TOPICS SURROUNDING THE COMBINATORIAL ANABELIAN GEOMETRY OF HYPERBOLIC CURVES III: TRIPODS AND TEMPERED FUNDAMENTAL GROUPS

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ABSTRACT. Let Σ be a subset of the set of prime numbers which is either equal to the entire set of prime numbers or of cardinality one. In the present paper, we continue our study of the pro- Σ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of Σ are invertible. The focus of the present paper is on **appli**cations of the theory developed in previous papers to the theory of tempered fundamental groups, in the style of André. These applications are motivated by the goal of surmounting two fundamental technical difficulties that appear in previous work of André, namely: (a) the fact that the characterization of the local Galois groups in the global Galois image associated to a hyperbolic curve that is given in earlier work of André is only proven for a *quite limited* class of hyperbolic curves, i.e., a class that is "far from generic"; (b) the proof given in earlier work of André of a certain key injectivity result, which is of central importance in establishing the theory of a "p-adic local analogue" of the well-known "global" theory of the Grothendieck-Teichmüller group, contains a fundamental gap. In the present paper, we surmount these technical difficulties by introducing the notion of an "M-admissible", or "metric-admissible", outer automorphism of the profinite geometric fundamental group of a *p*-adic hyperbolic curve. Roughly speaking, M-admissible outer automorphisms are outer automorphisms that are compatible with the data constituted by the *indices* at the various *nodes* of the special fiber of the *p*-adic curve under consideration. By combining this notion with combinatorial anabelian results and techniques developed in earlier papers by the authors, together with the theory of cyclotomic synchronization [also developed in earlier papers by the authors], we obtain a generalization of André's characterization of the local Galois groups in the global Galois image associated to a hyperbolic curve to the case of arbitrary hyperbolic curves [cf. (a)]. Moreover, by applying the theory of *local contractibility* of *p*-adic analytic spaces developed by Berkovich, we show that the techniques developed in the present and earlier papers by the authors allow one to relate the groups of M-admissible outer automorphisms treated in the present paper to the groups of outer automorphisms of tempered fundamental groups of higher-dimensional configuration

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spaces [associated to the given *p*-adic hyperbolic curve]. These considerations allow one to "repair" the gap in André's proof — albeit at the expense of working with *M*-admissible outer automorphisms — and hence to realize the goal of obtaining a "local analogue of the Grothendieck-Teichmüller group" [cf. (b)].

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INTRODUCTION

Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset of the set of prime numbers \mathfrak{Primes} which is either equal to \mathfrak{Primes} or of cardinality one. In the present paper, we continue our study of the *pro-* Σ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of Σ are invertible [cf. [MzTa], [CmbCsp], [NodNon], [CbTpI], [CbTpII]]. The focus of the present paper is on **applications** of the theory developed in previous papers to the theory of **tempered fundamental groups**, in the style of [André].

Just as in previous papers, the main technical result that underlies our approach is a certain *combinatorial anabelian result* [cf. Theorem 1.11; Corollary 1.12], which may be summarized as a *generalization* of results obtained in earlier papers [cf., e.g., [NodNon], Theorem A; [CbTpII], Theorem 1.9] in the case of pro- Σ fundamental groups to the case of **almost pro-\Sigma fundamental groups** [i.e., maximal almost pro- Σ quotients of profinite fundamental groups — cf. Definition 1.1]. The technical details surrounding this generalization occupy the bulk of §1.

In §2, we observe that the theory of §1 may be applied, via a similar argument to the argument applied in [NodNon] to derive [NodNon], Theorem B, from [NodNon], Theorem A, to obtain almost pro- Σ generalizations [cf. Theorem 2.9; Corollary 2.10; Remark 2.10.1] of the **injectivity** portion of the theory of combinatorial cuspidalization [i.e., [NodNon], Theorem B]. In the final portion of §2, we discuss the theory of almost pro-l commensurators of tripods [i.e., copies of the [geometric fundamental group of the] projective line minus three points — cf. Lemma 2.12, Corollary 2.13], in the context of the theory of the tripod homomorphism developed in [CbTpII], §3. Just as in the case of the

theory of \$1, the theory of \$2 is *conceptually* not very difficult, but *technically* quite involved.

Before proceeding, we recall that a substantial portion of the theory of [André] revolves around the study of outomorphism [i.e., outer automorphism] groups of the **tempered geometric fundamental group** of a *p*-adic hyperbolic curve, from the point of view of the goal of establishing

> a *p***-adic local analogue** of the well-known theory of the **Grothendieck-Teichmüller group** [i.e., which appears in the context of *hyperbolic curves over number fields*].

¿From the point of view of the theory of the present series of papers, outomorphisms of such tempered fundamental groups may be thought of as [i.e., are equivalent to — cf. Remark 3.3.1; Proposition 3.6, (iii); Remark 3.13.1, (i)] outomorphisms of the profinite geometric fundamental group that are "G-admissible" [cf. Definition 3.7, (i)], i.e., preserve the graph-theoretic structure on the profinite geometric fundamental group. In a word, the essential thrust of the applications to the theory of tempered fundamental groups given in the present paper may be summarized as follows:

By replacing, in effect, the *G*-admissible outomorphism groups that [modulo the "translation" discussed above] appear throughout the theory of [André] by "**M**-admissible" outomorphism groups — i.e., groups of outomorphisms of the profinite geometric fundamental group that preserve not only the graph-theoretic structure on the profinite geometric fundamental group, but also the [somewhat finer] metric structure on the various dual graphs that appear [i.e., the various *indices* at the nodes of the special fiber of the p-adic curve under consideration — cf. Definition 3.7, (ii)] — it is possible to overcome various significant technical difficulties that appear in the theory of [André].

Here, we recall that the two main technical difficulties that appear in the theory of [André] may be described as follows:

- The characterization of the local Galois groups in the global Galois image associated to a hyperbolic curve that is given in [André], Theorems 7.2.1, 7.2.3, is only proven for a quite limited class of hyperbolic curves [i.e., a class that is "far from generic" cf. [MzTa], Corollary 5.7], which are "closely related to tripods".
- The proof given in [André] of a certain key injectivity result, which is of central importance in establishing the theory of a

"local analogue of the Grothendieck-Teichmüller group", contains a fundamental gap [cf. Remark 3.19.1].

In the present paper, our approach to surmounting the *first* technical difficulty consists of the following result [cf. Theorems 3.17, (iv); 3.18, (i)], which asserts, roughly speaking, that the theory of the **tripod homomorphism** developed in [CbTpII], §3, is **compatible** with the property of **M-admissibility**.

Theorem A (Metric-admissible outomorphisms and the tripod homomorphism). Let $n \ge 3$ be an integer; (g, r) a pair of nonnegative integers such that 2g - 2 + r > 0; p a prime number; Σ a set of prime numbers such that $\Sigma \neq \{p\}$, and, moreover, is either equal to the set of all prime numbers or of cardinality one; R a mixed characteristic complete discrete valuation ring of residue characteristic p whose residue field is separably closed; K the field of fractions of R; \overline{K} an algebraic closure of K;

$$X_{K}^{\log}$$

a smooth log curve of type (q, r) over K. Write

$$(X_K)_n^{\log}$$

for the n-th log configuration space [cf. the discussion entitled "Curves" in [CbTpII], §0] of X_K^{\log} over K; $(X_{\overline{K}})_n^{\log} \stackrel{\text{def}}{=} (X_K)_n^{\log} \times_K \overline{K}$;

$$\prod_n \stackrel{\text{def}}{=} \pi_1 ((X_{\overline{K}})_n^{\log})^{\Sigma}$$

for the maximal pro- Σ quotient of the log fundamental group of $(X_{\overline{K}})_n^{\log}$. Let Π^{tpd} be a 1-central {1, 2, 3}-tripod of Π_n [cf. [CbTpII], Definitions 3.3, (i); 3.7, (ii)]. Then the restriction of the tripod homomorphism associated to Π_n

$$\mathfrak{T}_{\Pi^{\mathrm{tpd}}} \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{tpd}})$$

[cf. [CbTpII], Definition 3.19] to the subgroup $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\mathrm{M}} \subseteq \operatorname{Out}^{\operatorname{FC}}(\Pi_n)$ of **M-admissible** outomorphisms [cf. Definition 3.7, (iii)] factors through the subgroup $\operatorname{Out}(\Pi^{\operatorname{tpd}})^{\mathrm{M}} \subseteq \operatorname{Out}^{\mathrm{C}}(\Pi^{\operatorname{tpd}})$ [cf. Definition 3.7, (i), (ii); Remark 3.13.1, (i), (ii)], i.e., we have a natural commutative diagram of profinite groups

Theorem A has the following formal consequence, namely, a generalization of the *characterization* of the *local Galois groups* in the *global Galois image* associated to a hyperbolic curve that is given in [André], Theorems 7.2.1, 7.2.3, to **arbitrary hyperbolic curves**, albeit at the expense of, in effect, replacing "G-admissibility" by the stronger condition of "**M-admissibility**" [cf. Corollary 3.20; Remark 3.20.1]. This generalization may also be regarded as a sort of strong version of the *Galois injectivity* result given in [NodNon], Theorem C [cf. Remark 3.20.2].

Theorem B (Characterization of the local Galois groups in the global Galois image associated to a hyperbolic curve). Let F be a number field, i.e., a finite extension of the field of rational numbers; \mathfrak{p} a nonarchimedean prime of F; $\overline{F}_{\mathfrak{p}}$ an algebraic closure of the \mathfrak{p} -adic completion $F_{\mathfrak{p}}$ of F; $\overline{F} \subseteq \overline{F}_{\mathfrak{p}}$ the algebraic closure of F in $\overline{F}_{\mathfrak{p}}$; X_F^{\log} a smooth log curve over F. Write $\overline{F}_{\mathfrak{p}}^{\wedge}$ for the completion of $\overline{F}_{\mathfrak{p}}$; $G_{\mathfrak{p}} \stackrel{\text{def}}{=} \operatorname{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \subseteq G_F \stackrel{\text{def}}{=} \operatorname{Gal}(\overline{F}/F)$; $X_{\overline{F}}^{\log} \stackrel{\text{def}}{=} X_F^{\log} \times_F \overline{F}$;

$$\pi_1(X_{\overline{F}}^{\log})$$

for the log fundamental group of $X_{\overline{F}}^{\log}$ [which, in the following, we identify with the log fundamental groups of $X_{F}^{\log} \times_{F} \overline{F}_{\mathfrak{p}}, X_{F}^{\log} \times_{F} \overline{F}_{\mathfrak{p}}^{\wedge}$ — cf. the definition of \overline{F} !];

$$\pi_1^{\text{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})$$

for the **tempered fundamental group** of $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}$ [cf. [André], §4];

$$\rho_{X_{\overline{\nu}}^{\log}} \colon G_F \longrightarrow \operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))$$

for the natural outer Galois action associated to X_F^{\log} ;

$$\rho_{X_F^{\log},\mathfrak{p}}^{\text{temp}} \colon G_{\mathfrak{p}} \longrightarrow \text{Out}(\pi_1^{\text{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}))$$

for the natural outer Galois action associated to $X_F^{\log} \times_F F_{\mathfrak{p}}$ [cf. [André], Proposition 5.1.1];

$$\operatorname{Out}(\pi_1(X_{\overline{F}}^{\operatorname{log}}))^{\mathsf{M}} \subseteq (\operatorname{Out}(\pi_1^{\operatorname{temp}}(X_F^{\operatorname{log}} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})) \subseteq) \operatorname{Out}(\pi_1(X_{\overline{F}}^{\operatorname{log}}))$$

for the subgroup of **M-admissible** outcomorphisms of $\pi_1(X_{\overline{F}}^{\log})$ [cf. Definition 3.7, (i), (ii); Proposition 3.6, (i)]. Then the following hold:

- (i) The outer Galois action $\rho_{X_F^{\log},\mathfrak{p}}^{\operatorname{temp}}$ factors through the subgroup $\operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{M}} \subseteq \operatorname{Out}(\pi_1^{\operatorname{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})).$
- (ii) We have a natural commutative diagram

$$\begin{array}{ccc} G_{\mathfrak{p}} & \longrightarrow & \operatorname{Out}(\pi_{1}(X_{\overline{F}}^{\operatorname{log}}))^{\operatorname{M}} \\ & & & \downarrow \\ & & & \downarrow \\ G_{F} & \xrightarrow{\rho_{X_{F}^{\operatorname{log}}}} & \operatorname{Out}(\pi_{1}(X_{\overline{F}}^{\operatorname{log}})) \end{array}$$

— where the vertical arrows are the natural inclusions, the upper horizontal arrow is the homomorphism arising from the factorization of (i), and all arrows are **injective**.

(iii) The diagram of (ii) is cartesian, i.e., if we regard the various groups involved as subgroups of $Out(\pi_1(X_{\overline{F}}^{log}))$, then we have an equality

$$G_{\mathfrak{p}} = G_F \cap \operatorname{Out}(\pi_1(X_{\overline{E}}^{\log}))^{\mathrm{M}}$$

One central technical aspect of the theory of the present paper lies in the equivalence [cf. Theorem 3.9] between the M-admissibility of outomorphisms of the profinite geometric fundamental group of the given *p*-adic hyperbolic curve and the **I-admissibility** [i.e., roughly speaking, compatibility with the outer action, by *some* open subgroup of the *inertia group* of the absolute Galois group of the base field, on an arbitrary almost pro-l quotient of the profinite geometric fundamental group — cf. Definition 3.8] of such outomorphisms. This equivalence is obtained by applying the theory of cyclotomic synchronization developed in [CbTpI], §5. Once this equivalence is established, the almost pro-l injectivity results obtained in §2 then allow us to conclude that this M-admissibility of outomorphisms of the profinite geometric fundamental group of the given *p*-adic hyperbolic curve is, in fact, equivalent to the *I-admissibility* of any [necessarily unique!] *lifting* of such an outomorphism to an outomorphism of the profinite geometric fundamental group of a higher-dimensional configuration space associated to the given p-adic hyperbolic curve [cf. Theorem 3.17, (ii)]. Finally, by combining this "higher-dimensional **I-admissibility**" with the *combinatorial anabelian theory* of [CbTpII], $\{1, we conclude [cf. Proposition 3.16, (i); Theorem 3.17, (ii)] that a$ certain "higher-dimensional G-admissibility" also holds, i.e., that the *lifted outomorphism* of the profinite geometric fundamental group of a higher-dimensional configuration space associated to the given padic hyperbolic curve preserves the *graph-theoretic* structure not only on the profinite geometric fundamental group of the original hyperbolic curve, but also on the profinite geometric fundamental groups of the various *successive fibers* of the higher-dimensional configuration space under consideration. In a word,

> it is precisely by applying this chain of equivalences which allows us to *control the* **graph-theoretic** *structure of the* **successive fibers** of the higher-dimensional configuration space under consideration — that allow us to surmount the two main technical difficulties discussed above that appear in the theory of [André].

Put another way, if, instead of considering M-admissible outcomorphisms [i.e., of the profinite geometric fundamental group of the given

p-adic hyperbolic curve], one considers arbitrary *G*-admissible outomorphisms [of the profinite geometric fundamental group of the given p-adic hyperbolic curve, as is done, in effect, in [André]], then there does not appear to exist, at least at the time of writing, any effective way to control the graph-theoretic structure on the successive fibers of higher-dimensional configuration spaces.

In this context, we recall that in the theory of [CbTpII], a result is obtained concerning the preservation of the graph-theoretic structure on the successive fibers of higher-dimensional configuration spaces [cf. [CbTpII], Theorem 4.7], in the context of *pro-l* geometric fundamental groups. The significance, however, of the theory of the present paper is that it may be applied to *almost pro-l* geometric fundamental groups, i.e., where the order of the finite quotient implicit in the term "almost" is allowed to be *divisible by p*.

Once one establishes the "higher-dimensional G-admissibility" discussed above, it is then possible to apply the theory of local contractibility of p-adic analytic spaces developed in [Brk] to construct from the given outomorphism of a profinite geometric fundamental group [of a higher-dimensional configuration space] an outomorphism of the corresponding tempered fundamental group [cf. Proposition 3.16, (ii)]. This portion of the theory may be summarized as follows [cf. Theorem 3.19, (ii)].

Theorem C (Metric-admissible outomorphisms and tempered fundamental groups). Let n be a positive integer; (g, r) a pair of nonnegative integers such that 2g - 2 + r > 0; p a prime number; Σ a nonempty set of prime numbers such that $\Sigma \neq \{p\}$, and, moreover, if $n \geq 2$, then Σ is either equal to the set of all prime numbers or of cardinality one; R a mixed characteristic complete discrete valuation ring of residue characteristic p whose residue field is separably closed; K the field of fractions of R; \overline{K} an algebraic closure of K;

 X_{κ}^{\log}

a smooth log curve of type (g, r) over K. Write

$$(X_K)_n^{\log}$$

for the n-th log configuration space [cf. the discussion entitled "Curves" in [CbTpII], §0] of X_K^{\log} over K; $(X_{\overline{K}})_n^{\log} \stackrel{\text{def}}{=} (X_K)_n^{\log} \times_K \overline{K}$;

$$\Pi_n \stackrel{\text{def}}{=} \pi_1 ((X_{\overline{K}})_n^{\log})^{\Sigma}$$

for the maximal pro- Σ quotient of the log fundamental group of $(X_{\overline{K}})_n^{\log}; \overline{K}^{\wedge}$ for the p-adic completion of $\overline{K};$

$$\pi_1^{\text{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})$$

for the **tempered fundamental group** [cf. [André], §4, as well as the discussion of Definition 3.1, (ii), of the present paper] of $(X_{\overline{K}})_n^{\log} \times_{\overline{K}}$

 $\overline{K}^{\wedge};$

$$\Pi_n^{\rm tp} \stackrel{\rm def}{=} \varprojlim_N \pi_1^{\rm temp} ((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})/N$$

for the Σ -tempered fundamental group of $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$ [cf. [CmbGC], Corollary 2.10, (iii)], i.e., the inverse limit given by allowing N to vary over the open normal subgroups of $\pi_1^{\text{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})$ such that the quotient by N corresponds to a **topological covering** [cf. [André], §4.2, as well as the discussion of Definition 3.1, (ii), of the present paper] of some finite log étale Galois covering of $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$ of degree a product of primes $\in \Sigma$. [Here, we recall that, when n = 1, such a "topological covering" corresponds to a "combinatorial covering", i.e., a covering determined by a covering of the dual semi-graph of the special fiber of the stable model of some finite log étale covering of $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$.] Write

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}} \subseteq \operatorname{Out}(\Pi_n^{\operatorname{tp}})$$

for the inverse image of $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\mathrm{M}} \subseteq \operatorname{Out}(\Pi_n)$ [cf. Definition 3.7, (iii)] via the natural homomorphism $\operatorname{Out}(\Pi_n^{\operatorname{tp}}) \to \operatorname{Out}(\Pi_n)$ [cf. Proposition 3.3, (i)]. Then the resulting natural homomorphism

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}} \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}}$$

is **split surjective**, *i.e.*, there exists a homomorphism

$$\Phi : \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}}$$

such that the composite

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} \xrightarrow{\Phi} \operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}} \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}}$$

is the identity automorphism of $Out^{FC}(\Pi_n)^M$.

Up till now, in the present discussion, the *p*-adic hyperbolic curve under consideration was arbitrary. If, however, one specializes the theory discussed above to the case of **tripods** [i.e., copies of the projective line minus three points], then one obtains the desired *p*-adic local analogue of the theory of the Grothendieck-Teichmüller group, by considering the **"metrized Grothendieck-Teichmüller group** GT^M" as follows [cf. Theorem 3.17, (iv); Theorem 3.18, (ii); Theorem 3.19, (ii); Remarks 3.19.2, 3.20.3].

Theorem D (Metric-admissible outomorphisms and tripods). In the notation of Theorem C, suppose that (g, r) = (0, 3). Write

$$\operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\Delta +} \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_n)$$

for the inverse image via the natural homomorphism $\operatorname{Out}^{\mathrm{F}}(\Pi_n) \to \operatorname{Out}(\Pi_1)$ [cf. [CbTpI], Theorem A, (i)] of $\operatorname{Out}^{\mathrm{C}}(\Pi_1)^{\Delta_+} \subseteq \operatorname{Out}(\Pi_1)$

[cf. [CbTpII], Definition 3.4, (i)];

 $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\Delta +} \stackrel{\operatorname{def}}{=} \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\Delta +} \cap \operatorname{Out}^{\operatorname{FC}}(\Pi_n)$

[cf. Remark 3.18.1];

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}\Delta +} \stackrel{\mathrm{def}}{=} \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\Delta +} \cap \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}};$$

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}\Delta+} \stackrel{\text{def}}{=} \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\Delta+} \cap \operatorname{Out}^{\operatorname{F}}(\Pi_n)^{\operatorname{M}}$$

Then the following hold:

(i) We have equalities

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\Delta +} = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\Delta +},$$
$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}\Delta +} = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{M}\Delta +}.$$

Moreover, the natural homomorphisms of profinite groups

$$\begin{array}{cccc} \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1})^{\Delta +} & \longrightarrow & \operatorname{Out}^{\operatorname{FC}}(\Pi_{n})^{\Delta +} \\ & & & & & \\ & & & & \\ & & & \\ \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1})^{\Delta +} & \longrightarrow & \operatorname{Out}^{\operatorname{FC}}(\Pi_{n})^{\Delta +} \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1})^{\operatorname{M\Delta} +} & \longrightarrow & \operatorname{Out}^{\operatorname{FC}}(\Pi_{n})^{\operatorname{M\Delta} +} \end{array}$$

are **bijective** for $n \ge 1$. In the following, we shall **identify** the various groups that occur for varying n by means of these natural isomorphisms and write

$$GT^{M} \stackrel{\text{def}}{=} Out^{F}(\Pi_{n})^{M\Delta +} = Out^{FC}(\Pi_{n})^{M\Delta +}$$
$$\subseteq GT \stackrel{\text{def}}{=} Out^{F}(\Pi_{n})^{\Delta +} = Out^{FC}(\Pi_{n})^{\Delta +}$$

[cf. [CmbCsp], Remark 1.11.1].

(ii) Write

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}\Delta+} \subseteq \operatorname{Out}(\Pi_n^{\operatorname{tp}})$$

for the inverse image of $GT^{M} \subseteq Out(\Pi_{n})$ [cf. (i)] via the natural homomorphism $Out(\Pi_{n}^{tp}) \rightarrow Out(\Pi_{n})$ [cf. Proposition 3.3, (i)]. Then the resulting natural homomorphism

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}\Delta +} \longrightarrow \operatorname{GT}^{\operatorname{M}}$$

is split surjective, i.e., there exists a homomorphism

 $\Phi_{\mathrm{GT}} \colon \mathrm{GT}^{\mathrm{M}} \longrightarrow \mathrm{Out}^{\mathrm{FC}} (\Pi_n^{\mathrm{tp}})^{\mathrm{M}\Delta +}$

such that the composite

$$\mathrm{GT}^{\mathrm{M}} \xrightarrow{\Phi_{\mathrm{GT}}} \mathrm{Out}^{\mathrm{FC}} (\Pi_n^{\mathrm{tp}})^{\mathrm{M}\Delta +} \longrightarrow \mathrm{GT}^{\mathrm{M}}$$

is the identity automorphism of GT^{M} .

In closing, we recall that "conventional research" concerning the Grothendieck-Teichmüller group GT tends to focus on the issue of whether or not the *natural inclusion* of the absolute Galois group of \mathbb{Q}

$$G_{\mathbb{O}} \hookrightarrow \mathrm{GT}$$

is, in fact, an *isomorphism* [cf. the discussion of [CbTpII], Remark 3.19.1]. By contrast, one important theme of the present series of papers lies in the point of view that, instead of pursuing the issue of whether or not GT is *literally isomorphic* to $G_{\mathbb{Q}}$, it is perhaps more natural to concentrate on the issue of verifying that

GT exhibits analogous behavior/properties to $G_{\mathbb{Q}}$ [or \mathbb{Q}].

¿From this point of view, the theory of **tripod synchronization** and **surjectivity** of the **tripod homomorphism** developed in [CbTpII] [cf. [CbTpII], Theorem C, (iii), (iv), as well as the following discussion] may be regarded as an *abstract combinatorial analogue* of the **scheme-theoretic** fact that Spec Q lies **under** all characteristic zero schemes/algebraic stacks in a **unique fashion** — i.e., put another way, that all morphisms between schemes and moduli stacks that occur in the theory of hyperbolic curves in characteristic zero are *compatible* with the respective *structure morphisms* to Spec Q. In a similar vein, the theory of the subgroup $\mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{GT}$ developed in the present paper may be regarded as an *abstract combinatorial analogue* of the various **decomposition subgroups** $G_{\mathfrak{p}} \subseteq G_F$ ($\subseteq G_{\mathbb{Q}}$) [cf. Theorem B] associated to **nonarchimedean primes**. In particular, from the point of view of pursuing "abstract behavioral similarities" to the subgroups $G_{\mathfrak{p}} \subseteq G_F$ ($\subseteq G_{\mathbb{Q}}$), it is natural to pose the question:

Is the subgroup $GT^M \subseteq GT$ commensurably terminal?

Unfortunately, in the present paper, we are only able to give a *partial* answer to this question. That is to say, we show [cf. Theorem 3.17, (v), and its proof; Remark 3.20.1] the following result. [Here, we remark that although this result is not stated *explicitly* in Theorem 3.17, (v), it follows by applying to GT^{M} the argument, involving *l*-graphically full actions, that was applied, in the proof of Theorem 3.17, (v), to "Out^{FC}(Π_n)^M".]

Theorem E (Commensurator of the metrized Grothendieck--Teichmüller group). In the notation of Theorem D [cf., especially, the bijections of Theorem D, (i)], the commensurator of GT^{M} in $Out^{F}(\Pi_{n})$ is contained in the subgroup

$$\operatorname{Out}^{\mathrm{G}}(\Pi_n) \subseteq \operatorname{Out}^{\mathrm{FC}}(\Pi_n)$$

of outomorphisms that satisfy the condition of "higher-dimensional G-admissibility" discussed above [cf. Definition 3.13, (iv); Remark 3.13.1, (ii)]. In particular, the commensurator of GT^{M} in GT is contained in

$$\mathrm{GT}^{\mathrm{G}} \stackrel{\mathrm{def}}{=} \mathrm{GT} \cap \left(\bigcap_{n \ge 1} \mathrm{Out}^{\mathrm{G}}(\Pi_n)\right) \subseteq \left(\bigcap_{n \ge 1} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)\right) \subseteq \mathrm{Out}(\Pi_1)$$

[cf. the injections $\operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1}) \hookrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n)$ of [NodNon], Theorem B].

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0. NOTATIONS AND CONVENTIONS

Topological groups: Let G be a profinite group and Σ a nonempty set of prime numbers. Then we shall write G^{Σ} for the maximal pro- Σ quotient of G.

Let G be a profinite group and $G \rightarrow Q$, Q' quotients of G. Then we shall say that the quotient Q dominates the quotient Q' if the natural surjection $G \rightarrow Q'$ factors through the natural surjection $G \rightarrow Q$.

1. Almost pro- Σ combinatorial anabelian geometry

In the present §1, we discuss almost $pro-\Sigma$ analogues of results on combinatorial anabelian geometry developed in earlier papers of the authors. In particular, we obtain almost $pro-\Sigma$ analogues of combinatorial versions of the Grothendieck Conjecture for outer representations of NN- and IPSC-type [cf. Theorem 1.11; Corollary 1.12 below].

In the present §1, let $\Sigma \subseteq \Sigma^{\dagger}$ be nonempty sets of prime numbers and \mathcal{G} a semi-graph of anabelioids of pro- Σ^{\dagger} PSC-type. Write \mathbb{G} for the underlying semi-graph of \mathcal{G} , $\Pi_{\mathcal{G}}$ for the [pro- Σ^{\dagger}] fundamental group of \mathcal{G} , and $\widetilde{\mathcal{G}} \to \mathcal{G}$ for the universal covering of \mathcal{G} corresponding to $\Pi_{\mathcal{G}}$.

Definition 1.1. Let G be a profinite group, $N \subseteq G$ a normal open subgroup of G, and $G \twoheadrightarrow Q$ a quotient of G. Then we shall say that Q is the maximal almost pro- Σ quotient of G with respect to N if the kernel of the surjection $G \twoheadrightarrow Q$ is the kernel of $N \twoheadrightarrow N^{\Sigma}$ [cf. the discussion entitled "Topological groups" in §0], i.e., $Q = G/\text{Ker}(N \twoheadrightarrow N^{\Sigma})$. Thus, Q fits into an exact sequence of profinite groups

$$1 \longrightarrow N^{\Sigma} \longrightarrow Q \longrightarrow G/N \longrightarrow 1.$$

[Note that since N is normal in G, and the kernel $\operatorname{Ker}(N \twoheadrightarrow N^{\Sigma})$ of the natural surjection $N \twoheadrightarrow N^{\Sigma}$ is characteristic in N, it holds that $\operatorname{Ker}(N \twoheadrightarrow N^{\Sigma})$ is normal in G.] We shall say that Q is a maximal almost pro- Σ quotient of G if Q is the maximal almost pro- Σ quotient of G with respect to some normal open subgroup of G.

Lemma 1.2 (Properties of maximal almost pro- Σ quotients). Let G be a profinite group. Then the following hold.

- (i) Let N ⊆ G be a normal open subgroup of G and G → J a quotient of G. Write N_J ⊆ J for the image of N in J. [Thus, N_J is a normal open subgroup of J.] Then the quotient of J determined by the maximal almost pro-Σ quotient [cf. Definition 1.1] of G with respect to N, i.e., the quotient of J by the image of Ker(N → N^Σ) in J, is the maximal almost pro-Σ quotient of J with respect to N_J.
- (ii) Let N ⊆ G be a normal open subgroup of G and H ⊆ G a closed subgroup of G. If the natural homomorphism (N ∩ H)^Σ → N^Σ is injective, then the image of H in the maximal almost pro-Σ quotient of G with respect to N is the maximal almost pro-Σ quotient of H with respect to N ∩ H.
- (iii) Let $H \subseteq G$ be a normal closed subgroup of G and $H \twoheadrightarrow H^*$ a maximal almost pro- Σ quotient of H. Suppose that H is topologically finitely generated. Then there exists a maximal

almost pro- Σ quotient $H \rightarrow H^{**}$ of H which dominates $H \rightarrow H^*$ [cf. the discussion entitled "Topological groups" in $\S{0}$] such that the kernel of $H \rightarrow H^{**}$ is normal in G.

Proof. Assertions (i), (ii) follow immediately from the various definitions involved. Next, we verify assertion (iii). Let $N \subseteq H$ be a normal open subgroup of H with respect to which H^* is the maximal almost pro- Σ quotient of H. Now since H is topologically finitely generated, and $N \subseteq H$ is open, it follows that there exists a characteristic open subgroup $J \subseteq H$ such that $J \subseteq N$. Observe that since H is normal in G, and J is characteristic in H, it holds that J is normal in G. Thus, if we write H^{**} for the maximal almost pro- Σ quotient of H with respect to J, then H^{**} satisfies the conditions of assertion (iii). This completes the proof of assertion (iii).

Definition 1.3. Let I be a profinite group and $\rho: I \to \operatorname{Aut}(\mathcal{G}) \subseteq \operatorname{Out}(\Pi_{\mathcal{G}})$ a continuous homomorphism. Then we shall say that ρ is of *PIPSC-type* [where the "PIPSC" stands for "potentially IPSC"] if the following conditions are satisfied:

- (i) I is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma^{\dagger}}$ as an abstract profinite group.
- (ii) there exists an open subgroup $J \subseteq I$ such that the restriction of ρ to J is of IPSC-type [cf. [NodNon], Definition 2.4, (i)].

Lemma 1.4 (Profinite Dehn multi-twists and finite étale coverings). Let $\alpha \in \text{Out}(\Pi_{\mathcal{G}})$, $\tilde{\alpha} \in \text{Aut}(\Pi_{\mathcal{G}})$ a lifting of α , and $\mathcal{H} \to \mathcal{G}$ a connected finite étale Galois subcovering of $\tilde{\mathcal{G}} \to \mathcal{G}$ such that $\tilde{\alpha}$ preserves the corresponding open subgroup $\Pi_{\mathcal{H}} \subseteq \Pi_{\mathcal{G}}$, hence induces an element $\alpha_{\mathcal{H}} \in \text{Out}(\Pi_{\mathcal{H}})$. Suppose that $\alpha_{\mathcal{H}} \in \text{Dehn}(\mathcal{H})$ [cf. [CbTpI], Definition 4.4]. Then $\alpha \in \text{Dehn}(\mathcal{G})$.

Proof. It follows immediately from [CmbGC], Propositions 1.2, (ii); 1.5, (ii), that $\alpha \in \operatorname{Aut}(\mathcal{G})$. The fact that $\alpha \in \operatorname{Dehn}(\mathcal{G})$ now follows from [CmbGC], Propositions 1.2, (i); 1.5, (i), together with the commensurable terminality of VCN-subgroups of $\Pi_{\mathcal{G}}$ [cf. [CmbGC], Proposition 1.2, (ii)] and the slimness of verticial subgroups of $\Pi_{\mathcal{G}}$ [cf. [CmbGC], Remark 1.1.3]. [Here, we recall that an automorphism of a slim profinite group is equal to the identity if and only if it preserves and induces the identity on an open subgroup.]

Lemma 1.5 (Outer representations of VA-, NN-, PIPSC-type and finite étale coverings). In the notation of Definition 1.3, suppose that I is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma^{\dagger}}$ as an abstract profinite group; let $\widetilde{\rho}_J \colon J \to \operatorname{Aut}(\Pi_{\mathcal{G}})$ be a lifting of the restriction of ρ to an open subgroup $J \subseteq I$ and $\mathcal{H} \to \mathcal{G}$ a connected finite étale Galois subcovering of $\widetilde{\mathcal{G}} \to \mathcal{G}$ such that the action of J on $\Pi_{\mathcal{G}}$, via $\widetilde{\rho}_J$, preserves the corresponding open subgroup $\Pi_{\mathcal{H}} \subseteq \Pi_{\mathcal{G}}$, hence induces a continuous homomorphism $J \to \operatorname{Aut}(\Pi_{\mathcal{H}})$. Then ρ is of VA-type [cf. [NodNon], Definition 2.4, (ii), as well as Remark 1.5.1 below] (respectively, NN-type [cf. [NodNon], Definition 2.4, (iii)]; PIPSC-type [cf. Definition 1.3]) if and only if the composite $J \to \operatorname{Aut}(\Pi_{\mathcal{H}}) \twoheadrightarrow \operatorname{Out}(\Pi_{\mathcal{H}})$ is of VA-type (respectively, NN-type; PIPSC-type).

Proof. Necessity in the case of outer representations of VA-type (respectively, *NN-type*; *PIPSC-type*) follows immediately from [NodNon], Lemma 2.6, (i) (respectively, [NodNon], Lemma 2.6, (i); the various definitions involved, together with the well-known properness of the moduli stack of pointed stable curves of a given type). To verify suf*ficiency*, let us first observe that it follows immediately from the various definitions involved that we may assume without loss of generality that J = I, and that the outer representation $J = I \rightarrow \text{Out}(\Pi_{\mathcal{H}})$ is of SVA-type (respectively, SNN-type; IPSC-type) [cf. [NodNon], Definition 2.4]. Then sufficiency in the case of outer representations of VA-type (respectively, NN-type; PIPSC-type) follows immediately, in light of the criterion of [CbTpI], Corollary 5.9, (i) (respectively, (ii); (iii)), from Lemma 1.4, together with the compatibility property of [CbTpI], Corollary 5.9, (v) [applied, via [CbTpI], Theorem 4.8, (ii), (iv), to each of the *Dehn coordinates* of the profinite Dehn multi-twists under consideration — cf. the proof of [CbTpII], Lemma 3.26, (ii)]. This completes the proof of Lemma 1.5.

Remark 1.5.1. Here, we take the opportunity to correct an *unfortunate misprint* in [NodNon]. The phrase "of VA-type" that appears near the beginning of [NodNon], Definition 2.4, (ii), should read "is of VA-type".

Definition 1.6. Let \mathcal{H} be a semi-graph of anabelioids of pro- Σ^{\dagger} PSCtype. Write \mathbb{H} for the underlying semi-graph of \mathcal{H} , $\Pi_{\mathcal{H}}$ for the [pro- Σ^{\dagger}] fundamental group of \mathcal{H} , and $\widetilde{\mathcal{H}} \to \mathcal{H}$ for the universal covering of \mathcal{H} corresponding to $\Pi_{\mathcal{H}}$. Let $\Pi_{\mathcal{G}}^*$ (respectively, $\Pi_{\mathcal{H}}^*$) be a maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$) [cf. Definition 1.1].

(i) For each $v \in \operatorname{Vert}(\mathcal{G})$ (respectively, $e \in \operatorname{Edge}(\mathcal{G})$; $e \in \operatorname{Node}(\mathcal{G})$; $e \in \operatorname{Cusp}(\mathcal{G})$; $z \in \operatorname{VCN}(\mathcal{G})$), we shall refer to the image of a verticial (respectively, an edge-like; a nodal; a cuspidal; a VCN-[cf. [CbTpI], Definition 2.1, (i)]) subgroup of $\Pi_{\mathcal{G}}$ associated to v (respectively, e; e; e; z) in the quotient $\Pi_{\mathcal{G}}^*$ as a verticial (respectively, an *edge-like*; a *nodal*; a *cuspidal*; a *VCN-*) *subgroup* of $\Pi_{\mathcal{G}}^*$ associated to v (respectively, e; e; e; z). For each element $\tilde{v} \in \operatorname{Vert}(\widetilde{\mathcal{G}})$ (respectively, $\tilde{e} \in \operatorname{Edge}(\widetilde{\mathcal{G}})$; $\tilde{e} \in \operatorname{Node}(\widetilde{\mathcal{G}})$; $\tilde{e} \in \operatorname{Cusp}(\widetilde{\mathcal{G}})$; $\tilde{z} \in \operatorname{VCN}(\widetilde{\mathcal{G}})$), we shall refer to the image of the verticial (respectively, edge-like; nodal; cuspidal; VCN-) subgroup of $\Pi_{\mathcal{G}}$ associated to \tilde{v} (respectively, \tilde{e} ; \tilde{e} ; \tilde{z}) in the quotient $\Pi_{\mathcal{G}}^*$ as the *verticial* (respectively, *edge-like*; *nodal*; *cuspidal*; *VCN-*) *subgroup* of $\Pi_{\mathcal{G}}^*$ associated to \tilde{v} (respectively, $\tilde{e}; \tilde{e}; \tilde{e}; \tilde{z}$).

- (ii) We shall say that an isomorphism $\Pi_{\mathcal{G}}^* \xrightarrow{\sim} \Pi_{\mathcal{H}}^*$ is group-theoretically verticial (respectively, group-theoretically nodal; grouptheoretically cuspidal) if the isomorphism induces a bijection between the set of the verticial (respectively, nodal; cuspidal) subgroups [cf. (i)] of $\Pi_{\mathcal{G}}^*$ and the set of the verticial (respectively, nodal; cuspidal) subgroups of $\Pi_{\mathcal{H}}^*$. We shall say that an outer isomorphism $\Pi_{\mathcal{G}}^* \xrightarrow{\sim} \Pi_{\mathcal{H}}^*$ is group-theoretically verticial (respectively, group-theoretically nodal; group-theoretically cuspidal) if the outer isomorphism arises from an isomorphism $\Pi_{\mathcal{G}}^* \xrightarrow{\sim} \Pi_{\mathcal{H}}^*$ which is group-theoretically verticial (respectively, group-theoretically nodal; group-theoretically cuspidal).
- (iii) We shall say that an isomorphism $\Pi_{\mathcal{G}}^* \xrightarrow{\sim} \Pi_{\mathcal{H}}^*$ is group-theoretically graphic if the isomorphism is group-theoretically verticial, group-theoretically nodal, and group-theoretically cuspidal [cf. (ii)]. We shall say that an outer isomorphism $\Pi_{\mathcal{G}}^* \xrightarrow{\sim} \Pi_{\mathcal{H}}^*$ is group-theoretically graphic if the outer isomorphism arises from an isomorphism $\Pi_{\mathcal{G}}^* \xrightarrow{\sim} \Pi_{\mathcal{H}}^*$ which is group-theoretically graphic. We shall write

$$\operatorname{Aut}^{\operatorname{grph}}(\Pi^*_{\mathcal{C}}) \subseteq \operatorname{Aut}(\Pi^*_{\mathcal{C}})$$

for the subgroup of group-theoretically graphic automorphisms of $\Pi^*_{\mathcal{G}}$ and

$$\operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*) \stackrel{\text{def}}{=} \operatorname{Aut}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*) / \operatorname{Inn}(\Pi_{\mathcal{G}}^*) \subseteq \operatorname{Out}(\Pi_{\mathcal{G}}^*)$$

for the subgroup of group-theoretically graphic outomorphisms of $\Pi^*_{\mathcal{G}}.$

(iv) Let I be a profinite group. Then we shall say that a continuous homomorphism $\rho: I \to \operatorname{Aut}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*) \subseteq \operatorname{Aut}(\Pi_{\mathcal{G}}^*)$ [cf. (iii)] is of VA-type (respectively, NN-type; PIPSC-type) if the following condition is satisfied: Let $N \subseteq \Pi_{\mathcal{G}}$ be a normal open subgroup of $\Pi_{\mathcal{G}}$ with respect to which $\Pi_{\mathcal{G}}^*$ is the maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$. [Thus, $N^{\Sigma} \subseteq \Pi_{\mathcal{G}}^*$.] Then there exists a characteristic open subgroup $M \subseteq \Pi_{\mathcal{G}}^*$ of $\Pi_{\mathcal{G}}^*$ such that the following conditions are satisfied:

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- (1) $M \subseteq N^{\Sigma}$. [Thus, M may be regarded as the [pro- Σ] fundamental group of the pro- Σ completion \mathcal{G}_{M}^{Σ} cf. [SemiAn], Definition 2.9, (ii) of the connected finite étale Galois subcovering $\mathcal{G}_{M} \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$ corresponding to $M \subseteq \Pi_{\mathcal{G}}^{*}$, i.e., $M = \Pi_{\mathcal{G}_{M}^{\Sigma}}$.]
- (2) The composite $I \to \operatorname{Aut}(M) \to \operatorname{Out}(M) = \operatorname{Out}(\Pi_{\mathcal{G}_M^{\Sigma}})$, where the first arrow is the homomorphism induced by ρ , is of VA-type (respectively, NN-type; PIPSC-type) in the sense of [NodNon], Definition 2.4, (ii) [cf. also Remark 1.5.1 of the present paper] (respectively, [NodNon], Definition 2.4, (iii); Definition 1.3 of the present paper) [i.e., as an outer representation of $pro-\Sigma$ PSC-type — cf. [NodNon], Definition 2.1, (i)].

[Here, we observe that it follows immediately from Lemma 1.5 that condition (2) is *independent* of the choice of M — cf. Lemma 1.9 below.] We shall say that a continuous homomorphism $\rho: I \to \operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*) \subseteq \operatorname{Out}(\Pi_{\mathcal{G}}^*)$ [cf. (iii)] is of VA-type (respectively, NN-type; PIPSC-type) if ρ arises from a homomorphism $I \to \operatorname{Aut}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*) \subseteq \operatorname{Aut}(\Pi_{\mathcal{G}}^*)$ which is of VA-type (respectively, NN-type; PIPSC-type). [Here, we observe that it follows immediately from Lemma 1.5, together with the *slimness* of $\Pi_{\mathcal{G}}^*$ [cf. Proposition 1.7, (i), below], that this condition on $\rho: I \to \operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*)$ is *independent* of the choice of the homomorphism $I \to \operatorname{Aut}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*)$.]

(v) Let $\alpha \in \text{Out}(\Pi_{\mathcal{G}}^*)$. Then we shall say that α is a *profinite Dehn* multi-twist of $\Pi_{\mathcal{G}}^*$ if, for each $\widetilde{v} \in \text{Vert}(\widetilde{\mathcal{G}})$, there exists a lifting $\alpha[\widetilde{v}] \in \text{Aut}(\Pi_{\mathcal{G}}^*)$ of α which preserves the verticial subgroup $\Pi_{\widetilde{v}}^* \subseteq \Pi_{\mathcal{G}}^*$ associated to $\widetilde{v} \in \text{Vert}(\widetilde{\mathcal{G}})$ [cf. (i)] and induces the identity automorphism of $\Pi_{\widetilde{v}}^*$. We shall write

$\operatorname{Dehn}(\Pi^*_{\mathcal{G}}) \subseteq \operatorname{Out}(\Pi^*_{\mathcal{G}})$

for the subgroup of profinite Dehn multi-twists of $\Pi_{\mathcal{C}}^*$.

