CmbCsp}] \), Proposition 1.2, (iii), together with the \( C\text{-admissibility} \) of \( (\alpha_{v_1})_1 \), that \( (\alpha_{v_1})_2 \) is \( C\text{-admissible} \), i.e., \( \in \text{Aut}(G|_v) \).
Now we compute:
\[
\begin{align*}
\chi_{G|_v}((\alpha_{v_1})_1) &= \chi_{G|_v}((\alpha_{v_1})_{2}) \quad \text{[cf. \( [\text{CmbCsp}] \), Proposition 1.2, (iii)]} \\
&= \chi_{(G|_v)_{2 \in \{1,2\}, x}}((\alpha_{v_1})_{2/1}) \quad \text{[cf. \( [\text{CbTnP}] \), Corollary 3.9, (iv)]} \\
&= \chi_{T_v}((\alpha_{v_1})_{2/1}|_{T_v}) \quad \text{[cf. \( [\text{CbTnP}] \), Corollary 3.9, (iv)]}
\end{align*}
\]
[where we refer to Lemma 3.12, (i), concerning “\( (\alpha_{v_1})_{2/1}|_{T_v} \)”, and we write \( \chi_{T_v} \) for the “\( \chi \)” associated to the vertex of \( (G|_v)_{2 \in \{1,2\}, x} \) corresponding to \( T_v \). Moreover, by applying a similar argument to the above argument, we conclude that there exists a lifting \( \tilde{\alpha}_w \) of \( \alpha_w \) such that the automorphism \( (\alpha_{w})_{2/1} \) of \( (G|_w)_{2 \in \{1,2\}, x} \) is determined by \( \tilde{\alpha}_w \) is graphic and fixes each of the vertices of \( (G|_w)_{2 \in \{1,2\}, x} \), and, moreover, if we write \( T_w \subseteq (\Pi_w)_2 \) for the \( \{1,2\} \)-tripod arising from the cusp \( x_{\{1\}} \) of \( G|_w \), then it holds that \( \chi_{G|_w}((\alpha_{w})_{1}) = \chi_{T_w}((\alpha_{w})_{2/1}|_{T_w}) \). On the other
hand, since \((\alpha_v)_{w\in \text{Vert}(G)} \in \text{Glu}(\Pi_2)\), it holds that 
\[\chi_{T_v}(\vert \alpha_v\vert_2/1_{T_v}) = \chi_{T_w}(\vert \alpha_w\vert_2/1_{T_w})\]. In particular, we obtain that 
\[\chi_{G_{w'}}((\alpha_v)_1) = \chi_{G_{w'}}((\alpha_w)_1)\]. This completes the proof of assertion (i).

Next, we verify assertion (ii). If \(n = 1\), then assertion (ii) amounts to 
Theorem 4.2, (ii) [cf. also Remark 4.9.1]. If \(n \geq 2\), then assertion (ii) follows immediately from Lemma 4.8, (ii), together with the fact that the homomorphism “\(\mathcal{F}_{T}\)” of Theorem 3.16, (i), does not depend on the choice of “\(T\)” among its conjugates. This completes the proof of assertion (ii).

\[\square\]

**Definition 4.11.** We shall write \(\rho_n^{brch}\) for the homomorphism

\[\text{Out}^{\text{FC}}(\Pi_n)^{brch} \rightarrow \text{Glu}(\Pi_n)\]

determined by the factorization of Lemma 4.10, (ii).

**Lemma 4.12 (Glueable collections in the case of precisely one node).** Suppose that \(n = 2\), and that \(\text{Node}(G) = 1\). Let \(v, \tilde{v}, \tilde{w} \in \text{Vert}(\tilde{G})\) be distinct elements such that \(N(\tilde{v}) \cap N(\tilde{w}) \neq \emptyset\). Write \(\tilde{e} \in \text{Node}(\tilde{G})\) for the unique element of \(N(\tilde{v}) \cap N(\tilde{w})\) [cf. [NodNon], Lemma 1.8]; \(\Pi_{\tilde{v}}, \Pi_{\tilde{w}}, \Pi_{e} \subseteq \Pi_{G} \sim \Pi_{1}\) for the VCN-subgroups of \(\Pi_{G} \sim \Pi_{1}\) associated to \(\tilde{v}, \tilde{w}, \tilde{e} \in \text{VCN}(\tilde{G})\), respectively; \(v \equiv \tilde{v}(G)\); \(w \equiv \tilde{w}(G)\); \(e \equiv \tilde{e}(G)\). [Thus, one verifies easily that \(\Pi_e = \Pi_{\tilde{v}} \cap \Pi_{\tilde{w}}\) [cf. [NodNon], Lemma 1.9, (i)], that \(\text{Vert}(G) = \{v, w\}\), and that if \(G\) is noncyclically primitive (respectively, cyclically primitive) [cf. [CbTpI], Definition 4.1], then \(v \neq w\) (respectively, \(v = w\)).] Let \(x \in X_2(k)\) be a \(k\)-valued geometric point of \(X_2\) such that \(x_{(1)} \in X(k)\) [cf. Definition 3.1, (ii)] lies on the unique node of \(X^{\log}\) [i.e., which corresponds to \(e\)]. Write \(G_{2/1} \equiv G_{2x(1),x}\) [cf. Definition 3.1, (iii)]; \(\tilde{G}_{2/1} \rightarrow G_{2/1}\) for the profinite étale covering corresponding to \(\Pi_{G_{2/1}} \sim \Pi_{2/1}\); \(v_{new}\) for the “\(v_{new}\)” of Lemma 3.6, (iv). For each \(z \in \text{Vert}(\tilde{G})\), write \(z^o \in \text{Vert}(\tilde{G}_{2/1})\) for the vertex of \(G_{2/1}\) that corresponds to \(z\) via the bijection of Lemma 3.6, (iv). [Thus, it follows from Lemma 3.6, (iv), that \(\text{Vert}(\tilde{G}_{2/1}) = \{v_{new}, v^o, w^o\}\).] Then the following hold [cf. also Figures 2, 3, below]:

(i) Let \((\Pi_{\tilde{v}})_2 \subseteq \Pi_{G}\) be a configuration space subgroup of \(\Pi_{G}\) associated to \(v\) [cf. Definition 4.3] such that the image of the composite \((\Pi_{\tilde{v}})_2 \hookrightarrow \Pi_2 \twoheadrightarrow \Pi_1\) coincides with \(\Pi_{\tilde{v}} \subseteq \Pi_{G} \sim \Pi_{1}\). Also, let us fix a vertical subgroup \(\Pi_{new} \subseteq \Pi_{G_{2/1}} \sim \Pi_{2/1}\) of \(\Pi_{G_{2/1}} \sim \Pi_{2/1}\) associated to \(v_{new} \in \text{Vert}(\tilde{G}_{2/1})\) that lies over \(v_{new} \in \text{Vert}(G_{2/1})\) and is contained in \((\Pi_{\tilde{v}})_2\). Then there
exists a unique configuration space subgroup \((\Pi_{\bar{w}})^2 \subseteq \Pi_2\) of \(\Pi_2\) associated to \(w\) [cf. Definition 4.3] such that \(\Pi_{\text{new}} = (\Pi_{\bar{w}})^2_1 \cap (\Pi_{\bar{w}})^2_2\) — where we write \((\Pi_{\bar{w}})^2_2 \defeq \Pi_{2/1} \cap (\Pi_{\bar{w}})^2_2\); \((\Pi_{\bar{w}})^2_1 \defeq \Pi_{2/1} \cap (\Pi_{\bar{w}})^2_1\) — and, moreover, the image of the composite \((\Pi_{\bar{w}})^2_2 \hookrightarrow \Pi_2 \twoheadrightarrow \Pi_1\) coincides with \(\Pi_{\bar{w}} \subseteq \Pi_1\).

(ii) In the situation of (i), the natural homomorphism

\[
\lim_{\to} (\Pi_{\bar{w}} \hookrightarrow \Pi_{\bar{z}} \hookrightarrow \Pi_0) \longrightarrow \Pi_1
\]

— where the inductive limit is taken in the category of pro-\(\Sigma\) groups — is injective, and its image is commensurably terminal in \(\Pi_1\). Write \(\Pi_{\bar{v}, \bar{w}} \subseteq \Pi_1\) for the image of the above homomorphism; \(\Pi_{2|\bar{v}, \bar{w}} \subseteq \Pi_2\) for the fiber product of \(\Pi_2 \twoheadrightarrow \Pi_1\) and \(\Pi_{\bar{v}, \bar{w}} \hookrightarrow \Pi_1\). Thus, we have an exact sequence of profinite groups

\[
1 \longrightarrow \Pi_{2/1} \longrightarrow \Pi_{2|\bar{v}, \bar{w}} \longrightarrow \Pi_{\bar{v}, \bar{w}} \longrightarrow 1.
\]

Finally, if \(G\) is noncyclically primitive, then \(\Pi_{\bar{v}, \bar{w}} = \Pi_1\), \(\Pi_{2|\bar{v}, \bar{w}} = \Pi_2\).

(iii) In the situation of (ii), for each \(\bar{v} \in \{\bar{v}, \bar{w}\}\), let \(\Pi_{\bar{v}} \subseteq \Pi_{\bar{v}/1} \defeq \Pi_{\bar{v}/1}\) be a vertical subgroup of \(\Pi_{\bar{v}/1} \defeq \Pi_{2/1}\) associated to \(v^o \defeq \bar{z}(G)^o \in \{v^o, w^o\} \subseteq \text{Vert}(G_{2/1})\) such that \(\Pi_{\bar{v}} \subseteq (\Pi_{\bar{z}})^{2/1}_2\) [cf. (i)], and, moreover, \(\Pi_{\bar{v}} \cap \Pi_{\text{new}} \neq \{1\}\). Thus, \(\Pi_{\bar{v}} \cap \Pi_{\text{new}}\) is the nodal subgroup of \(\Pi_{\bar{v}/1} \defeq \Pi_{2/1}\) associated to the unique element \(\bar{e}_{\bar{v}}\) of \(\mathcal{N}(\bar{v}^o) \cap \mathcal{N}(\bar{v}_{\text{new}})\) [cf. [NodNon], Lemma 1.9, (i)]. Write \(e_{\bar{v}}^o \defeq \bar{e}_{\bar{v}}^o(G_{2/1})\). Then the natural homomorphism

\[
\lim_{\to} (\Pi_{\bar{v}} \hookrightarrow \Pi_{\bar{v}/1} \hookrightarrow \Pi_{\text{new}}) \longrightarrow (\Pi_{\bar{z}})^{2/1}_2
\]

— where the inductive limit is taken in the category of pro-\(\Sigma\) groups — is an isomorphism. Write \(G^o_{\bar{v}/1}\) for the subsemi-graph of PSC-type [cf. [CbTpI], Definition 2.2, (i)] of the underlying semi-graph of \(G_{2/1}\) whose set of vertices = \(\{\bar{z}(G)^o, v_{\text{new}}\}; T_{\bar{v}}^o \defeq (\text{Node}(G_{2/1}) \setminus \{e_{\bar{v}}^o\}) \cap \text{Node}(G_{2/1}|_{G^o_{\bar{v}/1}}) \subseteq \text{Node}(G_{2/1})\) [cf. [CbTpI], Definition 2.2, (ii)]. Then the natural homomorphism of the above display allows one to identify \((\Pi_{\bar{z}})^{2/1}_2\) with the [pro-\(\Sigma\)] fundamental group \(H_{\bar{v}}^o\) of

\[
H_{\bar{v}}^o \defeq (G_{2/1}|_{G^o_{\bar{v}/1}})_{\sim T_{\bar{v}}^o}
\]

[cf. [CbTpI], Definition 2.5, (ii)].
\[ \mathcal{G}_{2/1} \]
\[ \Pi \mathcal{H}_{\tilde{v}^\circ} \quad \Pi \mathcal{H}_{\tilde{w}^\circ} \]
\[ (\Pi \tilde{v})_{2/1} \quad (\Pi \tilde{w})_{2/1} \]
\[ \tilde{e}_{\tilde{v}^\circ} \quad \tilde{e}_{\tilde{w}^\circ} \]
\[ \tilde{v}^\circ \quad \tilde{v} \quad \tilde{v} \quad \tilde{v} \quad \tilde{w} \quad \tilde{w} \quad \tilde{w} \]

\[ \mathcal{G} = \mathcal{G}_1 \]
\[ \Pi_{\tilde{v}, \tilde{w}} \]
\[ \tilde{v} \quad \tilde{e} \quad \tilde{w} \]
\[ \tilde{v} \quad \tilde{e} \quad \tilde{w} \quad \tilde{w} \quad \tilde{w} \quad \tilde{w} \quad \tilde{w} \]

\[ \mathcal{G} = \mathcal{G}_1 \]
\[ \Pi_{\tilde{v}, \tilde{w}} \]
\[ \tilde{v} \quad \tilde{e} \quad \tilde{w} \]
\[ \tilde{v} \quad \tilde{e} \quad \tilde{w} \quad \tilde{w} \quad \tilde{w} \quad \tilde{w} \quad \tilde{w} \]

Figure 2: the noncyclically primitive case
Figure 3: the cyclically primitive case
(iv) In the situation of (iii), let \((\alpha_z)_{z \in \text{Vert}(\mathcal{G})} \in \text{Glu}(\Pi_2)\). Write \(((\alpha_z)_{1z} \in \text{Vert}(\mathcal{G}) \in \text{Glu}(\Pi_1))\) for the image of \((\alpha_z)_{z \in \text{Vert}(\mathcal{G})} \in \text{Glu}(\Pi_2)\) via the injection of Lemma 4.10, (i). Let \(\alpha_1 \in \text{Aut}^{\text{Bch}(\mathcal{G})}(\mathcal{G})\) be such that \(\rho^{\text{Bch}}(\alpha_1) = ((z)_{1z} \in \text{Vert}(\mathcal{G}) \in \text{Glu}(\Pi_1))\); cf. Theorem 4.2, (iii); Definition 4.11. Then the automorphism \(\alpha_1\) of \(\Pi_1\) preserves the \(\Pi_1\)-conjugacy class of \(\Pi_{\bar{v}, \bar{w}} \subseteq \Pi_1\). Thus, by applying the portion of (ii) concerning commensurable terminality, we obtain [cf. Lemma 3.10, (i)] a restricted automorphism \(\alpha_1|_{\Pi_{\bar{v}, \bar{w}}} \in \text{Out}(\Pi_{\bar{v}, \bar{w}})\).

(v) In the situation of (iv), there exists an automorphism \(\beta_{\bar{v}, \bar{w}}[\alpha_1]\) of \(\Pi_2|_{\Pi_{\bar{v}, \bar{w}}}\) that satisfies the following conditions:

1. \(\beta_{\bar{v}, \bar{w}}[\alpha_1]\) preserves \(\Pi_2/2 \subseteq \Pi_2|_{\Pi_{\bar{v}, \bar{w}}}\) and the \(\Pi_2|_{\Pi_{\bar{v}, \bar{w}}}\)-conjugacy classes of \((\Pi_2)_2, (\Pi_2)_2 \subseteq \Pi_2|_{\Pi_{\bar{v}, \bar{w}}}\).

2. There exists an automorphism \(\beta_{\bar{v}, \bar{w}}[\alpha_1]\) of \(\Pi_2|_{\Pi_{\bar{v}, \bar{w}}}\) that lifts the automorphism \(\beta_{\bar{v}, \bar{w}}[\alpha_1]\) such that the automorphism of \(\Pi_{2,1} \to \Pi_{2,1}\) determined by \(\beta_{\bar{v}, \bar{w}}[\alpha_1]\) [cf. (1)] is contained in \(\text{Aut}^{\text{Bch}(\mathcal{G}_{2,1})}(\mathcal{G}_{2,1}) \subseteq \text{Out}(\Pi_{2,1})\).

3. For each \(z \in \{\bar{v}, \bar{w}\}\), the automorphism \(\beta_{\bar{v}, \bar{w}}[\alpha_1]|_{(\Pi_2)_2}\) of \((\Pi_2)_2\) determined by \(\beta_{\bar{v}, \bar{w}}[\alpha_1]\) [i.e., obtained by applying (1) and Lemma 3.10, (i)] — where we note that \((\Pi_2)_2\) is commensurably terminal in \(\Pi_2\) [cf. Lemma 4.4], hence also in \(\Pi_2|_{\Pi_{\bar{v}, \bar{w}}}\) — coincides with \(\alpha_{\bar{z}}(\mathcal{G})\) [cf. the notation of (iv)].

4. The automorphism of \(\Pi_{\bar{v}, \bar{w}}\) induced by \(\beta_{\bar{v}, \bar{w}}[\alpha_1]\) [cf. (1)] coincides with \(\alpha_1|_{\Pi_{\bar{v}, \bar{w}}}\) [cf. (iv)].

