TOPICS SURROUNDING THE COMBINATORIAL ANABELIAN GEOMETRY OF HYPERBOLIC CURVES III: TRIPODS AND TEMPERED FUNDAMENTAL GROUPS

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OCTOBER 2016

Abstract. Let Σ be a subset of the set of prime numbers which is either equal to the entire set of prime numbers or of cardinality one. In the present paper, we continue our study of the pro-Σ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of Σ are invertible. The focus of the present paper is on **applications** of the theory developed in previous papers to the theory of **tempered fundamental groups**, in the style of André. These applications are motivated by the goal of surmounting *two fundamental technical difficulties* that appear in previous work of André, namely: (a) the fact that the characterization of the local Galois groups in the global Galois image associated to a hyperbolic curve that is given in earlier work of André is only proven for a *quite limited class* of hyperbolic curves, i.e., a class that is *"far from generic"*; (b) the proof given in earlier work of André of a certain *key injectivity result*, which is of central importance in establishing the theory of a "p-adic local analogue" of the well-known "global" theory of the Grothendieck-Teichmüller group, contains a *fundamental gap*. In the present paper, we surmount these technical difficulties by introducing the notion of an **"M-admissible"**, or **"metric-admissible"**, outer automorphism of the profinite geometric fundamental group of a p-adic hyperbolic curve. Roughly speaking, M-admissible outer automorphisms are outer automorphisms that are compatible with the data constituted by the *indices* at the various *nodes* of the special fiber of the p-adic curve under consideration. By combining this notion with **combinatorial anabelian** results and techniques developed in earlier papers by the authors, together with the theory of **cyclotomic synchronization** [also developed in earlier papers by the authors], we obtain a *generalization* of André's **characterization** of the **local Galois groups** in the **global Galois image** associated to a hyperbolic curve to the case of **arbitrary hyperbolic curves** [cf. (a)]. Moreover, by applying the theory of *local contractibility of p-adic analytic spaces* developed by Berkovich, we show that the techniques developed in the present and earlier papers by the authors allow one to relate the groups of M-admissible outer automorphisms treated in the present paper to the groups of outer automorphisms

2010 Mathematics Subject Classification. Primary 14H30; Secondary 14H10.
Key words and phrases. anabelian geometry, combinatorial anabelian geometry, tempered fundamental group, tripod, Grothendieck-Teichmüller group, semi-graph of anabelioids, hyperbolic curve, configuration space.
The first author was supported by Grant-in-Aid for Scientific Research (C), No. 24540016, Japan Society for the Promotion of Science.

of *tempered fundamental groups* of *higher-dimensional configuration spaces* [associated to the given p-adic hyperbolic curve]. These considerations allow one to "repair" the gap in André's proof — albeit at the expense of working with *M-admissible* outer automorphisms — and hence to realize the goal of obtaining a **"local analogue of the Grothendieck-Teichmüller group"** [cf. (b)].

Contents

Introduction

Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset of the set of prime numbers $\mathfrak{Primes}$ which is either equal to $\mathfrak{Primes}$ or of cardinality one. In the present paper, we continue our study of the *pro-Σ fundamental groups* of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of Σ are invertible [cf. [MzTa], [CmbCsp], [NodNon], [CbTpI], [CbTpII]]. The focus of the present paper is on **applications** of the theory developed in previous papers to the theory of **tempered fundamental groups**, in the style of [André].

Just as in previous papers, the main technical result that underlies our approach is a certain *combinatorial anabelian result* [cf. Theorem 1.11; Corollary 1.12], which may be summarized as a *generalization* of results obtained in earlier papers [cf., e.g., [NodNon], Theorem A; [CbTpII], Theorem 1.9] in the case of pro-Σ fundamental groups to the case of **almost pro-Σ fundamental groups** [i.e., *maximal almost pro-Σ quotients* of profinite fundamental groups — cf. Definition 1.1]. The technical details surrounding this generalization occupy the bulk of §1.

In §2, we observe that the theory of §1 may be applied, via a similar argument to the argument applied in [NodNon] to derive [NodNon], Theorem B, from [NodNon], Theorem A, to obtain *almost pro-Σ generalizations* [cf. Theorem 2.9; Corollary 2.10; Remark 2.10.1] of the **injectivity** portion of the theory of combinatorial cuspidalization [i.e., [NodNon], Theorem B]. In the final portion of §2, we discuss the theory of *almost pro-l commensurators* of tripods [i.e., copies of the [geometric fundamental group of the] projective line minus three points — cf. Corollary 2.13], in the context of the theory of the *tripod homomorphism* developed in [CbTpII], §3. Just as in the case of the theory of

§1, the theory of §2 is *conceptually* not very difficult, but *technically* quite involved.

Before proceeding, we recall that a substantial portion of the theory of [André] revolves around the study of outomorphism [i.e., outer automorphism] groups of the **tempered geometric fundamental group** of a *p-adic hyperbolic curve*, from the point of view of the goal of establishing

> a ***p*-adic local analogue** of the well-known theory of the **Grothendieck-Teichmüller group** [i.e., which appears in the context of *hyperbolic curves over number fields*].

From the point of view of the theory of the present series of papers, outomorphisms of such tempered fundamental groups may be thought of as [i.e., are equivalent to — cf. Remark 3.3.1; Proposition 3.6, (iii); Remark 3.13.1, (i)] outomorphisms of the profinite geometric fundamental group that are *"G-admissible"* [cf. Definition 3.7, (i)], i.e., preserve the *graph-theoretic* structure on the profinite geometric fundamental group. In a word, the essential thrust of the applications to the theory of tempered fundamental groups given in the present paper may be summarized as follows:

> By replacing, in effect, the *G-admissible* outomorphism groups that [modulo the "translation" discussed above] appear throughout the theory of [André] by **"M-admissible"** outomorphism groups — i.e., groups of outomorphisms of the profinite geometric fundamental group that preserve not only the *graph-theoretic* structure on the profinite geometric fundamental group, but also the [somewhat finer] **metric** structure on the various dual graphs that appear [i.e., the various *indices* at the *nodes* of the special fiber of the p-adic curve under consideration — cf. Definition 3.7, (ii)] — it is possible to **overcome** various **significant technical difficulties** that appear in the theory of [André].

Here, we recall that the two main technical difficulties that appear in the theory of [André] may be described as follows:

- The *characterization* of the *local Galois groups* in the *global Galois image* associated to a hyperbolic curve that is given in [André], Theorems 7.2.1, 7.2.3, is only proven for a *quite limited class* of hyperbolic curves [i.e., a class that is *"far from generic"* — cf. [MzTa], Corollary 5.7], which are "closely related to tripods".
- The proof given in [André] of a certain *key injectivity result*, which is of central importance in establishing the theory of a

"local analogue of the Grothendieck-Teichmüller group", contains a *fundamental gap* [cf. Remark 3.19.1].

In the present paper, our approach to surmounting the *first* technical difficulty consists of the following result [cf. Theorems 3.17, (iv); 3.18, (i)], which asserts, roughly speaking, that the theory of the **tripod homomorphism** developed in [CbTpII], §3, is **compatible** with the property of **M-admissibility**.

Theorem A (**Metric-admissible outomorphisms and the tripod homomorphism).** *Let $n \geq 3$ be an integer; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; p a prime number; Σ a set of prime numbers such that $\Sigma \neq \{p\}$, and, moreover, is either equal to the set of all prime numbers or of cardinality one; R a mixed characteristic complete discrete valuation ring of residue characteristic p whose residue field is separably closed; K the field of fractions of R; $\overline{K}$ an algebraic closure of K;*

$$X_K^{\log}$$

a **smooth log curve** *of type (g, r) over K. Write*

$$(X_K)_n^{\log}$$

for the n-th **log configuration space** *[cf. the discussion entitled "Curves" in* [CbTpI], *§0] of $X_K^{\log}$ over K; $(X_{\overline{K}})_n^{\log} \stackrel{\text{def}}{=} (X_K)_n^{\log} \times_K \overline{K}$;*

$$\Pi_n \stackrel{\text{def}}{=} \pi_1((X_{\overline{K}})_n^{\log})^{\Sigma}$$

for the maximal pro-Σ quotient of the **log fundamental group** *of $(X_{\overline{K}})_n^{\log}$. Let Π^{tpd} be a* **central $\{1,2,3\}$-tripod** *of Π_n [cf.* [CbTpII], *Definitions 3.3, (i); 3.7, (ii)]. Then the restriction of the* **tripod homomorphism** *associated to Π_n*

$$\mathfrak{T}_{\Pi^{\text{tpd}}} : \text{Out}^{\text{FC}}(\Pi_n) \longrightarrow \text{Out}^{\text{C}}(\Pi^{\text{tpd}})$$

[cf. [CbTpII], *Definition 3.19] to the subgroup $\text{Out}^{\text{FC}}(\Pi_n)^{\text{M}} \subseteq \text{Out}^{\text{FC}}(\Pi_n)$ of* **M-admissible** *outomorphisms [cf. Definition 3.7, (iii)]* **factors** *through the subgroup $\text{Out}(\Pi^{\text{tpd}})^{\text{M}} \subseteq \text{Out}^{\text{C}}(\Pi^{\text{tpd}})$ [cf. Definition 3.7, (ii)], i.e., we have a* **natural commutative diagram** *of* **profinite groups**

$$\begin{array}{ccc} \text{Out}^{\text{FC}}(\Pi_n)^{\text{M}} & \longrightarrow & \text{Out}(\Pi^{\text{tpd}})^{\text{M}} \\ \downarrow & & \downarrow \\ \text{Out}^{\text{FC}}(\Pi_n) & \xrightarrow[\mathfrak{T}_{\Pi^{\text{tpd}}}]{} & \text{Out}^{\text{C}}(\Pi^{\text{tpd}}) . \end{array}$$

Theorem A has the following formal consequence, namely, a generalization of the *characterization* of the *local Galois groups* in the *global Galois image* associated to a hyperbolic curve that is given in [André], Theorems 7.2.1, 7.2.3, to **arbitrary hyperbolic curves**, albeit at the expense of, in effect, replacing "G-admissibility" by the stronger condition of **"M-admissibility"** [cf. Corollary 3.20; Remark 3.20.1]. This generalization may also be regarded as a sort of strong version of the *Galois injectivity* result given in [NodNon], Theorem C [cf. Remark 3.20.2].

Theorem B (**Characterization of the local Galois groups in the global Galois image associated to a hyperbolic curve**). *Let F be a* **number field**, *i.e., a finite extension of the field of rational numbers; $\mathfrak{p}$ a nonarchimedean prime of F; $\overline{F}_{\mathfrak{p}}$ an algebraic closure of the $\mathfrak{p}$-adic completion $F_{\mathfrak{p}}$ of F; $\overline{F} \subseteq \overline{F}_{\mathfrak{p}}$ the algebraic closure of F in $\overline{F}_{\mathfrak{p}}$; $X_F^{\log}$ a* **smooth log curve** *over F. Write $\overline{F}_{\mathfrak{p}}^{\wedge}$ for the completion of $\overline{F}_{\mathfrak{p}}$; $G_{\mathfrak{p}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \subseteq G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$; $X_{\overline{F}}^{\log} \stackrel{\mathrm{def}}{=} X_F^{\log} \times_F \overline{F}$;*

$$\pi_1(X_{\overline{F}}^{\log})$$

for the **log fundamental group** *of $X_{\overline{F}}^{\log}$ [which, in the following, we* **identify** *with the log fundamental groups of $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}$, $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}$ — cf. the definition of $\overline{F}$!];*

$$\pi_1^{\mathrm{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})$$

for the **tempered fundamental group** *of $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}$ [cf.* [André], *§4];*

$$\rho_{X_F^{\log}} \colon G_F \longrightarrow \mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))$$

for the natural outer Galois action associated to $X_F^{\log}$;

$$\rho_{X_F^{\log},\mathfrak{p}}^{\mathrm{temp}} \colon G_{\mathfrak{p}} \longrightarrow \mathrm{Out}(\pi_1^{\mathrm{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}))$$

for the natural outer Galois action associated to $X_F^{\log} \times_F F_{\mathfrak{p}}$ [cf. [André], *Proposition 5.1.1];*

$$\mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{M}} \subseteq \;(\;\mathrm{Out}(\pi_1^{\mathrm{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})) \subseteq \;)\; \mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))$$

for the subgroup of **M-admissible** *outomorphisms of $\pi_1(X_{\overline{F}}^{\log})$ [cf. Definition 3.7, (i), (ii); Proposition 3.6, (i)]. Then the following hold:*

(i) *The outer Galois action $\rho_{X_F^{\log},\mathfrak{p}}^{\mathrm{temp}}$ factors through the subgroup $\mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{M}} \subseteq \mathrm{Out}(\pi_1^{\mathrm{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}))$.*

(ii) *We have a natural commutative diagram*

$$\begin{array}{ccc} G_{\mathfrak{p}} & \longrightarrow & \mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{M}} \\ \downarrow & & \downarrow \\ G_F & \xrightarrow{\rho_{X_F^{\log}}} & \mathrm{Out}(\pi_1(X_{\overline{F}}^{\log})) \end{array}$$

— where the vertical arrows are the natural inclusions, the upper horizontal arrow is the homomorphism arising from the factorization of (i), and all arrows are **injective***.*

(iii) *The diagram of (ii) is* **cartesian***, i.e., if we regard the various groups involved as subgroups of* $\mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$*, then we have an equality*

$$G_{\mathfrak{p}} = G_F \cap \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))^{\mathrm{M}}\,.$$

One central technical aspect of the theory of the present paper lies in the **equivalence** [cf. Theorem 3.9] between the **M-admissibility** of outomorphisms of the profinite geometric fundamental group of the given p-adic hyperbolic curve and the **I-admissibility** [i.e., roughly speaking, compatibility with the outer action, by *some* open subgroup of the *inertia group* of the absolute Galois group of the base field, on an arbitrary almost pro-l quotient of the profinite geometric fundamental group — cf. Definition 3.8] of such outomorphisms. This equivalence is obtained by applying the theory of **cyclotomic synchronization** developed in [CbTpI], §5. Once this equivalence is established, the *almost pro-l injectivity* results obtained in §2 then allow us to conclude that this M-admissibility of outomorphisms of the profinite geometric fundamental group of the given p-adic hyperbolic curve is, in fact, equivalent to the *I-admissibility* of any [necessarily unique!] *lifting* of such an outomorphism to an outomorphism of the profinite geometric fundamental group of a *higher-dimensional configuration space* associated to the given p-adic hyperbolic curve [cf. Theorem 3.17, (ii)]. Finally, by combining this **"higher-dimensional I-admissibility"** with the *combinatorial anabelian theory* of [CbTpII], §1, we conclude [cf. Proposition 3.16, (i); Theorem 3.17, (ii)] that a certain **"higher-dimensional G-admissibility"** also holds, i.e., that the *lifted outomorphism* of the profinite geometric fundamental group of a higher-dimensional configuration space associated to the given p-adic hyperbolic curve preserves the *graph-theoretic* structure not only on the profinite geometric fundamental group of the original hyperbolic curve, but also on the profinite geometric fundamental groups of the various *successive fibers* of the higher-dimensional configuration space under consideration. In a word,

> it is precisely by applying this chain of equivalences — which allows us to *control the* **graph-theoretic** *structure of the* **successive fibers** of the higher-dimensional configuration space under consideration — that allow us to surmount the two main technical difficulties discussed above that appear in the theory of [André].

Put another way, if, instead of considering *M-admissible* outomorphisms [i.e., of the profinite geometric fundamental group of the given

p-adic hyperbolic curve], one considers arbitrary *G-admissible* outomorphisms [of the profinite geometric fundamental group of the given p-adic hyperbolic curve, as is done, in effect, in [André]], then there does not appear to exist, at least at the time of writing, any effective way to control the graph-theoretic structure on the successive fibers of higher-dimensional configuration spaces.

In this context, we recall that in the theory of [CbTpII], a result is obtained concerning the preservation of the graph-theoretic structure on the successive fibers of higher-dimensional configuration spaces [cf. [CbTpII], Theorem 4.7], in the context of *pro-l* geometric fundamental groups. The significance, however, of the theory of the present paper is that it may be applied to *almost pro-l* geometric fundamental groups, i.e., where the order of the finite quotient implicit in the term "almost" is allowed to be *divisible by p*.

Once one establishes the *"higher-dimensional G-admissibility"* discussed above, it is then possible to apply the theory of *local contractibility of p-adic analytic spaces* developed in [Brk] to construct from the given outomorphism of a profinite geometric fundamental group [of a higher-dimensional configuration space] an outomorphism of the corresponding *tempered fundamental group* [cf. Proposition 3.16, (ii)]. This portion of the theory may be summarized as follows [cf. Theorem 3.19, (ii)].

Theorem C (Metric-admissible outomorphisms and tempered fundamental groups). *Let n be a positive integer; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; p a prime number; Σ a nonempty set of prime numbers such that $\Sigma \neq \{p\}$, and, moreover, if $n \geq 2$, then Σ is either equal to the set of all prime numbers or of cardinality one; R a mixed characteristic complete discrete valuation ring of residue characteristic p whose residue field is separably closed; K the field of fractions of R; $\overline{K}$ an algebraic closure of K;*

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for the n-th **log configuration space** *[cf. the discussion entitled "Curves" in* [CbTpI], *§0] of $X_K^{\log}$ over K; $(X_{\overline{K}})_n^{\log} \stackrel{\text{def}}{=} (X_K)_n^{\log} \times_K \overline{K}$;*

$$\Pi_n \stackrel{\text{def}}{=} \pi_1((X_{\overline{K}})_n^{\log})^{\Sigma}$$

for the maximal pro-Σ quotient of the **log fundamental group** *of $(X_{\overline{K}})_n^{\log}$; $\overline{K}^{\wedge}$ for the p-adic completion of $\overline{K}$;*

$$\pi_1^{\text{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})$$

for the **tempered fundamental group** *[cf.* [André], *§4] of* $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$*;*

$$\Pi_n^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \varprojlim_N \pi_1^{\mathrm{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})/N$$

for the **Σ-tempered fundamental group** *of* $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$ *[cf.* [CmbGC] *Corollary 2.10, (iii)], i.e., the inverse limit given by allowing* N *to vary over the open normal subgroups of* $\pi_1^{\mathrm{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})$ *such that the quotient by* N *corresponds to a* **topological covering** *[cf.* [André]*, §4.2] of some* **finite étale Galois covering** *of* $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$ *of degree a product of primes* $\in \Sigma$*. [Here, we recall that, when* $n = 1$*, such a "topological covering" corresponds to a "combinatorial covering", i.e., a covering determined by a covering of the dual semi-graph of the special fiber of the stable model of some finite étale covering of* $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$*.] Write*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}} \subseteq \mathrm{Out}(\Pi_n^{\mathrm{tp}})$$

for the inverse image of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \subseteq \mathrm{Out}(\Pi_n)$ *[cf. Definition 3.7, (iii)] via the natural homomorphism* $\mathrm{Out}(\Pi_n^{\mathrm{tp}}) \to \mathrm{Out}(\Pi_n)$ *[cf. Proposition 3.3, (i)]. Then the resulting natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$$

is **split surjective***, i.e., there exists a homomorphism*

$$\Phi\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}}$$

such that the composite

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \xrightarrow{\Phi} \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$$

is the **identity automorphism** *of* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$.

Up till now, in the present discussion, the *p-adic hyperbolic curve* under consideration was *arbitrary.* If, however, one *specializes* the theory discussed above to the case of **tripods** [i.e., copies of the projective line minus three points], then one obtains the desired *p-adic local analogue* of the theory of the *Grothendieck-Teichmüller group*, by considering the **"metrized Grothendieck-Teichmüller group** GT^{M}**"** as follows [cf. Theorem 3.17, (iv); Theorem 3.18, (ii); Theorem 3.19, (ii); Remarks 3.19.2, 3.20.3].

Theorem D (Metric-admissible outomorphisms and tripods). *In the notation of Theorem C, suppose that* $(g, r) = (0, 3)$*. Write*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

for the inverse image via the natural homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{Out}(\Pi_1)$ *[cf.* [CbTpI]*, Theorem A, (i)] of* $\mathrm{Out}^{\mathrm{C}}(\Pi_1)^{\Delta+} \subseteq \mathrm{Out}(\Pi_1)$

[cf. [CbTpII]*, Definition 3.4, (i)];*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} \overset{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

[cf. Remark 3.18.1];

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} \overset{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}}\,;$$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+} \overset{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}}\,.$$

Then the following hold:

(i) *We have equalities*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+}\,,$$
$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+}\,.$$

Moreover, the natural homomorphisms of **profinite groups**

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{\Delta+} & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} \\ \| & & \| \\ \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1})^{\Delta+} & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} \end{array}$$

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{\mathrm{M}\Delta+} & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+} \\ \| & & \| \\ \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1})^{\mathrm{M}\Delta+} & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} \end{array}$$

are **bijective** *for $n \geq 1$. In the following, we shall* **identify** *the various groups that occur for varying n by means of these natural isomorphisms and write*

$$\mathrm{GT}^{\mathrm{M}} \overset{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+}$$
$$\subseteq \mathrm{GT} \overset{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+}$$

[cf. [CmbCsp]*, Remark 1.11.1].*

(ii) *Write*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}\Delta+} \subseteq \mathrm{Out}(\Pi_n^{\mathrm{tp}})$$

for the inverse image of $\mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{Out}(\Pi_n)$ *[cf. (i)] via the natural homomorphism* $\mathrm{Out}(\Pi_n^{\mathrm{tp}}) \to \mathrm{Out}(\Pi_n)$ *[cf. Proposition 3.3, (i)]. Then the resulting natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}\Delta+} \longrightarrow \mathrm{GT}^{\mathrm{M}}$$

is **split surjective***, i.e., there exists a homomorphism*

$$\Phi_{\mathrm{GT}}\colon \mathrm{GT}^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}\Delta+}$$

such that the composite

$$\mathrm{GT}^{\mathrm{M}} \overset{\Phi_{\mathrm{GT}}}{\longrightarrow} \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}\Delta+} \longrightarrow \mathrm{GT}^{\mathrm{M}}$$

is the **identity automorphism** *of* GT^{M}.

In closing, we recall that *"conventional research"* concerning the Grothendieck-Teichmüller group GT tends to focus on the issue of whether or not the *natural inclusion* of the absolute Galois group of $\mathbb{Q}$

$$G_{\mathbb{Q}} \hookrightarrow \mathrm{GT}$$

is, in fact, an *isomorphism* [cf. the discussion of [CbTpII], Remark 3.19.1]. By contrast, one important theme of the present series of papers lies in the point of view that, instead of pursuing the issue of whether or not GT is *literally isomorphic* to $G_{\mathbb{Q}}$, it is perhaps more natural to concentrate on the issue of verifying that

GT *exhibits* **analogous behavior/properties** *to* $G_{\mathbb{Q}}$ *[or* $\mathbb{Q}$*].*

From this point of view, the theory of **tripod synchronization** and **surjectivity** of the **tripod homomorphism** developed in [CbTpII] [cf. [CbTpII], Theorem C, (iii), (iv), as well as the following discussion] may be regarded as an *abstract combinatorial analogue* of the **scheme-theoretic** fact that $\mathrm{Spec}\,\mathbb{Q}$ lies **under** all characteristic zero schemes/algebraic stacks in a **unique fashion** — i.e., put another way, that all morphisms between schemes and moduli stacks that occur in the theory of hyperbolic curves in characteristic zero are *compatible* with the respective *structure morphisms* to $\mathrm{Spec}\,\mathbb{Q}$. In a similar vein, the theory of the subgroup $\mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{GT}$ developed in the present paper may be regarded as an *abstract combinatorial analogue* of the various **decomposition subgroups** $G_{\mathfrak{p}} \subseteq G_F$ $(\subseteq G_{\mathbb{Q}})$ [cf. Theorem B] associated to **nonarchimedean primes**. In particular, from the point of view of pursuing "abstract behavioral similarities" to the subgroups $G_{\mathfrak{p}} \subseteq G_F$ $(\subseteq G_{\mathbb{Q}})$, it is natural to pose the question:

Is the subgroup $\mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{GT}$ *commensurably terminal*?

Unfortunately, in the present paper, we are only able to give a *partial answer* to this question. That is to say, we show [cf. Theorem 3.17, (v), and its proof; Remark 3.20.1] the following result. [Here, we remark that although this result is not stated *explicitly* in Theorem 3.17, (v), it follows by applying to GT^{M} the argument, involving *l-graphically full* actions, that was applied, in the proof of Theorem 3.17, (v), to "$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$".]

Theorem E (**Commensurator of the metrized Grothendieck--Teichmüller group**)**.** *In the notation of Theorem D [cf., especially, the* **bijections** *of Theorem D, (i)], the* **commensurator** *of* GT^{M} *in* $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$ *is* **contained** *in the subgroup*

$$\mathrm{Out}^{\mathrm{G}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

of outomorphisms that satisfy the condition of **"higher-dimensional G-admissibility"** *discussed above [cf. Definition 3.13, (iv); Remark 3.13.1, (ii)]. In particular, the* **commensurator** *of* GT^{M} *in* GT *is* **contained** *in*

$$\mathrm{GT}^{\mathrm{G}} \overset{\mathrm{def}}{=} \mathrm{GT} \cap \Big(\bigcap_{n\geq 1} \mathrm{Out}^{\mathrm{G}}(\Pi_n)\Big) \subseteq \Big(\bigcap_{n\geq 1} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)\Big) \subseteq \mathrm{Out}(\Pi_1)$$

[cf. the **injections** $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *of* [NodNon]*, Theorem B].*

Acknowledgment

The authors would like to thank *E. Lepage* for helpful discussions concerning the theory of Berkovich spaces and *Y. Iijima* for informing us of [Prs].

0. Notations and Conventions

Topological groups: Let G be a profinite group and Σ a nonempty set of prime numbers. Then we shall write G^{Σ} for the *maximal pro-*Σ *quotient* of G.

Let G be a profinite group and $G \twoheadrightarrow Q$, Q' quotients of G. Then we shall say that the quotient Q *dominates* the quotient Q' if the natural surjection $G \twoheadrightarrow Q'$ factors through the natural surjection $G \twoheadrightarrow Q$.

1. Almost pro-Σ combinatorial anabelian geometry

In the present §1, we discuss *almost pro-Σ analogues* of results on combinatorial anabelian geometry developed in earlier papers of the authors. In particular, we obtain *almost pro-Σ analogues* of combinatorial versions of the Grothendieck Conjecture for outer representations of *NN*- and *IPSC*-type [cf. Theorem 1.11; Corollary 1.12 below].

In the present §1, let $\Sigma \subseteq \Sigma^\dagger$ be nonempty sets of prime numbers, $\mathcal{G}$ a semi-graph of anabelioids of pro-$\Sigma^\dagger$ PSC-type. Write $\mathbb{G}$ for the underlying semi-graph of $\mathcal{G}$, $\Pi_\mathcal{G}$ for the [pro-$\Sigma^\dagger$] fundamental group of $\mathcal{G}$, and $\widetilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of $\mathcal{G}$ corresponding to $\Pi_\mathcal{G}$.

Definition 1.1. Let G be a profinite group, $N \subseteq G$ a normal open subgroup of G, and $G \twoheadrightarrow Q$ a quotient of G. Then we shall say that Q is the *maximal almost pro-Σ quotient of G with respect to N* if the kernel of the surjection $G \twoheadrightarrow Q$ is the kernel of $N \twoheadrightarrow N^\Sigma$ [cf. the discussion entitled "*Topological groups*" in §0], i.e., $Q = G/\mathrm{Ker}(N \twoheadrightarrow N^\Sigma)$. Thus, Q fits into an exact sequence of profinite groups

$$1 \longrightarrow N^\Sigma \longrightarrow Q \longrightarrow G/N \longrightarrow 1\,.$$

[Note that since N is *normal* in G, and the kernel $\mathrm{Ker}(N \twoheadrightarrow N^\Sigma)$ of the natural surjection $N \twoheadrightarrow N^\Sigma$ is *characteristic* in N, it holds that $\mathrm{Ker}(N \twoheadrightarrow N^\Sigma)$ is *normal* in G.] We shall say that Q is a *maximal almost pro-Σ quotient of G* if Q is the maximal almost pro-Σ quotient of G with respect to some normal open subgroup of G.

Lemma 1.2 (**Properties of maximal almost pro-$\boldsymbol{\Sigma}$ quotients**). *Let G be a profinite group. Then the following hold.*

(i) *Let $N \subseteq G$ be a normal open subgroup of G and $G \twoheadrightarrow J$ a quotient of G. Write $N_J \subseteq J$ for the image of N in J. [Thus, N_J is a normal open subgroup of J.] Then the quotient of J determined by the maximal almost pro-Σ quotient [cf. Definition 1.1] of G with respect to N, i.e., the quotient of J by the image of* $\mathrm{Ker}(N \twoheadrightarrow N^\Sigma)$ *in J, is the* **maximal almost pro-$\boldsymbol{\Sigma}$ quotient** *of J with respect to N_J.*

(ii) *Let $N \subseteq G$ be a normal open subgroup of G and $H \subseteq G$ a closed subgroup of G. If the natural homomorphism $(N \cap H)^\Sigma \to N^\Sigma$ is* **injective**, *then the image of H in the maximal almost pro-Σ quotient of G with respect to N is the* **maximal almost pro-$\boldsymbol{\Sigma}$ quotient** *of H with respect to $N \cap H$.*

(iii) *Let $H \subseteq G$ be a normal closed subgroup of G and $H \twoheadrightarrow H^*$ a maximal almost pro-Σ quotient of H. Suppose that H is* **topologically finitely generated**. *Then there exists a* **maximal**

almost pro-Σ quotient $H \twoheadrightarrow H^{**}$ *of* H *which* **dominates** $H \twoheadrightarrow H^*$ *[cf. the discussion entitled "Topological groups" in §0] such that the kernel of* $H \twoheadrightarrow H^{**}$ *is normal in* G.

Proof. Assertions (i), (ii) follow immediately from the various definitions involved. Next, we verify assertion (iii). Let $N \subseteq H$ be a normal open subgroup of H with respect to which H^* is the maximal almost pro-Σ quotient of H. Write $M \subseteq G$ for the normal closed subgroup of G obtained by forming the intersection of all G-conjugates of N. Note that $M \subseteq H$. Moreover, since H is *topologically finitely generated*, and $N \subseteq H$ is *open*, it follows that there exists a *characteristic open* subgroup $J \subseteq H$ such that $J \subseteq N$. Thus, $J \subseteq M$, so M is *open* in H. In particular, if we write H^{**} for the maximal almost pro-Σ quotient of H with respect to M, then H^{**} satisfies the conditions of assertion (iii). This completes the proof of assertion (iii). □

Definition 1.3. Let I be a profinite group and $\rho\colon I \to \mathrm{Aut}(\mathcal{G}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$ a continuous homomorphism. Then we shall say that ρ is *of PIPSC-type* [where the "PIPSC" stands for "potentially IPSC"] if the following conditions are satisfied:

(i) I is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma^\dagger}$ as an abstract profinite group.

(ii) there exists an open subgroup $J \subseteq I$ such that the restriction of ρ to J is of IPSC-type [cf. [NodNon], Definition 2.4, (i)].

Lemma 1.4 (Profinite Dehn multi-twists and finite étale coverings). *Let* $\alpha \in \mathrm{Out}(\Pi_{\mathcal{G}})$, $\widetilde{\alpha} \in \mathrm{Aut}(\Pi_{\mathcal{G}})$ *a lifting of* α, *and* $\mathcal{H} \to \mathcal{G}$ *a connected finite étale Galois subcovering of* $\widetilde{\mathcal{G}} \to \mathcal{G}$ *such that* $\widetilde{\alpha}$ *preserves the corresponding open subgroup* $\Pi_{\mathcal{H}} \subseteq \Pi_{\mathcal{G}}$, *hence induces an element* $\alpha_{\mathcal{H}} \in \mathrm{Out}(\Pi_{\mathcal{H}})$. *Suppose that* $\alpha_{\mathcal{H}} \in \mathrm{Dehn}(\mathcal{H})$ *[cf.* [CbTpI], *Definition 4.4]. Then* $\alpha \in \mathrm{Dehn}(\mathcal{G})$.

Proof. It follows immediately from [CmbGC], Propositions 1.2, (ii); 1.5, (ii), that $\alpha \in \mathrm{Aut}(\mathcal{G})$. The fact that $\alpha \in \mathrm{Dehn}(\mathcal{G})$ now follows from [CmbGC], Propositions 1.2, (i); 1.5, (i), together with the *commensurable terminality* of VCN-subgroups of $\Pi_{\mathcal{G}}$ [cf. [CmbGC], Proposition 1.2, (ii)] and the *slimness* of verticial subgroups of $\Pi_{\mathcal{G}}$ [cf. [CmbGC], Remark 1.1.3]. [Here, we recall that an automorphism of a slim profinite group is equal to the identity if and only if it preserves and induces the identity on an open subgroup.] □

Lemma 1.5 (Outer representations of VA-, NN-, PIPSC-type and finite étale coverings). *In the notation of Definition 1.3, suppose that* I *is isomorphic to* $\widehat{\mathbb{Z}}^{\Sigma^\dagger}$ *as an abstract profinite group; let*

$\widetilde{\rho}_J : J \to \mathrm{Aut}(\Pi_{\mathcal{G}})$ *be a lifting of the restriction of* ρ *to an open subgroup* $J \subseteq I$ *and* $\mathcal{H} \to \mathcal{G}$ *a connected finite étale Galois subcovering of* $\widetilde{\mathcal{G}} \to \mathcal{G}$ *such that the action of* J *on* $\Pi_{\mathcal{G}}$*, via* $\widetilde{\rho}_J$*, preserves the corresponding open subgroup* $\Pi_{\mathcal{H}} \subseteq \Pi_{\mathcal{G}}$*, hence induces a continuous homomorphism* $J \to \mathrm{Aut}(\Pi_{\mathcal{H}})$*. Then* ρ *is of* **VA-type** *[cf.* [NodNon]*, Definition 2.4, (ii), as well as Remark 1.5.1 below] (respectively,* **NN-type** *[cf.* [NodNon]*, Definition 2.4, (iii)];* **PIPSC-type** *[cf. Definition 1.3]) if and only if the composite* $J \to \mathrm{Aut}(\Pi_{\mathcal{H}}) \twoheadrightarrow \mathrm{Out}(\Pi_{\mathcal{H}})$ *is of* **VA-type** *(respectively,* **NN-type***;* **PIPSC-type***).*

Proof. Necessity in the case of outer representations of *VA-type* (respectively, *NN-type*; *PIPSC-type*) follows immediately from [NodNon], Lemma 2.6, (i) (respectively, [NodNon], Lemma 2.6, (i); the various definitions involved, together with the well-known *properness* of the moduli stack of pointed stable curves of a given type). To verify *sufficiency*, let us first observe that it follows immediately from the various definitions involved that we may assume without loss of generality that $J = I$, and that the outer representation $J = I \to \mathrm{Out}(\Pi_{\mathcal{H}})$ is of *SVA-type* (respectively, *SNN-type*; *IPSC-type*) [cf. [NodNon], Definition 2.4]. Then sufficiency in the case of outer representations of *VA-type* (respectively, *NN-type*; *PIPSC-type*) follows immediately, in light of the criterion of [CbTpI], Corollary 5.9, (i) (respectively, (ii); (iii)), from Lemma 1.4, together with the compatibility property of [CbTpI], Corollary 5.9, (v) [applied, via [CbTpI], Theorem 4.8, (ii), (iv), to each of the *Dehn coordinates* of the profinite Dehn multi-twists under consideration — cf. the proof of [CbTpII], Lemma 3.26, (ii)]. This completes the proof of Lemma 1.5. □

Remark 1.5.1. Here, we take the opportunity to correct an *unfortunate misprint* in [NodNon]. The phrase "*of VA-type*" that appears near the beginning of [NodNon], Definition 2.4, (ii), should read "is *of VA-type*".

Definition 1.6. let $\mathcal{H}$ be a semi-graph of anabelioids of pro-$\Sigma^\dagger$ PSC-type. Write $\mathbb{H}$ for the underlying semi-graph of $\mathcal{H}$, $\Pi_{\mathcal{H}}$ for the [pro-$\Sigma^\dagger$] fundamental group of $\mathcal{H}$, and $\widetilde{\mathcal{H}} \to \mathcal{H}$ for the universal covering of $\mathcal{H}$ corresponding to $\Pi_{\mathcal{H}}$. Let $\Pi^*_{\mathcal{G}}$ (respectively, $\Pi^*_{\mathcal{H}}$) be a *maximal almost pro-*Σ *quotient* of $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$) [cf. Definition 1.1].

(i) For each $v \in \mathrm{Vert}(\mathcal{G})$ (respectively, $e \in \mathrm{Edge}(\mathcal{G})$; $e \in \mathrm{Node}(\mathcal{G})$; $e \in \mathrm{Cusp}(\mathcal{G})$; $z \in \mathrm{VCN}(\mathcal{G})$), we shall refer to the image of a verticial (respectively, an edge-like; a nodal; a cuspidal; a VCN- [cf. [CbTpI], Definition 2.1, (i)]) subgroup of $\Pi_{\mathcal{G}}$ associated to v (respectively, e; e; e; z) in the quotient $\Pi^*_{\mathcal{G}}$ as a *verticial* (respectively, an *edge-like*; a *nodal*; a *cuspidal*; a *VCN-*) *subgroup* of $\Pi^*_{\mathcal{G}}$ associated to v (respectively, e; e; e; z). For each element $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ (respectively, $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$; $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$; $\widetilde{e} \in \mathrm{Cusp}(\widetilde{\mathcal{G}})$; $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$), we shall refer to the image of the verticial (respectively, edge-like; nodal; cuspidal; VCN-) subgroup of $\Pi_{\mathcal{G}}$ associated to $\widetilde{v}$ (respectively, $\widetilde{e}$; $\widetilde{e}$; $\widetilde{e}$; $\widetilde{z}$) in the quotient $\Pi^*_{\mathcal{G}}$ as the *verticial* (respectively, *edge-like*; *nodal*; *cuspidal*; *VCN-*) *subgroup* of $\Pi^*_{\mathcal{G}}$ associated to $\widetilde{v}$ (respectively, $\widetilde{e}$; $\widetilde{e}$; $\widetilde{e}$; $\widetilde{z}$).

(ii) We shall say that an isomorphism $\Pi^*_{\mathcal{G}} \xrightarrow{\sim} \Pi^*_{\mathcal{H}}$ is *group-theoretically verticial* (respectively, *group-theoretically nodal*; *group-theoretically cuspidal*) if the isomorphism induces a bijection between the set of the verticial (respectively, nodal; cuspidal) subgroups [cf. (i)] of $\Pi^*_{\mathcal{G}}$ and the set of the verticial (respectively, nodal; cuspidal) subgroups of $\Pi^*_{\mathcal{H}}$. We shall say that an outer isomorphism $\Pi^*_{\mathcal{G}} \xrightarrow{\sim} \Pi^*_{\mathcal{H}}$ is *group-theoretically verticial* (respectively, *group-theoretically nodal*; *group-theoretically cuspidal*) if the outer isomorphism arises from an isomorphism $\Pi^*_{\mathcal{G}} \xrightarrow{\sim} \Pi^*_{\mathcal{H}}$ which is group-theoretically verticial (respectively, group-theoretically nodal; group-theoretically cuspidal).

(iii) We shall say that an isomorphism $\Pi^*_{\mathcal{G}} \xrightarrow{\sim} \Pi^*_{\mathcal{H}}$ is *group-theoretically graphic* if the isomorphism is group-theoretically verticial, group-theoretically nodal, and group-theoretically cuspidal [cf. (ii)]. We shall say that an outer isomorphism $\Pi^*_{\mathcal{G}} \xrightarrow{\sim} \Pi^*_{\mathcal{H}}$ is *group-theoretically graphic* if the outer isomorphism arises from an isomorphism $\Pi^*_{\mathcal{G}} \xrightarrow{\sim} \Pi^*_{\mathcal{H}}$ which is group-theoretically graphic. We shall write

$$\mathrm{Aut}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}}) \subseteq \mathrm{Aut}(\Pi^*_{\mathcal{G}})$$

for the subgroup of group-theoretically graphic automorphisms of $\Pi^*_{\mathcal{G}}$ and

$$\mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}})/\mathrm{Inn}(\Pi^*_{\mathcal{G}}) \subseteq \mathrm{Out}(\Pi^*_{\mathcal{G}})$$

for the subgroup of group-theoretically graphic outomorphisms of $\Pi^*_{\mathcal{G}}$.

(iv) Let I be a profinite group. Then we shall say that a continuous homomorphism $\rho\colon I \to \mathrm{Aut}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}}) \subseteq \mathrm{Aut}(\Pi^*_{\mathcal{G}})$ [cf. (iii)] is of *VA-type* (respectively, *NN-type*; *PIPSC-type*) if the following condition is satisfied: Let $N \subseteq \Pi_{\mathcal{G}}$ be a normal open subgroup of $\Pi_{\mathcal{G}}$ with respect to which $\Pi^*_{\mathcal{G}}$ is the maximal almost pro-Σ quotient of $\Pi_{\mathcal{G}}$. [Thus, $N^{\Sigma} \subseteq \Pi^*_{\mathcal{G}}$.] Then there exists a characteristic open subgroup $M \subseteq \Pi^*_{\mathcal{G}}$ of $\Pi^*_{\mathcal{G}}$ such that the following conditions are satisfied:

(1) $M \subseteq N^{\Sigma}$. [Thus, M may be regarded as the [pro-Σ] fundamental group of the pro-Σ completion $\mathcal{G}_M^{\Sigma}$ — cf. [SemiAn], Definition 2.9, (ii) — of the connected finite étale Galois subcovering $\mathcal{G}_M \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$ corresponding to $M \subseteq \Pi_{\mathcal{G}}^*$, i.e., $M = \Pi_{\mathcal{G}_M^{\Sigma}}$.]

(2) The composite $I \to \mathrm{Aut}(M) \twoheadrightarrow \mathrm{Out}(M) = \mathrm{Out}(\Pi_{\mathcal{G}_M^{\Sigma}})$, where the first arrow is the homomorphism induced by ρ, is of VA-type (respectively, NN-type; PIPSC-type) in the sense of [NodNon], Definition 2.4, (ii) [cf. also Remark 1.5.1 of the present paper] (respectively, [NodNon], Definition 2.4, (iii); Definition 1.3 of the present paper) [i.e., as an outer representation of *pro-*Σ PSC-type — cf. [NodNon], Definition 2.1, (i)].

[Here, we observe that it follows immediately from Lemma 1.5 that condition (2) is *independent* of the choice of M — cf. Lemma 1.9 below.] We shall say that a continuous homomorphism $\rho\colon I \to \mathrm{Out}^{\mathrm{grph}}(\Pi_{\mathcal{G}}^*) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}}^*)$ [cf. (iii)] is of *VA-type* (respectively, *NN-type*; *PIPSC-type*) if ρ arises from a homomorphism $I \to \mathrm{Aut}^{\mathrm{grph}}(\Pi_{\mathcal{G}}^*) \subseteq \mathrm{Aut}(\Pi_{\mathcal{G}}^*)$ which is of VA-type (respectively, NN-type; PIPSC-type). [Here, we observe that it follows immediately from Lemma 1.5 that this condition on $\rho\colon I \to \mathrm{Out}^{\mathrm{grph}}(\Pi_{\mathcal{G}}^*)$ is *independent* of the choice of the homomorphism $I \to \mathrm{Aut}^{\mathrm{grph}}(\Pi_{\mathcal{G}}^*)$.]

(v) Let $\alpha \in \mathrm{Out}(\Pi_{\mathcal{G}}^*)$. Then we shall say that α is a *profinite Dehn multi-twist* of $\Pi_{\mathcal{G}}^*$ if, for each $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, there exists a lifting $\alpha[\widetilde{v}] \in \mathrm{Aut}(\Pi_{\mathcal{G}}^*)$ of α which preserves the verticial subgroup $\Pi_{\widetilde{v}}^* \subseteq \Pi_{\mathcal{G}}^*$ associated to $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ [cf. (i)] and induces the identity automorphism of $\Pi_{\widetilde{v}}^*$. We shall write

$$\mathrm{Dehn}(\Pi_{\mathcal{G}}^*) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}}^*)$$

for the subgroup of profinite Dehn multi-twists of $\Pi_{\mathcal{G}}^*$.

Remark 1.6.1. In the notation of Definition 1.6, if $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{H}}^*$ are the respective maximal almost pro-Σ quotients of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ with respect to $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$, then it follows immediately from the various definitions involved that $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{H}}^*$ are the respective *maximal pro-*Σ *quotients* of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$. In particular, it follows immediately that one may regard $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{H}}^*$ as the [pro-Σ] fundamental groups of the semi-graphs of anabelioids of pro-Σ PSC-type $\mathcal{G}^{\Sigma}$, $\mathcal{H}^{\Sigma}$ obtained by forming the pro-Σ completions [cf. [SemiAn] Definition 2.9, (ii)] of $\mathcal{G}$, $\mathcal{H}$, respectively, i.e., $\Pi_{\mathcal{G}}^* = \Pi_{\mathcal{G}^{\Sigma}}$, $\Pi_{\mathcal{H}}^* = \Pi_{\mathcal{H}^{\Sigma}}$. Moreover, one verifies immediately that, relative to these identifications, the notions defined in Definition 1.6, (i), (ii), (iii), (iv), are *compatible* with their counterparts defined [for the most part] in earlier papers of the authors:

- VCN-subgroups [cf. [CbTpI], Definition 2.1, (i)];
- group-theoretically verticial/nodal/cuspidal/graphic (outer) isomorphisms [cf. [CmbGC], Definition 1.4, (i), (iv); [NodNon], Definition 1.12];
- outer representations of VA-/NN-/PIPSC-type [cf. [NodNon], Definition 2.4, (ii), (iii); Remark 1.5.1 of the present paper; Definition 1.3 of the present paper; Lemma 1.5 of the present paper];
- profinite Dehn multi-twists [cf. [CbTpI], Definition 4.4], i.e., so $\mathrm{Dehn}(\mathcal{G}^{\Sigma}) = \mathrm{Dehn}(\Pi^*_{\mathcal{G}}) \subseteq \mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}})$.