Remark 1.6.1. In the notation of Definition 1.6, if $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{H}}^*$ are the respective maximal almost pro- Σ quotients of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$ with respect to $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$, then it follows immediately from the various definitions involved that $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{H}}^*$ are the respective maximal pro- Σ quotients of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{H}}$. In particular, it follows immediately that one may regard $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{H}}^*$ as the [pro- Σ] fundamental groups of the semi-graphs of anabelioids of pro- Σ PSC-type \mathcal{G}^{Σ} , \mathcal{H}^{Σ} obtained by forming the pro- Σ completions [cf. [SemiAn] Definition 2.9, (ii)] of \mathcal{G} , \mathcal{H} , respectively, i.e., $\Pi_{\mathcal{G}}^* = \Pi_{\mathcal{G}^{\Sigma}}$, $\Pi_{\mathcal{H}}^* = \Pi_{\mathcal{H}^{\Sigma}}$. Moreover, one verifies immediately that, relative to these identifications, the notions defined in Definition 1.6, (i), (ii), (iii), (iv), are *compatible* with their counterparts defined [for the most part] in earlier papers of the authors:

- VCN-subgroups [cf. [CbTpI], Definition 2.1, (i)];
- group-theoretically verticial/nodal/cuspidal/graphic (outer) isomorphisms [cf. [CmbGC], Definition 1.4, (i), (iv); [NodNon], Definition 1.12];
- outer representations of VA-/NN-/PIPSC-type [cf. [NodNon], Definition 2.4, (ii), (iii); Remark 1.5.1 of the present paper; Definition 1.3 of the present paper; Lemma 1.5 of the present paper];
- profinite Dehn multi-twists [cf. [CbTpI], Definition 4.4], i.e., so $\text{Dehn}(\mathcal{G}^{\Sigma}) = \text{Dehn}(\Pi_{\mathcal{G}}^*) \subseteq \text{Out}^{\text{grph}}(\Pi_{\mathcal{G}}^*).$

Remark 1.6.2. In the situation of Definition 1.6, (iv), it follows immediately from Lemma 1.5, together with [NodNon], Remark 2.4.2, that we have implications

$$PIPSC-type \implies NN-type \implies VA-type.$$

Proposition 1.7 (Properties of VCN-subgroups). Let $\Pi_{\mathcal{G}}^*$ be a maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$ [cf. Definition 1.1]. For \tilde{v} , $\tilde{w} \in \operatorname{Vert}(\tilde{\mathcal{G}})$; $\tilde{e} \in \operatorname{Edge}(\tilde{\mathcal{G}})$, write $\mathcal{G}^* \to \mathcal{G}$ for the connected profinite étale subcovering of $\tilde{\mathcal{G}} \to \mathcal{G}$ corresponding to $\Pi_{\mathcal{G}}^*$;

$$\operatorname{Vert}(\mathcal{G}^*) \stackrel{\text{def}}{=} \operatorname{\underline{\lim}} \operatorname{Vert}(\mathcal{G}'), \ \operatorname{Edge}(\mathcal{G}^*) \stackrel{\text{def}}{=} \operatorname{\underline{\lim}} \operatorname{Edge}(\mathcal{G}')$$

— where the projective limits range over all connected finite étale subcoverings $\mathcal{G}' \to \mathcal{G}$ of $\mathcal{G}^* \to \mathcal{G}$;

$$\widetilde{v}(\mathcal{G}^*) \in \operatorname{Vert}(\mathcal{G}^*), \ \widetilde{e}(\mathcal{G}^*) \in \operatorname{Edge}(\mathcal{G}^*)$$

for the images of $\widetilde{v} \in \operatorname{Vert}(\widetilde{\mathcal{G}}), \ \widetilde{e} \in \operatorname{Edge}(\widetilde{\mathcal{G}})$ via the natural maps $\operatorname{Vert}(\widetilde{\mathcal{G}}) \twoheadrightarrow \operatorname{Vert}(\mathcal{G}^*), \operatorname{Edge}(\widetilde{\mathcal{G}}) \twoheadrightarrow \operatorname{Edge}(\mathcal{G}^*), respectively;$

$$\mathcal{E}_{\mathcal{G}^*} \colon \operatorname{Vert}(\mathcal{G}^*) \longrightarrow 2^{\operatorname{Edge}(\mathcal{G}^*)}$$

[cf. the discussion entitled "Sets" in [CbTpI], §0, concerning the notation $2^{\text{Edge}(\mathcal{G}^*)}$] for the map induced by the various \mathcal{E} 's involved [cf. [NodNon], Definition 1.1, (iv)];

$$\delta(\widetilde{v}(\mathcal{G}^*), \widetilde{w}(\mathcal{G}^*)) \stackrel{\text{def}}{=} \sup_{\mathcal{G}'} \left\{ \delta(\widetilde{v}(\mathcal{G}'), \widetilde{w}(\mathcal{G}')) \right\} \in \mathbb{N} \cup \{\infty\}$$

[cf. [NodNon], Definition 1.1, (vii)] — where \mathcal{G}' ranges over the connected finite étale subcoverings $\mathcal{G}' \to \mathcal{G}$ of $\mathcal{G}^* \to \mathcal{G}$. Then the following hold:

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- (i) Π^{*}_G is topologically finitely generated, slim [cf. the discussion entitled "Topological groups" in [CbTpI], §0], and almost torsion-free [cf. the discussion entitled "Topological groups" in [CbTpI], §0]. In particular, every VCN-subgroup of Π^{*}_G [cf. Definition 1.6, (i)] is almost torsion-free.
- (ii) Let $z \in VCN(\mathcal{G})$ and $\Pi_z \subseteq \Pi_{\mathcal{G}}$ a VCN-subgroup of $\Pi_{\mathcal{G}}$ associated to $z \in VCN(\mathcal{G})$. Write $\Pi_z^* \subseteq \Pi_{\mathcal{G}}^*$ for the VCN-subgroup of $\Pi_{\mathcal{G}}^*$ obtained by forming the image of $\Pi_z \subseteq \Pi_{\mathcal{G}}$ in $\Pi_{\mathcal{G}}^*$. Then Π_z^* is a maximal almost pro- Σ quotient of Π_z . In particular, every verticial subgroup of $\Pi_{\mathcal{G}}^*$ is topologically finitely generated and slim.
- (iii) For i = 1, 2, let $\widetilde{v}_i \in \text{Vert}(\mathcal{G})$. Write $\Pi^*_{\widetilde{v}_i} \subseteq \Pi^*_{\mathcal{G}}$ for the verticial subgroup of $\Pi^*_{\mathcal{G}}$ associated to \widetilde{v}_i . Consider the following three [mutually exclusive] conditions:
 - (1) $\delta(\widetilde{v}_1(\mathcal{G}^*), \widetilde{v}_2(\mathcal{G}^*)) = 0.$
 - (2) $\delta(\widetilde{v}_1(\mathcal{G}^*), \widetilde{v}_2(\mathcal{G}^*)) = 1.$
 - (3) $\delta(\widetilde{v}_1(\mathcal{G}^*), \widetilde{v}_2(\mathcal{G}^*)) \ge 2.$

Then we have equivalences

$$(1) \iff (1') ; (2) \iff (2') ; (3) \iff (3')$$

with the following three conditions:

- (1') $\Pi^*_{\tilde{v}_1} = \Pi^*_{\tilde{v}_2}.$
- (2') $\Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2}$ is infinite, but $\Pi^*_{\widetilde{v}_1} \neq \Pi^*_{\widetilde{v}_2}$.
- (3') $\Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2}$ is finite.
- (iv) In the situation of (iii), if condition (2), hence also condition (2), holds, then it holds that $(\mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_1(\mathcal{G}^*)) \cap \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_2(\mathcal{G}^*)))^{\sharp} = 1$, and, moreover, $\Pi^*_{\widetilde{v}_1} \cap \Pi^*_{\widetilde{v}_2} = \Pi^*_{\widetilde{e}}$, for any element $\widetilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ such that $\widetilde{e}(\mathcal{G}^*) \in \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_1(\mathcal{G}^*)) \cap \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}_2(\mathcal{G}^*))$.
- (v) For i = 1, 2, let $\tilde{e}_i \in \text{Edge}(\tilde{\mathcal{G}})$. Write $\Pi^*_{\tilde{e}_i} \subseteq \Pi^*_{\mathcal{G}}$ for the edge-like subgroup of $\Pi^*_{\mathcal{G}}$ associated to \tilde{e}_i . Then $\Pi^*_{\tilde{e}_1} \cap \Pi^*_{\tilde{e}_2}$ is infinite if and only if $\tilde{e}_1(\mathcal{G}^*) = \tilde{e}_2(\mathcal{G}^*)$. In particular, $\Pi^*_{\tilde{e}_1} \cap \Pi^*_{\tilde{e}_2}$ is infinite if and only if $\Pi^*_{\tilde{e}_1} = \Pi^*_{\tilde{e}_2}$.
- (vi) Let $\widetilde{v} \in \operatorname{Vert}(\widetilde{\mathcal{G}})$, $\widetilde{e} \in \operatorname{Edge}(\widetilde{\mathcal{G}})$. Write $\Pi_{\widetilde{v}}^*$, $\Pi_{\widetilde{e}}^* \subseteq \Pi_{\mathcal{G}}^*$ for the VCN-subgroups of $\Pi_{\mathcal{G}}^*$ associated to \widetilde{v} , \widetilde{e} , respectively. Then $\Pi_{\widetilde{e}}^* \cap \Pi_{\widetilde{v}}^*$ is infinite if and only if $\widetilde{e}(\mathcal{G}^*) \in \mathcal{E}_{\mathcal{G}^*}(\widetilde{v}(\mathcal{G}^*))$. In particular, $\Pi_{\widetilde{e}}^* \cap \Pi_{\widetilde{v}}^*$ is infinite if and only if $\Pi_{\widetilde{e}}^* \subseteq \Pi_{\widetilde{v}}^*$.
- (vii) Every VCN-subgroup of $\Pi_{\mathcal{G}}^*$ is commensurably terminal [cf. the discussion entitled "Topological groups" in [CbTpI], §0] in $\Pi_{\mathcal{G}}^*$.

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(viii) Let $z \in VCN(\mathcal{G})$, $\Pi_z \subseteq \Pi_{\mathcal{G}}$ a VCN-subgroup of $\Pi_{\mathcal{G}}$ associated to $z \in VCN(\mathcal{G})$, and $\Pi_z \twoheadrightarrow \Pi_z^{\ddagger}$ an almost pro- Σ quotient of Π_z^{\ddagger} . Then there exists a maximal almost pro- Σ quotient $\Pi_{\mathcal{G}}^{\ast\ast}$ of $\Pi_{\mathcal{G}}$ such that the quotient of Π_z determined by the quotient $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\ast\ast}$ dominates the quotient $\Pi_z \twoheadrightarrow \Pi_z^{\ddagger}$ [cf. the discussion entitled "Topological groups" in §0].

Proof. Let $N \subseteq \Pi_{\mathcal{G}}$ be a normal open subgroup of $\Pi_{\mathcal{G}}$ with respect to which $\Pi_{\mathcal{G}}^*$ is the maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$. [Thus, $N^{\Sigma} \subseteq \Pi_{\mathcal{G}}^*$.] Write $\mathcal{G}_N \to \mathcal{G}$ for the connected finite étale Galois subcovering of $\widetilde{\mathcal{G}} \to \mathcal{G}$ corresponding to $N \subseteq \Pi_{\mathcal{G}}$. Thus, $N^{\Sigma} \subseteq \Pi_{\mathcal{G}}^*$ may be regarded as the [pro- Σ] fundamental group of the pro- Σ completion \mathcal{G}_N^{Σ} [cf. [SemiAn], Definition 2.9, (ii)] of \mathcal{G}_N , i.e., $N^{\Sigma} = \Pi_{\mathcal{G}_N^{\Sigma}}$.

First, we verify assertion (i). Since $\Pi_{\mathcal{G}}$ is topologically finitely generated, it is immediate that $\Pi_{\mathcal{G}}^*$ is topologically finitely generated. Now let us recall [cf. [MzTa], Remark 1.2.2; [MzTa], Proposition 1.4] that $N^{\Sigma} = \Pi_{\mathcal{G}_N^{\Sigma}}$ is torsion-free and slim. Thus, the fact that $\Pi_{\mathcal{G}}^*$ is almost torsion-free is immediate; the slimness of $\Pi_{\mathcal{G}}^*$ follows immediately, by considering the natural outer action $\Pi_{\mathcal{G}}/N \to \operatorname{Out}(N^{\Sigma})$, from the well-known fact that any nontrivial automorphism of a stable log curve over an algebraically closed field of characteristic $\notin \Sigma$ induces a nontrivial outomorphism of the maximal pro- Σ quotient of the geometric log fundamental group of the stable curve [cf. [CmbGC], Proposition 1.2, (i), (ii); [MzTa], Proposition 1.4, applied to the verticial subgroups of the geometric log fundamental group under consideration]. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us recall that since $\Pi_z \cap N \subseteq N = \Pi_{\mathcal{G}_N}$ is a VCN-subgroup of $\Pi_{\mathcal{G}_N}$, the natural homomorphism $(\Pi_z \cap N)^{\Sigma} \to \Pi_{\mathcal{G}_N}^{\Sigma}$ is *injective* [cf., e.g., the proof of [SemiAn], Proposition 2.5, (i); [SemiAn], Example 2.10]. Thus, it follows immediately from Lemma 1.2, (ii), that Π_z^* is a maximal almost pro- Σ quotient of Π_z . In particular, if $z \in \operatorname{Vert}(\mathcal{G})$, then it follows immediately from assertion (i) that Π_z^* is topologically finitely generated and slim. This completes the proof of assertion (ii).

Next, we verify assertions (iii), (v), and (vi). Since $N^{\Sigma} = \Pi_{\mathcal{G}_{N}^{\Sigma}}$, one verifies easily — by considering the intersections of $N^{\Sigma} = \Pi_{\mathcal{G}_{N}^{\Sigma}}$ with the various VCN-subgroups of $\Pi_{\mathcal{G}}^{*}$ under consideration and applying [NodNon], Lemma 1.9, (ii) (respectively, [NodNon], Lemma 1.5; [NodNon], Lemma 1.7), together with the well-known fact that every VCN-subgroup of $\Pi_{\mathcal{G}_{N}^{\Sigma}}$ is *nontrivial* and *torsion-free* [hence also *infinite*] — that assertion (iii) (respectively, (v); (vi)) holds. This completes the proof of assertions (iii), (v), and (vi). Assertion (vii) follows formally from assertions (iii), (v). Indeed, let $\Pi_{\tilde{z}}^{*} \subseteq \Pi_{\mathcal{G}}^{*}$ be the VCN-subgroup of $\Pi_{\mathcal{G}}^{*}$ associated to an element $\tilde{z} \in \text{VCN}(\tilde{\mathcal{G}})$ and $\gamma \in C_{\Pi_{\mathcal{C}}^{*}}(\Pi_{\tilde{z}}^{*})$. Then it follows immediately from assertions (iii), (v) that $\tilde{z} = \tilde{z}^{\gamma}$; we thus conclude that $\gamma \in \Pi_{\tilde{z}}^*$. This completes the proof of assertion (vii).

Next, we verify assertion (iv). By applying [NodNon], Lemma 1.8, to \mathcal{G}_N^{Σ} , one verifies immediately that $(\mathcal{E}_{\mathcal{G}^*}(\tilde{v}_1(\mathcal{G}^*)) \cap \mathcal{E}_{\mathcal{G}^*}(\tilde{v}_2(\mathcal{G}^*)))^{\sharp} = 1$. Thus, it follows immediately from assertion (vi); [NodNon], Lemma 1.5; [NodNon], Lemma 1.9, (ii), that $\Pi_{\tilde{e}}^* \cap N^{\Sigma} = \Pi_{\tilde{v}_1}^* \cap \Pi_{\tilde{v}_2}^* \cap N^{\Sigma}$. Since N^{Σ} is open in $\Pi_{\mathcal{G}}^*$, we conclude from assertion (vi) that $\Pi_{\tilde{e}}^*$ is an open subgroup of $\Pi_{\tilde{v}_1}^* \cap \Pi_{\tilde{v}_2}^*$, hence that $\Pi_{\tilde{v}_1}^* \cap \Pi_{\tilde{v}_2}^* \subseteq C_{\Pi_{\mathcal{G}}^*}(\Pi_{\tilde{e}}^*) = \Pi_{\tilde{e}}^*$ [cf. assertion (vii)]. This completes the proof of assertion (iv).

Finally, we verify assertion (viii). It follows from the definition of an almost pro- Σ quotient that the natural surjection $\Pi_z \twoheadrightarrow \Pi_z^{\ddagger}$ factors through a maximal almost pro- Σ quotient of Π_z . Thus, by replacing Π_z^{\ddagger} by a suitable maximal almost pro- Σ quotient of Π_z , we may assume without loss of generality that Π_z^{\ddagger} is a maximal almost pro- Σ quotient of Π_z . Let $N_z \subseteq \Pi_z$ be a normal open subgroup of Π_z with respect to which Π_z^{\ddagger} is the maximal almost pro- Σ quotient of Π_z and $N_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}$ a normal open subgroup of $\Pi_{\mathcal{G}}$ such that $N_{\mathcal{G}} \cap \Pi_z \subseteq N_z$. Here, we recall that the existence of such a subgroup $N_{\mathcal{G}}$ follows immediately from the fact that the natural profinite topology on Π_z coincides with the topology on Π_z induced by the topology of $\Pi_{\mathcal{G}}$. Then one verifies easily that the maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$ as in the statement of assertion (viii). This completes the proof of assertion (viii).

Definition 1.8. Let $\Pi_{\mathcal{G}}^*$ be a maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$ [cf. Definition 1.1]. Then we shall write

 $\operatorname{Aut}^{|\operatorname{grph}|}(\Pi^*_{\mathcal{G}}) \subseteq \operatorname{Aut}^{\operatorname{grph}}(\Pi^*_{\mathcal{G}})$

for the subgroup of group-theoretically graphic [cf. Definition 1.6, (iii)] automorphisms α of $\Pi_{\mathcal{G}}^*$ such that the natural action of α on the underlying semi-graph \mathbb{G} [determined by the group-theoretic graphicity of α , together with Proposition 1.7, (iii), (v), (vi)] is the identity automorphism. Also, we shall write

$$\operatorname{Out}^{|\operatorname{grph}|}(\Pi_{\mathcal{G}}^*) \stackrel{\text{def}}{=} \operatorname{Aut}^{|\operatorname{grph}|}(\Pi_{\mathcal{G}}^*) / \operatorname{Inn}(\Pi_{\mathcal{G}}^*) \subseteq \operatorname{Out}(\Pi_{\mathcal{G}}^*).$$

for the image of Aut^{|grph|}($\Pi_{\mathcal{C}}^*$) in Out($\Pi_{\mathcal{C}}^*$).

Remark 1.8.1. In the notation of Definition 1.8, one verifies easily that

$$\operatorname{Dehn}(\Pi^*_{\mathcal{G}}) \subseteq \operatorname{Out}^{|\operatorname{grph}|}(\Pi^*_{\mathcal{G}})$$

[cf. Definitions 1.6, (v); 1.8; [CmbGC], Proposition 1.2, (i)].

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Remark 1.8.2. In the spirit of Remark 1.6.1, one verifies immediately that the notation of Definition 1.8 is *consistent* with the the notation of [CbTpI], Definition 2.6, (i) [cf. also [CbTpII], Remark 4.1.2].

Lemma 1.9 (Alternative characterization of outer representations of VA-, NN-, PIPSC-type). Let $\Pi_{\mathcal{G}}^*$ be a maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$ [cf. Definition 1.1], I a profinite group, and $\rho: I \to \operatorname{Aut}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*)$ a continuous homomorphism. Then the following conditions are equivalent:

- (i) ρ is of VA-type (respectively, NN-type; PIPSC-type) [cf. Definition 1.6, (iv)].
- (ii) Let $N \subseteq \Pi_{\mathcal{G}}$ be a normal open subgroup of $\Pi_{\mathcal{G}}$ with respect to which $\Pi_{\mathcal{G}}^*$ is the maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$. [Thus, $N^{\Sigma} \subseteq \Pi_{\mathcal{G}}^*$.] Let $M \subseteq \Pi_{\mathcal{G}}^*$ be a characteristic open subgroup of $\Pi_{\mathcal{G}}^*$ such that $M \subseteq N^{\Sigma}$. [Thus, M may be regarded as the [pro- Σ] fundamental group of the pro- Σ completion \mathcal{G}_{M}^{Σ} cf. [SemiAn], Definition 2.9, (ii) — of the connected finite étale Galois subcovering $\mathcal{G}_{M} \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$ corresponding to $M \subseteq \Pi_{\mathcal{G}}^*$, i.e., $M = \Pi_{\mathcal{G}_{M}^{\Sigma}}$.] Then it holds that the composite of the resulting homomorphism $I \to \operatorname{Aut}(M) = \operatorname{Aut}(\Pi_{\mathcal{G}_{M}^{\Sigma}})$ with the natural projection $\operatorname{Aut}(\Pi_{\mathcal{G}_{M}^{\Sigma}}) \to \operatorname{Out}(\Pi_{\mathcal{G}_{M}^{\Sigma}})$ is an outer representation of VA-type (respectively, NN-type; PIPSCtype) in the sense of [NodNon], Definition 2.4, (ii) [cf. also Remark 1.5.1 of the present paper] (respectively, [NodNon], Definition 2.4, (iii); Definition 1.3 of the present paper).

Proof. The implication (ii) \Rightarrow (i) is immediate; the implication (i) \Rightarrow (ii) follows immediately from Lemma 1.5. This completes the proof of Lemma 1.9.

Lemma 1.10 (Automorphisms of semi-graphs of anabelioids of PSC-type with prescribed underlying semi-graphs). Let $\Pi_{\mathcal{G}}^*$ be a maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}}$ [cf. Definition 1.1] and $\alpha \in \operatorname{Out}(\Pi_{\mathcal{G}}^*)$. Suppose that there exist distinct elements $v_1, v_2,$ $v_3 \in \operatorname{Vert}(\mathcal{G})$; $e_1, e_2 \in \operatorname{Node}(\mathcal{G})$ such that $\operatorname{Vert}(\mathcal{G}) = \{v_1, v_2, v_3\}$; $\operatorname{Node}(\mathcal{G}) = \{e_1, e_2\}; \mathcal{V}(e_i) = \{v_i, v_{i+1}\}$ [where $i \in \{1, 2\}$]. For each $i \in \{1, 2\}$, write $\Pi_{\mathcal{G}_{\rightarrow}\{e_i\}}^*$ for the maximal almost pro- Σ quotient of $\Pi_{\mathcal{G}_{\rightarrow}\{e_i\}}$ [cf. [CbTpI], Definition 2.8] determined by the natural outer isomorphism $\Phi_{\mathcal{G}_{\rightarrow}\{e_i\}} : \Pi_{\mathcal{G}_{\rightarrow}\{e_i\}} \xrightarrow{\sim} \Pi_{\mathcal{G}}$ [cf. [CbTpI], Definition 2.10] and the maximal almost pro- Σ quotient $\Pi_{\mathcal{G}}^*$ of $\Pi_{\mathcal{G}}; \Phi_{\mathcal{G}_{\rightarrow}\{e_i\}}^* : \Pi_{\mathcal{G}_{\rightarrow}\{e_i\}} \xrightarrow{\sim} \Pi_{\mathcal{G}}^*$ for the outer isomorphism determined by $\Phi_{\mathcal{G}_{\rightarrow}\{e_i\}}$. Suppose, moreover, that, for each $i \in \{1, 2\}$, the outomorphism of $\Pi^*_{\mathcal{G}_{\sim}\{e_i\}}$ obtained by conjugating α by $\Phi^*_{\mathcal{G}_{\sim}\{e_i\}}$ is a **profinite Dehn multi-twist** of $\Pi^*_{\mathcal{G}_{\sim}\{e_i\}}$ [cf. Definition 1.6, (v)]. Then α is the **identity outomorphism**.

Proof. First, let us observe that it follows immediately from the definition of a profinite Dehn multi-twist that α is a profinite Dehn multitwist of $\Pi_{\mathcal{G}}^*$. Let us fix a verticial subgroup $\Pi_{v_2} \subseteq \Pi_{\mathcal{G}}$ associated to v_2 . Let $\Pi_{e_1}, \Pi_{e_2} \subseteq \Pi_{\mathcal{G}}$ be nodal subgroups associated to e_1, e_2 , respectively, which are contained in $\Pi_{v_2}; \Pi_{v_1} \subseteq \Pi_{\mathcal{G}}$ a verticial subgroup associated to v_1 which contains $\Pi_{e_1}; \Pi_{v_3} \subseteq \Pi_{\mathcal{G}}$ a verticial subgroup associated to v_3 which contains Π_{e_2} . Thus, we have inclusions

$$\Pi_{v_1} \supseteq \Pi_{e_1} \subseteq \Pi_{v_2} \supseteq \Pi_{e_2} \subseteq \Pi_{v_3} \,.$$

For each $z \in \{v_1, v_2, v_3, e_1, e_2\}$, write $\prod_z^* \subseteq \prod_{\mathcal{G}}^*$ for the VCN-subgroup of $\Pi^*_{\mathcal{G}}$ associated to z obtained by forming the image of $\Pi_z \subseteq \Pi_{\mathcal{G}}$ in $\Pi_{\mathcal{G}}^*$. Then since α is a profinite Dehn multi-twist, there exists a lifting $\alpha[v_2] \in \operatorname{Aut}(\Pi^*_{\mathcal{G}})$ of α which preserves and induces the *identity auto*morphism of $\Pi_{v_2}^*$; in particular, $\alpha[v_2]$ preserves and induces the *identity* automorphisms of $\Pi_{e_1}^*$, $\Pi_{e_2}^*$. Moreover, by applying a similar argument to the argument given in the proof of [CbTpI], Lemma 4.6, (i), where we replace [CmbGC], Remark 1.1.3 (respectively, [CmbGC], Proposition 1.2, (ii); [CbTpI], Proposition 4.5; [NodNon], Lemma 1.7), in the proof of [CbTpI], Lemma 4.6, (i), by Proposition 1.7, (ii) (respectively, Proposition 1.7, (vii); Remark 1.8.1; Proposition 1.7, (vi)), we conclude that $\alpha[v_2](\Pi_{v_1}^*) = \Pi_{v_1}^*, \, \alpha[v_2](\Pi_{v_3}^*) = \Pi_{v_3}^*$, and, moreover, that there exist unique elements $\gamma_1 \in \Pi_{e_1}^*, \ \gamma_2 \in \Pi_{e_2}^*$ such that the restrictions of $\alpha[v_2]$ to $\Pi_{v_1}^*$, $\Pi_{v_3}^*$ are the inner automorphisms determined by γ_1 , γ_2 , respectively. Thus, since, for each $i \in \{1, 2\}$, the outomorphism of $\Pi^*_{\mathcal{G}_{\neg \{e_i\}}}$ obtained by conjugating α by $\Phi^*_{\mathcal{G}_{\neg \{e_i\}}}$ is a *profinite Dehn multi*twist of $\Pi^*_{\mathcal{G}_{\to\{e_i\}}}$, one verifies easily — by considering the restriction of this outomorphism of $\Pi^*_{\mathcal{G}_{\to\{e_i\}}}$ to the *unique* conjugacy class of verticial subgroups of $\Pi^*_{\mathcal{G}_{\neg \neg \{e_i\}}}$ that does *not* arise from a conjugacy class of verticial subgroups of $\Pi_{\mathcal{C}}^*$ [cf. also Proposition 1.7, (ii), (vii)] — that γ_1 and γ_2 are *trivial*. On the other hand, it follows immediately from a similar argument to the argument applied in the proof of [CmbCsp], Proposition 1.5, (iii), that $\Pi_{\mathcal{G}}$ is topologically generated by Π_{v_1} , Π_{v_2} , and Π_{v_3} ; in particular, $\Pi^*_{\mathcal{G}}$ is topologically generated by $\Pi^*_{v_1}$, $\Pi^*_{v_2}$, and $\Pi_{v_3}^*$. Thus, we conclude that $\alpha[v_2]$ is the *identity automorphism* of $\Pi_{\mathcal{G}}^*$. This completes the proof of Lemma 1.10.

Theorem 1.11 (Group-theoretic verticiality/nodality of isomorphisms of outer representations of NN-, PIPSC-type). Let $\Sigma \subseteq \Sigma^{\dagger}$ be nonempty sets of prime numbers, \mathcal{G} (respectively, \mathcal{H}) a semi-graph of anabelioids of pro- Σ^{\dagger} PSC-type, $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$) the [pro- Σ^{\dagger}] fundamental group of \mathcal{G} (respectively, \mathcal{H}), $\Pi_{\mathcal{G}}^{*}$ (respectively, $\Pi_{\mathcal{H}}^{*}$) a maximal almost pro- Σ quotient [cf. Definition 1.1] of $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$), $\alpha \colon \Pi_{\mathcal{G}}^{*} \xrightarrow{\sim} \Pi_{\mathcal{H}}^{*}$ an isomorphism of profinite groups, I(respectively, J) a profinite group, $\rho_{I} \colon I \to \operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^{*})$ (respectively, $\rho_{J} \colon J \to \operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{H}}^{*})$) [cf. Definition 1.6, (iii)] a continuous homomorphism, and $\beta \colon I \xrightarrow{\sim} J$ an isomorphism of profinite groups. Suppose that the diagram

— where the right-hand vertical arrow is the isomorphism obtained by conjugating by α — commutes. Then the following hold:

- (i) Suppose, moreover, that ρ_I, ρ_J are of NN-type [cf. Definition 1.6, (iv)]. Then the following three conditions are equivalent:
 - The isomorphism α is group-theoretically verticial [i.e., roughly speaking, preserves verticial subgroups — cf. Definition 1.6, (ii)].
 - (2) The isomorphism α is group-theoretically nodal [i.e., roughly speaking, preserves nodal subgroups — cf. Definition 1.6, (ii)].
 - (3) There exists an infinite subgroup $H \subseteq \Pi_{\mathcal{G}}^*$ of $\Pi_{\mathcal{G}}^*$ such that $H \subseteq \Pi_{\mathcal{G}}^*$, $\alpha(H) \subseteq \Pi_{\mathcal{H}}^*$ are contained in verticial subgroups of $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{H}}^*$, respectively [cf. Definition 1.6, (i)].
- (ii) Suppose, moreover, that ρ_I is of **NN-type**, and that ρ_J is of **PIPSC-type** [cf. Definition 1.6, (iv)]. [For example, this will be the case if both ρ_I and ρ_J are of **PIPSC-type** cf. Remark 1.6.2.] Then α is group-theoretically verticial, hence also group-theoretically nodal.

Proof. The implication $(1) \Rightarrow (2)$ of assertion (i) and the final portion of assertion (ii) [i.e., the portion concerning group-theoretic nodality] follow immediately from Proposition 1.7, (iv). The implication $(2) \Rightarrow (3)$ of assertion (i) is immediate. Finally, we verify assertion (ii) (respectively, the implication $(3) \Rightarrow (1)$ of assertion (i)). Suppose that ρ_I , ρ_J are as in assertion (ii) (respectively, condition (3) of assertion (i)). Let $N_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}, N_{\mathcal{H}} \subseteq \Pi_{\mathcal{H}}$ be normal open subgroups of $\Pi_{\mathcal{G}}, \Pi_{\mathcal{H}}$ with respect to which $\Pi_{\mathcal{G}}^*, \Pi_{\mathcal{H}}^*$ are the maximal almost pro- Σ quotients of $\Pi_{\mathcal{G}}, \Pi_{\mathcal{H}}$, respectively. [Thus, $N_{\mathcal{G}}^{\Sigma} \subseteq \Pi_{\mathcal{G}}^*, N_{\mathcal{H}}^{\Sigma} \subseteq \Pi_{\mathcal{H}}^*$.] Now it follows immediately from the fact that $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{H}}^*$ are topologically finitely generated [cf. Proposition 1.7, (i)] that there exists a characteristic open subgroup $M_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}^*$ of $\Pi_{\mathcal{G}}^*$ such that $M_{\mathcal{G}} \subseteq N_{\mathcal{G}}^{\Sigma}$, $M_{\mathcal{H}} \stackrel{\text{def}}{=} \alpha(M_{\mathcal{G}}) \subseteq N_{\mathcal{H}}^{\Sigma}$. Thus, it follows immediately, in light of Lemma 1.9, from [CbTpII], Theorem 1.9, (ii) (respectively, the implication (3) \Rightarrow (1) of [CbTpII], Theorem 1.9, (i)), together with Proposition 1.7, (vii), that α is grouptheoretically verticial. This completes the proof of Theorem 1.11.

Corollary 1.12 (Group-theoretic graphicity of group-theoretically cuspidal isomorphisms of outer representations of NN-, PIPSC-type). Let $\Sigma \subseteq \Sigma^{\dagger}$ be nonempty sets of prime numbers, \mathcal{G} (respectively, \mathcal{H}) a semi-graph of anabelioids of pro- Σ^{\dagger} PSC-type, $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$) the [pro- Σ^{\dagger}] fundamental group of \mathcal{G} (respectively, \mathcal{H}), $\Pi_{\mathcal{G}}^*$ (respectively, $\Pi_{\mathcal{H}}$) a maximal almost pro- Σ quotient [cf. Definition 1.1] of $\Pi_{\mathcal{G}}$ (respectively, $\Pi_{\mathcal{H}}$), $\alpha \colon \Pi_{\mathcal{G}}^* \xrightarrow{\sim} \Pi_{\mathcal{H}}^*$ an isomorphism of profinite groups, I (respectively, J) a profinite group, $\rho_I \colon I \rightarrow$ $\operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{G}}^*)$ (respectively, $\rho_J \colon J \rightarrow \operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{H}}^*)$) [cf. Definition 1.6, (iii)] a continuous homomorphism, and $\beta \colon I \xrightarrow{\sim} J$ an isomorphism of profinite groups. Suppose that the following conditions are satisfied:

(i) The diagram

— where the right-hand vertical arrow is the isomorphism obtained by conjugating by α — commutes.

- (ii) α is group-theoretically cuspidal [cf. Definition 1.6, (ii)].
- (iii) ρ_I , ρ_J are of **NN-type** [cf. Definition 1.6, (iv)].

Suppose, moreover, that one of the following conditions is satisfied:

- (1) $\operatorname{Cusp}(\mathcal{G}) \neq \emptyset$.
- (2) Either ρ_I or ρ_J is of **PIPSC-type** [cf. Definition 1.6, (iv)].

Then α is group-theoretically graphic [cf. Definition 1.6, (iii)].

Proof. This follows immediately from Theorem 1.11, (i) (respectively, (ii)), whenever condition (1) (respectively, (2)) is satisfied. \Box

2. Almost pro- Σ injectivity

In the present §2, we develop an almost $pro-\Sigma$ version of the injectivity portion of the theory of combinatorial cuspidalization [cf. Theorem 2.9, Corollary 2.10 below]. We also discuss an almost proanalogue [cf. Corollary 2.13 below] of the tripod homomorphism of [CbTpII], Definition 3.19.

In the present §2, let Σ be a nonempty set of prime numbers.

Definition 2.1. Let l be a prime number; n a positive integer; (g, r) a pair of nonnegative integers such that 2g - 2 + r > 0; k an algebraically closed field of characteristic zero; $(\operatorname{Spec} k)^{\log}$ the log scheme obtained by equipping $\operatorname{Spec} k$ with the log structure determined by the fs chart $\mathbb{N} \to$ k that maps $1 \mapsto 0$; X^{\log} a stable log curve of type (g, r) over $(\operatorname{Spec} k)^{\log}$. For each positive integer i, write X_i^{\log} for the i-th log configuration space of X^{\log} [cf. the discussion entitled "Curves" in [CbTpII], §0]; Π_i for the pro-**\mathfrak{Primes}** configuration space group [cf. [ExtFam], Theorem B; [MzTa], Definition 2.3, (i)] given by the kernel of the natural outer surjection $\pi_1(X_i^{\log}) \twoheadrightarrow \pi_1((\operatorname{Spec} k)^{\log})$. Let $\Pi_n \twoheadrightarrow \Pi_n^*$ be a quotient of Π_n . Write

$$\{1\} = \prod_{n/n} \subseteq \prod_{n/n-1} \subseteq \cdots \subseteq \prod_{n/m} \subseteq \cdots \subseteq \prod_{n/2} \subseteq \prod_{n/1} \subseteq \prod_{n/0} = \prod_n$$

for the standard fiber filtration on Π_n — i.e., $\Pi_{n/m} \subseteq \Pi_n$ is the kernel of some fixed surjection [that belongs to the collection of surjections that constitutes the outer surjection] $p_{n/m}^{\Pi} \colon \Pi_n \twoheadrightarrow \Pi_m$ induced by the projection $p_{n/m}^{\log} \colon X_n^{\log} \to X_m^{\log}$ obtained by forgetting the factors labeled by indices > m [cf. [CmbCsp], Definition 1.1, (i)];

 $\{1\} = \prod_{n/n}^* \subseteq \prod_{n/n-1}^* \subseteq \cdots \subseteq \prod_{n/m}^* \subseteq \cdots \subseteq \prod_{n/2}^* \subseteq \prod_{n/1}^* \subseteq \prod_{n/0}^* = \prod_n^*$ for the induced filtration on \prod_n^* .

- (i) For each $1 \le m \le n$, we shall refer to the subquotient $\Pi_{n/m-1}^*/\Pi_{n/m}^*$ of Π_n^* as a standard-adjacent subquotient of Π_n^* .
- (ii) We shall say that Π_n^* is an *SA-maximal almost pro-l quotient* of Π_n [where the "SA" stands for "standard-adjacent"] if, for every $1 \le m \le n$, the natural quotient $\Pi_{n/m-1}/\Pi_{n/m} \twoheadrightarrow$ $\Pi_{n/m-1}^*/\Pi_{n/m}^*$ is a maximal almost pro-*l* quotient of $\Pi_{n/m-1}/\Pi_{n/m}$ [cf. Definition 1.1].
- (iii) We shall say that Π_n^* is *F*-characteristic if every F-admissible automorphism [cf. [CmbCsp], Definition 1.1, (ii)] of Π_n preserves the kernel of the quotient $\Pi_n \to \Pi_n^*$.
- (iv) We shall refer to the image of a fiber subgroup [cf. [MzTa], Definition 2.3, (iii)] of Π_n in Π_n^* as a *fiber subgroup* of Π_n^* . For each $1 \le m \le n$, we shall refer to the image of a cuspidal

inertia subgroup of $\Pi_{n/m-1}/\Pi_{n/m}$ in $\Pi_{n/m-1}^*/\Pi_{n/m}^*$ as a cuspidal inertia subgroup of $\Pi_{n/m-1}^*/\Pi_{n/m}^*$.

- (v) Let α be an automorphism of Π_n^* . Then we shall say that α is *F-admissible* if α preserves each fiber subgroup [cf. (iv)] of Π_n^* . We shall say that α is *C-admissible* if α preserves the filtration
- $\{1\} = \Pi_{n/n}^* \subseteq \Pi_{n/n-1}^* \subseteq \cdots \subseteq \Pi_{n/m}^* \subseteq \cdots \subseteq \Pi_{n/2}^* \subseteq \Pi_{n/1}^* \subseteq \Pi_{n/0}^* = \Pi_n^*,$ and, moreover, α induces a bijection of the set of cuspidal inertia subgroups [cf. (iv)] of every standard-adjacent subquotient [cf. (i)] of Π_n^* . We shall say that α is *FC-admissible* if α is F-admissible and C-admissible.
 - (vi) Let α be an outomorphism of Π_n^* . Then we shall say that α is *F*-admissible (respectively, *C*-admissible; *FC*-admissible) if α arises from an automorphism of Π_n^* that is F-admissible (respectively, C-admissible; FC-admissible) [cf. (v)].
- (vii) Write

$$\operatorname{Aut}^{\operatorname{F}}(\Pi_n^*)$$
, $\operatorname{Aut}^{\operatorname{C}}(\Pi_n^*)$, $\operatorname{Aut}^{\operatorname{FC}}(\Pi_n^*) \subseteq \operatorname{Aut}(\Pi_n^*)$

for the respective subgroups of F-, C-, and FC-admissible automorphisms of Π_n^* [cf. (v)];

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n}^{*}) \stackrel{\text{def}}{=} \operatorname{Aut}^{\mathrm{F}}(\Pi_{n}^{*}) / \operatorname{Inn}(\Pi_{n}^{*}) ,$$
$$\operatorname{Out}^{\mathrm{C}}(\Pi_{n}^{*}) \stackrel{\text{def}}{=} \operatorname{Aut}^{\mathrm{C}}(\Pi_{n}^{*}) / \operatorname{Inn}(\Pi_{n}^{*}) ,$$

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n^*) \stackrel{\text{def}}{=} \operatorname{Aut}^{\operatorname{FC}}(\Pi_n^*) / \operatorname{Inn}(\Pi_n^*) \subseteq \operatorname{Out}(\Pi_n^*)$$

for the respective subgroups of F-, C-, and FC-admissible outomorphisms of Π_n^* [cf. (vi)].

(viii) Let $\Pi_n \to \Pi_n^{**}$ be a quotient of Π_n that *dominates* the quotient $\Pi_n \to \Pi_n^*$ [cf. the discussion entitled "*Topological groups*" in §0]. Then we shall write

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n}^{**} \twoheadrightarrow \Pi_{n}^{*}) \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_{n}^{**}),$$
$$\operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{**} \twoheadrightarrow \Pi_{n}^{*}) \subseteq \operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{**})$$

[cf. (vii)] for the respective subgroups of F-, FC-admissible outomorphisms of Π_n^{**} that preserve the kernel of the natural surjection $\Pi_n^{**} \to \Pi_n^*$. Thus, we have natural homomorphisms

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n}^{**} \twoheadrightarrow \Pi_{n}^{*}) \longrightarrow \operatorname{Out}^{\mathrm{F}}(\Pi_{n}^{*}),$$
$$\operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{**} \twoheadrightarrow \Pi_{n}^{*}) \longrightarrow \operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{*}).$$

We shall write

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n}^{*} \leftarrow \Pi_{n}^{**}) \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_{n}^{*}),$$
$$\operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{*} \leftarrow \Pi_{n}^{**}) \subseteq \operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{*})$$

for the respective images of these natural homomorphisms. Thus, we have natural surjections

$$\operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{**} \twoheadrightarrow \Pi_{n}^{*}) \twoheadrightarrow \operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{*} \twoheadleftarrow \Pi_{n}^{**}),$$
$$\operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{**} \twoheadrightarrow \Pi_{n}^{*}) \twoheadrightarrow \operatorname{Out}^{\mathrm{FC}}(\Pi_{n}^{*} \twoheadleftarrow \Pi_{n}^{**}).$$

Remark 2.1.1. In the notation of Definition 2.1, suppose that Π_n^* is *F*-characteristic [cf. Definition 2.1, (iii)]. Then it follows from the various definitions involved that

 $\operatorname{Out}^{\mathrm{F}}(\Pi_n \twoheadrightarrow \Pi_n^*) = \operatorname{Out}^{\mathrm{F}}(\Pi_n), \ \operatorname{Out}^{\mathrm{FC}}(\Pi_n \twoheadrightarrow \Pi_n^*) = \operatorname{Out}^{\mathrm{FC}}(\Pi_n)$

[cf. Definition 2.1, (viii)]; thus, we have natural surjections

 $\operatorname{Out}^{\mathrm{F}}(\Pi_n) \twoheadrightarrow \operatorname{Out}^{\mathrm{F}}(\Pi_n^* \twoheadleftarrow \Pi_n), \ \operatorname{Out}^{\mathrm{FC}}(\Pi_n) \twoheadrightarrow \operatorname{Out}^{\mathrm{FC}}(\Pi_n^* \twoheadleftarrow \Pi_n)$ [cf. Definition 2.1, (viii)].



be a commutative diagram of profinite groups — where the horizontal sequences are \mathbf{exact} , and the vertical arrows are $\mathbf{surjective}$. Write

$$G^* \stackrel{\text{def}}{=} \operatorname{Ker}(G \twoheadrightarrow Q \twoheadrightarrow \overline{Q}) / \operatorname{Ker}(N \twoheadrightarrow \overline{N})$$

and N^* for the image of N in G^* . Suppose that \overline{N} is center-free. Then the image of $\operatorname{Ker}(G \twoheadrightarrow \overline{G})$ in G^* is equal to the centralizer $Z_{G^*}(N^*)$.

Proof. Observe that, by replacing G by $\operatorname{Ker}(G \twoheadrightarrow Q \twoheadrightarrow \overline{Q}) (= G \times_Q \operatorname{Ker}(Q \twoheadrightarrow \overline{Q}))$, we may assume without loss of generality that $\overline{Q} = \{1\}$. In a similar vein, by replacing G by $G/\operatorname{Ker}(N \twoheadrightarrow \overline{N})$, we may assume without loss of generality that $N = \overline{N}$, which [since $\overline{Q} = \{1\}$] implies that $G = G^*$, $N = N^* = \overline{N}$. Then one verifies easily that the natural inclusions N, $\operatorname{Ker}(G \twoheadrightarrow \overline{G}) \hookrightarrow G$ determine an *isomorphism* $N \times \operatorname{Ker}(G \twoheadrightarrow \overline{G}) \xrightarrow{\sim} G$. Thus, since N is *center-free*, we obtain that $\operatorname{Ker}(G \twoheadrightarrow \overline{G}) = Z_G(N)$. This completes the proof of Lemma 2.2.