Here, we observe, in the context of (2), that the outer isomorphism \(\Pi_{2,1} \to \Pi_{2,1}\) [i.e., which gives rise to “the” closed subgroup \(\text{Aut}^{\text{Bch}(\mathcal{G}_{2,1})}(\mathcal{G}_{2,1}) \subseteq \text{Out}(\Pi_{2,1})\)] may be characterized, up to composition with elements of \(\text{Aut}^{\text{Bch}(\mathcal{G}_{2,1})}(\mathcal{G}_{2,1}) \subseteq \text{Out}(\Pi_{2,1})\), as the group-theoretically cuspidal [cf. [CmbGC], Definition 1.4, (iv)] outer isomorphism such that the semi-graph of anabeloids structure on \(\mathcal{G}_{2,1}\) is the semi-graph of anabeloids structure determined [cf. [NodNon], Theorem A] by the resulting composite

\[
\Pi_2 \leftrightarrow \Pi_0 \cong \Pi_1 \to \text{Out}(\Pi_{2,1}) \to \text{Out}(\Pi_{2,1})
\]

— where the third arrow is the outer action determined by the exact sequence \(1 \to \Pi_{2,1} \to \Pi_2 \xrightarrow{p_1^\Pi} \Pi_1 \to 1\) — in a fashion compatible with the projection \(p_{2,1}^{\Pi_{1,2}/(2)}|_{\Pi_{2,1}}: \Pi_{2,1} \to \Pi_{2}\) and the given outer isomorphisms \(\Pi_{2} \to \Pi_{1} \to \Pi_{0}\).
Proof. First, we verify assertion (i). The existence of such a \((\Pi_{\bar{w}})_2 \subseteq \Pi_2\) follows immediately from the various definitions involved. Thus, it remains to verify the uniqueness of such a \((\Pi_{\bar{w}})_2\). Let \((\Pi_{\bar{w}})_2 \subseteq \Pi_2\) be as in assertion (i) and \(\gamma \in \Pi_2\) an element such that the conjugate \((\Pi_{\bar{w}})_2^\gamma\) of \((\Pi_{\bar{w}})_2\) by \(\gamma\) satisfies the condition on “\((\Pi_{\bar{w}})_2\)” stated in assertion (i). Then since \(\Pi_{\bar{w}}\) is commensurably terminal in \(\Pi_1\) [cf. [CmbGC], Proposition 1.2, (ii)], it holds that the image of \(\gamma\) via \(p_2^{\Pi}\) is contained in \(\Pi_{\bar{w}}\). Thus — by replacing \(\gamma\) with a suitable element of \((\Pi_{\bar{w}})_2\) — we may assume without loss of generality that \(\gamma \in \Pi_{2/1}\). In particular, since \(\Pi_{\bar{w}}^{\text{new}} \subseteq (\Pi_{\bar{w}})_{2/1} \cap (\Pi_{\bar{w}})_{2/1}^{\gamma}\) — where we write \((\Pi_{\bar{w}})_{2/1}^\gamma \overset{\text{def}}{=} \Pi_{2/1} \cap (\Pi_{\bar{w}})_2^\gamma\) — is not abelian [cf. [CmbGC], Remark 1.1.3], it follows immediately from [NodNon], Lemma 1.9, (i), that \((\Pi_{\bar{w}})_2/1 = (\Pi_{\bar{w}})_2/1^\gamma\). Thus, since \((\Pi_{\bar{w}})_2/1\) is commensurably terminal in \(\Pi_{2/1}\) [cf. [CmbGC], Proposition 1.2, (ii)], it holds that \(\gamma \in (\Pi_{\bar{w}})_2/1\). This completes the proof of assertion (i).

Assertions (ii), (iii), (iv) follow immediately from the various definitions involved [cf. also [CmbGC], Propositions 1.2, (ii), and 1.5, (i), as well as the proofs of [CmbCsp], Proposition 1.5, (iii); [CbTpI], Proposition 2.11].

Finally, we verify assertion (v). It follows immediately from the definition of “\(\text{Out}^{\text{FC}}((\Pi_{\langle - \rangle})_{2})^{G\text{-node}}\)” [cf. Definition 4.6, (ii)] that, for each \(\tilde{z} \in \{\tilde{v}, \tilde{w}\}\), there exists a lifting \(\tilde{\alpha}_z \in \text{Aut}((\Pi_{\bar{z}})_2)\) of \(\alpha_{z(G)}\) such that if we write \((\tilde{\alpha}_z)\), for the automorphism of \(\Pi_{\bar{z}}\) determined by \(\tilde{\alpha}_z\), then \((\tilde{\alpha}_z)_{1}(\Pi_{\bar{z}}) = \Pi_{\bar{z}}\). Now we claim that the following assertion holds:

Claim 4.12.A: Write \((\alpha_z)_{2/1}\) for the automorphism of \((\Pi_{\bar{z}})_{2/1}\) determined by \(\alpha_z\). Then — relative to the natural identification \(\Pi_{\bar{z}}^{\text{new}} \sim (\Pi_{\bar{z}})_{2/1}\) of assertion (iii) — it holds that

\[
(\alpha_z)_{2/1} \in \text{Aut}^{|\text{Brch}((H_{\bar{z}})_{2})|}(\Pi_{\bar{z}})^{\gamma} \\
\subseteq \text{Out}(\Pi_{\bar{z}}) \sim \text{Out}((\Pi_{\bar{z}})_{2/1}).
\]

Indeed, careful inspection of the various definitions involved reveals that Claim 4.12.A follows immediately from Theorem 4.7 [together with the commensurability of the subgroup \(\Pi_{\bar{z}} \subseteq \Pi_{\bar{z}}\). Moreover, since [cf. Claim 4.12.A] \(\alpha_z\) preserves the \((\Pi_{\bar{z}})_{2/1}\)-conjugacy classes of \(\Pi_{\bar{z}}\) and \(\Pi_{\bar{z}}^{\text{new}}\), and the vertical subgroups \(\Pi_{\bar{z}}^{\text{new}} \subseteq \Pi_{\bar{z}}^{\text{new}} \subseteq \Pi_{\bar{z}}^{\sim} \subseteq \Pi_{2/1}\) are the unique vertical subgroups of \(\Pi_{2/1} \sim \Pi_{2/1}\) associated to \(\tilde{z}(G)^{\circ}, v^{\text{new}} \in \text{Vert}(G_{2/1})\), respectively, such that \(\Pi_{\bar{z}}^{\text{new}} = \Pi_{\bar{z}} \cap \Pi_{\bar{z}}^{\text{new}}\) [cf. [CmbGC], Proposition 1.5, (i)], we thus conclude that \(\alpha_z(\Pi_{\bar{z}}^{\text{new}}) = \Pi_{\bar{z}}^{\text{new}}, \alpha_z(\Pi_{\bar{z}}^{\text{new}}) = \Pi_{\bar{z}}^{\text{new}}\).

Next, write \((\alpha_z)_{\bar{z}}^{\text{new}}, (\alpha_z)_{\bar{z}}^{\text{new}}\) for the respective automorphisms of \(\Pi_{\bar{z}}, \Pi_{\bar{z}}^{\text{new}}\) determined by \(\tilde{\alpha}_z\). Now we claim that the following assertion holds:
Claim 4.12.B: It holds that
\[(\alpha_{\bar{v}})_{\text{new}} = (\alpha_{\bar{w}})_{\text{new}}.\]

Moreover, if \(v = w\), i.e., \(G\) is cyclically primitive, then — relative to the natural outer isomorphism \(\Pi_{\bar{v}} \rightarrow \Pi_{\bar{w}}\) [where we note that if \(v = w\), then \(\Pi_{\bar{v}}\) is a \(2/1\)-conjugate of \(\Pi_{\bar{w}}\)] — it holds that
\[(\alpha_{\bar{v}})_{\bar{v}} = (\alpha_{\bar{w}})_{\bar{w}}.\]

Indeed, the equality \((\alpha_{\bar{v}})_{\text{new}} = (\alpha_{\bar{w}})_{\text{new}}\) follows from the definition of \(\text{Glu}(\Pi_2)\). Next, suppose that \(G\) is cyclically primitive. To verify the equality \((\alpha_{\bar{v}})_{\bar{v}} = (\alpha_{\bar{w}})_{\bar{w}}\), let us observe that, for each \(\bar{v} \in \{\bar{v}, \bar{w}\}\), the composite \(\Pi_{\bar{v}} \rightarrow \Pi_2 \rightarrow \Pi_{(2)} \Rightarrow \Pi_G\) is injective [and its image is a vertical subgroup of \(\Pi_G\) associated to \(\bar{z}(G) \in \text{Vert}(G)\)]. Thus, to verify the equality \((\alpha_{\bar{v}})_{\bar{v}} = (\alpha_{\bar{w}})_{\bar{w}}\), it suffices to verify that the automorphism of the image of \(\Pi_{\bar{v}}\) in \(\Pi_{(2)}\) induced by \((\alpha_{\bar{v}})_{\bar{v}}\) coincides with the automorphism of the image of \(\Pi_{\bar{w}}\) in \(\Pi_{(2)}\) induced by \((\alpha_{\bar{w}})_{\bar{w}}\). On the other hand, this follows immediately from the fact that both \(\bar{z}\) and \(\bar{z}\) are liftings of the same automorphism \(\alpha_v = \alpha_w\) of \(\langle \Pi_{(2)} \rangle = \langle \Pi_{(2)} \rangle\) [cf. \([\text{CmbCsp}], \text{Proposition} 1.2, \text{(iii)}\)]]. This completes the proof of Claim 4.12.B.

Next, let us observe that it follows immediately from the various definitions involved that if \(G\) is noncyclically primitive (respectively, cyclically primitive), then \(\text{Vert}((\mathcal{G}_{(2)})_{\rightarrow(\varepsilon_{v})})^j = 2\) (respectively, \(= 1\)), and that, relative to the correspondence discussed in \([\text{CbTpi}], \text{Proposition} 2.9, \text{(i)}, \text{(iii)}\), \(H_{v^o}, G_{v^o, (\mathcal{G}_{(2)})_{\rightarrow(\varepsilon_{v})}} \) (respectively, \(H_{w^o}, G_{w^o, (\mathcal{G}_{(2)})_{\rightarrow(\varepsilon_{w})}}\)) correspond(s) to the two vertices (respectively, the unique vertex) of \((\mathcal{G}_{(2)})_{\rightarrow(\varepsilon_{v})}\).

Next, let us observe the following equalities [cf. the notation of \([\text{CbTpi}], \text{Definition} 3.8, \text{(ii)}\)]:
\[
\chi_{H_{v^o}}((\alpha_{\bar{v}})_{(2)}) = \chi_{H_{v^o, \text{new}}}((\alpha_{\bar{v}})_{\text{new}}) \quad \text{[cf. \([\text{CbTpi}], \text{Corollary} 3.9, \text{(iv)}\)]}
\]
\[
\chi_{H_{w^o}}((\alpha_{\bar{w}})_{(2)}) = \chi_{H_{w^o, \text{new}}}((\alpha_{\bar{w}})_{\text{new}}) \quad \text{[cf. Claim 4.12.B]}
\]
\[
\chi_{H_{w^o}}((\alpha_{\bar{w}})_{(2)}) = \chi_{G_{w^o, (\mathcal{G}_{(2)})_{\rightarrow(\varepsilon_{v})}}}((\alpha_{\bar{w}})_{\text{new}}) \quad \text{[cf. \([\text{CbTpi}], \text{Corollary} 3.9, \text{(iv)}\)]}.
\]

Now it follows immediately from these equalities, together with Claim 4.12.A, that the data
\[
((\alpha_{\bar{v}})_{(2)}, (\alpha_{\bar{w}})_{\text{new}}) \in \text{Aut}(H_{v^o}) \times \text{Aut}(G_{w^o, (\mathcal{G}_{(2)})_{\rightarrow(\varepsilon_{v})}})
\]
(respectively, \((\alpha_{\bar{v}})_{(2)} \in \text{Aut}(H_{v^o})\))

may be regarded as an element of \(\text{Glu}^{\text{brch}}((\mathcal{G}_{(2)})_{\rightarrow(\varepsilon_{v})})\) [cf. \([\text{Definition} 4.1, \text{(iii)}\)]]. Thus, by applying the exact sequence of Theorem 4.2, (iii) [cf. also Remark 4.9.1], we obtain an element
\[
\alpha_{(2)}[\bar{v}] \in \text{Aut}^{\text{brch}}((\mathcal{G}_{(2)})_{\rightarrow(\varepsilon_{v})})(G_{(2)})_{\rightarrow(\varepsilon_{v})})
\]
that [cf. [CbTpI], Definition 2.10] belongs to a collection of automorphisms of

\[ \Phi_{(G_2/1)_{\to\{e,v\}}} : \Pi_{(G_2/1)_{\to\{e,v\}}} \xrightarrow{\sim} \Pi_{G_2/1} \xrightarrow{\sim} \Pi_{2/1} \]

[i.e., contained in \( \text{Aut}((G_{2/1})_{\to\{e,v\}}) \hookrightarrow \text{Out}(\Pi_{2/1}) \)] that admits a natural structure of torsor over

\[ \text{Dehn}((G_{2/1})_{\to\{e,v\}}) (\subseteq \text{Aut}((G_{2/1})_{\to\{e,v\}})) \]

A similar argument yields an element

\[ \alpha_{2/1}[\tilde{w}] \in \text{Aut}^{[\text{Bran}((G_{2/1})_{\to\{e,w\}})]((G_{2/1})_{\to\{e,w\}})} \]

that [cf. [CbTpI], Definition 2.10] belongs to a collection of automorphisms of

\[ \Phi_{(G_2/1)_{\to\{e,w\}}} : \Pi_{(G_2/1)_{\to\{e,w\}}} \xrightarrow{\sim} \Pi_{G_2/1} \xrightarrow{\sim} \Pi_{2/1} \]

[i.e., contained in \( \text{Aut}((G_{2/1})_{\to\{e,w\}}) \hookrightarrow \text{Out}(\Pi_{2/1}) \)] that admits a natural structure of torsor over

\[ \text{Dehn}((G_{2/1})_{\to\{e,w\}}) (\subseteq \text{Aut}((G_{2/1})_{\to\{e,w\}})) \]

Now we claim that the following assertion holds:

**Claim 4.12.C:** For each \( \tilde{z} \in \{\tilde{v}, \tilde{w}\} \), the automorphism \( (\tilde{\alpha}_z)_{1} \) of \( \Pi_{\tilde{z}} \) is compatible with the automorphism \( \alpha_{2/1}[\tilde{z}] \) of \( \Pi_{2/1} \) relative to the homomorphism \( \Pi_{\tilde{z}} \hookrightarrow \Pi_{1} \to \text{Out}(\Pi_{2/1}) \) — where the second arrow is the natural outer action determined by the exact sequence

\[ 1 \rightarrow \Pi_{2/1} \rightarrow \Pi_{2} \xrightarrow{\rho_{2/1}} \Pi_{1} \rightarrow 1. \]

Indeed, to verify the compatibility of \( (\tilde{\alpha}_z)_{1} \) and \( \alpha_{2/1}[\tilde{v}] \), it follows immediately from the various definitions involved that it suffices to verify that, for each \( \sigma \in \Pi_{\tilde{v}} \), if we write \( \tau \overset{\text{def}}{=} (\tilde{\alpha}_z)_{1}(\sigma) \in \Pi_{\tilde{z}} \), then there exist liftings \( \hat{\sigma}, \hat{\tau} \in \Pi_{2} \) of \( \sigma, \tau \in \Pi_{\tilde{z}} \), respectively, such that the equality [which is in fact independent of the choice of liftings]

\[ \alpha_{2/1}[\tilde{v}] \circ [\text{Inn}(\hat{\sigma})] \circ \alpha_{2/1}[\tilde{v}]^{-1} = [\text{Inn}(\hat{\tau})] \in \text{Out}(\Pi_{2/1}) \]

— where we write “Inn(−)” for the automorphism of \( \Pi_{2/1} \) determined by conjugation by “(−)” and “[Inn(−)]” for the automorphism of \( \Pi_{2/1} \) determined by this automorphism — holds. To this end, let \( \tilde{\sigma} \in (\Pi_{\tilde{v}})_{2} \) be a lifting of \( \sigma \in \Pi_{\tilde{v}} \). Then since \( (\Pi_{\tilde{v}})_{2/1} \subseteq (\Pi_{\tilde{v}})_{2} \) is normal, \( \text{Inn}(\tilde{\sigma}) \) preserves \( (\Pi_{\tilde{v}})_{2/1} \).