Remark 1.6.2. In the situation of Definition 1.6, (iv), it follows immediately from Lemma 1.5, together with [NodNon], Remark 2.4.2, that we have implications

$$\text{PIPSC-type} \quad \Longrightarrow \quad \text{NN-type} \quad \Longrightarrow \quad \text{VA-type}\,.$$

Proposition 1.7 (Properties of VCN-subgroups). *Let* $\Pi^*_{\mathcal{G}}$ *be a* **maximal almost pro-Σ quotient** *of* $\Pi_{\mathcal{G}}$ *[cf. Definition 1.1]. For* $\widetilde{v}$, $\widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$*;* $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$*, write* $\mathcal{G}^* \to \mathcal{G}$ *for the connected profinite étale subcovering of* $\widetilde{\mathcal{G}} \to \mathcal{G}$ *corresponding to* $\Pi^*_{\mathcal{G}}$*;*

$$\mathrm{Vert}(\mathcal{G}^*) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{Vert}(\mathcal{G}'),\ \ \mathrm{Edge}(\mathcal{G}^*) \stackrel{\mathrm{def}}{=} \varprojlim \mathrm{Edge}(\mathcal{G}')$$

— where the projective limits range over all connected finite étale subcoverings $\mathcal{G}' \to \mathcal{G}$ *of* $\mathcal{G}^* \to \mathcal{G}$*;*

$$\widetilde{v}(\mathcal{G}^*) \in \mathrm{Vert}(\mathcal{G}^*),\ \ \widetilde{e}(\mathcal{G}^*) \in \mathrm{Edge}(\mathcal{G}^*)$$

for the images of $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ *via the natural maps* $\mathrm{Vert}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{Vert}(\mathcal{G}^*)$, $\mathrm{Edge}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{Edge}(\mathcal{G}^*)$*, respectively;*

$$\mathcal{E}_{\mathcal{G}^*} \colon \mathrm{Vert}(\mathcal{G}^*) \longrightarrow 2^{\mathrm{Edge}(\mathcal{G}^*)}$$

[cf. the discussion entitled "Sets" in [CbTpI]*, §0, concerning the notation* $2^{\mathrm{Edge}(\mathcal{G}^*)}$*] for the map induced by the various* $\mathcal{E}$*'s involved [cf.* [NodNon]*, Definition 1.1, (iv)];*

$$\delta(\widetilde{v}(\mathcal{G}^*), \widetilde{w}(\mathcal{G}^*)) \stackrel{\mathrm{def}}{=} \sup_{\mathcal{G}'} \{\delta(\widetilde{v}(\mathcal{G}'), \widetilde{w}(\mathcal{G}'))\} \in \mathbb{N} \cup \{\infty\}$$

[cf. [NodNon]*, Definition 1.1, (vii)] — where* $\mathcal{G}'$ *ranges over the connected finite étale subcoverings* $\mathcal{G}' \to \mathcal{G}$ *of* $\mathcal{G}^* \to \mathcal{G}$*. Then the following hold:*

(i) $\Pi^*_{\mathcal{G}}$ *is* **topologically finitely generated**, **slim** *[cf. the discussion entitled "Topological groups" in* [CbTpI], *§0], and* **almost torsion-free** *[cf. the discussion entitled "Topological groups" in* [CbTpI], *§0]. In particular, every VCN-subgroup of* $\Pi^*_{\mathcal{G}}$ *[cf. Definition 1.6, (i)] is* **almost torsion-free**.

(ii) *Let* $z \in \mathrm{VCN}(\mathcal{G})$ *and* $\Pi_z \subseteq \Pi_{\mathcal{G}}$ *a VCN-subgroup of* $\Pi_{\mathcal{G}}$ *associated to* $z \in \mathrm{VCN}(\mathcal{G})$. *Write* $\Pi^*_z \subseteq \Pi^*_{\mathcal{G}}$ *for the VCN-subgroup of* $\Pi^*_{\mathcal{G}}$ *obtained by forming the image of* $\Pi_z \subseteq \Pi_{\mathcal{G}}$ *in* $\Pi^*_{\mathcal{G}}$. *Then* Π^*_z *is a* **maximal almost pro-Σ quotient** *of* Π_z. *In particular, every verticial subgroup of* $\Pi^*_{\mathcal{G}}$ *is* **topologically finitely generated** *and* **slim**.

(iii) *For* $i = 1, 2$, *let* $\widetilde{v}_i \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. *Write* $\Pi^*_{\widetilde{v}_i} \subseteq \Pi^*_{\mathcal{G}}$ *for the verticial subgroup of* $\Pi^*_{\mathcal{G}}$ *associated to* $\widetilde{v}_i$. *Consider the following three [mutually exclusive] conditions:*

(1) $\delta(\widetilde{v}_1(\mathcal{G}^*), \widetilde{v}_2(\mathcal{G}^*)) = 0$.

(2) $\delta(\widetilde{v}_1(\mathcal{G}^*), \widetilde{v}_2(\mathcal{G}^*)) = 1$.

(3) $\delta(\widetilde{v}_1(\mathcal{G}^*), \widetilde{v}_2(\mathcal{G}^*)) \geq 2$.

Then we have equivalences

$$(1) \iff (1') \ ; \ (2) \iff (2') \ ; \ (3) \iff (3')$$

with the following three conditions:

(1′) $\Pi^*_{\widetilde{v}_1} = \Pi^*_{\widetilde{v}_2}$.

(2′) $\Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2}$ *is* **infinite**, *but* $\Pi^*_{\widetilde{v}_1} \neq \Pi^*_{\widetilde{v}_2}$.

(3′) $\Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2}$ *is* **finite**.

(iv) *In the situation of (iii), if condition (2), hence also condition (2′), holds, then it holds that* $(\mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_1(\mathcal{G}^*)) \cap \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_2(\mathcal{G}^*)))^\sharp = 1$, *and, moreover,* $\Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2} = \Pi^*_{\widetilde{e}}$, *for any element* $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ *such that* $\widetilde{e}(\mathcal{G}^*) \in \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_1(\mathcal{G}^*)) \cap \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_2(\mathcal{G}^*))$.

(v) *For* $i = 1, 2$, *let* $\widetilde{e}_i \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. *Write* $\Pi^*_{\widetilde{e}_i} \subseteq \Pi^*_{\mathcal{G}}$ *for the edge-like subgroup of* $\Pi^*_{\mathcal{G}}$ *associated to* $\widetilde{e}_i$. *Then* $\Pi^*_{\widetilde{e}_1} \cap \Pi^*_{\widetilde{e}_2}$ *is* **infinite** *if and only if* $\widetilde{e}_1(\mathcal{G}^*) = \widetilde{e}_2(\mathcal{G}^*)$. *In particular,* $\Pi^*_{\widetilde{e}_1} \cap \Pi^*_{\widetilde{e}_2}$ *is* **infinite** *if and only if* $\Pi^*_{\widetilde{e}_1} = \Pi^*_{\widetilde{e}_2}$.

(vi) *Let* $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. *Write* $\Pi^*_{\widetilde{v}}, \Pi^*_{\widetilde{e}} \subseteq \Pi^*_{\mathcal{G}}$ *for the VCN-subgroups of* $\Pi^*_{\mathcal{G}}$ *associated to* $\widetilde{v}$, $\widetilde{e}$, *respectively. Then* $\Pi^*_{\widetilde{e}} \cap \Pi^*_{\widetilde{v}}$ *is* **infinite** *if and only if* $\widetilde{e}(\mathcal{G}^*) \in \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}(\mathcal{G}^*))$. *In particular,* $\Pi^*_{\widetilde{e}} \cap \Pi^*_{\widetilde{v}}$ *is* **infinite** *if and only if* $\Pi^*_{\widetilde{e}} \subseteq \Pi^*_{\widetilde{v}}$.

(vii) *Every VCN-subgroup of* $\Pi^*_{\mathcal{G}}$ *is* **commensurably terminal** *[cf. the discussion entitled "Topological groups" in* [CbTpI], *§0] in* $\Pi^*_{\mathcal{G}}$.

(viii) *Let $z \in \mathrm{VCN}(\mathcal{G})$, $\Pi_z \subseteq \Pi_{\mathcal{G}}$ a VCN-subgroup of $\Pi_{\mathcal{G}}$ associated to $z \in \mathrm{VCN}(\mathcal{G})$, and $\Pi_z \twoheadrightarrow \Pi_z^{\ddagger}$ an* **almost pro-Σ quotient** *of $\Pi_z^{\ddagger}$. Then there exists a* **maximal almost pro-Σ quotient** *$\Pi_{\mathcal{G}}^{**}$ of $\Pi_{\mathcal{G}}$ such that the quotient of Π_z determined by the quotient $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{**}$* **dominates** *the quotient $\Pi_z \twoheadrightarrow \Pi_z^{\ddagger}$ [cf. the discussion entitled "Topological groups" in §0].*

Proof. Let $N \subseteq \Pi_{\mathcal{G}}$ be a normal open subgroup of $\Pi_{\mathcal{G}}$ with respect to which $\Pi_{\mathcal{G}}^{*}$ is the maximal almost pro-Σ quotient of $\Pi_{\mathcal{G}}$. [Thus, $N^{\Sigma} \subseteq \Pi_{\mathcal{G}}^{*}$.] Write $\mathcal{G}_N \to \mathcal{G}$ for the connected finite étale Galois subcovering of $\widetilde{\mathcal{G}} \to \mathcal{G}$ corresponding to $N \subseteq \Pi_{\mathcal{G}}$. Thus, $N^{\Sigma} \subseteq \Pi_{\mathcal{G}}^{*}$ may be regarded as the [pro-Σ] fundamental group of the pro-Σ completion $\mathcal{G}_N^{\Sigma}$ [cf. [SemiAn], Definition 2.9, (ii)] of $\mathcal{G}_N$, i.e., $N^{\Sigma} = \Pi_{\mathcal{G}_N^{\Sigma}}$.

First, we verify assertion (i). Since $\Pi_{\mathcal{G}}$ is *topologically finitely generated*, it is immediate that $\Pi_{\mathcal{G}}^{*}$ is *topologically finitely generated*. Now let us recall [cf. [CmbGC], Remark 1.1.3] that $N^{\Sigma} = \Pi_{\mathcal{G}_N^{\Sigma}}$ is *torsion-free* and *slim*. Thus, the fact that $\Pi_{\mathcal{G}}^{*}$ is *almost torsion-free* is immediate; the *slimness* of $\Pi_{\mathcal{G}}^{*}$ follows immediately, by considering the natural outer action $\Pi_{\mathcal{G}}/N \to \mathrm{Out}(N^{\Sigma})$, from the well-known fact that any *nontrivial* automorphism of a stable log curve over an algebraically closed field of characteristic $\notin \Sigma$ induces a *nontrivial* outomorphism of the maximal pro-Σ quotient of the geometric log fundamental group of the stable curve [cf. [CmbGC], Proposition 1.2, (i), (ii); [MzTa], Proposition 1.4, applied to the *verticial subgroups* of the geometric log fundamental group under consideration]. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us recall that since $\Pi_z \cap N \subseteq N = \Pi_{\mathcal{G}_N}$ is a VCN-subgroup of $\Pi_{\mathcal{G}_N}$, the natural homomorphism $(\Pi_z \cap N)^{\Sigma} \to \Pi_{\mathcal{G}_N}^{\Sigma}$ is *injective* [cf., e.g., the proof of [SemiAn], Proposition 2.5, (i); [SemiAn], Example 2.10]. Thus, it follows immediately from Lemma 1.2, (ii), that Π_z^{*} is a *maximal almost pro-Σ quotient* of Π_z. In particular, if $z \in \mathrm{Vert}(\mathcal{G})$, then it follows immediately from assertion (i) that Π_z^{*} is *topologically finitely generated* and *slim*. This completes the proof of assertion (ii).

Next, we verify assertions (iii), (v), and (vi). Since $N^{\Sigma} = \Pi_{\mathcal{G}_N^{\Sigma}}$, by considering the intersections of $N^{\Sigma} = \Pi_{\mathcal{G}_N^{\Sigma}}$ with the various VCN-subgroups of $\Pi_{\mathcal{G}}^{*}$ under consideration, one verifies easily, by applying [NodNon], Lemma 1.9, (ii) (respectively, [NodNon], Lemma 1.5; [NodNon], Lemma 1.7), together with the well-known fact that every VCN-subgroup of $\Pi_{\mathcal{G}_N^{\Sigma}}$ is *nontrivial* and *torsion-free* [hence also *infinite*], that assertion (iii) (respectively, (v); (vi)) holds. This completes the proof of assertions (iii), (v), and (vi). Assertion (vii) follows formally from assertions (iii), (v). Indeed, let $\Pi_{\widetilde{z}}^{*} \subseteq \Pi_{\mathcal{G}}^{*}$ be the VCN-subgroup of $\Pi_{\mathcal{G}}^{*}$ associated to an element $\widetilde{z} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ and $\gamma \in C_{\Pi_{\mathcal{G}}^{*}}(\Pi_{\widetilde{z}}^{*})$.

Then it follows immediately from assertions (iii), (v) that $\widetilde{z} = \widetilde{z}^{\gamma}$; we thus conclude that $\gamma \in \Pi^*_{\widetilde{z}}$. This completes the proof of assertion (vii).

Next, we verify assertion (iv). By applying [NodNon], Lemma 1.8, to $\mathcal{G}^{\Sigma}_N$, one verifies immediately that $(\mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_1(\mathcal{G}^*)) \cap \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_2(\mathcal{G}^*)))^{\sharp} = 1$. Thus, it follows immediately from assertion (vi); [NodNon], Lemma 1.9, (ii), that $\Pi^*_{\widetilde{e}} \cap N^{\Sigma} = \Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2} \cap N^{\Sigma}$. Since N^{Σ} is *open* in $\Pi^*_{\mathcal{G}}$, we conclude from assertion (vi) that $\Pi^*_{\widetilde{e}}$ is an *open subgroup* of $\Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2}$, hence that $\Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2} \subseteq C_{\Pi^*_{\mathcal{G}}}(\Pi^*_{\widetilde{e}}) = \Pi^*_{\widetilde{e}}$ [cf. assertion (vii)]. This completes the proof of assertion (iv).

Finally, we verify assertion (viii). It follows from the definition of an almost pro-Σ quotient that the natural surjection $\Pi_z \twoheadrightarrow \Pi^{\ddagger}_z$ factors through a maximal almost pro-Σ quotient of Π_z. Thus, by replacing $\Pi^{\ddagger}_z$ by a suitable maximal almost pro-Σ quotient of Π_z, we may assume without loss of generality that $\Pi^{\ddagger}_z$ is a *maximal almost pro-*Σ *quotient* of Π_z. Let $N_z \subseteq \Pi_z$ be a normal open subgroup of Π_z with respect to which $\Pi^{\ddagger}_z$ is the maximal almost pro-Σ quotient of Π_z and $N_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}$ a normal open subgroup of $\Pi_{\mathcal{G}}$ such that $N_{\mathcal{G}} \cap \Pi_z \subseteq N_z$. Here, we recall that the existence of such a subgroup $N_{\mathcal{G}}$ follows immediately from the fact that the natural profinite topology on Π_z coincides with the topology on Π_z induced by the topology of $\Pi_{\mathcal{G}}$. Then one verifies easily that the maximal almost pro-Σ quotient of $\Pi_{\mathcal{G}}$ with respect to $N_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}$ is a maximal almost pro-Σ quotient of $\Pi_{\mathcal{G}}$ as in the statement of assertion (viii). This completes the proof of assertion (viii). □

Definition 1.8. Let $\Pi^*_{\mathcal{G}}$ be a *maximal almost pro-*Σ *quotient* of $\Pi_{\mathcal{G}}$ [cf. Definition 1.1]. Then we shall write

$$\mathrm{Aut}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}}) \subseteq \mathrm{Aut}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}})$$

for the subgroup of group-theoretically graphic [cf. Definition 1.6, (iii)] automorphisms α of $\Pi^*_{\mathcal{G}}$ such that the natural action of α on the underlying semi-graph $\mathbb{G}$ [determined by the group-theoretic graphicity of α, together with Proposition 1.7, (iii), (v), (vi)] is the identity automorphism. Also, we shall write

$$\mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}})/\mathrm{Inn}(\Pi^*_{\mathcal{G}}) \subseteq \mathrm{Out}(\Pi^*_{\mathcal{G}})\,.$$

for the image of $\mathrm{Aut}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}})$ in $\mathrm{Out}(\Pi^*_{\mathcal{G}})$.

Remark 1.8.1. In the notation of Definition 1.8, one verifies easily that

$$\mathrm{Dehn}(\Pi^*_{\mathcal{G}}) \subseteq \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}})$$

[cf. Definitions 1.6, (v); 1.8; [CmbGC], Proposition 1.2, (i)].

Remark 1.8.2. In the spirit of Remark 1.6.1, one verifies immediately that the notation of Definition 1.8 is *consistent* with the the notation of [CbTpI], Definition 2.6, (i) [cf. also [CbTpII], Remark 4.1.2].

Lemma 1.9 (Alternative characterization of outer representations of VA-, NN-, PIPSC-type). *Let* $\Pi^*_{\mathcal{G}}$ *be a* **maximal almost pro-Σ quotient** *of* $\Pi_{\mathcal{G}}$ *[cf. Definition 1.1], I a profinite group, and* $\rho\colon I \to \mathrm{Aut}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}})$ *a continuous homomorphism. Then the following conditions are equivalent:*

(i) *ρ is of* **VA-type** *(respectively,* **NN-type***;* **PIPSC-type***) [cf. Definition 1.6, (iv)].*

(ii) *Let $N \subseteq \Pi_{\mathcal{G}}$ be a normal open subgroup of $\Pi_{\mathcal{G}}$ with respect to which $\Pi^*_{\mathcal{G}}$ is the maximal almost pro-Σ quotient of $\Pi_{\mathcal{G}}$. [Thus, $N^{\Sigma} \subseteq \Pi^*_{\mathcal{G}}$.] Let $M \subseteq \Pi^*_{\mathcal{G}}$ be a characteristic open subgroup of $\Pi^*_{\mathcal{G}}$ such that $M \subseteq N^{\Sigma}$. [Thus, M may be regarded as the [pro-Σ] fundamental group of the pro-Σ completion $\mathcal{G}^{\Sigma}_{M}$ — cf.* [SemiAn]*, Definition 2.9, (ii) — of the connected finite étale Galois subcovering $\mathcal{G}_M \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$ corresponding to $M \subseteq \Pi^*_{\mathcal{G}}$, i.e., $M = \Pi_{\mathcal{G}^{\Sigma}_{M}}$.] Then it holds that the composite of the resulting homomorphism $I \to \mathrm{Aut}(M) = \mathrm{Aut}(\Pi_{\mathcal{G}^{\Sigma}_{M}})$ with the natural projection $\mathrm{Aut}(\Pi_{\mathcal{G}^{\Sigma}_{M}}) \to \mathrm{Out}(\Pi_{\mathcal{G}^{\Sigma}_{M}})$ is an outer representation of* **VA-type** *(respectively,* **NN-type***;* **PIPSC-type***) in the sense of* [NodNon]*, Definition 2.4, (ii) [cf. also Remark 1.5.1 of the present paper] (respectively,* [NodNon]*, Definition 2.4, (iii); Definition 1.3 of the present paper).*

Proof. The implication (ii) ⇒ (i) is immediate; the implication (i) ⇒ (ii) follows immediately from Lemma 1.5. This completes the proof of Lemma 1.9. ☐

Lemma 1.10 (Automorphisms of semi-graphs of anabelioids of PSC-type with prescribed underlying semi-graphs). *Let* $\Pi^*_{\mathcal{G}}$ *be a* **maximal almost pro-Σ quotient** *of* $\Pi_{\mathcal{G}}$ *[cf. Definition 1.1] and $\alpha \in \mathrm{Out}(\Pi^*_{\mathcal{G}})$. Suppose that there exist distinct elements v_1, v_2, $v_3 \in \mathrm{Vert}(\mathcal{G})$; e_1, $e_2 \in \mathrm{Node}(\mathcal{G})$ such that $\mathrm{Vert}(\mathcal{G}) = \{v_1, v_2, v_3\}$; $\mathrm{Node}(\mathcal{G}) = \{e_1, e_2\}$; $\mathcal{V}(e_i) = \{v_i, v_{i+1}\}$ [where $i \in \{1, 2\}$]. For each $i \in \{1, 2\}$, write $\Pi^*_{\mathcal{G}_{\rightsquigarrow\{e_i\}}}$ for the maximal almost pro-Σ quotient of $\Pi_{\mathcal{G}_{\rightsquigarrow\{e_i\}}}$ [cf.* [CbTpI]*, Definition 2.8] determined by the natural outer isomorphism $\Phi_{\mathcal{G}_{\rightsquigarrow\{e_i\}}}\colon \Pi_{\mathcal{G}_{\rightsquigarrow\{e_i\}}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$ [cf.* [CbTpI]*, Definition 2.10] and the maximal almost pro-Σ quotient $\Pi^*_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$; $\Phi^*_{\mathcal{G}_{\rightsquigarrow\{e_i\}}}\colon \Pi^*_{\mathcal{G}_{\rightsquigarrow\{e_i\}}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}}$ for the outer isomorphism determined by $\Phi_{\mathcal{G}_{\rightsquigarrow\{e_i\}}}$. Suppose, moreover,* *that, for each* $i \in \{1,2\}$*, the outomorphism of* $\Pi^*_{\mathcal{G}\rightsquigarrow\{e_i\}}$ *obtained by conjugating* α *by* $\Phi^*_{\mathcal{G}\rightsquigarrow\{e_i\}}$ *is a* **profinite Dehn multi-twist** *of* $\Pi^*_{\mathcal{G}\rightsquigarrow\{e_i\}}$ *[cf. Definition 1.6, (v)]. Then* α *is the* **identity outomorphism***.*

Proof. First, let us observe that it follows immediately from the definition of a profinite Dehn multi-twist that α is a *profinite Dehn multi-twist* of $\Pi^*_{\mathcal{G}}$. Let us fix a verticial subgroup $\Pi_{v_2} \subseteq \Pi_{\mathcal{G}}$ associated to v_2. Let $\Pi_{e_1}, \Pi_{e_2} \subseteq \Pi_{\mathcal{G}}$ be nodal subgroups associated to e_1, e_2, respectively, which are contained in Π_{v_2}; $\Pi_{v_1} \subseteq \Pi_{\mathcal{G}}$ a verticial subgroup associated to v_1 which contains Π_{e_1}; $\Pi_{v_3} \subseteq \Pi_{\mathcal{G}}$ a verticial subgroup associated to v_3 which contains Π_{e_2}. Thus, we have inclusions

$$\Pi_{v_1} \supseteq \Pi_{e_1} \subseteq \Pi_{v_2} \supseteq \Pi_{e_2} \subseteq \Pi_{v_3} .$$

For each $z \in \{v_1, v_2, v_3, e_1, e_2\}$, write $\Pi^*_z \subseteq \Pi^*_{\mathcal{G}}$ for the VCN-subgroup of $\Pi^*_{\mathcal{G}}$ associated to z obtained by forming the image of $\Pi_z \subseteq \Pi_{\mathcal{G}}$ in $\Pi^*_{\mathcal{G}}$. Then since α is a *profinite Dehn multi-twist*, there exists a lifting $\alpha[v_2] \in \mathrm{Aut}(\Pi^*_{\mathcal{G}})$ of α which preserves and induces the *identity automorphism* of $\Pi^*_{v_2}$; in particular, $\alpha[v_2]$ preserves and induces the *identity automorphisms* of $\Pi^*_{e_1}$, $\Pi^*_{e_2}$. Moreover, by applying a similar argument to the argument given in the proof of [CbTpI], Lemma 4.6, (i), where we replace [CmbGC], Remark 1.1.3 (respectively, [CmbGC], Proposition 1.2, (ii); [CbTpI], Proposition 4.5; [NodNon], Lemma 1.7), in the proof of [CbTpI], Lemma 4.6, (i), by Proposition 1.7, (ii) (respectively, Proposition 1.7, (vii); Remark 1.8.1; Proposition 1.7, (vi)), we conclude that $\alpha[v_2](\Pi^*_{v_1}) = \Pi^*_{v_1}$, $\alpha[v_2](\Pi^*_{v_3}) = \Pi^*_{v_3}$, and, moreover, that there exist *unique* elements $\gamma_1 \in \Pi^*_{e_1}$, $\gamma_2 \in \Pi^*_{e_2}$ such that the restrictions of $\alpha[v_2]$ to $\Pi^*_{v_1}$, $\Pi^*_{v_3}$ are the *inner automorphisms determined by* γ_1, γ_2, respectively. Thus, since, for each $i \in \{1,2\}$, the outomorphism of $\Pi^*_{\mathcal{G}\rightsquigarrow\{e_i\}}$ obtained by conjugating α by $\Phi^*_{\mathcal{G}\rightsquigarrow\{e_i\}}$ is a *profinite Dehn multi-twist* of $\Pi^*_{\mathcal{G}\rightsquigarrow\{e_i\}}$, one verifies easily — by considering the restriction of this outomorphism of $\Pi^*_{\mathcal{G}\rightsquigarrow\{e_i\}}$ to the *unique* conjugacy class of verticial subgroups of $\Pi^*_{\mathcal{G}\rightsquigarrow\{e_i\}}$ that does *not* arise from a conjugacy class of verticial subgroups of $\Pi^*_{\mathcal{G}}$ [cf. also Proposition 1.7, (ii), (vii)] — that γ_1 and γ_2 are *trivial*. On the other hand, it follows immediately from a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.5, (iii), that $\Pi_{\mathcal{G}}$ is topologically generated by Π_{v_1}, Π_{v_2}, and Π_{v_3}; in particular, $\Pi^*_{\mathcal{G}}$ is topologically generated by $\Pi^*_{v_1}$, $\Pi^*_{v_2}$, and $\Pi^*_{v_3}$. Thus, we conclude that $\alpha[v_2]$ is the *identity automorphism* of $\Pi^*_{\mathcal{G}}$. This completes the proof of Lemma 1.10. □

Theorem 1.11 (**Group-theoretic verticiality/nodality of isomorphisms of outer representations of NN-, PIPSC-type**)**.** *Let* $\Sigma \subseteq \Sigma^\dagger$ *be nonempty sets of prime numbers,* $\mathcal{G}$ *(respectively,* $\mathcal{H}$*) a semi-graph of anabelioids of pro-*$\Sigma^\dagger$ *PSC-type,* $\Pi_{\mathcal{G}}$ *(respectively,* $\Pi_{\mathcal{H}}$*)*

the [pro-$\Sigma^\dagger$*] fundamental group of* $\mathcal{G}$ *(respectively,* $\mathcal{H}$*),* $\Pi^*_{\mathcal{G}}$ *(respectively,* $\Pi^*_{\mathcal{H}}$*) a* **maximal almost pro-Σ quotient** *[cf. Definition 1.1] of* $\Pi_{\mathcal{G}}$ *(respectively,* $\Pi_{\mathcal{H}}$*),* $\alpha\colon \Pi^*_{\mathcal{G}} \xrightarrow{\sim} \Pi^*_{\mathcal{H}}$ *an isomorphism of profinite groups,* I *(respectively,* J*) a profinite group,* $\rho_I\colon I \to \mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}})$ *(respectively,* $\rho_J\colon J \to \mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{H}})$*) [cf. Definition 1.6, (iii)] a continuous homomorphism, and* $\beta\colon I \xrightarrow{\sim} J$ *an isomorphism of profinite groups. Suppose that the diagram*

$$\begin{array}{ccc} I & \xrightarrow{\rho_I} & \mathrm{Out}(\Pi^*_{\mathcal{G}}) \\ \beta\downarrow & & \downarrow\mathrm{Out}(\alpha) \\ J & \xrightarrow{\rho_J} & \mathrm{Out}(\Pi^*_{\mathcal{H}}) \end{array}$$

— where the right-hand vertical arrow is the isomorphism induced by α *—* **commutes***. Then the following hold:*

(i) *Suppose, moreover, that* ρ_I*,* ρ_J *are of* **NN-type** *[cf. Definition 1.6, (iv)]. Then the following three conditions are equivalent:*

  (1) *The isomorphism* α *is* **group-theoretically verticial** *[i.e., roughly speaking, preserves verticial subgroups — cf. Definition 1.6, (ii)].*

  (2) *The isomorphism* α *is* **group-theoretically nodal** *[i.e., roughly speaking, preserves nodal subgroups — cf. Definition 1.6, (ii)].*

  (3) *There exists an* **infinite** *subgroup* $H \subseteq \Pi^*_{\mathcal{G}}$ *of* $\Pi^*_{\mathcal{G}}$ *such that* $H \subseteq \Pi^*_{\mathcal{G}}$*,* $\alpha(H) \subseteq \Pi^*_{\mathcal{H}}$ *are* **contained** *in* **verticial subgroups** *of* $\Pi^*_{\mathcal{G}}$*,* $\Pi^*_{\mathcal{H}}$*, respectively [cf. Definition 1.6, (i)].*

(ii) *Suppose, moreover, that* ρ_I *is of* **NN-type***, and that* ρ_J *is of* **PIPSC-type** *[cf. Definition 1.6, (iv)]. [For example, this will be the case if both* ρ_I *and* ρ_J *are of* **PIPSC-type** *— cf. Remark 1.6.2.] Then* α *is* **group-theoretically verticial***, hence also* **group-theoretically nodal***.*

Proof. The implication (1) ⇒ (2) of assertion (i) and the final portion of assertion (ii) [i.e., the portion concerning *group-theoretic nodality*] follow immediately from Proposition 1.7, (iv). The implication (2) ⇒ (3) of assertion (i) is immediate. Finally, we verify assertion (ii) (respectively, the implication (3) ⇒ (1) of assertion (i)). Suppose that ρ_I, ρ_J are as in assertion (ii) (respectively, condition (3) of assertion (i)). Let $N_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}$, $N_{\mathcal{H}} \subseteq \Pi_{\mathcal{H}}$ be normal open subgroups of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ with respect to which $\Pi^*_{\mathcal{G}}$, $\Pi^*_{\mathcal{H}}$ are the *maximal almost pro-*Σ *quotients* of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$, respectively. [Thus, $N^{\Sigma}_{\mathcal{G}} \subseteq \Pi^*_{\mathcal{G}}$, $N^{\Sigma}_{\mathcal{H}} \subseteq \Pi^*_{\mathcal{H}}$.] Now it follows immediately from the fact that $\Pi^*_{\mathcal{G}}$, $\Pi^*_{\mathcal{H}}$ are *topologically finitely generated* [cf. Proposition 1.7, (i)] that there exists a characteristic open subgroup $M_{\mathcal{G}} \subseteq \Pi^*_{\mathcal{G}}$ of $\Pi^*_{\mathcal{G}}$ such that $M_{\mathcal{G}} \subseteq N^{\Sigma}_{\mathcal{G}}$, $M_{\mathcal{H}} \stackrel{\mathrm{def}}{=} \alpha(M_{\mathcal{G}}) \subseteq N^{\Sigma}_{\mathcal{H}}$. Thus, it follows immediately, in light of Lemma 1.9, from [CbTpII], Theorem 1.9, (ii) (respectively, the implication (3) ⇒ (1) of [CbTpII], Theorem 1.9, (i)), together with Proposition 1.7, (vii), that α is *group-theoretically verticial*. This completes the proof of Theorem 1.11. □

Corollary 1.12 (**Group-theoretic graphicity of group-theoretically cuspidal isomorphisms of outer representations of NN-, PIPSC-type).** *Let $\Sigma \subseteq \Sigma^\dagger$ be nonempty sets of prime numbers, $\mathcal{G}$ (respectively, $\mathcal{H}$) a semi-graph of anabelioids of pro-$\Sigma^\dagger$ PSC-type, $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$) the [pro-$\Sigma^\dagger$] fundamental group of $\mathcal{G}$ (respectively, $\mathcal{H}$), $\Pi^*_{\mathcal{G}}$ (respectively, $\Pi^*_{\mathcal{H}}$) a* **maximal almost pro-Σ quotient** *[cf. Definition 1.1] of $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$), $\alpha\colon \Pi^*_{\mathcal{G}} \xrightarrow{\sim} \Pi^*_{\mathcal{H}}$ an isomorphism of profinite groups, I (respectively, J) a profinite group, $\rho_I\colon I \to \mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}})$ (respectively, $\rho_J\colon J \to \mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{H}})$) [cf. Definition 1.6, (iii)] a continuous homomorphism, and $\beta\colon I \xrightarrow{\sim} J$ an isomorphism of profinite groups. Suppose that the following conditions are satisfied:*

(i) *The diagram*

$$\begin{array}{ccc} I & \xrightarrow{\rho_I} & \mathrm{Out}(\Pi^*_{\mathcal{G}}) \\ \beta \downarrow & & \downarrow \mathrm{Out}(\alpha) \\ J & \xrightarrow{\rho_J} & \mathrm{Out}(\Pi^*_{\mathcal{H}}) \end{array}$$

— where the right-hand vertical arrow is the isomorphism induced by α — **commutes**.

(ii) *α is* **group-theoretically cuspidal** *[cf. Definition 1.6, (ii)].*

(iii) *ρ_I, ρ_J are of* **NN-type** *[cf. Definition 1.6, (iv)].*

Suppose, moreover, that one of the following conditions is satisfied:

(1) $\mathrm{Cusp}(\mathcal{G}) \neq \emptyset$.

(2) *Either ρ_I or ρ_J is of* **PIPSC-type** *[cf. Definition 1.6, (iv)].*

Then α is **group-theoretically graphic** *[cf. Definition 1.6, (iii)].*

Proof. This follows immediately from Theorem 1.11, (i) (respectively, (ii)), whenever condition (1) (respectively, (2)) is satisfied. □

2. Almost pro-Σ injectivity

In the present §2, we develop an *almost pro-Σ version* of the *injectivity portion* of the theory of combinatorial cuspidalization [cf. Theorem 2.9, Corollary 2.10 below]. We also discuss an *almost pro-l analogue* [cf. Corollary 2.13 below] of the *tripod homomorphism* of [CbTpII], Definition 3.19.

In the present §2, let Σ be a nonempty set of prime numbers.

Definition 2.1. Let l be a prime number; n a positive integer; (g, r) a pair of nonnegative integers such that $2g-2+r > 0$; k an algebraically closed field of characteristic zero; $(\mathrm{Spec}\, k)^{\log}$ the log scheme obtained by equipping $\mathrm{Spec}\, k$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$; $X^{\log}$ a *stable log curve* of type (g, r) over $(\mathrm{Spec}\, k)^{\log}$. For each positive integer i, write $X_i^{\log}$ for the i-th *log configuration space* of $X^{\log}$ [cf. the discussion entitled "*Curves*" in [CbTpI], §0]; Π_i for the pro-$\mathfrak{Primes}$ configuration space group [cf. [MzTa], Definition 2.3, (i)] given by the kernel of the natural surjection $\pi_1(X_i^{\log}) \twoheadrightarrow \pi_1((\mathrm{Spec}\, k)^{\log})$. Let $\Pi_n \twoheadrightarrow \Pi_n^*$ be a *quotient* of Π_n. Write

$$\{1\} = \Pi_{n/n} \subseteq \Pi_{n/n-1} \subseteq \cdots \subseteq \Pi_{n/m} \subseteq \cdots \subseteq \Pi_{n/2} \subseteq \Pi_{n/1} \subseteq \Pi_{n/0} = \Pi_n$$

for the *standard fiber filtration* on Π_n — i.e., $\Pi_{n/m} \subseteq \Pi_n$ is the kernel of the surjection $p_{n/m}^{\Pi} \colon \Pi_n \twoheadrightarrow \Pi_m$ induced by the projection $p_{n/m}^{\log} \colon X_n^{\log} \to X_m^{\log}$ obtained by forgetting the factors labeled by indices $> m$ [cf. [CmbCsp], Definition 1.1, (i)];

$$\{1\} = \Pi^*_{n/n} \subseteq \Pi^*_{n/n-1} \subseteq \cdots \subseteq \Pi^*_{n/m} \subseteq \cdots \subseteq \Pi^*_{n/2} \subseteq \Pi^*_{n/1} \subseteq \Pi^*_{n/0} = \Pi^*_n$$

for the induced filtration on Π_n^*.

(i) For each $1 \leq m \leq n$, we shall refer to the subquotient $\Pi^*_{n/m-1}/\Pi^*_{n/m}$ of Π_n^* as a *standard-adjacent subquotient* of Π_n^*.

(ii) We shall say that Π_n^* is an *SA-maximal almost pro-l quotient* of Π_n [where the "SA" stands for "standard-adjacent"] if, for every $1 \leq m \leq n$, the natural quotient $\Pi_{n/m-1}/\Pi_{n/m} \twoheadrightarrow \Pi^*_{n/m-1}/\Pi^*_{n/m}$ is a maximal almost pro-l quotient of $\Pi_{n/m-1}/\Pi_{n/m}$ [cf. Definition 1.1].

(iii) We shall say that Π_n^* is *F-characteristic* if every F-admissible automorphism [cf. [CmbCsp], Definition 1.1, (ii)] of Π_n preserves the kernel of the quotient $\Pi_n \twoheadrightarrow \Pi_n^*$.

(iv) We shall refer to the image of a fiber subgroup [cf. [MzTa], Definition 2.3, (iii)] of Π_n in Π_n^* as a *fiber subgroup* of Π_n^*. For each $1 \leq m \leq n$, we shall refer to the image of a cuspidal inertia subgroup of $\Pi_{n/m-1}/\Pi_{n/m}$ in $\Pi^*_{n/m-1}/\Pi^*_{n/m}$ as a *cuspidal inertia subgroup* of $\Pi^*_{n/m-1}/\Pi^*_{n/m}$.

(v) Let α be an automorphism of Π^*_n. Then we shall say that α is *F-admissible* if α preserves each fiber subgroup [cf. (iv)] of Π^*_n. We shall say that α is *C-admissible* if α preserves the filtration

$$\{1\} = \Pi^*_{n/n} \subseteq \Pi^*_{n/n-1} \subseteq \cdots \subseteq \Pi^*_{n/m} \subseteq \cdots \subseteq \Pi^*_{n/2} \subseteq \Pi^*_{n/1} \subseteq \Pi^*_{n/0} = \Pi^*_n \,,$$

and, moreover, α induces a bijection of the set of cuspidal inertia subgroups [cf. (iv)] of every standard-adjacent subquotient [cf. (i)] of Π^*_n. We shall say that α is *FC-admissible* if α is F-admissible and C-admissible.

(vi) Let α be an outomorphism of Π^*_n. Then we shall say that α is *F-admissible* (respectively, *C-admissible*; *FC-admissible*) if α arises from an automorphism of Π^*_n that is F-admissible (respectively, C-admissible; FC-admissible) [cf. (v)].

(vii) Write

$$\mathrm{Aut}^{\mathrm{F}}(\Pi^*_n) \ , \ \ \mathrm{Aut}^{\mathrm{C}}(\Pi^*_n) \ , \ \ \mathrm{Aut}^{\mathrm{FC}}(\Pi^*_n) \ \subseteq \ \mathrm{Aut}(\Pi^*_n)$$

for the respective subgroups of F-, C-, and FC-admissible automorphisms of Π^*_n [cf. (v)];

$$\mathrm{Out}^{\mathrm{F}}(\Pi^*_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{F}}(\Pi^*_n)/\mathrm{Inn}(\Pi^*_n) \ ,$$

$$\mathrm{Out}^{\mathrm{C}}(\Pi^*_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{C}}(\Pi^*_n)/\mathrm{Inn}(\Pi^*_n) \ ,$$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi^*_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{FC}}(\Pi^*_n)/\mathrm{Inn}(\Pi^*_n) \ \subseteq \ \mathrm{Out}(\Pi^*_n)$$

for the respective subgroups of F-, C-, and FC-admissible outomorphisms of Π^*_n [cf. (vi)].

(viii) Let $\Pi_n \twoheadrightarrow \Pi^{**}_n$ be a quotient of Π_n that *dominates* the quotient $\Pi_n \twoheadrightarrow \Pi^*_n$ [cf. the discussion entitled "*Topological groups*" in §0]. Then we shall write

$$\mathrm{Out}^{\mathrm{F}}(\Pi^{**}_n \twoheadrightarrow \Pi^*_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi^{**}_n) \,,$$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi^{**}_n \twoheadrightarrow \Pi^*_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi^{**}_n)$$

[cf. (vii)] for the respective subgroups of F-, FC-admissible outomorphisms of Π^{**}_n that preserve the kernel of the natural surjection $\Pi^{**}_n \twoheadrightarrow \Pi^*_n$. Thus, we have natural homomorphisms

$$\mathrm{Out}^{\mathrm{F}}(\Pi^{**}_n \twoheadrightarrow \Pi^*_n) \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi^*_n) \,,$$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi^{**}_n \twoheadrightarrow \Pi^*_n) \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi^*_n) \,.$$

We shall write

$$\mathrm{Out}^{\mathrm{F}}(\Pi^*_n \twoheadleftarrow \Pi^{**}_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi^*_n) \,,$$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi^*_n \twoheadleftarrow \Pi^{**}_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi^*_n)$$

for the respective images of these natural homomorphisms. Thus, we have natural surjections

$$\mathrm{Out}^{\mathrm{F}}(\Pi^{**}_n \twoheadrightarrow \Pi^*_n) \twoheadrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi^*_n \twoheadleftarrow \Pi^{**}_n) \,,$$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{**} \twoheadrightarrow \Pi_n^*) \twoheadrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n^* \twoheadleftarrow \Pi_n^{**})\,.$$

Remark 2.1.1. In the notation of Definition 2.1, suppose that Π_n^* is *F-characteristic* [cf. Definition 2.1, (iii)]. Then it follows from the various definitions involved that

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n \twoheadrightarrow \Pi_n^*) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)\ ,\ \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n \twoheadrightarrow \Pi_n^*) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

[cf. Definition 2.1, (viii)]; thus, we have natural surjections

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n) \twoheadrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n^* \twoheadleftarrow \Pi_n)\ ,\ \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \twoheadrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n^* \twoheadleftarrow \Pi_n)$$

[cf. Definition 2.1, (viii)].

Lemma 2.2 (**Preservation of quotients of extensions).** *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \overline{N} & \longrightarrow & \overline{G} & \longrightarrow & \overline{Q} & \longrightarrow & 1 \end{array}$$

be a commutative diagram of profinite groups — where the horizontal sequences are **exact***, and the vertical arrows are* **surjective***. Write*

$$G^* \stackrel{\mathrm{def}}{=} \mathrm{Ker}(G \twoheadrightarrow Q \twoheadrightarrow \overline{Q})/\mathrm{Ker}(N \twoheadrightarrow \overline{N})$$

and N^* for the image of N in G^*. Suppose that $\overline{N}$ is **center-free***. Then the image of* $\mathrm{Ker}(G \twoheadrightarrow \overline{G})$ *in G^* is equal to the* **centralizer** *$Z_{G^*}(N^*)$.*

Proof. Observe that, by replacing G by $\mathrm{Ker}(G \twoheadrightarrow Q \twoheadrightarrow \overline{Q})$ ($= G \times_Q \mathrm{Ker}(Q \twoheadrightarrow \overline{Q})$), we may assume without loss of generality that $\overline{Q} = \{1\}$. In a similar vein, by replacing G by $G/\mathrm{Ker}(N \twoheadrightarrow \overline{N})$, we may assume without loss of generality that $N = \overline{N}$, which [since $\overline{Q} = \{1\}$] implies that $G = G^*$, $N = N^* = \overline{N}$. Then one verifies easily that the natural inclusions N, $\mathrm{Ker}(G \twoheadrightarrow \overline{G}) \hookrightarrow G$ determine an *isomorphism* $N \times \mathrm{Ker}(G \twoheadrightarrow \overline{G}) \xrightarrow{\sim} G$. Thus, since N is *center-free*, we obtain that $\mathrm{Ker}(G \twoheadrightarrow \overline{G}) = Z_G(N)$. This completes the proof of Lemma 2.2. ☐

Proposition 2.3 (**Existence of F-characteristic SA-maximal almost pro-*l* quotients).** *In the notation of Definition 2.1, let $\Pi_n \twoheadrightarrow \Pi_n^*$ be a quotient of Π_n. Then the following hold:*

(i) *If Π_n^* is an* **SA-maximal almost pro-*l* quotient** *of Π_n [cf. Definition 2.1, (ii)], then Π_n^* is* **topologically finitely generated***,* **almost pro-*l*** *[cf. the discussion entitled "Topological groups" in* [CbTpI]*, §0], and* **slim** *[cf. the discussion entitled "Topological groups" in* [CbTpI]*, §0].*

(ii) *Let* $0 \leq m_1 \leq m_2 \leq n$ *be integers and* $(\Pi_{n/m_1}/\Pi_{n/m_2})^{\ddagger}$ *an* **almost pro-l quotient** *of* $\Pi_{n/m_1}/\Pi_{n/m_2}$. *Then there exists an* **F-characteristic** *[cf. Definition 2.1, (iii)]* **SA-maximal almost pro-l quotient** Π_n^{**} *of* Π_n *such that the quotient of* $\Pi_{n/m_1}/\Pi_{n/m_2}$ *determined by the quotient* $\Pi_n \twoheadrightarrow \Pi_n^{**}$ **dominates** *the quotient* $\Pi_{n/m_1}/\Pi_{n/m_2} \twoheadrightarrow (\Pi_{n/m_1}/\Pi_{n/m_2})^{\ddagger}$ *[cf. the discussion entitled "Topological groups" in §0].*

(iii) *Let* $1 \leq m \leq n$ *be an integer,* $H \subseteq \Pi_{n/m-1}/\Pi_{n/m}$ *a VCN-subgroup of* $\Pi_{n/m-1}/\Pi_{n/m}$ *[cf.* [CbTpII], *Definition 3.1, (iv)], and* $H \twoheadrightarrow H^{\ddagger}$ *an* **almost pro-l quotient** *of* H. *Then there exists an* **F-characteristic SA-maximal almost pro-l quotient** Π_n^{**} *of* Π_n *such that the quotient of* H *determined by the quotient* $\Pi_n \twoheadrightarrow \Pi_n^{**}$ **dominates** *the quotient* $H \twoheadrightarrow H^{\ddagger}$.