Proposition 2.3 (Existence of F-characteristic SA-maximal almost pro-l quotients). In the notation of Definition 2.1, let $\Pi_n \twoheadrightarrow$ Π_n^* be a quotient of Π_n . Then the following hold:

- (i) If Π_n^{*} is an SA-maximal almost pro-l quotient of Π_n [cf. Definition 2.1, (ii)], then Π_n^{*} is topologically finitely generated, almost pro-l [cf. the discussion entitled "Topological groups" in [CbTpI], §0], and slim [cf. the discussion entitled "Topological groups" in [CbTpI], §0].
- (ii) Let $0 \le m_1 \le m_2 \le n$ be integers and $(\prod_{n/m_1}/\prod_{n/m_2})^{\ddagger}$ an almost pro-*l* quotient of $\prod_{n/m_1}/\prod_{n/m_2}$. Then there exists an F-characteristic [cf. Definition 2.1, (iii)] SA-maximal almost pro-*l* quotient \prod_{n}^{**} of \prod_n such that the quotient of $\prod_{n/m_1}/\prod_{n/m_2}$ determined by the quotient $\prod_n \twoheadrightarrow \prod_{n}^{**}$ dominates the quotient $\prod_{n/m_1}/\prod_{n/m_2} \twoheadrightarrow (\prod_{n/m_1}/\prod_{n/m_2})^{\ddagger}$ [cf. the discussion entitled "Topological groups" in §0].
- (iii) Let $1 \leq m \leq n$ be an integer, $H \subseteq \prod_{n/m-1}/\prod_{n/m} a$ VCNsubgroup of $\prod_{n/m-1}/\prod_{n/m}$ [cf. [CbTpII], Definition 3.1, (iv)], and $H \twoheadrightarrow H^{\ddagger}$ an almost pro-*l* quotient of *H*. Then there exists an **F**-characteristic SA-maximal almost pro-*l* quotient \prod_{n}^{**} of \prod_{n} such that the quotient of *H* determined by the quotient $\prod_{n} \twoheadrightarrow \prod_{n}^{**}$ dominates the quotient $H \twoheadrightarrow H^{\ddagger}$.

Proof. First, we verify assertion (i). Observe that it follows immediately from Proposition 1.7, (i), together with the definition of an SA-maximal almost pro-*l* quotient, that Π_n^* is a successive extension of almost pro-*l*, topologically finitely generated, slim profinite groups. Thus, one verifies immediately that Π_n^* is almost pro-*l*, topologically finitely generated, and slim. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, observe that since $(\prod_{n/m_1}/\prod_{n/m_2})^{\ddagger}$ may be regarded as an *almost pro-l quotient* of \prod_{n/m_1} , we may assume without loss of generality that $m_2 = n$. Write $m \stackrel{\text{def}}{=} m_1$. If m = n, then one may take the quotient \prod_n^{**} to be the *maximal pro-l quotient* of \prod_n [cf. [MzTa], Proposition 2.2, (i)]. Thus, we may assume without loss of generality that $m \leq n - 1$.

Let us verify assertion (ii) by induction on n. If n = 1, then assertion (ii) follows immediately from the fact that Π_1 is topologically finitely generated, which implies that the topology of Π_1 admits a basis of characteristic open subgroups. Thus, we suppose that $n \ge 2$, and that the induction hypothesis is in force. Then observe that since the subgroup $\Pi_{n/n-1} \subseteq \Pi_n$ may be regarded as the " Π_1 " associated to some stable log curve of type (g, r + n - 1), by applying the induction hypothesis to the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/n-1}^{\ddagger}$ determined by the quotient $\Pi_{n/m} \twoheadrightarrow \Pi_{n/m}^{\ddagger}$, we obtain an *F*-characteristic SA-maximal almost proquotient $\Pi_{n/n-1}^{**}$ of $\Pi_{n/n-1}$ which dominates $\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/n-1}^{\ddagger}$. In particular, since the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/n-1}^{**}$ is *F*-characteristic, hence arises from a subgroup of $\Pi_{n/n-1}$ which is normal in Π_n , we thus obtain

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a natural outer action

 $\Pi_n/\Pi_{n/n-1} (\stackrel{\sim}{\to} \Pi_{n-1}) \to \operatorname{Out}(\Pi_{n/n-1}^{**}).$

Since the profinite group $\Pi_{n/n-1}^{**}$ is almost pro-l and topologically finitely generated [cf. assertion (i)], it follows immediately that the outer action $\Pi_n/\Pi_{n/n-1} \to \operatorname{Out}(\Pi_{n/n-1}^{**})$ factors through an almost pro-l quotient

$$\Pi_n / \Pi_{n/n-1} \twoheadrightarrow Q$$

of $\Pi_n/\Pi_{n/n-1}$. In particular, it follows that the natural outer action $\Pi_{n/m}/\Pi_{n/n-1} \subseteq \Pi_n/\Pi_{n/n-1} \to \operatorname{Out}(\Pi_{n/n-1}^{**})$ factors through an *almost pro-l* quotient of $\Pi_{n/m}/\Pi_{n/n-1}$. Note that this implies [cf. the *slimness* of $\Pi_{n/n-1}^{**}$ proved in assertion (i)] that there exists an *almost pro-l* quotient $\Pi_{n/m} \twoheadrightarrow Q^{**}$ of $\Pi_{n/m}$ that induces the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/n-1}^{**}$ of $\Pi_{n/n-1}$. Now one verifies immediately that the quotient Q^{***} determined by the *intersection* of the kernels of the two quotients $\Pi_{n/m} \twoheadrightarrow \Pi_{n/m}^{\ddagger}$, $\Pi_{n/m} \twoheadrightarrow Q^{**}$ is an *almost pro-l* quotient of $\Pi_{n/m}$ that induces the quotient $\Pi_{n/m-1}$ of $\Pi_{n/n-1}$. Thus, we conclude that by replacing the quotient $\Pi_{n/m-1} \twoheadrightarrow \Pi_{n/m}^{\ddagger}$ by this quotient Q^{***} , we may assume without loss of generality that the quotient $\Pi_{n/m} \twoheadrightarrow \Pi_{n/m}^{\ddagger}$ induces the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/n-1}^{**}$ of $\Pi_{n/n-1}$.

Next, let us observe that if we regard $\Pi_n/\Pi_{n/n-1}$ as the " Π_{n-1} " associated to some stable log curve of type (g, r), then:

• If we apply the induction hypothesis to the almost pro-l quotient $\Pi_{n/m}/\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/m}^{\ddagger}/\Pi_{n/n-1}^{\ddagger}$, then we obtain an [Fcharacteristic SA-maximal] almost pro-l quotient

$$\Pi_n/\Pi_{n/n-1} \twoheadrightarrow Q^{\ddagger}$$

of $\Pi_n/\Pi_{n/n-1}$ which induces a quotient of $\Pi_{n/m}/\Pi_{n/n-1}$ that dominates the quotient $\Pi_{n/m}/\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/m}^{\ddagger}/\Pi_{n/n-1}^{\ddagger}$.

• If we apply the induction hypothesis to any almost pro-l quotient of $\Pi_n/\Pi_{n/n-1}$ that dominates both Q and Q^{\ddagger} [e.g., the quotient determined by the intersection of the kernels determined by the quotients Q, Q^{\ddagger}], then we obtain an *F*-characteristic SA-maximal almost pro-l quotient

$$\Pi_n/\Pi_{n/n-1} \twoheadrightarrow (\Pi_n/\Pi_{n/n-1})^{*}$$

of $\Pi_n/\Pi_{n/n-1}$ that dominates Q and, moreover, induces a quotient of $\Pi_{n/m}/\Pi_{n/n-1}$ that dominates $\Pi_{n/m}^{\ddagger}/\Pi_{n/n-1}^{\ddagger}$. In particular, the above outer action $\Pi_n/\Pi_{n/n-1} \to \operatorname{Out}(\Pi_{n/n-1}^{**})$ factors through the natural surjection $\Pi_n/\Pi_{n/n-1} \to (\Pi_n/\Pi_{n/n-1})^{**}$.

Now let us write $\Pi_n^{**} \stackrel{\text{def}}{=} \Pi_{n/n-1}^{**} \stackrel{\text{out}}{\rtimes} (\Pi_n/\Pi_{n/n-1})^{**}$ [cf. the discussion entitled "*Topological groups*" in [CbTpI], §0 — where we note that

 $\Pi_{n/n-1}^{**}$ is center-free by assertion (i)]. Then it follows immediately from Lemma 2.2 [which allows one to reduce an *inclusion* assertion concerning "Ker(-)'s" to an inclusion assertion concerning centralizers] and the various definitions involved, together with our assumption that the quotient $\Pi_{n/m} \twoheadrightarrow \Pi_{n/m}^{\ddagger}$ induces the quotient $\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/n-1}^{**}$ of $\Pi_{n/n-1}$, that Π_n^{**} is an SA-maximal almost pro-l quotient of Π_n such that the quotient of $\Pi_{n/m}$ determined by $\Pi_n \twoheadrightarrow \Pi_n^{**}$ dominates the quotient $\Pi_{n/m} \twoheadrightarrow \Pi_{n/m}^{\ddagger}$. Finally, it follows immediately from Lemma 2.2 [which allows one to reduce an *F*-characteristicity assertion concerning "Ker(-)" to an F-characteristicity assertion concerning a certain centralizer], together with the fact that the quotients $\Pi_{n/n-1} \twoheadrightarrow \Pi_{n/n-1}^{**}$ and $\Pi_n/\Pi_{n/n-1} \twoheadrightarrow (\Pi_n/\Pi_{n/n-1})^{**}$ are *F*-characteristic, that Π_n^{**} is *F*characteristic. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (ii), together with Proposition 1.7, (viii). This completes the proof of Proposition 2.3. \Box

Definition 2.4. In the notation of Definition 2.1, write $\Pi_{\rm F} \stackrel{\text{def}}{=} \Pi_{2/1}$, $\Pi_{\rm T} \stackrel{\text{def}}{=} \Pi_2$, $\Pi_{\rm B} \stackrel{\text{def}}{=} \Pi_1$; thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\rm F} \longrightarrow \Pi_{\rm T} \longrightarrow \Pi_{\rm B} \longrightarrow 1$$

[cf. the notation introduced in [CbTpI], Definition 6.3]. Let $\Pi_{\rm F} \rightarrow \Pi_{\rm F}^*$ be a maximal almost pro- Σ quotient of $\Pi_{\rm F}$ [cf. Definition 1.1]. Then we shall say that $\Pi_{\rm F} \rightarrow \Pi_{\rm F}^*$ is base-admissible if the kernel of $\Pi_{\rm F} \rightarrow \Pi_{\rm F}^*$ is normal in $\Pi_{\rm T}$. Thus, if $\Pi_{\rm F} \rightarrow \Pi_{\rm F}^*$ is base-admissible, then the quotient $\Pi_{\rm F} \rightarrow \Pi_{\rm F}^*$ determines a quotient $\Pi_{\rm T} \rightarrow \Pi_{\rm T}^*$ of $\Pi_{\rm T}$ which fits into a natural commutative diagram of profinite groups



— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*.

Definition 2.5. In the notation of Definition 2.4, suppose that

$$\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^*$$

is *base-admissible* [cf. Definition 2.4]; thus, we have a quotient

$$\Pi_{\mathrm{T}} \twoheadrightarrow \Pi_{\mathrm{T}}^{*}$$

of Π_{T} that fits into the commutative diagram of Definition 2.4. Let $x \in X(k)$ be a k-valued point of the underlying scheme X of X^{\log} .

(i) We shall write

$$\Pi_{\mathcal{G}_x} \twoheadrightarrow \Pi^*_{\mathcal{G}_x}$$

[cf. [CbTpI], Definition 6.3, (i)] for the maximal almost $pro-\Sigma$ quotient of $\Pi_{\mathcal{G}_x}$ determined by the quotient $\Pi_F \twoheadrightarrow \Pi_F^*$ and the isomorphism $\Pi_F \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ fixed in [CbTpI], Definition 6.3, (i). [Here, we note that this quotient $\Pi_{\mathcal{G}_x} \twoheadrightarrow \Pi_{\mathcal{G}_x}^*$ is independent of the choice of isomorphism $\Pi_F \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ in [CbTpI], Definition 6.3, (i).] Thus, the fixed isomorphism $\Pi_F \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ induces an isomorphism of profinite groups $\Pi_F^* \xrightarrow{\sim} \Pi_{\mathcal{G}_x}^*$.

- (ii) For $c \in \text{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. [CbTpI], Definition 6.5, (i)], we shall refer to a closed subgroup of Π_{F}^* obtained by forming the image — via the isomorphism $\Pi_{\mathcal{G}_x}^* \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}^*$ [cf. (i)] for some kvalued point $x \in X(k)$ — of a cuspidal subgroup of $\Pi_{\mathcal{G}_x}^*$ associated to the cusp of \mathcal{G}_x corresponding to $c \in \text{Cusp}^{\mathrm{F}}(\mathcal{G})$ [cf. [CbTpI], Lemma 6.4, (ii)] as a cuspidal subgroup of Π_{F}^* associated to $c \in \text{Cusp}^{\mathrm{F}}(\mathcal{G})$. Note that it follows immediately from [CbTpI], Lemma 6.4, (ii), that the Π_{F}^* -conjugacy class of a cuspidal subgroup of Π_{F}^* associated to $c \in \text{Cusp}^{\mathrm{F}}(\mathcal{G})$ depends only on $c \in \text{Cusp}^{\mathrm{F}}(\mathcal{G})$, i.e., it does not depend on the choice of x or on the choices of isomorphisms made in [CbTpI], Definition 6.3, (i).
- (iii) Recall that $\Pi_{\rm T} = \Pi_2$, $\Pi_{\rm F} = \Pi_{2/1}$ [cf. Definition 2.4]. In particular, it makes sense to speak of *F*-/*C*-/*FC*-admissible automorphisms or outomorphisms of $\Pi_{\rm T}^*$, $\Pi_{\rm F}^*$ [cf. Definition 2.1, (v), (vi)].

Lemma 2.6 (Maximal almost pro- Σ quotients of VCN-subgroups). In the notation of Definition 2.5, let $\Pi_{c_{\text{diag}}}^* \subseteq \Pi_{\text{F}}^*$ be a cuspidal subgroup of Π_{F}^* [cf. Definition 2.5, (ii)] associated to $c_{\text{diag}}^{\text{F}} \in \text{Cusp}^{\text{F}}(\mathcal{G})$ [cf. [CbTpI], Definition 6.5, (i)]. Write $N_{\text{diag}}^* \subseteq \Pi_{\text{F}}^*$ for the normal closed subgroup of Π_{F}^* topologically normally generated by $\Pi_{c_{\text{diag}}}^* \subseteq \Pi_{\text{F}}^*$. [Note that it follows immediately from [CbTpI], Lemma 6.4, (i), (ii), that N_{diag}^* is normal in Π_{T}^* .] Then the following hold:

- (i) If we regard Π_F^{*}/N_{diag}^{*} as a quotient of Π_G by means of the natural outer isomorphism Π_F/N_{diag} → Π_G of [CbTpI], Lemma 6.6, (i), and the natural surjection Π_F/N_{diag} → Π_F^{*}/N_{diag}^{*}, then Π_F^{*}/N_{diag}^{*} is a maximal almost pro-Σ quotient of Π_G [cf. Definition 1.1].
- (ii) Let $z^{\mathrm{F}} \in \mathrm{VCN}(\mathcal{G}_x)$, $\Pi_{z^{\mathrm{F}}} \subseteq \Pi_{\mathcal{G}_x}$ a VCN-subgroup of $\Pi_{\mathcal{G}_x}$ associated to z^{F} , and $\Pi_{z^{\mathrm{F}}} \twoheadrightarrow \Pi_{z^{\mathrm{F}}}^{\ddagger}$ an almost pro- Σ quotient of

 $\Pi_{z^{\mathrm{F}}}$. Then there exists a base-admissible [cf. Definition 2.4] maximal almost pro- Σ quotient Π_{F}^{**} of Π_{F} such that the quotient $\Pi_{z^{\mathrm{F}}} \twoheadrightarrow \Pi_{z^{\mathrm{F}}}^{**}$ determined by the quotient $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}^{**}$ dominates the quotient $\Pi_{z^{\mathrm{F}}} \twoheadrightarrow \Pi_{z^{\mathrm{F}}}^{\ddagger}$ [cf. the discussion entitled "Topological groups" in §0].

(iii) Let $z^{\mathrm{F}} \in \mathrm{VCN}(\mathcal{G}_x) \setminus \{c^{\mathrm{F}}_{\mathrm{diag}}\}$ and $\Pi^*_{z^{\mathrm{F}}} \subseteq \Pi^*_{\mathcal{G}_x}$ a VCN-subgroup of $\Pi^*_{\mathcal{G}_x}$ associated to z^{F} [cf. Definition 1.6, (i)]. Suppose that either

•
$$z^{\mathrm{F}} \in \mathrm{Edge}(\mathcal{G}_x)$$

or

• $z^{\mathrm{F}} = v_x^{\mathrm{F}}$ for $v \in \mathrm{Vert}(\mathcal{G})$ [cf. [CbTpI], Definition 6.3, (ii)] such that x does **not lie** on v [cf. [CbTpI], Definition 6.3, (iii)].

Then there exist a maximal almost pro- Σ quotient Π_{F}^{**} of Π_{F} and a VCN-subgroup $\Pi_{z^{\mathrm{F}}}^{**} \subseteq \Pi_{\mathcal{G}_x}^{**}$ of $\Pi_{\mathcal{G}_x}^{**}$ associated to z^{F} such that the following conditions are satisfied:

- (a) $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{**}$ dominates $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{*}$.
- (b) $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{**}$ is base-admissible.
- (c) The quotient of Π_{zF}^{**} determined by the composite

 $\Pi_{z^{\mathrm{F}}}^{**} \hookrightarrow \Pi_{\mathcal{C}_{x}}^{**} \xleftarrow{\sim} \Pi_{\mathrm{F}}^{**} \twoheadrightarrow \Pi_{\mathrm{F}}^{*}$

factors through the quotient of $\Pi^{**}_{z^{\rm F}}$ determined by the composite

 $\Pi_{z^{\mathrm{F}}}^{**} \hookrightarrow \Pi_{\mathcal{G}_x}^{**} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}^{**} \twoheadrightarrow \Pi_{\mathrm{F}}^{**} / N_{\mathrm{diag}}^{**}$

— where we write N_{diag}^{**} for the normal closed subgroup of Π_{F}^{**} topologically normally generated by the cuspidal subgroups of Π_{F}^{**} associated to $c_{\text{diag}}^{\text{F}} \in \text{Cusp}^{\text{F}}(\mathcal{G})$.

Proof. Assertion (i) follows immediately from Lemma 1.2, (i). Assertion (ii) follows immediately from Proposition 1.7, (viii), together with Lemma 1.2, (iii) [cf. also Proposition 1.7, (i)]. In a similar vein, assertion (iii) follows immediately, in light of the *injectivity* assertion of [CbTpI], Lemma 6.6, (iii), from Proposition 1.7, (viii) [applied to $\Pi_{\rm F}/N_{\rm diag}$], together with Lemma 1.2, (iii) [cf. also Proposition 1.7, (i)]. This completes the proof of Lemma 2.6.

Lemma 2.7 (Outomorphisms that preserve the diagonal). In the notation of Lemma 2.6, let $\tilde{\alpha}^*$ be an automorphism of Π_T^* over Π_B [i.e., which preserves and induces the identity automorphism on the quotient $\Pi_T^* \to \Pi_B$]. Write $\alpha_F^* \in \text{Out}(\Pi_F^*)$ for the outomorphism of Π_F^* determined by of $\tilde{\alpha}^*$. Then the following hold:

- (i) Suppose that $\tilde{\alpha}^*$ preserves $\Pi_{c_{\text{diag}}^*}^* \subseteq \Pi_{\text{F}}^*$. Then the automorphism of $\Pi_{\text{F}}^*/N_{\text{diag}}^*$ induced by $\tilde{\alpha}^*$ is the identity automorphism.
- (ii) Let $e \in \operatorname{Edge}(\mathcal{G})$, $x \in X(k)$ be such that $x \frown e$ [cf. [CbTpI], Definition 6.3, (iii)]. Suppose that α_{F}^* is C-admissible [cf. Definition 2.5, (iii)], and that $\operatorname{Edge}(\mathcal{G}) = \{e\} \cup \operatorname{Cusp}(\mathcal{G})$. Then it holds that $\alpha_{\mathrm{F}}^* \in \operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{G}_x}^*) (\subseteq \operatorname{Out}(\Pi_{\mathcal{G}_x}^*) \leftarrow \operatorname{Out}(\Pi_{\mathrm{F}}^*))$ [cf. Definition 1.6, (iii)]. If, moreover, $\widetilde{\alpha}^*$ preserves $\Pi_{c_{\mathrm{diag}}}^* \subseteq \Pi_{\mathrm{F}}^*$, then $\alpha_{\mathrm{F}}^* \in \operatorname{Out}^{|\operatorname{grph}|}(\Pi_{\mathcal{G}_x}^*) (\subseteq \operatorname{Out}^{\operatorname{grph}}(\Pi_{\mathcal{G}_x}^*))$ [cf. Definition 1.8].
- (iii) If $\tilde{\alpha}^*$ is **FC-admissible** [cf. Definition 2.5, (iii)], then $\tilde{\alpha}^*$ **preserves** the Π^*_{F} -conjugacy class of $\Pi^*_{c^{\mathrm{F}}_{\mathrm{diag}}} \subseteq \Pi^*_{\mathrm{F}}$.

Proof. First, we verify assertion (i). Write $D^* \stackrel{\text{def}}{=} N_{\Pi_{\mathrm{T}}^*}(\Pi_{c_{\mathrm{diag}}^{\mathrm{F}}}) \subseteq \Pi_{\mathrm{T}}^*$. Then it follows immediately from Proposition 1.7, (vii), that the natural inclusion $D^* \hookrightarrow \Pi_{\mathrm{T}}^*$ fits into the following exact sequence



— where the horizontal sequences are *exact*. Thus, assertion (i) follows immediately from a similar argument to the argument applied in the proof of the first assertion of [CbTpI], Lemma 6.7, (i) [cf. also the proof of [CmbCsp], Proposition 1.2, (iii)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). The fact that $\alpha_{\rm F}^* \in {\rm Out}^{{\rm grph}}(\Pi_{\mathcal{G}_x}^*)$ ($\subseteq {\rm Out}(\Pi_{\mathcal{G}_x}^*) \leftarrow {\rm Out}(\Pi_{\rm F}^*)$) follows immediately from Corollary 1.12, together with a similar argument to the argument applied in the proof of the first assertion of [CbTpI], Lemma 6.7, (ii). Now suppose, moreover, that $\tilde{\alpha}^*$ preserves $\Pi_{c_{\rm diag}}^* \subseteq \Pi_{\rm F}^*$. Then the fact that $\alpha_{\rm F}^* \in {\rm Out}^{|{\rm grph}|}(\Pi_{\mathcal{G}_x}^*)$ ($\subseteq {\rm Out}^{\rm grph}(\Pi_{\mathcal{G}_x}^*)$) follows immediately from assertion (i); Lemma 2.6, (i); Proposition 1.7, (iii), (v), together with a similar argument to the argument applied in the proof of the second assertion of [CbTpI], Lemma 6.7, (ii). This completes the proof of assertion (ii).

Finally, assertion (iii) follows immediately, in light of Lemma 2.6, (i), from the definition of FC-admissibility [cf. also Proposition 1.7, (v)]. This completes the proof of Lemma 2.7.

Lemma 2.8 (Triviality of certain outomorphisms). In the notation of Definition 2.5, there exists a base-admissible maximal almost pro- Σ quotient $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{**}$ [cf. Definitions 1.1; 2.4] of $\Pi_{\rm F}$ that **dominates** $\Pi_{\rm F} \rightarrow \Pi_{\rm F}^*$ [cf. the discussion entitled "Topological groups" in §0] such that the following condition (‡) is satisfied:

> (‡): Let $\tilde{\alpha}^*$ be an automorphism of Π^*_{T} . Then for any base-admissible maximal almost pro- Σ quotient $\Pi_{\mathrm{F}} \rightarrow \Pi^{***}_{\mathrm{F}}$ that dominates $\Pi_{\mathrm{F}} \rightarrow \Pi^{**}_{\mathrm{F}}$, if $\tilde{\alpha}^*$ arises from an FC-admissible automorphism [cf. Definition 2.5, (iii)] of Π^{***}_{T} [where we write Π^{***}_{T} for the " Π^*_{T} " determined by Π^{***}_{F}] over $\Pi^{***}_{\mathrm{T}}/\Pi^{***}_{\mathrm{F}} \rightarrow \Pi_{\mathrm{B}}$, then $\tilde{\alpha}^*$ is Π^*_{F} -inner.

Proof. The following argument is essentially the same as the argument applied in [CmbCsp], [NodNon], [CbTpI] to prove [CmbCsp], Corollary 2.3, (ii); [NodNon], Corollary 5.3; [CbTpI], Lemma 6.8, respectively.

Let us fix a cuspidal subgroup $\Pi_{c_{\text{fiag}}}^* \subseteq \Pi_F^*$ of Π_F^* [cf. Definition 2.5, (ii)] associated to $c_{\text{diag}}^F \in \text{Cusp}^F(\mathcal{G})$ [cf. [CbTpI], Definition 6.5, (i)]. Let $\Pi_F \twoheadrightarrow \Pi_F^{**}$ be a base-admissible maximal almost pro- Σ quotient of Π_F that dominates $\Pi_F \twoheadrightarrow \Pi_F^*$; $\Pi_F \twoheadrightarrow \Pi_F^{***}$ a base-admissible maximal almost pro- Σ quotient of Π_F that dominates $\Pi_F \twoheadrightarrow \Pi_F^{**}$; $\tilde{\alpha}^*$ an automorphism of Π_T^* that arises from an *FC*-admissible automorphism $\tilde{\alpha}^{***}$ of Π_T^{***} over $\Pi_T^{***}/\Pi_F^{***} \xrightarrow{\sim} \Pi_B$. Here, let us observe that one verifies easily that $\tilde{\alpha}^*$ is an *FC*-admissible automorphism of Π_T^* over $\Pi_T^*/\Pi_F^* \xrightarrow{\sim} \Pi_B$. Write α_F^* for the outomorphism of Π_F^* determined by $\tilde{\alpha}^*$. Observe that since α_F^* preserves the Π_F^* -conjugacy class of $\Pi_{c_{\text{diag}}}^* \subseteq \Pi_F^*$ [cf. Lemma 2.7, (iii)], we may assume without loss of generality — by replacing $\tilde{\alpha}^{***}$ by a suitable Π_F^{***} -conjugate of $\tilde{\alpha}^{***}$ — that $\tilde{\alpha}^*$ preserves $\Pi_{c_{\text{diag}}}^* \subseteq \Pi_F^*$, and hence [cf. Lemma 2.7, (i), (ii)] that

- (a) the automorphism of $\Pi_{\rm F}^*/N_{\rm diag}^*$ induced by $\tilde{\alpha}^*$ is the *identity* automorphism;
- (b) for $e \in \text{Edge}(\mathcal{G})$, $x \in X(k)$ such that $x \curvearrowright e$ [cf. [CbTpI], Definition 6.3, (iii)], if $\text{Edge}(\mathcal{G}) = \{e\} \cup \text{Cusp}(\mathcal{G})$, then $\alpha_{\text{F}}^* \in \text{Out}^{|\text{grph}|}(\Pi_{\mathcal{G}_x}^*) (\subseteq \text{Out}(\Pi_{\mathcal{G}_x}^*) \xleftarrow{} \text{Out}(\Pi_{\text{F}}^*))$ [cf. Definition 1.8].

Now we claim that the following assertion holds:

Claim 2.8.A: Lemma 2.8 holds if (g, r) = (0, 3).

Indeed, write $c_1, c_2, c_3 \in \text{Cusp}(\mathcal{G})$ for the three distinct cusps of \mathcal{G} ; $v \in \text{Vert}(\mathcal{G})$ for the unique vertex of \mathcal{G} . For $i \in \{1, 2, 3\}$, let $x_i \in X(k)$ be such that $x_i \curvearrowright c_i$. Next, let us observe that since our assumption that (g, r) = (0, 3) implies that $\text{Node}(\mathcal{G}) = \emptyset$, it follows immediately from (b) that, for $i \in \{1, 2, 3\}$, the outomorphism α_F^* of $\Pi_{\mathcal{G}_{x_i}}^* \leftarrow \Pi_F^*$ is $\in \text{Out}^{|\text{grph}|}(\Pi_{\mathcal{G}_{x_i}}^*) (\subseteq \text{Out}(\Pi_{\mathcal{G}_{x_i}}^*) \leftarrow \text{Out}(\Pi_F^*))$. Next, let us fix a verticial subgroup $\Pi_{v_{x_2}}^* \subseteq \Pi_{\mathcal{G}_{x_2}}^* \leftarrow \Pi_F^*$ associated to $v_{x_2}^F \in \text{Vert}(\mathcal{G}_{x_2})$ [cf. [CbTpI], Definition 6.3, (ii)]. Then since $\alpha_{\rm F}^* \in {\rm Out}^{|{\rm grph}|}(\Pi_{\mathcal{G}_{x_2}}^*)$, it follows immediately from the [easily verified] surjectivity of the composite $\Pi_{v_{x_2}}^* \hookrightarrow \Pi_{\mathcal{G}_{x_2}}^* \widetilde{\leftarrow} \Pi_{\rm F}^* \twoheadrightarrow \Pi_{\rm F}^*/N_{\rm diag}^*$ that there exists an $N_{\rm diag}^*$ -conjugate $\widetilde{\beta}^*$ of $\widetilde{\alpha}^*$ such that $\widetilde{\beta}^*(\Pi_{v_{x_2}}^*) = \Pi_{v_{x_2}}^*$. Thus, it follows immediately from Lemma 2.6, (iii) — by replacing $\Pi_{\rm F}^{**}$ by a suitable base-admissible maximal almost pro- Σ quotient $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{**}$ [i.e., a quotient as in Lemma 2.6, (iii)] that dominates the original $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{**}$ and applying the conclusion " $\widetilde{\beta}^*(\Pi_{v_{x_2}}^*) = \Pi_{v_{x_2}}^*$ ", together with the property (a) discussed above, in the case where " $\widetilde{\alpha}^{***}$ " is taken to be $\widetilde{\alpha}^{***} \in {\rm Aut}(\Pi_{\rm T}^{***})$ — that we may assume without loss of generality that

(‡₁): $\widetilde{\beta}^*$ fixes and induces the *identity automorphism* on $\Pi_{v_{x_0}^F} \subseteq \Pi_{\mathcal{G}_{x_2}}^* \widetilde{\leftarrow} \Pi_F^*$.

Next, let $\Pi_{c^{\mathrm{F}}}^* \subseteq \Pi_{\mathrm{F}}^*$ be a cuspidal subgroup of Π_{F}^* associated to $c_1^{\rm F} \in {\rm Cusp}^{\rm F}(\mathcal{G})$ [cf. [CbTpI], Definition 6.5, (i)] that is contained in $\Pi^*_{v_{x_2}^{\mathsf{F}}} \subseteq \Pi^*_{\mathcal{G}_{x_2}} \stackrel{\sim}{\leftarrow} \Pi^*_{\mathsf{F}}; \, \Pi^*_{v_{x_3}^{\mathsf{F}}} \subseteq \Pi^*_{\mathcal{G}_{x_3}} \stackrel{\sim}{\leftarrow} \Pi^*_{\mathsf{F}} \text{ a verticial subgroup associated}$ to $\tilde{v}_{x_3}^{\mathrm{F}} \in \operatorname{Vert}(\mathcal{G}_{x_3})$ that contains $\Pi_{c_1}^* \subseteq \Pi_{\mathrm{F}}^*$. Then it follows from the inclusion $\Pi_{c_1^{\mathrm{F}}}^* \subseteq \Pi_{v_{x_2}^{\mathrm{F}}}^*$, together with (\ddagger_1) , that $\widetilde{\beta}^*(\Pi_{c_1^{\mathrm{F}}}^*) = \Pi_{c_1^{\mathrm{F}}}^*$. Thus, since the verticial subgroup $\Pi_{v_{x_3}}^* \subseteq \Pi_{\mathcal{G}_{x_3}}^* \stackrel{\sim}{\leftarrow} \Pi_F^*$ is the *unique* verticial subgroup of $\Pi_{\mathcal{G}_{x_3}}^* \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}^*$ associated to $v_{x_3}^{\mathrm{F}} \in \mathrm{Vert}(\mathcal{G}_{x_3})$ that contains $\Pi_{c_{\mathbf{i}}}^{*}$ [cf. Proposition 1.7, (v), (vi)], it follows immediately from the fact that $\alpha_{\mathrm{F}}^* \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi_{\mathcal{G}_{x_3}}^*)$ that $\widetilde{\beta}^*(\Pi_{v_{x_3}}^*) = \Pi_{v_{x_3}}^*$. In particular, it follows immediately from Lemma 2.6, (iii) — by replacing $\Pi_{\rm F}^{**}$ by a suitable base-admissible maximal almost pro- Σ quotient $\Pi_{\rm F} \rightarrow \Pi_{\rm F}^{**}$ [i.e., a quotient as in Lemma 2.6, (iii)] that dominates the original $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{**}$ and applying the conclusion " $\widetilde{\beta}^*(\Pi_{v_{x_3}}^{*}) = \Pi_{v_{x_3}}^{**}$ ", together with the property (a) discussed above, in the case where " $\tilde{\alpha}^{*,"}$ is taken to be $\tilde{\alpha}^{***} \in \operatorname{Aut}(\Pi_{\mathrm{T}}^{***})$ — that we may assume without loss of generality that

> (\ddagger_2): $\widetilde{\beta}^*$ fixes and induces the *identity automorphism* on $\Pi^*_{v_{x_3}^{\mathrm{F}}} \subseteq \Pi^*_{\mathcal{G}_{x_3}} \stackrel{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$.

On the other hand, since $\Pi_{\rm F}^*$ is topologically generated by $\Pi_{v_{x_2}}^* \subseteq \Pi_{\mathcal{G}_{x_2}}^* \widetilde{\leftarrow} \Pi_{\rm F}^*$ and $\Pi_{v_{x_3}}^* \subseteq \Pi_{\mathcal{G}_{x_3}}^* \widetilde{\leftarrow} \Pi_{\rm F}^*$ [cf. [CmbCsp], Lemma 1.13], (\ddagger_1) and (\ddagger_2) imply that $\widetilde{\beta}^*$ induces the *identity automorphism* on $\Pi_{\rm F}^*$. This completes the proof of Claim 2.8.A.

Next, we claim that the following assertion holds:

Claim 2.8.B: Lemma 2.8 holds if (g, r) = (1, 1).

Indeed, let us first observe that by working with 2-cuspidalizable degeneration structures [cf. [CbTpII], Definition 3.23, (i), (v)] that arise scheme-theoretically via a specialization isomorphism as in the discussion preceding [CmbCsp], Definition 2.1 [cf. also [CbTpI], Remark 5.6.1], we may switch back and forth, at will, between the case of smooth and non-smooth " X^{\log} ". In particular, we may assume without loss of generality that $(\operatorname{Vert}(\mathcal{G})^{\sharp}, \operatorname{Cusp}(\mathcal{G})^{\sharp}, \operatorname{Node}(\mathcal{G})^{\sharp}) = (1, 1, 1).$

Let v be the unique vertex of \mathcal{G} , c the unique cusp of \mathcal{G} , e the unique node of \mathcal{G} , $x \in X(k)$ such that $x \curvearrowright c$ [cf. [CbTpI], Definition 6.3, (iii)], and \mathbb{H} the sub-semi-graph of PSC-type [cf. [CbTpI], Definition 2.2, (i)] of the underlying semi-graph \mathbb{G}_x of \mathcal{G}_x whose set of vertices $= \{v_x^{\mathrm{F}}\}$ [cf. [CbTpI], Definition 6.3, (ii)]. Then it follows from [CbTpI], Lemma 6.4, (iv), that there exists a unique node $e_{\mathrm{new},x}^{\mathrm{F}}$ of \mathcal{G}_x such that $e_{\mathrm{new},x}^{\mathrm{F}} \in \mathcal{N}(v_{\mathrm{new},x}^{\mathrm{F}})$ [cf. [CbTpI], Lemma 6.4, (iii)]. Thus, one verifies easily that there exists a unique element $e_x^{\mathrm{F}} \in \mathcal{N}(v_x^{\mathrm{F}})$ such that $\mathcal{N}(v_x^{\mathrm{F}}) = \{e_{\mathrm{new},x}^{\mathrm{F}}, e_x^{\mathrm{F}}\}$. Let us fix

• a nodal subgroup $\Pi_{e_{\text{new},x}}^* \subseteq \Pi_{\mathcal{G}_x}^* \xleftarrow{} \Pi_{\text{F}}^*$ associated to $e_{\text{new},x}^{\text{F}}$

[cf. Figure 1 below].



Figure 1: \mathcal{G}_x

Then it follows immediately — by applying Proposition 1.7, (v), (vi), in the situation that arises in the case of a *smooth* " X^{\log} " of type (1, 1) [cf. the observations made above concerning degeneration structures] — that there exist

- a unique verticial subgroup $\Pi^*_{v_{\text{new},x}^{\text{F}}} \subseteq \Pi^*_{\mathcal{G}_x} \stackrel{\sim}{\leftarrow} \Pi^*_{\text{F}}$ associated to $v_{\text{new},x}^{\text{F}}$ and
- a unique subgroup $\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}} \subseteq \Pi^*_{\mathcal{G}_x} \stackrel{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}$ that belongs to the Π^*_{F} -conjugacy class of subgroups that arises as the *image* of the natural outer homomorphism $\Pi_{(\mathcal{G}_x)|_{\mathbb{H}}} \hookrightarrow \Pi_{\mathcal{G}_x} \twoheadrightarrow \Pi^*_{\mathcal{G}_x}$ [cf. [CbTpI], Definition 2.2, (ii)]
such that $\Pi^*_{e^{\mathrm{F}}_{\mathrm{new},x}} \subseteq \Pi^*_{v^{\mathrm{F}}_{\mathrm{new},x}}, \ \Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$. Moreover, one verifies easily by applying the property (b) discussed above in the situation that arises in the case of a smooth " X^{\log} " of type (1,1) [cf. the observations made above concerning degeneration structures] — that $\alpha_{\rm F}^*$ preserves the Π_{F}^* -conjugacy classes of $\Pi_{e_{\mathrm{new},x}}^*$, $\Pi_{v_{\mathrm{new},x}}^*$, $\Pi_{(\mathcal{G}_x)|_{\mathbb{H}}}^* \subseteq \Pi_{\mathcal{G}_x}^* \xleftarrow{} \Pi_{\mathrm{F}}^*$. Thus, it follows immediately from the *commensurable terminality* of the image of the composite $\Pi_{e_{\text{new},x}^*}^* \hookrightarrow \Pi_{\mathcal{G}_x}^* \stackrel{\sim}{\leftarrow} \Pi_{\text{F}}^* \twoheadrightarrow \Pi_{\text{F}}^*/N_{\text{diag}}^*$ [cf. Proposition 1.7, (vii); Lemma 2.6, (i)], together with the property (a) discussed above, that there exists an N^*_{diag} -conjugate $\tilde{\beta}^*$ of $\tilde{\alpha}^*$ such that $\widetilde{\beta}^*(\Pi_{e_{\text{new},x}}^*) = \Pi_{e_{\text{new},x}}^*$. In particular, in light of the uniqueness properties applied above to specify the subgroups $\Pi^*_{v_{\text{frew},x}}$ and $\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}}$, we conclude that $\widetilde{\beta}^*(\Pi_{v_{\text{new},x}}^*) = \Pi_{v_{\text{new},x}}^*, \ \widetilde{\beta}^*(\Pi_{(\mathcal{G}_x)|_{\mathbb{H}}}^*) = \Pi_{(\mathcal{G}_x)|_{\mathbb{H}}}^*.$ Thus, it follows immediately from Lemma 2.6, (iii) — by replacing $\Pi_{\rm F}^{**}$ by a suitable base-admissible maximal almost pro- Σ quotient $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{**}$ [i.e., a quotient as in Lemma 2.6, (iii), applied in the situation that arises in the case of a smooth " X^{\log} " of type (1,1) — cf. the observations made above concerning degeneration structures] that dominates the original $\Pi_{\rm F} \twoheadrightarrow \Pi_{\rm F}^{**}$ and applying the conclusion " $\widetilde{\beta}^*(\Pi_{(\mathcal{G}_x)|_{\mathbb{H}}}^*) = \Pi_{(\mathcal{G}_x)|_{\mathbb{H}}}^*$ ", together with the property (a) discussed above, in the case where " $\tilde{\alpha}^{*}$ " is taken to be $\tilde{\alpha}^{***} \in \operatorname{Aut}(\Pi^{***}_{T})$ — that we may assume without loss of generality that

> (\ddagger_3): β^* fixes and induces the *identity automorphism* on $\Pi^*_{(\mathcal{G}_x)|_{\mathbb{H}}} \subseteq \Pi^*_{\mathcal{G}_x} \stackrel{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}.$

Next, let us write

• $\Pi_{v_x^{\mathrm{F}}}^* \subseteq \Pi_{\mathcal{G}_x}^* \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{F}}^*$ for the *unique* [cf. Proposition 1.7, (v), (vi)] *verticial subgroup* associated to v_x^{F} [cf. [CbTpI], Definition 6.3, (ii)]

such that $\Pi_{e_{\mathrm{new},x}}^* \subseteq \Pi_{v_x}^* \subseteq \Pi_{(\mathcal{G}_x)|_{\mathbb{H}}}^*$. [Note that it follows immediately from the various definitions involved that such a verticial subgroup associated to v_x^{F} always exists.] Then since $\Pi_{v_x}^* \subseteq \Pi_{(\mathcal{G}_x)|_{\mathbb{H}}}^*$, it follows from (\ddagger_3) that $\tilde{\beta}^*$ fixes and induces the *identity automorphism* on $\Pi_{v_x}^* \subseteq \Pi_{\mathcal{G}_x}^* \leftarrow \Pi_{\mathrm{F}}^*$. Thus, since $\tilde{\beta}^*(\Pi_{v_{\mathrm{new},x}}^*) = \Pi_{v_{\mathrm{new},x}}^*$ [cf. the discussion preceding (\ddagger_3)], we conclude that $\tilde{\beta}^*$ preserves the closed subgroup $\Pi_{\mathrm{Fsub}}^* \subseteq \Pi_{\mathrm{F}}^*$ of Π_{F}^* obtained by forming the image of the natural homomorphism

$$\varinjlim \left(\Pi^*_{v_{\mathrm{new},x}^{\mathrm{F}}} \hookleftarrow \Pi^*_{e_{\mathrm{new},x}^{\mathrm{F}}} \hookrightarrow \Pi^*_{v_x^{\mathrm{F}}} \right) \longrightarrow \Pi^*_{\mathrm{F}}$$

— where the inductive limit is taken in the category of profinite groups.

Next, let us observe that the Π_{F}^* -conjugacy class of $\Pi_{\mathrm{Fsub}}^* \subseteq \Pi_{\mathrm{F}}^*$ coincides with the Π_{F}^* -conjugacy class of the image $\Pi_{(\mathcal{G}_x)_{\succ \{e_{\mathrm{F}}\}}}^*$ [cf. [CbTpI],

Definition 2.5, (ii)] of the composite

$$\Pi_{(\mathcal{G}_x)_{\succ \{e_{F}\}}} \hookrightarrow \Pi_{\mathcal{G}_x} \stackrel{\sim}{\leftarrow} \Pi_F \twoheadrightarrow \Pi_F^*$$

— where the first arrow is the natural outer injection discussed in [CbTpI], Proposition 2.11, and we recall that $e_x^{\rm F}$ is the node of \mathcal{G}_x that corresponds to the node e of \mathcal{G} . On the other hand, if we write $\Pi^*_{(\mathcal{G}_x)_{\to \{e_{\rm new}^{\rm F}\}}}$ for the maximal almost $pro-\Sigma$ quotient of $\Pi_{(\mathcal{G}_x)_{\to \{e_{\rm new}^{\rm F}\}}}$ [cf. [CbTpI], Definition 2.8] determined by the maximal almost $pro-\Sigma$ quotient $\Pi^*_{\mathcal{G}_x}$ and the natural outer isomorphism $\Phi_{(\mathcal{G}_x)_{\to \{e_{\rm new}^{\rm F}\}}}: \Pi_{(\mathcal{G}_x)_{\to \{e_{\rm new}^{\rm F}\}}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ [cf. [CbTpI], Definition 2.10], then $\Pi^*_{\rm Fsub}$ may be regarded as a verticial subgroup of $\Pi^*_{(\mathcal{G}_x)_{\to \{e_{\rm new}^{\rm F}\}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x} \subset \Pi^*_{\rm F}$ [cf. [CbTpI], Proposition 2.9, (i), (3)]. Thus, it follows from Proposition 1.7, (vii), that $\Pi^*_{\rm Fsub}$ is commensurably terminal in $\Pi^*_{\rm F}$.