Next, let us observe that the semi-graph of anabelioids structure of \( (G_{2/1})_{\to\{e,v\}} \) [with respect to which \( w^0 \)] is a vertex! may be thought of as the semi-graph of anabelioids structure on the fiber subgroup \( \Pi_{2/1} \) [cf. Definition 3.1, (iii)] arising from a point of \( X^{\log} \) that lies in the interior of the irreducible component of \( X^{\log} \) corresponding to \( v \). Now it follows immediately from this observation that \( \text{Inn}(\tilde{\sigma}) \) preserves
the $\Pi_{2/1}$-conjugacy class of $\Pi_{\tilde{w}^e}$, as well as the $\Pi_{2/1}$-conjugacy class of $\Pi_{\tilde{w}^e} = (\Pi_{\tilde{v}})_{2/1} \cap \Pi_{\tilde{w}^e}$. By considering the various $\Pi_{2/1}$-conjugates of $\Pi_{\tilde{w}^e}$ and $\Pi_{\tilde{w}^e}$ and applying [CmbGC], Propositions 1.2, (ii); 1.5, (i), we thus conclude that $\text{Inn}(\tilde{\sigma})$ preserves the $(\Pi_{\tilde{v}})_{2/1}$-conjugacy classes of $\Pi_{\tilde{w}^e}$, $\Pi_{\tilde{w}^e}$. In particular — by multiplying $\tilde{\sigma}$ by a suitable element of $(\Pi_{\tilde{v}})_{2/1}$ — we may assume without loss of generality that $\text{Inn}(\tilde{\sigma})$ preserves $(\Pi_{\tilde{v}})_{2/1}$, $\Pi_{\tilde{w}^e}$, and $\Pi_{\tilde{w}^e}$.

Next, let us observe that one verifies easily [cf. Lemma 3.6, (iv)] that the composite $\Pi_{\tilde{w}^e} \hookrightarrow \Pi_{\Pi(1,2)/2} \xrightarrow{\Pi_{\Pi(1,2)/2}} \Pi_{\Pi(1,2)/2}$ surjects onto a nodal subgroup of $\Pi_{\Pi(1,2)/2}$ associated to $e \in \text{Node}(\mathcal{G})$. Thus, since $\text{Inn}(\tilde{\sigma})$ preserves $\Pi_{\tilde{w}^e}$, it follows [cf. [CmbGC], Proposition 1.2, (ii)] that the image of $\tilde{\sigma} \in \Pi_{\Pi(1,2)/2}$ is contained in the image of the composite $\Pi_{\tilde{w}^e} \hookrightarrow \Pi_{\Pi(1,2)/2} \xrightarrow{\Pi_{\Pi(1,2)/2}} \Pi_{\Pi(1,2)/2}$. In particular — by multiplying $\tilde{\sigma}$ by a suitable element of $\Pi_{\tilde{w}^e}$ (subseteq $(\Pi_{\tilde{v}})_{2/1}$) — we may assume without loss of generality that $\tilde{\sigma} \in \text{Ker}(\Pi_{\Pi(1,2)/2})$. A similar argument implies that there exists a lifting $\tilde{\tau} \in (\Pi_{\tilde{v}})_{2/1}$ of $\tau = (\tilde{\alpha}_{\tilde{v}})_1(\sigma) \in \Pi_{\tilde{v}}$ such that $\text{Inn}(\tilde{\tau})$ preserves $(\Pi_{\tilde{v}})_{2/1}$, $\Pi_{\tilde{w}^e}$, $\Pi_{\tilde{w}^e}$, and, moreover, $\tilde{\tau} \in \text{Ker}(\Pi_{\Pi(1,2)/2})$.

Now since the automorphisms $(\tilde{\alpha}_{\tilde{v}})_{2/1}, (\tilde{\alpha}_{\tilde{v}})_1$ of $(\Pi_{\tilde{v}})_{2/1}$, $\Pi_{\tilde{v}}$, respectively, arise from the automorphism $\tilde{\alpha}_{\tilde{v}}$ of $(\Pi_{\tilde{v}})$, it follows immediately from the construction of $\alpha_{2/1}[\tilde{v}]$ that the equality

$$\alpha_{2/1}[\tilde{v}] \circ [\text{Inn}(\tilde{\sigma})] \circ \alpha_{2/1}[\tilde{v}]^{-1} = [\text{Inn}(\tilde{\tau})]$$

holds upon restriction to [an equality of automorphisms of] $(\Pi_{\tilde{v}})_{2/1}$.

Moreover, since the composite $\Pi_{\tilde{w}^e} \hookrightarrow \Pi_{\Pi(1,2)/2} \xrightarrow{\Pi_{\Pi(1,2)/2}} \Pi_{\Pi(1,2)/2}$ is injective [and its image is a vertical subgroup of $\Pi_{\Pi(1,2)/2}$ associated to $w \in \text{Vert}(\mathcal{G})$] — cf. Lemma 3.6, (iv)], to verify the restriction of the equality

$$\alpha_{2/1}[\tilde{v}] \circ [\text{Inn}(\tilde{\sigma})] \circ \alpha_{2/1}[\tilde{v}]^{-1} = [\text{Inn}(\tilde{\tau})]$$

to [an equality of automorphisms of] $\Pi_{\tilde{w}^e}$, it suffices to verify that the automorphism of the image of $\Pi_{\tilde{w}^e}$ in $\Pi_{\Pi(1,2)/2}$ induced by the product

$$\alpha_{2/1}[\tilde{v}] \circ [\text{Inn}(\tilde{\sigma})] \circ \alpha_{2/1}[\tilde{v}]^{-1} \circ [\text{Inn}(\tilde{\sigma})]^{-1}$$

is trivial. On the other hand, this follows immediately from the fact that $\tilde{\sigma}, \tilde{\tau} \in \text{Ker}(\Pi_{\Pi(1,2)/2})$.

Thus, in summary, the restrictions of the equality in question [i.e., in Claim 4.12.C] to [equalities of automorphisms of] $(\Pi_{\tilde{v}})_{2/1}$ and $\Pi_{\tilde{w}^e}$ hold. In particular, it follows immediately from the displayed exact sequence of Theorem 4.2, (iii) [cf. also Remark 4.9.1], that the product

$$\alpha_{2/1}[\tilde{v}] \circ [\text{Inn}(\tilde{\sigma})] \circ \alpha_{2/1}[\tilde{v}]^{-1} \circ [\text{Inn}(\tilde{\sigma})]^{-1}$$
is contained in $\text{Dehn}((\mathcal{G}_{2/1})_{-\{e_o\}})$. Thus — by considering the automorphism of $\Pi_{2/1}$ induced by the above product — one verifies easily from [CbTp], Theorem 4.8, (iv), together with the fact that $\bar{\sigma}$, $\bar{\tau} \in \text{Ker}(p^{H}_{1/2}(1/2))$, that the equality in question holds. This completes the proof of the compatibility of $(\bar{\alpha}_e\bar{\tau}^{-1})_1$ and $\alpha_{2/1}[\bar{v}]$. The compatibility of $(\bar{\alpha}_e\bar{w})_1$ and $\alpha_{2/1}[\bar{w}]$ follows from a similar argument. This completes the proof of Claim 4.12.D.

Next, we claim that the following assertion holds:

Claim 4.12.D: The difference $\alpha_{2/1}[\bar{v}] \circ \alpha_{2/1}[\bar{w}]^{-1} \in \text{Out}(\Pi_{2/1})$ is contained in $\text{Dehn}(\mathcal{G}_{2/1}) \subseteq \text{Out}(\Pi_{2/1}) \sim \text{Out}(\Pi_{2/1})$.

Indeed, this follows immediately from the two displayed equalities of Claim 4.12.B, together with the construction of $\alpha_{2/1}[\bar{v}]$, $\alpha_{2/1}[\bar{w}]$. This completes the proof of Claim 4.12.D.

Thus, it follows immediately from Claim 4.12.D, together with the existence of the natural isomorphism

$$\text{Dehn}((\mathcal{G}_{2/1})_{-\{e_o\}}) \oplus \text{Dehn}((\mathcal{G}_{2/1})_{-\{e_o\}}) \sim \text{Dehn}(\mathcal{G}_{2/1}),$$

cf. [CbTp], Theorem 4.8, (ii), (iv), that — by replacing $\alpha_{2/1}[\bar{v}]$, $\alpha_{2/1}[\bar{w}]$ by the composites of $\alpha_{2/1}[\bar{v}]$, $\alpha_{2/1}[\bar{w}]$ with suitable elements of $\text{Dehn}((\mathcal{G}_{2/1})_{-\{e_o\}})$, $\text{Dehn}((\mathcal{G}_{2/1})_{-\{e_o\}})$, respectively [where we recall that the outomorphisms $\alpha_{2/1}[\bar{v}]$, $\alpha_{2/1}[\bar{w}]$ belong to torsors over $\text{Dehn}((\mathcal{G}_{2/1})_{-\{e_o\}})$, $\text{Dehn}((\mathcal{G}_{2/1})_{-\{e_o\}})$, respectively] — we may assume without loss of generality that $\alpha_{2/1}[\bar{v}] = \alpha_{2/1}[\bar{w}]$.

Write $\beta_{2/1} \overset{\text{def}}{=} \alpha_{2/1}[\bar{v}] = \alpha_{2/1}[\bar{w}]$. Then it follows immediately from Claim 4.12.C, together with the fact that $\Pi_{\bar{v},\bar{w}}$ is topologically generated by $\Pi_{\bar{v}}$, $\Pi_{\bar{v}} \subseteq \Pi_{\bar{v},\bar{w}}$ [cf. assertion (ii)], that the automorphism $\beta_{2/1}$ of $\Pi_{2/1}$ is compatible with the automorphism $\tilde{\alpha}_{1}[\Pi_{\bar{v},\bar{w}}]$ of $\Pi_{\bar{v},\bar{w}}$ [i.e., the automorphism induced by $(\tilde{\alpha}_{e})_1$, $(\tilde{\alpha}_e)_1$ — cf. assertion (ii)], relative to the composite $\Pi_{\bar{v},\bar{w}} \ni \Pi_{\bar{v}} \rightarrow \Pi_{\bar{v}} \rightarrow \text{Out}(\Pi_{2/1})$ — where the second arrow is the outer action determined by the displayed exact sequence of Claim 4.12.C. In particular, by considering the natural isomorphism

$\Pi_{2/1}[\Pi_{\bar{v},\bar{w}}] \sim \Pi_{2/1} \times \Pi_{\bar{v},\bar{w}}$ [cf. the discussion entitled “Topological groups” in [CbTp], §8], we obtain an automorphism $\beta_{\bar{v},\bar{w}}$ of $\Pi_{2/1}[\Pi_{\bar{v},\bar{w}}]$ which, by construction, satisfies the four conditions listed in assertion (v). This completes the proof of assertion (v). \qed

Lemma 4.13 (Glueability of combinatorial cuspidalizations in the case of precisely one node). Suppose that $n = 2$, and that $\text{Node}(\mathcal{G})^b = 1$. Then $\rho_{2}^{\text{brch}}$ [cf. Definition 4.11] is surjective.

Proof. If $\mathcal{G}$ is noncyclically primitive [cf. [CbTp], Definition 4.1], then the surjectivity of $\rho_{2}^{\text{brch}}$ follows immediately from Lemma 4.12, (v),
Next, let us fix a vertical subgroup the evident pro-$\Sigma$ generalization of [SemiAn], Proposition 3.6, (iii). Let sequence each $a$ for the image of $\Pi$ temp $V$ verticial subgroup $\Pi_{\tilde{v}(a)} \subseteq \pi_1^{\mathrm{temp}}(G)$. Let $\gamma_\infty \in \pi_1^{\mathrm{top}}(G)$ be a generator of $\pi_1^{\mathrm{top}}(G)$ ($\cong \mathbb{Z}$) and $\tilde{\gamma}_\infty \in \pi_1^{\mathrm{temp}}(G)$ a lifting of $\gamma_\infty$. By abuse of notation, write $\tilde{\gamma}_\infty \in \Pi_G \tilde{\sim} \Pi_1$ for the image of $\tilde{\gamma}_\infty \in \pi_1^{\mathrm{temp}}(G)$ via the natural injection $\pi_1^{\mathrm{temp}}(G) \hookrightarrow \Pi_2 \tilde{\sim} \Pi_1$ [cf. the evident pro-$\Sigma$ generalization of [SemiAn], Proposition 3.6, (iii)].

Next, let us fix a vertical subgroup
\[
\Pi_{\tilde{v}(a)} \subseteq (\pi_1^{\mathrm{temp}}(G) \cong) \pi_1^{\mathrm{temp}}(G)
\]
of $\pi_1^{\mathrm{temp}}(G)$ that corresponds to a vertex $\tilde{v}(a) \in \operatorname{Vert}(\tilde{G})$ that lifts the vertex $V(a) \in \operatorname{Vert}(G)$ [cf. [CbTpI], Lemma 4.3, (iii)]. Thus, for each integer $a \in \mathbb{Z}$, by forming the conjugate of $\Pi_{\tilde{v}(a)}$ by $\tilde{\gamma}_\infty^a$, we obtain a vertical subgroup
\[
\Pi_{\tilde{v}(a)} \subseteq (\pi_1^{\mathrm{temp}}(G) \cong) \pi_1^{\mathrm{temp}}(G)
\]
of $\pi_1^{\mathrm{temp}}(G)$ associated to some vertex $\tilde{v}(a) \in \operatorname{Vert}(\tilde{G})$ that lifts the vertex $V(a) \in \operatorname{Vert}(G)$ [cf. [CbTpI], Lemma 4.3, (iii), (vi)]. Write
\[
\Pi_{\tilde{v}(a)} \subseteq \Pi_G
\]
for the image of $\Pi_{\tilde{v}(a)} \subseteq \pi_1^{\mathrm{temp}}(G)$ in $\Pi_G$.

Next, let us suppose that $\tilde{\gamma}_\infty$ was chosen in such a way that, for each $a \in \mathbb{Z}$, the intersection $\mathcal{N}(\tilde{v}(a)) \cap \mathcal{N}(\tilde{v}(a + 1))$ consists of a unique node $\tilde{n}(a + 1) \in \operatorname{Node}(\tilde{G})$ that lifts the node $N(a + 1) \in \operatorname{Node}(G)$ [cf. [CbTpI], Lemma 4.3, (iii)]. [One verifies easily that such a $\tilde{\gamma}_\infty$ always exists.] Then let us observe that, for each $a \leq b \in \mathbb{Z}$, we have a natural morphism of semi-graphs of anabelioids $\tilde{G}_{[a,b]} \to G_\infty$ [cf. [CbTpI], Lemma 4.3, (iv)], which induces injections [cf. the evident pro-$\Sigma$ generalizations of [SemiAn], Example 2.10; [SemiAn], Proposition 2.5, (i); [SemiAn], Proposition 3.6, (iii)]
\[
\pi_1^{\mathrm{temp}}(G_{[a,b]}) \hookrightarrow \pi_1^{\mathrm{temp}}(G_\infty), \quad \Pi_{G_{[a,b]}} \hookrightarrow \Pi_G
\]
— where we write $\pi_1^{\mathrm{temp}}(G_{[a,b]}), \Pi_{G_{[a,b]}}$ for the tempered, pro-$\Sigma$ fundamental groups of the semi-graph of anabelioids $G_{[a,b]}$ of pro-$\Sigma$ PSC-type, respectively — which are well-defined up to composition with inner...
automorphisms. By choosing appropriate basepoints, these inner automorphism indeterminacies may be eliminated in such a way that, for each \( a \leq c \leq b \), the resulting injections are compatible with one another and, moreover, their images contain the subgroups \( \Pi_{\bar{v}(c)}^{\text{temp}} \subseteq \pi_1^{\text{temp}}(G_{\infty}) \), \( \Pi_{\bar{v}(c)} \subseteq \Pi_G \), respectively. Then, relative to the resulting inclusions, \( \Pi_{\bar{v}(c)}^{\text{temp}} \), \( \Pi_{\bar{v}(c)} \) form vertical subgroups of \( \pi_1^{\text{temp}}(G_{[a,b]}) \), \( \Pi_{\bar{v}_{[a,b]}} \) associated to the vertex of \( G_{[a,b]} \) corresponding to \( V(c) \) [cf. [CbTpI], Lemma 4.3, (iii)]. In particular, we have a natural isomorphism

\[
\Pi_{[a,a+1]} \overset{\text{def}}{=} \Pi_{\bar{v}(a),\bar{v}(a+1)} \overset{\sim}{\longrightarrow} \Pi_{\bar{v}_{[a,a+1]}}
\]

[cf. Lemma 4.12, (ii)]. Let us write

\[
\Pi_2\mid_{[a+1]} \overset{\text{def}}{=} \Pi_2 \mid_{[a,a+1]} \subseteq \Pi_2
\]

[cf. Lemma 4.12, (ii)];

\[
\Pi_{[a]} \overset{\text{def}}{=} \Pi_{\bar{v}(a)} ;
\]

\[
\Pi_2\mid_{[a]} \overset{\text{def}}{=} \Pi_2 \times \Pi_1 \mid_{[a]} \subseteq \Pi_2 \mid_{[a-1,a]} ; \quad \Pi_2\mid_{[a,a+1]} .
\]

Next, we claim that the following assertion holds:

Claim 4.13.A: The profinite group \( \Pi_G \) is topologically generated by \( \Pi_{[0]} \subseteq \Pi_G \) and \( \bar{\gamma}_{\infty} \in \Pi_G \).