Proof. First, we verify assertion (i). Observe that it follows immediately from Proposition 1.7, (i), together with the definition of an SA-maximal almost pro-l quotient, that Π_n^{*} is a *successive extension* of *almost pro-l, topologically finitely generated, slim* profinite groups. Thus, one verifies immediately that Π_n^{*} is *almost pro-l*, *topologically finitely generated*, and *slim*. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, observe that since $(\Pi_{n/m_1}/\Pi_{n/m_2})^{\ddagger}$ may be regarded as an *almost pro-l quotient* of Π_{n/m_1}, we may assume without loss of generality that $m_2 = n$. Write $m \overset{\text{def}}{=} m_1$. If $m = n$, then one may take the quotient Π_n^{**} to be the *maximal pro-l quotient* of Π_n [cf. [MzTa], Proposition 2.2, (i)]. Thus, we may assume without loss of generality that $m \leq n-1$.

Let us verify assertion (ii) by *induction on n*. If $n = 1$, then assertion (ii) follows immediately from the fact that Π_1 is *topologically finitely generated*, which implies that the topology of Π_1 admits a basis of *characteristic open subgroups*. Thus, we suppose that $n \geq 2$, and that the *induction hypothesis* is in force. Then observe that since the subgroup $\Pi_{n/n-1} \subseteq \Pi_n$ may be regarded as the "Π_1" associated to some stable log curve of type $(g, r+n-1)$, by applying the *induction hypothesis* to the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi^{\ddagger}_{n/n-1}$ determined by the quotient $\Pi_{n/m} \twoheadrightarrow \Pi^{\ddagger}_{n/m}$, we obtain an *F-characteristic SA-maximal almost pro-l quotient* $\Pi^{**}_{n/n-1}$ of $\Pi_{n/n-1}$ which *dominates* $\Pi_{n/n-1} \twoheadrightarrow \Pi^{\ddagger}_{n/n-1}$. In particular, since the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi^{**}_{n/n-1}$ is *F-characteristic*, hence arises from a subgroup of $\Pi_{n/n-1}$ which is *normal* in Π_n, we thus obtain a *natural outer action*

$$\Pi_n/\Pi_{n/n-1} \, (\xrightarrow{\sim} \Pi_{n-1}) \to \mathrm{Out}(\Pi^{**}_{n/n-1}) \, .$$

Since the profinite group $\Pi^{**}_{n/n-1}$ is *almost pro-l*, *topologically finitely generated*, and *slim* [cf. assertion (i)], it follows immediately that the outer action $\Pi_n/\Pi_{n/n-1} \to \mathrm{Out}(\Pi^{**}_{n/n-1})$ factors through an *almost pro-l* quotient Q of $\Pi_n/\Pi_{n/n-1}$. In particular, it follows that the natural outer action $\Pi_{n/m}/\Pi_{n/n-1} \subseteq \Pi_n/\Pi_{n/n-1} \to \mathrm{Out}(\Pi^{**}_{n/n-1})$ factors through an *almost pro-l* quotient of $\Pi_m/\Pi_{n/n-1}$. Note that this implies that there exists an *almost pro-l* quotient $\Pi_{n/m} \twoheadrightarrow Q^{**}$ of $\Pi_{n/m}$ that induces the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi^{**}_{n/n-1}$ of $\Pi_{n/n-1}$. Now one verifies immediately that the quotient Q^{***} determined by the *intersection* of the kernels of the two quotients $\Pi_{n/m} \twoheadrightarrow \Pi^{\ddagger}_{n/m}$, $\Pi_{n/m} \twoheadrightarrow Q^{**}$ is an *almost pro-l* quotient of $\Pi_{n/m}$ that induces the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi^{**}_{n/n-1}$ of $\Pi_{n/n-1}$. Thus, we conclude that by replacing the quotient $\Pi_{n/m} \twoheadrightarrow \Pi^{\ddagger}_{n/m}$ by this quotient Q^{***}, we may assume without loss of generality that the quotient $\Pi_{n/m} \twoheadrightarrow \Pi^{\ddagger}_{n/m}$ induces the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi^{**}_{n/n-1}$ of $\Pi_{n/n-1}$.

Next, let us observe that if we regard $\Pi_n/\Pi_{n/n-1}$ as the "Π_{n-1}" associated to some stable log curve of type (g, r), then:

- If we apply the *induction hypothesis* to the *almost pro-l quotient* $\Pi_{n/m}/\Pi_{n/n-1} \twoheadrightarrow \Pi^{\ddagger}_{n/m}/\Pi^{\ddagger}_{n/n-1}$, then we obtain an [F-characteristic SA-maximal] *almost pro-l quotient*

$$\Pi_n/\Pi_{n/n-1} \twoheadrightarrow Q^{\ddagger}$$

  of $\Pi_n/\Pi_{n/n-1}$ which induces a quotient of $\Pi_{n/m}/\Pi_{n/n-1}$ that *dominates* the quotient $\Pi_{n/m}/\Pi_{n/n-1} \twoheadrightarrow \Pi^{\ddagger}_{n/m}/\Pi^{\ddagger}_{n/n-1}$.
- If we apply the *induction hypothesis* to any *almost pro-l quotient* of $\Pi_n/\Pi_{n/n-1}$ that *dominates both Q and $Q^{\ddagger}$* [e.g., the quotient determined by the *intersection* of the kernels determined by the quotients Q, $Q^{\ddagger}$], then we obtain an *F-characteristic SA-maximal almost pro-l quotient*

$$\Pi_n/\Pi_{n/n-1} \twoheadrightarrow (\Pi_n/\Pi_{n/n-1})^{**}$$

  of $\Pi_n/\Pi_{n/n-1}$ that *dominates* Q and, moreover, induces a quotient of $\Pi_{n/m}/\Pi_{n/n-1}$ that *dominates* $\Pi^{\ddagger}_{n/m}/\Pi^{\ddagger}_{n/n-1}$. In particular, the above outer action $\Pi_n/\Pi_{n/n-1} \to \mathrm{Out}(\Pi^{**}_{n/n-1})$ *factors* through the natural surjection $\Pi_n/\Pi_{n/n-1} \twoheadrightarrow (\Pi_n/\Pi_{n/n-1})^{**}$.

Now let us write $\Pi^{**}_n \stackrel{\mathrm{def}}{=} \Pi^{**}_{n/n-1} \stackrel{\mathrm{out}}{\rtimes} (\Pi_n/\Pi_{n/n-1})^{**}$ [cf. the discussion entitled "*Topological groups*" in [CbTpI], §0 — where we note that $\Pi^{**}_{n/n-1}$ is *center-free* by assertion (i)]. Then it follows immediately from Lemma 2.2 and the various definitions involved, together with our assumption that the quotient $\Pi_{n/m} \twoheadrightarrow \Pi^{\ddagger}_{n/m}$ induces the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi^{**}_{n/n-1}$ of $\Pi_{n/n-1}$, that Π^{**}_n is an *SA-maximal almost pro-l quotient* of Π_n such that the quotient of $\Pi_{n/m}$ determined by $\Pi_n \twoheadrightarrow \Pi^{**}_n$ *dominates* the quotient $\Pi_{n/m} \twoheadrightarrow \Pi^{\ddagger}_{n/m}$. Finally, it follows immediately from Lemma 2.2, together with the fact that the quotients $\Pi_{n/n-1} \twoheadrightarrow$

$\Pi^{**}_{n/n-1}$ and $\Pi_n/\Pi_{n/n-1} \twoheadrightarrow (\Pi_n/\Pi_{n/n-1})^{**}$ are *F-characteristic*, that Π^{**}_n is *F-characteristic*. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (ii), together with Proposition 1.7, (viii). This completes the proof of Proposition 2.3. □

Definition 2.4. In the notation of Definition 2.1, write $\Pi_{\mathrm{F}} \stackrel{\mathrm{def}}{=} \Pi_{2/1}$, $\Pi_{\mathrm{T}} \stackrel{\mathrm{def}}{=} \Pi_2$, $\Pi_{\mathrm{B}} \stackrel{\mathrm{def}}{=} \Pi_1$; thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathrm{F}} \longrightarrow \Pi_{\mathrm{T}} \longrightarrow \Pi_{\mathrm{B}} \longrightarrow 1$$

[cf. the notation introduced in [CbTpI], Definition 6.3]. Let $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$ be a *maximal almost pro-*Σ *quotient* of Π_{F} [cf. Definition 1.1]. Then we shall say that $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$ is *base-admissible* if the kernel of $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$ is *normal* in Π_{T}. Thus, if $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$ is *base-admissible*, then the quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$ determines a quotient $\Pi_{\mathrm{T}} \twoheadrightarrow \Pi^*_{\mathrm{T}}$ of Π_{T} which fits into a natural commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Pi^*_{\mathrm{F}} & \longrightarrow & \Pi^*_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*.

Definition 2.5. In the notation of Definition 2.4, suppose that

$$\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$$

is *base-admissible* [cf. Definition 2.4]; thus, we have a quotient

$$\Pi_{\mathrm{T}} \twoheadrightarrow \Pi^*_{\mathrm{T}}$$

of Π_{T} that fits into the commutative diagram of Definition 2.4. Let $x \in X(k)$ be a k-valued point of the underlying scheme X of $X^{\log}$.

(i) We shall write

$$\Pi_{\mathcal{G}_x} \twoheadrightarrow \Pi^*_{\mathcal{G}_x}$$

[cf. [CbTpI], Definition 6.3, (i)] for the *maximal almost pro-*Σ *quotient* of $\Pi_{\mathcal{G}_x}$ determined by the quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$ and the isomorphism $\Pi_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ *fixed* in [CbTpI], Definition 6.3, (i). [Here, we note that this quotient $\Pi_{\mathcal{G}_x} \twoheadrightarrow \Pi^*_{\mathcal{G}_x}$ is *independent* of the choice of isomorphism $\Pi_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ in [CbTpI], Definition 6.3, (i).] Thus, the fixed isomorphism $\Pi_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ induces an isomorphism of profinite groups $\Pi^*_{\mathrm{F}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x}$.

(ii) For $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. [CbTpI], Definition 6.5, (i)], we shall refer to a closed subgroup of Π^*_{F} obtained by forming the image — via the isomorphism $\Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ [cf. (i)] for some k-valued point $x \in X(k)$ — of a cuspidal subgroup of $\Pi^*_{\mathcal{G}_x}$ associated to the cusp of $\mathcal{G}_x$ corresponding to $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. [CbTpI], Lemma 6.4, (ii)] as a *cuspidal subgroup of* Π^*_{F} *associated to* $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$. Note that it follows immediately from [CbTpI], Lemma 6.4, (ii), that the Π^*_{F}-conjugacy class of a cuspidal subgroup of Π^*_{F} associated to $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ *depends only* on $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$, i.e., it does *not depend* on the choice of x or on the choices of isomorphisms made in [CbTpI], Definition 6.3, (i).

(iii) Recall that $\Pi_{\mathrm{T}} = \Pi_2$, $\Pi_{\mathrm{F}} = \Pi_{2/1}$ [cf. Definition 2.4]. In particular, it makes sense to speak of *F-/C-/FC-admissible* automorphisms or outomorphisms of Π^*_{T}, Π^*_{F} [cf. Definition 2.1, (v), (vi)].

Lemma 2.6 (**Maximal almost pro-Σ quotients of VCN-subgroups).** *In the notation of Definition 2.5, let* $\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi^*_{\mathrm{F}}$ *be a cuspidal subgroup of* Π^*_{F} *[cf. Definition 2.5, (ii)] associated to* $c^{\mathrm{F}}_{\mathrm{diag}} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ *[cf.* [CbTpI], *Definition 6.5, (i)]. Write* $N^*_{\mathrm{diag}} \subseteq \Pi^*_{\mathrm{F}}$ *for the normal closed subgroup of* Π^*_{F} *topologically normally generated by* $\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi^*_{\mathrm{F}}$. *[Note that it follows immediately from* [CbTpI], *Lemma 6.4, (i), (ii), that* N^*_{diag} *is* **normal** *in* Π^*_{T}.*] Then the following hold:*

(i) *If we regard* $\Pi^*_{\mathrm{F}}/N^*_{\mathrm{diag}}$ *as a quotient of* $\Pi_{\mathcal{G}}$ *by means of the natural outer isomorphism* $\Pi_{\mathrm{F}}/N_{\mathrm{diag}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$ *of* [CbTpI], *Lemma 6.6, (i), and the natural surjection* $\Pi_{\mathrm{F}}/N_{\mathrm{diag}} \twoheadrightarrow \Pi^*_{\mathrm{F}}/N^*_{\mathrm{diag}}$, *then* $\Pi^*_{\mathrm{F}}/N^*_{\mathrm{diag}}$ *is a* **maximal almost pro-Σ quotient** *of* $\Pi_{\mathcal{G}}$ *[cf. Definition 1.1].*

(ii) *Let* $z^{\mathrm{F}} \in \mathrm{VCN}(\mathcal{G}_x)$, $\Pi_{z^{\mathrm{F}}} \subseteq \Pi_{\mathcal{G}_x}$ *a VCN-subgroup of* $\Pi_{\mathcal{G}_x}$ *associated to* z^{F}, *and* $\Pi_{z^{\mathrm{F}}} \twoheadrightarrow \Pi^{\ddagger}_{z^{\mathrm{F}}}$ *an* **almost pro-Σ quotient** *of* $\Pi_{z^{\mathrm{F}}}$. *Then there exists a* **base-admissible** *[cf. Definition 2.4]* **maximal almost pro-Σ quotient** Π^{**}_{F} *of* Π_{F} *such that the quotient* $\Pi_{z^{\mathrm{F}}} \twoheadrightarrow \Pi^{**}_{z^{\mathrm{F}}}$ *determined by the quotient* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ **dominates** *the quotient* $\Pi_{z^{\mathrm{F}}} \twoheadrightarrow \Pi^{\ddagger}_{z^{\mathrm{F}}}$ *[cf. the discussion entitled "Topological groups" in §0].*

(iii) *Let* $z^{\mathrm{F}} \in \mathrm{VCN}(\mathcal{G}_x) \setminus \{c^{\mathrm{F}}_{\mathrm{diag}}\}$ *and* $\Pi^*_{z^{\mathrm{F}}} \subseteq \Pi^*_{\mathcal{G}_x}$ *a VCN-subgroup of* $\Pi^*_{\mathcal{G}_x}$ *associated to* z^{F} *[cf. Definition 1.6, (i)]. Suppose that either*

- $z^{\mathrm{F}} \in \mathrm{Edge}(\mathcal{G}_x)$

or

- $z^{\mathrm{F}} = v_x^{\mathrm{F}}$ *for* $v \in \mathrm{Vert}(\mathcal{G})$ *[cf.* [CbTpI], *Definition 6.3, (ii)] such that* x *does* **not lie** *on* v *[cf.* [CbTpI], *Definition 6.3, (iii)].*

Then there exist a **maximal almost pro-Σ quotient** Π_{F}^{**} *of* Π_{F} *and a VCN-subgroup* $\Pi_{z^{\mathrm{F}}}^{**} \subseteq \Pi_{\mathcal{G}_x}^{**}$ *of* $\Pi_{\mathcal{G}_x}^{**}$ *associated to* z^{F} *such that the following conditions are satisfied:*

(a) $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}^{**}$ **dominates** $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}^{*}$.

(b) $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}^{**}$ *is* **base-admissible**.

(c) *The quotient of* $\Pi_{z^{\mathrm{F}}}^{**}$ *determined by the composite*

$$\Pi_{z^{\mathrm{F}}}^{**} \hookrightarrow \Pi_{\mathcal{G}_x}^{**} \xleftarrow{\sim} \Pi_{\mathrm{F}}^{**} \twoheadrightarrow \Pi_{\mathrm{F}}^{*}$$

factors through the quotient of $\Pi_{z^{\mathrm{F}}}^{**}$ *determined by the composite*

$$\Pi_{z^{\mathrm{F}}}^{**} \hookrightarrow \Pi_{\mathcal{G}_x}^{**} \xleftarrow{\sim} \Pi_{\mathrm{F}}^{**} \twoheadrightarrow \Pi_{\mathrm{F}}^{**}/N_{\mathrm{diag}}^{**}$$

— where we write N_{diag}^{**} *for the normal closed subgroup of* Π_{F}^{**} *topologically normally generated by the cuspidal subgroups of* Π_{F}^{**} *associated to* $c_{\mathrm{diag}}^{\mathrm{F}} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$*.*

Proof. Assertion (i) follows immediately from Lemma 1.2, (i). Assertion (ii) follows immediately from Proposition 1.7, (viii), together with Lemma 1.2, (iii) [cf. also Proposition 1.7, (i)]. In a similar vein, assertion (iii) follows immediately, in light of the *injectivity* assertion of [CbTpI], Lemma 6.6, (iii), from Proposition 1.7, (viii) [applied to $\Pi_{\mathrm{F}}/N_{\mathrm{diag}}$], together with Lemma 1.2, (iii) [cf. also Proposition 1.7, (i)]. This completes the proof of Lemma 2.6. □

Lemma 2.7 (**Outomorphisms that preserve the diagonal**)**.** *In the notation of Lemma 2.6, let* $\widetilde{\alpha}^*$ *be an automorphism of* Π_{T}^* *over* Π_{B} *[i.e., which preserves and induces the identity automorphism on the quotient* $\Pi_{\mathrm{T}}^* \twoheadrightarrow \Pi_{\mathrm{B}}$*]. Write* $\alpha_{\mathrm{F}}^* \in \mathrm{Out}(\Pi_{\mathrm{F}}^*)$ *for the outomorphism of* Π_{F}^* *determined by of* $\widetilde{\alpha}^*$*. Then the following hold:*

(i) *Suppose that* $\widetilde{\alpha}^*$ *preserves* $\Pi_{c_{\mathrm{diag}}^{\mathrm{F}}}^* \subseteq \Pi_{\mathrm{F}}^*$*. Then the automorphism of* $\Pi_{\mathrm{F}}^*/N_{\mathrm{diag}}^*$ *induced by* $\widetilde{\alpha}^*$ *is the* **identity automorphism**.

(ii) *Let* $e \in \mathrm{Edge}(\mathcal{G})$*,* $x \in X(k)$ *be such that* $x \curvearrowright e$ *[cf.* [CbTpI], *Definition 6.3, (iii)]. Suppose that* α_{F}^* *is* **C-admissible** *[cf. Definition 2.5, (iii)], and that* $\mathrm{Edge}(\mathcal{G}) = \{e\} \cup \mathrm{Cusp}(\mathcal{G})$*. Then it holds that* $\alpha_{\mathrm{F}}^* \in \mathrm{Out}^{\mathrm{grph}}(\Pi_{\mathcal{G}_x}^*)$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}^*) \xleftarrow{\sim} \mathrm{Out}(\Pi_{\mathrm{F}}^*))$ *[cf. Definition 1.6, (iii)]. If, moreover,* $\widetilde{\alpha}^*$ *preserves* $\Pi_{c_{\mathrm{diag}}^{\mathrm{F}}}^* \subseteq$

Π^*_{F}, *then* $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_x})$ $(\subseteq \mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}_x}))$ *[cf. Definition 1.8].*

(iii) *If* $\widetilde{\alpha}^*$ *is* **FC-admissible** *[cf. Definition 2.5, (iii)], then* $\widetilde{\alpha}^*$ **preserves** *the* Π^*_{F}*-conjugacy class of* $\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi^*_{\mathrm{F}}$.

Proof. First, we verify assertion (i). Write $D^* \stackrel{\mathrm{def}}{=} N_{\Pi^*_{\mathrm{T}}}(\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}}) \subseteq \Pi^*_{\mathrm{T}}$. Then it follows immediately from Proposition 1.7, (vii), that the natural inclusion $D^* \hookrightarrow \Pi^*_{\mathrm{T}}$ fits into the following exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} & \longrightarrow & D^* & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Pi^*_{\mathrm{F}} & \longrightarrow & \Pi^*_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*. Thus, assertion (i) follows immediately from a similar argument to the argument applied in the proof of the first assertion of [CbTpI], Lemma 6.7, (i) [cf. also the proof of [CmbCsp], Proposition 1.2, (iii)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). The fact that $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}_x})$ $(\subseteq \mathrm{Out}(\Pi^*_{\mathcal{G}_x}) \stackrel{\sim}{\leftarrow} \mathrm{Out}(\Pi^*_{\mathrm{F}}))$ follows immediately from Corollary 1.12, together with a similar argument to the argument applied in the proof of the first assertion of [CbTpI], Lemma 6.7, (ii). Now suppose, moreover, that $\widetilde{\alpha}^*$ preserves $\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi^*_{\mathrm{F}}$. Then the fact that $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_x})$ $(\subseteq \mathrm{Out}^{\mathrm{grph}}(\Pi^*_{\mathcal{G}_x}))$ follows immediately from assertion (i); Lemma 2.6, (i); Proposition 1.7, (iii), (v), together with a similar argument to the argument applied in the proof of the second assertion of [CbTpI], Lemma 6.7, (ii). This completes the proof of assertion (ii).

Finally, assertion (iii) follows immediately, in light of Lemma 2.6, (i), from the definition of *FC-admissibility* [cf. also Proposition 1.7, (v)]. This completes the proof of Lemma 2.7. □

Lemma 2.8 (**Triviality of certain outomorphisms**). *In the notation of Definition 2.5, there exists a* **base-admissible maximal almost pro-Σ quotient** $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ *[cf. Definitions 1.1; 2.4] of* Π_{F} *that* **dominates** $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$ *[cf. the discussion entitled "Topological groups" in §0] such that the following condition* (‡) *is satisfied:*

> (‡)*: Let* $\widetilde{\alpha}^*$ *be an automorphism of* Π^*_{T}*. Then for any* **base-admissible maximal almost pro-Σ quotient** $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{***}_{\mathrm{F}}$ *that* **dominates** $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$*, if* $\widetilde{\alpha}^*$ *arises from an* **FC-admissible** *automorphism [cf. Definition 2.5, (iii)] of* Π^{***}_{T} *[where we write* Π^{***}_{T} *for the "*Π^*_{T}*" determined by* Π^{***}_{F}*] over* $\Pi^{***}_{\mathrm{T}}/\Pi^{***}_{\mathrm{F}} \stackrel{\sim}{\to} \Pi_{\mathrm{B}}$*, then* $\widetilde{\alpha}^*$ *is* Π^*_{F}**-inner**.

Proof. The following argument is essentially the same as the argument applied in [CmbCsp], [NodNon], [CbTpI] to prove [CmbCsp], Corollary 2.3, (ii); [NodNon], Corollary 5.3; [CbTpI], Lemma 6.8, respectively.

Let us fix a cuspidal subgroup $\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi^*_{\mathrm{F}}$ of Π^*_{F} [cf. Definition 2.5, (ii)] associated to $c^{\mathrm{F}}_{\mathrm{diag}} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. [CbTpI], Definition 6.5, (i)]. Let $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ be a *base-admissible* maximal almost pro-Σ quotient of Π_{F} that *dominates* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$; $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{***}_{\mathrm{F}}$ a *base-admissible* maximal almost pro-Σ quotient of Π_{F} that *dominates* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$; $\widetilde{\alpha}^*$ an automorphism of Π^*_{T} that arises from an *FC-admissible* automorphism $\widetilde{\alpha}^{***}$ of Π^{***}_{T} over $\Pi^{***}_{\mathrm{T}}/\Pi^{***}_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathrm{B}}$. Here, let us observe that one verifies easily that $\widetilde{\alpha}^*$ is an *FC-admissible* automorphism of Π^*_{T} over $\Pi^*_{\mathrm{T}}/\Pi^*_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathrm{B}}$. Write α^*_{F} for the outomorphism of Π^*_{F} determined by $\widetilde{\alpha}^*$. Observe that since α^*_{F} preserves the Π^*_{F}-conjugacy class of $\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi^*_{\mathrm{F}}$ [cf. Lemma 2.7, (iii)], we may assume without loss of generality — by replacing $\widetilde{\alpha}^{***}$ by a suitable Π^{***}_{F}-conjugate of $\widetilde{\alpha}^{***}$ — that $\widetilde{\alpha}^*$ *preserves* $\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi^*_{\mathrm{F}}$, and hence [cf. Lemma 2.7, (i), (ii)] that

(a) the automorphism of $\Pi^*_{\mathrm{F}}/N^*_{\mathrm{diag}}$ induced by $\widetilde{\alpha}^*$ is the *identity automorphism*;

(b) for $e \in \mathrm{Edge}(\mathcal{G})$, $x \in X(k)$ such that $x \curvearrowright e$ [cf. [CbTpI], Definition 6.3, (iii)], if $\mathrm{Edge}(\mathcal{G}) = \{e\} \cup \mathrm{Cusp}(\mathcal{G})$, then $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_x})$ $(\subseteq \mathrm{Out}(\Pi^*_{\mathcal{G}_x}) \xleftarrow{\sim} \mathrm{Out}(\Pi^*_{\mathrm{F}}))$ [cf. Definition 1.8].

Now we claim that the following assertion holds:

Claim 2.8.A: Lemma 2.8 holds if $(g, r) = (0, 3)$.

Indeed, write c_1, c_2, $c_3 \in \mathrm{Cusp}(\mathcal{G})$ for the three distinct cusps of $\mathcal{G}$; $v \in \mathrm{Vert}(\mathcal{G})$ for the *unique* vertex of $\mathcal{G}$. For $i \in \{1, 2, 3\}$, let $x_i \in X(k)$ be such that $x_i \curvearrowright c_i$. Next, let us observe that since our assumption that $(g, r) = (0, 3)$ implies that $\mathrm{Node}(\mathcal{G}) = \emptyset$, it follows immediately from (b) that, for $i \in \{1, 2, 3\}$, the outomorphism α^*_{F} of $\Pi^*_{\mathcal{G}_{x_i}} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ is $\in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_{x_i}})$ $(\subseteq \mathrm{Out}(\Pi^*_{\mathcal{G}_{x_i}}) \xleftarrow{\sim} \mathrm{Out}(\Pi^*_{\mathrm{F}}))$. Next, let us fix a verticial subgroup $\Pi^*_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi^*_{\mathcal{G}_{x_2}} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ associated to $v^{\mathrm{F}}_{x_2} \in \mathrm{Vert}(\mathcal{G}_{x_2})$ [cf. [CbTpI], Definition 6.3, (ii)]. Then since $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_{x_2}})$, it follows immediately from the *commensurable terminality* of the image of the composite $\Pi^*_{v^{\mathrm{F}}_{x_2}} \hookrightarrow \Pi^*_{\mathcal{G}_{x_2}} \xleftarrow{\sim} \Pi^*_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}/N^*_{\mathrm{diag}}$ [cf. Proposition 1.7, (vii); Lemma 2.6, (i)], together with the property (a) discussed above, that there exists an N^*_{diag}-conjugate $\widetilde{\beta}^*$ of $\widetilde{\alpha}^*$ such that $\widetilde{\beta}^*(\Pi^*_{v^{\mathrm{F}}_{x_2}}) = \Pi^*_{v^{\mathrm{F}}_{x_2}}$. Thus, it follows immediately from Lemma 2.6, (iii) — by replacing Π^{**}_{F} by a suitable *base-admissible* maximal almost pro-Σ quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ [i.e., a quotient as in Lemma 2.6, (iii)] that *dominates* the *original* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ and applying the conclusion "$\widetilde{\beta}^*(\Pi^*_{v^{\mathrm{F}}_{x_2}}) = \Pi^*_{v^{\mathrm{F}}_{x_2}}$", together with the property (a) discussed above, in the case where "α^*" is taken to be $\alpha^{***} \in \mathrm{Aut}(\Pi_{\mathrm{T}}^{***})$ — that we may assume without loss of generality that

$(\ddagger_1)$: $\widetilde{\beta}^*$ *fixes* and induces the *identity automorphism* on $\Pi^*_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi^*_{\mathcal{G}_{x_2}} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$.

Next, let $\Pi^*_{c^{\mathrm{F}}_1} \subseteq \Pi^*_{\mathrm{F}}$ be a cuspidal subgroup of Π^*_{F} associated to $c^{\mathrm{F}}_1 \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. [CbTpI], Definition 6.5, (i)] that is contained in $\Pi^*_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi^*_{\mathcal{G}_{x_2}} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$; $\Pi^*_{v^{\mathrm{F}}_{x_3}} \subseteq \Pi^*_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$ a verticial subgroup associated to $v^{\mathrm{F}}_{x_3} \in \mathrm{Vert}(\mathcal{G}_{x_3})$ that contains $\Pi^*_{c^{\mathrm{F}}_1} \subseteq \Pi^*_{\mathrm{F}}$. Then it follows from the inclusion $\Pi^*_{c^{\mathrm{F}}_1} \subseteq \Pi^*_{v^{\mathrm{F}}_{x_2}}$, together with $(\ddagger_1)$, that $\widetilde{\beta}^*(\Pi^*_{c^{\mathrm{F}}_1}) = \Pi^*_{c^{\mathrm{F}}_1}$. Thus, since the verticial subgroup $\Pi^*_{v^{\mathrm{F}}_{x_3}} \subseteq \Pi^*_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$ is the *unique* verticial subgroup of $\Pi^*_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$ associated to $v^{\mathrm{F}}_{x_3} \in \mathrm{Vert}(\mathcal{G}_{x_3})$ that contains $\Pi^*_{c^{\mathrm{F}}_1}$ [cf. Proposition 1.7, (v), (vi)], it follows immediately from the fact that $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_{x_3}})$ that $\widetilde{\beta}^*(\Pi^*_{v^{\mathrm{F}}_{x_3}}) = \Pi^*_{v^{\mathrm{F}}_{x_3}}$. In particular, it follows immediately from Lemma 2.6, (iii) — by replacing Π^{**}_{F} by a suitable *base-admissible* maximal almost pro-Σ quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ [i.e., a quotient as in Lemma 2.6, (iii)] that *dominates* the *original* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ and applying the conclusion "$\widetilde{\beta}^*(\Pi^*_{v^{\mathrm{F}}_{x_3}}) = \Pi^*_{v^{\mathrm{F}}_{x_3}}$", together with the property (a) discussed above, in the case where "α^*" is taken to be $\alpha^{***} \in \mathrm{Aut}(\Pi_{\mathrm{T}}^{***})$ — that we may assume without loss of generality that

$(\ddagger_2)$: $\widetilde{\beta}^*$ *fixes* and induces the *identity automorphism* on $\Pi^*_{v^{\mathrm{F}}_{x_3}} \subseteq \Pi^*_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$.

On the other hand, since Π^*_{F} is topologically generated by $\Pi^*_{v^{\mathrm{F}}_{x_2}} \subseteq \Pi^*_{\mathcal{G}_{x_2}} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$ and $\Pi^*_{v^{\mathrm{F}}_{x_3}} \subseteq \Pi^*_{\mathcal{G}_{x_3}} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$ [cf. [CmbCsp], Lemma 1.13], $(\ddagger_1)$ and $(\ddagger_2)$ imply that $\widetilde{\beta}^*$ induces the *identity automorphism* on Π^*_{F}. This completes the proof of Claim 2.8.A.

Next, we claim that the following assertion holds:

Claim 2.8.B: Lemma 2.8 holds if $(g, r) = (1, 1)$.

Indeed, let us first *observe* that by working with *2-cuspidalizable degeneration structures* [cf. [CbTpII], Definition 3.23, (i), (v)] that arise *scheme-theoretically* via a *specialization isomorphism* as in the discussion preceding [CmbCsp], Definition 2.1 [cf. also [CbTpI], Remark 5.6.1], we may switch back and forth, at will, between the case of *smooth* and *non-smooth* "$X^{\log}$". In particular, we may assume without loss of generality that $(\mathrm{Vert}(\mathcal{G})^\sharp, \mathrm{Cusp}(\mathcal{G})^\sharp, \mathrm{Node}(\mathcal{G})^\sharp) = (1, 1, 1)$.

Let v be the *unique* vertex of $\mathcal{G}$, c the *unique* cusp of $\mathcal{G}$, e the *unique* node of $\mathcal{G}$, $x \in X(k)$ such that $x \curvearrowright c$ [cf. [CbTpI], Definition 6.3, (iii)], and $\mathbb{H}$ the sub-semi-graph of PSC-type [cf. [CbTpI], Definition

2.2, (i)] of the underlying semi-graph $\mathbb{G}_x$ of $\mathcal{G}_x$ whose set of vertices $= \{v^{\mathrm{F}}_x\}$ [cf. [CbTpI], Definition 6.3, (ii)]. Then it follows from [CbTpI], Lemma 6.4, (iv), that there exists a *unique* node $e^{\mathrm{F}}_{\mathrm{new},x}$ of $\mathcal{G}_x$ such that $e^{\mathrm{F}}_{\mathrm{new},x} \in \mathcal{N}(v^{\mathrm{F}}_{\mathrm{new},x})$ [cf. [CbTpI], Lemma 6.4, (iii)]. Thus, one verifies easily that there exists a *unique* element $e^{\mathrm{F}}_x \in \mathcal{N}(v^{\mathrm{F}}_x)$ such that $\mathcal{N}(v^{\mathrm{F}}_x) = \{e^{\mathrm{F}}_{\mathrm{new},x}, e^{\mathrm{F}}_x\}$. Let us fix

- a *nodal subgroup* $\Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}} \subseteq \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ associated to $e^{\mathrm{F}}_{\mathrm{new},x}$

[cf. Figure 1 below].

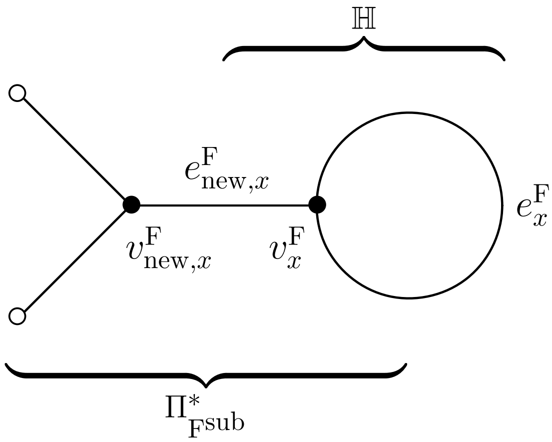

Figure 1: $\mathcal{G}_x$

Then it follows immediately — by applying Proposition 1.7, (v), (vi), in the situation that arises in the case of a *smooth* "$X^{\log}$" of type $(1,1)$ [cf. the observations made above concerning degeneration structures] — that there exist

- a *unique verticial subgroup* $\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}} \subseteq \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ associated to $v^{\mathrm{F}}_{\mathrm{new},x}$ and
- a *unique subgroup* $\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}} \subseteq \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ that belongs to the Π^*_{F}-conjugacy class of subgroups that arises as the *image* of the natural outer homomorphism $\Pi_{(\mathcal{G}_x)|_{\mathbb{H}}} \hookrightarrow \Pi_{\mathcal{G}_x} \twoheadrightarrow \Pi^*_{\mathcal{G}_x}$ [cf. [CbTpI], Definition 2.2, (ii)]

such that $\Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}} \subseteq \Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}, \Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$. Moreover, one verifies easily — by applying the property (b) discussed above in the situation that arises in the case of a *smooth* "$X^{\log}$" of type $(1,1)$ [cf. the observations made above concerning degeneration structures] — that α^*_{F} preserves the Π^*_{F}-conjugacy classes of $\Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}}, \Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}, \Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}} \subseteq \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$. Thus, it follows immediately from the *commensurable terminality* of the image of the composite $\Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}} \hookrightarrow \Pi^*_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}/N^*_{\mathrm{diag}}$ [cf. Proposition 1.7, (vii); Lemma 2.6, (i)], together with the property (a) discussed above, that there exists an N^*_{diag}-conjugate $\widetilde{\beta}^*$ of $\widetilde{\alpha}^*$ such that $\widetilde{\beta}^*(\Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}}) = \Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}}$. In particular, in light of the *uniqueness* properties applied above to specify the subgroups $\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}$ and $\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$, we conclude that $\widetilde{\beta}^*(\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}) = \Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}$, $\widetilde{\beta}^*(\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}) = \Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$. Thus, it follows immediately from Lemma 2.6, (iii) — by replacing Π^{**}_{F} by a suitable *base-admissible* maximal almost pro-Σ quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ [i.e., a quotient as in Lemma 2.6, (iii), applied in the situation that arises in the case of a *smooth* "$X^{\log}$" of type $(1,1)$ — cf. the observations made above concerning degeneration structures] that *dominates* the *original* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ and applying the conclusion "$\widetilde{\beta}^*(\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}) = \Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$", together with the property (a) discussed above, in the case where "α^*" is taken to be $\alpha^{***} \in \mathrm{Aut}(\Pi^{***}_{\mathrm{T}})$ — that we may assume without loss of generality that

($\ddagger_3$): $\widetilde{\beta}^*$ *fixes* and induces the *identity automorphism* on $\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}} \subseteq \Pi^*_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$.

Next, let us write

- $\Pi^*_{v^{\mathrm{F}}_x} \subseteq \Pi^*_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$ for the *unique* [cf. Proposition 1.7, (v), (vi)] *verticial subgroup* associated to v^{F}_x [cf. [CbTpI], Definition 6.3, (ii)]

such that $\Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}} \subseteq \Pi^*_{v^{\mathrm{F}}_x} \subseteq \Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$. [Note that it follows immediately from the various definitions involved that such a verticial subgroup associated to v^{F}_x always exists.] Then since $\Pi^*_{v^{\mathrm{F}}_x} \subseteq \Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$, it follows from ($\ddagger_3$) that $\widetilde{\beta}^*$ *fixes* and induces the *identity automorphism* on $\Pi^*_{v^{\mathrm{F}}_x} \subseteq \Pi^*_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$. Thus, since $\widetilde{\beta}^*(\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}) = \Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}$ [cf. the discussion preceding ($\ddagger_3$)], we conclude that $\widetilde{\beta}^*$ *preserves* the closed subgroup $\Pi^*_{\mathrm{Fsub}} \subseteq \Pi^*_{\mathrm{F}}$ of Π^*_{F} obtained by forming the image of the natural homomorphism

$$\varinjlim \left(\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}} \hookleftarrow \Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}} \hookrightarrow \Pi^*_{v^{\mathrm{F}}_x} \right) \longrightarrow \Pi^*_{\mathrm{F}}$$

— where the inductive limit is taken in the category of profinite groups.

Next, let us observe that the Π^*_{F}-conjugacy class of $\Pi^*_{\mathrm{Fsub}} \subseteq \Pi^*_{\mathrm{F}}$ *coincides* with the Π^*_{F}-conjugacy class of the image $\Pi^*_{(\mathcal{G}_x)_{\succ\{e^{\mathrm{F}}_x\}}}$ [cf. [CbTpI], Definition 2.5, (ii)] of the composite

$$\Pi_{(\mathcal{G}_x)_{\succ\{e^{\mathrm{F}}_x\}}} \hookrightarrow \Pi_{\mathcal{G}_x} \overset{\sim}{\leftarrow} \Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$$

— where the first arrow is the natural outer injection discussed in [CbTpI], Proposition 2.11, and we recall that e^{F}_x is the node of $\mathcal{G}_x$ that corresponds to the node e of $\mathcal{G}$. On the other hand, if we write

$\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e^{\mathrm{F}}_{\mathrm{new}}\}}}$ for the *maximal almost pro-*Σ *quotient* of $\Pi_{(\mathcal{G}_x)_{\rightsquigarrow\{e^{\mathrm{F}}_{\mathrm{new}}\}}}$ [cf. [CbTpI], Definition 2.8] determined by the *maximal almost pro-*Σ *quotient* $\Pi^*_{\mathcal{G}_x}$ and the natural outer isomorphism $\Phi_{(\mathcal{G}_x)_{\rightsquigarrow\{e^{\mathrm{F}}_{\mathrm{new}}\}}} : \Pi_{(\mathcal{G}_x)_{\rightsquigarrow\{e^{\mathrm{F}}_{\mathrm{new}}\}}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ [cf. [CbTpI], Definition 2.10], then $\Pi^*_{\mathrm{F^{sub}}}$ may be regarded as a *verticial subgroup* of $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e^{\mathrm{F}}_{\mathrm{new}}\}}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ [cf. [CbTpI], Proposition 2.9, (i), (3)]. Thus, it follows from Proposition 1.7, (vii), that $\Pi^*_{\mathrm{F^{sub}}}$ is *commensurably terminal* in Π^*_{F}.

Next, let us observe that, by applying a similar argument to the argument given in [CmbCsp], Definition 2.1, (iii), (vi), or [NodNon], Definition 5.1, (ix), (x) [i.e., roughly speaking, by considering the portion of the underlying scheme X_2 of $X_2^{\log}$ corresponding to the underlying scheme $(X_v)_2$ of the 2-nd log configuration space $(X_v)_2^{\log}$ of the stable log curve $X_v^{\log}$ determined by $\mathcal{G}|_v$ — cf. [CbTpI], Definition 2.1, (iii)], one concludes that there exists a verticial subgroup $\Pi_v \subseteq \Pi_{\mathcal{G}} \xleftarrow{\sim} \Pi_{\mathrm{B}}$ associated to $v \in \mathrm{Vert}(\mathcal{G})$ such that the outer action of Π_v on Π^*_{F} determined by the composite $\Pi_v \hookrightarrow \Pi_{\mathrm{B}} \xrightarrow{\rho^*_{2/1}} \mathrm{Out}(\Pi^*_{\mathrm{F}})$ — where we write $\rho^*_{2/1}$ for the outer action determined by the exact sequence of profinite groups

$$1 \longrightarrow \Pi^*_{\mathrm{F}} \longrightarrow \Pi^*_{\mathrm{T}} \longrightarrow \Pi_{\mathrm{B}} \longrightarrow 1$$

— preserves the Π^*_{F}-conjugacy class of $\Pi^*_{\mathrm{F^{sub}}} \subseteq \Pi^*_{\mathrm{F}}$ [so we obtain a natural outer representation $\Pi_v \to \mathrm{Out}(\Pi^*_{\mathrm{F^{sub}}})$ — cf. [CbTpI], Lemma 2.12, (iii)], and, moreover, that if we write $\Pi^*_{\mathrm{T^{sub}}} \overset{\mathrm{def}}{=} \Pi^*_{\mathrm{F^{sub}}} \overset{\mathrm{out}}{\rtimes} \Pi_v$ $(\subseteq \Pi^*_{\mathrm{T}})$ [cf. the discussion entitled "*Topological groups*" in [CbTpI], §0], then $\Pi^*_{\mathrm{T^{sub}}}$ is naturally isomorphic to a profinite group of the form "Π^*_{T}" obtained by taking "$\mathcal{G}$" to be $\mathcal{G}|_v$.