Next, let us observe that, by applying a similar argument to the argument given in [CmbCsp], Definition 2.1, (iii), (vi), or [NodNon], Definition 5.1, (ix), (x) [i.e., roughly speaking, by considering the portion of the underlying scheme X_2 of X_2^{\log} corresponding to the underlying scheme $(X_v)_2$ of the 2-nd log configuration space $(X_v)_2^{\log}$ of the stable log curve X_v^{\log} determined by $\mathcal{G}|_v$ — cf. [CbTpI], Definition 2.1, (iii)], one concludes that there exists a verticial subgroup $\Pi_v \subseteq \Pi_{\mathcal{G}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{B}}$ associated to $v \in \operatorname{Vert}(\mathcal{G})$ such that the outer action of Π_v on Π_{F}^* determined by the composite $\Pi_v \hookrightarrow \Pi_{\mathrm{B}} \stackrel{\rho_{2/1}^*}{\to} \operatorname{Out}(\Pi_{\mathrm{F}}^*)$ — where we write $\rho_{2/1}^*$ for the outer action determined by the exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\rm F}^* \longrightarrow \Pi_{\rm T}^* \longrightarrow \Pi_{\rm B} \longrightarrow 1$$

— preserves the $\Pi_{\rm F}^*$ -conjugacy class of the commensurably terminal subgroup $\Pi_{\rm F^{sub}}^* \subseteq \Pi_{\rm F}^*$ [so we obtain a natural outer representation $\Pi_v \to {\rm Out}(\Pi_{\rm F^{sub}}^*)$ — cf. [CbTpI], Lemma 2.12, (iii)], and, moreover, that if we write $\Pi_{\rm T^{sub}}^* \stackrel{\text{def}}{=} \Pi_{\rm F^{sub}}^* \rtimes \Pi_v \ (\subseteq \Pi_{\rm T}^*)$ [cf. the discussion entitled "Topological groups" in [CbTpI], §0], then it follows from Proposition 1.7, (ii), that $\Pi_{\rm T^{sub}}^*$ is naturally isomorphic to a profinite group of the form " $\Pi_{\rm T}$ " obtained by taking " \mathcal{G} " to be $\mathcal{G}|_v$.

Now since $\beta^*(\Pi_{F^{sub}}^*) = \Pi_{F^{sub}}^*$, and $\tilde{\alpha}^*$ is an automorphism over the quotient $\Pi_F^*/\Pi_T^* \to \Pi_B$, one verifies immediately that $\tilde{\beta}^*$ determines an automorphism $\tilde{\beta}_{T^{sub}}^*$ of $\Pi_{T^{sub}}^*$ over Π_v . Thus, since $\mathcal{G}|_v$ is of type (0,3) [cf. [CbTpI], Definition 2.3, (i)], by considering a diagram similar to the diagram of [CmbCsp], Definition 2.1, (vi), or [NodNon], Definition 5.1, (x), and applying Claim 2.8.A [cf. also Lemma 2.6, (ii)], we conclude — by replacing Π_F^{**} by a suitable *base-admissible* maximal almost pro- Σ quotient $\Pi_F \to \Pi_F^{**}$ that *dominates* the *original* $\Pi_F \to \Pi_F^{**}$ and applying the conclusion that $\tilde{\beta}^*$ determines an automorphism of $\Pi_{T^{sub}}^*$ over Π_v

in the case where " $\tilde{\alpha}^{*}$ " is taken to be $\tilde{\alpha}^{***} \in \operatorname{Aut}(\Pi_{\mathrm{T}}^{***})$ — that we may assume without loss of generality that

(\ddagger_4): $\widetilde{\beta}^*_{\text{Tsub}}$ is a Π^*_{Fsub} -inner automorphism.

Moreover, since $\widetilde{\beta}^*$ fixes and induces the *identity automorphism* on $\Pi_{v_x}^*$ [cf. the discussion following (\ddagger_3)], and $\Pi_{v_x}^*$ is commensurably terminal in [$\Pi_{\rm F}^*$, hence also in] $\Pi_{\rm Fsub}^*$ [cf. Proposition 1.7, (vii)] and slim [cf. Proposition 1.7, (ii)], we conclude that $\widetilde{\beta}_{\rm Tsub}^*$ is the *identity automorphism*; in particular, since $\Pi_{v_{\rm frew,x}}^* \subseteq \Pi_{\rm Fsub}^*$, $\widetilde{\beta}^*$ induces the *identity automorphism* on $\Pi_{v_{\rm frew,x}}^*$. Thus, since $\Pi_{\rm F}^*$ is topologically generated by $\Pi_{(\mathcal{G}_x)|_{\mathbb{H}}}^*$ and $\Pi_{v_{\rm frew,x}}^*$ [cf. [CmbCsp], Proposition 2.2, (iii)], it follows from (\ddagger_3) that $\widetilde{\beta}^*$ is the *identity automorphism*. This completes the proof of Claim 2.8.B.

Finally, we claim that the following assertion holds:

Claim 2.8.C: Lemma 2.8 holds for arbitrary (g, r).

We verify Claim 2.8.C by induction on 3g - 3 + r. If 3g - 3 + r = 0, i.e., (g,r) = (0,3), then Claim 2.8.C amounts to Claim 2.8.A. On the other hand, if (g,r) = (1,1), then Claim 2.8.C amounts to Claim 2.8.B. Thus, we suppose that 3g - 3 + r > 0, that $(g,r) \neq (1,1)$, and that the induction hypothesis is in force. Since 3g - 3 + r > 0 and $(g,r) \neq (1,1)$, one verifies easily that there exists a stable log curve Y^{\log} of type (g,r) over $(\operatorname{Spec} k)^{\log}$ such that Y^{\log} has precisely one node and precisely two vertices. Thus, by working with 2-cuspidalizable degeneration structures [cf. [CbTpII], Definition 3.23, (i), (v)] that arise scheme-theoretically via a specialization isomorphism as in the discussion preceding [CmbCsp], Definition 2.1 [cf. also [CbTpI], Remark 5.6.1], we may replace X^{\log} by Y^{\log} and assume without loss of generality that $(\operatorname{Vert}(\mathcal{G})^{\sharp}, \operatorname{Node}(\mathcal{G})^{\sharp}) = (2, 1)$.

Let e be the unique node of \mathcal{G} and $x \in X(k)$ such that $x \curvearrowright e$ [cf. [CbTpI], Definition 6.3, (iii)]. Next, let us observe that since Node $(\mathcal{G})^{\sharp} = \{e\}^{\sharp} = 1$, it follows from the property (b) discussed above that $\alpha_{\mathrm{F}}^* \in \mathrm{Out}^{|\mathrm{grph}|}(\Pi_{\mathcal{G}_x}^*) \subset \mathrm{Out}(\Pi_{\mathcal{G}_x}^*) \leftarrow \mathrm{Out}(\Pi_{\mathrm{F}}^*)$). Write $\{e_1^{\mathrm{F}}, e_2^{\mathrm{F}}\} = \mathcal{N}(v_{\mathrm{new},x}^{\mathrm{F}})$ [cf. [CbTpI], Lemma 6.4, (iv)]. Also, for $i \in \{1,2\}$, denote by $v_i \in \mathrm{Vert}(\mathcal{G})$ the vertex of \mathcal{G} such that $(v_i)_x^{\mathrm{F}} \in \mathrm{Vert}(\mathcal{G}_x)$ [cf. [CbTpI], Definition 6.3, (ii)] is the unique element of $\mathcal{V}(e_i^{\mathrm{F}}) \setminus \{v_{\mathrm{new},x}^{\mathrm{F}}\}$ [cf. [CbTpI], Lemma 6.4, (iv)]; by \mathbb{H}_i the sub-semi-graph of PSC-type [cf. [CbTpI], Definition 2.2, (i)] of the underlying semi-graph \mathbb{G}_x of \mathcal{G}_x whose set of vertices = $\{v_{\mathrm{new},x}^{\mathrm{F}}, (v_i)_x^{\mathrm{F}}\}$ [cf. Figure 2 below].

For $i \in \{1, 2\}$, let $\Pi^*_{(v_i)_x} \subseteq \Pi^*_{\mathcal{G}_x} \stackrel{\sim}{\leftarrow} \Pi^*_F$ be a verticial subgroup of $\Pi^*_{\mathcal{G}_x} \stackrel{\sim}{\leftarrow} \Pi^*_F$ associated to the vertex $(v_i)_x^F \in \mathcal{V}(e_i^F) \setminus \{v_{\text{new},x}^F\}$. Then since $\alpha^*_F \in \text{Out}^{|\text{grph}|}(\Pi^*_{\mathcal{G}_x})$, it follows that $\widetilde{\alpha}^*$ preserves the Π^*_F -conjugacy class of $\Pi^*_{(v_i)_x} \subseteq \Pi^*_{\mathcal{G}_x} \stackrel{\sim}{\leftarrow} \Pi^*_F$. Thus, since the image of the composite



Figure 2: \mathcal{G}_x

 $\Pi_{(v_i)_x}^* \hookrightarrow \Pi_F^* \to \Pi_F^*/N_{\text{diag}}^*$ is commensurably terminal [cf. Proposition 1.7, (vii); Lemma 2.6, (i)], it follows immediately from the property (a) discussed above that there exists an N_{diag}^* -conjugate $\widetilde{\beta}_i^*$ [which may depend on $i \in \{1, 2\}$!] of $\widetilde{\alpha}^*$ such that $\widetilde{\beta}_i^*(\Pi_{(v_i)_x}^*) = \Pi_{(v_i)_x}^*$. Therefore, it follows immediately from Lemma 2.6, (iii) — by replacing Π_F^{**} by a suitable base-admissible maximal almost pro- Σ quotient $\Pi_F \to \Pi_F^{**}$ [i.e., a quotient as in Lemma 2.6, (iii)] that dominates the original $\Pi_F \to \Pi_F^{**}$ and applying the conclusion " $\widetilde{\beta}_i^*(\Pi_{(v_i)_x}^*) = \Pi_{(v_i)_x}^*$ ", together with the property (a) discussed above, in the case where " $\widetilde{\alpha}^*$ " is taken to be $\widetilde{\alpha}^{***} \in \operatorname{Aut}(\Pi_T^{***})$ — that we may assume without loss of generality that

(\ddagger_5): $\widetilde{\beta}_i^*$ induces the *identity automorphism* of $\Pi^*_{(v_i)_T^{\mathrm{F}}}$.

Next, let $\Pi_{e_i^{\mathrm{F}}}^* \subseteq \Pi_{(v_i)_x^{\mathrm{F}}}^*$ be a nodal subgroup of $\Pi_{\mathcal{G}_x}^* \xleftarrow{} \Pi_{\mathrm{F}}^*$ associated to $e_i^{\mathrm{F}} \in \operatorname{Node}(\mathcal{G}_x)$ that is contained in $\Pi_{(v_i)_x^{\mathrm{F}}}^*$; $\Pi_{v_{\operatorname{New},x}^*;i}^* \subseteq \Pi_{\mathcal{G}_x}^* \xleftarrow{} \Pi_{\mathrm{F}}^*$ a verticial subgroup [which may depend on $i \in \{1, 2\}$!] associated to $v_{\operatorname{New},x}^{\mathrm{F}} \in \operatorname{Vert}(\mathcal{G}_x)$ that contains $\Pi_{e_i^{\mathrm{F}}}^*$:

$$\Pi^*_{v_{\mathrm{new},x}^{\mathrm{F}};i} \supseteq \Pi^*_{e_i^{\mathrm{F}}} \subseteq \Pi^*_{(v_i)_x^{\mathrm{F}}} \subseteq \Pi^*_{\mathcal{G}_x} \stackrel{\sim}{\leftarrow} \Pi^*_{\mathrm{F}}.$$

Then it follows from the inclusion $\Pi_{e_i^F}^* \subseteq \Pi_{(v_i)_x}^*$, together with (\ddagger_5) , that $\widetilde{\beta}_i^*(\Pi_{e_i^F}^*) = \Pi_{e_i^F}^*$. Since, moreover, $\Pi_{v_{\text{new},x}^;i}^*$ is the unique verticial subgroup of $\Pi_{\mathcal{G}_x}^* \leftarrow \Pi_F^*$ associated to $v_{\text{new},x}^F$ that contains $\Pi_{e_i^F}^*$ [cf. Proposition 1.7, (v), (vi)], it follows immediately from the fact that $\alpha_F^* \in \text{Out}^{|\text{grph}|}(\Pi_{\mathcal{G}_x}^*)$ that $\widetilde{\beta}_i^*(\Pi_{v_{\text{new},x}^;i}^*) = \Pi_{v_{\text{new},x}^;i}^*$. Thus, $\widetilde{\beta}_i^*$ preserves the closed subgroup $\Pi_{F_i}^* \subseteq \Pi_F^*$ of Π_F^* obtained by forming the image of the natural homomorphism

$$\underline{\lim} \left(\prod_{v_{\mathrm{new},x}^{\mathrm{F}};i}^{*} \longleftrightarrow \prod_{e_{i}^{\mathrm{F}}}^{*} \hookrightarrow \prod_{(v_{i})_{x}^{\mathrm{F}}}^{*} \right) \longrightarrow \prod_{\mathrm{F}}^{*}$$

— where the inductive limit is taken in the category of profinite groups.

Next, let us observe that the Π_{F}^* -conjugacy class of $\Pi_{\mathrm{F}_i}^* \subseteq \Pi_{\mathrm{F}}^*$ coincides with the Π_{F}^* -conjugacy class of the image $\Pi_{(\mathcal{G}_x)|_{\mathbb{H}_i}}^*$ [cf. [CbTpI], Definition 2.2, (ii)] of the composite

$$\Pi_{(\mathcal{G}_x)|_{\mathbb{H}_s}} \hookrightarrow \Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}^*$$

— where the first arrow is the natural outer injection discussed in [CbTpI], Proposition 2.11. On the other hand, if we write $\Pi^*_{(\mathcal{G}_x)_{\to \{e^{\mathbf{F}}\}}}$ for the maximal almost pro- Σ quotient of $\Pi_{(\mathcal{G}_x)_{\sim\{e_i^{\mathrm{F}}\}}}$ [cf. [CbTpI], Definition 2.8] determined by the maximal almost pro- Σ quotient $\Pi^*_{\mathcal{G}_r}$ and the natural outer isomorphism $\Phi_{(\mathcal{G}_x)_{\sim \{e_i^{\mathrm{F}}\}}} \colon \Pi_{(\mathcal{G}_x)_{\sim \{e_i^{\mathrm{F}}\}}} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ [cf. [CbTpI], Definition 2.10], then $\Pi_{\mathbf{F}_i}^*$ may be regarded as a *verticial subgroup* of $\Pi^*_{(\mathcal{G}_x)_{\to \{e^F\}}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x} \xleftarrow{\sim} \Pi^*_F$ [cf. [CbTpI], Proposition 2.9, (i), (3)]. Thus, it follows from Proposition 1.7, (vii), that $\Pi_{\mathbf{F}_i}^*$ is commensurably terminal in $\Pi_{\rm F}^*$. Moreover, by applying a similar argument to the argument given in [CmbCsp], Definition 2.1, (iii), (vi), or [NodNon], Definition 5.1, (ix), (x) [i.e., roughly speaking, by considering the portion of the underlying scheme X_2 of X_2^{\log} corresponding to the underlying scheme $(X_{v_i})_2$ of the 2-nd log configuration space $(X_{v_i})_2^{\log}$ of the stable log curve $X_{v_i}^{\log}$ determined by $\mathcal{G}|_{v_i}$ — cf. [CbTpI], Definition 2.1, (iii)], one concludes that there exists a verticial subgroup $\Pi_{v_i} \subseteq \Pi_{\mathcal{G}} \stackrel{\sim}{\leftarrow} \Pi_{\mathrm{B}}$ associated to $v_i \in \mathrm{Vert}(\mathcal{G})$ such that the outer action of Π_{v_i} on Π_{F}^* determined by the composite $\Pi_{v_i} \hookrightarrow \Pi_{\mathrm{B}} \xrightarrow{\rho_{2/1}^*} \mathrm{Out}(\Pi_{\mathrm{F}}^*)$ — where we write $\rho_{2/1}^*$ for the outer action determined by the exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\rm F}^* \longrightarrow \Pi_{\rm T}^* \longrightarrow \Pi_{\rm B} \longrightarrow 1$$

— preserves the $\Pi_{\rm F}^*$ -conjugacy class of the commensurably terminal subgroup $\Pi_{{\rm F}_i}^* \subseteq \Pi_{\rm F}^*$ [so we obtain a natural outer representation $\Pi_{v_i} \to {\rm Out}(\Pi_{{\rm F}_i}^*)$ — cf. [CbTpI], Lemma 2.12, (iii)], and, moreover, that if we write $\Pi_{{\rm T}_i}^* \stackrel{\text{def}}{=} \Pi_{{\rm F}_i}^* \stackrel{\text{out}}{\rtimes} \Pi_{v_i} (\subseteq \Pi_{\rm T}^*)$ [cf. the discussion entitled "Topological groups" in [CbTpI], §0], then it follows from Proposition 1.7, (ii), that $\Pi_{{\rm T}_i}^*$ is naturally isomorphic to a profinite group of the form " $\Pi_{\rm T}^*$ " obtained by taking " \mathcal{G} " to be $\mathcal{G}|_{v_i}$.

Now since $\widehat{\beta}_i^*(\Pi_{F_i}^*) = \Pi_{F_i}^*$, and $\widetilde{\alpha}^*$ is an automorphism over the quotient $\Pi_F^*/\Pi_T^* \xrightarrow{\sim} \Pi_B$, one verifies immediately that $\widetilde{\beta}_i$ determines an automorphism $\widetilde{\beta}_{T_i}^*$ of $\Pi_{T_i}^*$ over Π_{v_i} . Thus, since the quantity "3g - 3 + r" associated to $\mathcal{G}|_{v_i}$ is < 3g - 3 + r, by considering a diagram similar to the diagram of [CmbCsp], Definition 2.1, (vi), or [NodNon], Definition 5.1,

(x), and applying the *induction hypothesis* [cf. also Lemma 2.6, (ii)], we conclude — by replacing $\Pi_{\rm F}^{**}$ by a suitable *base-admissible* maximal almost pro- Σ quotient $\Pi_{\rm F} \to \Pi_{\rm F}^{**}$ that *dominates* the *original* $\Pi_{\rm F} \to \Pi_{\rm F}^{**}$ and applying the conclusion that $\tilde{\beta}_i^*$ determines an automorphism of $\Pi_{{\rm T}_i}^*$ over Π_{v_i} in the case where " $\tilde{\alpha}^{**}$ " is taken to be $\tilde{\alpha}^{***} \in {\rm Aut}(\Pi_{\rm T}^{***})$ that we may assume without loss of generality that

(\ddagger_6): $\widetilde{\beta}^*_{\mathbf{T}_i}$ is a $\Pi^*_{\mathbf{F}_i}$ -inner automorphism.

In particular, it follows immediately, by allowing $i \in \{1, 2\}$ to vary, from Proposition 1.7, (vii) [which implies the commensurable terminality of $\Pi^*_{(v_i)_T^F} \subseteq \Pi^*_{F_i}$], that the outomorphisms of $\Pi^*_{(\mathcal{G}_x)_{\longrightarrow \{e_1^F\}}}$, $\Pi^*_{(\mathcal{G}_x)_{\longrightarrow \{e_2^F\}}}$ obtained by conjugating α^*_F by the isomorphisms $\Pi^*_{(\mathcal{G}_x)_{\longrightarrow \{e_1^F\}}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x}$ [induced by $\Phi_{(\mathcal{G}_x)_{\longrightarrow \{e_1^F\}}}$], $\Pi^*_{(\mathcal{G}_x)_{\longrightarrow \{e_2^F\}}} \xrightarrow{\sim} \Pi^*_{\mathcal{G}_x}$ [induced by $\Phi_{(\mathcal{G}_x)_{\longrightarrow \{e_1^F\}}}$], $\Pi^*_{(\mathcal{G}_x)_{\longrightarrow \{e_1^F\}}}$, $\Pi^*_{(\mathcal{G}_x)_{\longrightarrow \{e_2^F\}}}$, respectively. Thus, it follows from Lemma 1.10 that α^*_F is the *identity outomorphism*. This completes the proof of Claim 2.8.C, hence also of Lemma 2.8.

Theorem 2.9 (Almost pro- Σ analogue of the injectivity portion of the theory of combinatorial cuspidalization). Let Σ be a nonempty set of prime numbers, n a positive integer, (q, r) a pair of nonnegative integers such that 2g - 2 + r > 0, and X a hyperbolic **curve** of type (q,r) over an algebraically closed field of characteristic zero. For each positive integer i, write X_i for the *i*-th configuration space of X [cf. [MzTa], Definition 2.1, (i)]; Π_i for the pro-Primes configuration space group [cf. [MzTa], Definition 2.3, (i)] given by the étale fundamental group $\pi_1(X_i)$ of X_i . Also, we shall write pr: $X_{n+1} \twoheadrightarrow X_n$ for the projection obtained by forgetting the (n+1)st factor and $\Pi_{n+1/n} \subseteq \Pi_{n+1}$ for the kernel of some fixed surjection $\mathrm{pr}^{\Pi} \colon \Pi_{n+1} \twoheadrightarrow \Pi_n$ [that belongs to the collection of surjections that constitutes the outer surjection induced by pr. Let $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^*$ be a quotient of Π_{n+1} such that the quotient $\Pi_{n+1/n}^*$ of $\Pi_{n+1/n} \subseteq \Pi_{n+1}$ determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^*$ is a maximal almost pro- Σ **quotient** of $\Pi_{n+1/n}$ [cf. Definition 1.1]. Then there exists a quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{**}$ of Π_{n+1} such that the following conditions are satisfied:

- (i) The quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{**}$ dominates [cf. the discussion entitled "Topological groups" in §0] the quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{*}$ [*i.e.*, $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{**} \longrightarrow \Pi_{n+1}^{*}$].
- (ii) The quotient $\Pi_{n+1/n}^{**}$ of $\Pi_{n+1/n} \subseteq \Pi_{n+1}$ determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{**}$ is a maximal almost pro- Σ quotient of $\Pi_{n+1/n}$.

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(iii) Let α* be an outomorphism of Π^{*}_{n+1} and Π_{n+1} → Π^{***}_{n+1} a quotient that dominates the quotient Π_{n+1} → Π^{***}_{n+1} and induces a maximal almost pro-Σ quotient Π^{***}_{n+1/n} of Π_{n+1/n}. Suppose that α* arises from an FC-admissible [cf. Definition 2.1, (v)] automorphism α̃^{***} of Π^{***}_{n+1} over Π^{***}_n [i.e., which induces the identity automorphism of Π^{***}_{n+1}] — where we write Π^{***}_n for the quotient of Π_n determined by the quotient Π_{n+1} → Π^{***}_{n+1}. Then α* is the identity outomorphism.

Proof. First, we claim that the following assertion holds:

Claim 2.9.A: To verify Theorem 2.9, it suffices to verify Theorem 2.9 in the case where the kernel of the natural surjection $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^*$ is contained in $\Pi_{n+1/n}$, i.e., the natural surjection

$$\Pi_n \stackrel{\sim}{\leftarrow} \Pi_{n+1} / \Pi_{n+1/n} \twoheadrightarrow \Pi_{n+1}^* / \Pi_{n+1/n}^*$$

— where the first arrow is the natural isomorphism — is an *isomorphism*.

Indeed, Claim 2.9.A follows immediately, by considering the objects obtained by base-changing the various objects that appear in the case of an *arbitrary* quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^*$ via the natural surjection $\Pi_n \stackrel{\sim}{\leftarrow} \Pi_{n+1}/\Pi_{n+1/n} \twoheadrightarrow \Pi_{n+1}^*/\Pi_{n+1/n}^*$. By Claim 2.9.A, we may assume without loss of generality that the kernel of $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^*$ is contained in $\Pi_{n+1/n}$.

Next, we claim that the following assertion holds:

Claim 2.9.B: To verify Theorem 2.9, it suffices to verify Theorem 2.9 in the case where n = 1.

Indeed, suppose that $n \geq 2$, and that Theorem 2.9 holds whenever n = 1. Write $\prod_{n+1/n-1} \subseteq \prod_{n+1}$ for the kernel of the outer surjection $\Pi_{n+1} \twoheadrightarrow \Pi_{n-1}$ induced by the projection $X_{n+1} \to X_{n-1}$ obtained by forgetting the (n+1)-st and n-th factors of X_{n+1} ; $\Pi_{n/n-1} \subseteq \Pi_n$ for the kernel of the outer surjection $\Pi_n \twoheadrightarrow \Pi_{n-1}$ induced by the projection $X_n \to X_{n-1}$ obtained by forgetting the *n*-th factor of X_n ; $\prod_{n+1/n-1}^*$ for the quotient of $\Pi_{n+1/n-1}$ determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^*$. Then let us recall [cf. [MzTa], Proposition 2.4, (i)] that one may interpret the surjection $\Pi_{n+1/n-1}^* \twoheadrightarrow \Pi_{n/n-1}$ induced by the fixed surjection $\mathrm{pr}^{\Pi} \colon \Pi_{n+1} \twoheadrightarrow \Pi_n$ as the surjection " $\mathrm{pr}^{\Pi} \colon \Pi_2^* \twoheadrightarrow \Pi_1$ " in the case where "X" is of type (g, r + n - 1). Thus, by applying Theorem 2.9 in the case where n = 1 to the quotient $\prod_{n+1/n-1} \twoheadrightarrow \prod_{n+1/n-1}^{*}$, we obtain a quotient $\Pi_{n+1/n-1}^{**}$ of $\Pi_{n+1/n-1}$ which satisfies conditions (i), (ii), (iii) in the statement of Theorem 2.9. [Here, we note that since the kernel of $\Pi_{n+1/n-1} \twoheadrightarrow \Pi_{n+1/n-1}^*$ is contained in $\Pi_{n+1/n}$, the kernel of $\Pi_{n+1/n-1} \twoheadrightarrow \Pi_{n+1/n-1}^{**}$ is also contained in $\Pi_{n+1/n}$.]

Next, let $N \subseteq \prod_{n+1/n}$ be a normal open subgroup of $\prod_{n+1/n}$ with respect to which $\Pi_{n+1/n}^{**}$ is a maximal almost pro- Σ quotient of $\Pi_{n+1/n}$. Then it follows immediately from Lemma 1.2, (iii) [cf. also [MzTa], Proposition 2.2, (ii)], that we may assume without loss of generality – by replacing N by a suitable normal open subgroup contained in N that the kernel of $\Pi_{n+1/n} \twoheadrightarrow \Pi_{n+1/n}^{**}$ is *normal* in Π_{n+1} . Write Π_{n+1}^{**} for the quotient of Π_{n+1} by the kernel of $\Pi_{n+1/n} \twoheadrightarrow \Pi_{n+1/n}^{**}$. Then it is immediate that this quotient Π_{n+1}^{**} satisfies conditions (i), (ii) in the statement of Theorem 2.9, and, moreover, that the kernel of $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{**}$ is contained in $\Pi_{n+1/n}$. To verify that Π_{n+1}^{**} satisfies condition (iii) in the statement of Theorem 2.9, let $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{***}$ be a quotient as in condition (iii) in the statement of Theorem 2.9 and $\tilde{\alpha}^*$ an automorphism of Π_{n+1}^* which arises from an *FC-admissible* automorphism $\widetilde{\alpha}^{***}$ of Π_{n+1}^{***} over Π_n . Then since $\tilde{\alpha}^{***}$ is *FC-admissible*, it is immediate that $\tilde{\alpha}^{***}$ preserves $\Pi_{n+1/n-1}^{***} \subseteq \Pi_{n+1}^{***}$, where we write $\Pi_{n+1/n-1}^{***}$ for the quotient of $\Pi_{n+1/n-1}$ determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{***}$. In particular, it follows from our choice of $\Pi_{n+1/n-1}^{**}$, together with the fact that $\tilde{\alpha}^{***}$ is an automorphism of Π_{n+1}^{***} over Π_n [which implies that $\tilde{\alpha}^*$ is an automorphism of Π_{n+1}^* over Π_n , that we may assume without loss of generality — i.e., by replacing $\widetilde{\alpha}^*$ by a suitable $\Pi^*_{n+1/n-1}$ -conjugate, which may in fact [in light of the *slimness* of $\Pi_{n/n-1}$ — cf., e.g., [CmbGC], Remark 1.1.3] be taken to be a $\Pi_{n+1/n}^*$ -conjugate — that the automorphism of $\Pi^*_{n+1/n}$ induced by $\widetilde{\alpha}^*$ is the *identity automorphism*. Thus, since $\widetilde{\alpha}^*$ is an automorphism of Π_{n+1}^* over Π_n , and $\Pi_{n+1/n}^*$ is *slim* [cf. Proposition 1.7,

(i)], we may apply the natural isomorphism $\Pi_{n+1}^* \xrightarrow{\sim} \Pi_{n+1/n}^* \xrightarrow{\text{out}} \Pi_n$ [cf. the discussion entitled "*Topological groups*" in [CbTpI], §0] to conclude [cf., e.g., [Hsh], Lemma 4.10] that the automorphism $\tilde{\alpha}^*$ of Π_{n+1}^* is the *identity automorphism*. In particular, we conclude that Π_{n+1}^{**} satisfies condition (iii) in the statement of Theorem 2.9. This completes the proof of Claim 2.9.B.

By Claim 2.9.B, we may assume without loss of generality that n = 1. On the other hand, if n = 1, then one verifies easily that Theorem 2.9 follows immediately from Lemma 2.8. This completes the proof of Theorem 2.9.

Corollary 2.10 (Almost pro-*l* analogue of the injectivity portion of the theory of combinatorial cuspidalization). Let *l* be a prime number, *n* a positive integer, (g, r) a pair of nonnegative integers such that 2g - 2 + r > 0, and *X* a hyperbolic curve of type (g, r) over an algebraically closed field of characteristic zero. For each positive integer *i*, write X_i for the *i*-th configuration space of X [cf. [MzTa], Definition 2.1, (i)]; Π_i for the pro- \mathfrak{Primes} configuration space group [cf. [MzTa], Definition 2.3, (i)] given by the étale fundamental group $\pi_1(X_i)$ of X_i . Let $\Pi_{n+1} \to \Pi_{n+1}^*$ be an **F**characteristic SA-maximal almost pro-*l* quotient of Π_{n+1} [cf. Definition 2.1, (ii), (iii)]. Then there exists an **F**-characteristic SAmaximal almost pro-*l* quotient $\Pi_{n+1} \to \Pi_{n+1}^{**}$ of Π_{n+1} such that $\Pi_{n+1} \to \Pi_{n+1}^{**}$ dominates [cf. the discussion entitled "Topological groups" in §0] the quotient $\Pi_{n+1} \to \Pi_{n+1}^*$, and, moreover, satisfies the following property: For any **F**-characteristic SA-maximal almost pro-*l* quotient $\Pi_{n+1} \to \Pi_{n+1}^{***}$ of Π_{n+1} that dominates the quotient $\Pi_{n+1} \to \Pi_{n+1}^{***}$, the image of the composite

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1}^{***} \twoheadrightarrow \Pi_{n+1}^{*}) \cap \operatorname{Ker}\left(\operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1}^{***}) \to \operatorname{Out}^{\operatorname{FC}}(\Pi_{n}^{***})\right) \hookrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1}^{***} \twoheadrightarrow \Pi_{n+1}^{*}) \twoheadrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1}^{*} \twoheadleftarrow \Pi_{n+1}^{***}) \hookrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1}^{*})$$

[cf. Definition 2.1, (vii), (viii)] — where we write Π_n^{***} for the quotient of Π_n determined by the quotient $\Pi_{n+1} \twoheadrightarrow \Pi_{n+1}^{***}$, and the homomorphism $\operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1}^{***}) \to \operatorname{Out}^{\operatorname{FC}}(\Pi_n^{***})$ [in large parentheses] is the homomorphism induced by the projection $X_{n+1} \to X_n$ obtained by forgetting the (n+1)-st factor — is trivial.

Proof. This follows immediately from Theorem 2.9, together with Proposition 2.3, (ii). \Box

Remark 2.10.1.

- (i) Theorem 2.9 and Corollary 2.10 may be regarded, respectively, as almost pro-Σ, almost pro-l versions of the injectivity portion of [NodNon], Theorem B. In this context, it is of interest to recall that the pro-l version of this sort of injectivity result may also be obtained by means of the Lie-theoretic approach of [Tk]. On the other hand, it does not appear, at the time of writing, that this Lie-theoretic approach may be extended so as to yield an alternate proof either of the profinite portion of the injectivity result of [NodNon], Theorem B, or of the almost pro-Σ/pro-l versions of this result given in Theorem 2.9, Corollary 2.10 of the present paper.
- (ii) In the context of the observations of (i), it is of interest to recall that the various injectivity results of [NodNon] and the present paper that are discussed in (i) are obtained as consequences of various combinatorial versions of the Grothendieck Conjecture. From this point of view, it seems natural to pose the following question:

Is it possible to prove a *Lie-theoretic combinatorial* version of the Grothendieck Conjecture that allows one to derive the *Lie-theoretic injectivity* results of

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[Tk] by means of techniques analogous to the techniques applied in [NodNon] and the present paper?At the time of writing, it is not clear to the authors whether or not this question may be answered in the affirmative.

In the remainder of §2, we consider an *almost pro-l analogue* of the *tripod homomorphism* of [CbTpII], Definition 3.19.

Lemma 2.11 (Commensurators of various subgroups of geometric origin). We shall apply the notational conventions established in §3 of [CbTpII]. In the notation of [CbTpII], Lemma 3.6, suppose that (j,i) = (1,2); $E = \{i,j\}$; $z_{i,j,x} \in \text{Edge}(\mathcal{G}_{j \in E \setminus \{i\},x})$. [Thus, $\mathcal{G}_{j\in E\setminus\{i\},x} = \mathcal{G}_{i\in E\setminus\{j\},x} = \mathcal{G}; \ \Pi_2 = \Pi_E; \ \Pi_1 = \Pi_{\{j\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{j\in E\setminus\{i\},x}} = \Pi_{\mathcal{G}};$ $\Pi_{2/1} = \Pi_{E/(E \setminus \{i\})} \xrightarrow{\sim} \Pi_{\mathcal{G}_{i \in E, x}}.] \quad Write \ \mathcal{G}_{2/1} \stackrel{\text{def}}{=} \mathcal{G}_{i \in E, x}; \ \mathcal{G}_{1 \setminus 2} \stackrel{\text{def}}{=} \mathcal{G}_{j \in E, x};$ $p_{1\backslash 2}^{\Pi} \stackrel{\text{def}}{=} p_{E/\{2\}}^{\Pi} \colon \Pi_2 \twoheadrightarrow \Pi_{\{2\}}; \ \Pi_{1\backslash 2} \stackrel{\text{def}}{=} \operatorname{Ker}(p_{1\backslash 2}^{\Pi}) = \Pi_{E/\{2\}} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_{1\backslash 2}};$ $z_x \stackrel{\text{def}}{=} z_{i,j,x} \in \text{Edge}(\mathcal{G}); \ c^{\text{diag}} \stackrel{\text{def}}{=} c^{\text{diag}}_{i,j,x} \in \text{Cusp}(\mathcal{G}_{2/1}) \ [cf. the notation]$ of [CbTpII], Lemma 3.6, (ii)]; $v^{\text{new}} \stackrel{\text{def}}{=} v^{\text{new}}_{i,j,x} \in \text{Vert}(\mathcal{G}_{2/1})$ [cf. the no-tation of [CbTpII], Lemma 3.6, (iv)]. Let $\Pi_{z_x} \subseteq \Pi_1$ be an edge-like subgroup associated to $z_x \in \text{Edge}(\mathcal{G})$; $\Pi_{v^{\text{new}}} \subseteq \Pi_{2/1}$ a verticial subgroup associated to v^{new} ; $\Pi_{c^{\text{diag}}} \subseteq \Pi_{2/1}$ a cuspidal subgroup associated to c^{diag} that is contained in $\Pi_{v^{\text{new}}}$ [CbTpII], Lemma 3.6, (iv)]. Let $\Pi_2 \twoheadrightarrow \Pi_2^*$ be an SA-maximal almost pro-*l* quotient of Π_2 [cf. Definition 2.1, (ii)]. Write $\Pi_{2/1}^*$, $\Pi_{1\backslash 2}^*$, Π_1^* , $\Pi_{\{2\}}^*$ for the respective quotients of $\Pi_{2/1}$, $\Pi_{1\backslash 2}$, Π_1 , $\Pi_{\{2\}}$ determined by the quotient $\Pi_2 \twoheadrightarrow \Pi_2^*$ of Π_2 ; $\Pi_{\mathcal{G}}^*$, $\Pi_{\mathcal{G}_{2/1}}^*$ for the respective quotients of $\Pi_{\mathcal{G}}$, $\Pi_{\mathcal{G}_{2/1}}$ determined by the quotients $\Pi_1 \twoheadrightarrow \Pi_1^*$, $\Pi_{2/1} \twoheadrightarrow \Pi_{2/1}^*$ and the isomorphisms $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}}$, $\Pi_{2/1} \xrightarrow{\sim} \Pi_{\mathcal{G}_{2/1}} \text{ fixed in [CbTpII], Definition 3.1, (iii); } (p_{2/1}^{\Pi})^* \colon \Pi_2^* \twoheadrightarrow \Pi_1^*,$ $(p_{1\backslash 2}^{\Pi})^* \colon \Pi_2^* \twoheadrightarrow \Pi_{\{2\}}^* \text{ for the respective natural surjections induced by}$ $p_{2/1}^{\Pi} \colon \Pi_2 \twoheadrightarrow \Pi_1, \ p_{1\backslash 2}^{\Pi} \colon \Pi_2 \twoheadrightarrow \Pi_{\{2\}}; \ \Pi_{z_x}^* \subseteq \Pi_1^*, \ \Pi_{c^{\text{diag}}}^* \subseteq \Pi_{z_1}^* \text{ in } \Pi_{z_1}^* \text{ for the respective images of } \Pi_{z_x} \subseteq \Pi_1, \ \Pi_{c^{\text{diag}}} \subseteq \Pi_{v^{\text{new}}} \subseteq \Pi_{2/1} \text{ in } \Pi_1^*,$ $\Pi_{2/1}^*; \ \Pi_2^*|_{z_x} \ \stackrel{\text{def}}{=} \ \Pi_2^* \times_{\Pi_1^*} \Pi_{z_x}^* \ \subseteq \ \Pi_2^*; \ D_{c^{\text{diag}}}^* \ \stackrel{\text{def}}{=} \ N_{\Pi_2^*}(\Pi_{c^{\text{diag}}}^*); \ I_{v^{\text{new}}}^*|_{z_x}$ $Z_{\Pi_2^*|z_r}(\Pi_{v^{\text{new}}}^*) \subseteq D_{v^{\text{new}}}^*|_{z_r} \stackrel{\text{def}}{=} N_{\Pi_2^*|z_r}(\Pi_{v^{\text{new}}}^*).$ Then the following hold:

- (i) It holds that $D^*_{c^{\text{diag}}} \cap \Pi^*_{2/1} = C_{\Pi^*_2}(\Pi^*_{c^{\text{diag}}}) \cap \Pi^*_{2/1} = \Pi^*_{c^{\text{diag}}}.$
- (ii) It holds that $C_{\Pi_2^*}(\Pi_{c^{\text{diag}}}^*) = D_{c^{\text{diag}}}^*$.
- (iii) The surjection $(p_{2/1}^{\Pi})^* \colon \Pi_2^* \twoheadrightarrow \Pi_1^*$ determines an isomorphism $D_{c^{\text{diag}}}^*/\Pi_{c^{\text{diag}}} \xrightarrow{\sim} \Pi_1^*$. Moreover, the composite

$$\Pi_1 \twoheadrightarrow \Pi_1^* \stackrel{\sim}{\leftarrow} D^*_{c^{\text{diag}}} / \Pi^*_{c^{\text{diag}}} \twoheadrightarrow \Pi^*_{\{2\}}$$

— where the first arrow is the natural surjection, the second arrow is the isomorphism obtained above, and the third arrow is the surjection determined by $(p_{1\backslash 2}^{\Pi})^* \colon \Pi_2^* \to \Pi_{\{2\}}^*$ — coincides, up to composition with an inner automorphism, with the natural surjection $\Pi_1 \to \Pi_{\{2\}}^*$.

- (iv) The composite $I_{v^{\text{new}}}^*|_{z_x} \hookrightarrow D_{v^{\text{new}}}^*|_{z_x} \to \Pi_{z_x}^*$ is an isomorphism.
- (v) The natural inclusions $\Pi_{v^{\text{new}}}^*$, $I_{v^{\text{new}}}^*|_{z_x} \hookrightarrow D_{v^{\text{new}}}^*|_{z_x}$ determine an isomorphism $\Pi_{v^{\text{new}}}^* \times I_{v^{\text{new}}}^*|_{z_x} \xrightarrow{\sim} D_{v^{\text{new}}}^*|_{z_x} = C_{\Pi_2^*|_{z_x}}(\Pi_{v^{\text{new}}}^*).$
- (vi) It holds that $C_{\Pi_2^*}(D_{v^{\text{new}}}^*|_{z_x}) \subseteq C_{\Pi_2^*}(\Pi_{v^{\text{new}}}^*).$
- (vii) $D_{v^{\text{new}}}^*|_{z_x}$ is commensurably terminal in Π_2^* .

Proof. First, we verify assertion (i). Observe that we have inclusions $\Pi^*_{c^{\text{diag}}} \subseteq D^*_{c^{\text{diag}}} \subseteq C_{\Pi^*_2}(\Pi^*_{c^{\text{diag}}})$. Thus, since $\Pi^*_{c^{\text{diag}}}$ is commensurably terminal in $\Pi^*_{2/1}$ [cf. Proposition 1.7, (vii)], we conclude that $\Pi^*_{c^{\text{diag}}} \subseteq D^*_{c^{\text{diag}}} \cap \Pi^*_{2/1} \subseteq C_{\Pi^*_2}(\Pi^*_{c^{\text{diag}}}) \cap \Pi^*_{2/1} = C_{\Pi^*_{2/1}}(\Pi^*_{c^{\text{diag}}}) = \Pi^*_{c^{\text{diag}}}$. This completes the proof of assertion (i). Assertions (ii), (iii) follow immediately from assertion (i), together with the [easily verified] fact that the composite $D^*_{c^{\text{diag}}} \hookrightarrow \Pi^*_2 \xrightarrow{(p^{\Pi}_{2/1})^*} \Pi^*_1$ is surjective.

Next, we verify assertion (iv). Since $\Pi_{v^{\text{new}}}^*$ is *slim* and *commensurably* terminal in $\Pi_{2/1}^*$ [cf. Proposition 1.7, (ii), (vii)], it follows that $I_{v^{\text{new}}}^*|_{z_x} \cap \Pi_{2/1}^* = \{1\}$, which implies the *injectivity* of the composite in question. On the other hand, since the composite $I_{v^{\text{new}}}|_{z_x} \hookrightarrow D_{v^{\text{new}}}|_{z_x} \hookrightarrow \Pi_2|_{z_x} \twoheadrightarrow \Pi_{z_x}$ is surjective [cf. [CbTpII], Lemma 3.11, (iv)], it follows immediately that the composite $I_{v^{\text{new}}}|_{z_x} \hookrightarrow D_{v^{\text{new}}}|_{z_x} \twoheadrightarrow \Pi_{z_x}^*$ is surjective. This completes the proof of assertion (iv).

Next, we verify assertion (v). It follows immediately from assertion (iv), together with the *commensurable terminality* of $\Pi_{v^{\text{new}}}^*$ in $\Pi_{2/1}^*$ [cf. Proposition 1.7, (vii)], that we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Pi_{v^{\text{new}}}^* \longrightarrow D_{v^{\text{new}}}^* |_{z_x} \longrightarrow \Pi_{z_x}^* \longrightarrow 1$$

— where we observe that the inclusion $I_{v^{new}}^*|_{z_x} \hookrightarrow D_{v^{new}}^*|_{z_x}$ determines a *splitting* of this exact sequence. Thus, it follows from the definition of $I_{v^{new}}^*|_{z_x}$ that the natural inclusions $\Pi_{v^{new}}^*$, $I_{v^{new}}^*|_{z_x} \hookrightarrow D_{v^{new}}^*|_{z_x}$ determine an *isomorphism* $\Pi_{v^{new}}^* \times I_{v^{new}}^*|_{z_x} \xrightarrow{\sim} D_{v^{new}}^*|_{z_x}$. On the other hand, again by the *commensurable terminality* of $\Pi_{v^{new}}^*$ in $\Pi_{2/1}^*$ [cf. Proposition 1.7, (vii)], the above displayed sequence implies that $D_{v^{new}}^*|_{z_x} = C_{\Pi_2^*|_{z_x}}(\Pi_{v^{new}}^*)$. This completes the proof of assertion (v).

Next, we verify assertion (vi). It follows from the *commensurable* terminality of $\Pi_{v^{\text{new}}}^*$ in $\Pi_{2/1}^*$ [cf. Proposition 1.7, (vii)] that $D_{v^{\text{new}}}^*|_{z_x} \cap$ $\Pi_{2/1}^* = \Pi_{v^{\text{new}}}^*$. Thus, since $\Pi_{2/1}^*$ is normal in Π_2^* , assertion (vi) follows immediately from [CbTpII], Lemma 3.9, (i). This completes the proof of assertion (vi).