Indeed, let us first observe that it follows immediately from a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.5, (iii) [i.e., in essence, from the “van Kampen Theorem” in elementary algebraic topology], that the image of the natural homomorphism

\[
\lim_{a \geq 0} \pi_1^{\text{temp}}(G_{[-a,a]}) \longrightarrow \pi_1^{\text{temp}}(G_{\infty})
\]

— where the inductive limit is taken in the category of tempered groups [cf. [SemiAn], Definition 3.1, (i); [SemiAn], Example 2.10; [SemiAn], Proposition 3.6, (i)] — is dense. In particular, it follows immediately from the exact sequence of [CbTpI], Lemma 4.3, (i), that the tempered group \( \pi_1^{\text{temp}}(G) \) [cf. [SemiAn], Example 2.10; [SemiAn], Proposition 3.6, (i)] is topologically generated by \( \Pi_{\bar{v}(0)}^{\text{temp}} \subseteq \pi_1^{\text{temp}}(G) \) and \( \bar{\gamma}_{\infty} \in \pi_1^{\text{temp}}(G) \). Thus, Claim 4.13.A follows immediately from the fact that the image of the natural injection \( \pi_1^{\text{temp}}(G) \hookrightarrow \Pi_G \) is dense. This completes the proof of Claim 4.13.A.

For \( a \in \mathbb{Z} \), let us write

\[
G_{2/1}^{[a,a+1]} \overset{\text{def}}{=} G_{2\in\{1,2\},x}
\]

[cf. Definition 3.1, (iii)], where we take the “fixed” outer isomorphism

\[
\Pi_{2/1} \overset{\sim}{\longrightarrow} \Pi_{G_{2/1}^{[a,a+1]}}
\]
of Definition 3.1, (iii), to be an outer isomorphism [cf. the discussion of the final portion of Lemma 4.12, (v)] such that the semi-graph of anabelioids structure on $G_{2/1}^{[a,a+1]}$ is the semi-graph of anabelioids structure determined by the resulting composite

$$\Pi_{\tilde{n}(a+1)} \hookrightarrow \Pi_G \overset{\sim}{\to} \Pi_1 \to \text{Out}(\Pi_{2/1}) \overset{\sim}{\to} \text{Out}(\Pi_{G_{2/1}^{[a,a+1]}})$$

— where we write $\Pi_{\tilde{n}(a+1)} \subseteq \Pi_G$ for the nodal subgroup of $\Pi_G$ associated to the unique element $\tilde{n}(a+1) \in \mathcal{N}(\tilde{v}(a)) \cap \mathcal{N}(\tilde{v}(a+1))$, and the third arrow arises from the outer action determined by the exact sequence $1 \to \Pi_{2/1} \to \Pi_2 \overset{p_{2/1}^1}{\to} \Pi_1 \to 1$ — in a fashion compatible with the projection $p_{\{1,2\}/\{2\}}: \Pi_{2/1} \to \Pi_{\{2\}}$ and the given outer isomorphisms $\Pi_{\{2\}} \overset{\sim}{\to} \Pi_1 \overset{\sim}{\to} \Pi_G$. Here, we note that, for $a, b \in \mathbb{Z}$, there exist isomorphisms $G_{2/1}^{[a,a+1]} \overset{\sim}{\to} G_{2\in[1,2],x} \overset{\sim}{\to} G_{2/1}^{[b,b+1]}$ of semi-graphs of anabelioids of pro-$\Sigma$ PSC-type. On the other hand, it is not difficult to show [although we shall not use this fact in the present proof!] that the well-known injectivity of the homomorphism $\Pi_1 \to \text{Out}(\Pi_{2/1})$ of the above display [cf. [Asd], Theorem 1; [Asd], the Remark following the proof of Theorem 1] implies that when $a \neq b$, the composite

$$\Pi_{G_{2/1}^{[a,a+1]}} \hookrightarrow \Pi_{2/1} \overset{\sim}{\to} \Pi_{G_{2/1}^{[b,b+1]}}$$

in fact fails to be graphic!

---

**Figure 4:** $G_{2/1}^{[a-1,a]}$, $G_{2/1}^{[a]}$, and $G_{2/1}^{[a,a+1]}$
For each \( a \in \mathbb{Z} \), let us write
\[
\mathcal{G}^{[a,a+1] \to [a]}_{2/1} \triangleq (\mathcal{G}^{[a,a+1]}_{2/1})_{\sim \{e_{v(a)}\}}, \quad \mathcal{G}^{[a,a+1] \to [a+1]}_{2/1} \triangleq (\mathcal{G}^{[a,a+1]}_{2/1})_{\sim \{e_{v(a+1)}\}}
\]
where we write \( e_{v(a)} \) for the nodes “\( e_v \)” of Lemma 4.12, (iii), that occur, respectively, in the cases where the pair “\( (\omega, v) \)” is taken to be \((\mathcal{G}^{[a,a+1]}_{2/1}, \bar{v}(a))\); \((\mathcal{G}^{[a,a+1]}_{2/1}, \bar{v}(a+1))\). Then one verifies easily that the composite
\[
\Pi^{[a-1,a] \to [a]}_{\mathcal{G}^{[a-1,a]}_{2/1}} \rightharpoonup \Pi^{[a,a+1]}_{\mathcal{G}^{[a,a+1]}_{2/1}} \hookrightarrow \Pi^{[a,a+1]}_{\mathcal{G}^{[a,a+1]}_{2/1}} \rightharpoonup \Pi^{[a+1,a+1] \to [a]}_{\mathcal{G}^{[a+1,a+1]}_{2/1}}
\]
where the first and fourth arrows are the natural specialization outer isomorphisms [cf. [CbTpI], Definition 2.10], and the second and third arrows are the outer isomorphisms fixed above — is graphic. In light of this observation, it makes sense to write
\[
\mathcal{G}^{[a]}_{2/1} \triangleq \mathcal{G}^{[a-1,a] \to [a]}_{2/1} \rightharpoonup \mathcal{G}^{[a,a+1] \to [a]}_{2/1}
\]
[cf. Figure 4 above].

Next, let us observe that one verifies easily that the composites
\[
\Pi^{[a]} \hookrightarrow \Pi^{[a]} \rightharpoonup \Pi^{(\Pi^{[a]}_2/1)} \rightharpoonup \Pi^{(\Pi^{[a+1]}_2/1)} \hookrightarrow \Pi^{(\Pi^{[a+1]}_2/1)} \rightharpoonup \Pi^{(\Pi^{[a+1]}_2/1)}
\]
where the third arrows on each line of the display arise from the outer action determined by the exact sequence \( 1 \to \Pi^{[a]}_2/1 \to \Pi^{[a]}_2 \to \Pi^{[a]}_1 \to 1 \), the fourth arrows are the isomorphisms induced by the outer isomorphism \( \Pi^{[a]}_2/1 \rightharpoonup \Pi^{[a+1]}_2/1 \) fixed above, and the fifth arrows are the isomorphisms induced by the natural specialization outer isomorphisms [cf. [CbTpI], Definition 2.10] — factor through
\[
\text{Aut}(\mathcal{G}^{[a]}_{2/1}) \subseteq \text{Out}(\Pi^{[a]}_{\mathcal{G}^{[a]}_{2/1}}), \quad \text{Aut}(\mathcal{G}^{[a+1]}_{2/1}) \subseteq \text{Out}(\Pi^{[a+1]}_{\mathcal{G}^{[a+1]}_{2/1}}),
\]
respectively.

Now we turn to the verification of the surjectivity of the homomorphism \( \rho_{2/1}^{\text{brch}} \). Let \( \alpha_v \in \text{Glu}(\Pi^{[a]}_2) \) be such that \( \rho^{\text{brch}}_{2/1}(\alpha_v)_1 \in \text{Glu}(\Pi^{[a]}_1) \) [cf. Theorem 4.2, (iii); Definition 4.11]. Now, by applying Lemma 4.12, (v), in the case where we take the pair “\( (\bar{v}, \bar{w}) \)” to be \((\bar{v}(0), \bar{v}(1))\), we obtain an automorphism \( \beta^{[0,1]} \triangleq \beta^{[0,1]}_{(0,1)} \) of \( \Pi^{[0,1]}_2 \) [cf. the notation of the discussion preceding Claim 4.13.A]. Let \( \bar{\beta}_{[0,1]} \in \text{Aut}(\Pi^{[0,1]}_2) \) be an automorphism that lifts \( \beta^{[0,1]} \in \text{Out}(\Pi^{[0,1]}_2) \) and \( \bar{\gamma}_\infty \in \Pi^{1} \) a lifting of \( \gamma_\infty \in \Pi^{1} \). Then since \( \bar{\beta}_{[0,1]} \) is easily verified to be a lifting of \( \gamma_\infty \) in \( \Pi^{1} \) [cf. the notation of the discussion preceding Claim 4.13.A] is the conjugate of \( \Pi^{[0,1]}_2 \) by \( \bar{\gamma}_\infty \), by conjugating \( \bar{\beta}_{[0,1]} \) by \( \bar{\gamma}_\infty \), we obtain
an automorphism $\widetilde{\beta}_{[1,2]}$ of $\Pi_2|_{[1,2]}$. On the other hand, it follows immediately from [CmbGC], Proposition 1.2, (ii), together with Lemma 4.5, that $\Pi_2|_{[1]}$ [cf. the notation of the discussion preceding Claim 4.13.A] is \textit{commensurably terminal} in $\Pi_2|_{[0,1]}$, $\Pi_2|_{[1,2]}$, which thus implies [cf. Lemma 3.10, (i); condition (4) of Lemma 4.12, (v)] that — by restricting $\widetilde{\beta}_{[0,1]}$, $\widetilde{\beta}_{[1,2]}$ to $\Pi_2|_{[1]} \subseteq \Pi_2|_{[0,1]}$, $\Pi_2|_{[1,2]}$ — we obtain two \textit{restricted automorphisms}

$$\beta_{[0,1]|_{[1]}}, \beta_{[1,2]|_{[1]}}$$

of $\Pi_2|_{[1]}$. Now we claim that the following assertion holds:

\textbf{Claim 4.13.B}: There exist automorphisms $\widetilde{\beta}_{[0,1]|_{[1]}}, \widetilde{\beta}_{[1,2]|_{[1]}}$ of $\Pi_2|_{[1]}$ that lift $\beta_{[0,1]|_{[1]}}, \beta_{[1,2]|_{[1]}}$, respectively, such that the automorphisms of $\Pi_{2/1} \subseteq \Pi_2|_{[1]}$ determined by $\widetilde{\beta}_{[0,1]|_{[1]}}, \widetilde{\beta}_{[1,2]|_{[1]}}$ \textit{coincide}.

Indeed, it follows from condition (2) of Lemma 4.12, (v), together with the definition of $\beta_{[1,2]}$, that there exist automorphisms $\widetilde{\beta}_{[0,1]|_{[1]}}, \widetilde{\beta}_{[1,2]|_{[1]}}$ of $\Pi_2|_{[1]}$ that lift $\beta_{[0,1]|_{[1]}}, \beta_{[1,2]|_{[1]}}$, respectively, such that the automorphisms $(\widetilde{\beta}_{[0,1]|_{[1]}}|_{2/1}, \widetilde{\beta}_{[1,2]|_{[1]}}|_{2/1})$ of $\Pi_{2/1}$ determined by $\widetilde{\beta}_{[0,1]|_{[1]}}, \widetilde{\beta}_{[1,2]|_{[1]}}$ are \textit{contained} in $\text{Aut}^{B_{\Pi(2), 1}}(G_{2/1}^{0,1}), \text{Aut}^{B_{\Pi(2), 1}}(G_{2/1}^{1,2}) (\subseteq \text{Out}(\Pi_{2/1}))$, respectively. In particular, it follows that, relative to the \textit{specialization outer isomorphisms} $\Pi_{g_{2/1}^{0,1}} \to \Pi_{g_{2/1}^{0,1}}, \Pi_{g_{2/1}^{1,2}} \to \Pi_{g_{2/1}^{0,1}}$ that appeared in the discussion following the proof of Claim 4.13.A, together with the natural inclusion of [CbTpl], Proposition 2.9, (ii), it holds that

$$(\widetilde{\beta}_{[0,1]|_{[1]}}|_{2/1}, \widetilde{\beta}_{[1,2]|_{[1]}}|_{2/1}) \in \text{Aut}^{B_{\Pi(2), 1}}(G_{2/1}^{0,1}), \text{Aut}^{B_{\Pi(2), 1}}(G_{2/1}^{1,2}) (\subseteq \text{Out}(\Pi_{2/1})).$$

Moreover, it follows immediately from condition (3) of Lemma 4.12, (v), applied in the case of $\beta_{[0,1]}$, together with the definition of $\beta_{[1,2]}$, that the automorphisms of the configuration space subgroup

$$\left(\Pi_2 \cong \Pi_2|_{[0,1]} \cong \Pi_{\tilde{v}(1)}|_{2} \subseteq \Pi_2|_{[1,2]} \subseteq \Pi_2\right)$$

associated to the vertex $\tilde{v}(1)$ determined by $\beta_{[0,1]}, \beta_{[1,2]}$ \textit{coincide} with $\alpha_v$. Now let us recall from the above discussion that the composite

$$\Pi_2|_{[1]} \hookrightarrow \Pi_2 \to \text{Out}(\Pi_{2/1}) \to \text{Out}(\Pi_{g_{2/1}^{0,1}})$$

\textit{factors} through

$$\text{Aut}(G_{2/1}^{0,1}) \subseteq \text{Out}(\Pi_{g_{2/1}^{0,1}}).$$

Thus, it follows immediately from the displayed exact sequence of Theorem 4.2, (iii) [cf. also Remark 4.9.1], that — after possibly replacing $\widetilde{\beta}_{[1,2]|_{[1]}}$ by a suitable $\Pi_2|_{[1]}$-conjugate — if we write

$$\delta \overset{\text{def}}{=} (\widetilde{\beta}_{[0,1]|_{[1]}}|_{2/1} \circ \widetilde{\beta}_{[1,2]|_{[1]}}|_{2/1})^{-1} \in \text{Aut}^{B_{\Pi(2), 1}}(G_{2/1}^{0,1}) (\subseteq \text{Out}(\Pi_{2/1})),$$

then it holds that $\delta \in \text{Dehn}(G_{2/1}^{0,1})$. 
Next, let us observe that, for \( a \in \{0, 1\} \), since \( \widetilde{\beta}_{[a,a+1]} \) preserves the \( \Pi_{2/1} \)-conjugacy class of cuspidal inertia subgroups associated to the diagonal cusp [cf. condition (2) of Lemma 4.12, (v)], it follows from a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.2, (iii), that the automorphism \( \left( \widetilde{\beta}_{[a,a+1]} \right)_{\{2\}} \) of \( \Pi_{\{2\}} \) induced by \( \widetilde{\beta}_{[a,a+1]} \) on the quotient

\[
\Pi_{\{2\}} \xleftarrow{\sim} \Pi_{2/1} \xrightarrow{\Pi_{\{2\}} \mapsto \Pi_{\{2\}}} \Pi_{\{2\}}
\]

is compatible, relative to the natural outer inclusion \( \Pi_{[a,a+1]} \hookrightarrow \Pi_{\{2\}} \), with the automorphism \( \alpha_1 |_{\Pi_{[a,a+1]}} \) [cf. condition (4) of Lemma 4.12, (v)]. Since an element of \( \text{Aut}^{\text{Brch}(\mathcal{G})}(\mathcal{G}) \) is completely determined by its restriction to \( \text{Aut}(\mathcal{G}_{[a,a+1]}) \) [cf. [CbTpI], Definition 4.4; [CbTpI], Remark 4.8.1], we thus conclude that, relative to the natural outer isomorphisms \( \Pi_{\{2\}} \xrightarrow{\sim} \Pi_1 \xrightarrow{\sim} \Pi_\mathcal{G} \), it holds that

\[
\left( \widetilde{\beta}_{[a,a+1]} \right)_{\{2\}} = \alpha_1 .
\]

In particular, we thus conclude that the element of \( \text{Aut}^{\text{Brch}(\mathcal{G})}(\mathcal{G}) \) induced by \( \delta \in \text{Aut}^{\text{Brch}(\mathcal{G}_{\{2\}})}(\mathcal{G}_{\{2\}}) \) on the quotient \( \Pi_{\{2\}} \xrightarrow{\sim} \Pi_{2/1} \xrightarrow{\Pi_{\{2\}} \mapsto \Pi_{\{2\}}} \Pi_{\mathcal{G}} \) is trivial. On the other hand, let us observe that one verifies easily from [CbTpI], Theorem 4.8, (iii), (iv), that this composite \( \Pi_{\{2\}} \xrightarrow{\sim} \Pi_{2/1} \xrightarrow{\Pi_{\{2\}} \mapsto \Pi_{\{2\}}} \Pi_{\mathcal{G}} \) determines an isomorphism

\[
\text{Dehn}(\mathcal{G}_{\{2\}}) \xrightarrow{\sim} \text{Dehn}(\mathcal{G}) .
\]

Thus, we conclude that \( \delta \) is the identity automorphism of \( \Pi_{2/1} \). This completes the proof of Claim 4.13.B. In the following, we shall suppose that the automorphism \( \widetilde{\beta}_{[0,1]} \) of \( \Pi_{2|[0,1]} \) was chosen so as to satisfy the following condition:

\( \widetilde{\beta}_{[0,1]} \) preserves the subgroup \( \Pi_{2|[1]} \subseteq \Pi_{2|[0,1]} \), and its restriction to \( \Pi_{2|[1]} \) is equal to the lifting \( \"\widetilde{\beta}_{[0,1]|\{1\}}" \) of Claim 4.13.B.