Now since $\widetilde{\beta}^*(\Pi^*_{\mathrm{F^{sub}}}) = \Pi^*_{\mathrm{F^{sub}}}$, and $\widetilde{\alpha}^*$ is an automorphism over the quotient $\Pi^*_{\mathrm{F}}/\Pi^*_{\mathrm{T}} \xrightarrow{\sim} \Pi_{\mathrm{B}}$, one verifies immediately that $\widetilde{\beta}^*$ determines an automorphism $\widetilde{\beta}^*_{\mathrm{T^{sub}}}$ of $\Pi^*_{\mathrm{T^{sub}}}$ over Π_v. Thus, since $\mathcal{G}|_v$ is of type $(0,3)$ [cf. [CbTpI], Definition 2.3, (i)], by considering a diagram similar to the diagram of [CmbCsp], Definition 2.1, (vi), or [NodNon], Definition 5.1, (x), and applying Claim 2.8.A [cf. also Lemma 2.6, (ii)], we conclude — by replacing Π^{**}_{F} by a suitable *base-admissible* maximal almost pro-Σ quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ that *dominates* the *original* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ — that we may assume without loss of generality that

($\ddagger_4$): $\widetilde{\beta}^*_{\mathrm{T^{sub}}}$ is a $\Pi^*_{\mathrm{F^{sub}}}$*-inner automorphism.*

Moreover, since $\widetilde{\beta}^*$ *fixes* and induces the *identity automorphism* on $\Pi^*_{v^{\mathrm{F}}_x}$ [cf. the discussion following ($\ddagger_3$)], and $\Pi^*_{v^{\mathrm{F}}_x}$ is *commensurably terminal* in [Π^*_{F}, hence also in] $\Pi^*_{\mathrm{F^{sub}}}$ [cf. Proposition 1.7, (vii)] and *slim* [cf. Proposition 1.7, (ii)], we conclude that $\widetilde{\beta}^*_{\mathrm{T^{sub}}}$ is the *identity automorphism*; in particular, since $\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}} \subseteq \Pi^*_{\mathrm{F^{sub}}}$, $\widetilde{\beta}^*$ induces the *identity* *automorphism* on $\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}$. Thus, since Π^*_{F} is topologically generated by $\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$ and $\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}$ [cf. [CmbCsp], Proposition 2.2, (iii)], it follows from ($\ddagger_3$) that $\widetilde{\beta}^*$ is the *identity automorphism*. This completes the proof of Claim 2.8.B.

Finally, we claim that the following assertion holds:

Claim 2.8.C: Lemma 2.8 holds for arbitrary (g, r).

We verify Claim 2.8.C by *induction on* $3g - 3 + r$. If $3g - 3 + r = 0$, i.e., $(g, r) = (0, 3)$, then Claim 2.8.C amounts to Claim 2.8.A. On the other hand, if $(g, r) = (1, 1)$, then Claim 2.8.C amounts to Claim 2.8.B. Thus, we suppose that $3g - 3 + r > 0$, that $(g, r) \neq (1, 1)$, and that the *induction hypothesis* is in force. Since $3g - 3 + r > 0$ and $(g, r) \neq (1, 1)$, one verifies easily that there exists a stable log curve $Y^{\log}$ of type (g, r) over $(\mathrm{Spec}\, k)^{\log}$ such that $Y^{\log}$ has *precisely one node* and *precisely two vertices*. Thus, by working with *2-cuspidalizable degeneration structures* [cf. [CbTpII], Definition 3.23, (i), (v)] that arise *scheme-theoretically* via a *specialization isomorphism* as in the discussion preceding [CmbCsp], Definition 2.1 [cf. also [CbTpI], Remark 5.6.1], we may replace $X^{\log}$ by $Y^{\log}$ and assume without loss of generality that $(\mathrm{Vert}(\mathcal{G})^\sharp, \mathrm{Node}(\mathcal{G})^\sharp) = (2, 1)$.

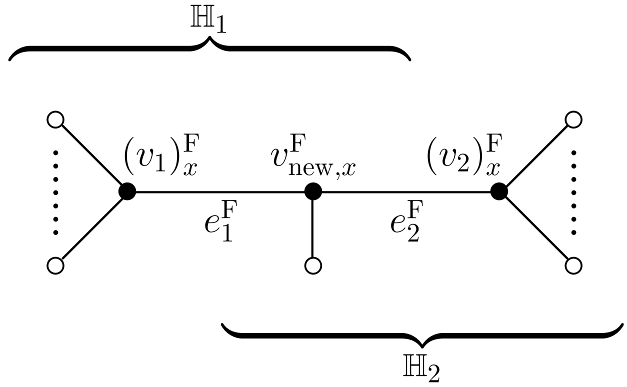

Figure 2: $\mathcal{G}_x$

Let e be the *unique* node of $\mathcal{G}$ and $x \in X(k)$ such that $x \curvearrowright e$ [cf. [CbTpI], Definition 6.3, (iii)]. Next, let us observe that since $\mathrm{Node}(\mathcal{G})^\sharp = \{e\}^\sharp = 1$, it follows from the property (b) discussed above that $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_x})$ ($\subseteq \mathrm{Out}(\Pi^*_{\mathcal{G}_x}) \xleftarrow{\sim} \mathrm{Out}(\Pi^*_{\mathrm{F}})$). Write $\{e^{\mathrm{F}}_1, e^{\mathrm{F}}_2\} = \mathcal{N}(v^{\mathrm{F}}_{\mathrm{new},x})$ [cf. [CbTpI], Lemma 6.4, (iv)]. Also, for $i \in \{1, 2\}$, denote by $v_i \in \mathrm{Vert}(\mathcal{G})$ the vertex of $\mathcal{G}$ such that $(v_i)^{\mathrm{F}}_x \in \mathrm{Vert}(\mathcal{G}_x)$ [cf. [CbTpI], Definition 6.3, (ii)] is the *unique* element of $\mathcal{V}(e^{\mathrm{F}}_i) \setminus \{v^{\mathrm{F}}_{\mathrm{new},x}\}$ [cf. [CbTpI], Lemma 6.4, (iv)]; by $\mathbb{H}_i$ the sub-semi-graph of PSC-type [cf. [CbTpI], Definition 2.2, (i)] of the underlying semi-graph $\mathbb{G}_x$ of $\mathcal{G}_x$ whose set of vertices $= \{v^{\mathrm{F}}_{\mathrm{new},x}, (v_i)^{\mathrm{F}}_x\}$ [cf. Figure 2 above].

For $i \in \{1,2\}$, let $\Pi^*_{(v_i)^{\mathrm{F}}_x} \subseteq \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ be a verticial subgroup of $\Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ associated to the vertex $(v_i)^{\mathrm{F}}_x \in \mathcal{V}(e^{\mathrm{F}}_i) \setminus \{v^{\mathrm{F}}_{\mathrm{new},x}\}$. Then since $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_x})$, it follows that $\widetilde{\alpha}^*$ preserves the Π^*_{F}-conjugacy class of $\Pi^*_{(v_i)^{\mathrm{F}}_x} \subseteq \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$. Thus, since the image of the composite $\Pi^*_{(v_i)^{\mathrm{F}}_x} \hookrightarrow \Pi^*_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}/N^*_{\mathrm{diag}}$ is *commensurably terminal* [cf. Proposition 1.7, (vii); Lemma 2.6, (i)], it follows immediately from the property (a) discussed above that there exists an N^*_{diag}-conjugate $\widetilde{\beta}^*_i$ [which *may depend* on $i \in \{1,2\}$!] of $\widetilde{\alpha}^*$ such that $\widetilde{\beta}^*_i(\Pi^*_{(v_i)^{\mathrm{F}}_x}) = \Pi^*_{(v_i)^{\mathrm{F}}_x}$. Therefore, it follows immediately from Lemma 2.6, (iii) — by replacing Π^{**}_{F} by a suitable *base-admissible* maximal almost pro-Σ quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ [i.e., a quotient as in Lemma 2.6, (iii)] that *dominates* the *original* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ and applying the conclusion "$\widetilde{\beta}^*_i(\Pi^*_{(v_i)^{\mathrm{F}}_x}) = \Pi^*_{(v_i)^{\mathrm{F}}_x}$", together with the property (a) discussed above, in the case where "α^*" is taken to be $\alpha^{***} \in \mathrm{Aut}(\Pi^{***}_{\mathrm{T}})$ — that we may assume without loss of generality that

$$(\ddagger_5)\colon\ \widetilde{\beta}^*_i \text{ induces the } \textit{identity automorphism} \text{ of } \Pi^*_{(v_i)^{\mathrm{F}}_x}.$$

Next, let $\Pi^*_{e^{\mathrm{F}}_i} \subseteq \Pi^*_{(v_i)^{\mathrm{F}}_x}$ be a nodal subgroup of $\Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ associated to $e^{\mathrm{F}}_i \in \mathrm{Node}(\mathcal{G}_x)$ that is contained in $\Pi^*_{(v_i)^{\mathrm{F}}_x}$; $\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x};i} \subseteq \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ a verticial subgroup [which *may depend* on $i \in \{1,2\}$!] associated to $v^{\mathrm{F}}_{\mathrm{new},x} \in \mathrm{Vert}(\mathcal{G}_x)$ that contains $\Pi^*_{e^{\mathrm{F}}_i}$:

$$\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x};i} \quad \supseteq \quad \Pi^*_{e^{\mathrm{F}}_i} \quad \subseteq \quad \Pi^*_{(v_i)^{\mathrm{F}}_x} \quad \subseteq \quad \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}\,.$$

Then it follows from the inclusion $\Pi^*_{e^{\mathrm{F}}_i} \subseteq \Pi^*_{(v_i)^{\mathrm{F}}_x}$, together with $(\ddagger_5)$, that $\widetilde{\beta}^*_i(\Pi^*_{e^{\mathrm{F}}_i}) = \Pi^*_{e^{\mathrm{F}}_i}$. Since, moreover, $\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x};i}$ is the *unique* verticial subgroup of $\Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ associated to $v^{\mathrm{F}}_{\mathrm{new},x}$ that contains $\Pi^*_{e^{\mathrm{F}}_i}$ [cf. Proposition 1.7, (v), (vi)], it follows immediately from the fact that $\alpha^*_{\mathrm{F}} \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi^*_{\mathcal{G}_x})$ that $\widetilde{\beta}^*_i(\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x};i}) = \Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x};i}$. Thus, $\widetilde{\beta}^*_i$ *preserves* the closed subgroup $\Pi^*_{\mathrm{F}_i} \subseteq \Pi^*_{\mathrm{F}}$ of Π^*_{F} obtained by forming the image of the natural homomorphism

$$\varinjlim \left(\Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x};i} \hookleftarrow \Pi^*_{e^{\mathrm{F}}_i} \hookrightarrow \Pi^*_{(v_i)^{\mathrm{F}}_x} \right) \longrightarrow \Pi^*_{\mathrm{F}}$$

— where the inductive limit is taken in the category of profinite groups.

Next, let us observe that the Π^*_{F}-conjugacy class of $\Pi^*_{\mathrm{F}_i} \subseteq \Pi^*_{\mathrm{F}}$ *coincides* with the Π^*_{F}-conjugacy class of the image $\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}_i}}$ [cf. [CbTpI], Definition 2.2, (ii)] of the composite

$$\Pi_{(\mathcal{G}_x)|_{\mathbb{H}_i}} \hookrightarrow \Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{\mathrm{F}} \twoheadrightarrow \Pi^*_{\mathrm{F}}$$

— where the first arrow is the natural outer injection discussed in [CbTpI], Proposition 2.11. On the other hand, if we write $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e^{\mathrm{F}}_i\}}}$ for the *maximal almost pro-*Σ *quotient* of $\Pi_{(\mathcal{G}_x)_{\rightsquigarrow\{e_i^{\mathrm{F}}\}}}$ [cf. [CbTpI], Definition 2.8] determined by the *maximal almost pro-*Σ *quotient* $\Pi^*_{\mathcal{G}_x}$ and the natural outer isomorphism $\Phi_{(\mathcal{G}_x)_{\rightsquigarrow\{e_i^{\mathrm{F}}\}}} : \Pi_{(\mathcal{G}_x)_{\rightsquigarrow\{e_i^{\mathrm{F}}\}}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ [cf. [CbTpI], Definition 2.10], then $\Pi^*_{\mathrm{F}_i}$ may be regarded as a *verticial subgroup* of $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e_i^{\mathrm{F}}\}}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_{\mathrm{F}}$ [cf. [CbTpI], Proposition 2.9, (i), (3)]. Thus, it follows from Proposition 1.7, (vii), that $\Pi^*_{\mathrm{F}_i}$ is *commensurably terminal* in Π^*_{F}. Moreover, by applying a similar argument to the argument given in [CmbCsp], Definition 2.1, (iii), (vi), or [NodNon], Definition 5.1, (ix), (x) [i.e., roughly speaking, by considering the portion of the underlying scheme X_2 of $X_2^{\log}$ corresponding to the underlying scheme $(X_{v_i})_2$ of the 2-nd log configuration space $(X_{v_i})_2^{\log}$ of the stable log curve $X_{v_i}^{\log}$ determined by $\mathcal{G}|_{v_i}$ — cf. [CbTpI], Definition 2.1, (iii)], one concludes that there exists a verticial subgroup $\Pi_{v_i} \subseteq \Pi_{\mathcal{G}} \xleftarrow{\sim} \Pi_{\mathrm{B}}$ associated to $v_i \in \mathrm{Vert}(\mathcal{G})$ such that the outer action of Π_{v_i} on Π^*_{F} determined by the composite $\Pi_{v_i} \hookrightarrow \Pi_{\mathrm{B}} \stackrel{\rho^*_{2/1}}{\to} \mathrm{Out}(\Pi^*_{\mathrm{F}})$ — where we write $\rho^*_{2/1}$ for the outer action determined by the exact sequence of profinite groups

$$1 \longrightarrow \Pi^*_{\mathrm{F}} \longrightarrow \Pi^*_{\mathrm{T}} \longrightarrow \Pi_{\mathrm{B}} \longrightarrow 1$$

— preserves the Π^*_{F}-conjugacy class of $\Pi^*_{\mathrm{F}_i} \subseteq \Pi^*_{\mathrm{F}}$ [so we obtain a natural outer representation $\Pi_{v_i} \to \mathrm{Out}(\Pi^*_{\mathrm{F}_i})$ — cf. [CbTpI], Lemma 2.12, (iii)], and, moreover, that if we write $\Pi^*_{\mathrm{T}_i} \stackrel{\mathrm{def}}{=} \Pi^*_{\mathrm{F}_i} \stackrel{\mathrm{out}}{\rtimes} \Pi_{v_i}$ $(\subseteq \Pi^*_{\mathrm{T}})$ [cf. the discussion entitled "*Topological groups*" in [CbTpI], §0], then $\Pi^*_{\mathrm{T}_i}$ is naturally isomorphic to a profinite group of the form "Π^*_{T}" obtained by taking "$\mathcal{G}$" to be $\mathcal{G}|_{v_i}$.

Now since $\widetilde{\beta}^*_i(\Pi^*_{\mathrm{F}_i}) = \Pi^*_{\mathrm{F}_i}$, and $\widetilde{\alpha}^*$ is an automorphism over the quotient $\Pi^*_{\mathrm{F}}/\Pi^*_{\mathrm{T}} \xrightarrow{\sim} \Pi_{\mathrm{B}}$, one verifies immediately that $\widetilde{\beta}_i$ determines an automorphism $\widetilde{\beta}^*_{\mathrm{T}_i}$ of $\Pi^*_{\mathrm{T}_i}$ over Π_{v_i}. Thus, since the quantity "$3g-3+r$" associated to $\mathcal{G}|_{v_i}$ is $< 3g-3+r$, by considering a diagram similar to the diagram of [CmbCsp], Definition 2.1, (vi), or [NodNon], Definition 5.1, (x), and applying the *induction hypothesis* [cf. also Lemma 2.6, (ii)], we conclude — by replacing Π^{**}_{F} by a suitable *base-admissible* maximal almost pro-Σ quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ that *dominates* the *original* $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi^{**}_{\mathrm{F}}$ — that we may assume without loss of generality that

($\ddagger_6$): $\widetilde{\beta}^*_{\mathrm{T}_i}$ is a $\Pi^*_{\mathrm{F}_i}$*-inner automorphism*.

In particular, it follows immediately, by allowing $i \in \{1,2\}$ to *vary*, that the outomorphisms of $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e_1^{\mathrm{F}}\}}}$, $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e_2^{\mathrm{F}}\}}}$ obtained by conjugating α^*_{F} by the isomorphisms $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e_1^{\mathrm{F}}\}}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x}$ [induced by $\Phi_{(\mathcal{G}_x)_{\rightsquigarrow\{e_1^{\mathrm{F}}\}}}$], $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e_2^{\mathrm{F}}\}}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x}$ [induced by $\Phi_{(\mathcal{G}_x)_{\rightsquigarrow\{e_2^{\mathrm{F}}\}}}$] are *profinite Dehn multi-twists* of $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e_1^{\mathrm{F}}\}}}$, $\Pi^*_{(\mathcal{G}_x)_{\rightsquigarrow\{e_2^{\mathrm{F}}\}}}$, respectively. Thus, it follows from Lemma 1.10

that α^*_{F} is the *identity outomorphism*. This completes the proof of Claim 2.8.C, hence also of Lemma 2.8. □

Theorem 2.9 (Almost pro-Σ analogue of the injectivity portion of the theory of combinatorial cuspidalization). *Let Σ be a nonempty set of prime numbers, n a positive integer, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and X a* **hyperbolic curve** *of type (g, r) over an algebraically closed field of characteristic zero. For each positive integer i, write X_i for the i-th* **configuration space** *of X [cf.* [MzTa], *Definition 2.1, (i)]; Π_i for the pro-$\mathfrak{Primes}$* **configuration space group** *[cf.* [MzTa], *Definition 2.3, (i)] given by the étale fundamental group $\pi_1(X_i)$ of X_i. Also, we shall write* pr: $X_{n+1} \twoheadrightarrow X_n$ *for the projection obtained by forgetting the $(n+1)$-st factor,* $\mathrm{pr}^\Pi \colon \Pi_{n+1} \twoheadrightarrow \Pi_n$ *for the surjection induced by* pr*, and $\Pi_{n+1/n} \subseteq \Pi_{n+1}$ for the kernel of the surjection* pr^Π. *Let $\Pi_{n+1} \twoheadrightarrow \Pi^*_{n+1}$ be a quotient of Π_{n+1} such that the quotient $\Pi^*_{n+1/n}$ of $\Pi_{n+1/n} \subseteq \Pi_{n+1}$ determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^*_{n+1}$ is a* **maximal almost pro-Σ quotient** *of $\Pi_{n+1/n}$ [cf. Definition 1.1]. Then there exists a quotient $\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1}$ of Π_{n+1} such that the following conditions are satisfied:*

(i) *The quotient $\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1}$* **dominates** *[cf. the discussion entitled "Topological groups" in §0] the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^*_{n+1}$ [i.e., $\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1} \twoheadrightarrow \Pi^*_{n+1}$].*

(ii) *The quotient $\Pi^{**}_{n+1/n}$ of $\Pi_{n+1/n} \subseteq \Pi_{n+1}$ determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1}$ is a* **maximal almost pro-Σ quotient** *of $\Pi_{n+1/n}$.*

(iii) *Let α^* be an outomorphism of Π^*_{n+1} and $\Pi_{n+1} \twoheadrightarrow \Pi^{***}_{n+1}$ a quotient that* **dominates** *the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1}$ and induces a* **maximal almost pro-Σ quotient** *$\Pi^{***}_{n+1/n}$ of $\Pi_{n+1/n}$. Suppose that α^* arises from an* **FC-admissible** *[cf. Definition 2.1, (v)] automorphism $\widetilde{\alpha}^{***}$ of Π^{***}_{n+1} over Π^{***}_n — where we write Π^{***}_n for the quotient of Π_n determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^{***}_{n+1}$. Then α^* is the* **identity outomorphism**.

Proof. First, we claim that the following assertion holds:

> Claim 2.9.A: To verify Theorem 2.9, it suffices to verify Theorem 2.9 in the case where the kernel of the natural surjection $\Pi_{n+1} \twoheadrightarrow \Pi^*_{n+1}$ is contained in $\Pi_{n+1/n}$, i.e., the natural surjection
>
> $$\Pi_n \stackrel{\sim}{\leftarrow} \Pi_{n+1}/\Pi_{n+1/n} \twoheadrightarrow \Pi^*_{n+1}/\Pi^*_{n+1/n}$$
>
> — where the first arrow is the natural isomorphism — is an *isomorphism*.

Indeed, Claim 2.9.A follows immediately, by considering the objects obtained by base-changing the various objects involved via the natural surjection $\Pi_n \overset{\sim}{\leftarrow} \Pi_{n+1}/\Pi_{n+1/n} \twoheadrightarrow \Pi^*_{n+1}/\Pi^*_{n+1/n}$. By Claim 2.9.A, we may assume without loss of generality that the kernel of $\Pi_{n+1} \twoheadrightarrow \Pi^*_{n+1}$ is contained in $\Pi_{n+1/n}$.

Next, we claim that the following assertion holds:

> Claim 2.9.B: To verify Theorem 2.9, it suffices to verify Theorem 2.9 in the case where $n = 1$.

Indeed, suppose that $n \geq 2$, and that Theorem 2.9 holds whenever $n = 1$. Write $\Pi_{n+1/n-1} \subseteq \Pi_{n+1}$ for the kernel of the surjection $\Pi_{n+1} \twoheadrightarrow \Pi_{n-1}$ induced by the projection $X_{n+1} \to X_{n-1}$ obtained by forgetting the $(n+1)$-st and n-th factors of X_{n+1}; $\Pi_{n/n-1} \subseteq \Pi_n$ for the kernel of the surjection $\Pi_n \twoheadrightarrow \Pi_{n-1}$ induced by the projection $X_n \to X_{n-1}$ obtained by forgetting the n-th factor of X_n; $\Pi^*_{n+1/n-1}$ for the quotient of $\Pi_{n+1/n-1}$ determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^*_{n+1}$. Then let us recall [cf. [MzTa], Proposition 2.4, (i)] that one may *interpret* the surjection $\Pi^*_{n+1/n-1} \twoheadrightarrow \Pi_{n/n-1}$ induced by the surjection $\mathrm{pr}^\Pi \colon \Pi_{n+1} \twoheadrightarrow \Pi_n$ as the surjection "$\mathrm{pr}^\Pi \colon \Pi^*_2 \twoheadrightarrow \Pi_1$" in the case where "$X$" is of type $(g, r+n-1)$. Thus, by applying Theorem 2.9 in the case where $n = 1$ to the quotient $\Pi_{n+1/n-1} \twoheadrightarrow \Pi^*_{n+1/n-1}$, we obtain a quotient $\Pi^{**}_{n+1/n-1}$ of $\Pi_{n+1/n-1}$ which satisfies conditions (i), (ii), (iii) in the statement of Theorem 2.9. [Here, we note that since the kernel of $\Pi_{n+1/n-1} \twoheadrightarrow \Pi^*_{n+1/n-1}$ is contained in $\Pi_{n+1/n}$, the kernel of $\Pi_{n+1/n-1} \twoheadrightarrow \Pi^{**}_{n+1/n-1}$ is also contained in $\Pi_{n+1/n}$.]

Next, let $N \subseteq \Pi_{n+1/n}$ be a normal open subgroup of $\Pi_{n+1/n}$ with respect to which $\Pi^{**}_{n+1/n}$ is a *maximal almost pro-*Σ *quotient* of $\Pi_{n+1/n}$. Then it follows immediately from Lemma 1.2, (iii), that we may assume without loss of generality — by replacing N by a suitable normal open subgroup contained in N — that the kernel of $\Pi_{n+1/n} \twoheadrightarrow \Pi^{**}_{n+1/n}$ is *normal* in Π_{n+1}. Write Π^{**}_{n+1} for the quotient of Π_{n+1} by the kernel of $\Pi_{n+1/n} \twoheadrightarrow \Pi^{**}_{n+1/n}$. Then it is immediate that this quotient Π^{**}_{n+1} satisfies conditions (i), (ii) in the statement of Theorem 2.9, and, moreover, that the kernel of $\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1}$ is contained in $\Pi_{n+1/n}$. To verify that Π^{**}_{n+1} satisfies condition (iii) in the statement of Theorem 2.9, let $\Pi_{n+1} \twoheadrightarrow \Pi^{***}_{n+1}$ be a quotient as in condition (iii) in the statement of Theorem 2.9, $\widetilde{\alpha}^*$ an automorphism of Π^*_{n+1} which arises from an *FC-admissible* automorphism $\widetilde{\alpha}^{***}$ of Π^{***}_{n+1} over Π_n. Then since $\widetilde{\alpha}^{***}$ is *FC-admissible*, it is immediate that $\widetilde{\alpha}^{***}$ preserves $\Pi^{***}_{n+1/n-1} \subseteq \Pi^{***}_{n+1}$, where we write $\Pi^{***}_{n+1/n-1}$ for the quotient of $\Pi_{n+1/n-1}$ determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^{***}_{n+1}$. In particular, it follows from our choice of $\Pi^{**}_{n+1/n-1}$, together with the fact that $\widetilde{\alpha}^{***}$ is an automorphism of Π^{***}_{n+1} over Π_n [which implies that $\widetilde{\alpha}^*$ is an automorphism of Π^*_{n+1} over Π_n], that we may assume without loss of generality — i.e., by replacing

$\widetilde{\alpha}^*$ by a suitable $\Pi^*_{n+1/n-1}$-conjugate, which may in fact [in light of the *slimness* of $\Pi_{n/n-1}$ — cf., e.g., [CmbGC], Remark 1.1.3] be taken to be a $\Pi^*_{n+1/n}$-conjugate — that the automorphism of $\Pi^*_{n+1/n}$ induced by $\widetilde{\alpha}^*$ is the *identity automorphism*. Thus, since $\widetilde{\alpha}^*$ is an automorphism of Π^*_{n+1} over Π_n, and $\Pi^*_{n+1/n}$ is *slim* [cf. Proposition 1.7, (i)], we may apply the natural isomorphism $\Pi^*_{n+1} \xrightarrow{\sim} \Pi^*_{n+1/n} \overset{\text{out}}{\rtimes} \Pi_n$ [cf. the discussion entitled "*Topological groups*" in [CbTpI], §0] to conclude [cf., e.g., [Hsh], Lemma 4.10] that the automorphism $\widetilde{\alpha}^*$ of Π^*_{n+1} is the *identity automorphism*. In particular, we conclude that Π^{**}_{n+1} satisfies condition (iii) in the statement of Theorem 2.9. This completes the proof of Claim 2.9.B.

By Claim 2.9.B, we may assume without loss of generality that $n = 1$. On the other hand, if $n = 1$, then one verifies easily that Theorem 2.9 follows immediately from Lemma 2.8. This completes the proof of Theorem 2.9. □

Corollary 2.10 (**Almost pro-*l* analogue of the injectivity portion of the theory of combinatorial cuspidalization**). *Let l be a prime number, n a positive integer, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and X a* **hyperbolic curve** *of type (g, r) over an algebraically closed field of characteristic zero. For each positive integer i, write X_i for the i-th* **configuration space** *of X [cf.* [MzTa], *Definition 2.1, (i)]; Π_i for the pro-$\mathfrak{Primes}$* **configuration space group** *[cf.* [MzTa], *Definition 2.3, (i)] given by the étale fundamental group $\pi_1(X_i)$ of X_i. Let $\Pi_{n+1} \twoheadrightarrow \Pi^*_{n+1}$ be an* **F-characteristic SA-maximal almost pro-*l* quotient** *of Π_{n+1} [cf. Definition 2.1, (ii), (iii)]. Then there exists an* **F-characteristic SA-maximal almost pro-*l* quotient** *$\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1}$ of Π_{n+1} such that $\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1}$* **dominates** *[cf. the discussion entitled "Topological groups" in §0] the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^*_{n+1}$, and, moreover, satisfies the following property: For any* **F-characteristic SA-maximal almost pro-*l* quotient** *$\Pi_{n+1} \twoheadrightarrow \Pi^{***}_{n+1}$ of Π_{n+1} that* **dominates** *the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^{**}_{n+1}$, the image of the composite*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi^{***}_{n+1} \twoheadrightarrow \Pi^*_{n+1}) \ \cap\ \mathrm{Ker}\Big(\mathrm{Out}^{\mathrm{FC}}(\Pi^{***}_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi^{***}_n)\Big)$$

$$\hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi^{***}_{n+1} \twoheadrightarrow \Pi^*_{n+1}) \twoheadrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi^*_{n+1} \twoheadleftarrow \Pi^{***}_{n+1}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi^*_{n+1})$$

[cf. Definition 2.1, (vii), (viii)] — where we write Π^{***}_n for the quotient of Π_n determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi^{***}_{n+1}$, and the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi^{***}_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi^{***}_n)$ [in large parentheses] is the homomorphism induced by the projection $X_{n+1} \to X_n$ obtained by forgetting the $(n+1)$-st factor — is **trivial**.

Proof. This follows immediately from Theorem 2.9, together with Proposition 2.3, (ii). □

Remark 2.10.1.

(i) Theorem 2.9 and Corollary 2.10 may be regarded, respectively, as *almost pro-*Σ, *almost pro-l* versions of the *injectivity portion* of [NodNon], Theorem B. In this context, it is of interest to recall that the *pro-l* version of this sort of injectivity result may also be obtained by means of the *Lie-theoretic approach* of [Tk]. On the other hand, it does not appear, at the time of writing, that this Lie-theoretic approach may be extended so as to yield an alternate proof *either* of the *profinite* portion of the injectivity result of [NodNon], Theorem B, *or* of the *almost pro-*Σ*/pro-l* versions of this result given in Theorem 2.9, Corollary 2.10 of the present paper.

(ii) In the context of the observations of (i), it is of interest to recall that the various injectivity results of [NodNon] and the present paper that are discussed in (i) are obtained as consequences of various *combinatorial versions of the Grothendieck Conjecture.* From this point of view, it seems natural to pose the following question:

> Is it possible to prove a *Lie-theoretic combinatorial version of the Grothendieck Conjecture* that allows one to derive the *Lie-theoretic injectivity* results of [Tk] by means of techniques analogous to the techniques applied in [NodNon] and the present paper?

At the time of writing, it is not clear to the authors whether or not this question may be answered in the affirmative.

In the remainder of §2, we consider an *almost pro-l analogue* of the *tripod homomorphism* of [CbTpII], Definition 3.19.

Lemma 2.11 (Commensurators of various subgroups of geometric origin). *We shall apply the notational conventions established in §3 of* [CbTpII]. *In the notation of* [CbTpII], *Lemma 3.6, suppose that* $(j, i) = (1, 2)$*;* $E = \{i, j\}$*;* $z_{i,j,x} \in \mathrm{Edge}(\mathcal{G}_{j\in E\setminus\{i\},x})$*. [Thus,* $\mathcal{G}_{j\in E\setminus\{i\},x} = \mathcal{G}_{i\in E\setminus\{j\},x} = \mathcal{G}$*;* $\Pi_2 = \Pi_E$*;* $\Pi_1 = \Pi_{\{j\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{j\in E\setminus\{i\},x}} = \Pi_{\mathcal{G}}$*;* $\Pi_{2/1} = \Pi_{E/(E\setminus\{i\})} \xrightarrow{\sim} \Pi_{\mathcal{G}_{i\in E,x}}$*.] Write* $\mathcal{G}_{2/1} \stackrel{\mathrm{def}}{=} \mathcal{G}_{i\in E,x}$*;* $\mathcal{G}_{1\setminus 2} \stackrel{\mathrm{def}}{=} \mathcal{G}_{j\in E,x}$*;* $p^{\Pi}_{1\setminus 2} \stackrel{\mathrm{def}}{=} p^{\Pi}_{E/\{2\}} \colon \Pi_2 \twoheadrightarrow \Pi_{\{2\}}$*;* $\Pi_{1\setminus 2} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(p^{\Pi}_{1\setminus 2}) = \Pi_{E/\{2\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{1\setminus 2}}$*;* $z_x \stackrel{\mathrm{def}}{=} z_{i,j,x} \in \mathrm{Edge}(\mathcal{G})$*;* $c^{\mathrm{diag}} \stackrel{\mathrm{def}}{=} c^{\mathrm{diag}}_{i,j,x} \in \mathrm{Cusp}(\mathcal{G}_{2/1})$ *[cf. the notation* *of* [CbTpII], *Lemma 3.6, (ii)]; $v^{\mathrm{new}} \stackrel{\mathrm{def}}{=} v^{\mathrm{new}}_{i,j,x} \in \mathrm{Vert}(\mathcal{G}_{2/1})$ [cf. the notation of* [CbTpII], *Lemma 3.6, (iv)]. Let $\Pi_{z_x} \subseteq \Pi_1$ be an edge-like subgroup associated to $z_x \in \mathrm{Edge}(\mathcal{G})$; $\Pi_{v^{\mathrm{new}}} \subseteq \Pi_{2/1}$ a verticial subgroup associated to v^{new}; $\Pi_{c^{\mathrm{diag}}} \subseteq \Pi_{2/1}$ a cuspidal subgroup associated to c^{diag} that is* **contained** *in $\Pi_{v^{\mathrm{new}}}$ [cf.* [CbTpII], *Lemma 3.6, (iv)]. Let $\Pi_2 \twoheadrightarrow \Pi_2^*$ be an* **SA-maximal almost pro-l quotient** *of Π_2 [cf. Definition 2.1, (ii)]. Write $\Pi_{2/1}^*$, $\Pi_{1\backslash 2}^*$, Π_1^*, $\Pi_{\{2\}}^*$ for the respective quotients of $\Pi_{2/1}$, $\Pi_{1\backslash 2}$, Π_1, $\Pi_{\{2\}}$ determined by the quotient $\Pi_2 \twoheadrightarrow \Pi_2^*$ of Π_2; $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{G}_{2/1}}^*$ for the respective quotients of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{G}_{2/1}}$ determined by the quotients $\Pi_1 \twoheadrightarrow \Pi_1^*$, $\Pi_{2/1} \twoheadrightarrow \Pi_{2/1}^*$ and the isomorphisms $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$, $\Pi_{2/1} \xrightarrow{\sim} \Pi_{\mathcal{G}_{2/1}}$ fixed in* [CbTpII], *Definition 3.1, (iii); $(p_{2/1}^{\Pi})^* \colon \Pi_2^* \twoheadrightarrow \Pi_1^*$, $(p_{1\backslash 2}^{\Pi})^* \colon \Pi_2^* \twoheadrightarrow \Pi_{\{2\}}^*$ for the respective natural surjections induced by $p_{2/1}^{\Pi} \colon \Pi_2 \twoheadrightarrow \Pi_1$, $p_{1\backslash 2}^{\Pi} \colon \Pi_2 \twoheadrightarrow \Pi_{\{2\}}$; $\Pi_{z_x}^* \subseteq \Pi_1^*$, $\Pi_{c^{\mathrm{diag}}}^* \subseteq \Pi_{v^{\mathrm{new}}}^* \subseteq \Pi_{2/1}^*$ for the respective images of $\Pi_{z_x} \subseteq \Pi_1$, $\Pi_{c^{\mathrm{diag}}} \subseteq \Pi_{v^{\mathrm{new}}} \subseteq \Pi_{2/1}$ in Π_1^*, $\Pi_{2/1}^*$; $\Pi_2^*|_{z_x} \stackrel{\mathrm{def}}{=} \Pi_2^* \times_{\Pi_1^*} \Pi_{z_x}^* \subseteq \Pi_2^*$; $D_{c^{\mathrm{diag}}}^* \stackrel{\mathrm{def}}{=} N_{\Pi_2^*}(\Pi_{c^{\mathrm{diag}}}^*)$; $I_{v^{\mathrm{new}}}^*|_{z_x} \stackrel{\mathrm{def}}{=} Z_{\Pi_2^*|_{z_x}}(\Pi_{v^{\mathrm{new}}}^*) \subseteq D_{v^{\mathrm{new}}}^*|_{z_x} \stackrel{\mathrm{def}}{=} N_{\Pi_2^*|_{z_x}}(\Pi_{v^{\mathrm{new}}}^*)$. Then the following hold:*

(i) *It holds that $D_{c^{\mathrm{diag}}}^* \cap \Pi_{2/1}^* = C_{\Pi_2^*}(\Pi_{c^{\mathrm{diag}}}^*) \cap \Pi_{2/1}^* = \Pi_{c^{\mathrm{diag}}}^*$.*

(ii) *It holds that $C_{\Pi_2^*}(\Pi_{c^{\mathrm{diag}}}^*) = D_{c^{\mathrm{diag}}}^*$.*

(iii) *The surjection $(p_{2/1}^{\Pi})^* \colon \Pi_2^* \twoheadrightarrow \Pi_1^*$ determines an* **isomorphism** *$D_{c^{\mathrm{diag}}}^*/\Pi_{c^{\mathrm{diag}}}^* \xrightarrow{\sim} \Pi_1^*$. Moreover, the composite*

$$\Pi_1 \twoheadrightarrow \Pi_1^* \xleftarrow{\sim} D_{c^{\mathrm{diag}}}^*/\Pi_{c^{\mathrm{diag}}}^* \twoheadrightarrow \Pi_{\{2\}}^*$$

— where the first arrow is the natural surjection, the second arrow is the isomorphism obtained above, and the third arrow is the surjection determined by $(p_{1\backslash 2}^{\Pi})^* \colon \Pi_2^* \twoheadrightarrow \Pi_{\{2\}}^*$ — **coincides**, *up to composition with an inner automorphism, with the natural surjection $\Pi_1 \twoheadrightarrow \Pi_{\{2\}}^*$.*

(iv) *The composite $I_{v^{\mathrm{new}}}^*|_{z_x} \hookrightarrow D_{v^{\mathrm{new}}}^*|_{z_x} \to \Pi_{z_x}^*$ is an* **isomorphism**.

(v) *The natural inclusions $\Pi_{v^{\mathrm{new}}}^*$, $I_{v^{\mathrm{new}}}^*|_{z_x} \hookrightarrow D_{v^{\mathrm{new}}}^*|_{z_x}$ determine an* **isomorphism** *$\Pi_{v^{\mathrm{new}}}^* \times I_{v^{\mathrm{new}}}^*|_{z_x} \xrightarrow{\sim} D_{v^{\mathrm{new}}}^*|_{z_x} = C_{\Pi_2^*|_{z_x}}(\Pi_{v^{\mathrm{new}}}^*)$.*

(vi) *It holds that $C_{\Pi_2^*}(D_{v^{\mathrm{new}}}^*|_{z_x}) \subseteq C_{\Pi_2^*}(\Pi_{v^{\mathrm{new}}}^*)$.*

(vii) *$D_{v^{\mathrm{new}}}^*|_{z_x}$ is* **commensurably terminal** *in Π_2^*.*

Proof. First, we verify assertion (i). Observe that we have inclusions $\Pi_{c^{\mathrm{diag}}}^* \subseteq D_{c^{\mathrm{diag}}}^* \subseteq C_{\Pi_2^*}(\Pi_{c^{\mathrm{diag}}}^*)$. Thus, since $\Pi_{c^{\mathrm{diag}}}^*$ is *commensurably terminal* in $\Pi_{2/1}^*$ [cf. Proposition 1.7, (vii)], we conclude that $\Pi_{c^{\mathrm{diag}}}^* \subseteq D_{c^{\mathrm{diag}}}^* \cap \Pi_{2/1}^* \subseteq C_{\Pi_2^*}(\Pi_{c^{\mathrm{diag}}}^*) \cap \Pi_{2/1}^* = C_{\Pi_{2/1}^*}(\Pi_{c^{\mathrm{diag}}}^*) = \Pi_{c^{\mathrm{diag}}}^*$. This completes the proof of assertion (i). Assertions (ii), (iii) follow immediately from assertion (i), together with the [easily verified] fact that the composite $D^*_{c_{\mathrm{diag}}} \hookrightarrow \Pi^*_2 \overset{(p^{\Pi}_{2/1})^*}{\twoheadrightarrow} \Pi^*_1$ is *surjective*.

Next, we verify assertion (iv). Since $\Pi^*_{v^{\mathrm{new}}}$ is *slim* and *commensurably terminal* in $\Pi^*_{2/1}$ [cf. Proposition 1.7, (ii), (vii)], it follows that $I^*_{v^{\mathrm{new}}}|_{z_x} \cap \Pi^*_{2/1} = \{1\}$, which implies the *injectivity* of the composite in question. On the other hand, since the composite $I_{v^{\mathrm{new}}}|_{z_x} \hookrightarrow D_{v^{\mathrm{new}}}|_{z_x} \hookrightarrow \Pi_2|_{z_x} \twoheadrightarrow \Pi_{z_x}$ is *surjective* [cf. [CbTpII], Lemma 3.11, (iv), and its proof], it follows immediately that the composite $I^*_{v^{\mathrm{new}}}|_{z_x} \hookrightarrow D^*_{v^{\mathrm{new}}}|_{z_x} \hookrightarrow \Pi^*_2|_{z_x} \twoheadrightarrow \Pi^*_{z_x}$ is *surjective*. This completes the proof of assertion (iv).

Next, we verify assertion (v). It follows immediately from assertion (iv), together with the *commensurable terminality* of $\Pi^*_{v^{\mathrm{new}}}$ in $\Pi^*_{2/1}$ [cf. Proposition 1.7, (vii)], that we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Pi^*_{v^{\mathrm{new}}} \longrightarrow D^*_{v^{\mathrm{new}}}|_{z_x} \longrightarrow \Pi^*_{z_x} \longrightarrow 1$$

— where we observe that the inclusion $I^*_{v^{\mathrm{new}}}|_{z_x} \hookrightarrow D^*_{v^{\mathrm{new}}}|_{z_x}$ determines a *splitting* of this exact sequence. Thus, it follows from the definition of $I^*_{v^{\mathrm{new}}}|_{z_x}$ that the natural inclusions $\Pi^*_{v^{\mathrm{new}}}, I^*_{v^{\mathrm{new}}}|_{z_x} \hookrightarrow D^*_{v^{\mathrm{new}}}|_{z_x}$ determine an *isomorphism* $\Pi^*_{v^{\mathrm{new}}} \times I^*_{v^{\mathrm{new}}}|_{z_x} \overset{\sim}{\to} D^*_{v^{\mathrm{new}}}|_{z_x}$. On the other hand, again by the *commensurable terminality* of $\Pi^*_{v^{\mathrm{new}}}$ in $\Pi^*_{2/1}$ [cf. Proposition 1.7, (vii)], the above displayed sequence implies that $D^*_{v^{\mathrm{new}}}|_{z_x} = C_{\Pi^*_2|_{z_x}}(\Pi^*_{v^{\mathrm{new}}})$. This completes the proof of assertion (v).

Next, we verify assertion (vi). It follows from the *commensurable terminality* of $\Pi^*_{v^{\mathrm{new}}}$ in $\Pi^*_{2/1}$ [cf. Proposition 1.7, (vii)] that $D^*_{v^{\mathrm{new}}}|_{z_x} \cap \Pi^*_{2/1} = \Pi^*_{v^{\mathrm{new}}}$. Thus, since $\Pi^*_{2/1}$ is *normal* in Π^*_2, assertion (vi) follows immediately from [CbTpII], Lemma 3.9, (i). This completes the proof of assertion (vi).