Finally, we verify assertion (vii). Since $\Pi_{z_x}^* \subseteq \Pi_1^*$ is commensurably terminal in Π_1^* [cf. Proposition 1.7, (vii)], it follows from the surjectivity of the composite $D_{v^{\text{new}}|z_x}^* \hookrightarrow \Pi_2^*|_{z_x} \to \Pi_{z_x}^*$ [cf. assertion (iv)] that $C_{\Pi_2^*}(D_{v^{\text{new}}|z_x}^*) \subseteq \Pi_2^*|_{z_x}$. In particular, it follows immediately from assertions (v), (vi) that $D_{v^{\text{new}}|z_x}^* \subseteq C_{\Pi_2^*}(D_{v^{\text{new}}|z_x}^*) \subseteq$ $C_{\Pi_2^*}(\Pi_{v^{\text{new}}}^*) \cap \Pi_2^*|_{z_x} = C_{\Pi_2^*|z_x}(\Pi_{v^{\text{new}}}^*) = D_{v^{\text{new}}|z_x}^*$. This completes the proof of assertion (vii).

Lemma 2.12 (Commensurator of a tripod arising from an edge). In the notation of Lemma 2.11, let $\Pi_2 \rightarrow \Pi_2^{**}$ be an SA-maximal almost pro-*l* quotient of Π_2 [cf. Definition 2.1, (ii)] that dominates $\Pi_2 \rightarrow \Pi_2^*$ [cf. the discussion entitled "Topological groups" in §0]. We shall use similar notation

$$\Pi_{2/1}^{**}; \ \Pi_{1\backslash 2}^{**}; \ \Pi_{1}^{**}; \ \Pi_{\{2\}}^{**}; \ \Pi_{\mathcal{G}}^{**}; \ \Pi_{\mathcal{G}_{2/1}}^{**}; (p_{2/1}^{\Pi})^{**}: \Pi_{2}^{**} \twoheadrightarrow \Pi_{1}^{**}; \ (p_{1\backslash 2}^{\Pi})^{**}: \Pi_{2}^{**} \twoheadrightarrow \Pi_{\{2\}}^{**}; \Pi_{z_{x}}^{**} \subseteq \Pi_{1}^{**}; \ \Pi_{c^{\text{diag}}}^{**} \subseteq \Pi_{v^{\text{new}}}^{**} \subseteq \Pi_{2/1}^{**}; \Pi_{2}^{**}|_{z_{x}}; \ D_{c^{\text{diag}}}^{**}; \ I_{v^{\text{new}}}^{**}|_{z_{x}} \subseteq D_{v^{\text{new}}}^{**}|_{z_{x}}$$

for objects associated to $\Pi_2 \twoheadrightarrow \Pi_2^{**}$ to the notation introduced in the statement of Lemma 2.11 for objects associated to $\Pi_2 \twoheadrightarrow \Pi_2^*$. Suppose that the natural [outer] surjection $\Pi_1 \twoheadrightarrow \Pi_{\{2\}}^{**}$ dominates the quotient $\Pi_1 \twoheadrightarrow \Pi_1^*$. Then the following hold:

- (i) The natural surjection $\Pi_2^{**} \twoheadrightarrow \Pi_2^*$ determines a surjection $I_{v^{\text{new}}|_{z_x}}^{**} \twoheadrightarrow I_{v^{\text{new}}|_{z_x}}^*$.
- (ii) The image of $Z_{\Pi_2^{\text{toc}}}^{\text{loc}}(\Pi_{v^{\text{new}}}^{**}) \subseteq \Pi_2^{**}$ [cf. the discussion entitled "Topological groups" in [CbTpII], §0] in Π_2^* coincides with $I_{v^{\text{new}}}^*|_{z_x}$.
- (iii) The image of $C_{\Pi_2^{**}}(\Pi_{v^{\text{new}}}) \subseteq \Pi_2^{**}$ in Π_2^* is contained in $D_{v^{\text{new}}}|_{z_x}$.
- (iv) The natural outer action, by conjugation, of $N_{\Pi_2^{**}}(\Pi_{v^{\text{new}}}^{**}) \subseteq \Pi_2^{**}$ on [not $\Pi_{v^{\text{new}}}^{**}$ but] $\Pi_{v^{\text{new}}}^{*}$ is trivial.

Proof. First, we verify assertion (i). Observe that it is immediate that the image of $I_{v^{new}}^{**}|_{z_x} \subseteq \Pi_2^{**}$ in Π_2^* is *contained* in $I_{v^{new}}^*|_{z_x}$. Thus, assertion (i) follows immediately from Lemma 2.11, (iv), together with the [easily verified] fact that the natural surjection $\Pi_2^{**} \to \Pi_2^*$ determines a *surjection* $\Pi_{z_x}^{**} \to \Pi_{z_x}^*$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $\operatorname{Im}(Z_{\Pi_{2}^{**}}^{\operatorname{loc}}(\Pi_{v^{\operatorname{new}}}^{**})) \subseteq \Pi_{2}^{*}$ for the image of $Z_{\Pi_{2}^{**}}^{\operatorname{loc}}(\Pi_{v^{\operatorname{new}}}^{**}) \subseteq \Pi_{2}^{**}$ in Π_{2}^{*} . Then it follows from assertion (i) that $I_{v^{\operatorname{new}}}^{*}|_{z_{x}} \subseteq \operatorname{Im}(Z_{\Pi_{2}^{**}}^{\operatorname{loc}}(\Pi_{v^{\operatorname{new}}}^{**}))$. Thus, to complete the verification of

assertion (ii), it suffices to verify that $\operatorname{Im}(Z_{\Pi_2^{**}}^{\operatorname{loc}}(\Pi_{v^{\operatorname{new}}}^{**})) \subseteq I_{v^{\operatorname{new}}}^*|_{z_x}$. To this end, let us observe that it follows immediately from the final portion of [CbTpII], Lemma 3.6, (iv), that the image $(p_{1\backslash 2}^{\Pi})^{**}(\Pi_{v^{\text{new}}}) \subseteq$ $\Pi_{\{2\}}^{**}$ coincides with the image of an edge-like subgroup of $\Pi_{\mathcal{G}} \stackrel{\sim}{\leftarrow} \Pi_1$ associated to $z_x \in \operatorname{Edge}(\mathcal{G})$ via the natural [outer] surjection $\Pi_1 \twoheadrightarrow$ $\Pi_{\{2\}}^{**}$, hence that the image [which is well-defined up to conjugacy] of $(p_{1\backslash 2}^{\Pi})^{**}(\Pi_{v^{new}}^{**}) \subseteq \Pi_{\{2\}}^{**}$ in Π_1^* [where we recall that we have assumed that $\Pi_1 \twoheadrightarrow \Pi_{\{2\}}^{**}$ dominates $\Pi_1 \twoheadrightarrow \Pi_1^*$ is an edge-like subgroup of $\Pi_{\mathcal{G}}^* \stackrel{\sim}{\leftarrow} \Pi_1^*$ associated to $z_x \in \operatorname{Edge}(\mathcal{G})$. Thus, since every edge-like subgroup of Π_1^* is commensurably terminal [cf. Proposition 1.7, (vii)], it follows that the image [which is well-defined up to conjugacy] of $(p_{1\backslash 2}^{\Pi})^{**}(Z_{\Pi_2^{**}}^{\mathrm{loc}}(\Pi_{v^{\mathrm{new}}}^{**})) \subseteq \Pi_{\{2\}}^{**}$ in Π_1^* is *contained* in an edge-like subgroup of $\Pi_{\mathcal{G}}^* \leftarrow \Pi_1^*$ associated to $z_x \in \operatorname{Edge}(\mathcal{G})$. On the other hand, since $\Pi_{c^{\text{diag}}}^{**} \subseteq \Pi_{v^{\text{new}}}^{**}, \text{ we have } Z_{\Pi_{2}^{**}}^{\text{loc}}(\Pi_{v^{\text{new}}}^{**}) \subseteq Z_{\Pi_{2}^{**}}^{\text{loc}}(\Pi_{c^{\text{diag}}}^{**}) \subseteq C_{\Pi_{2}^{**}}(\Pi_{c^{\text{diag}}}^{**}) =$ $D_{c^{\text{diag}}}^{**}$ [cf. Lemma 2.11, (ii)]. In particular, it follows immediately from Lemma 2.11, (iii), that the image of $(p_{2/1}^{\Pi})^{**}(Z_{\Pi_2^{**}}^{\text{loc}}(\Pi_{v^{\text{new}}}^{**})) \subseteq \Pi_1^{**}$ in Π_1^* is contained in some Π_1^* -conjugate of $\Pi_{z_x}^* \subseteq \Pi_1^*$, hence [since $I_{v^{\text{new}}}^{**}|_{z_x} \subseteq Z_{\Pi_2^{**}}^{\text{loc}}(\Pi_{v^{\text{new}}}^{**})$ surjects onto $\Pi_{z_x}^{**}$ — cf. Lemma 2.11, (iv)] that the image of $(p_{2/1}^{\Pi})^{**}(Z_{\Pi_2^{**}}^{\text{loc}}(\Pi_{v^{\text{new}}}^{**})) \subseteq \Pi_1^{**}$ in Π_1^* coincides with $\Pi_{z_x}^* \subseteq \Pi_1^*$ [cf. Proposition 1.7, (v)], i.e., $\operatorname{Im}(Z_{\Pi_2^{\mathrm{loc}}}^{\mathrm{loc}}(\Pi_{v^{\mathrm{new}}}^{**})) \subseteq \Pi_2^*|_{z_x}$. Thus, since [as is easily verified] $\operatorname{Im}(Z_{\Pi_2^{\mathrm{loc}}}^{\mathrm{loc}}(\Pi_{v^{\mathrm{new}}}^{**})) \subseteq Z_{\Pi_2^{\mathrm{toc}}}^{\mathrm{loc}}(\Pi_{v^{\mathrm{new}}}^{*})$, we conclude that

$$\mathrm{Im}(Z_{\Pi_{2}^{**}}^{\mathrm{loc}}(\Pi_{v^{\mathrm{new}}}^{**})) \subseteq \Pi_{2}^{*}|_{z_{x}} \cap Z_{\Pi_{2}^{*}}^{\mathrm{loc}}(\Pi_{v^{\mathrm{new}}}^{*}) = Z_{\Pi_{2}^{*}|_{z_{x}}}^{\mathrm{loc}}(\Pi_{v^{\mathrm{new}}}^{*}) = I_{v^{\mathrm{new}}}^{*}|_{z_{x}}$$

[where the final equality follows from Lemma 2.11, (v), together with the *slimness* portion of Proposition 1.7, (ii)]. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Write $\operatorname{Im}(C_{\Pi_{2}^{**}}(\Pi_{v^{\operatorname{new}}}^{**})) \subseteq \Pi_{2}^{*}$ for the image of $C_{\Pi_{2}^{**}}(\Pi_{v^{\operatorname{new}}}^{**}) \subseteq \Pi_{2}^{**}$ in Π_{2}^{*} . Then it follows from [CbTpII], Lemma 3.9, (ii), that $C_{\Pi_{2}^{**}}(\Pi_{v^{\operatorname{new}}}^{**}) \subseteq N_{\Pi_{2}^{**}}(Z_{\Pi_{2}^{**}}^{\operatorname{loc}}(\Pi_{v^{\operatorname{new}}}^{**}))$; thus, it follows from assertion (ii) that $\operatorname{Im}(C_{\Pi_{2}^{**}}(\Pi_{v^{\operatorname{new}}}^{**})) \subseteq N_{\Pi_{2}^{*}}(I_{v^{\operatorname{new}}}^{*}|_{z_{x}})$. In particular, since $D_{v^{\operatorname{new}}}^{*}|_{z_{x}}$ is topologically generated by $\Pi_{v^{\operatorname{new}}}^{*}$, $I_{v^{\operatorname{new}}}^{*}|_{z_{x}}$ [cf. Lemma 2.11, (v)], we conclude that

$$\operatorname{Im}(C_{\Pi_{2}^{**}}(\Pi_{v^{\operatorname{new}}}^{**})) \subseteq C_{\Pi_{2}^{*}}(D_{v^{\operatorname{new}}}^{*}|_{z_{x}}) = D_{v^{\operatorname{new}}}^{*}|_{z_{x}}$$

[cf. Lemma 2.11, (vii)]. This completes the proof of assertion (iii). Assertion (iv) follows immediately from assertion (iii), together with Lemma 2.11, (v). This completes the proof of Lemma 2.12. \Box

Corollary 2.13 (Almost pro-*l* quotients and tripod homomorphisms). In the notation of Definition 2.1, suppose that $n \geq 3$. Let

 $\Pi^{\text{tpd}} \subseteq \Pi_3$ be a **1-central** [{1, 2, 3}-]**tripod** of Π_3 [cf. [CbTpII], Definitions 3.3, (i); 3.7, (ii)]; $\Pi^{\text{tpd}} \rightarrow (\Pi^{\text{tpd}})^{\ddagger}$ an **almost pro-l** quotient. Then the following hold:

- (i) There exists an F-characteristic SA-maximal almost pro *l* quotient [cf. Definition 2.1, (ii), (iii)] Π_n^{*} of Π_n that satisfies the following condition: If we write Π₃^{*} for the quotient of Π₃ determined by the quotient Π_n → Π_n^{*} and (Π^{tpd})* ⊆ Π₃^{*} for the image of Π^{tpd} ⊆ Π₃ in Π₃^{*}, then the quotient Π^{tpd} → (Π^{tpd})* dominates the quotient Π^{tpd} → (Π^{tpd})[‡] [cf. the discussion entitled "Topological groups" in §0].
- (ii) Every element of the image $\operatorname{Im}(\mathfrak{T}_{\Pi^{\operatorname{tpd}}}) \subseteq \operatorname{Out}(\Pi^{\operatorname{tpd}})$ of the tripod homomorphism

$$\mathfrak{T}_{\Pi^{\mathrm{tpd}}} \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{tpd}})$$

associated to Π_n [cf. [CbTpII], Definition 3.19] **preserves** the kernel of the surjection $\Pi^{\text{tpd}} \twoheadrightarrow (\Pi^{\text{tpd}})^*$ of (i). Thus, we obtain a natural homomorphism

$$\operatorname{Im}(\mathfrak{T}_{\Pi^{\operatorname{tpd}}}) \longrightarrow \operatorname{Out}((\Pi^{\operatorname{tpd}})^*).$$

(iii) There exists an **F-characteristic SA-maximal almost pro** l quotient $\Pi_n \twoheadrightarrow \Pi_n^{**}$ of Π_n that dominates $\Pi_n \twoheadrightarrow \Pi_n^*$ [cf. (i)] such that the composite

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n) \twoheadrightarrow \operatorname{Im}(\mathfrak{T}_{\Pi^{\operatorname{tpd}}}) \to \operatorname{Out}((\Pi^{\operatorname{tpd}})^*)$$

— where the first arrow is the homomorphism induced by $\mathfrak{T}_{\Pi^{\text{tpd}}}$; the second arrow is the homomorphism of (ii) — factors through the natural surjection

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n) \twoheadrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n^{**} \twoheadleftarrow \Pi_n)$$

[cf. Definition 2.1, (viii); Remark 2.1.1]. Thus, we have a natural commutative diagram of profinite groups

$$\begin{array}{cccc} \operatorname{Out}^{\operatorname{FC}}(\Pi_n) & \longrightarrow & \operatorname{Im}(\mathfrak{T}_{\Pi^{\operatorname{tpd}}}) \\ & & & \downarrow \\ & & & \downarrow \\ \operatorname{Out}^{\operatorname{FC}}(\Pi_n^{**} \twoheadleftarrow \Pi_n) & \longrightarrow & \operatorname{Out}((\Pi^{\operatorname{tpd}})^*) \,. \end{array}$$

Proof. Assertion (i) is a consequence of Proposition 2.3, (iii). Assertion (ii) follows immediately from the fact that Π_n^* is *F*-characteristic, together with the definition of $\mathfrak{T}_{\Pi^{\text{tpd}}}$. Finally, we verify assertion (iii). Let us first observe that it follows immediately from the definition of $\mathfrak{T}_{\Pi^{\text{tpd}}}$, together with Proposition 2.3, (ii) [where we observe that any closed subgroup of a finite product of almost pro-l groups is almost pro-l], that, to verify assertion (iii), it suffices to verify the following assertion:

Claim 2.13.A: There exists an *F*-characteristic SAmaximal almost pro-l quotient $\Pi_3 \to \Pi_3^{**}$ of Π_3 that dominates $\Pi_3 \to \Pi_3^*$ such that if we write $(\Pi^{\text{tpd}})^{**} \subseteq$ Π_3^{**} for the image of $\Pi^{\text{tpd}} \subseteq \Pi_3$ in Π_3^{**} , then any automorphism of $(\Pi^{\text{tpd}})^*$ determined by conjugating by an element

$$\gamma^{**} \in N_{\Pi_3^{**}}((\Pi^{\text{tpd}})^{**})$$

is $(\Pi^{\text{tpd}})^*$ -inner.

To verify Claim 2.13.A, let $\Pi_3 \twoheadrightarrow \Pi_3^{**}$ be an F-characteristic SAmaximal almost pro-l quotient of Π_3 that dominates $\Pi_3 \twoheadrightarrow \Pi_3^*$ and $\gamma^{**} \in N_{\Pi_3^{**}}((\Pi^{\text{tpd}})^{**})$. Then it follows immediately from [CmbCsp], Proposition 1.9, (i), that $Z_{\Pi_3}(\Pi^{\text{tpd}}) \subseteq \Pi_3$ surjects onto Π_1 , hence also onto Π_1^{**} — where we write Π_1^{**} for the quotient of Π_1 determined by the quotient $\Pi_3 \twoheadrightarrow \Pi_3^{**}$. In particular, there exists an element $\tau \in Z_{\Pi_3}(\Pi^{\text{tpd}})$ such that the images of γ^{**} and τ in Π_1^{**} coincide. Thus, by replacing γ^{**} by the difference of γ^{**} and the image of τ in Π_3^{**} , we may assume without loss of generality that $\gamma^{**} \in \Pi_{3/1}^{**}$ — where we write $\Pi_{3/1}^{**}$ for the quotient of $\Pi_{3/1}$ [cf. Definition 2.1] induced by the quotient $\Pi_3 \twoheadrightarrow \Pi_3^{**}$. In particular, the existence of an F-characteristic SA-maximal almost pro-l quotient $\Pi_3 \twoheadrightarrow \Pi_3^{**}$ as in Claim 2.13.A follows immediately, in light of Proposition 2.3, (ii), from Lemma 2.12, (iv). This completes the proof of assertion (iii).

Finally, before proceeding, we review the following well-known result.

Lemma 2.14 (Automorphisms of stable log curves). Let l be a prime number. Write $l^{\text{aut}} \stackrel{\text{def}}{=} l$ if l is odd; $l^{\text{aut}} \stackrel{\text{def}}{=} 4$ if l is even. If G is a profinite group, then we shall refer to the tensor product with $\mathbb{Z}/l^{\text{aut}}\mathbb{Z}$ of the abelianization of G as the l^{aut} -abelianization of G. Let (g, r) be a pair of nonnegative integers such that 2g - 2 + r > 0.

- (i) Let k be an algebraically closed field such that l is invertible in k, $(\operatorname{Spec} k)^{\log}$ the log scheme obtained by equipping $\operatorname{Spec} k$ with the log structure determined by the fs chart $\mathbb{N} \to k$ that maps $1 \mapsto 0$, X^{\log} a stable log curve over $(\operatorname{Spec} k)^{\log}$, and α an automorphism of X^{\log} over $(\operatorname{Spec} k)^{\log}$. Write Π_1 for the maximal pro-l quotient of the kernel of the natural outer surjection $\pi_1(X^{\log}) \to \pi_1((\operatorname{Spec} k)^{\log})$. Suppose that α acts trivially on the l^{aut} -abelianization of Π_1 . Then α is the identity automorphism.
- (ii) Write M^{log} for the moduli stack of pointed stable curves of type (g,r) over Z[1/l], where we regard the marked points as unordered, equipped with the log structure determined by the divisor at infinity, and C^{log} → M^{log} for the tautological stable

log curve over \mathcal{M}^{\log} . Write $\mathcal{N}^{\log} \to \mathcal{M}^{\log}$ for the finite log étale morphism of log regular log stacks determined by the local system of trivializations of the l^{aut}-abelianizations of the log fundamental groups of the various logarithmic fibers of $\mathcal{C}^{\log} \to \mathcal{M}^{\log}$. Then the underlying algebraic stack \mathcal{N} of \mathcal{N}^{\log} is an algebraic space.

Proof. First, we consider assertion (i). We begin by recalling that when X^{\log} is a smooth log curve, and r < 1 [so q > 1], assertion (i) follows immediately from classical theory of endomorphisms of semi-abelian varieties and automorphisms of stable curves [cf., e.g., [Des], Lemme 5.17; [DM], Theorems 1.11, 1.13], together with [in the case where l = 2] the well-known fact that every root of unity ζ such that $(\zeta - 1)/l^{\text{aut}}$ is an algebraic integer is necessarily equal to 1. Now let us return to the case of an arbitrary stable log curve X^{\log} . Then it follows immediately from the description of the relationship between the abelianization of Π_1 and the abelianizations of verticial subgroups of Π_1 given in [NodNon], Lemma 1.4, together with the portion of assertion (i) that has already been verified, that α stabilizes and induces the identity automorphism on each of the irreducible components of X^{\log} of genus ≥ 1 . Next, let us observe that it follows immediately from the *definition* of l^{aut} , together with the well-known structure of the submodule of the abelianization of Π_1 generated by the cuspidal inertia subgroups, that α acts *trivially* on the set of cusps of X^{\log} . Thus, by considering the various connected components of the union of the genus zero irreducible components of X^{\log} , we conclude that, to complete the verification of assertion (i), it suffices to verify, in the case where q = 0, that any automorphism of X^{\log} over $(\operatorname{Spec} k)^{\log}$ that acts trivially on the set of cusps of X^{\log} is equal to the identity automorphism. But this follows immediately by *induction* on r, i.e., by considering, when r > 4, the stable log curve obtained from X^{\log} by "forgetting", successively, each of the cusps of X^{\log} . [Here, we apply the elementary combinatorial fact that every non-smooth pointed stable curve of genus 0 has at least two irreducible components that contain cusps.] This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with well-known generalities concerning algebraic stacks [cf., e.g., the discussion surrounding [FC], Chapter I, Theorem 4.10]. \square

3. Applications to the theory of tempered fundamental groups

In the present §3, we apply the technical tools developed in the preceding §2, together with the theory of [CbTpI], §5, to obtain applications to the theory of *tempered fundamental groups*. In particular, we prove a generalization of a result due to André [cf. [André], Theorems 7.2.1, 7.2.3] concerning the *characterization of the local Galois groups* in the image of the outer Galois action associated to a hyperbolic curve over a number field [cf. Corollary 3.20, (iii), below].

Definition 3.1. Let *n* be a nonnegative integer. For $\Box \in \{\circ, \bullet\}$, let \Box_p be a prime number; $\Box\Sigma$ a nonempty set of prime numbers such that $\Box\Sigma \neq \{\Box_p\}$; $\Box R$ a mixed characteristic complete discrete valuation ring of residue characteristic \Box_p whose residue field is separably closed; $\Box K$ the field of fractions of $\Box R$; $\Box \overline{K}$ an algebraic closure of $\Box K$. Write $I\Box_K \stackrel{\text{def}}{=} \operatorname{Gal}(\Box \overline{K}/\Box K)$ for the absolute Galois group of $\Box K$; $\Box \overline{R}$ for the ring of integers of $\Box \overline{K}$; $\Box \overline{R}^{\wedge}$ for the \Box_p -adic completion of $\Box \overline{R}$; $\Box \overline{K}^{\wedge}$ for the field of fractions of $\Box \overline{R}^{\wedge}$. If $n \geq 2$, then we suppose further that $\Box\Sigma$ is either equal to \mathfrak{P} rimes or of cardinality one. Let

$X^{\log}_{\Box K}$

be a smooth log curve over $\Box K$. Write $X_{\Box \overline{K}}^{\log} \stackrel{\text{def}}{=} X_{\Box K}^{\log} \times_{\Box K} \Box \overline{K};$

 $(X_{\Box \overline{K}})_n^{\log}$

for the *n*-th log configuration space [cf. the discussion entitled "Curves" in [CbTpII], §0] of the smooth log curve $X_{\Box \overline{K}}^{\log}$ over $\Box \overline{K}$.

(i) We shall write

$${}^{\Box}\Pi_n \stackrel{\text{def}}{=} \pi_1((X_{\Box\overline{K}})_n^{\log})^{\Box\Sigma}$$

for the maximal pro- Σ quotient of the log fundamental group of $(X_{\Box \overline{K}})_n^{\log}$. Thus, we have a natural outer Galois action

$${}^{\Box}\rho_n\colon I_{\Box_K}\longrightarrow \operatorname{Out}({}^{\Box}\Pi_n)\,.$$

Note that $\Box \Pi_n$ is equipped with a natural structure of pro- $\Box \Sigma$ configuration space group [cf. [MzTa], Definition 2.3, (i)].

(ii) We shall write

$$\pi_1^{\operatorname{temp}}((X_{\Box \overline{K}})_n^{\log} \times_{\Box \overline{K}} {}^{\Box} \overline{K}^{\wedge})$$

for the tempered fundamental group of $(X_{\Box \overline{K}})_n^{\log} \times_{\Box \overline{K}} \Box \overline{K}^{\wedge}$ [cf. [André], §4]. [Here, we note that [André], §4, only discusses the case where the base field $\Box \overline{K}^{\wedge}$ is a complete subfield of " \mathbb{C}_p ". On the other hand, let us recall from [AbsTpI], Proposition 2.2, that any profinite group of GFG-type [cf. [AbsTpI], Definition

2.1, (i)] is topologically finitely generated, which implies that the set of open subgroups of a profinite group of GFG-type [such as $\Box \Pi_n$] is countable. In particular, one verifies easily [cf. also [Brk], Corollary 9.5, and the following discussion] that the construction of the tempered fundamental group given in [André], §4, applies even in the case where the base field $\Box \overline{K}^{\wedge}$ is not a complete subfield of " \mathbb{C}_p ".] We shall write

$$\Box \Pi_n^{\text{tp}} \stackrel{\text{def}}{=} \varprojlim_N \pi_1^{\text{temp}}((X_{\Box \overline{K}})_n^{\log} \times_{\Box \overline{K}} {}^{\Box} \overline{K}^{\wedge})/N$$

for the $\Box \Sigma$ -tempered fundamental group of $(X_{\Box \overline{K}})_n^{\log} \times_{\Box \overline{K}} \Box \overline{K}^{\wedge}$ [cf. [CmbGC], Corollary 2.10, (iii)], i.e., the inverse limit given by allowing N to vary over the open normal subgroups of $\pi_1^{\text{temp}}((X_{\Box \overline{K}})_n^{\log} \times_{\Box \overline{K}} \Box \overline{K}^{\wedge})$ such that the quotient by N corresponds to a topological covering [cf. [André], §4.2] of some finite log étale Galois covering of $(X_{\Box \overline{K}})_n^{\log} \times_{\Box \overline{K}} \Box \overline{K}^{\wedge}$ of degree a product of primes $\in \Box \Sigma$. [Here, we recall that, when n = 1, such a "topological covering" corresponds to a "combinatorial covering", i.e., a covering determined by a covering of the dual semi-graph of the special fiber of the stable model of some finite log étale covering of $(X_{\Box \overline{K}})_n^{\log} \times_{\Box \overline{K}} \Box \overline{K}^{\wedge}$.] Thus, we have a natural outer Galois action

$${}^{\Box}\rho_n^{\operatorname{tp}}\colon I_{{}^{\Box}K}\longrightarrow \operatorname{Out}({}^{\Box}\Pi_n^{\operatorname{tp}})$$

[cf. [André], Proposition 5.1.1].

Lemma 3.2 (Pro- Σ completions of discrete free groups). Let Σ be a nonempty set of prime numbers and F a discrete free group. Then the following hold:

- (i) The natural homomorphism $F \to F^{\Sigma}$ from F to the pro- Σ completion F^{Σ} of F is injective.
- (ii) Suppose that F is **not of rank one**. Then the image of the injection $F \hookrightarrow F^{\Sigma}$ of (i) is **normally terminal** [cf. the discussion entitled "Topological groups" in [CbTpI], §0].

Proof. Assertion (i) follows immediately from [RZ], Proposition 3.3.15. Assertion (ii) follows immediately from the fact that F is *conjugacy l-separable* for every prime number l [cf. [Prs], Theorem 3.2], together with a similar argument to the argument applied in the proof of [André], Lemma 3.2.1. This completes the proof of Lemma 3.2.

Proposition 3.3 (Log and tempered fundamental groups). In the notation of Definition 3.1, the following hold:

- (i) Write $(\Box \Pi_n^{\text{tp}})^{\Box_{\Sigma}}$ for the pro- $\Box \Sigma$ completion of $\Box \Pi_n^{\text{tp}}$. Then there exists a **natural outer isomorphism** $(\Box \Pi_n^{\text{tp}})^{\Box_{\Sigma}} \xrightarrow{\sim} \Box \Pi_n$.
- (ii) The outer homomorphism $\Box \Pi_1^{\text{tp}} \to \Box \Pi_1$ determined by the outer isomorphism of (i) is **injective**.
- (iii) The image of the outer injection $\Box \Pi_1^{\text{tp}} \hookrightarrow \Box \Pi_1$ of (ii) is normally terminal.
- (iv) Write Isom(°Π₁^{tp}, •Π₁^{tp}) (respectively, Isom(°Π₁, •Π₁)) for the set of isomorphisms of °Π₁^{tp} (respectively, °Π₁) with •Π₁^{tp} (respectively, •Π₁) and Inn(-) for the group of inner automorphisms of "(-)". Then the natural map between sets of outer isomorphisms [i.e., sets of "Inn(-)-orbits"]

$$\operatorname{Isom}({}^{\circ}\Pi_{1}^{\operatorname{tp}}, {}^{\bullet}\Pi_{1}^{\operatorname{tp}})/\operatorname{Inn}({}^{\bullet}\Pi_{1}^{\operatorname{tp}}) \longrightarrow \operatorname{Isom}({}^{\circ}\Pi_{1}, {}^{\bullet}\Pi_{1})/\operatorname{Inn}({}^{\bullet}\Pi_{1})$$

induced by the natural outer isomorphism of (i) — hence also the natural homomorphism

$$\operatorname{Out}({}^{\Box}\Pi_1^{\operatorname{tp}}) \longrightarrow \operatorname{Out}({}^{\Box}\Pi_1)$$

-is injective.

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii) (respectively, (iii)). Let us first observe that it follows immediately from assertion (i) that, to verify assertion (ii) (respectively, (iii)), by replacing $X_{\Box \overline{K}}^{\log}$ by a suitable connected finite log étale covering of $X_{\Box \overline{K}}^{\log}$, we may assume without loss of generality that the first Betti number of the dual semi-graph of the special fiber of the stable model of every connected finite log étale covering of $X_{\Box \overline{K}}^{\log}$ is ≠ 1. Then since $\Box \Pi_1^{\text{tp}}$ is a projective limit of extensions of finite groups whose orders are products of primes ∈ $\Box \Sigma$ by discrete free groups whose ranks are ≠ 1, assertion (ii) (respectively, (iii)) follows immediately from Lemma 3.2, (i) (respectively, (ii)). This completes the proof of assertion (ii) (respectively, (iii)). This completes the proof of assertion (ii). This completes the proof of Proposition 3.3. □

Remark 3.3.1. The injections of Proposition 3.3, (iv), allow one to regard Isom(${}^{\circ}\Pi_{1}^{tp}, {}^{\bullet}\Pi_{1}^{tp}$)/Inn(${}^{\bullet}\Pi_{1}^{tp}$), (respectively, Out(${}^{\Box}\Pi_{1}^{tp}$)) as a subset (respectively, subgroup) of Isom(${}^{\circ}\Pi_{1}, {}^{\bullet}\Pi_{1}$)/Inn(${}^{\bullet}\Pi_{1}$) (respectively, Out(${}^{\Box}\Pi_{1}$)).

Remark 3.3.2. The *normal terminality* of Proposition 3.3, (iii), may also be verified by applying the theory of [SemiAn] and [NodNon]. We refer to the proof of [IUTeichI], Proposition 2.4, (iii), for more details concerning this approach.

Definition 3.4. Let \mathbb{G} be a [semi-]graph. Write Node(\mathbb{G}) for the set of closed edges of \mathbb{G} . Then we shall refer to a map

$$\mu\colon \operatorname{Node}(\mathbb{G})\to\mathbb{R}_{>0}\stackrel{\mathrm{def}}{=}\{a\in\mathbb{R}\,|\,a>0\,\}$$

as a *metric structure* on \mathbb{G} . Also, we shall refer to a [semi-]graph equipped with a metric structure as a *metric [semi-]graph*. Let Σ be a [possibly empty] set of prime numbers. Then we shall say that an isomorphism $\mathbb{G}_1 \xrightarrow{\sim} \mathbb{G}_2$ between two [semi-]graphs \mathbb{G}_1 , \mathbb{G}_2 equipped with metric structures μ_1, μ_2 is Σ -rationally compatible with the given metric structures if there exists an element

$$\xi \in (\widehat{\mathbb{Z}}^{\Sigma})^+ \ (\subseteq \mathbb{Q}_{>0} \stackrel{\text{def}}{=} \mathbb{Q} \cap \mathbb{R}_{>0})$$

— i.e., a positive rational number that is invertible, as an integer, at the primes of Σ [cf. the notation of [CbTpI], Corollary 5.9, (iv), if $\Sigma \neq \emptyset$; set $(\widehat{\mathbb{Z}}^{\Sigma})^+ \stackrel{\text{def}}{=} \mathbb{Q}_{>0}$ if $\Sigma = \emptyset$] — such that $\xi \cdot \mu_1$ coincides with the composite of the bijection Node(\mathbb{G}_1) $\xrightarrow{\sim}$ Node(\mathbb{G}_2) determined by the given isomorphism with μ_2 . [Thus, if $\mathbb{G}_1 = \mathbb{G}_2$ is a *finite* [semi-]graph, and $\mu_1 = \mu_2$, then such a ξ is *necessarily equal* to 1. Alternatively, if $\Sigma = \mathfrak{Primes}$, then such a ξ is *necessarily equal* to 1.]

Definition 3.5. In the notation of Definition 3.1, let $\Sigma \subseteq \Box \Sigma \setminus \{\Box p\}$ be a nonempty subset of $\Box \Sigma \setminus \{\Box p\}$ and $\Box H \subseteq \Box \Pi_1$ an open subgroup of $\Box \Pi_1$.

(i) We shall write

 $\mathcal{G}_{\Box_H}[\Sigma]$

for the semi-graph of anabelioids of pro- Σ PSC-type determined by the special fiber [cf. [CmbGC], Example 2.5] of the stable model over $\Box \overline{R}$ of the connected finite log étale covering of $X_{\Box \overline{K}}^{\log}$ corresponding to $\Box H \subseteq \Box \Pi_1$.

(ii) We shall write

 \mathbb{G}_{\square_H}

for the semi-graph associated to [i.e., the *dual semi-graph* of] the special fiber of the stable model over $\Box \overline{R}$ of the connected finite log étale covering of $X_{\Box \overline{K}}^{\log}$ corresponding to $\Box H \subseteq \Box \Pi_1$, i.e., the underlying semi-graph of $\mathcal{G}_{\Box H}[\Sigma]$ [cf. (i)]. Note that this semi-graph is *independent* of the choice of Σ .

(iii) We shall write

$$\mu_{\Box_H} \colon \operatorname{Node}(\mathbb{G}_{\Box_H}) \longrightarrow \mathbb{R}_{>0}$$

for the *metric structure* [cf. Definition 3.4] on \mathbb{G}_{\Box_H} associated to the stable model over $\Box \overline{R}$ of the connected finite log étale covering of $X_{\Box \overline{K}}^{\log}$ corresponding to $\Box H \subseteq \Box \Pi_1$, i.e., the metric structure defined as follows:

Write $v_{\Box \overline{K}^{\wedge}}$ for the $\Box p$ -adic valuation of $\Box \overline{K}^{\wedge}$ such that $v_{\Box \overline{K}^{\wedge}}(\Box p) = 1$. Let $e \in \operatorname{Node}(\mathbb{G}_{\Box H})$. Suppose that the $\Box \overline{R}^{\wedge}$ -algebra given by the completion at the node corresponding to e of the stable model of the connected covering of $X_{\Box \overline{K}}^{\log}$ determined by $\Box H \subseteq \Box \Pi_1$ is isomorphic to

$$\Box \overline{R}^{\wedge}[[s_1, s_2]]/(s_1s_2 - a_e)$$

— where $a_e \in \Box \overline{R}^{\wedge}$ is a nonzero non-unit, and s_1 and s_2 denote indeterminates. Then we set $\mu_{\Box H}(e) \stackrel{\text{def}}{=} v_{\Box \overline{K}^{\wedge}}(a_e)$:

$$\begin{array}{rccc} \mu_{\Box_H} \colon & \operatorname{Node}(\mathbb{G}_{\Box_H}) & \longrightarrow & \mathbb{R}_{>0} \\ & e & \mapsto & v_{\Box \overline{K}^{\wedge}}(a_e) \end{array}$$

Here, one verifies easily that " $\mu_{\Box_H}(a_e)$ " depends only on e, i.e., is *independent* of the choice of the local equation " $s_1s_2 - a_e$ ".

Remark 3.5.1. In the notation of Definition 3.5, it follows immediately from the various definitions involved that one has a *natural outer* isomorphism

$$(^{\Box}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\Box}_{H}[\Sigma]}$$

between the maximal pro- Σ quotient $(\Box H)^{\Sigma}$ of $\Box H$ and the [pro- Σ] fundamental group $\Pi_{\mathcal{G}_{\Box_H}[\Sigma]}$ of the semi-graph of anabelioids of pro- Σ PSC-type $\mathcal{G}_{\Box_H}[\Sigma]$.

Proposition 3.6 (Equivalences of properties of isomorphisms between fundamental groups). In the notation of Definition 3.1, let α : $^{\circ}\Pi_1 \xrightarrow{\sim} ^{\bullet}\Pi_1$ be an isomorphism of profinite groups. [Thus, it follows immediately that $^{\circ}\Sigma = ^{\bullet}\Sigma - cf.$, e.g., the proof of [CbTpI], Proposition 1.5, (i).] Consider the following conditions:

(a) The outer isomorphism ${}^{\circ}\Pi_{1} \xrightarrow{\sim} {}^{\bullet}\Pi_{1}$ determined by α is contained in

 $\text{Isom}(^{\circ}\Pi_1^{\text{tp}}, {}^{\bullet}\Pi_1^{\text{tp}})/\text{Inn}({}^{\bullet}\Pi_1^{\text{tp}}) \subseteq \text{Isom}(^{\circ}\Pi_1, {}^{\bullet}\Pi_1)/\text{Inn}({}^{\bullet}\Pi_1)$ [cf. Remark 3.3.1], and $^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq \{^{\circ}p, {}^{\bullet}p\}.$

(b^{\forall}) For any characteristic open subgroup $^{\circ}H \subseteq ^{\circ}\Pi_1$ of $^{\circ}\Pi_1$ and any nonempty subset $\Sigma \subseteq ^{\circ}\Sigma = ^{\bullet}\Sigma$ such that $^{\circ}p, ^{\bullet}p \notin \Sigma$, if we write $^{\bullet}H \stackrel{\text{def}}{=} \alpha(^{\circ}H) \subseteq ^{\bullet}\Pi_1$, then the outer isomorphism of $(^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{^{\circ}H}[\Sigma]}$ [cf. Remark 3.5.1] with $(^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{^{\bullet}H}[\Sigma]}$ induced by α is group-theoretically verticial [cf. [CmbGC], Definition 1.4, (iv)].

- (b[∃]) For any characteristic open subgroup °H ⊆ °Π₁ of °Π₁, there exists a nonempty subset Σ ⊆ °Σ = °Σ [which may depend on °H] such that °p, °p ∉ Σ, and, moreover, if we write °H ^{def} = α(°H) ⊆ °Π₁, then the outer isomorphism of (°H)^Σ → Π_{G°H}[Σ] [cf. Remark 3.5.1] with (°H)^Σ → Π_{G°H}[Σ] induced by α is group-theoretically verticial.
- (c^{\forall}) For any characteristic open subgroup $^{\circ}H \subseteq ^{\circ}\Pi_1$ of $^{\circ}\Pi_1$ and any nonempty subset $\Sigma \subseteq ^{\circ}\Sigma = ^{\bullet}\Sigma$ such that $^{\circ}p$, $^{\bullet}p \notin \Sigma$, if we write $^{\bullet}H \stackrel{\text{def}}{=} \alpha(^{\circ}H) \subseteq ^{\bullet}\Pi_1$, then the outer isomorphism of $(^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{^{\circ}H}[\Sigma]}$ [cf. Remark 3.5.1] with $(^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{^{\bullet}H}[\Sigma]}$ induced by α is graphic [cf. [CmbGC], Definition 1.4, (i)].
- (c[∃]) For any characteristic open subgroup $^{\circ}H \subseteq ^{\circ}\Pi_{1}$ of $^{\circ}\Pi_{1}$, there exists a nonempty subset $\Sigma \subseteq ^{\circ}\Sigma = ^{\bullet}\Sigma$ [which may depend on $^{\circ}H$] such that $^{\circ}p$, $^{\bullet}p \notin \Sigma$, and, moreover, if we write $^{\bullet}H \stackrel{\text{def}}{=} \alpha(^{\circ}H) \subseteq ^{\bullet}\Pi_{1}$, then the outer isomorphism of $(^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{^{\circ}H}[\Sigma]}$ [cf. Remark 3.5.1] with $(^{\bullet}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{^{\bullet}H}[\Sigma]}$ induced by α is graphic.

Then:

(i) We have implications:

$$(\mathbf{b}^{\forall}) \longleftrightarrow (\mathbf{c}^{\forall}) \longleftrightarrow (\mathbf{c}^{\exists}) \Longrightarrow (\mathbf{a}) \Longleftrightarrow (\mathbf{b}^{\exists}) \Longrightarrow (\mathbf{b}^{\forall}).$$

(ii) Suppose that $^{\circ}\Sigma = ^{\bullet}\Sigma \not\subseteq \{^{\circ}p, ^{\bullet}p\}$. [This condition is satisfied if, for instance, $^{\circ}p = ^{\bullet}p$.] Then we have equivalences:

$$(\mathbf{b}^{\exists}) \iff (\mathbf{b}^{\forall}) \quad and \quad (\mathbf{c}^{\exists}) \iff (\mathbf{c}^{\forall}).$$

(iii) Suppose that either ${}^{\circ}p \in {}^{\circ}\Sigma$ or ${}^{\bullet}p \in {}^{\bullet}\Sigma$. Then we have equivalences:

$$(a) \iff (b^{\exists}) \iff (c^{\exists}).$$

Moreover, (a), (b^{\exists}), and (c^{\exists}) imply that °p = •p.

Proof. First, we claim that the following assertion holds:

Claim 3.6.A: Suppose that (a) is satisfied, and that either ${}^{\circ}p \in {}^{\circ}\Sigma$ or ${}^{\bullet}p \in {}^{\bullet}\Sigma$. Then ${}^{\circ}p = {}^{\bullet}p \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$. Moreover, (c^{\exists}) is satisfied.

To verify Claim 3.6.A, suppose that (a) is satisfied, and that ${}^{\circ}p \in {}^{\circ}\Sigma$. Then it follows immediately from [SemiAn], Corollary 3.11 [cf., especially, the portion of the statement and proof of [SemiAn], Corollary 3.11, concerning, in the notation of *loc. cit.*, the assertion " $p_{\alpha} = p_{\beta}$ "]; [SemiAn], Remark 3.11.1 [cf. also [AbsTpII], Lemma 2.6, (i); the statement and proof of [AbsTpII], Corollary 2.11; [AbsTpII], Remark 2.11.1, (i)], that $^{\circ}p = ^{\bullet}p \in ^{\circ}\Sigma = ^{\bullet}\Sigma$, and, moreover, that (c[∃]) is satisfied. This completes the proof of Claim 3.6.A.