Next, let us fix an automorphism \( \widetilde{\alpha}_1 \in \text{Aut}(\Pi_1) \) that lifts \( \alpha_1 \in \text{Aut}_{\text{grph}}(\mathcal{G}) \subseteq \text{Out}(\Pi_\mathcal{G}) \xleftarrow{\sim} \text{Out}(\Pi_1) \) and preserves the subgroups \( \Pi_{[0]}, \Pi_{[1]}, \Pi_{[0,1]} \subseteq \Pi_1 \), and whose restriction to \( \Pi_{[0,1]} \subseteq \Pi_1 \) coincides with the automorphism of \( \Pi_{[0,1]} \) determined by the automorphism \( \widetilde{\beta}_{[0,1]} \) of \( \Pi_{[0,1]} \). [One verifies easily that such an \( \widetilde{\alpha}_1 \) always exists.] Write \( \beta_{2/1} \in \text{Out}(\Pi_{2/1}) \) for the automorphism of \( \Pi_{2/1} \subseteq \Pi_{2|[0,1]} \) determined by \( \widetilde{\beta}_{[0,1]} \). Now we claim that the following assertion holds:

Claim 4.13.C: Write \( \rho : \Pi_1 \rightarrow \text{Out}(\Pi_{2/1}) \) for the homomorphism determined by the exact sequence 1 →
Indeed, let us first observe that it follows from Claim 4.13.B, together with the definition of $\beta_{[1,2]}$, that there exists an element $\epsilon \in \Pi_{[1]}$ such that

$$\rho(\tilde{\gamma}_\infty) \cdot \beta_{2/1} \cdot \rho(\tilde{\gamma}^{-1}_\infty) \cdot \beta_{2/1}^{-1} = \rho(\epsilon^{-1}) \quad (\ast_1).$$

Next, let us observe that if we write

$$\eta \overset{\text{def}}{=} \tilde{\alpha}_1(\tilde{\gamma}_\infty) \cdot \tilde{\gamma}_\infty^{-1} \in \Pi_1 \quad (\ast_2),$$

[cf. our choice of $\tilde{\alpha}_1$], then it follows immediately from our choices of $\alpha_1$ and $\tilde{\gamma}_\infty$ that $\eta \in \Pi_{[1]}$. Thus, to verify Claim 4.13.C, it suffices to verify that $\rho(\epsilon) = \rho(\eta)$. To this end, let $\zeta \in \Pi_{[0]}$. Then, by our choice of $\tilde{\gamma}_\infty$, it follows that $\tilde{\gamma}_\infty \cdot \zeta \cdot \tilde{\gamma}_\infty^{-1} \in \Pi_{[1]}$. In particular, since the automorphism $\beta_{2/1}$ arises from an automorphism $\tilde{\beta}_{[0,1]}$ of $\Pi_{[0,1]}$, which is an automorphism over the restriction of $\tilde{\alpha}_1$ to $\Pi_{[0,1]}$, it follows immediately that

$$\beta_{2/1} \cdot \rho(\zeta) = \rho(\tilde{\alpha}_1(\zeta)) \cdot \beta_{2/1} \quad (\ast_3).$$

$$\beta_{2/1} \cdot \rho(\tilde{\gamma}_\infty \cdot \zeta \cdot \tilde{\gamma}_\infty^{-1}) = \rho(\tilde{\alpha}_1(\tilde{\gamma}_\infty \cdot \zeta \cdot \tilde{\gamma}_\infty^{-1})) \cdot \beta_{2/1} \quad (\ast_4).$$

Thus, if we write

$$\Theta_\epsilon \overset{\text{def}}{=} \rho(\epsilon \cdot \tilde{\gamma}_\infty \cdot \tilde{\alpha}_1(\zeta) \cdot \tilde{\gamma}_\infty^{-1} \cdot \epsilon^{-1}) \cdot \beta_{2/1} \in \Out(\Pi_{2/1}),$$

$$\Theta_\eta \overset{\text{def}}{=} \rho(\eta \cdot \tilde{\gamma}_\infty \cdot \tilde{\alpha}_1(\zeta) \cdot \tilde{\gamma}_\infty^{-1} \cdot \eta^{-1}) \cdot \beta_{2/1} \in \Out(\Pi_{2/1}),$$

then

$$\Theta_\epsilon \overset{\text{def}}{=} \rho(\epsilon \cdot \tilde{\gamma}_\infty \cdot \tilde{\alpha}_1(\zeta)) \cdot \beta_{2/1} \cdot \rho(\tilde{\gamma}_\infty) \quad \overset{(\ast_1)}{=} \rho(\epsilon \cdot \tilde{\gamma}_\infty) \cdot \beta_{2/1} \cdot \rho(\tilde{\gamma}_\infty \cdot \zeta \cdot \tilde{\gamma}_\infty^{-1}) \quad \overset{(\ast_4)}{=} \rho(\tilde{\alpha}_1(\tilde{\gamma}_\infty \cdot \zeta \cdot \tilde{\gamma}_\infty^{-1})) \cdot \beta_{2/1} \quad \overset{(\ast_4)}{=} \Theta_\eta$$

--- which thus implies that $\rho(\eta^{-1} \cdot \epsilon)$ commutes with $\rho(\tilde{\gamma}_\infty \cdot \tilde{\alpha}_1(\zeta) \cdot \tilde{\gamma}_\infty^{-1})$. In particular, since $\tilde{\gamma}_\infty \cdot \tilde{\alpha}_1(\Pi_{[0]}) \cdot \tilde{\gamma}_\infty^{-1} \overset{\text{def}}{=} \tilde{\gamma}_\infty \cdot \Pi_{[0]} \cdot \tilde{\gamma}_\infty^{-1} = \Pi_{[1]}$, by allowing “$\zeta$” to vary among the elements of $\Pi_{[0]}$, it follows that $\rho(\eta^{-1} \cdot \epsilon)$ centralizes $\rho(\Pi_{[1]})$. On the other hand, it follows from [Asd], Theorem 1; [Asd], the Remark following the proof of Theorem 1, that $\rho$ is injective. Thus, since $\epsilon, \eta \in \Pi_{[1]}$, we conclude that $\eta^{-1} \cdot \epsilon \in Z(\Pi_{[1]}) = \{1\}$ [cf. [CmbGC], Remark 1.1.3]. This completes the proof of Claim 4.13.C.

Now let us recall that the automorphism $\beta_{2/1}$ of $\Pi_{2/1}$ of Claim 4.13.C arises from an automorphism $\tilde{\beta}_{[0,1]}$ of $\Pi_{2[0,1]}$. Thus, it follows immediately from Claims 4.13.A, 4.13.C that the automorphism $\beta_{2/1}$ of $\Pi_{2/1}$ is compatible with the automorphism $\tilde{\alpha}_1 \in \Aut(\Pi_{[1]})$ relative to the homomorphism $\Pi_1 \rightarrow \Out(\Pi_{2/1})$ determined by the exact sequence
1 \to \Pi_{2/1} \to \Pi_2 \overset{\rho_{2/1}^n}{\longrightarrow} \Pi_1 \to 1. In particular — by considering the natural isomorphism \(\Pi_2 \cong \Pi_{2/1} \rtimes \Pi_1\) [cf. the discussion entitled “Topological groups” in [CbTpI], §30] — we conclude that the automorphism \(\beta_{2/1} \in \text{Out}(\Pi_{2/1})\) extends to an automorphism \(\alpha_2\) of \(\Pi_2\). On the other hand, it follows immediately from the various definitions involved that \(\alpha_2 \in \text{Out}^{\text{FC}}(\Pi_2)\), and that \(\rho_n^{\text{brch}}(\alpha_2) = \alpha_n \in \text{Glu}(\Pi_2)\) [cf. condition (3) of Lemma 4.12, (v)]. This completes the proof of Lemma 4.13 in the case where \(G\) is cyclically primitive, hence also of Lemma 4.13. \(\square\)

**Theorem 4.14 (Glueability of combinatorial cuspidalizations).** Let \((g, r)\) be a pair of nonnegative integers such that \(2g - 2 + r > 0\); \(n\) a positive integer; \(\Sigma\) a set of prime numbers which is either equal to the set of all prime numbers or of cardinality one; \(k\) an algebraically closed field of characteristic \(\not\equiv\Sigma\); \((\text{Spec} \; k)^{\log}\) the log scheme obtained by equipping \(\text{Spec} \; k\) with the log structure determined by the fs chart \(N \to k\) that maps \(1 \mapsto 0\); \(X^{\log} = X_1^{\log}\) a stable log curve of type \((g, r)\) over \((\text{Spec} \; k)^{\log}\). Write \(G\) for the semi-graph of anabelioids of pro-\(\Sigma\) PSC-type determined by the stable log curve \(X^{\log}\). For each positive integer \(i\), write \(X_i^{\log}\) for the \(i\)-th log configuration space of the stable log curve \(X^{\log}\) [cf. the discussion entitled “Curves” in [CbTpI], §30]; \(\Pi_i\) for the maximal pro-\(\Sigma\) quotient of the kernel of the natural surjection \(\pi_1(X_i^{\log}) \to \pi_1((\text{Spec} \; k)^{\log})\). Then the following hold:

(i) There exists a natural commutative diagram of profinite groups

\[
\begin{array}{ccc}
\text{Out}^{\text{FC}}(\Pi_{n+1})^{\text{brch}} & \overset{\rho_{n+1}^{\text{brch}}}{\longrightarrow} & \text{Glu}(\Pi_{n+1}) \\
\downarrow & & \downarrow \\
\text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}} & \overset{\rho_n^{\text{brch}}}{\longrightarrow} & \text{Glu}(\Pi_n)
\end{array}
\]

[cf. Definitions 4.6, (i); 4.9; 4.11] — where the vertical arrows [cf. Lemma 4.10, (i)] are injective.

(ii) The closed subgroup \(\text{Dehn}(G) \subseteq \langle \text{Aut}(G) \subseteq \text{Out}(\Pi_1)\) [cf. [CbTpI], Definition 4.4] is contained in the image of the injection \(\text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}} \hookrightarrow \text{Out}^{\text{FC}}(\Pi_1)^{\text{brch}}\) [cf. the left-hand vertical arrows of the diagrams of (i), for varying \(n\)]. Thus, one may regard \(\text{Dehn}(G)\) as a closed subgroup of \(\text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}}, \) i.e., \(\text{Dehn}(G) \subseteq \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}}\).

(iii) The homomorphism \(\rho_n^{\text{brch}} : \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}} \to \text{Glu}(\Pi_n)\) of (i) and the inclusion \(\text{Dehn}(G) \hookrightarrow \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}}\) of (ii) fit into
an exact sequence of profinite groups

\[ 1 \longrightarrow \text{Dehn}(\mathcal{G}) \longrightarrow \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}} \overset{\rho_n^{\text{brch}}}{\longrightarrow} \text{Glu}(\Pi_n) \longrightarrow 1. \]

In particular, the commutative diagram of (i) is cartesian, and the horizontal arrows of this diagram are surjective.

**Proof.** Assertion (i) follows immediately from Lemma 4.10, (i), together with the injectivity portion of [NodNon], Theorem B. Assertion (ii) follows immediately from Proposition 3.24, (ii); Theorem 4.2, (i).

Finally, we verify assertion (iii). First, we claim that the following assertion holds:

**Claim 4.14.A:** \( \ker(\rho_n^{\text{brch}}) = \text{Dehn}(\mathcal{G}) \) [cf. assertion (ii)].

Indeed, it follows immediately from Theorem 4.2, (iii) [cf. also Remark 4.9.1], together with assertion (i), that we have a natural commutative diagram

\[
\begin{array}{cccc}
1 & \longrightarrow & \ker(\rho_n^{\text{brch}}) & \longrightarrow & \text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}} \\
& & \downarrow & & \downarrow \\
1 & \longrightarrow & \text{Dehn}(\mathcal{G}) & \longrightarrow & \text{Out}^{\text{FC}}(\Pi_1)^{\text{brch}} \overset{\rho_1^{\text{brch}}}{\longrightarrow} \text{Glu}(\Pi_1) & \longrightarrow 1
\end{array}
\]

— where the horizontal sequences are exact, and the vertical arrows are injective. Thus, Claim 4.14.A follows immediately. In particular, to complete the verification of assertion (iii), it suffices to verify the surjectivity of \( \rho_n^{\text{brch}} \). The remainder of the proof of assertion (iii) is devoted to verifying this surjectivity.

Next, we claim that the following assertion holds:

**Claim 4.14.B:** If \( n = 2 \), then \( \rho_n^{\text{brch}} \) is surjective.

We verify Claim 4.14.B by induction on \( \text{Node}(\mathcal{G})^2 \). If \( \text{Node}(\mathcal{G})^2 = 0 \), then Claim 4.14.B is immediate. If \( \text{Node}(\mathcal{G})^2 = 1 \), then Claim 4.14.B follows from Lemma 4.13. Now suppose that \( \text{Node}(\mathcal{G})^2 > 1 \), and that the induction hypothesis is in force. Let \( (\alpha_v)_{v \in \text{Vert}(\mathcal{G})} \in \text{Glu}(\Pi_2) \). Write \( ((\alpha_v)_1)_{v \in \text{Vert}(\mathcal{G})} \in \text{Glu}(\Pi_1) \) for the element of \( \text{Glu}(\Pi_1) \) determined by \( (\alpha_v)_{v \in \text{Vert}(\mathcal{G})} \) [i.e., the image of \( (\alpha_v)_{v \in \text{Vert}(\mathcal{G})} \) via the right-hand vertical arrow of the diagram of assertion (i) in the case where \( n = 1 \)]. Let \( e \in \text{Node}(\mathcal{G}) \). Write \( \mathbb{H} \) for the unique sub-semi-graph of \( \text{PSC-type} \) [cf. [CbTpI], Definition 2.2, (i)] of the underlying semi-graph of \( \mathcal{G} \) whose set of vertices is \( \mathcal{V}(e) \). Then one verifies easily that \( S \overset{\text{def}}{=} \text{Node}(\mathcal{G}|_{\mathbb{H}}) \setminus \{e\} \) [cf. [CbTpI], Definition 2.2, (ii)] is not of separating type [cf. [CbTpI], Definition 2.5, (i)] as a subset of \( \text{Node}(\mathcal{G}|_{\mathbb{H}}) \). Thus, since \( (\mathcal{G}|_{\mathbb{H}})_S \) [cf. [CbTpI], Definition 2.5, (ii)] has precisely one node, and \( (\alpha_v)_{v \in \mathcal{V}(e)} \) may be regarded as an element of \( \text{Glu}((\mathcal{G}|_{\mathbb{H}})_S) \) — where we use the notation \( (\Pi_{\mathbb{H},S})_2 \) to denote a configuration space subgroup of \( \Pi_2 \) associated to \( (\mathbb{H},S) \) [cf. Definition 4.3], to which the notation “\( \text{Glu}(\mathcal{G}) \)” is applied.
in the evident sense — it follows from Lemma 4.13 that there exists an automorphism \( \beta_{\mathbb{E},S} \) of \((\mathbb{E}_{\mathbb{E},S})_2 \subseteq \mathbb{E}_2 \) that lifts \((\alpha_v)_{v \in \mathcal{V}(e)} \in \text{Glu}(\mathbb{E}_{\mathbb{E},S})_2 \).

Next, let us observe that it follows immediately from the various definitions involved that

\[
\gamma \overset{\text{def}}{=} (\beta_{\mathbb{E},S}, (\alpha_v)_{v \in \mathcal{V}(e)}) \in \text{Out}(\mathbb{E}_{\mathbb{E},S})_2 \times \prod_{v \in \mathcal{V}(e)} \text{Out}(\mathbb{E}_v)_2
\]

may be regarded as an element of the “\text{Glu}(\mathbb{E}_2)” that occurs in the case where we take the stable log curve “\( X^{\log} \)” to be a stable log curve over \((\text{Spec } k)^{\log} \) obtained by deforming the node corresponding to \( e \). Thus, since the number of nodes of such a stable log curve is \( = \text{Node}(\mathcal{G})^2 - 1 < \text{Node}(\mathcal{G})^2 \), by applying the induction hypothesis, we conclude that the above \( \gamma \) arises from an automorphism \( \alpha_\gamma \in \text{Out}^{\text{FC}}(\mathbb{E}_2)^{\text{brch}} \). On the other hand, it follows immediately from the various definitions involved that the image of \( \alpha_\gamma \) via \( \rho_n^{\text{brch}} \) coincides with \((\alpha_v)_{v \in \mathcal{V}(e)}\). This completes the proof of Claim 4.14.B.