Finally, we verify assertion (vii). Since $\Pi^*_{z_x} \subseteq \Pi^*_1$ is *commensurably terminal* in Π^*_1 [cf. Proposition 1.7, (vii)], it follows from the *surjectivity* of the composite $D^*_{v^{\mathrm{new}}}|_{z_x} \hookrightarrow \Pi^*_2|_{z_x} \twoheadrightarrow \Pi^*_{z_x}$ [cf. assertion (iv)] that $C_{\Pi^*_2}(D^*_{v^{\mathrm{new}}}|_{z_x}) \subseteq \Pi^*_2|_{z_x}$. In particular, it follows immediately from assertions (v), (vi) that $D^*_{v^{\mathrm{new}}}|_{z_x} \subseteq C_{\Pi^*_2}(D^*_{v^{\mathrm{new}}}|_{z_x}) \subseteq C_{\Pi^*_2}(\Pi^*_{v^{\mathrm{new}}}) \cap \Pi^*_2|_{z_x} = C_{\Pi^*_2|_{z_x}}(\Pi^*_{v^{\mathrm{new}}}) = D^*_{v^{\mathrm{new}}}|_{z_x}$. This completes the proof of assertion (vii). ☐

Lemma 2.12 (Commensurator of a tripod arising from an edge). *In the notation of Lemma 2.11, let* $\Pi_2 \twoheadrightarrow \Pi^{**}_2$ *be an* **SA-maximal almost pro-*l* quotient** *of* Π_2 *[cf. Definition 2.1, (ii)] that* **dominates** $\Pi_2 \twoheadrightarrow \Pi^*_2$ *[cf. the discussion entitled "Topological groups" in §0]. We shall use similar notation*

$$\Pi^{**}_{2/1}\ ,\ \ \Pi^{**}_{1\backslash 2}\ ,\ \ \Pi^{**}_{1}\ ,\ \ \Pi^{**}_{\{2\}}\ ,\ \ \Pi^{**}_{\mathcal{G}}\ ,\ \ \Pi^{**}_{\mathcal{G}_{2/1}}\ ,$$

$$(p^{\Pi}_{2/1})^{**}\colon \Pi^{**}_2 \twoheadrightarrow \Pi^{**}_1\ ,\ \ (p^{\Pi}_{1\backslash 2})^{**}\colon \Pi^{**}_2 \twoheadrightarrow \Pi^{**}_{\{2\}}\ ,$$

$$\Pi^{**}_{z_x} \subseteq \Pi^{**}_1 \ , \ \Pi^{**}_{c^{\mathrm{diag}}} \subseteq \Pi^{**}_{v^{\mathrm{new}}} \subseteq \Pi^{**}_{2/1} \ ,$$

$$\Pi^{**}_2|_{z_x} \ , \ D^{**}_{c^{\mathrm{diag}}} \ , \ I^{**}_{v^{\mathrm{new}}}|_{z_x} \subseteq D^{**}_{v^{\mathrm{new}}}|_{z_x}$$

for objects associated to $\Pi_2 \twoheadrightarrow \Pi^{**}_2$ *to the notation introduced in the statement of Lemma 2.11 for objects associated to* $\Pi_2 \twoheadrightarrow \Pi^*_2$*. Suppose that the natural [outer] surjection* $\Pi_1 \twoheadrightarrow \Pi^{**}_{\{2\}}$ **dominates** *the quotient* $\Pi_1 \twoheadrightarrow \Pi^*_1$*. Then the following hold:*

(i) *The natural surjection* $\Pi^{**}_2 \twoheadrightarrow \Pi^*_2$ *determines a* **surjection** $I^{**}_{v^{\mathrm{new}}}|_{z_x} \twoheadrightarrow I^*_{v^{\mathrm{new}}}|_{z_x}$*.*

(ii) *The image of* $Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq \Pi^{**}_2$ *[cf. the discussion entitled "Topological groups" in* [CbTpII]*, §0] in* Π^*_2 **coincides** *with* $I^*_{v^{\mathrm{new}}}|_{z_x}$*.*

(iii) *The image of* $C_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq \Pi^{**}_2$ *in* Π^*_2 *is* **contained** *in* $D^*_{v^{\mathrm{new}}}|_{z_x}$*.*

(iv) *The natural outer action, by conjugation, of* $N_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq \Pi^{**}_2$ *on [not* $\Pi^{**}_{v^{\mathrm{new}}}$ *but]* $\Pi^*_{v^{\mathrm{new}}}$ *is* **trivial***.*

Proof. First, we verify assertion (i). Observe that it is immediate that the image of $I^{**}_{v^{\mathrm{new}}}|_{z_x} \subseteq \Pi^{**}_2$ in Π^*_2 is *contained* in $I^*_{v^{\mathrm{new}}}|_{z_x}$. Thus, assertion (i) follows immediately from Lemma 2.11, (iv), together with the [easily verified] fact that the natural surjection $\Pi^{**}_2 \twoheadrightarrow \Pi^*_2$ determines a *surjection* $\Pi^{**}_{z_x} \twoheadrightarrow \Pi^*_{z_x}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $\mathrm{Im}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq \Pi^*_2$ for the image of $Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq \Pi^{**}_2$ in Π^*_2. Then it follows from assertion (i) that $I^*_{v^{\mathrm{new}}}|_{z_x} \subseteq \mathrm{Im}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}))$. Thus, to complete the verification of assertion (ii), it suffices to verify that $\mathrm{Im}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq I^*_{v^{\mathrm{new}}}|_{z_x}$. To this end, let us observe that it follows immediately from the final portion of [CbTpII], Lemma 3.6, (iv), that the image $(p^{\Pi}_{1\backslash 2})^{**}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq \Pi^{**}_{\{2\}}$ *coincides* with the image of an edge-like subgroup of $\Pi_{\mathcal{G}} \xleftarrow{\sim} \Pi_1$ associated to $z_x \in \mathrm{Edge}(\mathcal{G})$ via the natural [outer] surjection $\Pi_1 \twoheadrightarrow \Pi^{**}_{\{2\}}$, hence that the image [which is well-defined up to conjugacy] of $(p^{\Pi}_{1\backslash 2})^{**}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq \Pi^{**}_{\{2\}}$ in Π^*_1 [where we recall that we have assumed that $\Pi_1 \twoheadrightarrow \Pi^{**}_{\{2\}}$ *dominates* $\Pi_1 \twoheadrightarrow \Pi^*_1$] is an edge-like subgroup of $\Pi^*_{\mathcal{G}} \xleftarrow{\sim} \Pi^*_1$ associated to $z_x \in \mathrm{Edge}(\mathcal{G})$. Thus, since every edge-like subgroup of Π^*_1 is *commensurably terminal* [cf. Proposition 1.7, (vii)], it follows that the image [which is well-defined up to conjugacy] of $(p^{\Pi}_{1\backslash 2})^{**}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq \Pi^{**}_{\{2\}}$ in Π^*_1 is *contained* in an edge-like subgroup of $\Pi^*_{\mathcal{G}} \xleftarrow{\sim} \Pi^*_1$ associated to $z_x \in \mathrm{Edge}(\mathcal{G})$. On the other hand, since $\Pi^{**}_{c^{\mathrm{diag}}} \subseteq \Pi^{**}_{v^{\mathrm{new}}}$, we have $Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{c^{\mathrm{diag}}}) \subseteq C_{\Pi^{**}_2}(\Pi^{**}_{c^{\mathrm{diag}}}) = D^{**}_{c^{\mathrm{diag}}}$ [cf. Lemma 2.11, (ii)]. In particular, it follows immediately from Lemma 2.11, (iii), that the image of $(p^{\Pi}_{2/1})^{**}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq \Pi^{**}_1$ in Π^*_1 is *contained* in *some* Π^*_1-conjugate of $\Pi^*_{z_x} \subseteq \Pi^*_1$, hence [since

$I^{**}_{v^{\mathrm{new}}}|_{z_x} \subseteq Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})$ *surjects* onto $\Pi^{**}_{z_x}$ — cf. Lemma 2.11, (iv); Proposition 1.7, (v)] that the image of $(p^{\Pi}_{2/1})^{**}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq \Pi^{**}_1$ in Π^*_1 is contained in $\Pi^*_{z_x} \subseteq \Pi^*_1$, i.e., $\mathrm{Im}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq \Pi^*_2|_{z_x}$. Thus, since [as is easily verified] $\mathrm{Im}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq Z^{\mathrm{loc}}_{\Pi^*_2}(\Pi^*_{v^{\mathrm{new}}})$, we conclude that

$$\mathrm{Im}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq \Pi^*_2|_{z_x} \cap Z^{\mathrm{loc}}_{\Pi^*_2}(\Pi^*_{v^{\mathrm{new}}}) = Z^{\mathrm{loc}}_{\Pi^*_2|_{z_x}}(\Pi^*_{v^{\mathrm{new}}}) = I^*_{v^{\mathrm{new}}}|_{z_x}$$

[where the final equality follows from Lemma 2.11, (v), together with the *slimness* portion of Proposition 1.7, (ii)]. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Write $\mathrm{Im}(C_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq \Pi^*_2$ for the image of $C_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq \Pi^{**}_2$ in Π^*_2. Then it follows from [CbTpII], Lemma 3.9, (ii), that $C_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}) \subseteq N_{\Pi^{**}_2}(Z^{\mathrm{loc}}_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}}))$; thus, it follows from assertion (ii) that $\mathrm{Im}(C_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq N_{\Pi^*_2}(I^*_{v^{\mathrm{new}}}|_{z_x})$. In particular, since $D^*_{v^{\mathrm{new}}}|_{z_x}$ is *topologically generated by* $\Pi^*_{v^{\mathrm{new}}}$, $I^*_{v^{\mathrm{new}}}|_{z_x}$ [cf. Lemma 2.11, (v)], we conclude that

$$\mathrm{Im}(C_{\Pi^{**}_2}(\Pi^{**}_{v^{\mathrm{new}}})) \subseteq C_{\Pi^*_2}(D^*_{v^{\mathrm{new}}}|_{z_x}) = D^*_{v^{\mathrm{new}}}|_{z_x}$$

[cf. Lemma 2.11, (vii)]. This completes the proof of assertion (iii). Assertion (iv) follows immediately from assertion (iii), together with Lemma 2.11, (v). This completes the proof of Lemma 2.12. ☐

Corollary 2.13 (**Almost pro-l quotients and tripod homomorphisms**)**.** *In the notation of Definition 2.1, suppose that* $n \geq 3$. *Let* $\Pi^{\mathrm{tpd}} \subseteq \Pi_3$ *be a* **central** *[*$\{1,2,3\}$*-]***tripod** *of* Π_3 *[cf.* [CbTpII], *Definitions 3.3, (i); 3.7, (ii)];* $\Pi^{\mathrm{tpd}} \twoheadrightarrow (\Pi^{\mathrm{tpd}})^{\ddagger}$ *an* **almost pro-l quotient**. *Then the following hold:*

(i) *There exists an* **F-characteristic SA-maximal almost pro-l quotient** *[cf. Definition 2.1, (ii), (iii)]* Π^*_n *of* Π_n *that satisfies the following condition: If we write* Π^*_3 *for the quotient of* Π_3 *determined by the quotient* $\Pi_n \twoheadrightarrow \Pi^*_n$ *and* $(\Pi^{\mathrm{tpd}})^* \subseteq \Pi^*_3$ *for the image of* $\Pi^{\mathrm{tpd}} \subseteq \Pi_3$ *in* Π^*_3, *then the quotient* $\Pi^{\mathrm{tpd}} \twoheadrightarrow (\Pi^{\mathrm{tpd}})^*$ **dominates** *the quotient* $\Pi^{\mathrm{tpd}} \twoheadrightarrow (\Pi^{\mathrm{tpd}})^{\ddagger}$ *[cf. the discussion entitled "Topological groups" in §0].*

(ii) *Every element of the image* $\mathrm{Im}(\mathfrak{T}_{\Pi^{\mathrm{tpd}}}) \subseteq \mathrm{Out}(\Pi^{\mathrm{tpd}})$ *of the* **tripod homomorphism**

$$\mathfrak{T}_{\Pi^{\mathrm{tpd}}} \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{tpd}})$$

associated to Π_n *[cf.* [CbTpII], *Definition 3.19]* **preserves** *the kernel of the surjection* $\Pi^{\mathrm{tpd}} \twoheadrightarrow (\Pi^{\mathrm{tpd}})^*$ *of (i). Thus, we obtain a natural homomorphism*

$$\mathrm{Im}(\mathfrak{T}_{\Pi^{\mathrm{tpd}}}) \longrightarrow \mathrm{Out}((\Pi^{\mathrm{tpd}})^*)\,.$$

(iii) *There exists an* **F-characteristic SA-maximal almost pro-l quotient** $\Pi_n \twoheadrightarrow \Pi_n^{**}$ *of* Π_n *that* **dominates** $\Pi_n \twoheadrightarrow \Pi_n^*$ *[cf. (i)] such that the composite*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \twoheadrightarrow \mathrm{Im}(\mathfrak{T}_{\Pi^{\mathrm{tpd}}}) \to \mathrm{Out}((\Pi^{\mathrm{tpd}})^*)$$

— where the first arrow is the homomorphism induced by $\mathfrak{T}_{\Pi^{\mathrm{tpd}}}$*; the second arrow is the homomorphism of (ii) —* **factors** *through the natural surjection*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \twoheadrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{**} \twoheadleftarrow \Pi_n)$$

[cf. Definition 2.1, (viii); Remark 2.1.1]. Thus, we have a natural commutative diagram of profinite groups

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_n) & \longrightarrow & \mathrm{Im}(\mathfrak{T}_{\Pi^{\mathrm{tpd}}}) \\ \downarrow & & \downarrow \\ \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{**} \twoheadleftarrow \Pi_n) & \longrightarrow & \mathrm{Out}((\Pi^{\mathrm{tpd}})^*)\,. \end{array}$$

Proof. Assertion (i) is a consequence of Proposition 2.3, (iii). Assertion (ii) follows immediately from the fact that Π_n^* is *F-characteristic*, together with the definition of $\mathfrak{T}_{\Pi^{\mathrm{tpd}}}$. Finally, we verify assertion (iii). Let us first observe that it follows immediately from the definition of $\mathfrak{T}_{\Pi^{\mathrm{tpd}}}$, together with Proposition 2.3, (ii), that, to verify assertion (iii), it suffices to verify the following assertion:

> Claim 2.13.A: There exists an *F-characteristic SA-maximal almost pro-l quotient* $\Pi_3 \twoheadrightarrow \Pi_3^{**}$ of Π_3 that *dominates* $\Pi_3 \twoheadrightarrow \Pi_3^*$ such that if we write $(\Pi^{\mathrm{tpd}})^{**} \subseteq \Pi_3^{**}$ for the image of $\Pi^{\mathrm{tpd}} \subseteq \Pi_3$ in Π_3^{**}, then any automorphism of $(\Pi^{\mathrm{tpd}})^*$ determined by conjugating by an element
> $$\gamma^{**} \in N_{\Pi_3^{**}}((\Pi^{\mathrm{tpd}})^{**})$$
> is $(\Pi^{\mathrm{tpd}})^*$-*inner*.

To verify Claim 2.13.A, let $\Pi_3 \twoheadrightarrow \Pi_3^{**}$ be an F-characteristic SA-maximal almost pro-l quotient of Π_3 that *dominates* $\Pi_3 \twoheadrightarrow \Pi_3^*$ and $\gamma^{**} \in N_{\Pi_3^{**}}((\Pi^{\mathrm{tpd}})^{**})$. Then it follows immediately from [CmbCsp], Proposition 1.9, (i), that $Z_{\Pi_3}(\Pi^{\mathrm{tpd}}) \subseteq \Pi_3$ *surjects* onto Π_1, hence also onto Π_1^{**} — where we write Π_1^{**} for the quotient of Π_1 determined by the quotient $\Pi_3 \twoheadrightarrow \Pi_3^{**}$. In particular, there exists an element $\tau \in Z_{\Pi_3}(\Pi^{\mathrm{tpd}})$ such that the images of γ^{**} and τ in Π_1^{**} *coincide*. Thus, by replacing γ^{**} by the difference of γ^{**} and the image of τ in Π_3^{**}, we may assume without loss of generality that $\gamma^{**} \in \Pi_{3/1}^{**}$ — where we write $\Pi_{3/1}^{**}$ for the quotient of $\Pi_{3/1}$ [cf. Definition 2.1] induced by the quotient $\Pi_3 \twoheadrightarrow \Pi_3^{**}$. In particular, the existence of an F-characteristic SA-maximal almost pro-l quotient $\Pi_3 \twoheadrightarrow \Pi_3^{**}$ as in Claim 2.13.A follows immediately, in light of Proposition 2.3, (ii), from Lemma 2.12, (iv). This completes the proof of assertion (ii). □

Finally, before proceeding, we review the following well-known result.

Lemma 2.14 (**Automorphisms of stable log curves).** *Let l be a prime number. Write $l^{\mathrm{aut}} \stackrel{\mathrm{def}}{=} l$ if l is odd; $l^{\mathrm{aut}} \stackrel{\mathrm{def}}{=} 4$ if l is even. If G is a profinite group, then we shall refer to the tensor product with $\mathbb{Z}/l^{\mathrm{aut}}\mathbb{Z}$ of the abelianization of G as the* ***l^{aut}*-abelianization** *of G. Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$.*

(i) *Let k be an algebraically closed field such that l is invertible in k, $(\mathrm{Spec}\, k)^{\log}$ the log scheme obtained by equipping $\mathrm{Spec}\, k$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$, $X^{\log}$ a* **stable log curve** *over $(\mathrm{Spec}\, k)^{\log}$, α an* **automorphism** *of $X^{\log}$ over $(\mathrm{Spec}\, k)^{\log}$. Write Π_1 for the maximal pro-l quotient of the kernel of the natural surjection $\pi_1(X^{\log}) \twoheadrightarrow \pi_1((\mathrm{Spec}\, k)^{\log})$. Suppose that α acts* **trivially** *on the* ***l^{aut}*-abelianization** *of Π_1. Then α is the* **identity automorphism**.

(ii) *Write $\mathcal{M}^{\log}$ for the moduli stack of pointed stable curves of type (g, r) over $\mathbb{Z}[1/l]$, where we regard the marked points as* **unordered***, equipped with the log structure determined by the divisor at infinity, and $\mathcal{C}^{\log} \to \mathcal{M}^{\log}$ for the tautological stable log curve over $\mathcal{M}^{\log}$ [cf. the discussion entitled "Curves" in* [CbTpII]*, §0]. Write $\mathcal{N}^{\log} \to \mathcal{M}^{\log}$ for the finite log étale morphism of log regular log stacks determined by the local system of trivializations of the l^{aut}-abelianizations of the log fundamental groups of the various logarithmic fibers of $\mathcal{C}^{\log} \to \mathcal{M}^{\log}$. Then the underlying algebraic stack $\mathcal{N}$ of $\mathcal{N}^{\log}$ is an* **algebraic space**.

Proof. First, we consider assertion (i). We begin by recalling that when $X^{\log}$ is a *smooth log curve*, and $r \leq 1$ [so $g \geq 1$], assertion (i) follows immediately from classical theory of endomorphisms of abelian varieties [cf., e.g., [Des], Lemme 5.17], together with the well-known fact that every root of unity ζ such that $(\zeta - 1)/l^{\mathrm{aut}}$ is an *algebraic integer* is necessarily equal to 1. Now let us return to the case of an arbitrary *stable log curve* $X^{\log}$. Then it follows immediately from the description of the relationship between the abelianization of Π_1 and the abelianizations of verticial subgroups of Π_1 given in [NodNon], Lemma 1.4, together with the portion of assertion (i) that has already been verified, that α *stabilizes* and *induces the identity automorphism* on each of the irreducible components of $X^{\log}$ of genus ≥ 1. Next, let us observe that it follows immediately from the *definition* of l^{aut}, together with the well-known structure of the submodule of the abelianization of Π_1 generated by the cuspidal inertia subgroups, that α acts *trivially* on the set of *cusps* of $X^{\log}$. Thus, by considering the various connected components of the union of the genus zero irreducible components of $X^{\log}$, we conclude that, to complete the verification of assertion (i), it suffices to verify, in the case where $g = 0$, that any automorphism of $X^{\log}$ over $(\operatorname{Spec} k)^{\log}$ that acts trivially on the set of cusps of $X^{\log}$ is equal to the identity automorphism. But this follows immediately by *induction* on r, i.e., by considering, when $r \geq 4$, the stable log curve obtained from $X^{\log}$ by "forgetting" one of the cusps of $X^{\log}$. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with well-known generalities concerning algebraic stacks [cf., e.g., the discussion surrounding [FC], Chapter I, Theorem 4.10]. $\square$

3. Applications to the theory of tempered fundamental groups

In the present §3, we apply the technical tools developed in the preceding §2, together with the theory of [CbTpI], §5, to obtain applications to the theory of *tempered fundamental groups*. In particular, we prove a generalization of a result due to André [cf. [André], Theorems 7.2.1, 7.2.3] concerning the *characterization of the local Galois groups in the image of the outer Galois action* associated to a hyperbolic curve over a number field [cf. Corollary 3.20, (iii), below].

Definition 3.1. Let n be a nonnegative integer. For $\Box \in \{\circ, \bullet\}$, let ${}^{\Box}p$ be a prime number; ${}^{\Box}\Sigma$ a nonempty set of prime numbers such that ${}^{\Box}\Sigma \neq \{{}^{\Box}p\}$; ${}^{\Box}R$ a *mixed characteristic complete discrete valuation ring* of residue characteristic ${}^{\Box}p$ whose residue field is *separably closed*; ${}^{\Box}K$ the field of fractions of ${}^{\Box}R$; ${}^{\Box}\overline{K}$ an algebraic closure of ${}^{\Box}K$. Write $I_{{}^{\Box}K} \overset{\mathrm{def}}{=} \mathrm{Gal}({}^{\Box}\overline{K}/{}^{\Box}K)$ for the absolute Galois group of ${}^{\Box}K$; ${}^{\Box}\overline{R}$ for the ring of integers of ${}^{\Box}\overline{K}$; ${}^{\Box}\overline{R}^{\wedge}$ for the ${}^{\Box}p$-adic completion of ${}^{\Box}\overline{R}$; ${}^{\Box}\overline{K}^{\wedge}$ for the field of fractions of ${}^{\Box}\overline{K}^{\wedge}$. If $n \geq 2$, then we suppose further that ${}^{\Box}\Sigma$ is either *equal to* $\mathfrak{Primes}$ or *of cardinality one*. Let

$$X^{\log}_{{}^{\Box}K}$$

be a *smooth log curve* over ${}^{\Box}K$. Write $X^{\log}_{{}^{\Box}\overline{K}} \overset{\mathrm{def}}{=} X^{\log}_{{}^{\Box}K} \times_{{}^{\Box}K} {}^{\Box}\overline{K}$;

$$(X_{{}^{\Box}\overline{K}})^{\log}_{n}$$

for the n-th *log configuration space* [cf. the discussion entitled "*Curves*" in [CbTpI], §0] of the smooth log curve $X^{\log}_{{}^{\Box}\overline{K}}$ over ${}^{\Box}\overline{K}$.

(i) We shall write

$$ {}^{\Box}\Pi_n \overset{\mathrm{def}}{=} \pi_1((X_{{}^{\Box}\overline{K}})^{\log}_{n})^{{}^{\Box}\Sigma}$$

for the maximal pro-${}^{\Box}\Sigma$ quotient of the log fundamental group of $(X_{{}^{\Box}\overline{K}})^{\log}_{n}$. Thus, we have a natural outer Galois action

$${}^{\Box}\rho_n \colon I_{{}^{\Box}K} \longrightarrow \mathrm{Out}({}^{\Box}\Pi_n) \, .$$

Note that ${}^{\Box}\Pi_n$ is equipped with a natural structure of *pro-*${}^{\Box}\Sigma$ *configuration space group* [cf. [MzTa], Definition 2.3, (i)].

(ii) We shall write $\pi_1^{\mathrm{temp}}((X_{{}^{\Box}\overline{K}})^{\log}_{n} \times_{{}^{\Box}\overline{K}} {}^{\Box}\overline{K}^{\wedge})$ for the *tempered fundamental group* [cf. [André], §4] of $(X_{{}^{\Box}\overline{K}})^{\log}_{n} \times_{{}^{\Box}\overline{K}} {}^{\Box}\overline{K}^{\wedge}$ and

$${}^{\Box}\Pi^{\mathrm{tp}}_n \overset{\mathrm{def}}{=} \varprojlim_{N} \pi_1^{\mathrm{temp}}((X_{{}^{\Box}\overline{K}})^{\log}_{n} \times_{{}^{\Box}\overline{K}} {}^{\Box}\overline{K}^{\wedge})/N$$

for the ${}^{\Box}\Sigma$*-tempered fundamental group* of $(X_{{}^{\Box}\overline{K}})^{\log}_{n} \times_{{}^{\Box}\overline{K}} {}^{\Box}\overline{K}^{\wedge}$ [cf. [CmbGC] Corollary 2.10, (iii)], i.e., the inverse limit given by allowing N to vary over the open normal subgroups of

$\pi_1^{\mathrm{temp}}((X_{\Box\overline{K}})_n^{\log} \times_{\Box\overline{K}} {}^{\Box}\overline{K}^{\wedge})$ such that the quotient by N corresponds to a *topological covering* [cf. [André], §4.2] of some *finite étale Galois covering* of $(X_{\Box\overline{K}})_n^{\log} \times_{\Box\overline{K}} {}^{\Box}\overline{K}^{\wedge}$ of degree a product of primes $\in {}^{\Box}\Sigma$. [Here, we recall that, when $n = 1$, such a "topological covering" corresponds to a "combinatorial covering", i.e., a covering determined by a covering of the dual semi-graph of the special fiber of the stable model of some finite étale covering of $(X_{\Box\overline{K}})_n^{\log} \times_{\Box\overline{K}} {}^{\Box}\overline{K}^{\wedge}$.] Thus, we have a natural outer Galois action

$$ {}^{\Box}\rho_n^{\mathrm{tp}} \colon I_{\Box K} \longrightarrow \mathrm{Out}({}^{\Box}\Pi_n^{\mathrm{tp}}) $$

[cf. [André], Proposition 5.1.1].

Lemma 3.2 (**Pro-Σ completions of discrete free groups**). *Let Σ be a nonempty set of prime numbers and F a discrete free group. Then the following hold:*

(i) *The natural homomorphism $F \to F^{\Sigma}$ from F to the pro-Σ completion F^{Σ} of F is injective.*

(ii) *Suppose that F is* **not of rank one**. *Then the image of the injection $F \hookrightarrow F^{\Sigma}$ of (i) is* **normally terminal**.

Proof. Assertion (i) follows immediately from [RZ], Proposition 3.3.15. Assertion (ii) follows immediately from the fact that F is *conjugacy l-separable* for every prime number l [cf. [Prs], Theorem 3.2], together with a similar argument to the argument applied in the proof of [André], Lemma 3.2.1. This completes the proof of Lemma 3.2. ☐

Proposition 3.3 (**Log and tempered fundamental groups**). *In the notation of Definition 3.1, the following hold:*

(i) *Write $({}^{\Box}\Pi_n^{\mathrm{tp}})^{\Box\Sigma}$ for the pro-${}^{\Box}\Sigma$ completion of ${}^{\Box}\Pi_n^{\mathrm{tp}}$. Then there exists a* **natural outer isomorphism** $({}^{\Box}\Pi_n^{\mathrm{tp}})^{\Box\Sigma} \xrightarrow{\sim} {}^{\Box}\Pi_n$.

(ii) *The outer homomorphism ${}^{\Box}\Pi_1^{\mathrm{tp}} \to {}^{\Box}\Pi_1$ determined by the outer isomorphism of (i) is* **injective**.

(iii) *The image of the outer injection ${}^{\Box}\Pi_1^{\mathrm{tp}} \hookrightarrow {}^{\Box}\Pi_1$ of (ii) is* **normally terminal**.

(iv) *Write* $\mathrm{Isom}({}^{\circ}\Pi_1^{\mathrm{tp}}, {}^{\bullet}\Pi_1^{\mathrm{tp}})$ *(respectively,* $\mathrm{Isom}({}^{\circ}\Pi_1, {}^{\bullet}\Pi_1)$*) for the set of isomorphisms of ${}^{\circ}\Pi_1^{\mathrm{tp}}$ (respectively, ${}^{\circ}\Pi_1$) with ${}^{\bullet}\Pi_1^{\mathrm{tp}}$ (respectively, ${}^{\bullet}\Pi_1$) and* $\mathrm{Inn}(-)$ *for the group of inner automorphisms of "$(-)$". Then the natural map between sets of outer isomorphisms [i.e., sets of "*$\mathrm{Inn}(-)$*-orbits"]*

$$ \mathrm{Isom}({}^{\circ}\Pi_1^{\mathrm{tp}}, {}^{\bullet}\Pi_1^{\mathrm{tp}})/\mathrm{Inn}({}^{\bullet}\Pi_1^{\mathrm{tp}}) \longrightarrow \mathrm{Isom}({}^{\circ}\Pi_1, {}^{\bullet}\Pi_1)/\mathrm{Inn}({}^{\bullet}\Pi_1) $$

induced by the outer isomorphism of (i) — hence also the natural homomorphism

$$\mathrm{Out}({}^{\square}\Pi_1^{\mathrm{tp}}) \longrightarrow \mathrm{Out}({}^{\square}\Pi_1)$$

— *is* **injective**.

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii) (respectively, (iii)). Let us first observe that it follows immediately from assertion (i) that, to verify assertion (ii) (respectively, (iii)), by replacing $X^{\log}_{\square \overline{K}}$ by a suitable connected finite étale covering of $X^{\log}_{\square \overline{K}}$, we may assume without loss of generality that the *first Betti number of the dual semi-graph of the special fiber of the stable model of* $X^{\log}_{\square \overline{K}}$ *is* $\neq 1$. Then since ${}^{\square}\Pi_1^{\mathrm{tp}}$ is a projective limit of extensions of finite groups whose orders are products of primes $\in {}^{\square}\Sigma$ by discrete free groups whose ranks are $\neq 1$, assertion (ii) (respectively, (iii)) follows immediately from Lemma 3.2, (i) (respectively, (ii)). This completes the proof of assertion (ii) (respectively, (iii)). Assertion (iv) follows immediately from assertion (iii). This completes the proof of Proposition 3.3. ☐

Remark 3.3.1. The injections of Proposition 3.3, (iv), allow one to regard $\mathrm{Isom}({}^{\circ}\Pi_1^{\mathrm{tp}}, {}^{\bullet}\Pi_1^{\mathrm{tp}})/\mathrm{Inn}({}^{\bullet}\Pi_1^{\mathrm{tp}})$, (respectively, $\mathrm{Out}({}^{\square}\Pi_1^{\mathrm{tp}})$) as a subset (respectively, subgroup) of $\mathrm{Isom}({}^{\circ}\Pi_1, {}^{\bullet}\Pi_1)/\mathrm{Inn}({}^{\bullet}\Pi_1)$ (respectively, $\mathrm{Out}({}^{\square}\Pi_1)$).

Remark 3.3.2. The *normal terminality* of Proposition 3.3, (iii), may also be verified by applying the theory of [SemiAn] and [NodNon]. We refer to the proof of [IUTeichI], Proposition 2.4, (iii), for more details concerning this approach.

Definition 3.4. Let $\mathbb{G}$ be a [semi-]graph. Write $\mathrm{Node}(\mathbb{G})$ for the set of closed edges of $\mathbb{G}$. Then we shall refer to a map

$$\mu \colon \mathrm{Node}(\mathbb{G}) \to \mathbb{R}_{>0} \stackrel{\mathrm{def}}{=} \{ a \in \mathbb{R} \mid a > 0 \}$$

as a *metric structure* on $\mathbb{G}$. Also, we shall refer to a [semi-]graph equipped with a metric structure as a *metric [semi-]graph.* Let Σ be a [possibly empty] set of prime numbers. Then we shall say that an isomorphism $\mathbb{G}_1 \xrightarrow{\sim} \mathbb{G}_2$ between two [semi-]graphs $\mathbb{G}_1$, $\mathbb{G}_2$ equipped with metric structures μ_1, μ_2 is Σ-*rationally compatible* with the given metric structures if there exists an element

$$\xi \in (\widehat{\mathbb{Z}}^{\Sigma})^{+} \ (\subseteq \mathbb{Q}_{>0} \stackrel{\mathrm{def}}{=} \mathbb{Q} \cap \mathbb{R}_{>0})$$

— i.e., a positive rational number that is invertible, as an integer, at the primes of Σ [cf. the notation of [CbTpI], Corollary 5.9, (iv), if $\Sigma \neq \emptyset$; set $(\widehat{\mathbb{Z}}^{\Sigma})^{+} \overset{\text{def}}{=} \mathbb{Q}_{>0}$ if $\Sigma = \emptyset$] — such that $\xi \cdot \mu_1$ is compatible, relative to the given isomorphism, with μ_2. [Thus, if $\mathbb{G}_1 = \mathbb{G}_2$ is a *finite* [semi-]graph, and $\mu_1 = \mu_2$, then such a ξ is *necessarily equal* to 1. Alternatively, if $\Sigma = \mathfrak{Primes}$, then such a ξ is *necessarily equal* to 1.]

Definition 3.5. In the notation of Definition 3.1, let $\Sigma \subseteq {}^{\square}\Sigma \setminus \{{}^{\square}p\}$ be a nonempty subset of ${}^{\square}\Sigma \setminus \{{}^{\square}p\}$ and ${}^{\square}H \subseteq {}^{\square}\Pi_1$ an open subgroup of ${}^{\square}\Pi_1$.

(i) We shall write
$$\mathcal{G}_{{}^{\square}H}[\Sigma]$$
for the semi-graph of anabelioids of pro-Σ PSC-type determined by the special fiber [cf. [CmbGC], Example 2.5] of the stable model over ${}^{\square}\overline{R}$ of the connected finite log étale covering of $X^{\log}_{{}^{\square}\overline{K}}$ corresponding to ${}^{\square}H \subseteq {}^{\square}\Pi_1$.

(ii) We shall write
$$\mathbb{G}_{{}^{\square}H}$$
for the semi-graph associated to [i.e., the *dual semi-graph* of] the special fiber of the stable model over ${}^{\square}\overline{R}$ of the connected finite log étale covering of $X^{\log}_{{}^{\square}\overline{K}}$ corresponding to ${}^{\square}H \subseteq {}^{\square}\Pi_1$ — i.e., the underlying semi-graph of $\mathcal{G}_{{}^{\square}H}[\Sigma]$ [cf. (i)]. Note that this semi-graph is *independent* of the choice of Σ.

(iii) We shall write
$$\mu_{{}^{\square}H} \colon \mathrm{Node}(\mathbb{G}_{{}^{\square}H}) \longrightarrow \mathbb{R}_{>0}$$
for the *metric structure* [cf. Definition 3.4] on $\mathbb{G}_{{}^{\square}H}$ associated to the stable model over ${}^{\square}\overline{R}$ of the connected finite log étale covering of $X^{\log}_{{}^{\square}\overline{K}}$ corresponding to ${}^{\square}H \subseteq {}^{\square}\Pi_1$, i.e., the metric structure defined as follows:

Write $v_{{}^{\square}\overline{K}^{\wedge}}$ for the ${}^{\square}p$-adic valuation of ${}^{\square}\overline{K}^{\wedge}$ such that $v_{{}^{\square}\overline{K}^{\wedge}}({}^{\square}p) = 1$. Let $e \in \mathrm{Node}(\mathbb{G}_{{}^{\square}H})$. Suppose that the ${}^{\square}\overline{R}^{\wedge}$-algebra given by the completion at the node corresponding to e of the stable model of the connected covering of $X^{\log}_{{}^{\square}\overline{K}}$ determined by ${}^{\square}H \subseteq {}^{\square}\Pi_1$ is isomorphic to
$${}^{\square}\overline{R}^{\wedge}[[s_1, s_2]]/(s_1 s_2 - a_e)$$
— where $a_e \in {}^{\square}\overline{R}^{\wedge}$ is a nonzero non-unit, and s_1 and s_2 denote indeterminates. Then we set $\mu_{{}^{\square}H}(e) \overset{\text{def}}{=}$

$v_{\Box\overline{K}^\wedge}(a_e)$:

$$\begin{array}{cccc} \mu_{\Box H}: & \mathrm{Node}(\mathbb{G}_{\Box H}) & \longrightarrow & \mathbb{R}_{>0} \\ & e & \mapsto & v_{\Box\overline{K}^\wedge}(a_e)\,. \end{array}$$

Here, one verifies easily that "$\mu_{\Box H}(a_e)$" *depends only* on e, i.e., is *independent* of the choice of the local equation "$s_1 s_2 - a_e$".

Remark 3.5.1. In the notation of Definition 3.5, it follows immediately from the various definitions involved that one has a *natural outer isomorphism*

$$({}^{\Box}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\Box H}[\Sigma]}$$

between the maximal pro-Σ quotient $({}^{\Box}H)^{\Sigma}$ of ${}^{\Box}H$ and the [pro-Σ] fundamental group $\Pi_{\mathcal{G}_{\Box H}[\Sigma]}$ of the semi-graph of anabelioids of pro-Σ PSC-type $\mathcal{G}_{\Box H}[\Sigma]$.

Proposition 3.6 (Equivalences of properties of isomorphisms between fundamental groups). *In the notation of Definition 3.1, let $\alpha\colon {}^{\circ}\Pi_1 \xrightarrow{\sim} {}^{\bullet}\Pi_1$ be an isomorphism of profinite groups. [Thus, it follows immediately that ${}^{\circ}\Sigma = {}^{\bullet}\Sigma$ — cf., e.g., the proof of* [CbTpI], *Proposition 1.5, (i).] Consider the following conditions:*

(a) *The outer isomorphism ${}^{\circ}\Pi_1 \xrightarrow{\sim} {}^{\bullet}\Pi_1$ determined by α is* **contained** *in*

$$\mathrm{Isom}({}^{\circ}\Pi_1^{\mathrm{tp}}, {}^{\bullet}\Pi_1^{\mathrm{tp}})/\mathrm{Inn}({}^{\bullet}\Pi_1^{\mathrm{tp}}) \subseteq \mathrm{Isom}({}^{\circ}\Pi_1, {}^{\bullet}\Pi_1)/\mathrm{Inn}({}^{\bullet}\Pi_1)$$

[cf. Remark 3.3.1], and ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq \{{}^{\circ}p, {}^{\bullet}p\}$.

($\mathrm{b}^{\forall}$) *For any characteristic open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ of ${}^{\circ}\Pi_1$ and any nonempty subset $\Sigma \subseteq {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that ${}^{\circ}p$, ${}^{\bullet}p \notin \Sigma$, if we write ${}^{\bullet}H \stackrel{\mathrm{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$, then the outer isomorphism of $({}^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\circ}H}[\Sigma]}$ [cf. Remark 3.5.1] with $({}^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\bullet}H}[\Sigma]}$ induced by α is* **group-theoretically verticial** *[cf.* [CmbGC], *Definition 1.4, (iv)].*

($\mathrm{b}^{\exists}$) *For any characteristic open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ of ${}^{\circ}\Pi_1$, there exists a nonempty subset $\Sigma \subseteq {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ [which may depend on ${}^{\circ}H$] such that ${}^{\circ}p$, ${}^{\bullet}p \notin \Sigma$, and, moreover, if we write ${}^{\bullet}H \stackrel{\mathrm{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$, then the outer isomorphism of $({}^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\circ}H}[\Sigma]}$ [cf. Remark 3.5.1] with $({}^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\bullet}H}[\Sigma]}$ induced by α is* **group-theoretically verticial**.

($\mathrm{c}^{\forall}$) *For any characteristic open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ of ${}^{\circ}\Pi_1$ and any nonempty subset $\Sigma \subseteq {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that ${}^{\circ}p$, ${}^{\bullet}p \notin \Sigma$, if we write ${}^{\bullet}H \stackrel{\mathrm{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$, then the outer isomorphism of*

$({}^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\circ}H}[\Sigma]}$ *[cf. Remark 3.5.1] with* $({}^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\bullet}H}[\Sigma]}$ *induced by* α *is* **graphic** *[cf.* [CmbGC], *Definition 1.4, (i)].*

($c^{\exists}$) *For any characteristic open subgroup* ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ *of* ${}^{\circ}\Pi_1$*, there exists a nonempty subset* $\Sigma \subseteq {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ *[which may depend on* ${}^{\circ}H$*] such that* ${}^{\circ}p, {}^{\bullet}p \notin \Sigma$*, and, moreover, if we write* ${}^{\bullet}H \stackrel{\text{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$*, then the outer isomorphism of* $({}^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\circ}H}[\Sigma]}$ *[cf. Remark 3.5.1] with* $({}^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\bullet}H}[\Sigma]}$ *induced by* α *is* **graphic**.

Then:

(i) *We have implications:*

$$(b^{\forall}) \Longleftarrow (c^{\forall}) \Longleftarrow (c^{\exists}) \Longrightarrow (a) \Longleftrightarrow (b^{\exists}) \Longrightarrow (b^{\forall}).$$

(ii) *Suppose that* ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq \{{}^{\circ}p, {}^{\bullet}p\}$*. [This condition is satisfied if, for instance,* ${}^{\circ}p = {}^{\bullet}p$*.] Then we have equivalences:*

$$(b^{\exists}) \Longleftrightarrow (b^{\forall}) \quad \text{and} \quad (c^{\exists}) \Longleftrightarrow (c^{\forall}).$$

(iii) *Suppose that either* ${}^{\circ}p \in {}^{\circ}\Sigma$ *or* ${}^{\bullet}p \in {}^{\bullet}\Sigma$*. Then we have equivalences:*

$$(a) \Longleftrightarrow (b^{\exists}) \Longleftrightarrow (c^{\exists}).$$

Moreover, (a), ($b^{\exists}$)*, and* ($c^{\exists}$) *imply that* ${}^{\circ}p = {}^{\bullet}p$*.*

Proof. First, we claim that the following assertion holds:

> Claim 3.6.A: Suppose that (a) is satisfied, and that either ${}^{\circ}p \in {}^{\circ}\Sigma$ or ${}^{\bullet}p \in {}^{\bullet}\Sigma$. Then ${}^{\circ}p = {}^{\bullet}p \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$. Moreover, ($c^{\exists}$) is satisfied.

To verify Claim 3.6.A, suppose that (a) is satisfied, and that ${}^{\circ}p \in {}^{\circ}\Sigma$. Then it follows immediately from [SemiAn], Corollary 3.11 [cf., especially, the portion of the statement and proof of [SemiAn], Corollary 3.11, concerning, in the notation of *loc. cit.*, the assertion "$p_{\alpha} = p_{\beta}$"]; [SemiAn], Remark 3.11.1 [cf. also [AbsTpII], Corollary 2.11; [AbsTpII], Remark 2.11.1, (i)], that ${}^{\circ}p = {}^{\bullet}p \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$, and, moreover, that ($c^{\exists}$) is satisfied. This completes the proof of Claim 3.6.A.

Next, we verify assertion (i). Let us first observe that it follows from the fact that *graphicity* implies *group-theoretic verticiality* that the following implications hold: ($c^{\forall}$) $\Rightarrow$ ($b^{\forall}$) and ($c^{\exists}$) $\Rightarrow$ ($b^{\exists}$). Next, we verify the implication ($b^{\exists}$) $\Rightarrow$ ($b^{\forall}$) (respectively, ($c^{\exists}$) $\Rightarrow$ ($c^{\forall}$)). Suppose that ($b^{\exists}$) (respectively, ($c^{\exists}$)) is satisfied. Then it follows that ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq \{{}^{\circ}p, {}^{\bullet}p\}$. Next, let us observe that, to complete the verification of ($b^{\forall}$) (respectively, ($c^{\forall}$)), we may assume without loss of generality — by replacing the open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ in ($b^{\forall}$) (respectively, ($c^{\forall}$)) by ${}^{\circ}\Pi_1$ — that ${}^{\circ}H = {}^{\circ}\Pi_1$ and ${}^{\bullet}H = {}^{\bullet}\Pi_1$. Moreover, one verifies easily that, to complete the verification of ($b^{\forall}$) (respectively, ($c^{\forall}$)), we may assume without loss of generality — by replacing the subset Σ in ($b^{\forall}$)

(respectively, ($c^{\forall}$)) by ${}^{\circ}\Sigma \setminus ({}^{\circ}\Sigma \cap \{{}^{\circ}p, {}^{\bullet}p\}) = {}^{\bullet}\Sigma \setminus ({}^{\bullet}\Sigma \cap \{{}^{\circ}p, {}^{\bullet}p\})$ ($\neq \emptyset$) — that $\Sigma = {}^{\circ}\Sigma \setminus ({}^{\circ}\Sigma \cap \{{}^{\circ}p, {}^{\bullet}p\}) = {}^{\bullet}\Sigma \setminus ({}^{\bullet}\Sigma \cap \{{}^{\circ}p, {}^{\bullet}p\})$ ($\neq \emptyset$). Let ${}^{\circ}U \subseteq {}^{\circ}\Pi_1$ be a characteristic open subgroup. Write ${}^{\bullet}U \stackrel{\text{def}}{=} \alpha({}^{\circ}U) \subseteq {}^{\bullet}\Pi_1$. Then it follows immediately from ($b^{\exists}$) (respectively, ($c^{\exists}$)) that there exists a nonempty subset $\Sigma_{{}^{\circ}U} \subseteq \Sigma$ such that α induces a *functorial bijection*

$$\mathrm{Vert}(\mathcal{G}_{{}^{\circ}U}[\Sigma]) = \mathrm{Vert}(\mathcal{G}_{{}^{\circ}U}[\Sigma_{{}^{\circ}U}]) \xrightarrow{\sim} \mathrm{Vert}(\mathcal{G}_{{}^{\bullet}U}[\Sigma_{{}^{\circ}U}]) = \mathrm{Vert}(\mathcal{G}_{{}^{\bullet}U}[\Sigma])$$

(respectively,

$$\mathrm{VCN}(\mathcal{G}_{{}^{\circ}U}[\Sigma]) = \mathrm{VCN}(\mathcal{G}_{{}^{\circ}U}[\Sigma_{{}^{\circ}U}]) \xrightarrow{\sim} \mathrm{VCN}(\mathcal{G}_{{}^{\bullet}U}[\Sigma_{{}^{\circ}U}]) = \mathrm{VCN}(\mathcal{G}_{{}^{\bullet}U}[\Sigma])).$$

In particular, by considering these *functorial bijections* between the sets "Vert" (respectively, "VCN") associated to the connected finite étale coverings corresponding to the various characteristic open subgroups ${}^{\circ}U \subseteq {}^{\circ}\Pi_1$, ${}^{\bullet}U \stackrel{\text{def}}{=} \alpha({}^{\circ}U) \subseteq {}^{\bullet}\Pi_1$, we conclude that the isomorphism ${}^{\circ}\Pi_1^{\Sigma} \xrightarrow{\sim} {}^{\bullet}\Pi_1^{\Sigma}$ is *group-theoretically verticial* (respectively, *group-theoretically verticial* and *group-theoretically edge-like*, hence *graphic* [cf. [CmbGC], Proposition 1.5, (ii)]). This completes the proof of the implication ($b^{\exists}$) $\Rightarrow$ ($b^{\forall}$) (respectively, ($c^{\exists}$) $\Rightarrow$ ($c^{\forall}$)).