Next, we verify assertion (i). Let us first observe that it follows from the fact that *graphicity* implies *group-theoretic verticiality* that the following implications hold: $(c^{\forall}) \Rightarrow (b^{\forall})$ and $(c^{\exists}) \Rightarrow (b^{\exists})$. Next, we verify the implication $(b^{\exists}) \Rightarrow (b^{\forall})$ (respectively, $(c^{\exists}) \Rightarrow (c^{\forall})$). Suppose that (b^{\exists}) (respectively, (c^{\exists})) is satisfied. Then it follows that ° $\Sigma =$ $\Sigma \not\subseteq \{ p, p \}$. Next, let us observe that, to complete the verification of (\mathbf{b}^{\forall}) (respectively, (\mathbf{c}^{\forall})), we may assume without loss of generality by replacing the open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ in (\mathbf{b}^{\forall}) (respectively, (\mathbf{c}^{\forall})) by $\circ \Pi_1$ — that $\circ H = \circ \Pi_1$ and $\bullet H = \bullet \Pi_1$. Moreover, one verifies easily that, to complete the verification of (b^{\forall}) (respectively, (c^{\forall})), we may assume without loss of generality — by replacing the subset Σ in (b^{\forall}) (respectively, (c^{\forall})) by $^{\circ}\Sigma \setminus (^{\circ}\Sigma \cap \{^{\circ}p, ^{\bullet}p\}) = ^{\bullet}\Sigma \setminus (^{\bullet}\Sigma \cap \{^{\circ}p, ^{\bullet}p\}) \ (\neq \emptyset)$ that $\Sigma = {}^{\circ}\Sigma \setminus ({}^{\circ}\Sigma \cap \{{}^{\circ}p, {}^{\bullet}p\}) = {}^{\bullet}\Sigma \setminus ({}^{\bullet}\Sigma \cap \{{}^{\circ}p, {}^{\bullet}p\}) \ (\neq \emptyset).$ Let ${}^{\circ}U \subseteq {}^{\circ}\Pi_1$ be a characteristic open subgroup. Write ${}^{\bullet}U \stackrel{\text{def}}{=} \alpha({}^{\circ}U) \subseteq {}^{\bullet}\Pi_1$. Then it follows immediately from (b^{\exists}) (respectively, (c^{\exists})) that there exists a nonempty subset $\Sigma_{\circ U} \subseteq \Sigma$ such that α induces a functorial bijection

$$\operatorname{Vert}(\mathcal{G}_{\bullet U}[\Sigma]) = \operatorname{Vert}(\mathcal{G}_{\bullet U}[\Sigma_{\bullet U}]) \xrightarrow{\sim} \operatorname{Vert}(\mathcal{G}_{\bullet U}[\Sigma_{\bullet U}]) = \operatorname{Vert}(\mathcal{G}_{\bullet U}[\Sigma])$$

(respectively,

$$\operatorname{VCN}(\mathcal{G}_{\bullet U}[\Sigma]) = \operatorname{VCN}(\mathcal{G}_{\bullet U}[\Sigma_{\bullet U}]) \xrightarrow{\sim} \operatorname{VCN}(\mathcal{G}_{\bullet U}[\Sigma_{\bullet U}]) = \operatorname{VCN}(\mathcal{G}_{\bullet U}[\Sigma])).$$

In particular, by considering these functorial bijections between the sets "Vert" (respectively, "VCN") associated to the connected finite étale coverings corresponding to the various characteristic open subgroups ${}^{\circ}U \subseteq {}^{\circ}\Pi_1$, ${}^{\bullet}U \stackrel{\text{def}}{=} \alpha({}^{\circ}U) \subseteq {}^{\bullet}\Pi_1$, we conclude that the isomorphism ${}^{\circ}\Pi_1^{\Sigma} \xrightarrow{\rightarrow} {}^{\bullet}\Pi_1^{\Sigma}$ is group-theoretically verticial (respectively, group-theoretically verticial and group-theoretically edge-like, hence graphic [cf. [CmbGC], Proposition 1.5, (ii)]). This completes the proof of the implication (b^{\exists}) \Rightarrow (b^{\forall}) (respectively, (c^{\exists}) \Rightarrow (c[∀])).

Next, we observe that since (a) implies that ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq \{{}^{\circ}p, {}^{\bullet}p\}$, the implication (a) \Rightarrow (b^{\exists}) follows from [SemiAn], Theorem 3.7, (iv), together with [the evident Σ -tempered analogue of] the discussion of [SemiAn], Example 2.10. Thus, to complete the verification of assertion (i), it suffices to verify the implication (b^{\exists}) \Rightarrow (a). To this end, suppose that (b^{\exists}) is satisfied. Let ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ be a characteristic open subgroup of ${}^{\circ}\Pi_1$. Then it follows from (b^{\exists}) that there exists a nonempty subset $\Sigma \subseteq {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that ${}^{\circ}p, {}^{\bullet}p \notin \Sigma$, and, moreover, if we write ${}^{\bullet}H \stackrel{\text{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$, then the outer isomorphism of $({}^{\circ}H)^{\Sigma} \xrightarrow{} \Pi_{\mathcal{G}_{\circ}H}[\Sigma]$ [cf. Remark 3.5.1] with $({}^{\bullet}H)^{\Sigma} \xrightarrow{} \Pi_{\mathcal{G}_{\bullet}H}[\Sigma]$ induced by α is group-theoretically *verticial.* For each $\Box \in \{\circ, \bullet\}$, write

$$\mathcal{G}_{\Box_H}^{\neq c}[\Sigma]$$

for the graph of anabelioids obtained by omitting the cusps [i.e., open edges] of $\mathcal{G}_{\Box_H}[\Sigma]$;

$$\Pi^{\mathrm{tp}}_{\mathcal{G}_{\Box}_{H}[\Sigma]} , \ \Pi^{\mathrm{tp}\neq\mathrm{c}}_{\mathcal{G}_{\Box}_{H}[\Sigma]}$$

for the tempered fundamental groups of $\mathcal{G}_{\Box_H}[\Sigma]$, $\mathcal{G}_{\Box_H}^{\neq c}[\Sigma]$, respectively [cf. the discussion preceding [SemiAn], Proposition 3.6]. Here, let us observe that it follows immediately from the various definitions involved that we have a *natural commutative diagram*

$$\begin{split} \Pi^{\mathrm{tp}\neq\mathrm{c}}_{\mathcal{G}_{\Box_{H}}[\Sigma]} & \stackrel{\sim}{\longrightarrow} & \Pi^{\mathrm{tp}}_{\mathcal{G}_{\Box_{H}}[\Sigma]} \\ & \cap & & \cap \\ (\Pi^{\mathrm{tp}\neq\mathrm{c}}_{\mathcal{G}_{\Box_{H}}[\Sigma]})^{\Sigma} & \stackrel{\sim}{\longrightarrow} & (\Pi^{\mathrm{tp}}_{\mathcal{G}_{\Box_{H}}[\Sigma]})^{\Sigma} & \stackrel{\sim}{\longrightarrow} & \Pi_{\mathcal{G}_{\Box_{H}}[\Sigma]} \end{split}$$

— where we write $(\Pi_{\mathcal{G}_{\Box_{H}}[\Sigma]}^{\mathrm{tp}\neq c})^{\Sigma}$, $(\Pi_{\mathcal{G}_{\Box_{H}}[\Sigma]}^{\mathrm{tp}})^{\Sigma}$ for the pro- Σ completions of $\Pi_{\mathcal{G}_{\Box_{H}}[\Sigma]}^{\mathrm{tp}\neq c}$, $\Pi_{\mathcal{G}_{\Box_{H}}[\Sigma]}^{\mathrm{tp}}$, respectively; the horizontal arrows are outer isomorphisms; the lower right-hand horizontal arrow is the outer isomorphism of Proposition 3.3, (i); the vertical inclusions are the inclusions that arise from Proposition 3.3, (ii).

Now since the outer isomorphism of $({}^{\circ}H)^{\Sigma} \xrightarrow{\sim} \Pi_{\mathcal{G}_{{}^{\circ}H}[\Sigma]}$ with $({}^{\bullet}H)^{\Sigma} \xrightarrow{\sim}$ $\Pi_{\mathcal{G}_{\bullet_H}[\Sigma]}$ induced by α is group-theoretically verticial, it follows immediately from [NodNon], Proposition 1.13; the argument applied in the proof of the sufficiency portion of [CmbGC], Proposition 1.5, (ii), that α determines an isomorphism $\mathcal{G}_{\circ H}^{\neq c}[\Sigma] \xrightarrow{\sim} \mathcal{G}_{\circ H}^{\neq c}[\Sigma]$ of graphs of anabelioids. Thus, it follows immediately from the existence of the natural outer isomorphisms discussed above that the group-theoretically verticial] outer isomorphism $\Pi_{\mathcal{G}_{^\circ H}[\Sigma]} \xrightarrow{\sim} \Pi_{\mathcal{G}_{^\bullet H}[\Sigma]}$ induced by the isomorphism α maps the $\Pi_{\mathcal{G}_{^\circ H}[\Sigma]}$ -conjugacy class of $\Pi^{\mathrm{tp}}_{\mathcal{G}_{^\circ H}[\Sigma]}$ to the $\Pi_{\mathcal{G}_{^\bullet H}[\Sigma]}$ -conjugacy class of $\Pi^{\text{tp}}_{\mathcal{G} \bullet_{H}[\Sigma]}$. Moreover, it follows immediately from the normal terminality of Proposition 3.3, (iii), that the resulting conjugacy indeterminacies may be reduced to $\Pi^{\mathrm{tp}}_{\mathcal{G}_{\Box_{H}}[\Sigma]}$ -conjugacy indeterminacies. In particular, by applying these observations to the various characteristic open subgroups " ${}^{\circ}H$ " of ${}^{\circ}\Pi_1$, one verifies easily from the description of the tempered fundamental group as a [countably indexed!] projective limit given in [André], §4.5 [cf. also the discussion preceding [SemiAn], Proposition 3.6, as well as the discussion of Definition 3.1, (ii), of the present paper], that the outer isomorphism ${}^{\circ}\Pi_1 \xrightarrow{\sim} {}^{\circ}\Pi_1$ determined by α is contained in Isom($^{\circ}\Pi_{1}^{\text{tp}}, {}^{\bullet}\Pi_{1}^{\text{tp}}$)/Inn($^{\bullet}\Pi_{1}^{\text{tp}}$) \subseteq Isom($^{\circ}\Pi_{1}, {}^{\bullet}\Pi_{1}$)/Inn($^{\bullet}\Pi_{1}$), i.e., that (a) is satisfied. This completes the proof of the implication

 $(b^{\exists}) \Rightarrow (a)$, hence also of assertion (i). Assertion (ii) follows immediately from assertion (i), together with the various definitions involved. Assertion (iii) follows from assertion (i), together with Claim 3.6.A. This completes the proof of Proposition 3.6.

Definition 3.7. In the notation of Definition 3.1:

(i) Let $\alpha: {}^{\circ}\Pi_{1} \xrightarrow{\sim} {}^{\bullet}\Pi_{1}$ be an isomorphism of profinite groups. Then we shall say that α is *G-admissible* [i.e., "graph-admissible"] if α satisfies condition (c^{\exists}) — hence also conditions (a), (b^{\forall}), (b^{\exists}), (c^{\forall}) [cf. Proposition 3.6, (i)] — of Proposition 3.6. Write

$$\operatorname{Aut}(^{\circ}\Pi_1)^{\operatorname{G}} \subseteq \operatorname{Aut}(^{\circ}\Pi_1)$$

for the subgroup [cf. the equivalence $(c^{\forall}) \Leftrightarrow (c^{\exists})$ of Proposition 3.6, (ii))] of G-admissible automorphisms of $^{\circ}\Pi_1$ and

$$\operatorname{Out}(^{\circ}\Pi_1)^{\mathrm{G}} \stackrel{\text{def}}{=} \operatorname{Aut}(^{\circ}\Pi_1)^{\mathrm{G}} / \operatorname{Inn}(^{\circ}\Pi_1) \subseteq \operatorname{Out}(^{\circ}\Pi_1)$$

for the subgroup of G-admissible outomorphisms of $^{\circ}\Pi_1$.

(ii) Let α: °Π₁ → •Π₁ be an isomorphism of profinite groups [so °Σ = •Σ — cf., e.g., the proof of [CbTpI], Proposition 1.5, (i)]. Let Σ ⊆ °Σ = •Σ be a [possibly empty] subset such that °p, •p ∉ Σ. Then we shall say that α is Σ-M-admissible [i.e., "Σ-metric-admissible"] if α is G-admissible [cf. (i)], and, moreover, the following condition is satisfied:

Let ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ be a characteristic open subgroup of ${}^{\circ}\Pi_1$. Write ${}^{\bullet}H \stackrel{\text{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$. Then the isomorphism of $\mathbb{G}_{{}^{\circ}H}$ with $\mathbb{G}_{{}^{\bullet}H}$ induced by α [where we note that one verifies easily that the isomorphism of $\mathbb{G}_{{}^{\circ}H}$ with $\mathbb{G}_{{}^{\bullet}H}$ induced by α does not depend on the choice of " Σ " in condition (c^{\forall}) of Proposition 3.6] is Σ rationally compatible [cf. Definition 3.4] with respect to the metric structures $\mu_{{}^{\circ}H}$, $\mu_{{}^{\bullet}H}$ [cf. Definition 3.5, (iii)].

[Thus, if the collections of data labeled by \circ , • are *equal*, then the notion of Σ -M-admissibility is *independent* of the choice of Σ — cf. the final portion of Definition 3.4.] We shall say that α is *M*-admissible if α is \emptyset -M-admissible. Write

$$\operatorname{Aut}(^{\circ}\Pi_1)^{\mathrm{M}} \subseteq \operatorname{Aut}(^{\circ}\Pi_1)$$

for the subgroup of M-admissible automorphisms of $^{\circ}\Pi_1$ and

$$\operatorname{Out}(^{\circ}\Pi_1)^{\mathrm{M}} \stackrel{\text{def}}{=} \operatorname{Aut}(^{\circ}\Pi_1)^{\mathrm{M}} / \operatorname{Inn}(^{\circ}\Pi_1) \subseteq \operatorname{Out}(^{\circ}\Pi_1)$$

for the subgroup of M-admissible outomorphisms of $^{\circ}\Pi_1$.

(iii) We shall write

$$\operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})^{\mathrm{M}} \subseteq \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})$$

for the subgroup of the group $\operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_n)$ of F-admissible outomorphisms of $^{\circ}\Pi_n$ [cf. [CmbCsp], Definition 1.1, (ii)] obtained by forming the inverse image of $\operatorname{Out}(^{\circ}\Pi_1)^{\mathrm{M}} \subseteq \operatorname{Out}(^{\circ}\Pi_1)$ [cf. (ii)] via the natural homomorphism $\operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_n) \to \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_1) =$ $\operatorname{Out}(^{\circ}\Pi_1)$ [cf. [CbTpI], Theorem A, (i)];

$$\operatorname{Out}^{\operatorname{FC}}({}^{\circ}\Pi_n)^{\operatorname{M}} \stackrel{\operatorname{def}}{=} \operatorname{Out}^{\operatorname{F}}({}^{\circ}\Pi_n)^{\operatorname{M}} \cap \operatorname{Out}^{\operatorname{C}}({}^{\circ}\Pi_n) \subseteq \operatorname{Out}^{\operatorname{FC}}({}^{\circ}\Pi_n)$$

[cf. [CmbCsp], Definition 1.1, (ii)].

Definition 3.8. In the notation of Definition 3.1:

- (i) Let $\alpha: {}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ be an isomorphism of profinite groups [so ${}^{\circ}\Sigma = {}^{\bullet}\Sigma {}^{\circ}cf.$, e.g., the proof of [CbTpI], Proposition 1.5, (i)] and $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. Then we shall say that α is $\{l\}$ -*I*-admissible [i.e., " $\{l\}$ -inertia-admissible"] if α is PF-admissible whenever $n \geq 2$ [cf. [CbTpI], Definition 1.4, (i)], and, moreover, the following condition is satisfied:
 - Let ${}^{\circ}\Pi_n \rightarrow ({}^{\circ}\Pi_n)^*$ be an F-characteristic almost pro-lquotient of ${}^{\circ}\Pi_n (\leftarrow \pi_1((X_{\circ \overline{K}})_n^{\log}))$ [cf. Definition 2.1, (iii)]. If ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \neq \mathfrak{Primes}$, then we assume further that the quotient ${}^{\circ}\Pi_n \rightarrow ({}^{\circ}\Pi_n)^*$ is an *almost* maximal pro-l quotient relative to some characteristic open subgroup of ${}^{\circ}\Pi_n$ [cf. Definition 1.1]. Write ${}^{\bullet}\Pi_n \rightarrow ({}^{\bullet}\Pi_n)^*$ for the quotient of ${}^{\bullet}\Pi_n$ that corresponds to ${}^{\circ}\Pi_n \rightarrow ({}^{\circ}\Pi_n)^*$ via α . [Here, we observe that since α is PF-admissible whenever $n \geq 2$, one verifies immediately that the quotient ${}^{\bullet}\Pi_n \rightarrow ({}^{\bullet}\Pi_n)^*$ satisfies similar assumptions to the assumptions imposed on the quotient ${}^{\circ}\Pi_n \rightarrow ({}^{\circ}\Pi_n)^*$.] Then there exist open subgroups ${}^{\circ}J \subseteq I_{{}^{\circ}K}, {}^{\bullet}J \subseteq I_{{}^{\bullet}K}$ [which may depend on ${}^{\circ}\Pi_n \rightarrow ({}^{\circ}\Pi_n)^*$] such that the diagram

$$\begin{array}{ccc} \operatorname{Im}(^{\circ}J) & \longrightarrow & \operatorname{Out}((^{\circ}\Pi_{n})^{*}) \\ & & & \downarrow \\ & & & \downarrow \\ & & & \operatorname{Im}(^{\bullet}J) & \longrightarrow & \operatorname{Out}((^{\bullet}\Pi_{n})^{*}) \\ & - & \text{where, for } \Box \in \{\circ, \bullet\}, \text{ we write} \end{array}$$

 $\operatorname{Im}(^{\Box}J) \subseteq \operatorname{Out}((^{\Box}\Pi_n)^*)$

for the image of $\Box J$ via the homomorphism $\Box J \rightarrow Out((\Box \Pi_n)^*)$ induced [in light of our *assumptions* on

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the quotients under consideration!] by $\[\Box \rho_n\]$; the horizontal arrows are the natural inclusions; the righthand vertical arrow is the isomorphism induced by the isomorphism α — commutes for some [uniquely determined] isomorphism β : Im(°J) $\xrightarrow{\sim}$ Im(•J).

We shall say that an outer isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ is $\{l\}$ -*I*admissible if it arises from an isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ which is $\{l\}$ -I-admissible.

(ii) We shall say that an isomorphism of profinite groups ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ [so ${}^{\circ}\Sigma = {}^{\bullet}\Sigma - cf.$, e.g., the proof of [CbTpI], Proposition 1.5, (i)] is *I-admissible* [i.e., "inertia-admissible"] if ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq {}^{\circ}p, {}^{\bullet}p$ }, and, moreover, the isomorphism is ${l}$ -I-admissible [cf. (i)] for every prime number $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin {}^{\circ}p, {}^{\bullet}p$ }. We shall say that an outer isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ is *I-admissible* if it arises from an isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ which is I-admissible.

(iii) Let $l \in {}^{\circ}\Sigma$ be such that $l \neq {}^{\circ}p$. Then we shall write

$$\operatorname{Aut}^{\{l\}-\mathrm{I}}(^{\circ}\Pi_n) \subseteq \operatorname{Aut}(^{\circ}\Pi_n)$$

for the subgroup of $\{l\}$ -I-admissible automorphisms of $^{\circ}\Pi_n$ [cf. (i)];

$$\operatorname{Out}^{\{l\}-\mathrm{I}}(^{\circ}\Pi_n) \stackrel{\text{def}}{=} \operatorname{Aut}^{\{l\}-\mathrm{I}}(^{\circ}\Pi_n) / \operatorname{Inn}(^{\circ}\Pi_n) \subseteq \operatorname{Out}(^{\circ}\Pi_n)$$

for the subgroup of $\{l\}$ -I-admissible outomorphisms of $^{\circ}\Pi_n$;

$$\operatorname{Out}^{\mathrm{F} \{l\}-\mathrm{I}}(^{\circ}\Pi_{n}) \stackrel{\mathrm{def}}{=} \operatorname{Out}^{\{l\}-\mathrm{I}}(^{\circ}\Pi_{n}) \cap \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n}) \subseteq \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})$$

[cf. [CmbCsp], Definition 1.1, (ii)];

$$\operatorname{Out}^{\operatorname{FC}\{l\}-\operatorname{I}}(^{\circ}\Pi_n) \stackrel{\text{def}}{=} \operatorname{Out}^{\{l\}-\operatorname{I}}(^{\circ}\Pi_n) \cap \operatorname{Out}^{\operatorname{FC}}(^{\circ}\Pi_n) \subseteq \operatorname{Out}^{\operatorname{FC}}(^{\circ}\Pi_n)$$

[cf. [CmbCsp], Definition 1.1, (ii)]. Also, we shall write

$$\operatorname{Aut}^{\mathrm{I}}(^{\circ}\Pi_{n}) \stackrel{\mathrm{def}}{=} \bigcap_{l \in ^{\circ}\Sigma \setminus (^{\circ}\Sigma \cap \{^{\circ}p\})} \operatorname{Aut}^{\{l\}-\mathrm{I}}(^{\circ}\Pi_{n}) \subseteq \operatorname{Aut}(^{\circ}\Pi_{n})$$

for the subgroup of I-admissible automorphisms of $^{\circ}\Pi_n$ [cf. (ii)];

$$\operatorname{Out}^{\mathrm{I}}({}^{\circ}\Pi_n) \stackrel{\mathrm{def}}{=} \bigcap_{l \in {}^{\circ}\Sigma \backslash ({}^{\circ}\Sigma \cap \{{}^{\circ}p\})} \operatorname{Out}^{\{l\} \text{-}\mathrm{I}}({}^{\circ}\Pi_n) \subseteq \operatorname{Out}({}^{\circ}\Pi_n)$$

for the subgroup of I-admissible outomorphisms of $^{\circ}\Pi_n$;

$$\operatorname{Out}^{\mathrm{FI}}({}^{\circ}\Pi_{n}) \stackrel{\mathrm{def}}{=} \operatorname{Out}^{\mathrm{I}}({}^{\circ}\Pi_{n}) \cap \operatorname{Out}^{\mathrm{F}}({}^{\circ}\Pi_{n}) \subseteq \operatorname{Out}^{\mathrm{F}}({}^{\circ}\Pi_{n});$$
$$\operatorname{Out}^{\mathrm{FCI}}({}^{\circ}\Pi_{n}) \stackrel{\mathrm{def}}{=} \operatorname{Out}^{\mathrm{I}}({}^{\circ}\Pi_{n}) \cap \operatorname{Out}^{\mathrm{FC}}({}^{\circ}\Pi_{n}) \subseteq \operatorname{Out}^{\mathrm{FC}}({}^{\circ}\Pi_{n}).$$

(iv) Let $l \in {}^{\circ}\Sigma$ be such that $l \neq {}^{\circ}p$. Then we shall write

 $\operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})^{\{l\}-\mathrm{I}} \subseteq \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})$

for the subgroup of the group $\operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})$ of F-admissible outomorphisms of $^{\circ}\Pi_{n}$ obtained by forming the inverse image of $\operatorname{Out}^{\{l\}-\mathrm{I}}(^{\circ}\Pi_{1}) \subseteq \operatorname{Out}(^{\circ}\Pi_{1})$ [cf. (iii)] via the natural homomorphism $\operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n}) \to \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{1}) = \operatorname{Out}(^{\circ}\Pi_{1})$ [cf. [CbTpI], Theorem A, (i)];

$$\operatorname{Out}^{\operatorname{FC}}({}^{\circ}\Pi_n)^{\{l\}-\operatorname{I}} \stackrel{\operatorname{def}}{=} \operatorname{Out}^{\operatorname{F}}({}^{\circ}\Pi_n)^{\{l\}-\operatorname{I}} \cap \operatorname{Out}^{\operatorname{C}}({}^{\circ}\Pi_n) \subseteq \operatorname{Out}^{\operatorname{FC}}({}^{\circ}\Pi_n).$$

Also, we shall write

$$\operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})^{\mathrm{I}} \stackrel{\mathrm{def}}{=} \bigcap_{l \in ^{\circ}\Sigma \setminus (^{\circ}\Sigma \cap \{^{\circ}p\})} \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})^{\{l\}-\mathrm{I}} \subseteq \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n});$$
$$\operatorname{Out}^{\mathrm{FC}}(^{\circ}\Pi_{n})^{\mathrm{I}} \stackrel{\mathrm{def}}{=} \operatorname{Out}^{\mathrm{F}}(^{\circ}\Pi_{n})^{\mathrm{I}} \cap \operatorname{Out}^{\mathrm{C}}(^{\circ}\Pi_{n}) \subseteq \operatorname{Out}^{\mathrm{FC}}(^{\circ}\Pi_{n}).$$

Theorem 3.9 (Equivalence of metric-admissibility and inertia-admissibility). For $\Box \in \{\circ, \bullet\}$, let $\Box p$ be a prime number; $\Box \Sigma$ a nonempty set of prime numbers such that $\Box \Sigma \neq \{\Box p\}$; $\Box R$ a mixed characteristic complete discrete valuation ring of residue characteristic $\Box p$ whose residue field is separably closed; $\Box K$ the field of fractions of $\Box R$; $\Box \overline{K}$ an algebraic closure of $\Box K$;

$$X^{\log}_{\square_K}$$

a smooth log curve over $\Box K$. For $\Box \in \{\circ, \bullet\}$, write

$$X_{\Box\overline{K}}^{\log} \stackrel{\text{def}}{=} X_{\Box K}^{\log} \times_{\Box K} \Box\overline{K};$$
$$\Box \Pi_1 \stackrel{\text{def}}{=} \pi_1 (X_{\Box\overline{K}}^{\log})^{\Box\Sigma}$$

for the maximal pro- $\Box \Sigma$ quotient of the log fundamental group of $X_{\Box \overline{K}}^{\log}$. Let

$$\alpha \colon {}^{\circ}\Pi_1 \xrightarrow{\sim} {}^{\bullet}\Pi_1$$

be an isomorphism of profinite groups. [Thus, it follows immediately that $^{\circ}\Sigma = ^{\bullet}\Sigma - cf.$, e.g., the proof of [CbTpI], Proposition 1.5, (i).] If $^{\circ}p \notin ^{\circ}\Sigma$ and $^{\bullet}p \notin ^{\bullet}\Sigma$, then we assume further that α is grouptheoretically cuspidal [cf. [CmbGC], Definition 1.4, (iv)]. Then the following conditions are equivalent:

- (a) α is M-admissible [cf. Definition 3.7, (ii)].
- (b^{\forall}) α is **I-admissible** [cf. Definition 3.8, (ii)].
- (b³) There exists a prime number $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$, and, moreover, α is $\{l\}$ -I-admissible [cf. Definition 3.8, (i)].

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Proof. First, let us observe that it follows formally from the various definitions involved [cf. Definitions 3.7, (i), (ii); 3.8, (ii)] that conditions (a), (b^{\forall}) , and (b^{\exists}) all imply that there exists a prime number $l \in {}^{\circ}\Sigma = {}^{\circ}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. Now fix such a prime number l and consider the condition:

 $(b^{\{l\}})$: α is $\{l\}$ -*I*-admissible [cf. Definition 3.8, (i)].

Then [since l is *arbitrary*, and condition (a) is manifestly *independent* of the choice of l] it follows formally from the various definitions involved that to verify Theorem 3.9, it suffices to verify the equivalence

$$(a) \iff (b^{\{l\}}).$$

To this end, let ${}^{\circ}H \subseteq {}^{\circ}\Pi_1$ be a characteristic open subgroup of ${}^{\circ}\Pi_1$. Write ${}^{\bullet}H \stackrel{\text{def}}{=} \alpha({}^{\circ}H) \subseteq {}^{\bullet}\Pi_1$. Also, for each $\Box \in \{\circ, \bullet\}$, write $({}^{\Box}\Pi_1)^*$ for the maximal almost pro-l quotient of ${}^{\Box}\Pi_1$ with respect to ${}^{\Box}H$. [Thus, $({}^{\Box}H)^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\Box H}[\{l\}]} \subseteq ({}^{\Box}\Pi_1)^*$ — cf. Remark 3.5.1.]

Next, let us observe that, for each $\Box \in \{\circ, \bullet\}$, since $(\Box H)^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\Box_H}[\{l\}]} \subseteq (\Box \Pi_1)^*$ is open, and $(\Box \Pi_1)^*$ is topologically finitely generated, slim [cf. Proposition 1.7, (i)] and almost pro-l, there exist an open subgroup $\Box J \subseteq I_{\Box_K}$ of I_{\Box_K} and a homomorphism

$${}^{\Box}\rho_1[{}^{\Box}H]: {}^{\Box}J \longrightarrow \operatorname{Out}(({}^{\Box}H)^{\{l\}})$$

such that $\[\[\rho_1 \[\[\square H \] \]$ is compatible [in the evident sense] with the homomorphism $\[\square J \rightarrow \operatorname{Out}((\[\square \Pi \])^*)\]$ induced by $\[\square \rho_1 \colon I_{\square_K} \rightarrow \operatorname{Out}(\[\square \Pi \]),\]$ and, moreover, $\[\square \rho_1 \[\square H \]$ factors through the maximal pro-*l* quotient $(\[\square J \])^{\{l\}}$ of $\[\square J \]$, which [as is easily verified] is isomorphic to $\[\square L \]$ as an abstract profinite group. Moreover, it follows immediately from the various definitions involved, together with the well-known properness of the moduli stack of pointed stable curves of a given type, that the outer representation $(\[\square J \])^{\{l\}} \rightarrow \operatorname{Out}((\[\square H \])^{\{l\}}) \xrightarrow{\sim} \operatorname{Out}(\Pi_{\mathcal{G}_{\square H}[\{l\}]})$ arising from such a homomorphism $\[\square \rho_1 \[\square H \]$ is of *PIPSC-type* [cf. Definition 1.3]. In particular, it follows immediately from Theorem 1.11, (ii) [i.e., in essence, [CbTpII], Theorem 1.9, (ii)], that if α satisfies condition (b $\[l^{\}} \]$, i.e., α is $\[l^{\}}$ -*I*-admissible, then the isomorphism of $(\[\circ H \])^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\circ H}[\{l\}]}$ with $(\[\circ H \])^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\circ H}[\{l\}]}$ induced by α is group-theoretically verticial, hence also group-theoretically nodal.

Thus, by allowing " $\square H$ " to vary among the various characteristic open subgroups of $\square \Pi_1$, we conclude that if α satisfies condition $(b^{\{l\}})$, i.e., α is $\{l\}$ -*I*-admissible, then α satisfies condition (b^{\exists}) of Proposition 3.6, hence [cf. Proposition 3.6, (iii); our assumption that α is group-theoretically cuspidal if ${}^{\circ}p \notin {}^{\circ}\Sigma$, ${}^{\bullet}p \notin {}^{\circ}\Sigma$] that α is *G*-admissible [cf. [CmbGC], Proposition 1.5, (ii)]. In particular, it follows from either of the conditions (a), $(b^{\{l\}})$ that the isomorphism of $({}^{\circ}H)^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\circ H}[\{l\}]}$ with $({}^{\bullet}H)^{\{l\}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{\circ H}[\{l\}]}$ induced by α is graphic [cf. the implication $(c^{\exists}) \Rightarrow (c^{\forall})$ of Proposition 3.6, (i)], hence that α determines a commutative diagram of *isomorphisms* of profinite groups

$$\begin{array}{ccc} \operatorname{Dehn}(\mathcal{G}_{\circ_{H}}[\{l\}]) & \xrightarrow{\mathfrak{D}_{\mathcal{G}_{\circ_{H}}}[\{l\}]} & \bigoplus_{\operatorname{Node}(\mathcal{G}_{\circ_{H}}[\{l\}])} \Lambda_{\mathcal{G}_{\circ_{H}}}[\{l\}]) \\ & \downarrow & & \downarrow \\ \operatorname{Dehn}(\mathcal{G}_{\bullet_{H}}[\{l\}]) & \xrightarrow{\mathfrak{D}_{\mathcal{G}_{\bullet_{H}}}[\{l\}]} & \bigoplus_{\operatorname{Node}(\mathcal{G}_{\bullet_{H}}[\{l\}])} \Lambda_{\mathcal{G}_{\bullet_{H}}}[\{l\}]) \end{array}$$

[cf. [CbTpI], Definition 4.4; [CbTpI], Theorem 4.8, (iv)].

On the other hand, since, for each $\Box \in \{\circ, \bullet\}$, the outer representation $(\Box J)^{\{l\}} \to \operatorname{Out}((\Box H)^{\{l\}}) \xrightarrow{\sim} \operatorname{Out}(\Pi_{\mathcal{G}_{\Box_{H}}[\{l\}]})$ is of *PIPSC-type*, it follows — by replacing $\Box J$ by an open subgroup of $\Box J$ if necessary — from [CbTpI], Corollary 5.9, (iii), that we may assume without loss of generality that this outer representation factors through $\operatorname{Dehn}(\mathcal{G}_{\Box_{H}}[\{l\}]) \subseteq \operatorname{Out}(\Pi_{\mathcal{G}_{\Box_{H}}[\{l\}]})$. Thus, by considering the *Dehn coordinates* [cf. [CbTpI], Definition 5.8, (i)] of the image of a topological generator of $(\Box J)^{\{l\}}$ in $\operatorname{Dehn}(\mathcal{G}_{\Box_{H}}[\{l\}])$ [with respect to a topological generator of $\Lambda_{\mathcal{G}_{\Box_{H}}[\{l\}]}$], it follows immediately from [CbTpI], Theorem 5.7; [CbTpI], Lemma 5.4, (ii), together with the existence of the commutative diagram of the above display, that

> the isomorphism $\mathbb{G}_{\circ H} \xrightarrow{\sim} \mathbb{G}_{\bullet H}$ induced by α is \emptyset -rationally compatible [cf. Definition 3.4] with the metric structures $\mu_{\circ H}$, $\mu_{\bullet H}$ [cf. Definition 3.5, (iii)] if and only if the images of the homomorphisms $(^{\circ}J)^{\{l\}} \rightarrow$ Dehn $(\mathcal{G}_{\circ H}[\{l\}])$ and $(^{\bullet}J)^{\{l\}} \rightarrow$ Dehn $(\mathcal{G}_{\bullet H}[\{l\}])$ are compatible, up to a $\mathbb{Q}_{>0}$ -multiple, with the isomorphisms induced by α .

In particular, by applying this equivalence to the various characteristic open subgroups " $\Box H$ " $\subseteq \Box \Pi_1$ of $\Box \Pi_1$, we conclude that α satisfies condition (b^{l}), i.e., α is $\{l\}$ -*I*-admissible, if and only if α satisfies condition (a), i.e., α is *M*-admissible. This completes the proof of Theorem 3.9.

Definition 3.10. In the notation of Definition 3.1, let $l \in {}^{\Box}\Sigma$ be such that $l \neq {}^{\Box}p$ and ${}^{\Box}H \subseteq {}^{\Box}\Pi_n$ an open subgroup of ${}^{\Box}\Pi_n$. For each $i \in \{0, \dots, n\}$, write ${}^{\Box}H_i \subseteq {}^{\Box}\Pi_i$ for the open subgroup of the quotient ${}^{\Box}\Pi_n \twoheadrightarrow {}^{\Box}\Pi_i$ [induced by the projection $(X_{\Box\overline{K}})_n^{\log} \to (X_{\Box\overline{K}})_i^{\log}$ to the first i factors] determined by the image of ${}^{\Box}H \subseteq {}^{\Box}\Pi_n$; ${}^{\Box}Y_i^{\log} \to (X_{\Box\overline{K}})_i^{\log}$ for the connected finite log étale covering of $(X_{\Box\overline{K}})_i^{\log}$ corresponding to ${}^{\Box}H_i \subseteq {}^{\Box}\Pi_i$. Then we have a sequence of morphisms of log schemes

$${}^{\Box}Y_{n}^{\log} \longrightarrow {}^{\Box}Y_{n-1}^{\log} \longrightarrow \cdots \longrightarrow {}^{\Box}Y_{2}^{\log} \longrightarrow {}^{\Box}Y_{1}^{\log} \longrightarrow {}^{\Box}Y_{0}^{\log}.$$

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Thus, for $i \in \{0, \dots, n\}$, if we write $\Box U_i$ for the *interior* of $\Box Y_i^{\log}$ [cf. the discussion entitled "Log schemes" in [CbTpI], §0], we obtain a sequence of morphisms of schemes [each of which determines a family of hyperbolic curves]

$${}^{\Box}U_n \longrightarrow {}^{\Box}U_{n-1} \longrightarrow \cdots \longrightarrow {}^{\Box}U_2 \longrightarrow {}^{\Box}U_1 \longrightarrow {}^{\Box}U_0.$$

Then we shall say that $\Box H$ is of *l*-polystable type if the following conditions are satisfied:

- (a) For each $i \in \{0, \dots, n\}$, $\alpha \in \operatorname{Aut}^{\mathsf{F}}({}^{\Box}\Pi_i)$ [cf. [CmbCsp], Definition 1.1, (ii)], the open subgroup ${}^{\Box}H_i \subseteq {}^{\Box}\Pi_i$ is *preserved* by α . Here, for convenience, when n = 1, and ${}^{\Box}\Sigma$ is *arbitrary*, we set $\operatorname{Aut}^{\mathsf{F}}({}^{\Box}\Pi_1) \stackrel{\text{def}}{=} \operatorname{Aut}({}^{\Box}\Pi_1)$. [In particular, ${}^{\Box}H_i$ is normal.]
- (b) The [necessarily F-characteristic cf. condition (a) above; Definition 2.1, (iii)] maximal almost pro-l quotient

$$(\pi_1((X_{\Box\overline{K}})_n^{\log}) \twoheadrightarrow) \ \Box \Pi_n \twoheadrightarrow (\Box \Pi_n)^*$$

with respect to $\Box H = \Box H_n \subseteq \Box \Pi_n$ [cf. Definition 1.1] is *SA-maximal* [cf. Definition 2.1, (ii)].

(c) For each $i \in \{1, \dots, n\}$, if we write $(\Box H_{i/i-1})^{\{l\}}$ for the maximal pro-l quotient of the kernel $\Box H_{i/i-1} \stackrel{\text{def}}{=} \text{Ker}(\Box H_i \twoheadrightarrow \Box H_{i-1})$, then the natural action of $\Box H_{i-1}$ on the l^{aut} -abelianization [cf. Lemma 2.14] of $(\Box H_{i/i-1})^{\{l\}}$ is trivial.

Remark 3.10.1. In the notation of Definition 3.10:

- (i) Let us observe that [one verifies easily that] condition (c) of Definition 3.10 implies that the following condition holds:
 - (d) For each $i \in \{1, \dots, n\}$, the natural outer representation

$$^{\Box}H_{i-1} \longrightarrow \operatorname{Out}((^{\Box}H_{i/i-1})^{\{l\}})$$

factors through a pro-l quotient of $\Box H_{i-1}$.

Moreover, it follows from Lemma 2.14, (ii); [ExtFam], Corollary 7.4 [together with the well-known structure of the submodule of the abelianization of $(\Box H_{i/i-1})^{\{l\}}$ generated by the cuspidal inertia subgroups — cf. the proof of Lemma 2.14, (i)], that condition (c) of Definition 3.10 also implies that the following condition holds:

(e) The sequence of morphisms of log schemes in Definition 3.10

$${}^{\Box}Y_n^{\mathrm{log}} \longrightarrow {}^{\Box}Y_{n-1}^{\mathrm{log}} \longrightarrow \cdots \longrightarrow {}^{\Box}Y_2^{\mathrm{log}} \longrightarrow {}^{\Box}Y_1^{\mathrm{log}} \longrightarrow {}^{\Box}Y_0^{\mathrm{log}}$$

extends to the factorization

$${}^{\Box}\mathcal{Y}_n^{\log} \longrightarrow {}^{\Box}\mathcal{Y}_{n-1}^{\log} \longrightarrow \cdots \longrightarrow {}^{\Box}\mathcal{Y}_2^{\log} \longrightarrow {}^{\Box}\mathcal{Y}_1^{\log} \longrightarrow {}^{\Box}\mathcal{Y}_0^{\log}$$

associated to the base-change to $\Box \mathcal{Y}_0^{\log}$ of the *log polystable* morphism determined by a [uniquely determined!] *stable polycurve* over the integral closure of $\Box R$ in some finite subextension of $\Box K$ in $\Box \overline{K}$ [cf. [ExtFam], Definition 4.5]. Here, the log structure of $\Box \mathcal{Y}_0^{\log}$ is the log structure on $\Box \mathcal{Y}_0 = \operatorname{Spec} \Box \overline{R}$ determined by the multiplicative monoid of nonzero elements of $\Box \overline{R}$.

(ii) One verifies easily that, for each $i \in \{0, \dots, n\}$, if $\Box H \subseteq \Box \Pi_n$ is of *l*-polystable type, then $\Box H_i \subseteq \Box \Pi_i$ is of *l*-polystable type.

Definition 3.11. In the notation of Definition 3.10, suppose that $\Box H$ is of *l*-polystable type [cf. Definition 3.10].

(i) We shall write

$$\overline{\mathrm{VCN}}^{\mathrm{sch}}(^{\Box}H)$$

for the set of points $y \in {}^{\Box}\mathcal{Y}_n$ of the underlying scheme ${}^{\Box}\mathcal{Y}_n$ of ${}^{\Box}\mathcal{Y}_n^{\log}$ [cf. the notation of condition (e) of Remark 3.10.1, (i)] that satisfy the following condition: For $i \in \{0, \dots, n\}$, write $y_i \in {}^{\Box}\mathcal{Y}_i$ for the image of y in ${}^{\Box}\mathcal{Y}_i$ and $y_i^{\log} \stackrel{\text{def}}{=} {}^{\Box}\mathcal{Y}_i^{\log} \times_{{}^{\Box}\mathcal{Y}_i} y_i$. [Thus, for each $i \in \{1, \dots, n\}$, we have a stable log curve ${}^{\Box}\mathcal{Y}_i^{\log}|_{y_{i-1}^{\log}} \stackrel{\text{def}}{=} {}^{\Box}\mathcal{Y}_i^{\log} \times_{{}^{\Box}\mathcal{Y}_{i-1}^{\log}} y_{i-1}^{\log}$ over y_{i-1}^{\log} .] Then

- (a) y_0 is the closed point of ${}^{\Box}\mathcal{Y}_0 = \operatorname{Spec}^{\Box}\overline{R}$;
- (b) for each $i \in \{1, \dots, n\}$, the point of $\Box \mathcal{Y}_i^{\log}|_{y_{i-1}^{\log}}$ determined by y_i^{\log} is either a *cusp*, *node*, or *generic point* [i.e., the generic point of an irreducible component] of the *stable* $\log curve \Box \mathcal{Y}_i^{\log}|_{y_{i-1}^{\log}}$.

Moreover, we shall write

$$\mathrm{VCN}^{\mathrm{sch}}(\Box H)$$

for the set of elements $y \in \overline{\text{VCN}}^{\text{sch}}({}^{\Box}H)$ such that, in the above notation,

(c) for each $i \in \{1, \dots, n\}$, the residue field $k(y_{i-1})$ of y_{i-1} is separably closed in the residue field $k(y_i)$ of y_i .

Thus, for each $i \in \{1, \dots, n\}$, we have natural maps

$$\begin{split} \overline{\mathrm{VCN}}^{\mathrm{sch}}(^{\Box}H) &\twoheadrightarrow \overline{\mathrm{VCN}}^{\mathrm{sch}}(^{\Box}H_i), \\ \mathrm{VCN}^{\mathrm{sch}}(^{\Box}H) &\to \mathrm{VCN}^{\mathrm{sch}}(^{\Box}H_i), \end{split}$$

the first of which is *surjective*. Finally, we shall say that $\Box H$ is *VCN-complete* if the equality $VCN^{sch}(\Box H) = \overline{VCN}^{sch}(\Box H)$ holds.

(ii) We shall refer to a projective system $\Box \mathbb{H} = \{\Box H_{\lambda}\}_{\lambda \in \Lambda}$ of open subgroups of $\Box \Pi_n$ as an $\Box H$ -*l*-system if each $\Box H_{\lambda}$ is of *l*-polystable type, VCN-complete, and contained in $\Box H$ [i.e., $\Box H_{\lambda} \subseteq \Box H$], and, moreover,

$$\operatorname{Ker}\left({}^{\Box}\Pi_n \twoheadrightarrow ({}^{\Box}\Pi_n)^*\right) = \left(\operatorname{Ker}\left({}^{\Box}H \twoheadrightarrow ({}^{\Box}H)^{\{l\}}\right) = \right) \bigcap_{\lambda \in \Lambda}{}^{\Box}H_{\lambda}$$

[cf. condition (b) of Definition 3.10], i.e., the system $\Box \mathbb{H}$ arises from a basis of the topology of $(\Box H)^{\{l\}}$.