Finally, we verify the surjectivity of \( \rho_n^{\text{brch}} \) [for arbitrary \( n \)] by induction on \( n \). If \( n \leq 2 \), then the surjectivity of \( \rho_n^{\text{brch}} \) follows from Theorem 4.2, (iii) [cf. also Remark 4.9.1], Claim 4.14.B. Now suppose that \( n \geq 3 \), and that the induction hypothesis is in force. Let \((\alpha_v)_{v \in \mathcal{V}(e)} \in \text{Glu}(\mathbb{E}_n) \). First, let us observe that it follows from the induction hypothesis that there exists an element \( \alpha_{n-1} \in \text{Out}^{\text{FC}}(\mathbb{E}_{n-1})^{\text{brch}} \) such that \( \rho_{n-1}^{\text{brch}}(\alpha_{n-1}) \) coincides with the element of \( \text{Glu}(\mathbb{E}_{n-1}) \) determined by \((\alpha_v)_{v \in \mathcal{V}(e)} \in \text{Glu}(\mathbb{E}_n) \) [cf. assertion (i)]. Let \( \tilde{\alpha}_{n-1} \) be an automorphism of \( \mathbb{E}_{n-1} \) that lifts \( \alpha_{n-1} \). Write \( \alpha_{n-1}/n-2 \) for the automorphism of \( \mathbb{E}_{n-1}/n-2 \) determined by \( \tilde{\alpha}_{n-1} \) and \( \tilde{\alpha}_{n-2} \) for the automorphism of \( \mathbb{E}_{n-2} \) determined by \( \tilde{\alpha}_{n-1} \).

Next, let us observe that one verifies easily from the various definitions involved that \( \mathbb{E}_{n-2}/n \subseteq \mathbb{E}_n \) may be regarded as the “\( \mathbb{E}_2 \)” associated to some stable log curve “\( X^{\log} \)” over \((\text{Spec } k)^{\log} \). Moreover, this stable log curve may be taken to be a geometric fiber of the sort discussed in Definition 3.1, (iii), in the case of the projection \( p_{n-1}/n \), relative to a point “\( x \in X_n(k) \)” that maps to the interior of the same irreducible component of \( X^{\log} \), relative to the \( n \) projections to \( X^{\log} \). In particular, by fixing such a stable log curve, together with a suitable choice of lifting \( \tilde{\alpha}_{n-1} \) [cf. Theorem 4.7], it makes sense to speak of \( \text{Glu}(\mathbb{E}_{n-2}) \). Moreover, it follows immediately from our choice of “\( x \)” that every configuration space subgroup that appears in the definition [cf. Definition 4.9, (ii)] of \( \text{Glu}(\mathbb{E}_{n-2}) \) either

- occurs as a configuration space subgroup of the intersection with \( \mathbb{E}_{n-2}/n \) of some configuration space subgroup that appears in the definition [cf. Definition 4.9, (ii)] of \( \text{Glu}(\mathbb{E}_n) \) or
- projects isomorphically, via the projection \( \mathbb{E}_n \rightarrow \mathbb{E}_2 \) to the factors labeled \( n \) and \( n-1 \), to a configuration space subgroup
of $\Pi_2$, i.e., a configuration space subgroup that appears in the definition [cf. Definition 4.9, (ii)] of $\text{Glu}(\Pi_2)$.

In particular, every tripod that appears in the definition [cf. Definition 4.9, (ii)] of $\text{Glu}(\Pi_{n/n-2})$ occurs as a tripod of a configuration space subgroup that appears either in the definition [cf. Definition 4.9, (iii)] of $\text{Glu}(\Pi_n)$ or in the definition [cf. Definition 4.9, (ii)] of $\text{Glu}(\Pi_2)$. Moreover, it follows from Theorem 4.7; Lemma 3.2, (iv); Lemma 4.8, (i), that the various $\alpha_v$'s preserve these configuration space subgroups and tripods — as well as each conjugacy class of cuspidal inertia subgroups of each of these tripods! — that appear in the definition [cf. Definition 4.9, (ii)] of $\text{Glu}(\Pi_{n/n-2})$. Thus, we conclude from Theorem 3.18, (ii), together with Definition 4.9, (iii), in the case of $\text{Glu}(\Pi_n)$, and Definition 4.9, (ii), in the case of $\text{Glu}(\Pi_2)$, that $(\alpha_v)_{v \in \text{Vert}(\mathcal{G})}$ determines an element $\in \text{Glu}(\Pi_{n/n-2})$, hence, by Claim 4.14.B, an element

$$\alpha_{n/n-2} \in \text{Out}^{\text{FC}}(\Pi_{n/n-2})$$

that lifts the element $\alpha_{n-1/n-2} \in \text{Out}(\Pi_{n-1/n-2})$.

Now we claim that the following assertion holds:

**Claim 4.14.C:** This automorphism $\alpha_{n/n-2}$ of $\Pi_{n/n-2}$ is compatible with the automorphism $\overline{\alpha}_{n-2}$ of $\Pi_{n-2}$ relative to the homomorphism $\Pi_{n-2} \to \text{Out}(\Pi_{n-2})$ induced by the natural exact sequence of profinite groups

$$1 \to \Pi_{n-2/n} \to \Pi_n \xrightarrow{\rho_n^\Pi} \Pi_{n-2} \to 1.$$

Indeed, this follows immediately from the corresponding fact for $\alpha_{n-1/n-2}$ [which follows from the existence of $\overline{\alpha}_{n-1}$], together with the injectivity of the natural homomorphism $\text{Out}^{\text{FC}}(\Pi_{n-2}) \to \text{Out}^{\text{FC}}(\Pi_{n-1/n-2})$ [cf. [NodNon], Theorem B]. This completes the proof of Claim 4.14.C.

Thus, by applying Claim 4.14.C and the natural isomorphism $\Pi_n \xrightarrow{\text{out}} \Pi_{n-2}$ [cf. the discussion entitled “Topological groups” in [CbTpI], §0], we obtain an automorphism $\alpha_n$ of $\Pi_n$ that lifts the automorphism $\alpha_{n-1}$ of $\Pi_{n-1}$. Thus, it follows immediately from Lemma 4.10, (i), that $\rho_n^\text{brch}(\alpha_n) = (\alpha_v)_{v \in \text{Vert}(\mathcal{G})}$. This completes the proof of the surjectivity of $\rho_n^\text{brch}$, hence also of assertion (iii).

**Remark 4.14.1.** In the notation of Theorem 4.14, observe that the data of collections of smooth log curves that [by gluing at prescribed cusps] give rise to a stable log curve whose associated semi-graph of anabeloids [of pro-$\Sigma$ PSC-type] is isomorphic to $\mathcal{G}$ form a smooth, connected moduli stack. In particular, by considering a suitable path in the étale fundamental groupoid of this moduli stack, one verifies immediately that one may reduce the verification of an “isomorphism version” — i.e., concerning PFC-admissible [cf. [CbTpI], Definition
1.4, (iii)] outer isomorphisms between the pro-$\Sigma$ fundamental groups of the configuration spaces associated to two a priori distinct stable log curves $\mathcal{X}_{\log}$ and $\mathcal{Y}_{\log}$ of Theorem 4.14 to the “automorphism version” given in Theorem 4.14 [cf. [CmbCsp], Remark 4.1.4]. A similar statement may be made concerning Theorem 4.7. We leave the routine details to the interested reader. In the present paper, we restricted our attention to the “automorphism versions” of these results in order to simplify the [already somewhat complicated!] notation.

**Remark 4.14.2.** One may regard [CmbCsp], Corollary 3.3, as a special case of the surjectivity of $\rho_{n}^{\text{brch}}$ discussed in Theorem 4.14, i.e., the case in which $\mathcal{X}_{\log}$ is obtained by gluing a tripod to a smooth log curve along a cusp of the smooth log curve.

**Corollary 4.15 (Surjectivity result).** In the notation of Theorem 3.16, suppose that $n \geq 3$. If $r = 0$, then we suppose further that $n \geq 4$. Then the tripod homomorphism
\[ \Sigma_{\Pi_{\text{tpd}}} : \text{Out}^{F}(\Pi_{n}) \rightarrow \text{Out}^{C}(\Pi_{\text{tpd}})^{\Delta+} \]
[cf. Definition 3.19; Theorem 3.16, (v)] is surjective.

**Proof.** Let $\alpha \in \text{Out}^{C}(\Pi_{\text{tpd}})^{\Delta+}$. First, let us observe that — by considering a suitable stable log curve of type $(g, r)$ over $(\text{Spec } k)^{\log}$ and applying a suitable specialization isomorphism [cf. Proposition 3.24, (i); the discussion preceding [CmbCsp], Definition 2.1, as well as [CbTpI], Remark 5.6.1] — to verify Corollary 4.15, we may assume without loss of generality that $G$ is totally degenerate [cf. [CbTpI], Definition 2.3, (iv)], i.e., that every vertex of $G$ is a tripod of $\mathcal{X}_{\log}$ [cf. Definition 3.1, (v)]. Then since $\alpha \in \text{Out}^{C}(\Pi_{\text{tpd}})^{\Delta+}$, it follows immediately from [CmbCsp], Corollary 4.2, (ii), that there exists an element $\alpha_{n} \in \text{Out}^{FC}(\Pi_{n})$ — where we write $\Pi_{\text{tpd}}$ for the “$\Pi_{n}$” that occurs in the case where we take $\mathcal{X}_{\log}$ to be a tripod — such that $\alpha$ arises as the image of $\alpha_{n}$ via the natural injection $\text{Out}^{FC}(\Pi_{\text{tpd}}) \hookrightarrow \text{Out}^{FC}(\Pi_{\text{tpd}})$ of [NodNon], Theorem B. Thus, it follows immediately from Theorem 4.14, (iii), that there exists an element $\beta \in \text{Out}^{FC}(\Pi_{n})^{\text{brch}}$ that lifts — relative to $\rho_{n}^{\text{brch}}$ — the element of Glu($\Pi_{n}$) determined by $\alpha_{n} \in \text{Out}^{FC}(\Pi_{n})$. [Here, recall that we have assumed that $G$ is totally degenerate.] Now it follows from Theorem 3.18, (ii), that $\Sigma_{\Pi_{\text{tpd}}} (\beta) = \alpha$, i.e., that $\alpha$ is contained in the image of $\Sigma_{\Pi_{\text{tpd}}}$. This completes the proof of Corollary 4.15. \[\square\]

**Corollary 4.16 (Absolute anabelian cuspidalization for stable log curves over finite fields).** Let $p, l_{X}, l_{Y}$ be prime numbers such that $p \not\in \{l_{X}, l_{Y}\}$; $(g_{X}, r_{X})$, $(g_{Y}, r_{Y})$ pairs of nonnegative integers such
that $2g_X - 2 + r_X, 2g_Y - 2 + r_Y > 0$; $k_X, k_Y$ finite fields of characteristic $p$; $\mathbb{F}_X, \mathbb{F}_Y$ algebraic closures of $k_X, k_Y$; $(\text{Spec } k_X)^{\log}, (\text{Spec } k_Y)^{\log}$ the log schemes obtained by equipping $\text{Spec } k_X, \text{Spec } k_Y$ with the log structures determined by the fs charts $\mathbb{N} \to k_X, \mathbb{N} \to k_Y$ that map $1 \mapsto 0$; $X^{\log}, Y^{\log}$ stable log curves [cf. the discussion entitled “Curves” in [CbTpI], §0] of type $(g_X, r_X), (g_Y, r_Y)$ over $(\text{Spec } k_X)^{\log}, (\text{Spec } k_Y)^{\log}$;

$$G^{\log}_{k_X} \overset{\text{def}}{=} \pi_1((\text{Spec } k_X)^{\log}) \to G_{k_X} \overset{\text{def}}{=} \text{Gal}(\mathbb{F}_X/k_X);$$

$$G^{\log}_{k_Y} \overset{\text{def}}{=} \pi_1((\text{Spec } k_Y)^{\log}) \to G_{k_Y} \overset{\text{def}}{=} \text{Gal}(\mathbb{F}_Y/k_Y);$$

$s_X : G_{k_X} \to G^{\log}_{k_x}, s_Y : G_{k_Y} \to G^{\log}_{k_y}$ sections of the above natural surjections $G^{\log}_{k_X} \to G_{k_X}, G^{\log}_{k_Y} \to G_{k_Y}$. For each positive integer $n$, write $X^{\log}_n, Y^{\log}_n$ for the $n$-th log configuration spaces [cf. the discussion entitled “Curves” in [CbTpI], §0] of $X^{\log}, Y^{\log}; \chi^{\Pi_n}, \chi^{\Pi_n}$ for the maximal pro-$k_X, pro-$k_Y$ quotients of the kernels of the natural surjections $\pi_1(X^{\log}_n) \to G^{\log}_{k_X}, \pi_1(Y^{\log}_n) \to G^{\log}_{k_Y}$. Then the sections $s_X, s_Y$ determine outer actions of $G_{k_X}, G_{k_Y}$ on $\chi^{\Pi_n}, \chi^{\Pi_n}$. Thus, we obtain profinite groups

$$\chi^{\Pi_n}_{\text{out}} \rtimes_{s_X} G_{k_X}, \chi^{\Pi_n}_{\text{out}} \rtimes_{s_Y} G_{k_Y}$$

[cf. [MzTa], Proposition 2.2, (ii); the discussion entitled “Topological groups” in [CbTpI], §0]. Let

$$\alpha_1 : \chi^{\Pi_1}_{\text{out}} \rtimes_{s_X} G_{k_X} \overset{\sim}{\longrightarrow} \chi^{\Pi_1}_{\text{out}} \rtimes_{s_Y} G_{k_Y}$$

be an isomorphism of profinite groups. Then $l_X = l_Y$; there exists a unique collection of isomorphisms of profinite groups

$$\left\{ \alpha_n : \chi^{\Pi_n}_{\text{out}} \rtimes_{s_X} G_{k_X} \overset{\sim}{\longrightarrow} \chi^{\Pi_n}_{\text{out}} \rtimes_{s_Y} G_{k_Y} \right\}_{n \geq 1}$$

— well-defined up to composition with an inner automorphism of $\chi^{\Pi_n}_{\text{out}} \rtimes_{s_Y} G_{k_Y}$ by an element of the intersection $\chi^{\Pi_n}_{\text{out}} \subseteq \chi^{\Pi_n}_{\text{out}}$ of the fiber subgroups of $\chi^{\Pi_n}_{\text{out}}$ of co-length 1 [cf. [CmbCsp], Definition 1.1, (iii)] — such that each diagram

$$\chi^{\Pi_{n+1}}_{\text{out}} \rtimes_{s_X} G_{k_X} \overset{\alpha_{n+1}}{\longrightarrow} \chi^{\Pi_{n+1}}_{\text{out}} \rtimes_{s_Y} G_{k_Y}$$

$$\chi^{\Pi_n}_{\text{out}} \rtimes_{s_X} G_{k_X} \overset{\alpha_n}{\longrightarrow} \chi^{\Pi_{n}}_{\text{out}} \rtimes_{s_Y} G_{k_Y}$$

— where the vertical arrows are the surjections induced by the projections $X^{\log}_{n+1} \to X^{\log}_n, Y^{\log}_{n+1} \to Y^{\log}_n$ obtained by forgetting the factors labeled $j$, for some $j \in \{1, \cdots, n+1\} —$ commutes, up to composition with a $\chi^{\Xi_n}_{\text{out}}$-inner automorphism.
First, let us observe that it follows from [AbsTpI], Corollary 2.8, (i), that \( \alpha_1 \) maps \( X \Pi_1 \subset X \Pi_1 \times_{s_X} G_{k_X} \) bijectively onto \( Y \Pi_1 \subset Y \Pi_1 \times_{s_Y} G_{k_Y} \). In particular, \( l_X = l_Y \); \( \alpha_1 \) induces isomorphisms of profinite groups

\[
\alpha_1^\Pi : X \Pi_1 \xrightarrow{\sim} Y \Pi_1 , \quad \alpha_0 : G_{k_X} \xrightarrow{\sim} G_{k_Y} .
\]

Write \( l \overset{\text{def}}{=} l_X = l_Y \). For \( \square \in \{ X, Y \} \), write \( G^{(l)}_{k_\square} \subset G_{k_\square} \) for the maximal pro-\( l \) closed subgroup of \( G_{k_\square} \); \( G^{(\neq l)}_{k_\square} \) for the maximal pro-prime-to-\( l \) closed subgroup of \( G_{k_\square} \). Then since \( G_{k_\square} \) is isomorphic to \( \hat{\mathbb{Z}} \) as an abstract profinite group, we have a natural decomposition

\[
G^{(l)}_{k_\square} \times G^{(\neq l)}_{k_\square} \xrightarrow{\sim} G_{k_\square} .
\]

Thus, the isomorphism \( \alpha_0 \) naturally decomposes into a pair of isomorphisms

\[
\alpha_0^{(l)} : G^{(l)}_{k_X} \xrightarrow{\sim} G^{(l)}_{k_Y} , \quad \alpha_0^{(\neq l)} : G^{(\neq l)}_{k_X} \xrightarrow{\sim} G^{(\neq l)}_{k_Y} .
\]