Next, we observe that since (a) implies that ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq \{{}^{\circ}p, {}^{\bullet}p\}$, the implication (a) $\Rightarrow$ ($b^{\exists}$) follows from [SemiAn], Theorem 3.7, (iv), together with [the evident Σ-tempered analogue of] the discussion of [SemiAn], Example 2.10. Thus, to complete the verification of assertion (i), it suffices to verify the implication ($b^{\exists}$) $\Rightarrow$ (a). To this end, suppose that ($b^{\exists}$) is satisfied. Let ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ be a characteristic open subgroup of ${}^{\circ}\Pi_1$. Then it follows from ($b^{\exists}$) that there exists a nonempty subset $\Sigma \subseteq {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that ${}^{\circ}p$, ${}^{\bullet}p \notin \Sigma$, and, moreover, if we write ${}^{\bullet}H \stackrel{\text{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$, then the outer isomorphism of $({}^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\circ}H}[\Sigma]}$ [cf. Remark 3.5.1] with $({}^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\bullet}H}[\Sigma]}$ induced by α is *group-theoretically verticial.* For each $\square \in \{\circ, \bullet\}$, write

$$\mathcal{G}^{\neq c}_{{}^{\square}H}[\Sigma]$$

for the *graph of anabelioids* obtained by omitting the cusps [i.e., open edges] of $\mathcal{G}_{{}^{\square}H}[\Sigma]$;

$$\Pi^{\mathrm{tp}}_{\mathcal{G}_{{}^{\square}H}[\Sigma]}, \quad \Pi^{\mathrm{tp}\neq c}_{\mathcal{G}_{{}^{\square}H}[\Sigma]}$$

for the *tempered fundamental groups* of $\mathcal{G}_{{}^{\square}H}[\Sigma]$, $\mathcal{G}^{\neq c}_{{}^{\square}H}[\Sigma]$, respectively [cf. the discussion preceding [SemiAn], Proposition 3.6]. Here, let us observe that it follows immediately from the various definitions involved that we have a *natural commutative diagram*

$$\begin{array}{ccccc}
\Pi^{\mathrm{tp}\neq c}_{\mathcal{G}_{{}^{\square}H}[\Sigma]} & \xrightarrow{\sim} & \Pi^{\mathrm{tp}}_{\mathcal{G}_{{}^{\square}H}[\Sigma]} & & \\
\cap & & \cap & & \\
(\Pi^{\mathrm{tp}\neq c}_{\mathcal{G}_{{}^{\square}H}[\Sigma]})^{\Sigma} & \xrightarrow{\sim} & (\Pi^{\mathrm{tp}}_{\mathcal{G}_{{}^{\square}H}[\Sigma]})^{\Sigma} & \xrightarrow{\sim} & \Pi_{\mathcal{G}_{{}^{\square}H}[\Sigma]}
\end{array}$$

— where we write $(\Pi^{\mathrm{tp}\neq c}_{\mathcal{G}_{\Box H}[\Sigma]})^{\Sigma}$, $(\Pi^{\mathrm{tp}}_{\mathcal{G}_{\Box H}[\Sigma]})^{\Sigma}$ for the pro-Σ completions of $\Pi^{\mathrm{tp}\neq c}_{\mathcal{G}_{\Box H}[\Sigma]}$, $\Pi^{\mathrm{tp}}_{\mathcal{G}_{\Box H}[\Sigma]}$, respectively; the horizontal arrows are outer isomorphisms; the lower right-hand horizontal arrow is the outer isomorphism of Proposition 3.3, (i); the vertical inclusions are the inclusions that arise from Proposition 3.3, (ii).

Now since the outer isomorphism of $({}^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\circ H}[\Sigma]}$ with $({}^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\bullet H}[\Sigma]}$ induced by α is *group-theoretically verticial*, it follows immediately from [NodNon], Proposition 1.13; the argument applied in the proof of the *sufficiency* portion of [CmbGC], Proposition 1.5, (ii), that α determines an *isomorphism* $\mathcal{G}^{\neq c}_{\circ H}[\Sigma] \xrightarrow{\sim} \mathcal{G}^{\neq c}_{\bullet H}[\Sigma]$ of graphs of anabelioids. Thus, it follows immediately from the existence of the natural outer isomorphisms discussed above that the [group-theoretically verticial] outer isomorphism $\Pi_{\mathcal{G}_{\circ H}[\Sigma]} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\bullet H}[\Sigma]}$ induced by the isomorphism α *maps the* $\Pi_{\mathcal{G}_{\circ H}[\Sigma]}$*-conjugacy class of* $\Pi^{\mathrm{tp}}_{\mathcal{G}_{\circ H}[\Sigma]}$ *to the* $\Pi_{\mathcal{G}_{\bullet H}[\Sigma]}$*-conjugacy class of* $\Pi^{\mathrm{tp}}_{\mathcal{G}_{\bullet H}[\Sigma]}$. Moreover, it follows immediately from the *normal terminality* of Proposition 3.3, (iii), that the resulting conjugacy indeterminacies may be reduced to $\Pi^{\mathrm{tp}}_{\mathcal{G}_{\Box H}[\Sigma]}$-conjugacy indeterminacies. In particular, by applying these observations to the various characteristic open subgroups "${}^{\circ}H$" of ${}^{\circ}\Pi_1$, one verifies easily from the description of the tempered fundamental group as a projective limit given in [André], §4.5 [cf. also the discussion preceding [SemiAn], Proposition 3.6] that the outer isomorphism ${}^{\circ}\Pi_1 \xrightarrow{\sim} {}^{\circ}\Pi_1$ determined by α is *contained* in $\mathrm{Isom}({}^{\circ}\Pi_1^{\mathrm{tp}}, {}^{\bullet}\Pi_1^{\mathrm{tp}})/\mathrm{Inn}({}^{\bullet}\Pi_1^{\mathrm{tp}}) \subseteq \mathrm{Isom}({}^{\circ}\Pi_1, {}^{\bullet}\Pi_1)/\mathrm{Inn}({}^{\bullet}\Pi_1)$, i.e., that (a) is satisfied. This completes the proof of the implication ($\mathrm{b}^{\exists}$) $\Rightarrow$ (a), hence also of assertion (i). Assertion (ii) follows immediately from assertion (i), together with the various definitions involved. Assertion (iii) follows from assertion (i), together with Claim 3.6.A. This completes the proof of Proposition 3.6. $\Box$

Definition 3.7. In the notation of Definition 3.1:

(i) Let $\alpha\colon {}^{\circ}\Pi_1 \xrightarrow{\sim} {}^{\bullet}\Pi_1$ be an isomorphism of profinite groups. Then we shall say that α is *G-admissible* [i.e., "graph-admissible"] if α satisfies condition ($\mathrm{c}^{\exists}$) — hence also conditions (a), ($\mathrm{b}^{\forall}$), ($\mathrm{b}^{\exists}$), ($\mathrm{c}^{\forall}$) [cf. Proposition 3.6, (i)] — of Proposition 3.6. Write

$$\mathrm{Aut}({}^{\circ}\Pi_1)^{\mathrm{G}} \subseteq \mathrm{Aut}({}^{\circ}\Pi_1)$$

for the subgroup [cf. the equivalence ($\mathrm{c}^{\forall}$) $\Leftrightarrow$ ($\mathrm{c}^{\exists}$) of Proposition 3.6, (ii))] of G-admissible automorphisms of ${}^{\circ}\Pi_1$ and

$$\mathrm{Out}({}^{\circ}\Pi_1)^{\mathrm{G}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}({}^{\circ}\Pi_1)^{\mathrm{G}}/\mathrm{Inn}({}^{\circ}\Pi_1) \subseteq \mathrm{Out}({}^{\circ}\Pi_1)$$

for the subgroup of G-admissible outomorphisms of ${}^{\circ}\Pi_1$.

(ii) Let $\alpha\colon {}^\circ\Pi_1 \xrightarrow{\sim} {}^\bullet\Pi_1$ be an isomorphism of profinite groups [so ${}^\circ\Sigma = {}^\bullet\Sigma$ — cf., e.g., the proof of [CbTpI], Proposition 1.5, (i)]. Let $\Sigma \subseteq {}^\circ\Sigma = {}^\bullet\Sigma$ be a [possibly empty] subset such that ${}^\circ p$, ${}^\bullet p \notin \Sigma$. Then we shall say that α is *Σ-M-admissible* [i.e., "Σ-metric-admissible"] if α is G-admissible [cf. (i)], and, moreover, the following condition is satisfied:

   Let ${}^\circ H \subseteq {}^\circ\Pi_1$ be a characteristic open subgroup of ${}^\circ\Pi_1$. Write ${}^\bullet H \stackrel{\mathrm{def}}{=} \alpha({}^\circ H) \subseteq {}^\bullet\Pi_1$. Then the isomorphism of $\mathbb{G}_{{}^\circ H}$ with $\mathbb{G}_{{}^\bullet H}$ induced by α [where we note that one verifies easily that the isomorphism of $\mathbb{G}_{{}^\circ H}$ with $\mathbb{G}_{{}^\bullet H}$ induced by α does not depend on the choice of "Σ" in condition ($c^\forall$) of Proposition 3.6] is Σ-*rationally compatible* [cf. Definition 3.4] with respect to the metric structures $\mu_{{}^\circ H}$, $\mu_{{}^\bullet H}$ [cf. Definition 3.5, (iii)].

[Thus, if the collections of data labeled by $\circ$, $\bullet$ are *equal*, then the notion of Σ-M-admissibility is *independent* of the choice of Σ — cf. the final portion of Definition 3.4.] We shall say that α is *M-admissible* if α is $\emptyset$-M-admissible. Write

$$\mathrm{Aut}({}^\circ\Pi_1)^{\mathrm{M}} \subseteq \mathrm{Aut}({}^\circ\Pi_1)$$

for the subgroup of M-admissible automorphisms of ${}^\circ\Pi_1$ and

$$\mathrm{Out}({}^\circ\Pi_1)^{\mathrm{M}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}({}^\circ\Pi_1)^{\mathrm{M}}/\mathrm{Inn}({}^\circ\Pi_1) \subseteq \mathrm{Out}({}^\circ\Pi_1)$$

for the subgroup of M-admissible outomorphisms of ${}^\circ\Pi_1$.

(iii) We shall write

$$\mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)^{\mathrm{M}} \subseteq \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)$$

for the subgroup of the group $\mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)$ of F-admissible outomorphisms of ${}^\circ\Pi_n$ [cf. [CmbCsp], Definition 1.1, (ii)] obtained by forming the inverse image of $\mathrm{Out}({}^\circ\Pi_1)^{\mathrm{M}} \subseteq \mathrm{Out}({}^\circ\Pi_1)$ [cf. (ii)] via the natural homomorphism $\mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n) \to \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_1) = \mathrm{Out}({}^\circ\Pi_1)$ [cf. [CbTpI], Theorem A, (i)];

$$\mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)^{\mathrm{M}} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)^{\mathrm{M}} \cap \mathrm{Out}^{\mathrm{C}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)$$

[cf. [CmbCsp], Definition 1.1, (ii)].

Definition 3.8. In the notation of Definition 3.1:

(i) Let $\alpha\colon {}^\circ\Pi_n \xrightarrow{\sim} {}^\bullet\Pi_n$ be an isomorphism of profinite groups [so ${}^\circ\Sigma = {}^\bullet\Sigma$ — cf., e.g., the proof of [CbTpI], Proposition 1.5, (i)] and $l \in {}^\circ\Sigma = {}^\bullet\Sigma$ such that $l \notin \{{}^\circ p, {}^\bullet p\}$. Then we shall say that α is *$\{l\}$-I-admissible* [i.e., "$\{l\}$-inertia-admissible"] if α is PF-admissible whenever $n \geq 2$ [cf. [CbTpI], Definition 1.4, (i)], and, moreover, the following condition is satisfied:

Let ${}^\circ\Pi_n \twoheadrightarrow ({}^\circ\Pi_n)^*$ be an F-characteristic almost pro-l quotient of ${}^\circ\Pi_n$ ($\twoheadleftarrow \pi_1((X_{{}^\circ\overline{K}})^{\log}_n)$) [cf. Definition 2.1, (iii)]. If ${}^\circ\Sigma = {}^\bullet\Sigma \neq \mathfrak{Primes}$, then we assume further that the quotient ${}^\circ\Pi_n \twoheadrightarrow ({}^\circ\Pi_n)^*$ is an *almost maximal pro-l quotient* relative to some characteristic open subgroup of ${}^\circ\Pi_n$ [cf. Definition 1.1]. Write ${}^\bullet\Pi_n \twoheadrightarrow ({}^\bullet\Pi_n)^*$ for the quotient of ${}^\bullet\Pi_n$ that corresponds to ${}^\circ\Pi_n \twoheadrightarrow ({}^\circ\Pi_n)^*$ via α. [Here, we observe that since α is PF-admissible whenever $n \geq 2$, one verifies immediately that the quotient ${}^\bullet\Pi_n \twoheadrightarrow ({}^\bullet\Pi_n)^*$ satisfies *similar assumptions* to the assumptions imposed on the quotient ${}^\circ\Pi_n \twoheadrightarrow ({}^\circ\Pi_n)^*$.] Then there exist open subgroups ${}^\circ J \subseteq I_{{}^\circ K}$, ${}^\bullet J \subseteq I_{{}^\bullet K}$ [which may depend on ${}^\circ\Pi_n \twoheadrightarrow ({}^\circ\Pi_n)^*$] such that the diagram

$$\begin{array}{ccc} \mathrm{Im}({}^\circ J) & \longrightarrow & \mathrm{Out}(({}^\circ\Pi_n)^*) \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow \\ \mathrm{Im}({}^\bullet J) & \longrightarrow & \mathrm{Out}(({}^\bullet\Pi_n)^*) \end{array}$$

— where, for $\Box \in \{\circ, \bullet\}$, we write

$$\mathrm{Im}({}^\Box J) \subseteq \mathrm{Out}(({}^\Box\Pi_n)^*)$$

for the image of ${}^\Box J$ via the homomorphism ${}^\Box J \to \mathrm{Out}(({}^\Box\Pi_n)^*)$ induced [in light of our *assumptions* on the quotients under consideration!] by ${}^\Box\rho_n$; the horizontal arrows are the natural inclusions; the right-hand vertical arrow is the isomorphism induced by the isomorphism α — *commutes* for some [*uniquely determined*] isomorphism $\beta\colon \mathrm{Im}({}^\circ J) \xrightarrow{\sim} \mathrm{Im}({}^\bullet J)$.

We shall say that an outer isomorphism ${}^\circ\Pi_n \xrightarrow{\sim} {}^\bullet\Pi_n$ is $\{l\}$-*I-admissible* if it arises from an isomorphism ${}^\circ\Pi_n \xrightarrow{\sim} {}^\bullet\Pi_n$ which is $\{l\}$-I-admissible.

(ii) We shall say that an isomorphism of profinite groups ${}^\circ\Pi_n \xrightarrow{\sim} {}^\bullet\Pi_n$ [so ${}^\circ\Sigma = {}^\bullet\Sigma$ — cf., e.g., the proof of [CbTpI], Proposition 1.5, (i)] is *I-admissible* [i.e., "inertia-admissible"] if ${}^\circ\Sigma = {}^\bullet\Sigma \not\subseteq \{{}^\circ p, {}^\bullet p\}$, and, moreover, the isomorphism is $\{l\}$-I-admissible [cf. (i)] for every prime number $l \in {}^\circ\Sigma = {}^\bullet\Sigma$ such that $l \notin \{{}^\circ p, {}^\bullet p\}$. We shall say that an outer isomorphism ${}^\circ\Pi_n \xrightarrow{\sim} {}^\bullet\Pi_n$ is *I-admissible* if it arises from an isomorphism ${}^\circ\Pi_n \xrightarrow{\sim} {}^\bullet\Pi_n$ which is I-admissible.

(iii) Let $l \in {}^\circ\Sigma$ be such that $l \neq {}^\circ p$. Then we shall write

$$\mathrm{Aut}^{\{l\}\text{-I}}({}^\circ\Pi_n) \subseteq \mathrm{Aut}({}^\circ\Pi_n)$$

for the subgroup of $\{l\}$-I-admissible automorphisms of ${}^\circ\Pi_n$ [cf. (i)];

$$\mathrm{Out}^{\{l\}\text{-I}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\{l\}\text{-I}}({}^\circ\Pi_n)/\mathrm{Inn}({}^\circ\Pi_n) \subseteq \mathrm{Out}({}^\circ\Pi_n)$$

for the subgroup of $\{l\}$-I-admissible outomorphisms of ${}^\circ\Pi_n$;

$$\mathrm{Out}^{\mathrm{F}\,\{l\}\text{-I}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\{l\}\text{-I}}({}^\circ\Pi_n) \cap \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)$$

[cf. [CmbCsp], Definition 1.1, (ii)];

$$\mathrm{Out}^{\mathrm{FC}\,\{l\}\text{-I}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\{l\}\text{-I}}({}^\circ\Pi_n) \cap \mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)$$

[cf. [CmbCsp], Definition 1.1, (ii)]. Also, we shall write

$$\mathrm{Aut}^{\mathrm{I}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \bigcap_{l \in {}^\circ\Sigma \setminus ({}^\circ\Sigma \cap \{{}^\circ p\})} \mathrm{Aut}^{\{l\}\text{-I}}({}^\circ\Pi_n) \subseteq \mathrm{Aut}({}^\circ\Pi_n)$$

for the subgroup of I-admissible automorphisms of ${}^\circ\Pi_n$ [cf. (ii)];

$$\mathrm{Out}^{\mathrm{I}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \bigcap_{l \in {}^\circ\Sigma \setminus ({}^\circ\Sigma \cap \{{}^\circ p\})} \mathrm{Out}^{\{l\}\text{-I}}({}^\circ\Pi_n) \subseteq \mathrm{Out}({}^\circ\Pi_n)$$

for the subgroup of I-admissible outomorphisms of ${}^\circ\Pi_n$;

$$\mathrm{Out}^{\mathrm{FI}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{I}}({}^\circ\Pi_n) \cap \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)\,;$$

$$\mathrm{Out}^{\mathrm{FCI}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{I}}({}^\circ\Pi_n) \cap \mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)\,.$$

(iv) Let $l \in {}^\circ\Sigma$ be such that $l \neq {}^\circ p$. Then we shall write

$$\mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)^{\{l\}\text{-I}} \subseteq \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)$$

for the subgroup of the group $\mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)$ of F-admissible outomorphisms of ${}^\circ\Pi_n$ obtained by forming the inverse image of $\mathrm{Out}^{\{l\}\text{-I}}({}^\circ\Pi_1) \subseteq \mathrm{Out}({}^\circ\Pi_1)$ [cf. (iii)] via the natural homomorphism $\mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n) \to \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_1) = \mathrm{Out}({}^\circ\Pi_1)$ [cf. [CbTpI], Theorem A, (i)];

$$\mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)^{\{l\}\text{-I}} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)^{\{l\}\text{-I}} \cap \mathrm{Out}^{\mathrm{C}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)\,.$$

Also, we shall write

$$\mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)^{\mathrm{I}} \stackrel{\mathrm{def}}{=} \bigcap_{l \in {}^\circ\Sigma \setminus ({}^\circ\Sigma \cap \{{}^\circ p\})} \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)^{\{l\}\text{-I}} \subseteq \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)\,;$$

$$\mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)^{\mathrm{I}} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}({}^\circ\Pi_n)^{\mathrm{I}} \cap \mathrm{Out}^{\mathrm{C}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)\,.$$

Theorem 3.9 (**Equivalence of metric-admissibility and inertia-admissibility**). *For* $\square \in \{\circ, \bullet\}$*, let* ${}^{\square}p$ *be a prime number;* ${}^{\square}\Sigma$ *a nonempty set of prime numbers such that* ${}^{\square}\Sigma \neq \{{}^{\square}p\}$*;* ${}^{\square}R$ *a mixed characteristic complete discrete valuation ring of residue characteristic* ${}^{\square}p$ *whose residue field is separably closed;* ${}^{\square}K$ *the field of fractions of* ${}^{\square}R$*;* ${}^{\square}\overline{K}$ *an algebraic closure of* ${}^{\square}K$*;*

$$X^{\log}_{{}^{\square}K}$$

a **smooth log curve** *over* ${}^{\square}K$*. For* $\square \in \{\circ, \bullet\}$*, write*

$$X^{\log}_{{}^{\square}\overline{K}} \overset{\mathrm{def}}{=} X^{\log}_{{}^{\square}K} \times_{{}^{\square}K} {}^{\square}\overline{K}\,;$$

$${}^{\square}\Pi_1 \overset{\mathrm{def}}{=} \pi_1(X^{\log}_{{}^{\square}\overline{K}})^{{}^{\square}\Sigma}$$

for the maximal pro-${}^{\square}\Sigma$ *quotient of the log fundamental group of* $X^{\log}_{{}^{\square}\overline{K}}$*. Let*

$$\alpha\colon {}^{\circ}\Pi_1 \overset{\sim}{\to} {}^{\bullet}\Pi_1$$

be an isomorphism of profinite groups. [Thus, it follows immediately that ${}^{\circ}\Sigma = {}^{\bullet}\Sigma$ *— cf., e.g., the proof of* [CbTpI]*, Proposition 1.5, (i).] If* ${}^{\circ}p \notin {}^{\circ}\Sigma$ *and* ${}^{\bullet}p \notin {}^{\bullet}\Sigma$*, then we assume further that* α *is* **group-theoretically cuspidal** *[cf.* [CmbGC]*, Definition 1.4, (iv)]. Then the following conditions are equivalent:*

(a) α *is* **M-admissible** *[cf. Definition 3.7, (ii)].*

($\mathrm{b}^{\forall}$) α *is* **I-admissible** *[cf. Definition 3.8, (ii)].*

($\mathrm{b}^{\exists}$) *There exists a prime number* $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ *such that* $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$*, and, moreover,* α *is* $\boldsymbol{\{l\}}$**-I-admissible** *[cf. Definition 3.8, (i)].*

Proof. First, let us observe that it follows formally from the various definitions involved that conditions (a), ($\mathrm{b}^{\forall}$), and ($\mathrm{b}^{\exists}$) all imply that there exists a prime number $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. Now *fix* such a prime number l and consider the condition:

($\mathrm{b}^{\{l\}}$): α is $\{l\}$-*I-admissible* [cf. Definition 3.8, (i)].

Then [since l is *arbitrary*, and condition (a) is manifestly *independent* of the choice of l] it follows formally from the various definitions involved that to verify Theorem 3.9, it suffices to verify the equivalence

$$\text{(a)} \iff (\mathrm{b}^{\{l\}})\,.$$

To this end, let ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ be a characteristic open subgroup of ${}^{\circ}\Pi_1$. Write ${}^{\bullet}H \overset{\mathrm{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$. Also, for each $\square \in \{\circ, \bullet\}$, write $({}^{\square}\Pi_1)^*$ for the maximal almost pro-l quotient of ${}^{\square}\Pi_1$ with respect to ${}^{\square}H$. [Thus, $({}^{\square}H)^{\{l\}} \overset{\sim}{\to} \Pi_{\mathcal{G}_{{}^{\square}H}[\{l\}]} \subseteq ({}^{\square}\Pi_1)^*$ — cf. Remark 3.5.1.]

Next, let us observe that, for each $\square \in \{\circ, \bullet\}$, since $({}^{\square}H)^{\{l\}} \overset{\sim}{\to} \Pi_{\mathcal{G}_{{}^{\square}H}[\{l\}]} \subseteq ({}^{\square}\Pi_1)^*$ is *open*, and $({}^{\square}\Pi_1)^*$ is *topologically finitely generated*, *slim* [cf. Proposition 1.7, (i)] and *almost pro-l*, there exist an *open subgroup* ${}^{\square}J \subseteq I_{{}^{\square}K}$ of $I_{{}^{\square}K}$ and a *homomorphism*

$$ {}^{\square}\rho_1[{}^{\square}H] \colon {}^{\square}J \longrightarrow \mathrm{Out}(({}^{\square}H)^{\{l\}}) $$

such that ${}^{\square}\rho_1[{}^{\square}H]$ is *compatible* [in the evident sense] with the homomorphism ${}^{\square}J \to \mathrm{Out}(({}^{\square}\Pi_1)^*)$ induced by ${}^{\square}\rho_1 \colon I_{{}^{\square}K} \to \mathrm{Out}({}^{\square}\Pi_1)$, and, moreover, ${}^{\square}\rho_1[{}^{\square}H]$ factors through the maximal pro-l quotient $({}^{\square}J)^{\{l\}}$ of ${}^{\square}J$, which [as is easily verified] is isomorphic to $\mathbb{Z}_l$ as an abstract profinite group. Moreover, it follows immediately from the various definitions involved, together with the well-known *properness* of the moduli stack of pointed stable curves of a given type, that the outer representation $({}^{\square}J)^{\{l\}} \to \mathrm{Out}(({}^{\square}H)^{\{l\}}) \xrightarrow{\sim} \mathrm{Out}(\Pi_{\mathcal{G}_{{}^{\square}H}[\{l\}]})$ arising from such a homomorphism ${}^{\square}\rho_1[{}^{\square}H]$ is of *PIPSC-type* [cf. Definition 1.3]. In particular, it follows immediately from Theorem 1.11, (ii), that if α satisfies condition ($\mathrm{b}^{\{l\}}$), i.e., α is *$\{l\}$-I-admissible*, then the isomorphism of $({}^{\circ}H)^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\circ}H}[\{l\}]}$ with $({}^{\bullet}H)^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\bullet}H}[\{l\}]}$ induced by α is *group-theoretically verticial*, hence also *group-theoretically nodal*. Thus, by allowing "${}^{\square}H$" to vary among the various characteristic open subgroups of ${}^{\square}\Pi_1$, we conclude that if α satisfies condition ($\mathrm{b}^{\{l\}}$), i.e., α is *$\{l\}$-I-admissible*, then α satisfies condition ($\mathrm{b}^{\exists}$) of Proposition 3.6, hence [cf. Proposition 3.6, (iii); our assumption that α is *group-theoretically cuspidal* if ${}^{\circ}p \notin {}^{\circ}\Sigma$, ${}^{\bullet}p \notin {}^{\bullet}\Sigma$] that α is *G-admissible*. In particular, it follows from *either* of the conditions (a), ($\mathrm{b}^{\{l\}}$) that the isomorphism of $({}^{\circ}H)^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\circ}H}[\{l\}]}$ with $({}^{\bullet}H)^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\bullet}H}[\{l\}]}$ induced by α is *graphic* [cf. condition ($\mathrm{c}^{\forall}$) of Proposition 3.6], hence that α determines a commutative diagram of *isomorphisms* of profinite groups

$$ \begin{array}{ccc} \mathrm{Dehn}(\mathcal{G}_{{}^{\circ}H}[\{l\}]) & \xrightarrow{\mathfrak{D}_{\mathcal{G}_{{}^{\circ}H}[\{l\}]}} & \bigoplus_{\mathrm{Node}(\mathcal{G}_{{}^{\circ}H}[\{l\}])} \Lambda_{\mathcal{G}_{{}^{\circ}H}[\{l\}]} \\ \downarrow & & \downarrow \\ \mathrm{Dehn}(\mathcal{G}_{{}^{\bullet}H}[\{l\}]) & \xrightarrow[\mathfrak{D}_{\mathcal{G}_{{}^{\bullet}H}[\{l\}]}]{} & \bigoplus_{\mathrm{Node}(\mathcal{G}_{{}^{\bullet}H}[\{l\}])} \Lambda_{\mathcal{G}_{{}^{\bullet}H}[\{l\}]} \end{array} $$

[cf. [CbTpI], Definition 4.4; [CbTpI], Theorem 4.8, (iv)].

On the other hand, since, for each $\square \in \{\circ, \bullet\}$, the outer representation $({}^{\square}J)^{\{l\}} \to \mathrm{Out}(({}^{\square}H)^{\{l\}}) \xrightarrow{\sim} \mathrm{Out}(\Pi_{\mathcal{G}_{{}^{\square}H}[\{l\}]})$ is of *PIPSC-type*, it follows — by replacing ${}^{\square}J$ by an open subgroup of ${}^{\square}J$ if necessary — from [CbTpI], Corollary 5.9, (iii), that we may assume without loss of generality that this outer representation factors through $\mathrm{Dehn}(\mathcal{G}_{{}^{\square}H}[\{l\}]) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_{{}^{\square}H}[\{l\}]})$. Thus, by considering the *Dehn coordinates* [cf. [CbTpI], Definition 5.8, (i)] of the image of a topological generator of $({}^{\square}J)^{\{l\}}$ in $\mathrm{Dehn}(\mathcal{G}_{{}^{\square}H}[\{l\}])$ [with respect to a topological generator of $\Lambda_{\mathcal{G}_{{}^{\square}H}[\{l\}]}$], it follows immediately from [CbTpI], Theorem

5.7; [CbTpI], Lemma 5.4, (ii), together with the existence of the commutative diagram of the above display, that

> the isomorphism $\mathbb{G}_{\circ H} \xrightarrow{\sim} \mathbb{G}_{\bullet H}$ induced by α is $\emptyset$-*rationally compatible* [cf. Definition 3.4] with the *metric structures* $\mu_{\circ H}$, $\mu_{\bullet H}$ [cf. Definition 3.5, (iii)] if and only if the images of the homomorphisms $({}^{\circ}J)^{\{l\}} \to \mathrm{Dehn}(\mathcal{G}_{\circ H}[\{l\}])$ and $({}^{\bullet}J)^{\{l\}} \to \mathrm{Dehn}(\mathcal{G}_{\bullet H}[\{l\}])$ are *compatible*, up to a $\mathbb{Q}_{>0}$-multiple, with the isomorphisms induced by α.

In particular, by applying this equivalence to the various characteristic open subgroups "${}^{\Box}H$"$\subseteq {}^{\Box}\Pi_1$ of ${}^{\Box}\Pi_1$, we conclude that α satisfies condition ($\mathrm{b}^{\{l\}}$), i.e., α is $\{l\}$-*I-admissible*, if and only if α satisfies condition (a), i.e., α is *M-admissible*. This completes the proof of Theorem 3.9. $\Box$

Definition 3.10. In the notation of Definition 3.1, let $l \in {}^{\Box}\Sigma$ be such that $l \neq {}^{\Box}p$ and ${}^{\Box}H \subseteq {}^{\Box}\Pi_n$ an open subgroup of ${}^{\Box}\Pi_n$. For each $i \in \{0, \cdots, n\}$, write ${}^{\Box}H_i \subseteq {}^{\Box}\Pi_i$ for the open subgroup of the quotient ${}^{\Box}\Pi_n \twoheadrightarrow {}^{\Box}\Pi_i$ [induced by the projection $(X_{\Box\overline{K}})^{\log}_n \to (X_{\Box\overline{K}})^{\log}_i$ to the first i factors] determined by the image of ${}^{\Box}H \subseteq {}^{\Box}\Pi_n$; ${}^{\Box}Y^{\log}_i \to (X_{\Box\overline{K}})^{\log}_i$ for the connected finite log étale covering of $(X_{\Box\overline{K}})^{\log}_i$ corresponding to ${}^{\Box}H_i \subseteq {}^{\Box}\Pi_i$. Then we have a sequence of morphisms of log schemes

$$ {}^{\Box}Y^{\log}_n \longrightarrow {}^{\Box}Y^{\log}_{n-1} \longrightarrow \cdots \longrightarrow {}^{\Box}Y^{\log}_2 \longrightarrow {}^{\Box}Y^{\log}_1 \longrightarrow {}^{\Box}Y^{\log}_0 . $$

Thus, for $i \in \{0, \cdots, n\}$, if we write ${}^{\Box}U_i$ for the *interior* of ${}^{\Box}Y^{\log}_i$ [cf. the discussion entitled "*Log schemes*" in [CbTpI], §0], we obtain a sequence of morphisms of schemes [each of which determines a family of hyperbolic curves]

$$ {}^{\Box}U_n \longrightarrow {}^{\Box}U_{n-1} \longrightarrow \cdots \longrightarrow {}^{\Box}U_2 \longrightarrow {}^{\Box}U_1 \longrightarrow {}^{\Box}U_0 . $$

Then we shall say that ${}^{\Box}H$ is of *l-polystable type* if the following conditions are satisfied:

(a) For each $i \in \{0, \cdots, n\}$, $\alpha \in \mathrm{Aut}^{\mathrm{F}}({}^{\Box}\Pi_i)$ [cf. [CmbCsp], Definition 1.1, (ii)], the open subgroup ${}^{\Box}H_i \subseteq {}^{\Box}\Pi_i$ is *preserved* by α. Here, for convenience, when $n = 1$, and ${}^{\Box}\Sigma$ is *arbitrary*, we set $\mathrm{Aut}^{\mathrm{F}}({}^{\Box}\Pi_1) \stackrel{\mathrm{def}}{=} \mathrm{Aut}({}^{\Box}\Pi_1)$. [In particular, ${}^{\Box}H_i$ is normal.]

(b) The [necessarily *F-characteristic* — cf. condition (a) above; Definition 2.1, (iii)] maximal almost pro-l quotient

$$ (\pi_1((X_{\Box\overline{K}})^{\log}_n) \twoheadrightarrow)\ {}^{\Box}\Pi_n \twoheadrightarrow ({}^{\Box}\Pi_n)^* $$

with respect to ${}^{\Box}H \subseteq {}^{\Box}\Pi_n$ [cf. Definition 1.1] is *SA-maximal* [cf. Definition 2.1, (ii)].

(c) For each $i \in \{1, \cdots, n\}$, if we write $({}^{\Box}H_{i/i-1})^{\{l\}}$ for the maximal pro-l quotient of the kernel ${}^{\Box}H_{i/i-1} \stackrel{\text{def}}{=} \text{Ker}({}^{\Box}H_i \twoheadrightarrow {}^{\Box}H_{i-1})$, then the natural action of ${}^{\Box}H_{i-1}$ on the l^{aut}-*abelianization* [cf. Lemma 2.14] of $({}^{\Box}H_{i/i-1})^{\{l\}}$ is *trivial*.

Remark 3.10.1. In the notation of Definition 3.10:

(i) Let us observe that [one verifies easily that] condition (c) of Definition 3.10 implies that the following condition holds:

(d) For each $i \in \{1, \cdots, n\}$, the natural outer representation
$$ {}^{\Box}H_{i-1} \longrightarrow \text{Out}(({}^{\Box}H_{i/i-1})^{\{l\}}) $$
factors through a pro-l quotient of ${}^{\Box}H_{i-1}$.

Moreover, it follows from Lemma 2.14, (ii); [ExtFam], Corollary 7.4, that condition (c) of Definition 3.10 also implies that the following condition holds:

(e) The sequence of morphisms of log schemes in Definition 3.10
$$ {}^{\Box}Y_n^{\log} \longrightarrow {}^{\Box}Y_{n-1}^{\log} \longrightarrow \cdots \longrightarrow {}^{\Box}Y_2^{\log} \longrightarrow {}^{\Box}Y_1^{\log} \longrightarrow {}^{\Box}Y_0^{\log} $$
extends to the factorization
$$ {}^{\Box}\mathcal{Y}_n^{\log} \longrightarrow {}^{\Box}\mathcal{Y}_{n-1}^{\log} \longrightarrow \cdots \longrightarrow {}^{\Box}\mathcal{Y}_2^{\log} \longrightarrow {}^{\Box}\mathcal{Y}_1^{\log} \longrightarrow {}^{\Box}\mathcal{Y}_0^{\log} $$
associated to the *log polystable* morphism determined by a [uniquely determined!] *stable polycurve* over ${}^{\Box}\overline{R}$ [cf. [ExtFam], Definition 4.5].

(ii) One verifies easily that, for each $i \in \{0, \cdots, n\}$, if ${}^{\Box}H \subseteq {}^{\Box}\Pi_n$ is of l-*polystable type*, then ${}^{\Box}H_i \subseteq {}^{\Box}\Pi_i$ is of l-*polystable type*.

Definition 3.11. In the notation of Definition 3.10, suppose that ${}^{\Box}H$ is of l-*polystable type* [cf. Definition 3.10].

(i) We shall refer to a point $y \in {}^{\Box}\mathcal{Y}_n$ of the underlying scheme ${}^{\Box}\mathcal{Y}_n$ of ${}^{\Box}\mathcal{Y}_n^{\log}$ [cf. the notation of condition (e) of Remark 3.10.1, (i)] as a *VCN-point* if the following condition is satisfied: For $i \in \{0, \cdots, n\}$, write $y_i \in {}^{\Box}\mathcal{Y}_i$ for the image of y in ${}^{\Box}\mathcal{Y}_i$ and $y_i^{\log} \stackrel{\text{def}}{=} {}^{\Box}\mathcal{Y}_i^{\log} \times_{{}^{\Box}\mathcal{Y}_i} y_i$. [Thus, for each $i \in \{1, \cdots, n\}$, we have a stable log curve ${}^{\Box}\mathcal{Y}_i^{\log}|_{y_{i-1}^{\log}} \stackrel{\text{def}}{=} {}^{\Box}\mathcal{Y}_i^{\log} \times_{{}^{\Box}\mathcal{Y}_{i-1}^{\log}} y_{i-1}^{\log}$ over $y_{i-1}^{\log}$.] Then

y_0 is the closed point of ${}^{\Box}\mathcal{Y}_0 = \text{Spec}\,{}^{\Box}\overline{R}$; for each $i \in \{1, \cdots, n\}$, the point of ${}^{\Box}\mathcal{Y}_i^{\log}|_{y_{i-1}^{\log}}$ determined by $y_i^{\log}$ is either a *cusp*, *node*, or *generic point* [i.e., the generic point of an irreducible component] of the *stable log curve* ${}^{\Box}\mathcal{Y}_i^{\log}|_{y_{i-1}^{\log}}$.

We shall write

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}H)$$

for the set of VCN-points of ${}^{\Box}\mathcal{Y}_n$.

(ii) We shall refer to a projective system ${}^{\Box}\mathbb{H} = \{{}^{\Box}H_\lambda\}_{\lambda\in\Lambda}$ of open subgroups of ${}^{\Box}\Pi_n$ as an ${}^{\Box}H$*-l-system* if each ${}^{\Box}H_\lambda$ is of l-polystable type and contained in ${}^{\Box}H$ [i.e., ${}^{\Box}H_\lambda \subseteq {}^{\Box}H$], and, moreover,

$$\mathrm{Ker}\Big({}^{\Box}\Pi_n \twoheadrightarrow ({}^{\Box}\Pi_n)^*\Big) = \Big(\mathrm{Ker}\Big({}^{\Box}H \twoheadrightarrow ({}^{\Box}H)^{\{l\}}\Big) = \Big) \bigcap_{\lambda\in\Lambda} {}^{\Box}H_\lambda$$

[cf. condition (b) of Definition 3.10] — i.e., the system ${}^{\Box}\mathbb{H}$ arises from a basis of the topology of $({}^{\Box}H)^{\{l\}}$.

(iii) Let ${}^{\Box}\mathbb{H} = \{{}^{\Box}H_\lambda\}_{\lambda\in\Lambda}$ be an ${}^{\Box}H$*-l-system* [cf. (ii)]. Then we shall write

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}) \stackrel{\mathrm{def}}{=} \varprojlim_{\lambda\in\Lambda} \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}H_\lambda)$$

[cf. (i) above; the portion of [ExtFam], Corollary 7.4, concerning extensions of morphisms]. Here, we note that one verifies easily that, for each $i \in \{0, \cdots, n\}$, if ${}^{\Box}\mathbb{H} = \{{}^{\Box}H_\lambda\}_{\lambda\in\Lambda}$ is an ${}^{\Box}H$*-l-system*, and we write $({}^{\Box}H_\lambda)_i \subseteq {}^{\Box}\Pi_i$ for the image of ${}^{\Box}H_\lambda$ in ${}^{\Box}\Pi_i$, then the system ${}^{\Box}\mathbb{H}_i \stackrel{\mathrm{def}}{=} \{({}^{\Box}H_\lambda)_i\}_{\lambda\in\Lambda}$ is an ${}^{\Box}H_i$-*l-system* [cf. condition (b) of Definition 3.10; Remark 3.10.1, (ii)]. Thus, we have a natural map

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}) \longrightarrow \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i)\,.$$

Definition 3.12. In the notation of Definition 3.11, let ${}^{\Box}\mathbb{H} = \{{}^{\Box}H_\lambda\}_{\lambda\in\Lambda}$ be an ${}^{\Box}H$*-l-system* [cf. Definition 3.11, (ii)] and $\widetilde{y} \in \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H})$ [cf. Definition 3.11, (iii)]. For each $i \in \{0, \cdots, n\}$, write $\widetilde{y}_i \in \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i)$ for the image of $\widetilde{y}$ via the natural map of the final display of Definition 3.11, (iii). Let $i \in \{1, \cdots, n\}$.

(i) Write

$$\mathcal{G}_{i,\widetilde{y}_{i-1}}$$

for the *semi-graph of anabelioids of pro-l PSC-type* determined by the stable log curve constituted by the log geometric fiber of ${}^{\Box}\mathcal{Y}_i^{\log} \to {}^{\Box}\mathcal{Y}_{i-1}^{\log}$ [cf. Definition 3.11, (i)] at the point of ${}^{\Box}\mathcal{Y}_{i-1}^{\log}$ determined by $\widetilde{y}_{i-1}$;

$$\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}} \longrightarrow \mathcal{G}_{i,\widetilde{y}_{i-1}}$$

for the *universal covering* [corresponding to the [pro-l] fundamental group $\Pi_{\mathcal{G}_{i,\widetilde{y}_{i-1}}}$ of $\mathcal{G}_{i,\widetilde{y}_{i-1}}$ relative to the *basepoint* of $\mathcal{G}_{i,\widetilde{y}_{i-1}}$ determined by the various ${}^{\Box}H_\lambda$'s] obtained by considering the "$\mathcal{G}_{i,\widetilde{y}_{i-1}}$'s" arising from the various ${}^{\Box}H_\lambda$'s.

(ii) Write

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i)|_{\widetilde{y}_{i-1}} \stackrel{\mathrm{def}}{=} \{\, \widetilde{y}' \in \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i) \mid \widetilde{y}'_{i-1} = \widetilde{y}_{i-1} \,\}$$

[cf. Definition 3.11, (iii)]. Then one verifies easily from the various definitions involved that we have a *natural bijection*

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i)|_{\widetilde{y}_{i-1}} \xrightarrow{\sim} \mathrm{VCN}(\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}})$$

[cf. (i)]. In particular, the element $\widetilde{y}_i \in \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i)|_{\widetilde{y}_{i-1}}$ determines an element

$$\widetilde{z}_{i,\widetilde{y}} \in \mathrm{VCN}(\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}})$$

of $\mathrm{VCN}(\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}})$.

(iii) It follows immediately from the various definitions involved that we have a natural action of $({}^{\Box}H_i)^{\{l\}}$, hence also of $({}^{\Box}H_{i/i-1})^{\{l\}}$ [cf. the notation of condition (c) of Definition 3.10], on the set $\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i)$. Thus, we obtain a *tautological isomorphism*

$$\Pi_{\mathcal{G}_{i,\widetilde{y}_{i-1}}} \xrightarrow{\sim} ({}^{\Box}H_{i/i-1})^{\{l\}}$$

such that the various *VCN-subgroups* [cf. [CbTpI], Definition 2.1, (i)] on the left-hand side of this isomorphism correspond to the various *stabilizer subgroups* of $({}^{\Box}H_{i/i-1})^{\{l\}}$ associated to elements of $\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i)|_{\widetilde{y}_{i-1}}$ [cf. the notation of (ii); the natural bijection of the second display of (ii)] on the right-hand side of this isomorphism.

(iv) Let $(F_i)_{i\in\{1,\cdots,n\}}$ be a collection of closed subgroups $F_i \subseteq ({}^{\Box}H_i)^{\{l\}}$. Then we shall say that the collection $(F_i)_{i\in\{1,\cdots,n\}}$ is the *VCN-chain of* ${}^{\Box}H$ *associated to* $\widetilde{y} \in \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H})$ if, for each $i \in \{1,\cdots,n\}$, the closed subgroup F_i coincides with the image of the VCN-subgroup of $\Pi_{\mathcal{G}_{i,\widetilde{y}_{i-1}}}$ associated to $\widetilde{z}_{i,\widetilde{y}} \in \mathrm{VCN}(\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}})$ [cf. (ii)] via the isomorphism $\Pi_{\mathcal{G}_{i,\widetilde{y}_{i-1}}} \xrightarrow{\sim} ({}^{\Box}H_{i/i-1})^{\{l\}} \subseteq ({}^{\Box}H_i)^{\{l\}}$ of (iii). We shall say that the collection $(F_i)_{i\in\{1,\cdots,n\}}$ is an ${}^{\Box}\mathbb{H}$-*VCN-chain* of ${}^{\Box}H$ if $(F_i)_{i\in\{1,\cdots,n\}}$ is the VCN-chain of ${}^{\Box}H$ associated to an element of $\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H})$. Write

$$\mathrm{VCN}^{\mathrm{gp}}({}^{\Box}\mathbb{H})$$

for the set of ${}^{\Box}\mathbb{H}$-VCN-chains of ${}^{\Box}H$. Thus, we conclude from [CmbGC], Proposition 1.2, (i), that the natural bijections of (ii) determine a *bijection*

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}({}^{\Box}\mathbb{H})\,.$$

Definition 3.13. In the notation of Definition 3.1:

(i) We shall say that an isomorphism of profinite groups ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ is *SAF-admissible* [i.e., "standard-adjacent-fiber-admissible"] if it is PF-admissible whenever $n \geq 2$ [cf. [CbTpI], Definition 1.4, (i)] and, moreover, is compatible with the standard fiber filtrations on ${}^{\circ}\Pi_n$ and ${}^{\bullet}\Pi_n$ [cf. [CmbCsp], Definition 1.1, (i)]. We shall refer to an outer isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ as *SAF-admissible* if it arises from an SAF-admissible isomorphism. One verifies easily that, in the case of an *automorphism* or *outomorphism*, *SAF-admissibility* is equivalent to *F-admissibility* whenever $n \geq 2$.