(iii) Let $\Box \mathbb{H} = {\Box H_{\lambda}}_{\lambda \in \Lambda}$ be an $\Box H$ -*l*-system [cf. (ii)]. Then we shall write

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}) \stackrel{\mathrm{def}}{=} \varprojlim_{\lambda \in \Lambda} \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}H_{\lambda})$$

[cf. (i) above; the portion of [ExtFam], Corollary 7.4, concerning extensions of morphisms; our assumption that each $\Box H_{\lambda}$ arises from an open subgroup of $(\Box H)^{\{l\}}$, where $l \neq \Box p$]. In fact, we shall see below that VCN^{sch}($\Box \mathbb{H}$) is *independent* of the choice of $\Box \mathbb{H}$ [cf. Lemma 3.14, (iv)]. Here, we note that one verifies easily that, for each $i \in \{0, \dots, n\}$, if $\Box \mathbb{H} = \{\Box H_{\lambda}\}_{\lambda \in \Lambda}$ is an $\Box H$ -*l*-system, and we write $(\Box H_{\lambda})_i \subseteq \Box \Pi_i$ for the image of $\Box H_{\lambda}$ in $\Box \Pi_i$, then the system $\Box \mathbb{H}_i \stackrel{\text{def}}{=} \{(\Box H_{\lambda})_i\}_{\lambda \in \Lambda}$ is an $\Box H_i$ *l*-system [cf. (i), (ii) above; condition (b) of Definition 3.10; Remark 3.10.1, (ii)]. Thus, for each $i \in \{0, \dots, n\}$, we have a natural map

$$\operatorname{VCN}^{\operatorname{sch}}(\Box \mathbb{H}) \longrightarrow \operatorname{VCN}^{\operatorname{sch}}(\Box \mathbb{H}_i).$$

Definition 3.12. In the notation of Definition 3.11, let $\Box \mathbb{H} = {\Box H_{\lambda}}_{\lambda \in \Lambda}$ be an $\Box H$ -*l*-system [cf. Definition 3.11, (ii)] and $\tilde{y} \in \text{VCN}^{\text{sch}}(\Box \mathbb{H})$ [cf. Definition 3.11, (iii)]. For each $i \in \{0, \dots, n\}$, write $\tilde{y}_i \in \text{VCN}^{\text{sch}}(\Box \mathbb{H}_i)$ for the image of \tilde{y} via the natural map of the final display of Definition 3.11, (iii). Let $i \in \{1, \dots, n\}$.

(i) Write

 $\mathcal{G}_{i,\widetilde{y}_{i-1}}$

for the semi-graph of anabelioids of pro-l PSC-type determined by the stable log curve constituted by the log geometric fiber of ${}^{\Box}\mathcal{Y}_{i}^{\log} \rightarrow {}^{\Box}\mathcal{Y}_{i-1}^{\log}$ [cf. Definition 3.11, (i)] at the point of ${}^{\Box}\mathcal{Y}_{i-1}^{\log}$ determined by \tilde{y}_{i-1} ;

$$\mathcal{G}_{i,\widetilde{y}_{i-1}}\longrightarrow \mathcal{G}_{i,\widetilde{y}_{i-1}}$$

for the universal covering [corresponding to the [pro-l] fundamental group $\Pi_{\mathcal{G}_{i,\tilde{y}_{i-1}}}$ of $\mathcal{G}_{i,\tilde{y}_{i-1}}$ relative to the basepoint of $\mathcal{G}_{i,\tilde{y}_{i-1}}$ determined by the various $\Box H_{\lambda}$'s] obtained by considering the " $\mathcal{G}_{i,\tilde{y}_{i-1}}$'s" arising from the various $\Box H_{\lambda}$'s.

(ii) Write

$$\mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i)|_{\widetilde{y}_{i-1}} \stackrel{\mathrm{def}}{=} \{ \widetilde{y}' \in \mathrm{VCN}^{\mathrm{sch}}({}^{\Box}\mathbb{H}_i) \, | \, \widetilde{y}'_{i-1} = \widetilde{y}_{i-1} \, \}$$

[cf. Definition 3.11, (iii)]. Then one verifies easily from the various definitions involved that we have a *natural bijection*

$$\operatorname{VCN}^{\operatorname{sch}}({}^{\Box}\mathbb{H}_i)|_{\widetilde{y}_{i-1}} \xrightarrow{\sim} \operatorname{VCN}(\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}})$$

[cf. (i)]. In particular, the element $\widetilde{y}_i \in \text{VCN}^{\text{sch}}(\square \mathbb{H}_i)|_{\widetilde{y}_{i-1}}$ determines an element

$$\widetilde{z}_{i,\widetilde{y}} \in \mathrm{VCN}(\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}})$$

of VCN($\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}}$).

(iii) It follows immediately from the various definitions involved that we have a natural action of $(\Box H_i)^{\{l\}}$, hence also of $(\Box H_{i/i-1})^{\{l\}}$ [cf. the notation of condition (c) of Definition 3.10], on the set VCN^{sch}($\Box \mathbb{H}_i$). Thus, we obtain a *tautological isomorphism*

$$\Pi_{\mathcal{G}_{i,\widetilde{y}_{i-1}}} \xrightarrow{\sim} (\Box H_{i/i-1})^{\{l\}}$$

such that the various *VCN-subgroups* [cf. [CbTpI], Definition 2.1, (i)] on the left-hand side of this isomorphism correspond to the various *stabilizer subgroups* of $(\Box H_{i/i-1})^{\{l\}}$ associated to elements of VCN^{sch} $(\Box \mathbb{H}_i)|_{\tilde{y}_{i-1}}$ [cf. the notation of (ii); the natural bijection of the second display of (ii)] on the right-hand side of this isomorphism.

(iv) Let $(F_i)_{i \in \{1, \dots, n\}}$ be a collection of closed subgroups $F_i \subseteq (\Box H_i)^{\{l\}}$. Then we shall say that the collection $(F_i)_{i \in \{1, \dots, n\}}$ is the VCNchain of $\Box H$ associated to $\widetilde{y} \in \text{VCN}^{\text{sch}}(\Box \mathbb{H})$ if, for each $i \in \{1, \dots, n\}$, the closed subgroup F_i coincides with the image of the VCN-subgroup of $\Pi_{\mathcal{G}_{i,\widetilde{y}_{i-1}}}$ associated to $\widetilde{z}_{i,\widetilde{y}} \in \text{VCN}(\widetilde{\mathcal{G}}_{i,\widetilde{y}_{i-1}})$ [cf. (ii)] via the isomorphism $\Pi_{\mathcal{G}_{i,\widetilde{y}_{i-1}}} \xrightarrow{\sim} (\Box H_{i/i-1})^{\{l\}} \subseteq (\Box H_i)^{\{l\}}$ of (iii). We shall say that the collection $(F_i)_{i \in \{1,\dots, n\}}$ is an $\Box \mathbb{H}$ -VCN-chain of $\Box H$ if $(F_i)_{i \in \{1,\dots, n\}}$ is the VCN-chain of $\Box H$ associated to an element of VCN^{sch}($\Box \mathbb{H}$). Write

$$\mathrm{VCN^{gp}}(\square\mathbb{H})$$

for the set of $\Box \mathbb{H}$ -VCN-chains of $\Box H$. In fact, we shall see below that VCN^{gp}($\Box \mathbb{H}$) is *independent* of the choice of $\Box \mathbb{H}$ [cf. Lemma 3.14, (iv)]. Thus, we conclude from [CmbGC], Proposition 1.2, (i), that the natural bijections of (ii) determine a *bijection*

 $\operatorname{VCN}^{\operatorname{sch}}(\Box \mathbb{H}) \xrightarrow{\sim} \operatorname{VCN}^{\operatorname{gp}}(\Box \mathbb{H}).$

Definition 3.13. In the notation of Definition 3.1:

- (i) We shall say that an isomorphism of profinite groups °Π_n → °Π_n is SAF-admissible [i.e., "standard-adjacent-fiber-admissible"] if it is PF-admissible whenever n ≥ 2 [cf. [CbTpI], Definition 1.4, (i)] and, moreover, is compatible with the standard fiber filtrations on °Π_n and °Π_n [cf. [CmbCsp], Definition 1.1, (i)]. We shall refer to an outer isomorphism °Π_n → °Π_n as SAF-admissible if it arises from an SAF-admissible isomorphism. One verifies easily that, in the case of an automorphism or outomorphism, SAF-admissibility is equivalent to F-admissibility whenever n ≥ 2.
- (ii) Let $\alpha: {}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ be an isomorphism of profinite groups [so ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \text{cf.}$, e.g., the proof of [CbTpI], Proposition 1.5, (i)] and $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. Then we shall say that α is $\{l\}$ -*G*-admissible [i.e., $\{l\}$ -graph-admissible] if α is SAF-admissible [cf. (i)], and, moreover, the following condition is satisfied:

Let ${}^{\circ}J \subseteq {}^{\circ}\Pi_n$ be an open subgroup of ${}^{\circ}\Pi_n$. Then there exist an open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_n$ of ${}^{\circ}\Pi_n$ of *l-polystable type* [cf. Definition 3.10] and an ${}^{\circ}H$ -*l-system* ${}^{\circ}\mathbb{H} = \{{}^{\circ}H_{\lambda}\}_{\lambda \in \Lambda}$ [cf. Definition 3.11, (ii)] such that ${}^{\circ}H \subseteq {}^{\circ}J$, ${}^{\bullet}H \stackrel{\text{def}}{=} \alpha({}^{\circ}H)$ is of *l-polystable type*, ${}^{\bullet}\mathbb{H} = \{{}^{\bullet}H_{\lambda} \stackrel{\text{def}}{=} \alpha({}^{\circ}H_{\lambda})\}_{\lambda \in \Lambda}$ is an ${}^{\bullet}H$ -*l-system*, and, moreover, the isomorphism ${}^{\circ}H \stackrel{\sim}{\to} {}^{\bullet}H$ determined by α induces a bijection

 $\mathrm{VCN}^{\mathrm{gp}}(^{\circ}\mathbb{H}) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}(^{\bullet}\mathbb{H})$

[cf. Definition 3.12, (iv)].

We shall say that an outer isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ is $\{l\}$ -*G*-*admissible* if it arises from an $\{l\}$ -G-admissible isomorphism.

(iii) We shall say that an isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ [so ${}^{\circ}\Sigma = {}^{\bullet}\Sigma$ — cf., e.g., the proof of [CbTpI], Proposition 1.5, (i)] is *Gadmissible* [i.e., graph-admissible] if ${}^{\circ}\Sigma = {}^{\bullet}\Sigma \not\subseteq \{{}^{\circ}p, {}^{\bullet}p\}$, and, moreover, the isomorphism is $\{l\}$ -G-admissible [cf. (ii)] for every prime number $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. We shall say that an outer isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ is *G*-admissible if it arises from a G-admissible isomorphism. (iv) We shall write

$$\operatorname{Aut}^{\{l\}\text{-}G}(^{\circ}\Pi_n) \subseteq \operatorname{Aut}(^{\circ}\Pi_n)$$

for the subgroup [cf. Lemma 3.14, (ii), (iii), below] of $\{l\}$ -Gadmissible automorphisms of ${}^{\circ}\Pi_n$ [cf. (ii)];

$$\operatorname{Out}^{\{l\}-\mathrm{G}}(^{\circ}\Pi_n) \stackrel{\text{def}}{=} \operatorname{Aut}^{\{l\}-\mathrm{G}}(^{\circ}\Pi_n) / \operatorname{Inn}(^{\circ}\Pi_n) \subseteq \operatorname{Out}(^{\circ}\Pi_n)$$

for the subgroup of $\{l\}$ -G-admissible outomorphisms of $^{\circ}\Pi_n$;

$$\operatorname{Aut}^{\operatorname{G}}({}^{\circ}\Pi_{n}) \stackrel{\operatorname{def}}{=} \bigcap_{l \in {}^{\circ}\Sigma \setminus ({}^{\circ}\Sigma \cap \{{}^{\circ}p\})} \operatorname{Aut}^{\{l\}-\operatorname{G}}({}^{\circ}\Pi_{n}) \subseteq \operatorname{Aut}({}^{\circ}\Pi_{n})$$

for the subgroup of G-admissible automorphisms of $^{\circ}\Pi_n$ [cf. (iii)];

$$\operatorname{Out}^{\mathsf{G}}(^{\circ}\Pi_{n}) \stackrel{\text{def}}{=} \bigcap_{l \in ^{\circ}\Sigma \setminus (^{\circ}\Sigma \cap \{^{\circ}p\})} \operatorname{Out}^{\{l\}-\mathsf{G}}(^{\circ}\Pi_{n}) \subseteq \operatorname{Out}(^{\circ}\Pi_{n})$$

for the subgroup of G-admissible outomorphisms of $^{\circ}\Pi_n$.

Remark 3.13.1.

- (i) In the notation of Definition 3.13, suppose that n = 1. Then it follows immediately from Proposition 3.6, (ii); Lemma 3.14, (ii), (iii), below; [CmbGC], Proposition 1.5, (ii), that the following conditions are equivalent:
 - α is *G*-admissible in the sense of Definition 3.7, (i).
 - There exists a prime number $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$, and, moreover, α is $\{l\}$ -*G*-admissible in the sense of Definition 3.13, (ii).
 - α is *G*-admissible in the sense of Definition 3.13, (iii).

In particular, for any prime number $l \in {}^{\circ}\Sigma$ such that $l \neq {}^{\circ}p$, we have equalities

$$\operatorname{Out}({}^{\circ}\Pi_{1})^{\mathrm{G}} = \operatorname{Out}^{\mathrm{G}}({}^{\circ}\Pi_{1}) = \operatorname{Out}^{\{l\}-\mathrm{G}}({}^{\circ}\Pi_{1})$$

[cf. Definitions 3.7, (i); 3.13, (iv)].

(ii) In the notation of Definition 3.13, (iv), one verifies easily from the various definitions involved that

$$\operatorname{Out}^{\mathrm{G}}(^{\circ}\Pi_{n}) \subseteq \operatorname{Out}^{\{l\}-\mathrm{G}}(^{\circ}\Pi_{n}) \subseteq \operatorname{Out}^{\mathrm{FC}}(^{\circ}\Pi_{n})$$

[cf. [CmbCsp], Definition 1.1, (ii)].

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Lemma 3.14 (Subgroups of *l*-polystable type). In the notation of Definition 3.1, let α : ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ be an isomorphism of profinite groups [so ${}^{\circ}\Sigma = {}^{\bullet}\Sigma - cf.$, e.g., the proof of [CbTpI], Proposition 1.5, (i)] and $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. Suppose that α is SAF-admissible [cf. Definition 3.13, (i)]. Then the following hold:

- (i) Let $^{\circ}J \subseteq ^{\circ}\Pi_n$ be an open subgroup of $^{\circ}\Pi_n$. Then there exists an open subgroup $^{\circ}H \subseteq ^{\circ}\Pi_n$ of $^{\circ}\Pi_n$ of *l*-polystable type [cf. Definition 3.10] such that $^{\circ}H \subseteq ^{\circ}J$.
- (ii) Let $^{\circ}H \subseteq ^{\circ}\Pi_n$ be an open subgroup of $^{\circ}\Pi_n$ of *l*-polystable type. Then $^{\bullet}H \stackrel{\text{def}}{=} \alpha(^{\circ}H)$ is an open subgroup of $^{\bullet}\Pi_n$ of *l*-polystable type.
- (iii) Let $^{\circ}H \subseteq ^{\circ}\Pi_n$ be an open subgroup of *l*-polystable type of $^{\circ}\Pi_n$. Then there exists an $^{\circ}H$ -*l*-system $^{\circ}\mathbb{H} = \{^{\circ}H_{\lambda}\}_{\lambda \in \Lambda}$ [cf. Definition 3.11, (ii)] such that $^{\bullet}\mathbb{H} = \{^{\bullet}H_{\lambda} \stackrel{\text{def}}{=} \alpha(^{\circ}H_{\lambda})\}_{\lambda \in \Lambda}$ is an $^{\bullet}H$ -*l*-system [cf. (ii)].
- (iv) Let $^{\circ}H \subseteq ^{\circ}\Pi_n$ be an open subgroup of *l*-polystable type of $^{\circ}\Pi_n$; $^{\circ}\mathbb{H} = \{^{\circ}H_{\lambda}\}_{\lambda \in \Lambda}$, $^{\circ}\mathbb{H}^{\dagger} = \{^{\circ}H_{\lambda^{\dagger}}^{\dagger}\}_{\lambda^{\dagger} \in \Lambda^{\dagger}}$ $^{\circ}H$ -*l*-systems. Then there exists an $^{\circ}H$ -*l*-system $^{\circ}\mathbb{H}^{\ddagger} = \{^{\circ}H_{\lambda^{\ddagger}}^{\ddagger}\}_{\lambda^{\ddagger} \in \Lambda^{\ddagger}}$ that satisfies the condition that, for each $(\lambda, \lambda^{\dagger}) \in \Lambda \times \Lambda^{\dagger}$, there exists a $\lambda^{\ddagger} \in \Lambda^{\ddagger}$ such that $^{\circ}H_{\lambda^{\ddagger}}^{\ddagger} \subseteq ^{\circ}H_{\lambda} \cap ^{\circ}H_{\lambda^{\dagger}}^{\dagger}$. In particular, the sets VCN^{sch}($^{\Box}\mathbb{H}$) [cf. Definition 3.11, (iii)] and VCN^{sp}($^{\Box}\mathbb{H}$) [cf. Definition 3.12, (iv)] are independent of the choice of $^{\Box}\mathbb{H}$ [cf. Definition 3.11, (ii)], i.e., depend only on $^{\circ}H$, respectively.
- (v) Let ${}^{\circ}H, {}^{\circ}H^{\dagger} \subseteq {}^{\circ}\Pi_{n}$ be open subgroups of *l*-polystable type of ${}^{\circ}\Pi_{n}; {}^{\circ}\mathbb{H} = \{{}^{\circ}H_{\lambda}\}_{\lambda \in \Lambda}$ an ${}^{\circ}H$ -*l*-system; ${}^{\circ}\mathbb{H}^{\dagger} = \{{}^{\circ}H_{\lambda^{\dagger}}^{\dagger}\}_{\lambda^{\dagger} \in \Lambda^{\dagger}}$ an ${}^{\circ}H^{\dagger}$ -*l*-system. Suppose that the inclusion ${}^{\circ}H^{\dagger} \subseteq {}^{\circ}H$, hence also the inclusion ${}^{\bullet}H^{\dagger} \stackrel{\text{def}}{=} \alpha({}^{\circ}H^{\dagger}) \subseteq {}^{\bullet}H \stackrel{\text{def}}{=} \alpha({}^{\circ}H)$, holds. Suppose, moreover, that ${}^{\bullet}\mathbb{H} = \{{}^{\bullet}H_{\lambda} \stackrel{\text{def}}{=} \alpha({}^{\circ}H_{\lambda})\}_{\lambda \in \Lambda}$ is an ${}^{\bullet}H$ *l*-system [cf. (ii)], and that ${}^{\bullet}\mathbb{H}^{\dagger} = \{{}^{\bullet}H_{\lambda^{\dagger}} \stackrel{\text{def}}{=} \alpha({}^{\circ}H_{\lambda^{\dagger}})\}_{\lambda^{\dagger} \in \Lambda^{\dagger}}$ is an ${}^{\bullet}H^{\dagger}$ -*l*-system [cf. (ii)]. Then if the isomorphism ${}^{\circ}H^{\dagger} \xrightarrow{}$ ${}^{\bullet}H^{\dagger}$ determined by α induces a bijection

 $\mathrm{VCN}^{\mathrm{gp}}(^{\circ}\mathbb{H}^{\dagger}) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}(^{\bullet}\mathbb{H}^{\dagger}),$

then the isomorphism ${}^{\circ}H \xrightarrow{\sim} {}^{\bullet}H$ determined by α induces a bijection

 $\operatorname{VCN^{gp}}(^{\circ}\mathbb{H}) \xrightarrow{\sim} \operatorname{VCN^{gp}}(^{\bullet}\mathbb{H}).$

Proof. First, we verify assertion (i) by *induction* on *n*. Write ${}^{\circ}J_{n-1}$ for the image of ${}^{\circ}J$ in ${}^{\circ}\Pi_{n-1}$ and $({}^{\circ}J_{n/n-1})^{\{l\}}$ for the maximal pro-*l* quotient of the kernel ${}^{\circ}J_{n/n-1} \stackrel{\text{def}}{=} \text{Ker}({}^{\circ}J \twoheadrightarrow {}^{\circ}J_{n-1})$. Now let us observe

that if n = 1, then assertion (i) follows immediately from the various definitions involved [cf. also the fact that ${}^{\circ}\Pi_n$ is topologically finitely generated — cf. [MzTa], Proposition 2.2, (ii)]]. Thus, suppose that $n \geq 2$, and that the *induction hypothesis* is in force.

Next, let us observe that since ${}^{\circ}\Pi_n$ is topologically finitely generated [cf. [MzTa], Proposition 2.2, (ii)], we may assume without loss of generality — by replacing ${}^{\circ}J$ by a suitable open subgroup of ${}^{\circ}J$ — that ${}^{\circ}J$ satisfies condition (a) of Definition 3.10 in the case where we take "i" to be *n*. Next, by applying the *induction hypothesis* to ${}^{\circ}J_{n-1}$, we obtain an open subgroup ${}^{\circ}H_{n-1} \subseteq {}^{\circ}\Pi_{n-1}$ of ${}^{\circ}\Pi_{n-1}$ that is contained in ${}^{\circ}J_{n-1}$ and of *l-polystable type*. Write ${}^{\circ}H \stackrel{\text{def}}{=} {}^{\circ}H_{n-1} \times {}^{\circ}J_{n-1} {}^{\circ}J$. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow {}^{\circ}J_{n/n-1} \longrightarrow {}^{\circ}H \longrightarrow {}^{\circ}H_{n-1} \longrightarrow 1.$$

Then it follows immediately from the condition imposed above on ${}^{\circ}J$, together with the *induction hypothesis*, that [by taking ${}^{\circ}J_{n-1}$ to be sufficiently small] we may assume without loss of generality that ${}^{\circ}H$ satisfies conditions (a) and (c) of Definition 3.10 [hence also (d) of Remark 3.10.1, (i)]. On the other hand, by considering the quotient ${}^{\circ}H \rightarrow ({}^{\circ}J_{n/n-1})^{\{l\}} \stackrel{\text{out}}{\rtimes} ({}^{\circ}H_{n-1})^{\{l\}}$ [i.e., that arises from the fact that ${}^{\circ}H$ satisfies condition (d) of Remark 3.10.1, (i) — cf. also the discussion entitled "Topological groups" in [CbTpI], §0], we conclude that the natural homomorphism $({}^{\circ}J_{n/n-1})^{\{l\}} \rightarrow ({}^{\circ}H)^{\{l\}}$ induced by the natural inclusion ${}^{\circ}J_{n/n-1} \leftrightarrow {}^{\circ}H$ is injective. Thus, one verifies easily from Lemma 1.2, (i) [where we take "(G, N, J)" to be $({}^{\circ}\Pi_n, {}^{\circ}H, {}^{\circ}\Pi_{n-1})$], (ii) [where we take "(G, N, H)" to be $({}^{\circ}\Pi_n, {}^{\circ}H, {}^{\circ}\Pi_{n-1})$], together with our choice of ${}^{\circ}H_{n-1}$, that ${}^{\circ}H$ satisfies condition (b) of Definition 3.10, i.e., that ${}^{\circ}H$ is *l*-polystable type. This completes the proof of assertion (i). Assertion (ii) follows immediately from the various definitions involved.

Next, we verify assertions (iii), (iv). Let us first observe that, to verify assertions (iii), (iv), it suffices to verify the following assertion:

Claim 3.14.A: Let ${}^{\circ}J \subseteq {}^{\circ}H$ be an open subgroup that arises from an open subgroup of the maximal pro-*l* quotient $({}^{\circ}H)^{\{l\}}$ of ${}^{\circ}H$. Then there exists an open subgroup ${}^{\circ}N \subseteq {}^{\circ}J$ such that ${}^{\circ}N$, ${}^{\bullet}N \stackrel{\text{def}}{=} \alpha({}^{\circ}N)$ are of *l*-polystable type, VCN-complete, and, moreover, arise from open subgroups of $({}^{\circ}H)^{\{l\}}$, $({}^{\bullet}H)^{\{l\}}$, respectively.

In the remainder of the proofs of assertions (iii), (iv), we verify Claim 3.14.A by *induction* on n. Write $\bullet J \stackrel{\text{def}}{=} \alpha(\circ J)$. For each $\Box \in \{\circ, \bullet\}$, write $\Box J_{n-1}$ for the image of $\Box J$ in $\Box \Pi_{n-1}$ and $(\Box J_{n/n-1})^{\{l\}}$ for the maximal pro-l quotient of the kernel $\Box J_{n/n-1} \stackrel{\text{def}}{=} \text{Ker}(\Box J \twoheadrightarrow \Box J_{n-1})$. Now let us observe that if n = 1, then Claim 3.14.A follows immediately

from the various definitions involved [cf. also the fact that $^{\circ}\Pi_n$ is topologically finitely generated — cf. [MzTa], Proposition 2.2, (ii)]]. Thus, suppose that $n \geq 2$, and that the *induction hypothesis* is in force.

Next, let us observe that since ${}^{\circ}\Pi_n$ is topologically finitely generated [cf. [MzTa], Proposition 2.2, (ii)], and ${}^{\circ}H$ satisfies condition (a) of Definition 3.10, we may assume without loss of generality — by replacing ${}^{\circ}J$ by a suitable open subgroup of ${}^{\circ}J$ — that ${}^{\circ}J$ satisfies condition (a) of Definition 3.10 in the case where we take "i" to be n. Next, let us observe that since ${}^{\circ}J$ arises from an open subgroup of $({}^{\circ}H)^{\{l\}}$, by considering the natural isomorphism $({}^{\circ}H)^{\{l\}} \xrightarrow{\sim} ({}^{\circ}H_{n/n-1})^{\{l\}} \xrightarrow{\text{out}} ({}^{\circ}H_{n-1})^{\{l\}}$ [i.e., that arises from the fact that ${}^{\circ}H$ satisfies condition (d) of Remark 3.10.1, (i)], we conclude that ${}^{\circ}J$ satisfies condition (d) of Remark 3.10.1, (i), in the case where we take "i" to be n. In particular, since the natural action of ${}^{\circ}J_{n-1}$ on $(({}^{\circ}J_{n/n-1})^{\{l\}})^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{Z}/l^{\text{aut}}\mathbb{Z}$ factors through a pro-l quotient of ${}^{\circ}J_{n-1}$, we may assume without loss of generality — by replacing ${}^{\circ}J$ by the inverse image in ${}^{\circ}J$ of a suitable open subgroup of ${}^{\circ}J_{n-1}$ — that ${}^{\circ}J$ satisfies condition (c) of Definition 3.10 in the case where we take "i" to be n.

Thus, by applying the *induction hypothesis* to ${}^{\circ}J_{n-1} \subseteq {}^{\circ}H_{n-1}$, we obtain an open subgroup ${}^{\circ}N_{n-1} \subseteq {}^{\circ}J_{n-1}$ such that ${}^{\circ}N_{n-1}$, ${}^{\bullet}N_{n-1} \stackrel{\text{def}}{=} \alpha({}^{\circ}N_{n-1})$ are of *l*-polystable type, VCN-complete, and arise from open subgroups of $({}^{\circ}H_{n-1})^{\{l\}}$, $({}^{\bullet}H_{n-1})^{\{l\}}$, respectively. Write

$$^{\circ}N \stackrel{\text{def}}{=} ^{\circ}N_{n-1} \times_{^{\circ}J_{n-1}} ^{\circ}J.$$

Then one verifies immediately, by a similar argument to the argument applied in the final portion of the proof of assertion (i), that ${}^{\circ}N$, ${}^{\circ}N \stackrel{\text{def}}{=} \alpha({}^{\circ}N)$ are of *l*-polystable type and, moreover, arise from open subgroups of $({}^{\circ}H)^{\{l\}}$, $({}^{\circ}H)^{\{l\}}$, respectively. In particular, since ${}^{\circ}N$, ${}^{\circ}N$ satisfy condition (d) of Remark 3.10.1, (i), in the case where we take "*i*" to be *n*, we may assume without loss of generality — by replacing ${}^{\circ}N$ by the inverse image in ${}^{\circ}N$ of a suitable open subgroup of ${}^{\circ}N_{n-1}$ [cf. the *induction hypothesis*] — that ${}^{\circ}N$, ${}^{\circ}N$ satisfy the condition that each of the elements of $\overline{\mathrm{VCN}^{\mathrm{sch}}}({}^{\circ}N)$, $\overline{\mathrm{VCN}^{\mathrm{sch}}}({}^{\circ}N)$ satisfies condition (c) of Definition 3.11, (i), in the case where we take "*i*" to be *n*. Thus, we conclude [cf. the fact that ${}^{\circ}N_{n-1}$, ${}^{\circ}N_{n-1}$ are *VCN-complete*] that ${}^{\circ}N$, ${}^{\circ}N$ are *VCN-complete*. This completes the proof of Claim 3.14.A, hence also the proofs of assertions (iii), (iv).

Finally, we verify assertion (v). For each $\Box \in \{\circ, \bullet\}$ and each $i \in \{1, \ldots, n\}$, write $\Box H_{i/i-1}$, $\Box H_{i/i-1}^{\dagger}$ for the respective subquotients of $\Box H$, $\Box H^{\dagger}$ determined by the subquotient $\Box \Pi_{i/i-1}$ of $\Box \Pi_n$; $(\Box H_{i/i-1})^{\{l\}}$, $(\Box H_{i/i-1}^{\dagger})^{\{l\}}$ for the respective maximal pro-*l* quotients of $\Box H_{i/i-1}, \Box H_{i/i-1}^{\dagger}$ [cf. Definition 3.10, (c)]. Then let us observe that it follows immediately from [CmbGC], Proposition 1.2, (ii), together with the various

definitions involved that, for each $\Box \in \{\circ, \bullet\}$ and each $i \in \{1, \ldots, n\}$, every *VCN-subgroup* of $(\Box H_{i/i-1})^{\{l\}}$ [i.e., discussed as in Definition 3.12, (iii), (iv)] may be obtained as the *commensurator of the image of a VCN-subgroup* of $(\Box H_{i/i-1}^{\dagger})^{\{l\}}$ [by the homomorphism $(\Box H_{i/i-1}^{\dagger})^{\{l\}} \rightarrow$ $(\Box H_{i/i-1})^{\{l\}}$ determined by the natural inclusion $\Box H^{\dagger} \subseteq \Box H$]. Moreover, one also verifies easily that every proper closed subgroup of $(\Box H_{i/i-1})^{\{l\}}$ obtained as the *commensurator of the image of a VCN-subgroup* of $(\Box H_{i/i-1}^{\dagger})^{\{l\}}$ is a *VCN-subgroup* of $(\Box H_{i/i-1})^{\{l\}}$. Assertion (v) now follows formally. This completes the proof of Lemma 3.14. \Box

Definition 3.15. In the notation of Definition 3.12, write $(F_i)_{i \in \{1, \dots, n\}} \in$ VCN^{gp}($^{\Box}\mathbb{H}$) for the VCN-chain of $^{\Box}H$ associated to $\tilde{y} \in$ VCN^{sch}($^{\Box}\mathbb{H}$) [cf. Definition 3.12, (iv)]. Now since ($^{\Box}H$) $^{\{l\}} \subseteq (^{\Box}\Pi_n)^*$ [cf. the notation of condition (b) of Definition 3.10] is *open*, and ($^{\Box}\Pi_n$)^{*} is *topologically finitely generated, slim* [cf. Proposition 2.3, (i)] and *almost pro-l*, there exist an *open subgroup* $^{\Box}J \subseteq I_{\Box_K}$ of I_{\Box_K} and a *homomorphism*

$${}^{\Box}\rho\colon{}^{\Box}J\longrightarrow\operatorname{Out}(({}^{\Box}H)^{\{l\}})$$

that

- is compatible [in the evident sense] with the homomorphism $\Box J \to \operatorname{Out}((\Box \Pi_n)^*)$ induced [cf. condition (a) of Definition 3.10] by $\Box \rho_n \colon I_{\Box K} \to \operatorname{Out}(\Box \Pi_n)$,
- induces, for each $i \in \{1, \dots, n\}$, a homomorphism

 $\Box J \longrightarrow \operatorname{Out}((\Box H_i)^{\{l\}})$

— relative to the natural surjection $(\Box H)^{\{l\}} \twoheadrightarrow (\Box H_i)^{\{l\}}$ — and, moreover,

• factors through the maximal pro-l quotient $(\Box J)^{\{l\}}$ of $\Box J$, which [as is easily verified] is isomorphic to \mathbb{Z}_l as an abstract profinite group.

Write $I_{\tilde{y}_0} \stackrel{\text{def}}{=} (\Box J)^{\{l\}}$. Then, for $i \in \{1, \dots, n\}$, we define closed subgroups

$$I_{\widetilde{y}_i} \subseteq {}^{\square}H_i^{\rho}|_{\widetilde{y}_{i-1}} \subseteq {}^{\square}H_i^{\rho} \stackrel{\text{def}}{=} ({}^{\square}H_i)^{\{l\}} \stackrel{\text{out}}{\rtimes} ({}^{\square}J)^{\{l\}}$$

[cf. the discussion entitled "Topological groups" in [CbTpI], 0] as follows [inductively on i]:

(i) Set

$${}^{\Box}H_1^{\rho}|_{\widetilde{y}_0} \stackrel{\text{def}}{=} {}^{\Box}H_1^{\rho}, \ I_{\widetilde{y}_1} \stackrel{\text{def}}{=} Z_{{}^{\Box}H_1^{\rho}|_{\widetilde{y}_0}}(F_1).$$

(ii) Suppose that $n \ge i \ge 2$. Then, by the *induction hypothesis*, we have already constructed closed subgroups

$$I_{\widetilde{y}_{i-1}} \subseteq {}^{\Box}H^{\rho}_{i-1}|_{\widetilde{y}_{i-2}} \subseteq {}^{\Box}H^{\rho}_{i-1},$$

hence also a natural outer representation

$$I_{\widetilde{y}_{i-1}} \hookrightarrow {}^{\square}H_{i-1}^{\rho} \to \operatorname{Out}(({}^{\square}H_{i/i-1})^{\{l\}})$$

— where the second arrow is the natural outer representation arising from the exact sequence of profinite groups

$$1 \longrightarrow ({}^{\square}H_{i/i-1})^{\{l\}} \longrightarrow {}^{\square}H_i^{\rho} \longrightarrow {}^{\square}H_{i-1}^{\rho} \longrightarrow 1.$$

Then we set

$${}^{\Box}H_i^{\rho}|_{\widetilde{y}_{i-1}} \stackrel{\text{def}}{=} ({}^{\Box}H_{i/i-1})^{\{l\}} \stackrel{\text{out}}{\rtimes} I_{\widetilde{y}_{i-1}}, \ I_{\widetilde{y}_i} \stackrel{\text{def}}{=} Z_{{}^{\Box}H_i^{\rho}|_{\widetilde{y}_{i-1}}}(F_i).$$

Remark 3.15.1. In the situation of Definition 3.15, it follows immediately from the definition of $I_{\tilde{y}_i}$ [cf. also [CmbGC], Remark 1.1.3; [CmbGC], Proposition 1.2, (ii)] that $I_{\tilde{y}_i}$ is isomorphic to a profinite group of the form $\mathbb{Z}_l^{\oplus j}$, where j is a positive integer $\leq i + 1$.

Proposition 3.16 (Graph-admissible isomorphisms). In the notation of Definition 3.1, let $\alpha : {}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ be an isomorphism of profinite groups [so ${}^{\circ}\Sigma = {}^{\bullet}\Sigma - cf.$, e.g., the proof of [CbTpI], Proposition 1.5, (i)] and $l \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$ such that $l \notin \{{}^{\circ}p, {}^{\bullet}p\}$. Then the following hold:

- (i) If °p ∉ °Σ and •p ∉ •Σ, then suppose that α is PC-admissible [cf. [CbTpI], Definition 1.4, (ii)]. If α is SAF-admissible [cf. Definition 3.13, (i)] and {l}-I-admissible [cf. Definition 3.8, (i)], then α is {l}-G-admissible [cf. Definition 3.13, (ii)].
- (ii) Suppose that α is $\{l\}$ -G-admissible. Then there exists an algorithm, which is functorial with respect to α , for constructing an isomorphism of topological groups

$$\alpha^{\mathrm{tp}} \colon {}^{\circ}\Pi_n^{\mathrm{tp}} \xrightarrow{\sim} {}^{\bullet}\Pi_n^{\mathrm{tp}}$$

such that the isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ induced by α^{tp} [cf. Proposition 3.3, (i)] coincides with α .

Proof. First, we verify assertion (i). Let ${}^{\circ}J \subseteq {}^{\circ}\Pi_n$ be an open subgroup of ${}^{\circ}\Pi_n$. Then it follows from Lemma 3.14, (i), (ii), (iii), that there exist an open subgroup ${}^{\circ}H \subseteq {}^{\circ}\Pi_n$ of ${}^{\circ}\Pi_n$ of l-polystable type [cf. Definition 3.10] and an ${}^{\circ}H$ -l-system ${}^{\circ}\mathbb{H} = \{{}^{\circ}H_{\lambda}\}_{\lambda \in \Lambda}$ [cf. Definition 3.11, (ii)] such that ${}^{\circ}H \subseteq {}^{\circ}J$, ${}^{\bullet}H \stackrel{\text{def}}{=} \alpha({}^{\circ}H)$ is of l-polystable type, and ${}^{\bullet}\mathbb{H} = \{{}^{\bullet}H_{\lambda} \stackrel{\text{def}}{=} \alpha({}^{\circ}H_{\lambda})\}_{\lambda \in \Lambda}$ is an ${}^{\bullet}H$ -*l*-system. Now it follows immediately from the various definitions involved that, to complete the verification of assertion (i), it suffices to verify the following assertion:

Claim 3.16.A: For each $i \in \{1, \dots, n\}$, the isomorphism ${}^{\circ}H_i \xrightarrow{\sim} {}^{\bullet}H_i$ [cf. the notation of Definition 3.10] determined by α induces a bijection

 $\operatorname{VCN^{gp}}(^{\circ}\mathbb{H}_i) \xrightarrow{\sim} \operatorname{VCN^{gp}}(^{\bullet}\mathbb{H}_i)$

[cf. Definitions 3.11, (iii); 3.12, (iv)].

We verify Claim 3.16.A by *induction* on *i*. If i = 1, then Claim 3.16.A follows immediately from the equivalence (a) \Leftrightarrow (b^{\exists}) of Theorem 3.9, together with Remark 3.13.1, (i). Now suppose that $i \geq 2$, and that the *induction hypothesis* is in force. Then it follows immediately from the *induction hypothesis* that, for each $j \in \{1, \dots, i-1\}$, the isomorphism ${}^{\circ}H_{j} \xrightarrow{\sim} {}^{\bullet}H_{j}$ determined by α induces a bijection

$$\operatorname{VCN^{gp}}(^{\circ}\mathbb{H}_{i}) \xrightarrow{\sim} \operatorname{VCN^{gp}}(^{\bullet}\mathbb{H}_{i}).$$

Let $\widetilde{y}_{i-1} \in \text{VCN}^{\text{sch}}(\widetilde{H}_{i-1}), \widetilde{y}_{i-1} \in \text{VCN}^{\text{sch}}(\bullet \mathbb{H}_{i-1})$ [cf. Definition 3.11, (iii)] be elements that correspond via the above bijection, relative to the \circ -, \bullet -versions of the displayed bijection of Definition 3.12, (iv).

Now since α is $\{l\}$ -*I*-admissible, for $\Box \in \{\circ, \bullet\}$, there exist an open subgroup $\Box J \subseteq I_{\Box_K}$ of I_{\Box_K} and an outer representation $\Box \rho \colon \Box J \to Out((\Box H)^{\{l\}})$ as in Definition 3.15 such that $\circ \rho$ is compatible, relative to α , with $\bullet \rho$. Thus, it follows immediately from the various definitions involved that the isomorphism $\circ H_i \xrightarrow{\sim} \bullet H_i$ determined by α induces an isomorphism of profinite groups

$${}^{\circ}H_{i}^{\rho}|_{{}^{\circ}\widetilde{y}_{i-1}} \xrightarrow{\sim} {}^{\bullet}H_{i}^{\rho}|_{{}^{\bullet}\widetilde{y}_{i-1}}$$

that lies over an isomorphism $\beta \colon I_{\circ \widetilde{y}_{i-1}} \xrightarrow{\sim} I_{\bullet \widetilde{y}_{i-1}}$ [cf. Definition 3.15]. In particular, we obtain a commutative diagram of profinite groups

— where the right-hand vertical arrow is the isomorphism induced by α . Moreover, one verifies immediately from the various definitions involved [cf. also Remark 3.15.1] that, for each $\Box \in \{\circ, \bullet\}$, the *positive definite profinite Dehn multi-twists* [cf. [CbTpI], Definition 4.4; [CbTpI], Definition 5.8, (iii)] in the *image* of the composite

$$I_{\square_{\widetilde{y}_{i-1}}} \longrightarrow \operatorname{Out}((\square H_{i/i-1})^{\{l\}}) \xleftarrow{\sim} \operatorname{Out}(\Pi_{\mathcal{G}_{i,\square_{\widetilde{y}_{i-1}}}})$$

— where the second arrow is the isomorphism induced by the isomorphism of Definition 3.12, (iii) — form a *dense* subset of this image [cf. [CbTpI], Lemma 5.4, (i), (ii), (iii); [CbTpI], Proposition 5.6, (ii)]. In

particular, it follows immediately [cf. the easily verified elementary fact that no dense subset of a nonzero finitely generated free \mathbb{Z}_l -module is contained in a finite union of proper \mathbb{Z}_l -submodules of the given finitely generated free \mathbb{Z}_l -module] that there exists an element $\gamma \in I_{\circ \tilde{y}_{i-1}}$ such that if we write $\gamma \stackrel{\text{def}}{=} \beta(\gamma) \in I_{\bullet \tilde{y}_{i-1}}$, then, for $\Box = \circ$ (respectively, $\Box = \bullet$), the image of $\Box \gamma$ via the composite of the above display is a positive definite profinite Dehn multi-twist (respectively, nondegenerate profinite Dehn multi-twist [cf. [CbTpI], Definition 4.4; [CbTpI], Definition 5.8, (ii)]). Thus, it follows immediately from [CbTpII], Theorem 1.9, (ii), together with the equivalences of [CbTpI], Corollary 5.9, (ii), (iii), that the isomorphism

$$\alpha_{i/i-1} \colon \Pi_{\mathcal{G}_{i,}\circ \widetilde{y}_{i-1}} \xrightarrow{\sim} (^{\circ}H_{i/i-1})^{\{l\}} \xrightarrow{\sim} (^{\bullet}H_{i/i-1})^{\{l\}} \xleftarrow{\sim} \Pi_{\mathcal{G}_{i,}\bullet \widetilde{y}_{i-1}}$$

induced by α is group-theoretically verticial, hence also group-theoretically nodal.

Next, let us observe that it follows from the fact that $\alpha_{i/i-1}$ is grouptheoretically verticial [hence also group-theoretically nodal], together with our assumption concerning *PC-admissibility*, that if $^{\circ}p \notin ^{\circ}\Sigma$ and • $p \notin \Sigma$, then [cf. [CmbGC], Proposition 1.5, (ii)] $\alpha_{i/i-1}$ is graphic. On the other hand, if either ${}^{\circ}p \in {}^{\circ}\Sigma$ or ${}^{\bullet}p \in {}^{\bullet}\Sigma$, then it follows from Proposition 3.6, (iii) [applied to " (c^{\exists}) " — cf. Remark 3.13.1, (i)], together with Claim 3.16. A in the case where i = 1, that ${}^{\circ}p = {}^{\bullet}p \in {}^{\circ}\Sigma = {}^{\bullet}\Sigma$. In particular, if either ${}^{\circ}p \in {}^{\circ}\Sigma$ or ${}^{\bullet}p \in {}^{\bullet}\Sigma$, then, by allowing the open subgroup " $^{\circ}H$ " of $^{\circ}\Pi_n$ to vary and applying the group-theoretic nodality of the resulting isomorphisms " $\alpha_{i/i-1}$ ", one concludes from the "existence of irreducible components that *collapse* to arbitrary cusps" [cf. the proof of "observation (iv)" given in the proof of [SemiAn], Corollary 3.11; [SemiAn], Remark 3.11.1; [AbsTpII], Corollary 2.11; [AbsTpII], Remark 2.11.1, (i)] that $\alpha_{i/i-1}$ is group-theoretically cuspidal, hence also [cf. [CmbGC], Proposition 1.5, (ii)] graphic. Thus, by allowing \widetilde{y}_{i-1} , \widetilde{y}_{i-1} to vary, we conclude immediately from the various definitions involved that Claim 3.16.A holds. This completes the proof of Claim 3.16.A, hence also of assertion (i).

Next, we verify assertion (ii). The theory of [Brk] yields

- a functorial homotopy [indeed, a proper strong deformation retraction!] between the skeleton of a polystable fibration [cf. [Brk], Definitions 1.2, 1.3] over the ring of integers of a complete nonarchimedean field and the analytic space associated to the polystable fibration [cf. [Brk], Theorem 8.1], as well as
- a *functorial homeomorphism* between the *skeleton* of a *polystable fibration* over the ring of integers of a complete nonarchimedean field and the geometric realization of a certain *polysimplicial set* associated to the *special fiber* of the polystable fibration [cf. [Brk], Theorem 8.2].

In particular,

the theory of [Brk] gives rise to a *functorial homotopy* between the *analytic space* associated to a *polystable fibration* over the ring of integers of a complete nonarchimedean field and the geometric realization of a certain *polysimplicial set* associated to the *special fiber* of the polystable fibration.

Here, we recall further that this polysimplicial set is completely determined by the set of *strata* of the special fiber, together with the *specialization/generization* relations between these strata [cf. the discussion surrounding [Brk], Proposition 2.1, and its proof; [Brk], Lemma 3.13; [Brk], Lemma 6.7].