Next, let us observe that since \( \square \Pi_1 \) is topologically finitely generated and pro-\( l \), one verifies easily that [by replacing \( G_{k_\square} \) by a suitable open subgroup] we may assume without loss of generality that the outer action of \( G_{k_\square} \) on \( \square \Pi_1 \) — hence [cf. the injectivity portion of [NodNon], Theorem B] also on \( \square \Pi_n \) for each positive integer \( n \) — factors through the quotient \( G_{k_\square} \xrightarrow{\sim} G^{(l)}_{k_\square} \times G^{(\neq l)}_{k_\square} \rightarrow G^{(l)}_{k_\square} . \) Thus, it follows immediately from the slimness of \( \square \Pi_n \) [cf. [MzTa], Proposition 2.2, (ii)] that the composite

\[
Z_{\square \Pi_n^\text{out} \times_{s_\square} G_{k_\square}} (\square \Pi_n) \hookrightarrow \square \Pi_n^\text{out} \times_{s_\square} G_{k_\square} \twoheadrightarrow G_{k_\square}
\]

determines an isomorphism

\[
Z_{\square \Pi_n^\text{out} \times_{s_\square} G_{k_\square}} (\square \Pi_n) \xrightarrow{\sim} G^{(\neq l)}_{k_\square} .
\]

In particular, if we identify \( Z_{\square \Pi_n^\text{out} \times_{s_\square} G_{k_\square}} (\square \Pi_n) \) with \( G^{(\neq l)}_{k_\square} \) by means of this isomorphism, then we obtain a natural isomorphism

\[
\left( \square \Pi_n^\text{out} \times_{s_\square} G^{(l)}_{k_\square} \times G^{(\neq l)}_{k_\square} \right) \xrightarrow{\sim} \square \Pi_n^\text{out} \times_{s_\square} G_{k_\square} .
\]

Next, let us observe that the following assertion holds:

Claim 4.16.A: There exists a power \( q \) of \( p \) such that

\[
\log_p(q) \text{ is divisible by } \log_p(k^{l_X}_X), \log_p(k^{l_Y}_Y), \text{ and, moreover,}
\]

\[
\alpha_0^{(l)} ((\text{Fr}_q)_{k_X}) = (\text{Fr}_q)_{k_Y},
\]

— where we write \( (\text{Fr}_q)_{k_X} \in G_{k_X}, (\text{Fr}_q)_{k_Y} \in G_{k_Y} \) for the \( q \)-power Frobenius elements of \( G_{k_X}, G_{k_Y} ; (\text{Fr}_q)^{(l)}_{k_X} \in \)
Indeed, let us first observe that it follows immediately from [CmbGC], Corollary 2.7, (ii) [cf. also the proof of [CmbGC], Proposition 2.4, (v)], that $\alpha^l_1$ is graphic. In particular, we have an equality $r_X = r_Y$, which thus implies [cf. the well-known fact that, for $\square \in \{X,Y\}$, the abelianization of $\square \Pi$ is a free $\mathbb{Z}_l$-module of rank $2q_{\square} + \max\{0, r_{\square} - 1\}$ — cf., e.g., [CmbGC], Remark 1.1.3] that $(g_X, r_X) = (g_Y, r_Y)$. Next, let us observe that, for $\square \in \{X,Y\}$, it follows immediately from the definition of the filtration on the abelianization of $\square \Pi$ given in the second display of [CmbGC], Definition 1.1, (ii) [cf. also the duality property reviewed in [CmbGC], Proposition 1.3], that the character $\det_{\square}: G_{k_{\square}}^l \to \mathbb{Z}_l^*$ determined by the square of the determinant of the abelianization [which is a free $\mathbb{Z}_l$-module of finite rank] of $\square \Pi$ coincides with the $2r_{\square}$-th tensor power of the $l$-adic cyclotomic character of $G_{k_{\square}}$, where we write $t_{\square} \overset{def}{=} g_{\square} + \max\{0, r_{\square} - 1\}$. Thus, for a suitable power $q$ of $p$ such that $\log_p(q)$ is divisible by $\log_p(k^2_{X,l})$, $\log_p(k^2_{Y,l})$, it follows immediately from the [easily verified] injectivity of $\det_{\square}$ that $\det_{\square}((Fr_q)_l^l) = q^{2r_{\square}}$. In particular, since $\det_X$ is compatible, relative to $\alpha_0$, with $\det_Y$, and $t_X = t_Y$, we conclude that $\alpha^l_0((Fr_q)_l^l) = (Fr_q)_l^l$. This completes the proof of Claim 4.16.A.

Write $H_{k_{\square}} \subseteq G_{k_{\square}}, H_{k_Y} \subseteq G_{k_Y}$ for the open subgroups of $G_{k_{\square}}, G_{k_Y}$ topologically generated by $(Fr_q)_k \subseteq G_{k_{\square}}, (Fr_q)_k \subseteq G_{k_Y}$ [cf. Claim 4.16.A]; $U_{k_{\square}} \subseteq G_{k_{\square}}$ for the open subgroup of $G_{k_{\square}}$ topologically generated by $\alpha_0((Fr_q)_k) \subseteq G_{k_Y}$; $H_{k_{\square}}^l \subseteq G_{k_{\square}}^l$ for the image of $H_{k_{\square}} \subseteq G_{k_{\square}}$ in $G_{k_{\square}}^l$; $H_{k_Y}^l \subseteq G_{k_Y}^l$ for the images of $H_{k_Y}, U_{k_{\square}} \subseteq G_{k_Y}$ in $G_{k_Y}^l$. Then it follows from Claim 4.16.A that we have an equality $H_{k_{\square}}^l = U_{k_{\square}}^l$, and, moreover, that the isomorphism $H_{k_{\square}}^l \cong U_{k_{\square}}^l$ induced by $\alpha_0$ induces an isomorphism $H_{k_{\square}}^l \cong U_{k_{\square}}^l = H_{k_{\square}}^l$. Thus, again by Claim 4.16.A, one verifies easily that if we write $\alpha^H_0: H_{k_{\square}} \cong H_{k_{\square}}^l$ for the [uniquely determined] isomorphism of profinite groups which

(a) preserves the respective $q$-power Frobenius elements of $H_{k_{\square}}, H_{k_Y}$,

then

(b) the isomorphism $H_{k_{\square}}^l \cong H_{k_Y}^l$ induced by $\alpha^H_0$ coincides with the above isomorphism $H_{k_{\square}}^l \cong U_{k_{\square}}^l = H_{k_{\square}}^l$ induced by $\alpha_0$. 


Moreover, it follows immediately from condition (b), together with the existence of the natural isomorphisms
\[
\left( \times^{\Pi_n}_{sx} G_{kX}^{(l)} \times G_{kX}^{(\neq l)} \right) \sim \times^{\Pi_n}_{sx} G_{kX}^1, \\
\left( \times^{\Pi_n}_{sy} G_{kY}^{(l)} \times G_{kY}^{(\neq l)} \right) \sim \times^{\Pi_n}_{sy} G_{kY}^1,
\]
that there exists an isomorphism
\[
\alpha_1^H : \times^{\Pi_1}_{sx} H_{kX} \sim \times^{\Pi_1}_{sy} H_{kY}
\]
such that

(c) the isomorphism “\(\alpha_0\)” of \(H_{kX}\) with \(H_{kY}\) that occurs in the case where we take the “\(\alpha_1\)” to be \(\alpha_1^H\) coincides with \(\alpha_0^H\), and, moreover,

(d) the isomorphism “\(\alpha_1^\Pi\)” of \(\times^{\Pi}_1\) with \(\times^{\Pi}_1\) that occurs in the case where we take the “\(\alpha_1\)” to be \(\alpha_1^H\) coincides with [the original] \(\alpha_1^\Pi\).

In particular, we conclude, again by the existence of the natural isomorphisms
\[
\left( \times^{\Pi_n}_{sx} G_{kX}^{(l)} \times G_{kX}^{(\neq l)} \right) \sim \times^{\Pi_n}_{sx} G_{kX}^1, \\
\left( \times^{\Pi_n}_{sy} G_{kY}^{(l)} \times G_{kY}^{(\neq l)} \right) \sim \times^{\Pi_n}_{sy} G_{kY}^1,
\]

that, together with the injectivity portion of [NodNon], Theorem B, that, to verify Corollary 4.16 — by replacing \(G_{kX}, G_{kY}, \alpha_1, H_{kX}, H_{kY}, \alpha_1^H\) — we may assume without loss of generality that \(\alpha_0^\Pi\) preserves the respective Frobenius elements of \(G_{kX}, G_{kY}\) [cf. condition (a)]. By choosing the power \(q\) of \(p\) in Claim 4.16.A in an appropriate fashion, we may also assume without loss of generality that the following condition holds:

(e) for \(\square \in \{X, Y\}\), \(G_{k\square}\) acts trivially on the underlying semi-graph of the semi-graph of anabelioids of pro-\(l\) PSC-type determined by \(\square^\log\).

Next, let us recall that the isomorphism \(\alpha_1^\Pi\) is graphic [cf. the proof of Claim 4.16.A]. In particular, by applying the observation of Remark 4.14.1, we reduce immediately to the case where \(X^\log = Y^\log\), and the automorphism \(\beta_1\) of \(\Pi_1\) determined by \(\alpha_1\) determines an element of \(\text{Aut}^{(\text{Brch}(|\mathcal{G}|)\text{Out}(\Pi_G) \sim \text{Out}(\Pi_1))}\) [where we omit the various superscript “\(X\)’s” that occur in the notation of the statement of Corollary 4.16]. Then the uniqueness portion of Corollary 4.16 follows immediately from the injectivity portion of [NodNon], Theorem B, together with the slimness of \(\Pi_1\).
Thus, it remains to verify the existence of a collection of $\alpha_n$’s as in the statement of Corollary 4.16. To this end, for each positive integer $i$ and each $v \in \text{Vert}(G)$, let us fix a configuration space subgroup $(\Pi_v)_i \subseteq \Pi_n$ of $\Pi_n$ associated to $v \in \text{Vert}(G)$. Write $(\beta_v)_{v \in \text{Vert}(G)} \overset{\text{def}}{=} \rho_G^{\text{brch}}(\beta_1) \in \text{Glu}^{\text{brch}}(G)$ [cf. Definition 4.1, (iii); Theorem 4.2, (iii)]. Then it follows immediately from the various definitions involved that, for each $v \in \text{Vert}(G)$, the automorphism $\beta_v$ of $(\Pi_v)_i$ is compatible with the natural outer action of $G_k$ [cf. condition (e)]. Thus, by applying [Wkb], Theorem C, we obtain an automorphism $\beta_{v,n}$ of $(\Pi_v)_n$ which is compatible with the natural outer action of $G_k$. Moreover, since $(\beta_v)_{v \in \text{Vert}(G)} \in \text{Glu}^{\text{brch}}(G)$, one verifies easily from the injectivity discussed in [Hsh], Remark 6, (iv) [i.e., applied to the automorphisms of the various tripods of $(\Pi_v)_n$ induced by $\beta_{v,n}$] that $(\beta_{v,n})_{v \in \text{Vert}(G)} \in \text{Glu}(\Pi_n)$ [cf. Definition 4.9]. In particular, since the diagram of Theorem 4.14, (i), is cartesian [cf. Theorem 4.14, (iii)], it follows that $\beta_1 \in \text{Aut}^{\text{brch}(G)}(G)$ and $(\beta_{v,n})_{v \in \text{Vert}(G)} \in \text{Glu}(\Pi_n)$ determine an element of $\text{Out}^{\text{FC}}(\Pi_n)^{\text{brch}}$, which — by the injectivity portion of [NodNon], Theorem B — is compatible with the natural outer action of $G_k$ on $\Pi_n$ determined by $s$. Finally, one verifies immediately that the resulting $\alpha_n$’s satisfy the properties stated in Corollary 4.16. This completes the proof of the existence of the $\alpha_n$’s, hence also of Corollary 4.16. □

**Remark 4.16.1.** Corollary 4.16 may be regarded as a generalization of [AbsCsp], Theorem 3.1; [Hsh], Theorem 0.1; [Wkb], Theorem C.

**Corollary 4.17** (Commensurator of the image of the absolute Galois group of a finite field in the totally degenerate case).

Let $n$ be a positive integer; $p$, $l$ two distinct prime numbers; $(g, r)$ a pair of nonnegative integers $\neq (0, 3)$ such that $2g - 2 + r > 0$; $k$ a finite field of characteristic $p$; $\overline{k}$ an algebraic closure of $k$; $(\text{Spec} \ k)^{\text{log}}$ the log scheme obtained by equipping $\text{Spec} \ k$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$; $X^{\text{log}}$ a stable log curve [cf. the discussion entitled “Curves” in [CbTpi], §0] of type $(g, r)$ over $(\text{Spec} \ k)^{\text{log}}$. Write $G$ for the semi-graph of anabelioids of pro-$l$ PSC-type associated to the stable log curve $X^{\text{log}}$; $\text{G}$ for the underlying semi-graph of $G$; $\Pi_0$ for the [pro-$l$] fundamental group of $G$;

$$G_k^{\text{log}} \overset{\text{def}}{=} \pi_1((\text{Spec} \ k)^{\text{log}}) \to G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$$

for the natural surjection. For each positive integer $i$, write $X_i^{\text{log}}$ for the $i$-th log configuration space [cf. the discussion entitled “Curves” in [CbTpi], §0] of $X^{\text{log}}$; $\Pi_i$ for the maximal pro-$l$ quotient of the kernel of the natural surjection $\pi_1(X_i^{\text{log}}) \to G_k^{\text{log}}$. Thus, we have a natural outer
isomorphism $\Pi_1 \sim \Pi_G$ and a natural outer action
$$
\rho_{X_{\log}^1} : G_k^{\log} \to \text{Out}^\text{FC}(\Pi_1)
$$
[cf. the notation of [CmbCsp], Definition 1.1, (ii)]. Let $H \subseteq G_k^{\log}$ be a closed subgroup of $G_k^{\log}$ whose image in $G_k$ is open. Write $I_H \subseteq H$ for the kernel of the composite $H \hookrightarrow G_k^{\log} \twoheadrightarrow G_k$. We shall say that $H$ is of $l$-Dehn type if the maximal pro-$l$ quotient of $I_H$ is nontrivial.

Suppose that the stable log curve $X_{\log}$ is totally degenerate [i.e., that the complement in $X$ of the nodes and cusps is a disjoint union of tripods]. Then the following hold:

(i) The image $\rho_{X_{\log}^1}(I_H) \subseteq \text{Out}(\Pi_1)$ is contained in $\text{Dehn}(G) \subseteq \text{Out}(\Pi_G) \sim \text{Out}(\Pi_1)$ [cf. the notation of [CbTpI], Definition 4.4]. Moreover, the image $\rho_{X_{\log}^1}(I_H)$ is nontrivial if and only if $H$ is of $l$-Dehn type. Write
$$
I_H^{C(\rho)} \overset{\text{def}}{=} (\rho_{X_{\log}^1}(I_H) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \cap \text{Dehn}(G) \subseteq \text{Dehn}(G)
$$
[considered in $\text{Dehn}(G) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ — cf. [CbTpI], Theorem 4.8, (iv)].

(ii) For any positive integer $m \leq n$, the natural injection $\text{Out}^\text{FC}(\Pi_n) \hookrightarrow \text{Out}^\text{FC}(\Pi_m)$ of [NodNon], Theorem B, induces isomorphisms
$$
\text{Z}_{\text{Out}^\text{FC}(\Pi_n)}(\rho_{X_{\log}^1}(H)) \sim \text{Z}_{\text{Out}^\text{FC}(\Pi_m)}(\rho_{X_{\log}^1}(H)),
$$
$$
\text{Z}_{\text{Out}^\text{FC}(\Pi_n)}^{\text{loc}}(\rho_{X_{\log}^1}(H)) \sim \text{Z}_{\text{Out}^\text{FC}(\Pi_m)}^{\text{loc}}(\rho_{X_{\log}^1}(H)),
$$
[cf. the discussion entitled “Topological groups” in §30],
$$
\text{N}_{\text{Out}^\text{FC}(\Pi_n)}(\rho_{X_{\log}^1}(H)) \sim \text{N}_{\text{Out}^\text{FC}(\Pi_m)}(\rho_{X_{\log}^1}(H)),
$$
$$
\text{C}_{\text{Out}^\text{FC}(\Pi_n)}(\rho_{X_{\log}^1}(H)) \sim \text{C}_{\text{Out}^\text{FC}(\Pi_m)}(\rho_{X_{\log}^1}(H)).
$$

(iii) Relative to the natural inclusion $\text{Aut}(G) \subseteq \text{Out}(\Pi_G) \sim \text{Out}(\Pi_1)$, the following equality holds:
$$
\text{C}_{\text{Out}^\text{FC}(\Pi_1)}(\rho_{X_{\log}^1}(H)) = \text{C}_{\text{Aut}(G)}(\rho_{X_{\log}^1}(H)).
$$
In particular, we have natural homomorphisms of profinite groups
$$
\text{C}_{\text{Out}^\text{FC}(\Pi_n)}(\rho_{X_{\log}^1}(H)) \sim \text{C}_{\text{Out}^\text{FC}(\Pi_1)}(\rho_{X_{\log}^1}(H)) \to \text{Aut}(G),
$$
$$
\text{C}_{\text{Out}^\text{FC}(\Pi_n)}(\rho_{X_{\log}^1}(H)) \sim \text{C}_{\text{Out}^\text{FC}(\Pi_1)}(\rho_{X_{\log}^1}(H)) \times_{\mathbb{Z}_l^*} \mathbb{Z}_l^*
$$
[cf. the notation of [CbTpI], Definition 3.8, (ii)] — where the first arrow on each line is the isomorphism of (ii). By abuse of notation [i.e., since $\rho_{X_{\log}^1}(H)$ is not necessarily contained in
\( \text{Aut}^{[\text{grph}]}(G) \) — cf. the notation of \([\text{CbTpI}], \text{Definition 2.6}, (i); \text{Remark 4.1.2 of the present paper}\), write

\[
\begin{align*}
Z_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H)) & \subseteq Z_{\text{Out}^{FC}(\Pi_n)}(\rho_{X_n}^{\log}(H)) , \\
Z_{\text{Aut}^{[\text{grph}]}(G)}^{\text{loc}}(\rho_{X_n}^{\log}(H)) & \subseteq Z_{\text{Out}^{FC}(\Pi_n)}^{\text{loc}}(\rho_{X_n}^{\log}(H)) , \\
N_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H)) & \subseteq N_{\text{Out}^{FC}(\Pi_n)}(\rho_{X_n}^{\log}(H)) , \\
C_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H)) & \subseteq C_{\text{Out}^{FC}(\Pi_n)}(\rho_{X_n}^{\log}(H)) ,
\end{align*}
\]

for the kernels of the restrictions of the composite homomorphism of the first line of the second display \([\text{of the present (iii)}]\) to

\[
\begin{align*}
Z_{\text{Out}^{FC}(\Pi_n)}(\rho_{X_n}^{\log}(H)) , & \quad Z_{\text{Out}^{FC}(\Pi_n)}^{\text{loc}}(\rho_{X_n}^{\log}(H)) , \\
N_{\text{Out}^{FC}(\Pi_n)}(\rho_{X_n}^{\log}(H)) , & \quad C_{\text{Out}^{FC}(\Pi_n)}(\rho_{X_n}^{\log}(H)) ,
\end{align*}
\]

respectively.