(ii) Let $\alpha\colon {}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ be an isomorphism of profinite groups [so ${}^{\circ}\Sigma = {}^{\bullet}\Sigma$ — cf., e.g., the proof of [CbTpI], Proposition 1.5, (i)] and $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. Then we shall say that α is $\{l\}$-*G-admissible* [i.e., $\{l\}$-graph-admissible] if α is SAF-admissible [cf. (i)], and, moreover, the following condition is satisfied:

Let ${}^{\circ}J \subseteq {}^{\circ}\Pi_n$ be an open subgroup of ${}^{\circ}\Pi_n$. Then there exist an open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_n$ of ${}^{\circ}\Pi_n$ of *l-polystable type* [cf. Definition 3.10] and an ${}^{\circ}H$-*l-system* ${}^{\circ}\mathbb{H} = \{{}^{\circ}H_\lambda\}_{\lambda\in\Lambda}$ [cf. Definition 3.11, (ii)] such that ${}^{\circ}H \subseteq {}^{\circ}J$, ${}^{\bullet}H \stackrel{\mathrm{def}}{=} \alpha({}^{\circ}H)$ is of *l-polystable type*, ${}^{\bullet}\mathbb{H} = \{{}^{\bullet}H_\lambda \stackrel{\mathrm{def}}{=} \alpha({}^{\circ}H_\lambda)\}_{\lambda\in\Lambda}$ is an ${}^{\bullet}H$-*l-system*, and, moreover, the isomorphism ${}^{\circ}H \xrightarrow{\sim} {}^{\bullet}H$ determined by α induces a bijection

$$\mathrm{VCN}^{\mathrm{gp}}({}^{\circ}\mathbb{H}) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}({}^{\bullet}\mathbb{H})$$

[cf. Definition 3.12, (iv)].

We shall say that an outer isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ is $\{l\}$-*G-admissible* if it arises from an $\{l\}$-G-admissible isomorphism.

(iii) We shall say that an isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ [so ${}^{\circ}\Sigma = {}^{\bullet}\Sigma$ — cf., e.g., the proof of [CbTpI], Proposition 1.5, (i)] is *G-admissible* [i.e., graph-admissible] if ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq \{{}^{\circ}p, {}^{\bullet}p\}$, and, moreover, the isomorphism is $\{l\}$-G-admissible [cf. (ii)] for every prime number $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. We shall say that an outer isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ is *G-admissible* if it arises from a G-admissible isomorphism.

(iv) We shall write

$$\mathrm{Aut}^{\{l\}\text{-}\mathrm{G}}({}^{\circ}\Pi_n) \subseteq \mathrm{Aut}({}^{\circ}\Pi_n)$$

for the subgroup [cf. Lemma 3.14, (iii), below; [CmbGC], Proposition 1.2, (ii)] of $\{l\}$-G-admissible automorphisms of ${}^{\circ}\Pi_n$

[cf. (ii)];

$$\mathrm{Out}^{\{l\}\text{-G}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\{l\}\text{-G}}({}^\circ\Pi_n)/\mathrm{Inn}({}^\circ\Pi_n) \subseteq \mathrm{Out}({}^\circ\Pi_n)$$

for the subgroup of $\{l\}$-G-admissible outomorphisms of ${}^\circ\Pi_n$;

$$\mathrm{Aut}^{\mathrm{G}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \bigcap_{l\in{}^\circ\Sigma\setminus({}^\circ\Sigma\cap\{{}^\circ p\})} \mathrm{Aut}^{\{l\}\text{-G}}({}^\circ\Pi_n) \subseteq \mathrm{Aut}({}^\circ\Pi_n)$$

for the subgroup of G-admissible automorphisms of ${}^\circ\Pi_n$ [cf. (iii)];

$$\mathrm{Out}^{\mathrm{G}}({}^\circ\Pi_n) \stackrel{\mathrm{def}}{=} \bigcap_{l\in{}^\circ\Sigma\setminus({}^\circ\Sigma\cap\{{}^\circ p\})} \mathrm{Out}^{\{l\}\text{-G}}({}^\circ\Pi_n) \subseteq \mathrm{Out}({}^\circ\Pi_n)$$

for the subgroup of G-admissible outomorphisms of ${}^\circ\Pi_n$.

Remark 3.13.1.

(i) In the notation of Definition 3.13, suppose that $n = 1$. Then it follows immediately from Proposition 3.6, (ii); [CmbGC], Proposition 1.5, (ii), that the following conditions are equivalent:

- α is *G-admissible* in the sense of Definition 3.7, (i).
- There exists a prime number $l \in {}^\circ\Sigma = {}^\bullet\Sigma$ such that $l \notin \{{}^\circ p, {}^\bullet p\}$, and, moreover, α is $\{l\}$-*G-admissible* in the sense of Definition 3.13, (ii).
- α is *G-admissible* in the sense of Definition 3.13, (iii).

In particular, for any prime number $l \in {}^\circ\Sigma$ such that $l \neq {}^\circ p$, we have equalities

$$\mathrm{Out}({}^\circ\Pi_1)^{\mathrm{G}} = \mathrm{Out}^{\mathrm{G}}({}^\circ\Pi_1) = \mathrm{Out}^{\{l\}\text{-G}}({}^\circ\Pi_1)$$

[cf. Definitions 3.7, (i); 3.13, (iv)].

(ii) In the notation of Definition 3.13, (iv), one verifies easily from the various definitions involved that

$$\mathrm{Out}^{\mathrm{G}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\{l\}\text{-G}}({}^\circ\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}({}^\circ\Pi_n)$$

[cf. [CmbCsp], Definition 1.1, (ii)].

Lemma 3.14 (**Subgroups of l-polystable type**)**.** *In the notation of Definition 3.1, let* $\alpha\colon {}^\circ\Pi_n \xrightarrow{\sim} {}^\bullet\Pi_n$ *be an isomorphism of profinite groups [so* ${}^\circ\Sigma = {}^\bullet\Sigma$ *— cf., e.g., the proof of* [CbTpI], *Proposition 1.5, (i)] and* $l \in {}^\circ\Sigma = {}^\bullet\Sigma$ *such that* $l \notin \{{}^\circ p, {}^\bullet p\}$*. Suppose that* α *is* **SAF-admissible** *[cf. Definition 3.13, (i)]. Then the following hold:*

(i) *Let* $^\circ J \subseteq {^\circ\Pi_n}$ *be an open subgroup of* $^\circ\Pi_n$. *Then there exists an open subgroup* $^\circ H \subseteq {^\circ\Pi_n}$ *of* $^\circ\Pi_n$ *of* ***l*-polystable type** *[cf. Definition 3.10] such that* $^\circ H \subseteq {^\circ J}$.

(ii) *Let* $^\circ H \subseteq {^\circ\Pi_n}$ *be an open subgroup of* $^\circ\Pi_n$ *of* ***l*-polystable type**. *Then there exists an* $^\circ$***H*-*l*-system** $^\circ\mathbb{H} = \{^\circ H_\lambda\}_{\lambda\in\Lambda}$ *[cf. Definition 3.11, (ii)].*

(iii) *Let* $^\circ H \subseteq {^\circ\Pi_n}$ *be an open subgroup of* ***l*-polystable type** *of* $^\circ\Pi_n$ *and* $^\circ\mathbb{H} = \{^\circ H_\lambda\}_{\lambda\in\Lambda}$ *an* $^\circ$***H*-*l*-system**. *Then* $^\bullet H \stackrel{\text{def}}{=} \alpha(^\circ H)$ *is an open subgroup of* ***l*-polystable type** *of* $^\bullet\Pi_n$, *and* $^\bullet\mathbb{H} = \{^\bullet H_\lambda \stackrel{\text{def}}{=} \alpha(^\circ H_\lambda)\}_{\lambda\in\Lambda}$ *is an* $^\bullet$***H*-*l*-system**.

Proof. First, we verify assertion (i) by *induction* on n. Write $^\circ J_{n-1}$ for the image of $^\circ J$ in $^\circ\Pi_{n-1}$ and $(^\circ J_{n/n-1})^{\{l\}}$ for the maximal pro-l quotient of the kernel $^\circ J_{n/n-1} \stackrel{\text{def}}{=} \mathrm{Ker}(^\circ J \twoheadrightarrow {^\circ J_{n-1}})$. Now let us observe that if $n = 1$, then assertion (i) follows immediately from the various definitions involved. Thus, suppose that $n \geq 2$, and that the *induction hypothesis* is in force.

Next, let us observe that since $^\circ\Pi_n$ is *topologically finitely generated* [cf. [MzTa], Proposition 2.2, (ii)], we may assume without loss of generality — by replacing $^\circ J$ by a suitable characteristic open subgroup of $^\circ J$ — that $^\circ J$ satisfies condition (a) of Definition 3.10 in the case where we take "i" to be n. Also, we observe that we may assume without loss of generality — by replacing $^\circ J$ by the inverse image in $^\circ J$ of a suitable characteristic open subgroup of $^\circ J_{n-1}$ — that $^\circ J$ satisfies condition (c) of Definition 3.10, hence also condition (d) of Remark 3.10.1, (i), in the case where we take "i" to be n.

Now, by applying the *induction hypothesis* to $^\circ J_{n-1}$, we obtain an open subgroup $^\circ H_{n-1} \subseteq {^\circ\Pi_{n-1}}$ of $^\circ\Pi_{n-1}$ that is contained in $^\circ J_{n-1}$ and of *l-polystable type*. Write $^\circ H \stackrel{\text{def}}{=} {^\circ H_{n-1}} \times_{^\circ J_{n-1}} {^\circ J}$. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow {^\circ J_{n/n-1}} \longrightarrow {^\circ H_n} \longrightarrow {^\circ H_{n-1}} \longrightarrow 1\,.$$

Then it follows immediately from the conditions imposed on $^\circ J$ in the preceding paragraph, together with the *induction hypothesis*, that $^\circ H$ satisfies conditions (a) and (c) of Definition 3.10 [hence also (d) of Remark 3.10.1, (i)]. On the other hand, by considering the quotient $^\circ H \twoheadrightarrow (^\circ J_{n/n-1})^{\{l\}} \stackrel{\text{out}}{\rtimes} (^\circ H_{n-1})^{\{l\}}$ [i.e., that arises from the fact that $^\circ H$ satisfies condition (d) of Remark 3.10.1, (i) — cf. also the discussion entitled "*Topological groups*" in [CbTpI], §0], we conclude that the natural homomorphism $(^\circ J_{n/n-1})^{\{l\}} \to (^\circ H)^{\{l\}}$ induced by the natural inclusion $^\circ J_{n/n-1} \hookrightarrow {^\circ H}$ is *injective*. Thus, one verifies easily from Lemma 1.2, (i), (ii), together with our *choice* of $^\circ H_{n-1}$, that $^\circ H$ satisfies condition (b) of Definition 3.10, i.e., that $^\circ H$ is *l-polystable type*. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that, to verify assertion (ii), it suffices to verify the following assertion:

> Claim 3.14.A: Let ${}^\circ J \subseteq {}^\circ H$ be an open subgroup that arises from an open subgroup of the maximal pro-l quotient $({}^\circ H)^{\{l\}}$ of ${}^\circ H$. Then there exists an open subgroup ${}^\circ N \subseteq {}^\circ J$ of *l-polystable type* that arises from an open subgroup of $({}^\circ H)^{\{l\}}$.

In the remainder of the proof of assertion (ii), we verify Claim 3.14.A by *induction* on n. Let us first observe that if $n = 1$, then Claim 3.14.A follows immediately from the various definitions involved. Thus, suppose that $n \geq 2$, and that the *induction hypothesis* is in force.

Now let us observe that since ${}^\circ\Pi_n$ is *topologically finitely generated* [cf. [MzTa], Proposition 2.2, (ii)], and ${}^\circ H$ satisfies condition (a) of Definition 3.10, we may assume without loss of generality — by replacing ${}^\circ J$ by a suitable characteristic open subgroup of ${}^\circ J$ — that ${}^\circ J$ satisfies condition (a) of Definition 3.10 in the case where we take "i" to be n. Next, let us observe that since ${}^\circ J$ arises from an open subgroup of $({}^\circ H)^{\{l\}}$, by considering the natural isomorphism $({}^\circ H)^{\{l\}} \xrightarrow{\sim} ({}^\circ H_{n/n-1})^{\{l\}} \overset{\text{out}}{\rtimes} ({}^\circ H_{n-1})^{\{l\}}$ [i.e., that arises from the fact that ${}^\circ H$ satisfies condition (d) of Remark 3.10.1, (i)], we conclude that ${}^\circ J$ satisfies condition (d) of Remark 3.10.1, (i), in the case where we take "i" to be n. In particular, since the natural action of ${}^\circ J_{n-1}$ on $(({}^\circ J_{n/n-1})^{\{l\}})^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Z}/l^{\mathrm{aut}}\mathbb{Z}$ *factors* through a pro-l quotient of ${}^\circ J_{n-1}$, we may assume without loss of generality — by replacing ${}^\circ J$ by the inverse image in ${}^\circ J$ of a suitable characteristic open subgroup of ${}^\circ J_{n-1}$ — that ${}^\circ J$ satisfies condition (c) of Definition 3.10 in the case where we take "i" to be n. Then one verifies immediately, by applying the *induction hypothesis*, together with a similar argument to the argument applied in the final portion of the proof of assertion (i), that Claim 3.14.A holds. This completes the proof of assertion (ii). Finally, assertion (iii) follows immediately from the various definitions involved. This completes the proof of Lemma 3.14. $\square$

Definition 3.15. In the notation of Definition 3.12, write $(F_i)_{i\in\{1,\cdots,n\}} \in \mathrm{VCN}^{\mathrm{gp}}({}^\square\mathbb{H})$ for the VCN-chain of ${}^\square H$ associated to $\widetilde{y} \in \mathrm{VCN}^{\mathrm{sch}}({}^\square\mathbb{H})$ [cf. Definition 3.12, (iv)]. Now since $({}^\square H)^{\{l\}} \subseteq ({}^\square\Pi_n)^*$ [cf. the notation of condition (b) of Definition 3.10] is *open*, and $({}^\square\Pi_n)^*$ is *topologically finitely generated*, *slim* [cf. Proposition 2.3, (i)] and *almost pro-l*, there exist an *open subgroup* ${}^\square J \subseteq I_{{}^\square K}$ of $I_{{}^\square K}$ and a *homomorphism*

$$^\square\rho\colon {}^\square J \longrightarrow \mathrm{Out}(({}^\square H)^{\{l\}})$$

that

- is *compatible* [in the evident sense] with the homomorphism ${}^{\Box}J \to \mathrm{Out}(({}^{\Box}\Pi_n)^*)$ induced [cf. condition (a) of Definition 3.10] by ${}^{\Box}\rho_n \colon I_{{}^{\Box}K} \to \mathrm{Out}({}^{\Box}\Pi_n)$,
- induces, for each $i \in \{1, \cdots, n\}$, a *homomorphism*

$$ {}^{\Box}J \longrightarrow \mathrm{Out}(({}^{\Box}H_i)^{\{l\}}) $$

— relative to the natural surjection $({}^{\Box}H)^{\{l\}} \twoheadrightarrow ({}^{\Box}H_i)^{\{l\}}$ — and, moreover,
- *factors* through the *maximal pro-l quotient* $({}^{\Box}J)^{\{l\}}$ of ${}^{\Box}J$, which [as is easily verified] is isomorphic to $\mathbb{Z}_l$ as an abstract profinite group.

Write $I_{\widetilde{y}_0} \stackrel{\mathrm{def}}{=} ({}^{\Box}J)^{\{l\}}$. Then, for $i \in \{1, \cdots, n\}$, we define closed subgroups

$$ I_{\widetilde{y}_i} \quad \subseteq \quad {}^{\Box}H_i^{\rho}|_{\widetilde{y}_{i-1}} \quad \subseteq \quad {}^{\Box}H_i^{\rho} \stackrel{\mathrm{def}}{=} ({}^{\Box}H_i)^{\{l\}} \stackrel{\mathrm{out}}{\rtimes} ({}^{\Box}J)^{\{l\}} $$

[cf. the discussion entitled "*Topological groups*" in [CbTpI], §0] as follows [*inductively* on i]:

(i) Set

$$ {}^{\Box}H_1^{\rho}|_{\widetilde{y}_0} \stackrel{\mathrm{def}}{=} {}^{\Box}H_1^{\rho}\ , \quad I_{\widetilde{y}_1} \stackrel{\mathrm{def}}{=} Z_{{}^{\Box}H_1^{\rho}|_{\widetilde{y}_0}}(F_1)\,. $$

(ii) Suppose that $n \geq i \geq 2$. Then, by the *induction hypothesis*, we have already constructed closed subgroups

$$ I_{\widetilde{y}_{i-1}} \quad \subseteq \quad {}^{\Box}H_{i-1}^{\rho}|_{\widetilde{y}_{i-2}} \quad \subseteq \quad {}^{\Box}H_{i-1}^{\rho}\,, $$

hence also a natural outer representation

$$ I_{\widetilde{y}_{i-1}} \hookrightarrow {}^{\Box}H_{i-1}^{\rho} \to \mathrm{Out}(({}^{\Box}H_{i/i-1})^{\{l\}}) $$

— where the second arrow is the natural outer representation arising from the exact sequence of profinite groups

$$ 1 \longrightarrow ({}^{\Box}H_{i/i-1})^{\{l\}} \longrightarrow {}^{\Box}H_i^{\rho} \longrightarrow {}^{\Box}H_{i-1}^{\rho} \longrightarrow 1\,. $$

Then we set

$$ {}^{\Box}H_i^{\rho}|_{\widetilde{y}_{i-1}} \stackrel{\mathrm{def}}{=} ({}^{\Box}H_{i/i-1})^{\{l\}} \stackrel{\mathrm{out}}{\rtimes} I_{\widetilde{y}_{i-1}}\ , \quad I_{\widetilde{y}_i} \stackrel{\mathrm{def}}{=} Z_{{}^{\Box}H_i^{\rho}|_{\widetilde{y}_{i-1}}}(F_i)\,. $$

Remark 3.15.1. In the situation of Definition 3.15, it follows immediately from the definition of $I_{\widetilde{y}_i}$ [cf. also [CmbGC], Remark 1.1.3; [CmbGC], Proposition 1.2, (ii)] that $I_{\widetilde{y}_i}$ is isomorphic to a profinite group of the form $\mathbb{Z}_l^{\oplus j}$, where j is a positive integer $\leq i+1$.

Proposition 3.16 (Graph-admissible isomorphisms). *In the notation of Definition 3.1, let* $\alpha\colon {}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ *be an isomorphism of profinite groups [so* ${}^{\circ}\Sigma = {}^{\bullet}\Sigma$ *— cf., e.g., the proof of* [CbTpI]*, Proposition 1.5, (i)] and* $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ *such that* $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$*. Then the following hold:*

(i) *If* ${}^{\circ}p \notin {}^{\circ}\Sigma$ *and* ${}^{\bullet}p \notin {}^{\bullet}\Sigma$*, then suppose that* α *is* **PC-admissible** *[cf.* [CbTpI]*, Definition 1.4, (ii)]. If* α *is* **SAF-admissible** *[cf. Definition 3.13, (i)] and* **$\{l\}$-I-admissible** *[cf. Definition 3.8, (i)], then* α *is* **$\{l\}$-G-admissible** *[cf. Definition 3.13, (ii)].*

(ii) *Suppose that* α *is* **$\{l\}$-G-admissible**. *Then there exists an* **algorithm**, *which is* **functorial** *with respect to* α*, for constructing an isomorphism of topological groups*

$$\alpha^{\mathrm{tp}}\colon {}^{\circ}\Pi_n^{\mathrm{tp}} \xrightarrow{\sim} {}^{\bullet}\Pi_n^{\mathrm{tp}}$$

such that the isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ *induced by* α^{tp} *[cf. Proposition 3.3, (i)]* **coincides** *with* α*.*

Proof. First, we verify assertion (i). Let ${}^{\circ}J \subseteq {}^{\circ}\Pi_n$ be an open subgroup of ${}^{\circ}\Pi_n$. Then it follows from Lemma 3.14, (i), (ii), (iii), that there exist an open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_n$ of ${}^{\circ}\Pi_n$ of *l-polystable type* [cf. Definition 3.10] and an ${}^{\circ}H$*-l-system* ${}^{\circ}\mathbb{H} = \{{}^{\circ}H_\lambda\}_{\lambda\in\Lambda}$ [cf. Definition 3.11, (ii)] such that ${}^{\circ}H \subseteq {}^{\circ}J$, ${}^{\bullet}H \stackrel{\mathrm{def}}{=} \alpha({}^{\circ}H)$ is of *l-polystable type*, and ${}^{\bullet}\mathbb{H} = \{{}^{\bullet}H_\lambda \stackrel{\mathrm{def}}{=} \alpha({}^{\circ}H_\lambda)\}_{\lambda\in\Lambda}$ is an ${}^{\bullet}H$*-l-system.* Now it follows immediately from the various definitions involved that, to complete the verification of assertion (i), it suffices to verify the following assertion:

> Claim 3.16.A: For each $i \in \{1, \cdots, n\}$, the isomorphism ${}^{\circ}H_i \xrightarrow{\sim} {}^{\bullet}H_i$ [cf. the notation of Definition 3.10] determined by α induces a bijection
>
> $$\mathrm{VCN}^{\mathrm{gp}}({}^{\circ}\mathbb{H}_i) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}({}^{\bullet}\mathbb{H}_i)$$
>
> [cf. Definition 3.12, (iv)].

We verify Claim 3.16.A by *induction* on i. If $i = 1$, then Claim 3.16.A follows immediately from the equivalence (a) $\Leftrightarrow$ ($\mathrm{b}^{\exists}$) of Theorem 3.9, together with Remark 3.13.1, (i). Now suppose that $i \geq 2$, and that the *induction hypothesis* is in force. Then it follows immediately from the *induction hypothesis* that, for each $j \in \{1, \cdots, i-1\}$, the isomorphism ${}^{\circ}H_j \xrightarrow{\sim} {}^{\bullet}H_j$ determined by α induces a bijection

$$\mathrm{VCN}^{\mathrm{gp}}({}^{\circ}\mathbb{H}_j) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}({}^{\bullet}\mathbb{H}_j)\,.$$

Let ${}^{\circ}\widetilde{y}_{i-1} \in \mathrm{VCN}^{\mathrm{sch}}({}^{\circ}\mathbb{H}_{i-1})$, ${}^{\bullet}\widetilde{y}_{i-1} \in \mathrm{VCN}^{\mathrm{sch}}({}^{\bullet}\mathbb{H}_{i-1})$ [cf. Definition 3.11, (iii)] be elements that correspond via the above bijection, relative to the $\circ$-, $\bullet$-versions of the displayed bijection of Definition 3.12, (iv).

Now since α is $\{l\}$-*I-admissible*, for $\square \in \{\circ, \bullet\}$, there exist an open subgroup ${}^{\square}J \subseteq I_{{}^{\square}K}$ of $I_{{}^{\square}K}$ and an outer representation ${}^{\square}\rho \colon {}^{\square}J \to \mathrm{Out}(({}^{\square}H)^{\{l\}})$ as in Definition 3.15 such that ${}^{\circ}\rho$ is *compatible*, relative to α, with ${}^{\bullet}\rho$. Thus, it follows immediately from the various definitions involved that the isomorphism ${}^{\circ}H_i \xrightarrow{\sim} {}^{\bullet}H_i$ determined by α induces an isomorphism of profinite groups

$$ {}^{\circ}H_i^{\rho}|_{{}^{\circ}\widetilde{y}_{i-1}} \xrightarrow{\sim} {}^{\bullet}H_i^{\rho}|_{{}^{\bullet}\widetilde{y}_{i-1}} $$

that lies over an isomorphism $\beta \colon I_{{}^{\circ}\widetilde{y}_{i-1}} \xrightarrow{\sim} I_{{}^{\bullet}\widetilde{y}_{i-1}}$ [cf. Definition 3.15]. In particular, we obtain a commutative diagram of profinite groups

$$ \begin{array}{ccc} I_{{}^{\circ}\widetilde{y}_{i-1}} & \longrightarrow & \mathrm{Out}(({}^{\circ}H_{i/i-1})^{\{l\}}) \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ I_{{}^{\bullet}\widetilde{y}_{i-1}} & \longrightarrow & \mathrm{Out}(({}^{\bullet}H_{i/i-1})^{\{l\}}) \end{array} $$

— where the right-hand vertical arrow is the isomorphism induced by α. Moreover, one verifies immediately from the various definitions involved [cf. also Remark 3.15.1] that, for each $\square \in \{\circ, \bullet\}$, the *positive definite profinite Dehn multi-twists* [cf. [CbTpI], Definition 4.4; [CbTpI], Definition 5.8, (iii)] in the *image* of the composite

$$ I_{{}^{\square}\widetilde{y}_{i-1}} \longrightarrow \mathrm{Out}(({}^{\square}H_{i/i-1})^{\{l\}}) \xleftarrow{\sim} \mathrm{Out}(\Pi_{\mathcal{G}_{i,{}^{\square}\widetilde{y}_{i-1}}}) $$

— where the second arrow is the isomorphism induced by the isomorphism of Definition 3.12, (iii) — form a *dense* subset of this image. In particular, it follows immediately that there exists an element ${}^{\circ}\gamma \in I_{{}^{\circ}\widetilde{y}_{i-1}}$ such that if we write ${}^{\bullet}\gamma \stackrel{\mathrm{def}}{=} \beta({}^{\circ}\gamma) \in I_{{}^{\bullet}\widetilde{y}_{i-1}}$, then, for $\square = \circ$ (respectively, $\square = \bullet$), the image of ${}^{\square}\gamma$ via the composite of the above display is a *positive definite profinite Dehn multi-twist* (respectively, *nondegenerate profinite Dehn multi-twist* [cf. [CbTpI], Definition 4.4; [CbTpI], Definition 5.8, (ii)]). Thus, it follows immediately from [CbTpII], Theorem 1.9, (ii), together with the *equivalences* of [CbTpI], Corollary 5.9, (ii), (iii), that the isomorphism

$$ \alpha_{i/i-1} \colon \Pi_{\mathcal{G}_{i,{}^{\circ}\widetilde{y}_{i-1}}} \xrightarrow{\sim} ({}^{\circ}H_{i/i-1})^{\{l\}} \xrightarrow{\sim} ({}^{\bullet}H_{i/i-1})^{\{l\}} \xleftarrow{\sim} \Pi_{\mathcal{G}_{i,{}^{\bullet}\widetilde{y}_{i-1}}} $$

induced by α is *group-theoretically verticial*, hence also *group-theoretically nodal*.

Next, let us observe that it follows from the fact that $\alpha_{i/i-1}$ is *group-theoretically verticial* [hence also *group-theoretically nodal*], together with our assumption concerning *PC-admissibility*, that if ${}^{\circ}p \notin {}^{\circ}\Sigma$ and ${}^{\bullet}p \notin {}^{\bullet}\Sigma$, then [cf. [CmbGC], Proposition 1.5, (ii)] $\alpha_{i/i-1}$ is *graphic*. On the other hand, if either ${}^{\circ}p \in {}^{\circ}\Sigma$ or ${}^{\bullet}p \in {}^{\bullet}\Sigma$, then it follows from Proposition 3.6, (iii), together with Claim 3.16.A in the case where $i = 1$, that ${}^{\circ}p = {}^{\bullet}p \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$. In particular, if either ${}^{\circ}p \in {}^{\circ}\Sigma$ or ${}^{\bullet}p \in {}^{\bullet}\Sigma$, then, by allowing the open subgroup "${}^{\circ}H$" of ${}^{\circ}\Pi_n$ to *vary* and applying the *group-theoretic nodality* of the resulting isomorphisms

"$\alpha_{i/i-1}$", one concludes from the "existence of irreducible components that *collapse* to arbitrary cusps" [cf. the proof of "assertion (iv)" given in the proof of [SemiAn], Corollary 3.11; [SemiAn], Remark 3.11.1; [AbsTpII], Corollary 2.11; [AbsTpII], Remark 2.11.1, (i)] that $\alpha_{i/i-1}$ is *group-theoretically cuspidal*, hence also [cf. [CmbGC], Proposition 1.5, (ii)] *graphic*. Thus, by allowing ${}^{\circ}\widetilde{y}_{i-1}$, ${}^{\bullet}\widetilde{y}_{i-1}$ to *vary*, we conclude immediately from the various definitions involved that Claim 3.16.A holds. This completes the proof of Claim 3.16.A, hence also of assertion (i).

Next, we verify assertion (ii). The theory of [Brk] yields

- a *functorial homotopy* [indeed, a proper strong deformation retraction!] between the *skeleton* of a *polystable fibration* [cf. [Brk], Definitions 1.2, 1.3] over the ring of integers of a complete nonarchimedean field and the *analytic space* associated to the polystable fibration [cf. [Brk], Theorem 8.1], as well as
- a *functorial homeomorphism* between the *skeleton* of a *polystable fibration* over the ring of integers of a complete nonarchimedean field and the geometric realization of a certain *polysimplicial set* associated to the *special fiber* of the polystable fibration [cf. [Brk], Theorem 8.2].

In particular,

> the theory of [Brk] gives rise to a *functorial homotopy* between the *analytic space* associated to a *polystable fibration* over the ring of integers of a complete nonarchimedean field and the geometric realization of a certain *polysimplicial set* associated to the *special fiber* of the polystable fibration.

Here, we recall further that this polysimplicial set is completely determined by the set of *strata* of the special fiber, together with the *specialization/generization* relations between these strata [cf. the discussion surrounding [Brk], Proposition 2.1, and its proof; [Brk], Lemma 3.13; [Brk], Lemma 6.7].

Next, let us observe that the various *bijections*

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\circ}\mathbb{H}) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}({}^{\circ}\mathbb{H}) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}({}^{\bullet}\mathbb{H}) \xleftarrow{\sim} \mathrm{VCN}^{\mathrm{sch}}({}^{\bullet}\mathbb{H})$$

[cf. Definitions 3.12, (iv); 3.13, (ii)] induced by an $\{l\}$-*G-admissible isomorphism* ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ induce *bijections* between the respective sets of *strata* of the special fibers of ${}^{\circ}\mathcal{Y}_n$, ${}^{\bullet}\mathcal{Y}_n$ [cf. the notation of condition (e) of Remark 3.10.1, (i)], which, in light of the *group-theoretic descriptions* of specialization/generization relations given in [CbTpI], Proposition 2.9, (i) [cf. also [CbTpI], Proposition 5.6, (iii), (iv)], are [easily seen to be] *compatible* with these specialization/generization relations. In particular, since each log scheme ${}^{\square}\mathcal{Y}_n^{\log}$ gives rise to a *polystable fibration* as in the above discussion of [Brk] [cf. condition (e) of Remark 3.10.1, (i)], we thus conclude, in light of the theory of [Brk], from the definition of the *tempered fundamental group* given in [André], §4.2, that any *$\{l\}$-G-admissible isomorphism* ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ determines an isomorphism

$${}^{\circ}\Pi_n^{\mathrm{tp}} \xrightarrow{\sim} {}^{\bullet}\Pi_n^{\mathrm{tp}}$$

between the respective *tempered fundamental groups*, which gives back the original isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ upon passing to the respective ${}^{\circ}\Sigma = {}^{\bullet}\Sigma$-completions [cf. Proposition 3.3, (i)]. This completes the proof of assertion (ii). □

Theorem 3.17 (Metric-, inertia-admissible outomorphisms of fundamental groups). *Let n be a positive integer; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; p a prime number; Σ a nonempty set of prime numbers such that $\Sigma \neq \{p\}$, and, moreover, if $n \geq 2$, then Σ is either equal to the set of all prime numbers or of cardinality one; R a mixed characteristic complete discrete valuation ring of residue characteristic p whose residue field is separably closed; K the field of fractions of R; $\overline{K}$ an algebraic closure of K;*

$$X_K^{\log}$$

a **smooth log curve** *of type (g, r) over K. Write*

$$(X_K)_n^{\log}$$

for the n-th **log configuration space** *[cf. the discussion entitled "Curves" in* [CbTpI], *§0] of $X_K^{\log}$ over K; $(X_{\overline{K}})_n^{\log} \stackrel{\mathrm{def}}{=} (X_K)_n^{\log} \times_K \overline{K}$;*

$$\Pi_n \stackrel{\mathrm{def}}{=} \pi_1((X_{\overline{K}})_n^{\log})^{\Sigma}$$

for the maximal pro-Σ quotient of the **log fundamental group** *of $(X_{\overline{K}})_n^{\log}$;*

$$\rho_n \colon I_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Out}(\Pi_n)$$

for the natural outer pro-Σ Galois action associated to $(X_K)_n^{\log}$; $(\mathrm{Spec}\, R)^{\log}$ *for the log scheme obtained by equipping* $\mathrm{Spec}\, R$ *with the log structure determined by the closed point of* $\mathrm{Spec}\, R$*. Then the following hold:*

(i) *Let $l \in \Sigma$ be such that $l \neq p$. Then we have equalities and an inclusion*

$$\begin{aligned} \mathrm{Out}(\Pi_1)^{\mathrm{M}} &= \mathrm{Out}^{\mathrm{I}}(\Pi_1) \cap \mathrm{Out}^{\mathrm{C}}(\Pi_1) \\ &= \mathrm{Out}^{\{l\}\text{-}\mathrm{I}}(\Pi_1) \cap \mathrm{Out}^{\mathrm{C}}(\Pi_1) \subseteq \mathrm{Out}(\Pi_1)^{\mathrm{G}} \end{aligned}$$

[cf. Definitions 3.7, (i), (ii); 3.8, (iii)]. If, moreover, $p \in \Sigma$, then we have equalities and inclusions

$$\mathrm{Out}(\Pi_1)^{\mathrm{M}} = \mathrm{Out}^{\mathrm{I}}(\Pi_1) = \mathrm{Out}^{\{l\}\text{-}\mathrm{I}}(\Pi_1) \subseteq \mathrm{Out}(\Pi_1)^{\mathrm{G}} \subseteq \mathrm{Out}(\Pi_1)\,.$$

(ii) *Let $l \in \Sigma$ be such that $l \neq p$. Then we have equalities and inclusions*

$$\begin{array}{rcccl} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} & = & \mathrm{Out}^{\mathrm{FCI}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}} \\ & = & \mathrm{Out}^{\mathrm{FC}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}} \\ & \subseteq & \mathrm{Out}^{\mathrm{G}}(\Pi_n) & \subseteq & \mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n)\,, \end{array}$$

$$\begin{array}{ccccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} & \subseteq & \mathrm{Out}^{\mathrm{FI}}(\Pi_n) & \subseteq & \mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) \\ \cap & & \cap & & \cap \\ \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} & \subseteq & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} & \subseteq & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}} \end{array}$$

[cf. Definitions 3.7, (iii); 3.8, (iii), (iv); 3.13, (iv)]. Moreover, the following hold:

(ii-a) *If $p \in \Sigma$, then we have:*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} = \mathrm{Out}^{\mathrm{FI}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n)\,,$$

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} = \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} = \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}}\,.$$

(ii-b) *If $n \neq 1$, then we have:*

$$\mathrm{Out}^{\mathrm{FI}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n)\,,$$

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} = \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} = \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}}\,.$$

(ii-c) *If $n \neq 2$, $(r, n) \neq (0, 3)$, and either $p \in \Sigma$ or $n \neq 1$, then we have:*

$$\begin{array}{rcccl} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} & = & \mathrm{Out}^{\mathrm{FI}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} \\ & = & \mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}} \\ = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} & = & \mathrm{Out}^{\mathrm{FCI}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}} \\ & = & \mathrm{Out}^{\mathrm{FC}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}}\,. \end{array}$$

(iii) *Suppose that $p \notin \Sigma$, and that $X_K^{\log}$ extends to a* **stable log curve** *over* $(\mathrm{Spec}\, R)^{\log}$. *Let $l \in \Sigma$. Write $\rho_n(I_K)[l] \subseteq \rho_n(I_K)$ for the maximal pro-l subgroup of the image $\rho_n(I_K)$. Then the* **normalizers** *of $\rho_n(I_K)$, $\rho_n(I_K)[l]$ in $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$ satisfy the following equalities:*

(iii-a) *If $(r, n) \neq (0, 2)$, then*

$$\mathrm{Out}^{\mathrm{FI}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} = N_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K))\,,$$

$$\mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}} = N_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)[l])\,.$$

(iii-b) *For arbitrary $r \geq 0$, $n \geq 1$,*

$$\mathrm{Out}^{\mathrm{FI}}(\Pi_n) = N_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K))\,,$$

$$\mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) = N_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)[l])\,.$$

(iv) *Let* $l \in \Sigma$ *be such that* $l \neq p$. *Then the subgroups*

$$\mathrm{Out}(\Pi_1)^{\mathrm{M}}\ ,\ \ \mathrm{Out}^{\mathrm{I}}(\Pi_1)\ ,\ \ \mathrm{Out}^{\{l\}\text{-}\mathrm{I}}(\Pi_1)\ ,\ \ \mathrm{Out}(\Pi_1)^{\mathrm{G}}$$

of $\mathrm{Out}(\Pi_1)$ *are* **closed** *in* $\mathrm{Out}(\Pi_1)$. *Moreover, the subgroups*

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}}, & \mathrm{Out}^{\mathrm{FI}}(\Pi_n), & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}}, \\ & \mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n), & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}}, \\ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}, & \mathrm{Out}^{\mathrm{FCI}}(\Pi_n), & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}}, \\ & \mathrm{Out}^{\mathrm{FC}\,\{l\}\text{-}\mathrm{I}}(\Pi_n), & \mathrm{Out}(\Pi_n)^{\mathrm{FC}\,\{l\}\text{-}\mathrm{I}}, \\ & \mathrm{Out}^{\mathrm{G}}(\Pi_n), & \mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n) \end{array}$$

of $\mathrm{Out}(\Pi_n)$ *are* **closed** *in* $\mathrm{Out}(\Pi_n)$. *In particular, these subgroups are* **compact**.

(v) *Let* $l \in \Sigma$ *be such that* $l \neq p$. *Then the closed subgroups* $\mathrm{Out}^{\mathrm{G}}(\Pi_n)$, $\mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ *[cf. (iv); Remark 3.13.1, (ii)] are* **commensurably terminal** *in* $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$. *Moreover, we have an inclusion*

$$C_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}) \subseteq \mathrm{Out}^{\mathrm{G}}(\Pi_n)\,.$$

(vi) *The natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$$

$$(\textit{respectively},\ \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1})^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}})$$

induced by the projection $(X_K)^{\log}_{n+1} \to (X_K)^{\log}_n$ *obtained by forgetting any one of the* $n+1$ *factors is* **injective** *(respectively,* **injective** *if* $(r,n) \neq (0,1)$*). If, moreover, either*

$$n \geq 4$$

or

$$n \geq 3 \text{ and } r \neq 0,$$

then this homomorphism is **bijective** *(respectively,* **bijective***).*

Proof. Assertion (i) follows immediately from Theorem 3.9. Next, we verify assertion (ii). First, we claim that the following assertion holds:

Claim 3.17.A: We have equalities

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}} = \mathrm{Out}^{\mathrm{FCI}}(\Pi_n)\ ;\ \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}} = \mathrm{Out}^{\mathrm{FC}\,\{l\}\text{-}\mathrm{I}}(\Pi_n)\,.$$

Indeed, this follows immediately from Corollary 2.10 [when $\Sigma = \mathfrak{Primes}$]; the *injectivity portion* of [NodNon], Theorem B [when $\Sigma = \{l\}$], together with the definition of *I-admissibility*, $\{l\}$*-I-admissibility* [cf. Definition 3.8]. This completes the proof of Claim 3.17.A.

Next, we claim that the following assertion holds:

Claim 3.17.B: We have equalities

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \ = \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}} \ = \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}}\,.$$

Indeed, this follows immediately from assertion (i), together with the various definitions involved. This completes the proof of Claim 3.17.B.

Next, we claim that the following assertion holds:

Claim 3.17.C: We have equalities and an inclusion

$$\begin{aligned}\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} &= \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}} \\ &= \mathrm{Out}^{\mathrm{FCI}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{G}}(\Pi_n)\,.\end{aligned}$$

Indeed, the first four equalities follow from Claims 3.17.A, 3.17.B. On the other hand, the final inclusion follows immediately from Proposition 3.16, (i) [cf. also the final portion of Definition 3.13, (i)]. This completes the proof of Claim 3.17.C.

Next, we claim that the following assertion holds:

Claim 3.17.D: We have inclusions

$$\begin{array}{ccccc}\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} & \subseteq & \mathrm{Out}^{\mathrm{FI}}(\Pi_n) & \subseteq & \mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) \\ \cap & & \cap & & \cap \\ \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} & \subseteq & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} & \subseteq & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}}\,.\end{array}$$

Indeed, let us observe that the left-hand upper inclusion follows immediately from Claim 3.17.C. Next, let us observe that the left-hand lower inclusion follows immediately from assertion (i). On the other hand, the remaining inclusions follow immediately from the various definitions involved. This completes the proof of Claim 3.17.D. The various equalities and inclusions of assertion (ii) that precede assertion (ii-a) all follow from Claims 3.17.C, 3.17.D.

Next, we consider assertion (ii-a). It follows immediately from Proposition 3.16, (i), that the inclusion $\mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) \subseteq \mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n)$ holds. In particular, it follows from Remark 3.13.1, (ii), that the inclusion $\mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$, hence also the equality $\mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}\,\{l\}\text{-}\mathrm{I}}(\Pi_n)$, holds. Thus, the first two equalities of assertion (ii-a) follow immediately from Claims 3.17.C, 3.17.D. On the other hand, the final two equalities of assertion (ii-a) follow immediately from the final portion of assertion (i). This completes the proof of assertion (ii-a).

Next, we consider assertion (ii-b). If $p \in \Sigma$, then assertion (ii-b) follows from assertion (ii-a). Thus, we may assume without loss of generality that $p \notin \Sigma$. Then since [by assumption!] $\Sigma = \{l\}$, the first equality of assertion (ii-b) follows immediately from the various definitions involved. On the other hand, the final two equalities follow immediately from assertion (i), together with [CbTpI], Theorem A, (ii). This completes the proof of assertion (ii-b).

Next, we consider assertion (ii-c). If $n \geq 3$ and $(r, n) \neq (0, 3)$, then assertion (ii-c) follows immediately from [CbTpII], Theorem A, (ii), together with Claims 3.17.C, 3.17.D. On the other hand, if $p \in \Sigma$ and $n = 1$, then assertion (ii-c) follows immediately from the final portion of assertion (i), together with Claims 3.17.C, 3.17.D. This completes the proof of assertion (ii-c), hence also of assertion (ii).

Next, we verify assertion (iii). First, we claim that the following assertion holds:

Claim 3.17.E: We have an equality

$$\mathrm{Out}^{\{l\}\text{-I}}(\Pi_1) = N_{\mathrm{Out}(\Pi_1)}(\rho_1(I_K)[l])\,.$$

Indeed, let us first observe that since $p \notin \Sigma$, we have a natural outer isomorphism $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}$ [cf. Remark 3.5.1]. Next, let us observe that, in light of our assumption that $X_K^{\log}$ extends to a *stable log curve* over $(\mathrm{Spec}\, R)^{\log}$, it follows from [CbTpI], Corollary 5.9, (iii), that the image of ρ_1 is *contained* in

$$\mathrm{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma]) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}) \xleftarrow{\sim} \mathrm{Out}(\Pi_1)$$

[cf. [CbTpI], Definition 4.4] and, moreover, is *isomorphic* to $\widehat{\mathbb{Z}}^{\Sigma}$ as an abstract profinite group. Next, let us observe that it follows immediately from the definition of $\{l\}$*-I-admissibility* that $N_{\mathrm{Out}(\Pi_1)}(\rho_1(I_K)[l]) \subseteq \mathrm{Out}^{\{l\}\text{-I}}(\Pi_1)$. Thus, to complete the verification of Claim 3.17.E, it suffices to verify that $\mathrm{Out}^{\{l\}\text{-I}}(\Pi_1) \subseteq N_{\mathrm{Out}(\Pi_1)}(\rho_1(I_K)[l])$.