(iv) Suppose that \( H \) is not of \( l \)-Dehn type. Then we have equalities

\[
Z_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H)) = Z_{\text{Aut}^{[\text{grph}]}(G)}^{\text{loc}}(\rho_{X_n}^{\log}(H)) = N_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H)) = C_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H))
\]

[cf. the notation of (iii)]. Moreover, each of the four groups appearing in these equalities is, in fact, independent of \( n \) [cf. (ii)].

(v) Suppose that \( H \) is of \( l \)-Dehn type. Then the composite homomorphism of the first line of the second display of (iii) determines an injection of profinite groups

\[
Z_{\text{Out}^{FC}(\Pi_n)}^{\text{loc}}(\rho_{X_n}^{\log}(H)) \hookrightarrow \text{Aut}(G).
\]

(vi) Write \( k_{[\text{grph}]}(\subseteq \overline{k}) \) for the [finite] subfield of \( \overline{k} \) consisting of the invariants of \( k \) with respect to [the natural action on \( \overline{k} \) of the kernel of the natural action of \( H \) on \( G \)]. Then the composite homomorphism of the second line of the second display of (iii) determines natural exact sequences of profinite groups

\[
\begin{align*}
1 & \longrightarrow I_H^{N(\rho)} \longrightarrow N_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H)) \longrightarrow \mathbb{Z}_l^* , \\
1 & \longrightarrow I_H^{C(\rho)} \longrightarrow C_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H)) \longrightarrow \mathbb{Z}_l^* ,
\end{align*}
\]

[cf. the notation of (i), (iii)] — where

\[
(\rho_{X_n}^{\log}(I_H) \subseteq \text{I}_H^{N(\rho)} \overset{\text{def}}{=} N_{\text{Aut}^{[\text{grph}]}(G)}(\rho_{X_n}^{\log}(H)) \cap \text{Dehn}(G) \]

[cf. (ii), (iii)] is an open subgroup of \( I_H^{C(\rho)} \); the image of the third arrow on each line contains \( k_{[\text{grph}]}^{\sharp} \in \mathbb{Z}_l^* \) and does not depend on the choice of \( n \). In particular, these images are
open; if, moreover, $k^{2}_{\text{graph}} \in \mathbb{Z}^*_l$ topologically generates $\mathbb{Z}^*_l$, then the third arrows on each line are surjective.

(vii) The closed subgroup $\rho_{X^\log_n}(H)$, hence also $N_{\text{Out}^{\text{FC}}(\Pi_n)}(\rho_{X^\log_n}(H))$, is open in $C_{\text{Out}^{\text{FC}}(\Pi_n)}(\rho_{X^\log_n}(H))$.

(viii) Consider the following conditions [cf. Remark 4.17.1 below]:

1. Write $\text{Aut}_{(\text{Spec }k)^{\log}}(X^\log)$ for the group of automorphisms of $X^\log$ over $(\text{Spec }k)^{\log}$. Then the natural homomorphism $\text{Aut}_{(\text{Spec }k)^{\log}}(X^\log) \to \text{Aut}(\mathcal{G})$ is surjective.

2. $k^{2}_{\text{graph}} \in \mathbb{Z}^*_l$ topologically generates $\mathbb{Z}^*_l$.

If condition (1) is satisfied, and $H$ is of $l$-Dehn type, then we have an equality

$$Z_{\text{Out}^{\text{FC}}(\Pi_n)}(\rho_{X^\log_n}(H)) = Z_{\text{Out}^{\text{FC}}(\Pi_n)}^\text{loc}(\rho_{X^\log_n}(H)),$$

and, moreover, the composite homomorphism of the first line of the second display of (iii) determines an isomorphism

$$Z_{\text{Out}^{\text{FC}}(\Pi_n)}^\text{loc}(\rho_{X^\log_n}(H)) \sim \to \text{Aut}(\mathcal{G}).$$

If conditions (1) and (2) are satisfied, then the composite homomorphisms of the two lines of the second display of (iii) determine natural exact sequences of profinite groups

$$1 \to I_H^{N(\rho)} \to N_{\text{Out}^{\text{FC}}(\Pi_n)}(\rho_{X^\log_n}(H)) \to \text{Aut}(\mathcal{G}) \times \mathbb{Z}^*_l \to 1,$$

$$1 \to I_H^{C(\rho)} \to C_{\text{Out}^{\text{FC}}(\Pi_n)}(\rho_{X^\log_n}(H)) \to \text{Aut}(\mathcal{G}) \times \mathbb{Z}^*_l \to 1.$$

Proof. Assertion (i) follows immediately from the various definitions involved, together with [CbTpI], Proposition 5.6, (ii). Assertion (ii) follows immediately from Corollary 4.16, together with the openness of the image of $H$ in $G_k$. Assertion (iii) follows immediately from [CmbGC], Corollary 2.7, (ii) [cf. also the proof of [CmbGC], Proposition 2.4, (v)], together with the openness of the image of $H$ in $G_k$.

For $\square \in \{Z, Z^\text{loc}, N, C\}$ and $v \in \text{Vert}(\mathcal{G})$, write

$$\square \text{ def } = \square_{\text{Out}^{\text{FC}}(\Pi_1)}(\rho_{X^\log_1}(H)) \subseteq \text{Out}(\Pi_1) \sim \to \text{Out}(\Pi_G);$$

$$\square_{\text{graph}} \text{ def } = \square \cap \text{Aut}_{\text{graph}}(\mathcal{G}) \subseteq \text{Out}(\Pi_G)$$

[cf. the notation of [CbTpI], Definition 2.6, (i); Remark 4.1.2 of the present paper];

$$\text{pr}_v: \text{Aut}_{\text{graph}}(\mathcal{G}) \to \text{Aut}_{\text{graph}}(\mathcal{G}^v).$$
for the homomorphism determined by \textit{restriction} to $\mathcal{G}|_v$ [cf. [CbTpI], Definition 2.14, (ii); [CbTpI], Remark 2.5.1, (ii)];

\[ \Box_v \subseteq \text{Aut}^{\text{grph}}(\mathcal{G}|_v) \]

for the image of $\Box^{\text{grph}}|_v \subseteq \text{Aut}^{\text{grph}}(\mathcal{G})$ via $\text{pr}_v$. Then we claim that the following assertion holds:

Claim 4.17.A: Let $v \in \text{Vert}(\mathcal{G})$. Then

\[ C_v \cap \text{Ker}(\chi_{\mathcal{G}|_v}) = \{1\} \]

[cf. the notation of [CbTpI], Definition 3.8, (ii)].

Indeed, let us first observe that it follows immediately from a similar argument to the argument applied in the proof of Claim 4.16.A [in the proof of Corollary 4.16] that $C_v \subseteq \text{Aut}^{\text{grph}}(\mathcal{G}|_v)$ is contained in the local centralizer [cf. the discussion entitled “Topological groups” in §0] of the natural image of $G_k$ in $\text{Aut}^{\text{grph}}(\mathcal{G}|_v)$ [cf. the fact that $\mathcal{G}|_v$ is of type $(0,3)$]. Thus, Claim 4.17.A follows immediately from the injectivity discussed in [Hsh], Remark 6, (iv). This completes the proof of Claim 4.17.A.

Next, we claim that the following assertion holds:

Claim 4.17.B: Let $v \in \text{Vert}(\mathcal{G})$. Then

\[ C^{\text{grph}}|_v \cap \text{Ker}(\text{pr}_v) = C^{\text{grph}}|_v \cap \text{Dehn}(\mathcal{G}) ; \]

\[ Z^{\text{grph}}|_v \cap \text{Ker}(\text{pr}_v) = Z^{\text{loc}}|^{\text{grph}}_v \cap \text{Ker}(\text{pr}_v) = \{1\}. \]

In particular, we obtain natural isomorphisms

\[ Z^{\text{grph}}|_v \cong Z_v, \quad Z^{\text{loc}}|^{\text{grph}}_v \cong Z^{\text{loc}}_v \]

and a natural exact sequence of profinite groups [cf. [CbTpI], Corollary 3.9, (iv)]

\[ 1 \longrightarrow C^{\text{grph}}|_v \cap \text{Dehn}(\mathcal{G}) \longrightarrow C^{\text{grph}}|_v \xrightarrow{\chi_v} Z^*_v. \]

Indeed, let us first observe that the first displayed equality of Claim 4.17.B follows immediately from Claim 4.17.A, together with [CbTpI], Corollary 3.9, (iv). On the other hand, since the image of $H$ in $G_k$ is open, the second displayed equality of Claim 4.17.B follows immediately from [CbTpI], Theorem 4.8, (iv), (v), together with the first displayed equality of Claim 4.17.B. This completes the proof of Claim 4.17.B.

Next, we verify assertion (iv). Let us first observe that it follows from Lemma 3.9, (ii), that $C^{\text{grph}}|_v \subseteq N_{\text{Out}^{\text{fc}}(\Pi_1)}(Z^{\text{loc}})$, which thus implies that we have a natural action of $C^{\text{grph}}|_v$ on $Z^{\text{loc}}$, hence also on $Z^{\text{loc}}|_v$, as well as a natural [trivial!] action of $C^{\text{grph}}|_v$ on $\text{Aut}(\mathcal{G})$. Moreover, by considering the inclusion

\[ (C^{\text{grph}}|_v \supseteq) Z^{\text{loc}}|^{\text{grph}}_v \cong Z^{\text{loc}}_v \hookrightarrow Z^*_v \]
induced by $\chi_G$ [cf. Claims 4.17.A, 4.17.B], we conclude that the homomorphisms of the two lines of the second display of assertion (iii) determine a natural $[C_{[\text{graph}]}]$ injection

$$\begin{align*}
Z^\text{loc} & \hookrightarrow \text{Aut}(G) \times Z_i^*.
\end{align*}$$

Thus, since $Z_i^*$ is abelian, it follows that $C_{[\text{graph}]}$ acts trivially on $Z^\text{loc}$, i.e., that $C_{[\text{graph}]} \subseteq Z_{\text{Out}^FC(\Pi_1)}(Z^\text{loc})$. On the other hand, since $H$ is not of $l$-Dehn type, one verifies easily from assertion (i) that $\rho_{X_1}^{\text{log}}(H)$ is abelian, hence that $\rho_{X_1}^{\text{log}}(H) \subseteq Z \subseteq Z^\text{loc}$. Thus, we conclude that

$$\begin{align*}
C_{[\text{graph}]} & \subseteq Z_{\text{Out}^FC(\Pi_1)}(Z^\text{loc}) \cap \text{Aut}_{[\text{graph}]}(G) \\
& \subseteq Z_{\text{Out}^FC(\Pi_1)}(\rho_{X_1}^{\text{log}}(H)) \cap \text{Aut}_{[\text{graph}]}(G) \\
& = Z \cap \text{Aut}_{[\text{graph}]}(G) = Z_{[\text{graph}]}.
\end{align*}$$

This completes the proof of assertion (iv).

Next, we verify assertion (v). First, let us observe that it follows immediately from Claims 4.17.A, 4.17.B, together with assertion (ii), that, to verify assertion (v), it suffices to verify that $G(C_{[\text{graph}]} \cap \text{Dehn}(G)) = 1$. On the other hand, since $H$ is of $l$-Dehn type, by considering the conjugation action of $Z_{[\text{graph}]}$ on $\rho_{X_1}^{\text{log}}(I_H)$ (which is nontrivial by assertion (i)), we conclude from [CbTpI], Theorem 4.8, (iv), (v), that $G(C_{[\text{graph}]} \cap \text{Dehn}(G)) = 1$, as desired. This completes the proof of assertion (v).

Next, we verify assertion (vi). First, we observe that it follows from assertions (ii), (iii) that the definition of $I_n^{\rho}(\rho)$ is indeed independent of $n$ [as the notation suggests!]. Next, we claim that the following assertion holds:

**Claim 4.17.C:**

$$\rho_{X_1}^{\text{log}}(I_H) \subseteq N_{[\text{graph}]} \cap \text{Dehn}(G) = I^{N(\rho)}_H \subseteq C_{[\text{graph}]} \cap \text{Dehn}(G) = I^{C(\rho)}_H.$$  

Indeed, the final equality follows immediately from an elementary computation [in which we apply [CbTpI], Theorem 4.8, (iv), (v)], together with assertion (i); the remainder of Claim 4.17.C follows immediately from the various definitions involved, together with assertion (i). This completes the proof of Claim 4.17.C. Now it follows immediately from Claims 4.17.B, 4.17.C, together with assertion (ii), that the composite homomorphism of the second line of the second display of (iii) determines the two displayed exact sequences of assertion (vi), and that $\rho_{X_1}^{\text{log}}(I_H)$, hence also $I^{N(\rho)}_H$, is an open subgroup of $I^{C(\rho)}_H$. The fact that the image of the third arrow on each line of the displayed sequences of assertion (vi) contains $k_{[\text{graph}]}^\rho \in Z_i^*$ follows immediately from the fact that the image, via $\rho_{X_1}^{\text{log}}$, of the kernel of the natural action of $H$ on $G$ is contained in $N_{[\text{graph}]}$. The fact that the image of the third arrow on each line of the displayed sequences of assertion (vi) does not depend
on the choice of \( n \) follows from assertion (ii). This completes the proof of assertion (vi).

Assertion (vii) follows immediately from assertions (iii) and (vi), together with the finiteness of \( \text{Aut}(G) \). Assertion (viii) follows immediately from assertions (v) and (vi). This completes the proof of Corollary 4.17.

\[ \square \]

Remark 4.17.1.

(i) One verifies easily that condition (1) of Corollary 4.17, (viii), holds if, for instance, \( k = k_{\text{grph}} \), and, moreover, the lengths [cf. [CbTpI], Definition 5.3, (ii)] of the various nodes of \( X^{\text{log}} \) [whose base-change from \( k \) to \( \overline{k} \)] may be thought of as the special fiber log stable curve of [CbTpI], Definition 5.3 coincide.

(ii) In a similar vein, one verifies easily that condition (2) of Corollary 4.17, (viii), holds if, for instance, \( k_{\text{grph}} = F_p \), and, moreover, \( p \) remains prime in the cyclotomic extension \( \mathbb{Q}(e^{2\pi i/\ell}) \), where \( i = \sqrt{-1} \), and we assume that \( \ell \) is odd.

Remark 4.17.2. The computation of the centralizer (respectively, normalizer and commensurator) in Corollary 4.17, (viii), may be thought of as a sort of relative geometrically pro-\( \ell \) (respectively, [semi-] absolute geometrically pro-\( \ell \)) version of the Grothendieck Conjecture for totally degenerate log stable curves over finite fields. In fact, the proofs of these computations of Corollary 4.17, (viii), only involve the theory of [CbTpI]. On the other hand, these computations of Corollary 4.17, (viii), can only be performed under certain relatively restrictive conditions [cf. Remark 4.17.1]. It is precisely for this reason that Corollary 4.17, (ii), which may be thought of as an application of the theory of the present paper, is of interest in the context of these computations of Corollary 4.17, (viii).
References