Let $\alpha \in \mathrm{Out}^{\{l\}\text{-I}}(\Pi_1)$ and $H \subseteq \Pi_1$ a characteristic open subgroup of Π_1. Write $\Pi_1 \twoheadrightarrow \Pi_1^*$ for the maximal almost pro-l quotient of Π_1 with respect to H [cf. Definition 1.1]; $\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]} \twoheadrightarrow \Pi^*_{\mathcal{G}_{\Pi_1}[\Sigma]}$ for the [necessarily maximal almost pro-l] quotient of $\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}$ corresponding to $\Pi_1 \twoheadrightarrow \Pi_1^*$ [relative to the above natural outer isomorphism $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}$]. Then it follows immediately from the definition of $\{l\}$*-I-admissibility* that there exists an open subgroup $J \subseteq I_K$ such that the image of $\rho_1(J)$ in $\mathrm{Out}(\Pi_1^*)$ is *normalized* by the outomorphism $\alpha^* \in \mathrm{Out}(\Pi_1^*)$ determined by $\alpha \in \mathrm{Out}(\Pi_1)$. On the other hand, it follows immediately from the above discussion that the outer representation

$$\rho_1(J) \longrightarrow \mathrm{Out}(\Pi_1^*) \xrightarrow{\sim} \mathrm{Out}(\Pi^*_{\mathcal{G}_{\Pi_1}[\Sigma]})$$

is of *PIPSC-type* [cf. Definition 1.6, (iv)]. Thus, it follows from Theorem 1.11, (ii), that $\alpha^* \in \mathrm{Out}(\Pi_1^*)$ is *group-theoretical verticial* [cf. Definition 1.6, (ii)]. In particular, by allowing H to *vary*, we conclude that $\alpha \in \mathrm{Out}(\Pi_1)$ is *group-theoretical verticial*. Thus, it follows immediately from the definition of a *profinite Dehn multi-twist* that $\alpha \in \mathrm{Out}(\Pi_1)$ *normalizes* $\mathrm{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma]) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}) \xleftarrow{\sim} \mathrm{Out}(\Pi_1)$, hence also [cf. [CbTpI], Theorem 4.8, (iv)] the maximal pro-l subgroup $\mathrm{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma])[l]$ of $\mathrm{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma])$. On the other hand, one verifies immediately again from [CbTpI], Theorem 4.8, (iv), that $\mathrm{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma])[l]$ is a *free* $\mathbb{Z}_l$*-module of finite rank*, and that the composite

$$\mathrm{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma])[l] \hookrightarrow \mathrm{Out}(\Pi_1) \to \mathrm{Out}(\Pi_1^*)$$

is *injective*. Thus, since *some* open subgroup of the maximal pro-l subgroup of the image of I_K in $\mathrm{Out}(\Pi_1^*)$ is *normalized* by $\alpha^* \in \mathrm{Out}(\Pi_1^*)$

[cf. the above discussion concerning "J"!], one verifies immediately that $\alpha \in N_{\mathrm{Out}(\Pi_1)}(\rho_1(I_K)[l])$. This completes the proof of Claim 3.17.E.

Now let us observe that one verifies easily [cf. also the discussion of the inclusion "$N_{\mathrm{Out}(\Pi_1)}(\rho_1(I_K)[l]) \subseteq \mathrm{Out}^{\{l\}\text{-I}}(\Pi_1)$" in the proof of Claim 3.17.E] that the inclusions

$$N_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)) \subseteq \mathrm{Out}^{\mathrm{FI}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}},$$

$$N_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)[l]) \subseteq \mathrm{Out}^{\mathrm{F}\{l\}\text{-I}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-I}}$$

hold. In particular, assertion (iii-a) follows immediately from Claim 3.17.E, together with the *injectivity portion* of [CbTpII], Theorem A, (i) [cf. also [CbTpI], Theorem A, (ii); [NodNon], Theorem B, in the case where $r = 0$]. Thus, to complete the proof of assertion (iii), it suffices to verify the two equalities of assertion (iii-b) in the case where $(r, n) = (0, 2)$. Suppose that $(r, n) = (0, 2)$, hence that $\Sigma = \{l\}$. Then one verifies easily that, to complete the proof of assertion (iii), it suffices to verify that $\mathrm{Out}^{\mathrm{F}\{l\}\text{-I}}(\Pi_n) \subseteq N_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)[l])$.

Thus, let $\widetilde{\alpha} \in \mathrm{Aut}(\Pi_n)$ be a lifting of an element $\alpha \in \mathrm{Out}^{\mathrm{F}\{l\}\text{-I}}(\Pi_n)$. Write $\alpha_1 \in \mathrm{Out}^{\mathrm{FC}\{l\}\text{-I}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{G}}(\Pi_1)$ [cf. assertion (i); [CbTpI], Theorem A, (i), (ii)] for the image of α in $\mathrm{Out}(\Pi_1)$. Then let us observe that it follows immediately from Claim 3.17.E that $\widetilde{\alpha}$ induces an automorphism $\widetilde{\beta}$ of the extension group $\Pi_1 \overset{\mathrm{out}}{\rtimes} \rho_1(I_K)$ [i.e., arising from the *outer representation of IPSC-type* $\rho_1(I_K) \to \mathrm{Out}(\Pi_1)$ *implicit* in the discussion surrounding Claim 3.17.E above], whose restriction to Π_1 is *G-admissible*. In particular, it follows that $\widetilde{\beta}$ maps *verticial inertia groups* [cf. [NodNon], Lemma 2.5, (i)] of $\Pi_1 \overset{\mathrm{out}}{\rtimes} \rho_1(I_K)$ to verticial inertia groups of $\Pi_1 \overset{\mathrm{out}}{\rtimes} \rho_1(I_K)$. Moreover, let us observe that it follows immediately from the fact that $\alpha \in \mathrm{Out}^{\mathrm{F}\{l\}\text{-I}}(\Pi_n)$ that $\widetilde{\beta}$ is *compatible* with the natural outer representations of *suitable open subgroups* of such verticial inertia groups of $\Pi_1 \overset{\mathrm{out}}{\rtimes} \rho_1(I_K)$ on $\Pi_{2/1}$. Thus, since the natural outer representation of such a verticial inertia group of $\Pi_1 \overset{\mathrm{out}}{\rtimes} \rho_1(I_K)$ on $\Pi_{2/1}$ is [easily verified to be] an *outer representation of IPSC-type*, one concludes from a similar argument to the argument applied above to verify Claim 3.17.E that $\widetilde{\beta}$ is *compatible* with these natural outer representations of verticial inertia groups of $\Pi_1 \overset{\mathrm{out}}{\rtimes} \rho_1(I_K)$ on $\Pi_{2/1}$. Now it follows *formally* that $\alpha \in N_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\rho_1(I_K)[l])$, as desired. This completes the proof of assertion of (iii).

Next, we verify assertion (iv). The *closedness* of $\mathrm{Out}(\Pi_1)^{\mathrm{G}}$ in $\mathrm{Out}(\Pi_1)$ follows immediately from condition ($\mathrm{c}^{\forall}$) of Proposition 3.6 [cf. Proposition 3.6, (ii)]. Thus, the *closedness* of $\mathrm{Out}(\Pi_1)^{\mathrm{M}}$ in $\mathrm{Out}(\Pi_1)$ follows from the easily verified fact that $\mathrm{Out}^{\mathrm{M}}(\Pi_1)$ is *closed* in $\mathrm{Out}(\Pi_1)^{\mathrm{G}}$. The fact that the subgroup $\mathrm{Out}^{\{l\}\text{-I}}(\Pi_1)$, hence also $\mathrm{Out}^{\mathrm{I}}(\Pi_1)$, is *closed* in $\mathrm{Out}(\Pi_1)$ may be verified as follows: If $p \in \Sigma$, then the *closedness* in question follows from the *closedness* of $\mathrm{Out}(\Pi_1)^{\mathrm{M}}$ [verified above], together with the final portion of assertion (i). On the other hand, if $p \notin \Sigma$, then the *closedness* in question follows immediately from assertion (iii). This completes the proof of the *closedness* of $\mathrm{Out}^{\{l\}\text{-I}}(\Pi_1)$ and $\mathrm{Out}^{\mathrm{I}}(\Pi_1)$ in $\mathrm{Out}(\Pi_1)$.

The *closedness* of

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}\ ,\ \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}}\ ,\ \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}\text{-I}}\ ,$$

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}}\ ,\ \ \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}}\ ,\ \ \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-I}}$$

in $\mathrm{Out}(\Pi_n)$ follows immediately from the various definitions involved, together with the *closedness* of $\mathrm{Out}(\Pi_1)^{\mathrm{M}}$, $\mathrm{Out}^{\mathrm{I}}(\Pi_1)$, and $\mathrm{Out}^{\{l\}\text{-I}}(\Pi_1)$ in $\mathrm{Out}(\Pi_1)$ [verified above]. The *closedness* of

$$\mathrm{Out}^{\mathrm{FCI}}(\Pi_n)\ ,\ \ \mathrm{Out}^{\mathrm{FC}\,\{l\}\text{-I}}(\Pi_n)$$

in $\mathrm{Out}(\Pi_n)$ follow from the *closedness* of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$ in $\mathrm{Out}(\Pi_n)$ [verified above], together with the equalities at the beginning of assertion (ii). The *closedness* of

$$\mathrm{Out}^{\mathrm{G}}(\Pi_n)\ ,\ \ \mathrm{Out}^{\{l\}\text{-G}}(\Pi_n)$$

in $\mathrm{Out}(\Pi_n)$ follow immediately from the definition of *G-admissibility*, *$\{l\}$-G-admissibility* [cf. Definition 3.13, (ii), (iii); Lemma 3.14, (iii)]. The fact that the subgroup $\mathrm{Out}^{\mathrm{F}\,\{l\}\text{-I}}(\Pi_n)$, hence also $\mathrm{Out}^{\mathrm{FI}}(\Pi_n)$, is *closed* in $\mathrm{Out}(\Pi_n)$ may be verified as follows: If $n = 1$, then the *closedness* in question has already been verified. If $p \in \Sigma$, then the *closedness* in question follows from the *closedness* of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$ [verified above], together with assertion (ii-a). On the other hand, if $p \notin \Sigma$, then the *closedness* in question follows from assertion (iii-b). This completes the proof of the *closedness* of $\mathrm{Out}^{\mathrm{F}\,\{l\}\text{-I}}(\Pi_n)$, $\mathrm{Out}^{\mathrm{FI}}(\Pi_n)$ in $\mathrm{Out}(\Pi_n)$, hence also of assertion (iv).

Next, we verify assertion (v). Let $\alpha \in C_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\mathrm{Out}^{\mathrm{G}}(\Pi_n))$ (respectively, $C_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\mathrm{Out}^{\{l\}\text{-G}}(\Pi_n))$; $C_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}})$) and $\widetilde{\alpha} \in \mathrm{Aut}^{\mathrm{F}}(\Pi_n)$ a lifting of α. Now observe that to complete the verification of assertion (v), it suffices to verify that $\alpha \in \mathrm{Out}^{\{l\}\text{-G}}(\Pi_n)$. To this end, let $J \subseteq \Pi_n$ be an open subgroup of Π_n. Then it follows from Lemma 3.14, (i), (ii), that there exist an open subgroup $H \subseteq J \subseteq \Pi_n$ of Π_n of *l-polystable type* [cf. Definition 3.10] and an *H-l-system* $\mathbb{H} = \{H_\lambda\}_{\lambda\in\Lambda}$ [cf. Definition 3.11, (ii)]. Note that it follows from condition (a) of Definition 3.10 that the subgroups H, H_λ of Π_n are *stabilized* by $\widetilde{\alpha}$. Then it follows immediately from the various definitions involved that, to complete the verification of the fact that $\alpha \in \mathrm{Out}^{\{l\}\text{-G}}(\Pi_n)$, it suffices to verify the following assertion:

> Claim 3.17.F: For each $i \in \{0, \cdots, n\}$, the outomorphism of the image H_i of H in Π_i determined by α

induces a bijection

$$\mathrm{VCN}^{\mathrm{gp}}(\mathbb{H}_i) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}(\mathbb{H}_i)$$

[cf. Definition 3.12, (iv)] — where, for convenience, we set $\Pi_0 \stackrel{\mathrm{def}}{=} \{1\}$, $\mathrm{VCN}^{\mathrm{gp}}(\mathbb{H}_0) \stackrel{\mathrm{def}}{=} \{\Pi_0\}$, and we write $(H_\lambda)_i$ for the image of H_λ in Π_i and $\mathbb{H}_i \stackrel{\mathrm{def}}{=} \{(H_\lambda)_i\}_{\lambda\in\Lambda}$.

We verify Claim 3.17.F by *induction* on i. If $i = 0$, then Claim 3.17.F is immediate. Now suppose that $i \geq 1$, and that the *induction hypothesis* is in force. Then it follows immediately from the *induction hypothesis* that, for each $j \in \{0, \cdots, i-1\}$, the outomorphism of H_j determined by α induces a bijection

$$\mathrm{VCN}^{\mathrm{gp}}(\mathbb{H}_j) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}(\mathbb{H}_j)\,.$$

Let $\widetilde{y}$, $\widetilde{y}' \in \mathrm{VCN}^{\mathrm{sch}}(\mathbb{H}_{i-1})$ [cf. Definition 3.11, (iii)] be elements that correspond via the bijection obtained by conjugating the above bijection by the displayed bijection of Definition 3.12, (iv). Here, for convenience, we set $\mathrm{VCN}^{\mathrm{sch}}(\mathbb{H}_0) \stackrel{\mathrm{def}}{=} \{\mathcal{Y}_0\}$.

Next, let us observe that since $\alpha \in C_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\mathrm{Out}^{\mathrm{G}}(\Pi_n))$ (respectively, $C_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n))$; $C_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}})$), there exist open subgroups N_1 and N_2 of $\mathrm{Out}^{\mathrm{G}}(\Pi_n)$ (respectively, $\mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n)$; $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$) such that the automorphism of H_i induced by $\widetilde{\alpha}$ extends to an *isomorphism of profinite groups* [cf. assertion (iv)]

$$H_i \overset{\mathrm{out}}{\rtimes} N_1 \xrightarrow{\sim} H_i \overset{\mathrm{out}}{\rtimes} N_2$$

[cf. the discussion entitled "*Topological groups*" in [CbTpI], §0] that lies over an isomorphism of profinite groups $N_1 \xrightarrow{\sim} N_2$. In particular, by considering the respective outer actions [by conjugation] of $H_{i-1} \overset{\mathrm{out}}{\rtimes} N_1$, $H_{i-1} \overset{\mathrm{out}}{\rtimes} N_2$ on the maximal pro-l quotient $(H_{i/i-1})^{\{l\}}$ of the kernel $H_{i/i-1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(H_i \twoheadrightarrow H_{i-1})$ [cf. the notation of Remark 3.10.1, (i)], we obtain a *commutative diagram of profinite groups*

$$\begin{array}{ccccc}
H_{i-1} \overset{\mathrm{out}}{\rtimes} N_1 & \longrightarrow & \mathrm{Out}((H_{i/i-1})^{\{l\}}) & \xleftarrow{\sim} & \mathrm{Out}(\Pi_{\mathcal{G}_{i,\widetilde{y}}}) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
H_{i-1} \overset{\mathrm{out}}{\rtimes} N_2 & \longrightarrow & \mathrm{Out}((H_{i/i-1})^{\{l\}}) & \xleftarrow{\sim} & \mathrm{Out}(\Pi_{\mathcal{G}_{i,\widetilde{y}'}})
\end{array}$$

— where the left-hand vertical arrow is the isomorphism induced by the *isomorphism of profinite groups* discussed above; the central vertical arrow is the isomorphism induced by $\widetilde{\alpha}$; the right-hand horizontal arrows are the isomorphisms induced by the $\widetilde{y}$-, $\widetilde{y}'$-versions of the isomorphism of Definition 3.12, (iii); the right-hand vertical arrow is the isomorphism induced by the composite

$$\widetilde{\alpha}_{\widetilde{y},\widetilde{y}'} \colon \Pi_{\mathcal{G}_{i,\widetilde{y}}} \xrightarrow{\sim} (H_{i/i-1})^{\{l\}} \xrightarrow{\sim} (H_{i/i-1})^{\{l\}} \xleftarrow{\sim} \Pi_{\mathcal{G}_{i,\widetilde{y}'}}$$

of the isomorphism $\Pi_{\mathcal{G}_{i,\widetilde{y}}} \xrightarrow{\sim} (H_{i/i-1})^{\{l\}}$ of Definition 3.12, (iii), the automorphism of $(H_{i/i-1})^{\{l\}}$ determined by $\widetilde{\alpha}$, and the isomorphism $(H_{i/i-1})^{\{l\}} \xleftarrow{\sim} \Pi_{\mathcal{G}_{i,\widetilde{y}'}}$ of Definition 3.12, (iii).

Now *suppose* that the smooth log curve $X_K^{\log}$ in fact arises, via base-change, from a smooth log curve over a complete discrete valuation field whose residue field is *finitely generated over a finite field*. Then one verifies immediately from the *openness* of N_1, N_2 in $\mathrm{Out}^{\mathrm{G}}(\Pi_n)$ (respectively, $\mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n)$; $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}} \subseteq \mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n)$ [cf. assertion (ii)]) that the composite horizontal arrows of the above commutative diagram *factor* through $\mathrm{Aut}(\mathcal{G}_{i,\widetilde{y}})$, $\mathrm{Aut}(\mathcal{G}_{i,\widetilde{y}'})$, respectively, and, moreover, are *l-graphically full* [i.e., in the sense of [CmbGC], Definition 2.3, (iii)] — cf. the argument applied in the proof of [CmbGC], Proposition 2.4, (v). Thus, it follows from Corollary 2.7, (ii), that the isomorphism $\widetilde{\alpha}_{\widetilde{y},\widetilde{y}'} \colon \Pi_{\mathcal{G}_{i,\widetilde{y}}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{i,\widetilde{y}'}}$ is *graphic*. In particular, by allowing $\widetilde{y}$, $\widetilde{y}'$ to *vary*, it follows immediately from the various definitions involved that Claim 3.17.F holds.

In fact, it will *not*, in general, be the case that the smooth log curve $X_K^{\log}$ in fact arises from a smooth log curve over a complete discrete valuation field whose residue field is finitely generated over a finite field. On the other hand, one verifies immediately that one may always *p-adically approximate* an arbitrary given smooth log curve $X_K^{\log}$ by a smooth log curve which is

- defined over a complete discrete valuation field whose residue field is *finitely generated over a finite field*, and, moreover,
- for a given *fixed choice* of H, $\mathbb{H}$, gives rise to a *commutative diagram of profinite groups* as discussed above that is *isomorphic* [in the evident sense] to the commutative diagram of profinite groups associated to the original given data.

In particular, there is, in fact, no loss of generality in assuming that the smooth log curve $X_K^{\log}$ arises from a smooth log curve over a complete discrete valuation field whose residue field is *finitely generated over a finite field*. This completes the proof of Claim 3.17.F, hence also of assertion (v). Assertion (vi) follows from [NodNon], Theorem B; [CbTpII], Theorem A, (i). This completes the proof of Theorem 3.17. □

Remark 3.17.1. In the notation of Theorem 3.17, it follows from Theorem 3.17, (v), that

$$C_{\mathrm{Out}^{\mathrm{F}}(\Pi_n)}(\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}) \subseteq \mathrm{Out}^{\mathrm{G}}(\Pi_n)\,.$$

On the other hand, $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$ is *not*, in general, *commensurably terminal* in $\mathrm{Out}^{\mathrm{G}}(\Pi_n)$ [or indeed in $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$ or $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$!]. Indeed, suppose that we are in the situation of Theorem 3.17, (iii) [so $p \notin \Sigma$], and that the semi-graph of anabelioids $\mathcal{G}$ of pro-Σ PSC type determined by the geometric special fiber of the stable model of $X_K^{\log}$ satisfies the following conditions:

- $\mathrm{Vert}(\mathcal{G})^\sharp = \mathrm{Node}(\mathcal{G})^\sharp = 2$. Write $\mathrm{Vert}(\mathcal{G}) = \{v_1, v_2\}$, $\mathrm{Node}(\mathcal{G}) = \{e_1, e_2\}$.
- For each $i \in \{1, 2\}$, $\mathcal{V}(e_i) = \mathrm{Vert}(\mathcal{G}) = \{v_1, v_2\}$.
- There exists an automorphism of $\mathcal{G}$ that induces a *nontrivial* automorphism of $\mathrm{Node}(\mathcal{G})$.

Finally, suppose that if we write $\mu_{X_K^{\log}}$ for the metric structure on the underlying semi-graph of $\mathcal{G}$ associated to the stable model of $X_K^{\log}$ [cf. Definition 3.5, (iii)], then $\mu_{X_K^{\log}}(e_1) \neq \mu_{X_K^{\log}}(e_2)$. [Here, we note that one verifies easily that such a smooth log curve $X_K^{\log}$ exists.] Then it follows immediately from the various assumptions imposed on the objects under consideration that $\mathrm{Out}^{\mathrm{FC}}(\Pi_1)^{\mathrm{M}}$ is of *index* 2, hence also *normal*, in $\mathrm{Out}^{\mathrm{G}}(\Pi_1)$. In particular, $\mathrm{Out}^{\mathrm{FC}}(\Pi_1)^{\mathrm{M}}$ is *not normally terminal*, hence, *a fortiori*, *not commensurably terminal*, in $\mathrm{Out}^{\mathrm{G}}(\Pi_1)$.

Remark 3.17.2. In the notation of Theorem 3.17, suppose that $p \in \Sigma$.

(i) It follows from Theorem 3.17, (ii-c), that if either

$$(\dagger_1): \quad n \geq 4 \quad \text{or} \quad n \geq 3 \text{ and } r \neq 0,$$

then we have equalities

$$\begin{array}{rcccc} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} & = & \mathrm{Out}^{\mathrm{FI}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} \\ & = & \mathrm{Out}^{\mathrm{F}\{l\}\text{-I}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-I}} \\ = \ \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} & = & \mathrm{Out}^{\mathrm{FCI}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}} \\ & = & \mathrm{Out}^{\mathrm{FC}\{l\}\text{-I}}(\Pi_n) & = & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}\text{-I}}. \end{array}$$

(ii) In Corollary 2.10, the authors gave what may be regarded as an *almost pro-l version* of the *injectivity portion* of [NodNon], Theorem B [i.e., the *injectivity* of the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$]. In fact, however, although a detailed exposition lies beyond the scope of the present paper [cf. the discussion of (iii) below], it seems quite likely that it should be possible to verify an *almost pro-l version* of the *injectivity portion* of [CbTpII], Theorem A, (i) [i.e., the *injectivity* of the natural homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ for $(r, n) \neq (0, 1)$]. Such an almost pro-l version would then *imply*, via a similar argument to the argument applied in the proof of the equalities

$$\mathrm{Out}^{\mathrm{FCI}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}}, \quad \mathrm{Out}^{\mathrm{FC}\{l\}\text{-I}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}\text{-I}}$$

[cf. Claim 3.17.A in the proof of Theorem 3.17, (ii)], that if either

$$(\dagger_2): \quad n \geq 3 \qquad \text{or} \qquad n \geq 2 \text{ and } r \neq 0,$$

then the equalities

$$\mathrm{Out}^{\mathrm{FI}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}}\ , \ \ \mathrm{Out}^{\mathrm{F}\,\{l\}\text{-}\mathrm{I}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}\text{-}\mathrm{I}}\,,$$

hence also [cf. Theorem 3.17, (ii); Theorem 3.17, (ii-a)] the *nine equalities* of the display of (i), hold.

(iii) The main reason that the authors did not go to the trouble to verify the *nine equalities* of the display of (i) under the *more general hypotheses* [i.e., $(\dagger_2)$] discussed in (ii) is the following. The main applications of the theory developed in the present paper are the following:

(1) the *generalization*, given in Corollary 3.20 below [cf. also Remark 3.20.1 below], of a result due to Andre [cf. [André], Theorems 7.2.1, 7.2.3] concerning the *characterization of local Galois groups* in the *global Galois image* associated to a hyperbolic curve over a number field and

(2) the establishment of an appropriate *local analogue*, satisfying various expected properties, of the *Grothendieck-Teichmüller group* [cf. Remark 3.19.2 below].

The theory surrounding these applications [cf. Theorem 3.18 below] revolves around the theory of the *tripod homomorphism* developed in [CbTpII], §3. On the other hand, this theory of the tripod homomorphism is only *well-behaved* [cf. [CbTpII], Definition 3.19] under the *more restrictive hypotheses* [i.e., $(\dagger_1)$] discussed in (i).

Theorem 3.18 (**Metric-admissible outomorphisms and tripods).** *In the notation of Theorem 3.17, the following hold:*

(i) *Suppose that* $n \geq 3$*. Let* Π^{tpd} *be a* **central** $\{\mathbf{1,2,3}\}$**-tripod** *of* Π_n *[cf.* [CbTpII]*, Definitions 3.3, (i); 3.7, (ii)]. Then the restriction of the* **tripod homomorphism** *associated to* Π_n

$$\mathfrak{T}_{\Pi^{\mathrm{tpd}}} : \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{tpd}})$$

[cf. [CbTpII]*, Definition 3.19] to the subgroup* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *[cf. Definition 3.7, (iii)]* **factors** *through the subgroup* $\mathrm{Out}(\Pi^{\mathrm{tpd}})^{\mathrm{M}} \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{tpd}})$ *[cf. Definition 3.7, (ii)], i.e.,*

we have a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} & \longrightarrow & \mathrm{Out}(\Pi^{\mathrm{tpd}})^{\mathrm{M}} \\ \downarrow & & \downarrow \\ \mathrm{Out}^{\mathrm{FC}}(\Pi_n) & \xrightarrow[\mathfrak{T}_{\Pi^{\mathrm{tpd}}}]{} & \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{tpd}}) . \end{array}$$

(ii) *Suppose that* $n \geq 1$*, and that* $(g, r) = (0, 3)$*. Write*

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

for the inverse image via the natural homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{Out}(\Pi_1)$ *[cf.* [CbTpI], *Theorem A, (i)] of* $\mathrm{Out}^{\mathrm{C}}(\Pi_1)^{\Delta+} \subseteq \mathrm{Out}(\Pi_1)$ *[cf.* [CbTpII], *Definition 3.4, (i)];*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

[cf. Remark 3.18.1 below];

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} ;$$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} .$$

Then we have equalities

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} ,$$

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+} .$$

Moreover, the natural homomorphisms

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{\Delta+} & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} \\ \| & & \| \\ \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1})^{\Delta+} & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} \end{array}$$

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{\mathrm{M}\Delta+} & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+} \\ \| & & \| \\ \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1})^{\mathrm{M}\Delta+} & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} \end{array}$$

are **bijective***.*

Proof. Assertion (i) follows immediately — in light of the equalities

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} = \mathrm{Out}^{\mathrm{FCI}}(\Pi_n) \ , \ \ \mathrm{Out}(\Pi^{\mathrm{tpd}})^{\mathrm{M}} = \mathrm{Out}^{\mathrm{I}}(\Pi^{\mathrm{tpd}}) \cap \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{tpd}})$$

[cf. Theorem 3.17, (i), (ii)] — from the definition of *I-admissibility*, together with [in the case where $\Sigma = \mathfrak{Primes}$] Corollary 2.13, (iii). Next, we verify assertion (ii). The equalities

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} \ , \ \ \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+}$$

follow immediately from [CbTpII], Theorem A, (ii), together with the various definitions involved. Next, let us observe that, to verify the *bijectivity* of the various homomorphisms in question, it suffices to verify the *bijectivity* of the natural homomorphism

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{\Delta+} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+} .$$

On the other hand, this *bijectivity* follows immediately, in light of the various definitions involved, from [CmbCsp], Corollary 4.2, (i), (ii). This completes the proof of assertion (ii), hence also of Theorem 3.18. □

Remark 3.18.1. In the notation of Theorem 3.18, suppose that $n \geq 2$. Then in [CmbCsp], Definition 1.11, (ii), a definition was given for the notation "$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+}$", in the case of *arbitrary* (g, r), that *differs* somewhat from the definition given for this notation in Theorem 3.18, (ii), when $(g, r) = (0, 3)$. On the other hand, one verifies easily, by applying the theory of [CbTpII], §3, that, when $(g, r) = (0, 3)$, these two definitions are in fact *equivalent*. Indeed, when $n = 2$ (respectively, $n \geq 3$), this follows immediately from [CbTpII], Lemma 3.15, (ii) (respectively, [CbTpII], Theorems 3.16, (v); 3.18, (ii)).

Theorem 3.19 (Metric-, graph-admissible outomorphisms and tempered fundamental groups). *In the notation of Theorem 3.17, write* $\overline{K}^{\wedge}$ *for the p-adic completion of* $\overline{K}$*;*

$$\pi_1^{\mathrm{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})$$

for the **tempered fundamental group** *[cf.* [André]*, §4] of* $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$*;*

$$\Pi_n^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \varprojlim_{N} \pi_1^{\mathrm{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})/N$$

for the **Σ-tempered fundamental group** *of* $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$ *[cf.* [CmbGC] *Corollary 2.10, (iii)], i.e., the inverse limit given by allowing N to vary over the open normal subgroups of* $\pi_1^{\mathrm{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})$ *such that the quotient by N corresponds to a* **topological covering** *[cf.* [André]*, §4.2] of some* **finite étale Galois covering** *of* $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$ *of degree a product of primes* $\in \Sigma$*. [Here, we recall that, when* $n = 1$*, such a "topological covering" corresponds to a "combinatorial covering", i.e., a covering determined by a covering of the dual semi-graph of the special fiber of the stable model of some finite étale covering of* $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$*.] Then the following hold:*

(i) *Let* $l \in \Sigma$ *be such that* $l \neq p$*. Then the natural inclusion*

$$\mathrm{Out}^{\{l\}\text{-}\mathrm{G}}(\Pi_n) \hookrightarrow \mathrm{Out}(\Pi_n)$$

[cf. Definition 3.13, (iv)] **factors** *as a composite of homomorphisms*

$$\mathrm{Out}^{\{l\}\text{-G}}(\Pi_n) \longrightarrow \mathrm{Out}(\Pi_n^{\mathrm{tp}}) \longrightarrow \mathrm{Out}(\Pi_n)$$

— where the second arrow is the natural homomorphism [cf. Proposition 3.3, (i)]. In particular, the image of the natural homomorphism $\mathrm{Out}(\Pi_n^{\mathrm{tp}}) \to \mathrm{Out}(\Pi_n)$ **contains** *the subgroup* $\mathrm{Out}^{\{l\}\text{-G}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$*, hence also the subgroup* $\mathrm{Out}^{\mathrm{G}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$ *[cf. Definition 3.13, (iv)].*

(ii) *Write*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}} \subseteq \mathrm{Out}(\Pi_n^{\mathrm{tp}})$$

for the inverse image of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \subseteq \mathrm{Out}(\Pi_n)$ *[cf. Definition 3.7, (iii)] via the natural homomorphism* $\mathrm{Out}(\Pi_n^{\mathrm{tp}}) \to \mathrm{Out}(\Pi_n)$ *[cf. (i)]. Then the resulting natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$$

is **split surjective***, i.e., there exists a homomorphism*

$$\Phi\colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}}$$

such that the composite

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \xrightarrow{\Phi} \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tp}})^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$$

is the **identity automorphism** *of* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$.

Proof. Assertion (i) follows immediately from Proposition 3.16, (ii). Assertion (ii) follows immediately from assertion (i), together with the fact that $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} \subseteq \mathrm{Out}^{\{l\}\text{-G}}(\Pi_n)$ [cf. Theorem 3.17, (ii)]. This completes the proof of Theorem 3.19. ☐

Remark 3.19.1. In the fourth line of the proof of [André], Proposition 8.6.2, it is *asserted* that one has an *injection*

$$\mathrm{Aut}^{\flat}(\Gamma_{0,r+1}^{\mathrm{alg}}) \hookrightarrow \mathrm{Aut}^{\flat}(\Gamma_{0,r}^{\mathrm{alg}})\,.$$

In the notation of the present series of papers [cf. [CmbCsp], Proposition 1.3, (vii)], this homomorphism corresponds to the natural homomorphism

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_{n+1})^{\mathrm{cusp}} \longrightarrow \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$$

in the case where $(g, r, \Sigma) = (0, 3, \mathfrak{Primes})$, and $n \geq 1$ corresponds to "$r - 3$" in the notation of [André], Proposition 8.6.2. However, this assertion is *false*. Indeed, since $\Gamma_{0,r+1}^{\mathrm{alg}}$ and $\Gamma_{0,r}^{\mathrm{alg}}$ are *center-free* [cf., e.g., [MzTa], Proposition 2.2, (ii)], it follows that the respective subgroups of inner automorphisms determine compatible injections $\Gamma_{0,r+1}^{\mathrm{alg}} \hookrightarrow \mathrm{Aut}^{\flat}(\Gamma_{0,r+1}^{\mathrm{alg}})$, $\Gamma_{0,r}^{\mathrm{alg}} \hookrightarrow \mathrm{Aut}^{\flat}(\Gamma_{0,r}^{\mathrm{alg}})$. On the other hand, since the natural surjection $\Gamma_{0,r+1}^{\mathrm{alg}} \twoheadrightarrow \Gamma_{0,r}^{\mathrm{alg}}$ is *far from injective*, it thus follows that the natural homomorphism $\mathrm{Aut}^\flat(\Gamma^{\mathrm{alg}}_{0,r+1}) \to \mathrm{Aut}^\flat(\Gamma^{\mathrm{alg}}_{0,r})$ also fails to be injective. In particular, the proof given in [André] of the *injectivity* of the first displayed homomorphism

$$\mathrm{GT}_p^{(r+1)} \longrightarrow \mathrm{GT}_p^{(r)}$$

of [André], Proposition 8.6.2, (1) — hence also of

- [André], Proposition 8.6.2, (2),
- [André], Corollary 8.6.4,
- the final portion of [André], Theorem 8.7.1, and
- the portion of [André], Corollary 8.7.2, concerning "$\mathrm{GT}_p^{(r)}$"

— must be considered *incomplete*.

Remark 3.19.2. Recall that, relative to the notation of the present series of papers, the usual *Grothendieck-Teichmüller group* corresponds to the group

$$\mathrm{GT} \overset{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta+}$$

discussed in Theorem 3.18, (ii) [cf. also Remark 3.18.1], in the case where $(g, r, \Sigma) = (0, 3, \mathfrak{Primes})$ [cf. [CmbCsp], Remark 1.11.1]. Thus, from the point of view of the present paper, it seems that one natural candidate for the notion of a *local version of the Grothendieck-Teichmüller group* is the *"metrized Grothendieck-Teichmüller group"*

$$\mathrm{GT}^{\mathrm{M}} \overset{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta+} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta+} \subseteq \mathrm{GT}$$

discussed in Theorem 3.18, (ii), again in the case where $(g, r, \Sigma) = (0, 3, \mathfrak{Primes})$. Here, we recall that each of these groups GT^{M}, GT admits a natural *profinite topology*, hence, in particular, is *compact* [cf. Theorem 3.17, (iv)], and, moreover, is *independent*, up to *canonical isomorphism*, of the choice of $n \geq 1$ [cf. Theorem 3.18, (ii)]. Finally, one verifies immediately from the existence of the *natural splitting* of the split surjection discussed in Theorem 3.19, (ii) [cf. also the discussion of the construction of this splitting in the proof of Proposition 3.16, (ii); Remark 3.19.3 below] that, for any positive integer n, one has a *natural inclusion*

$$\mathrm{GT}^{\mathrm{M}} \hookrightarrow \mathrm{GT}_p^{(n+3)}$$

[cf. [André], Notation 8.6.1], hence also a *natural inclusion*

$$\mathrm{GT}^{\mathrm{M}} \hookrightarrow \mathrm{GT}_p$$

[cf. [André], Definition 8.6.3]. In particular, one obtains a *natural outer action* of GT^{M} on the "tower" of tempered fundamental groups "$(\Gamma^{\mathrm{temp}}_{0,r})_{r\geq 4}$" discussed in [André], Corollary 8.6.4, i.e., in the notation of Theorem 3.19 of the present paper, on the system of tempered fundamental groups $\{\Pi_n^{\mathrm{tp}}\}_{n\geq 1}$ that is manifestly compatible with the quotients $\Pi_n^{\mathrm{tp}} \twoheadrightarrow \Gamma_{0,n+3}^{\mathrm{temp}}$ [cf. [André], §8.5].

Remark 3.19.3. The construction of the *splitting* Φ given in the proof of Theorem 3.19, (ii), appears, at first glance, to depend on the choice of the prime l, as well as on the *ordering* of the n factors of the configuration spaces that give rise to Π_n, Π_n^{tp}. In fact, however, it is not difficult to verify — by applying the *functoriality* of the various constructions involved [cf. the discussion of *"functorial bijections"* in the proof of Proposition 3.6] to relate the *"decomposition groups"* of the various *strata* that appear in the proof of Proposition 3.16, (ii) — that Φ is *independent* of the choice of l, as well as of the ordering of the n factors of the configuration spaces that give rise to Π_n, Π_n^{tp}.

Corollary 3.20 (**Characterization of the local Galois groups in the global Galois image associated to a hyperbolic curve**)**.** *Let F be a* **number field***, i.e., a finite extension of the field of rational numbers; $\mathfrak{p}$ a nonarchimedean prime of F; $\overline{F}_{\mathfrak{p}}$ an algebraic closure of the $\mathfrak{p}$-adic completion $F_{\mathfrak{p}}$ of F; $\overline{F} \subseteq \overline{F}_{\mathfrak{p}}$ the algebraic closure of F in $\overline{F}_{\mathfrak{p}}$; $X_F^{\log}$ a* **smooth log curve** *over F. Write $\overline{F}_{\mathfrak{p}}^{\wedge}$ for the completion of $\overline{F}_{\mathfrak{p}}$; $G_{\mathfrak{p}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \subseteq G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$; $X_{\overline{F}}^{\log} \stackrel{\mathrm{def}}{=} X_F^{\log} \times_F \overline{F}$;*

$$\pi_1(X_{\overline{F}}^{\log})$$

for the **log fundamental group** *of $X_{\overline{F}}^{\log}$ [which, in the following, we* **identify** *with the log fundamental groups of $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}$, $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}$ — cf. the definition of $\overline{F}$!];*

$$\pi_1^{\mathrm{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})$$

for the **tempered fundamental group** *of $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}$ [cf.* [André], *§4];*

$$\rho_{X_F^{\log}} \colon G_F \longrightarrow \mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))$$

for the natural outer Galois action associated to $X_F^{\log}$;

$$\rho_{X_F^{\log},\mathfrak{p}}^{\mathrm{temp}} \colon G_{\mathfrak{p}} \longrightarrow \mathrm{Out}(\pi_1^{\mathrm{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}))$$

for the natural outer Galois action associated to $X_F^{\log} \times_F F_{\mathfrak{p}}$ [cf. [André], *Proposition 5.1.1];*

$$\mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{M}} \subseteq \ (\ \mathrm{Out}(\pi_1^{\mathrm{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})) \subseteq \ )\ \mathrm{Out}(\pi_1(X_{\overline{F}}^{\log}))$$

for the subgroup of **M-admissible** *outomorphisms of $\pi_1(X_{\overline{F}}^{\log})$ [cf. Definition 3.7, (i), (ii); Proposition 3.6, (i)]. Then the following hold:*

(i) *The outer Galois action* $\rho^{\mathrm{temp}}_{X^{\log}_F,\mathfrak{p}}$ *factors through the subgroup* $\mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))^{\mathrm{M}} \subseteq \mathrm{Out}(\pi_1^{\mathrm{temp}}(X^{\log}_F \times_F \overline{F}^{\wedge}_{\mathfrak{p}}))$*.*

(ii) *We have a natural commutative diagram*

$$\begin{array}{ccc} G_{\mathfrak{p}} & \longrightarrow & \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))^{\mathrm{M}} \\ \downarrow & & \downarrow \\ G_F & \xrightarrow{\rho_{X^{\log}_F}} & \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}})) \end{array}$$

— where the vertical arrows are the natural inclusions, the upper horizontal arrow is the homomorphism arising from the factorization of (i), and all arrows are **injective***.*

(iii) *The diagram of (ii) is* **cartesian***, i.e., if we regard the various groups involved as subgroups of* $\mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$*, then we have an equality*

$$G_{\mathfrak{p}} = G_F \cap \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))^{\mathrm{M}}\,.$$

Proof. Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from the *injectivity* of the lower horizontal arrow $\rho_{X^{\log}_F}$ [cf. [NodNon], Theorem C], together with the various definitions involved. Finally, we verify assertion (iii). First, let us observe that if the smooth log curve "$X^{\log}_F$" is the smooth log curve associated to $\mathbb{P}^1_F \setminus \{0,1,\infty\}$, then assertion (iii) follows immediately from [André], Theorem 7.2.1. Write $(X_{\overline{F}})^{\log}_3$ for the 3-rd log configuration space of $X^{\log}_{\overline{F}}$. Then it follows immediately from [NodNon], Theorem B, that the group $\mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})^{\log}_3))$ of FC-admissible outomorphisms of the log fundamental group $\pi_1((X_{\overline{F}})^{\log}_3)$ of $(X_{\overline{F}})^{\log}_3$ [which, in the following, we *identify* with the log fundamental groups of $(X_{\overline{F}})^{\log}_3 \times_{\overline{F}} \overline{F}_{\mathfrak{p}}$, $(X_{\overline{F}})^{\log}_3 \times_{\overline{F}} \overline{F}^{\wedge}_{\mathfrak{p}}$ — cf. the definition of $\overline{F}$!] may be regarded as a closed subgroup of $\mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$. Moreover, it follows immediately from the various definitions involved that the respective images $\mathrm{Im}(\rho_{X^{\log}_F})$, $\mathrm{Im}(\rho^{\mathrm{temp}}_{X^{\log}_F,\mathfrak{p}})$ of the natural outer Galois actions $\rho_{X^{\log}_F}$, $\rho^{\mathrm{temp}}_{X^{\log}_F,\mathfrak{p}}$ associated to $X^{\log}_F$, $X^{\log}_F \times_F F_{\mathfrak{p}}$ are *contained* in this closed subgroup $\mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})^{\log}_3)) \subseteq \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$. Thus, to verify assertion (iii), one verifies easily that it suffices to verify the equality

$$\mathrm{Im}(\rho^{\mathrm{temp}}_{X^{\log}_F,\mathfrak{p}}) \;=\; \mathrm{Im}(\rho_{X^{\log}_F}) \cap \mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})^{\log}_3))^{\mathrm{M}}$$

[cf. Definition 3.7, (iii)]. On the other hand, since the "$\rho_{X^{\log}_F}$" that occurs in the case where we take "$X^{\log}_F$" to be the smooth log curve associated to $\mathbb{P}^1_F \setminus \{0,1,\infty\}$ is *injective* [cf. assertion (ii)], this equality follows immediately — by considering the images of the subgroups

$$\mathrm{Im}(\rho^{\mathrm{temp}}_{X^{\log}_F,\mathfrak{p}}) \subseteq \mathrm{Im}(\rho_{X^{\log}_F}) \cap \mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})^{\log}_3))^{\mathrm{M}}$$

of $\mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})^{\log}_3))^{\mathrm{M}}$ via the *tripod homomorphism* associated to $\mathrm{Out}^{\mathrm{F}}(\pi_1((X_{\overline{F}})^{\log}_3))$ [cf. [CbTpII], Definition 3.19] — from Theorem 3.18, (i), together with assertion (iii) in the case where we take "$X^{\log}_F$" to be the smooth log curve associated to $\mathbb{P}^1_F \setminus \{0, 1, \infty\}$ [which was verified above]. This completes the proof of assertion (iii), hence also of Corollary 3.20. □

Remark 3.20.1. Corollary 3.20, (iii), may be regarded as a *generalization* of [André], Theorems 7.2.1, 7.2.3, obtained at the cost of *replacing*, in effect, $\mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))^{\mathrm{G}}$ by the *possibly smaller* group $\mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))^{\mathrm{M}} \subseteq \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$. Here, we note that *unlike* the subgroups $G_{\mathfrak{p}} \subseteq G_F$ [cf., e.g., [AbsHyp], Theorem 1.1.1, (i)] and $\mathrm{Out}(\pi_1^{\mathrm{temp}}(X^{\log}_F \times_F \overline{F}^{\wedge}_{\mathfrak{p}})) \xrightarrow{\sim} \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))^{\mathrm{G}} \subseteq \mathrm{Out}(\pi_1(X^{\log}_{\overline{F}}))$ [cf. Definition 3.7, (i); Proposition 3.6, (i); Remark 3.13.1, (i); Theorem 3.17, (v)], which are *commensurably terminal*, the subgroup $\mathrm{Out}(-)^{\mathrm{M}} \subseteq \mathrm{Out}(-)$ *fails*, in general [at least in the pro-l case], even to be *normally terminal* [cf. Remark 3.17.1].

Remark 3.20.2. Let us recall that, in the proof of [NodNon], Theorem C, the authors applied

- the *injectivity portion* of the theory of combinatorial cuspidalization, together with
- the *injectivity* of the outer Galois representation associated to a **tripod**, to prove
- the *injectivity* of the outer Galois representation associated to an **arbitrary hyperbolic curve**.

On the other hand, in the proof of Corollary 3.20, the authors applied

- the [almost pro-l] *injectivity portion* of the theory of combinatorial cuspidalization, together with
- the *characterization* of the *local Galois groups* in the *global Galois image* for **tripods**, to prove
- an analogous *characterization* of the *local Galois groups* in the *global Galois image* for **arbitrary hyperbolic curves**.

The formal similarity of these two proofs suggests that it is perhaps natural to think of the *injectivity portion* of the theory of combinatorial cuspidalization as a sort of *tool for reducing* certain problems concerning **arbitrary hyperbolic curves** to the case of **tripods**.

Remark 3.20.3. By comparison to André's original characterization of the local Galois groups in the global Galois image [cf. [André], Theorems 7.2.1, 7.2.3], from the point of view of a researcher who is interested only in *tripods* [i.e., not in *arbitrary hyperbolic curves*], the motivation for the theory developed in the present paper concerning $\mathrm{Out}(-)^{\mathrm{M}}$ may at first glance appear insufficient. In fact, however, as discussed in Remarks 3.19.1 3.19.2, even if one is interested only in tripods, it is necessary to apply the extensive theory developed in the present paper concerning $\mathrm{Out}(-)^{\mathrm{M}}$ in order to repair the mistake in [André] and realize the original goal of this paper, i.e., of defining a suitable *local analogue of the Grothendieck-Teichmüller group.*

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