Next, let us observe that the various *bijections*

 $\mathrm{VCN}^{\mathrm{sch}}(^{\circ}\mathbb{H}) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}(^{\circ}\mathbb{H}) \xrightarrow{\sim} \mathrm{VCN}^{\mathrm{gp}}(^{\bullet}\mathbb{H}) \xleftarrow{\sim} \mathrm{VCN}^{\mathrm{sch}}(^{\bullet}\mathbb{H})$

[cf. Definitions 3.12, (iv); 3.13, (ii)] induced by an $\{l\}$ -G-admissible isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ induce bijections between the respective sets of strata of the special fibers of ${}^{\circ}\mathcal{Y}_n$, ${}^{\circ}\mathcal{Y}_n$ [cf. the notation of condition (e) of Remark 3.10.1, (i)], which, in light of the group-theoretic descriptions of specialization/generization relations given in [CbTpI], Proposition 2.9, (i) [cf. also [CbTpI], Proposition 5.6, (iii), (iv)], are [easily seen to be] compatible with these specialization/generization relations. In particular, since each log scheme ${}^{\Box}\mathcal{Y}_n^{\log}$ gives rise to a polystable fibration as in the above discussion of [Brk] [cf. condition (e) of Remark 3.10.1, (i)], we thus conclude, in light of the theory of [Brk], from the definition of the tempered fundamental group given in [André], §4.2 [cf. also the discussion of Definition 3.1, (ii), of the present paper], that any $\{l\}$ -G-admissible isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ determines an isomorphism

$$^{\circ}\Pi_{n}^{\mathrm{tp}} \xrightarrow{\sim} {}^{\bullet}\Pi_{n}^{\mathrm{tp}}$$

between the respective tempered fundamental groups, which gives back the original isomorphism ${}^{\circ}\Pi_n \xrightarrow{\sim} {}^{\bullet}\Pi_n$ upon passing to the respective ${}^{\circ}\Sigma = {}^{\bullet}\Sigma$ -completions [cf. Proposition 3.3, (i)]. This completes the proof of assertion (ii).

Theorem 3.17 (Metric-, inertia-admissible outomorphisms of fundamental groups). Let n be a positive integer; (g, r) a pair of nonnegative integers such that 2g - 2 + r > 0; p a prime number; Σ a nonempty set of prime numbers such that $\Sigma \neq \{p\}$, and, moreover, if $n \geq 2$, then Σ is either equal to the set of all prime numbers or of cardinality one; R a mixed characteristic complete discrete valuation ring of residue characteristic p whose residue field is separably closed; K the field of fractions of R; \overline{K} an algebraic closure of K;

a smooth log curve of type (g, r) over K. Write

 $(X_K)_n^{\log}$

for the n-th log configuration space [cf. the discussion entitled "Curves" in [CbTpII], §0] of X_K^{\log} over K; $(X_{\overline{K}})_n^{\log} \stackrel{\text{def}}{=} (X_K)_n^{\log} \times_K \overline{K}$;

$$\Pi_n \stackrel{\text{def}}{=} \pi_1 ((X_{\overline{K}})_n^{\log})^{\Sigma}$$

for the maximal pro- Σ quotient of the log fundamental group of $(X_{\overline{K}})_n^{\log}$;

$$\rho_n \colon I_K \stackrel{\text{def}}{=} \operatorname{Gal}(\overline{K}/K) \longrightarrow \operatorname{Out}(\Pi_n)$$

for the natural outer pro- Σ Galois action associated to $(X_K)_n^{\log}$; (Spec R)^{log} for the log scheme obtained by equipping Spec R with the log structure determined by the closed point of Spec R. Then the following hold:

(i) Let $l \in \Sigma$ be such that $l \neq p$. Then we have equalities and an inclusion

$$\begin{array}{rcl} \operatorname{Out}(\Pi_1)^{M} &=& \operatorname{Out}^{I}(\Pi_1) \cap \operatorname{Out}^{C}(\Pi_1) \\ &=& \operatorname{Out}^{\{l\} \cdot I}(\Pi_1) \cap \operatorname{Out}^{C}(\Pi_1) &\subseteq& \operatorname{Out}(\Pi_1)^{G} \end{array}$$

[cf. Definitions 3.7, (i), (ii); 3.8, (iii)]. If, moreover, $p \in \Sigma$, then we have equalities and inclusions

$$\operatorname{Out}(\Pi_1)^{\mathrm{M}} = \operatorname{Out}^{\mathrm{I}}(\Pi_1) = \operatorname{Out}^{\{l\}-\mathrm{I}}(\Pi_1) \subseteq \operatorname{Out}(\Pi_1)^{\mathrm{G}} \subseteq \operatorname{Out}(\Pi_1).$$

(ii) Let $l \in \Sigma$ be such that $l \neq p$. Then we have equalities and inclusions

$$\begin{array}{rcl} \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} &=& \operatorname{Out}^{\operatorname{FCI}}(\Pi_n) &=& \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{I}} \\ &=& \operatorname{Out}^{\operatorname{FC}\{l\}-\operatorname{I}}(\Pi_n) &=& \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\{l\}-\operatorname{I}} \\ &\subseteq& \operatorname{Out}^{\operatorname{G}}(\Pi_n) &\subseteq& \operatorname{Out}^{\{l\}-\operatorname{G}}(\Pi_n), \\ \\ \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} &\subseteq& \operatorname{Out}^{\operatorname{FI}}(\Pi_n) &\subseteq& \operatorname{Out}^{\operatorname{F}\{l\}-\operatorname{I}}(\Pi_n) \\ &\cap&& \cap \\ \\ \operatorname{Out}^{\operatorname{F}}(\Pi_n)^{\operatorname{M}} &\subseteq& \operatorname{Out}^{\operatorname{F}}(\Pi_n)^{\operatorname{I}} &\subseteq& \operatorname{Out}^{\operatorname{F}}(\Pi_n)^{\{l\}-\operatorname{I}} \end{array}$$

[cf. Definitions 3.7, (iii); 3.8, (iii), (iv); 3.13, (iv)]. Moreover, the following hold:

(ii-a) If $p \in \Sigma$, then we have:

$$\operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} = \operatorname{Out}^{\mathrm{FI}}(\Pi_n) = \operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_n),$$
$$\operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} = \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} = \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}-\mathrm{I}}.$$

(ii-b) If $n \neq 1$, then we have:

$$\operatorname{Out}^{\mathrm{FI}}(\Pi_n) = \operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_n),$$
$$\operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}} = \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} = \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}-\mathrm{I}}.$$

- (ii-c) If $n \neq 2$, $(r, n) \neq (0, 3)$, and either $p \in \Sigma$ or $n \neq 1$, then we have: $\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}} = \operatorname{Out}^{\mathrm{FI}}(\Pi_{n}) = \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{I}}$ $= \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{M}} = \operatorname{Out}^{\mathrm{FCI}}(\Pi_{n}) = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{I}}$ $= \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{M}} = \operatorname{Out}^{\mathrm{FCI}}(\Pi_{n}) = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{I}}$ $= \operatorname{Out}^{\mathrm{FC}}\{l\}^{-1}(\Pi_{n}) = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{I}}.$
- (iii) Suppose that $p \notin \Sigma$, and that X_K^{\log} extends to a stable log curve over $(\operatorname{Spec} R)^{\log}$. Let $l \in \Sigma$. Write $\rho_n(I_K)[l] \subseteq \rho_n(I_K)$ for the maximal pro-l subgroup of the [necessarily pro-cyclic cf. the injectivity portion of [NodNon], Theorem B; the discussion of [CbTpI], Definition 5.3] image $\rho_n(I_K)$. Then the normalizers of $\rho_n(I_K)$, $\rho_n(I_K)[l]$ in $\operatorname{Out}^{\mathrm{F}}(\Pi_n)$ satisfy the following equalities:

(iii-a) If
$$(r, n) \neq (0, 2)$$
, then
 $\operatorname{Out}^{\mathrm{FI}}(\Pi_n) = \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}} = N_{\operatorname{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)),$
 $\operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_n) = \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}-\mathrm{I}} = N_{\operatorname{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)[l]).$

(iii-b) For arbitrary $r \ge 0, n \ge 1$,

$$\operatorname{Out}^{\mathrm{FI}}(\Pi_n) = N_{\operatorname{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)),$$
$$\operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_n) = N_{\operatorname{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)[l]).$$

(iv) Let $l \in \Sigma$ be such that $l \neq p$. Then the subgroups

$$Out(\Pi_1)^M$$
, $Out^{I}(\Pi_1)$, $Out^{\{l\}-I}(\Pi_1)$, $Out(\Pi_1)^G$

of $Out(\Pi_1)$ are closed in $Out(\Pi_1)$. Moreover, the subgroups

of $Out(\Pi_n)$ are closed in $Out(\Pi_n)$. In particular, these subgroups are compact.

(v) Let $l \in \Sigma$ be such that $l \neq p$. Suppose, moreover, that the smooth log curve X_K^{\log} arises, via base-change, from a smooth log curve over a complete discrete valuation field whose residue field is finitely generated over a finite field. Then the closed subgroups $\operatorname{Out}^{\mathrm{G}}(\Pi_n)$, $\operatorname{Out}^{\{l\}-\mathrm{G}}(\Pi_n) \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_n)$ [cf. (iv); Remark 3.13.1, (ii)] are commensurably terminal in $\operatorname{Out}^{\mathrm{F}}(\Pi_n)$. Moreover, we have an inclusion

$$C_{\operatorname{Out}^{\mathrm{F}}(\Pi_n)}(\operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}) \subseteq \operatorname{Out}^{\mathrm{G}}(\Pi_n).$$

(vi) The natural homomorphism

 $\operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1})^{\operatorname{M}} \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}}$

(respectively, $\operatorname{Out}^{\mathrm{F}}(\Pi_{n+1})^{\mathrm{M}} \longrightarrow \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}}$)

induced by the projection $(X_K)_{n+1}^{\log} \to (X_K)_n^{\log}$ obtained by forgetting any one of the n+1 factors is **injective** (respectively, **injective** if $(r, n) \neq (0, 1)$). If, moreover, either

or

$$n > 3$$
 and $r \neq 0$,

n > 4

then this homomorphism is **bijective** (respectively, **bijective**).

Proof. Assertion (i) follows immediately from Theorem 3.9. Next, we verify assertion (ii). First, we claim that the following assertion holds:

Claim 3.17.A: We have equalities

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\mathrm{I}} = \operatorname{Out}^{\operatorname{FCI}}(\Pi_n); \quad \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\{l\}-\mathrm{I}} = \operatorname{Out}^{\operatorname{FC}\{l\}-\mathrm{I}}(\Pi_n).$$

Indeed, this follows immediately — in light of the definition of *I*admissibility, $\{l\}$ -*I*-admissibility [cf. Definition 3.8] — from Proposition 2.3, (ii), and Corollary 2.10 [when $\Sigma = \mathfrak{Primes}$]; the injectivity portion of [NodNon], Theorem B [when $\Sigma = \{l\}$]. This completes the proof of Claim 3.17.A.

Next, we claim that the following assertion holds:

Claim 3.17.B: We have equalities

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} = \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{I}} = \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\{l\}-\operatorname{I}}.$$

Indeed, this follows immediately from assertion (i), together with the various definitions involved. This completes the proof of Claim 3.17.B. Next, we claim that the following assertion holds:

Claim 3.17.C: We have equalities and an inclusion

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} = \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{I}} = \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\{l\}-\operatorname{I}}$$

=
$$\operatorname{Out}^{\operatorname{FCI}}(\Pi_n) = \operatorname{Out}^{\operatorname{FC}\{l\}-\operatorname{I}}(\Pi_n) \subseteq \operatorname{Out}^{\operatorname{G}}(\Pi_n).$$

Indeed, the first four equalities follow from Claims 3.17.A, 3.17.B. On the other hand, the final inclusion follows immediately from Proposition 3.16, (i) [cf. also the final portion of Definition 3.13, (i)]. This completes the proof of Claim 3.17.C.

Next, we claim that the following assertion holds:

Claim 3.17.D: We have inclusions

$$\operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{M}} \subseteq \operatorname{Out}^{\mathrm{FI}}(\Pi_{n}) \subseteq \operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_{n})$$

$$\cap \qquad \cap$$

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}} \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{I}} \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\{l\}-\mathrm{I}}.$$

Indeed, let us observe that the left-hand upper inclusion follows immediately from Claim 3.17.C. Next, let us observe that the left-hand lower inclusion follows immediately from assertion (i). On the other hand, the remaining inclusions follow immediately from the various definitions involved. This completes the proof of Claim 3.17.D. The various equalities and inclusions of assertion (ii) that precede assertion (ii-a) all follow from Claims 3.17.C, 3.17.D.

Next, we consider assertion (ii-a). It follows immediately from Proposition 3.16, (i), that the inclusion $\operatorname{Out}^{\operatorname{F}\{l\}-\operatorname{I}}(\Pi_n) \subseteq \operatorname{Out}^{\{l\}-\operatorname{G}}(\Pi_n)$ holds. In particular, it follows from Remark 3.13.1, (ii), that the inclusion $\operatorname{Out}^{\operatorname{F}\{l\}-\operatorname{I}}(\Pi_n) \subseteq \operatorname{Out}^{\operatorname{FC}}(\Pi_n)$, hence also the equality $\operatorname{Out}^{\operatorname{F}\{l\}-\operatorname{I}}(\Pi_n) = \operatorname{Out}^{\operatorname{FC}\{l\}-\operatorname{I}}(\Pi_n)$, holds. Thus, the first two equalities of assertion (ii-a) follow immediately from Claims 3.17.C, 3.17.D. On the other hand, the final two equalities of assertion (ii-a) follow immediately from the final portion of assertion (i). This completes the proof of assertion (ii-a).

Next, we consider assertion (ii-b). If $p \in \Sigma$, then assertion (ii-b) follows from assertion (ii-a). Thus, we may assume without loss of generality that $p \notin \Sigma$. Then since [by assumption!] $\Sigma = \{l\}$, the first equality of assertion (ii-b) follows immediately from the various definitions involved. On the other hand, the final two equalities follow immediately from assertion (i), together with [CbTpI], Theorem A, (ii). This completes the proof of assertion (ii-b).

Next, we consider assertion (ii-c). If $n \ge 3$ and $(r, n) \ne (0, 3)$, then assertion (ii-c) follows immediately from [CbTpII], Theorem A, (ii), together with Claim 3.17.C. On the other hand, if $p \in \Sigma$ and n = 1, then assertion (ii-c) follows immediately from the final portion of assertion (i), together with Claim 3.17.C [cf. also Remark 3.13.1, (ii)]. This completes the proof of assertion (ii-c), hence also of assertion (ii).

Next, we verify assertion (iii). First, we claim that the following assertion holds:

Claim 3.17.E: We have an equality

$$\operatorname{Out}^{\{l\}-1}(\Pi_1) = N_{\operatorname{Out}(\Pi_1)}(\rho_1(I_K)[l]).$$

Indeed, let us first observe that since $p \notin \Sigma$, we have a natural outer isomorphism $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}$ [cf. Remark 3.5.1]. Next, let us observe that, in light of our assumption that X_K^{\log} extends to a *stable log curve* over $(\operatorname{Spec} R)^{\log}$, it follows from [CbTpI], Definition 5.3, (i), that the image of ρ_1 is *contained* in

$$\operatorname{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma]) \subseteq \operatorname{Out}(\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}) \stackrel{\sim}{\leftarrow} \operatorname{Out}(\Pi_1)$$

[cf. [CbTpI], Definition 4.4] and, moreover, is *pro-cyclic*. Next, let us observe that it follows immediately from the definition of $\{l\}$ -*Iadmissibility* that $N_{\text{Out}(\Pi_1)}(\rho_1(I_K)[l]) \subseteq \text{Out}^{\{l\}-I}(\Pi_1)$. Thus, to complete the verification of Claim 3.17.E, it suffices to verify that we have an inclusion $\text{Out}^{\{l\}-I}(\Pi_1) \subseteq N_{\text{Out}(\Pi_1)}(\rho_1(I_K)[l])$. Let $\alpha \in \operatorname{Out}^{\{l\}-\mathrm{I}}(\Pi_1)$ and $H \subseteq \Pi_1$ a characteristic open subgroup of Π_1 . Write $\Pi_1 \twoheadrightarrow \Pi_1^*$ for the maximal almost pro-l quotient of Π_1 with respect to H [cf. Definition 1.1]; $\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]} \twoheadrightarrow \Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}^*$ for the [necessarily maximal almost pro-l] quotient of $\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}$ corresponding to $\Pi_1 \twoheadrightarrow \Pi_1^*$ [relative to the above natural outer isomorphism $\Pi_1 \xrightarrow{\sim} \Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}$]. Then it follows immediately [in light of the well-known simple structure of *pro-cyclic* profinite groups] from the definition of $\{l\}$ -*I-admissibility* that there exists an open subgroup $J \subseteq I_K$ such that the image of $\rho_1(J)$ in $\operatorname{Out}(\Pi_1^*)$ is *normalized* by the outomorphism $\alpha^* \in \operatorname{Out}(\Pi_1^*)$ determined by $\alpha \in \operatorname{Out}(\Pi_1)$. On the other hand, it follows immediately from the above discussion that the outer representation

$$\rho_1(J) \longrightarrow \operatorname{Out}(\Pi_1^*) \xrightarrow{\sim} \operatorname{Out}(\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}^*)$$

is of *PIPSC-type* [cf. Definition 1.6, (iv)]. Thus, it follows from Theorem 1.11, (ii), that $\alpha^* \in \text{Out}(\Pi_1^*)$ is group-theoretically verticial [cf. Definition 1.6, (ii)]. In particular, by allowing *H* to vary, we conclude that $\alpha \in \text{Out}(\Pi_1)$ is group-theoretically verticial. Thus, it follows immediately from the definition of a profinite Dehn multi-twist that $\alpha \in \text{Out}(\Pi_1)$ normalizes $\text{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma]) \subseteq \text{Out}(\Pi_{\mathcal{G}_{\Pi_1}[\Sigma]}) \stackrel{\sim}{\leftarrow} \text{Out}(\Pi_1)$, hence also [cf. [CbTpI], Theorem 4.8, (iv)] the maximal pro-*l* subgroup $\text{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma])[l]$ of $\text{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma])$. On the other hand, one verifies immediately again from [CbTpI], Theorem 4.8, (iv), that $\text{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma])[l]$ is a free \mathbb{Z}_l -module of finite rank, and that the composite

$$\operatorname{Dehn}(\mathcal{G}_{\Pi_1}[\Sigma])[l] \hookrightarrow \operatorname{Out}(\Pi_1) \to \operatorname{Out}(\Pi_1^*)$$

is *injective*. Thus, since *some* open subgroup of the maximal pro-l subgroup of the image of I_K in $Out(\Pi_1^*)$ is *normalized* by $\alpha^* \in Out(\Pi_1^*)$ [cf. the above discussion concerning "J"!], one verifies immediately [from well-known elementary properties of *free* \mathbb{Z}_l -modules of finite rank] that $\alpha \in N_{Out(\Pi_1)}(\rho_1(I_K)[l])$. This completes the proof of Claim 3.17.E.

Now let us observe that one verifies easily [cf. also the discussion of the inclusion " $N_{\text{Out}(\Pi_1)}(\rho_1(I_K)[l]) \subseteq \text{Out}^{\{l\}-1}(\Pi_1)$ " in the proof of Claim 3.17.E] that the inclusions

$$N_{\operatorname{Out}^{\mathrm{F}}(\Pi_{n})}(\rho_{n}(I_{K})) \subseteq \operatorname{Out}^{\mathrm{FI}}(\Pi_{n}) \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{I}},$$
$$N_{\operatorname{Out}^{\mathrm{F}}(\Pi_{n})}(\rho_{n}(I_{K})[l]) \subseteq \operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_{n}) \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\{l\}-\mathrm{I}}$$

hold. In particular, assertion (iii-a) follows immediately from Claim 3.17.E, together with the *injectivity portion* of [CbTpII], Theorem A, (i) [cf. also [CbTpI], Theorem A, (ii); [NodNon], Theorem B, in the case where r = 0]. Thus, to complete the proof of assertion (iii), it suffices to verify the two equalities of assertion (iii-b) in the case where (r, n) = (0, 2). Suppose that (r, n) = (0, 2), hence that $\Sigma = \{l\}$. Then one verifies easily that, to complete the proof of assertion (iii), it suffices to verify that $\operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_n) \subseteq N_{\operatorname{Out}^{\mathrm{F}}(\Pi_n)}(\rho_n(I_K)[l]).$

Thus, let $\widetilde{\alpha} \in \operatorname{Aut}(\Pi_n)$ be a lifting of an element $\alpha \in \operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_n)$. Then let us observe that it follows immediately from Claim 3.17.E that $\widetilde{\alpha}$ induces an automorphism $\widetilde{\beta}$ of the extension group $\Pi_1 \stackrel{\text{out}}{\rtimes} \rho_1(I_K)[l]$ i.e., arising from the outer representation of IPSC-type $\rho_1(I_K) \rightarrow$ $Out(\Pi_1)$ implicit in the discussion surrounding Claim 3.17.E above], whose restriction to Π_1 is *G*-admissible [cf. assertion (i)]. In particular, it follows that $\widetilde{\beta}$ maps verticial inertia groups of $\Pi_1 \stackrel{\text{out}}{\rtimes} \rho_1(I_K)[l]$ [each of which surjects onto $\rho_1(I_K)[l]$ — cf. [NodNon], Definition 2.2, (i); [NodNon], Definition 2.4, (ii); [NodNon], Remark 2.4.2] to verticial inertia groups of $\Pi_1 \stackrel{\text{out}}{\rtimes} \rho_1(I_K)[l]$. Moreover, let us observe that it follows immediately from the fact that $\alpha \in \text{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_n)$ that $\widetilde{\beta}$ is compatible with the natural outer representations of suitable open subgroups of such verticial inertia groups of $\Pi_1 \stackrel{\text{out}}{\rtimes} \rho_1(I_K)[l]$ on $\Pi_{2/1}$ [relative to the action of $\tilde{\alpha}$ on $\Pi_{2/1}$]. Thus, since the natural outer representation of such a verticial inertia group of $\Pi_1 \stackrel{\text{out}}{\rtimes} \rho_1(I_K)[l]$ on $\Pi_{2/1}$ is [easily verified to be] an outer representation of IPSC-type, one concludes from a similar argument to the [final portion of the] argument applied above to verify Claim 3.17.E that β is *compatible* with these natural outer representations of verticial inertia groups of $\Pi_1 \stackrel{\text{out}}{\rtimes} \rho_1(I_K)[l]$ on $\Pi_{2/1}$ [relative to the action of $\tilde{\alpha}$ on $\Pi_{2/1}$]. Now it follows formally that $\alpha \in N_{\text{Out}^{F}(\Pi_{n})}(\rho_{n}(I_{K})[l])$, as desired. This completes the proof of assertion of (iii).

Next, we verify assertion (iv). The closedness of $\operatorname{Out}(\Pi_1)^{\mathrm{G}}$ in $\operatorname{Out}(\Pi_1)$ follows immediately from condition (c^{\forall}) of Proposition 3.6 [cf. Proposition 3.6, (ii)]. Thus, the closedness of $\operatorname{Out}(\Pi_1)^{\mathrm{M}}$ in $\operatorname{Out}(\Pi_1)$ follows from the easily verified fact that $\operatorname{Out}^{\mathrm{M}}(\Pi_1)$ is closed in $\operatorname{Out}(\Pi_1)^{\mathrm{G}}$. The fact that the subgroup $\operatorname{Out}^{\{l\}-\mathrm{I}}(\Pi_1)$, hence also $\operatorname{Out}^{\mathrm{I}}(\Pi_1)$, is closed in $\operatorname{Out}(\Pi_1)$ may be verified as follows: If $p \in \Sigma$, then the closedness in question follows from the closedness of $\operatorname{Out}(\Pi_1)^{\mathrm{M}}$ [verified above], together with the final portion of assertion (i). On the other hand, if $p \notin \Sigma$, then the closedness in question follows immediately from assertion (iii). This completes the proof of the closedness of $\operatorname{Out}^{\{l\}-\mathrm{I}}(\Pi_1)$ and $\operatorname{Out}^{\mathrm{I}}(\Pi_1)$ in $\operatorname{Out}(\Pi_1)$.

The *closedness* of

$$\operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}, \quad \operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}}, \quad \operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\{l\}-\mathrm{I}},$$
$$\operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}}, \quad \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}}, \quad \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}-\mathrm{I}}$$

in $\operatorname{Out}(\Pi_n)$ follows immediately from the various definitions involved, together with the *closedness* of $\operatorname{Out}(\Pi_1)^M$, $\operatorname{Out}^{\mathrm{I}}(\Pi_1)$, and $\operatorname{Out}^{\{l\}-\mathrm{I}}(\Pi_1)$ in $\operatorname{Out}(\Pi_1)$ [verified above]. The *closedness* of

$$\operatorname{Out}^{\operatorname{FCI}}(\Pi_n), \operatorname{Out}^{\operatorname{FC}\{l\}-\operatorname{I}}(\Pi_n)$$

in $\operatorname{Out}(\Pi_n)$ follows from the *closedness* of $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}}$ in $\operatorname{Out}(\Pi_n)$ [verified above], together with the equalities at the beginning of assertion (ii).

Next, we verify the *closedness* of

$$\operatorname{Out}^{\mathrm{G}}(\Pi_n), \operatorname{Out}^{\{l\}-\mathrm{G}}(\Pi_n)$$

in $\operatorname{Out}(\Pi_n)$. Let us first observe that it is immediate that, to verify the desired *closedness*, it suffices to verify the *closedness* of $\operatorname{Out}^{\{l\}-G}(\Pi_n)$ in $\operatorname{Out}(\Pi_n)$. Next, to verify the *closedness* of $\operatorname{Out}^{\{l\}-G}(\Pi_n)$ in $\operatorname{Out}(\Pi_n)$, let $(\alpha_{\xi})_{\xi\geq 1}$ be a sequence [indexed by the positive integers] of elements $\in \operatorname{Out}^{\{l\}-G}(\Pi_n)$ that *converges* to an element $\alpha_{\infty} \in \operatorname{Out}(\Pi_n)$. Then since [one verifies easily that] the subgroup of $\operatorname{Out}(\Pi_n)$ consisting of *SAF-admissible* outomorphisms is *closed*, α_{∞} is *SAF-admissible*. Next, to verify that α_{∞} satisfies the condition of Definition 3.13, (ii), let us fix an open subgroup $J \subseteq \Pi_n$ of Π_n . Now we define open subgroups $H_{\xi} \subseteq J_{\xi} \subseteq \Pi_n$ of Π_n inductively on ξ as follows:

- Set $J_1 \stackrel{\text{def}}{=} J$.
- Suppose that J_ξ ⊆ Π_n has already been defined. Then since α_ξ is {l}-G-admissible, there exists an open subgroup H ⊆ °Π_n of °Π_n of *l*-polystable type such that H ⊆ J_ξ, and, moreover, α_ξ satisfies the condition of Definition 3.13, (ii), in the case where we take the "(°J, °H)" of Definition 3.13, (ii), to be (J_ξ, H). Then define H_ξ ^{def} H.
 Suppose that ξ ≥ 2, and that H_{ξ-1} ⊆ Π_n has already been
- Suppose that $\xi \geq 2$, and that $H_{\xi-1} \subseteq \Pi_n$ has already been defined. Then set $J_{\xi} \stackrel{\text{def}}{=} H_{\xi-1}$.

Then it follows immediately from Lemma 3.14, (iii), (v), that α_{∞} satisfies the condition of Definition 3.13, (ii), in the case where we take the "(°J, °H)" of Definition 3.13, (ii), to be $(J = J_1, H = H_1)$. In particular, the *SAF-admissible* outomorphism α_{∞} is $\{l\}$ -*G-admissible*, as desired. This completes the proof of the *closedness* of Out^{{l}-G}(Π_n) in Out(Π_n).

The fact that the subgroup $\operatorname{Out}^{\operatorname{F} \{l\}-\operatorname{I}}(\Pi_n)$, hence also $\operatorname{Out}^{\operatorname{FI}}(\Pi_n)$, is closed in $\operatorname{Out}(\Pi_n)$ may be verified as follows: If n = 1, then the closedness in question has already been verified. If $p \in \Sigma$, then the closedness in question follows from the closedness of $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}}$ [verified above], together with assertion (ii-a). On the other hand, if $p \notin \Sigma$, then the closedness in question follows from assertion (iii-b). This completes the proof of the closedness of $\operatorname{Out}^{\operatorname{F} \{l\}-\operatorname{I}}(\Pi_n)$, $\operatorname{Out}^{\operatorname{FI}}(\Pi_n)$ in $\operatorname{Out}(\Pi_n)$, hence also of assertion (iv).

Next, we verify assertion (v). Let $\alpha \in C_{\text{Out}^{F}(\Pi_{n})}(\text{Out}^{G}(\Pi_{n}))$ (respectively, $C_{\text{Out}^{F}(\Pi_{n})}(\text{Out}^{\{l\}-G}(\Pi_{n}))$; $C_{\text{Out}^{F}(\Pi_{n})}(\text{Out}^{\text{FC}}(\Pi_{n})^{M})$) and $\widetilde{\alpha} \in \text{Aut}^{F}(\Pi_{n})$ a lifting of α . Now observe that to complete the verification of assertion (v), it suffices to verify that $\alpha \in \text{Out}^{\{l\}-G}(\Pi_{n})$.

To this end, let $J \subseteq \Pi_n$ be an open subgroup of Π_n . Then it follows from Lemma 3.14, (i), (iii), that there exist an open subgroup $H \subseteq J \subseteq \Pi_n$ of Π_n of *l*-polystable type [cf. Definition 3.10] and an H-*l*-system $\mathbb{H} = \{H_\lambda\}_{\lambda \in \Lambda}$ [cf. Definition 3.11, (ii)]. Note that it follows from condition (a) of Definition 3.10 that the subgroups H, H_λ of Π_n are stabilized by $\tilde{\alpha}$. Then it follows immediately from the various definitions involved that, to complete the verification of the fact that $\alpha \in \text{Out}^{\{l\}-G}(\Pi_n)$, it suffices to verify the following assertion:

Claim 3.17.F: For each $i \in \{0, \dots, n\}$, the outomorphism of the image H_i of H in Π_i determined by α induces a bijection

$$\operatorname{VCN^{gp}}(\mathbb{H}_i) \xrightarrow{\sim} \operatorname{VCN^{gp}}(\mathbb{H}_i)$$

[cf. Definition 3.11, (iii); Definition 3.12, (iv)] — where, for convenience, we set $\Pi_0 \stackrel{\text{def}}{=} \{1\}$, VCN^{gp}(\mathbb{H}_0) $\stackrel{\text{def}}{=} \{\Pi_0\}$, and we write $(H_{\lambda})_i$ for the image of H_{λ} in Π_i and $\mathbb{H}_i \stackrel{\text{def}}{=} \{(H_{\lambda})_i\}_{\lambda \in \Lambda}$.

We verify Claim 3.17.F by *induction* on *i*. If i = 0, then Claim 3.17.F is immediate. Now suppose that $i \ge 1$, and that the *induction hypothesis* is in force. Then it follows immediately from the *induction hypothesis* that, for each $j \in \{0, \dots, i-1\}$, the outomorphism of H_j determined by α induces a bijection

$$\operatorname{VCN}^{\operatorname{gp}}(\mathbb{H}_{i}) \xrightarrow{\sim} \operatorname{VCN}^{\operatorname{gp}}(\mathbb{H}_{i}).$$

Let $\tilde{y}, \tilde{y}' \in \text{VCN}^{\text{sch}}(\mathbb{H}_{i-1})$ [cf. Definition 3.11, (iii)] be elements that correspond via the bijection obtained by conjugating the above bijection by the displayed bijection of Definition 3.12, (iv).

Next, let us observe that since $\alpha \in C_{\text{Out}^{\text{F}}(\Pi_n)}(\text{Out}^{\text{G}}(\Pi_n))$ (respectively, $C_{\text{Out}^{\text{F}}(\Pi_n)}(\text{Out}^{\{l\}-\text{G}}(\Pi_n))$; $C_{\text{Out}^{\text{F}}(\Pi_n)}(\text{Out}^{\text{FC}}(\Pi_n)^{\text{M}})$), there exist open subgroups N_1 and N_2 of $\text{Out}^{\text{G}}(\Pi_n)$ (respectively, $\text{Out}^{\{l\}-\text{G}}(\Pi_n)$; $\text{Out}^{\text{FC}}(\Pi_n)^{\text{M}}$) such that the automorphism of H_i induced by $\tilde{\alpha}$ extends to an *isomorphism of profinite groups* [cf. assertion (iv)]

$$H_i \stackrel{\text{out}}{\rtimes} N_1 \stackrel{\sim}{\longrightarrow} H_i \stackrel{\text{out}}{\rtimes} N_2$$

[cf. the discussion entitled "Topological groups" in [CbTpI], §0] that lies over an isomorphism of profinite groups $N_1 \xrightarrow{\sim} N_2$. In particular, by considering the respective outer actions [by conjugation] of $H_{i-1} \xrightarrow{\text{out}} N_1$, $H_{i-1} \xrightarrow{\text{out}} N_2$ on the maximal pro-*l* quotient $(H_{i/i-1})^{\{l\}}$ of the kernel $H_{i/i-1} \stackrel{\text{def}}{=} \text{Ker}(H_i \twoheadrightarrow H_{i-1})$ [cf. the notation of Remark 3.10.1, (i)], we obtain a commutative diagram of profinite groups

— where the left-hand vertical arrow is the isomorphism induced by the *isomorphism of profinite groups* discussed above; the central vertical arrow is the automorphism induced by $\tilde{\alpha}$; the right-hand horizontal arrows are the isomorphisms induced by the \tilde{y} -, \tilde{y}' -versions of the isomorphism of Definition 3.12, (iii); the right-hand vertical arrow is the isomorphism induced by the composite

$$\widetilde{\alpha}_{\widetilde{y},\widetilde{y}'} \colon \Pi_{\mathcal{G}_{i,\widetilde{y}}} \xrightarrow{\sim} (H_{i/i-1})^{\{l\}} \xrightarrow{\sim} (H_{i/i-1})^{\{l\}} \xleftarrow{\sim} \Pi_{\mathcal{G}_{i,\widetilde{y}'}}$$

of the isomorphism $\Pi_{\mathcal{G}_{i,\tilde{y}}} \xrightarrow{\sim} (H_{i/i-1})^{\{l\}}$ of Definition 3.12, (iii), the automorphism of $(H_{i/i-1})^{\{l\}}$ determined by $\tilde{\alpha}$, and the isomorphism $(H_{i/i-1})^{\{l\}} \xleftarrow{\sim} \Pi_{\mathcal{G}_{i,\tilde{y}'}}$ of Definition 3.12, (iii). Now let us recall that we have assumed that the smooth log curve

 X_K^{\log} arises, via base-change, from a smooth log curve over a complete discrete valuation field whose residue field is *finitely generated over a* finite field. In particular, one verifies immediately from the openness of N_1 , N_2 in $\operatorname{Out}^{\mathrm{G}}(\Pi_n)$ (respectively, $\operatorname{Out}^{\{l\}-\mathrm{G}}(\Pi_n)$; $\operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}} = \operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{I}} \subseteq \operatorname{Out}^{\{l\}-\mathrm{G}}(\Pi_n)$ [cf. assertion (ii)]) that the composite horizontal arrows of the above commutative diagram factor through $\operatorname{Aut}(\mathcal{G}_{i,\widetilde{v}}), \operatorname{Aut}(\mathcal{G}_{i,\widetilde{v}'}),$ respectively, and, moreover, are *l*-graphically full [i.e., in the sense of [CmbGC], Definition 2.3, (iii)] — cf. the argument applied in the proof of [CmbGC], Proposition 2.4, (v). Thus, it follows from [CmbGC], Corollary 2.7, (ii), that the isomorphism $\widetilde{\alpha}_{\widetilde{y},\widetilde{y}'} \colon \Pi_{\mathcal{G}_{i,\widetilde{y}'}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{i,\widetilde{y}'}}$ is graphic. In particular, by allowing $\widetilde{y}, \ \widetilde{y}'$ to vary, it follows immediately from the various definitions involved that Claim 3.17.F holds. This completes the proof of Claim 3.17.F, hence also of assertion (v). Assertion (vi) follows from [NodNon], Theorem B; [CbTpII], Theorem A, (i). This completes the proof of Theorem 3.17.

Remark 3.17.1. In the notation of Theorem 3.17, suppose that we are in the situation of Theorem 3.17, (v). Then it follows from Theorem 3.17, (v), that

$$C_{\operatorname{Out}^{\mathrm{FC}}(\Pi_n)}(\operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}) \subseteq \operatorname{Out}^{\mathrm{G}}(\Pi_n).$$

On the other hand, $\operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}$ is *not*, in general, *commensurably* terminal in $\operatorname{Out}^{\mathrm{G}}(\Pi_n)$ [or indeed in $\operatorname{Out}^{\mathrm{FC}}(\Pi_n)$ or $\operatorname{Out}^{\mathrm{FC}}(\Pi_n)!$]. Indeed, suppose, moreover, that we are in the situation of Theorem 3.17, (iii) [so $p \notin \Sigma$], and that the semi-graph of anabelioids \mathcal{G} of pro- Σ PSC type determined by the geometric special fiber of the stable model of X_K^{\log} satisfies the following conditions:

- Vert $(\mathcal{G})^{\sharp}$ = Node $(\mathcal{G})^{\sharp}$ = 2. Write Vert (\mathcal{G}) = { v_1, v_2 }, Node (\mathcal{G}) = { e_1, e_2 }.
- For each $i \in \{1, 2\}, \mathcal{V}(e_i) = \operatorname{Vert}(\mathcal{G}) = \{v_1, v_2\}.$
- There exists an automorphism of \mathcal{G} that induces a *nontrivial* automorphism of Node(\mathcal{G}).

Finally, suppose that if we write $\mu_{X_{K}^{\log}}$ for the metric structure on the underlying semi-graph of \mathcal{G} associated to the stable model of X_{K}^{\log} [cf. Definition 3.5, (iii)], then $\mu_{X_{K}^{\log}}(e_{1}) \neq \mu_{X_{K}^{\log}}(e_{2})$. [Here, we note that one verifies easily that such a smooth log curve X_{K}^{\log} exists.] Then it follows immediately from the various assumptions imposed on the objects under consideration that $\operatorname{Out}^{\mathrm{FC}}(\Pi_{1})^{\mathrm{M}}$ is of *index* 2, hence also *normal*, in $\operatorname{Out}^{\mathrm{G}}(\Pi_{1})$. In particular, $\operatorname{Out}^{\mathrm{FC}}(\Pi_{1})^{\mathrm{M}}$ is *not normally terminal*, hence, *a fortiori, not commensurably terminal*, in $\operatorname{Out}^{\mathrm{G}}(\Pi_{1})$.

Remark 3.17.2. In the notation of Theorem 3.17, suppose that $p \in \Sigma$.

(i) It follows from Theorem 3.17, (ii-c), that if either

 $(\dagger_1): n \ge 4$ or $n \ge 3$ and $r \ne 0$,

then we have equalities

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}} = \operatorname{Out}^{\mathrm{FI}}(\Pi_{n}) = \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{I}}$$
$$= \operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_{n}) = \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\{l\}-\mathrm{I}}$$
$$= \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{M}} = \operatorname{Out}^{\mathrm{FCI}}(\Pi_{n}) = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{I}}$$
$$= \operatorname{Out}^{\mathrm{FC}\{l\}-\mathrm{I}}(\Pi_{n}) = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\{l\}-\mathrm{I}}$$

(ii) In Corollary 2.10, the authors gave what may be regarded as an almost pro-l version of the injectivity portion of [NodNon], Theorem B [i.e., the injectivity of the natural homomorphism $\operatorname{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \operatorname{Out}^{\mathrm{FC}}(\Pi_n)$]. In fact, however, although a detailed exposition lies beyond the scope of the present paper [cf. the discussion of (iii) below], it seems quite likely that it should be possible to verify an almost pro-l version of the injectivity portion of [CbTpII], Theorem A, (i) [i.e., the injectivity of the natural homomorphism $\operatorname{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \operatorname{Out}^{\mathrm{F}}(\Pi_n)$ for $(r, n) \neq (0, 1)$]. Such an almost pro-l version would then imply, via a similar argument to the argument applied in the proof of the equalities

 $\operatorname{Out}^{\operatorname{FCI}}(\Pi_n) = \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{I}}, \ \operatorname{Out}^{\operatorname{FC}\{l\}-\operatorname{I}}(\Pi_n) = \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\{l\}-\operatorname{I}}$

[cf. Claim 3.17.A in the proof of Theorem 3.17, (ii)], that if either

 (\dagger_2) : $n \ge 3$ or $n \ge 2$ and $r \ne 0$,

then the equalities

$$\operatorname{Out}^{\mathrm{FI}}(\Pi_n) = \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{I}}, \ \operatorname{Out}^{\mathrm{F}\{l\}-\mathrm{I}}(\Pi_n) = \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\{l\}-\mathrm{I}},$$

hence also [cf. Theorem 3.17, (ii); Theorem 3.17, (ii-a)] the *nine equalities* of the display of (i), hold.

- (iii) The main reason that the authors did not go to the trouble to verify the nine equalities of the display of (i) under the more general hypotheses [i.e., (†₂)] discussed in (ii) is the following. The main applications of the theory developed in the present paper are the following:
 - (1) the generalization, given in Corollary 3.20 below [cf. also Remark 3.20.1 below], of a result due to Andre [cf. [André], Theorems 7.2.1, 7.2.3] concerning the characterization of local Galois groups in the global Galois image associated to a hyperbolic curve over a number field and
 - (2) the establishment of an appropriate *local analogue*, satisfying various expected properties, of the *Grothendieck*-*Teichmüller group* [cf. Remark 3.19.2 below].

The theory surrounding these applications [cf. Theorem 3.18 below] revolves around the theory of the *tripod homomorphism* developed in [CbTpII], §3. On the other hand, this theory of the tripod homomorphism is only *well-behaved* [cf. [CbTpII], Definition 3.19] under the *more restrictive hypotheses* [i.e., (\dagger_1)] discussed in (i).

Theorem 3.18 (Metric-admissible outomorphisms and tripods). In the notation of Theorem 3.17, the following hold:

(i) Suppose that $n \ge 3$. Let Π^{tpd} be a 1-central $\{1, 2, 3\}$ -tripod of Π_n [cf. [CbTpII], Definitions 3.3, (i); 3.7, (ii)]. Then the restriction of the tripod homomorphism associated to Π_n

$$\mathfrak{T}_{\Pi^{\mathrm{tpd}}} \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \longrightarrow \mathrm{Out}^{\mathrm{C}}(\Pi^{\mathrm{tpd}})$$

[cf. [CbTpII], Definition 3.19] to the subgroup $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\mathrm{M}} \subseteq \operatorname{Out}^{\operatorname{FC}}(\Pi_n)$ [cf. Definition 3.7, (iii)] factors through the subgroup $\operatorname{Out}(\Pi^{\operatorname{tpd}})^{\mathrm{M}} \subseteq \operatorname{Out}^{\mathrm{C}}(\Pi^{\operatorname{tpd}})$ [cf. Definition 3.7, (i), (ii); Remark 3.13.1, (i), (ii)], i.e., we have a natural commutative

diagram

(ii) Suppose that $n \ge 1$, and that (g, r) = (0, 3). Write $\operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\Delta +} \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_n)$

for the inverse image via the natural homomorphism $\operatorname{Out}^{\mathrm{F}}(\Pi_n) \rightarrow \operatorname{Out}(\Pi_1)$ [cf. [CbTpI], Theorem A, (i)] of $\operatorname{Out}^{\mathrm{C}}(\Pi_1)^{\Delta_+} \subseteq \operatorname{Out}(\Pi_1)$ [cf. [CbTpII], Definition 3.4, (i)];

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\Delta +} \stackrel{\operatorname{def}}{=} \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\Delta +} \cap \operatorname{Out}^{\operatorname{FC}}(\Pi_n)$$

[cf. Remark 3.18.1 below];

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}\Delta +} \stackrel{\mathrm{def}}{=} \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\Delta +} \cap \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}};$$

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}\Delta +} \stackrel{\operatorname{def}}{=} \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\Delta +} \cap \operatorname{Out}^{\operatorname{F}}(\Pi_n)^{\operatorname{M}}.$$

Then we have equalities

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\Delta +} = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\Delta +},$$
$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}\Delta +} = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{M}\Delta +}.$$

Moreover, the natural homomorphisms

$$\begin{array}{cccc} \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1})^{\Delta +} & \longrightarrow & \operatorname{Out}^{\operatorname{FC}}(\Pi_{n})^{\Delta +} \\ & & & & & \\ & & & & \\ & & & \\ \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1})^{\Delta +} & \longrightarrow & \operatorname{Out}^{\operatorname{FC}}(\Pi_{n})^{\Delta +} \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ \operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1})^{\operatorname{M\Delta} +} & \longrightarrow & \operatorname{Out}^{\operatorname{FC}}(\Pi_{n})^{\operatorname{M\Delta} +} \end{array}$$

are bijective.

Proof. Assertion (i) follows immediately — in light of the equalities $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\mathrm{M}} = \operatorname{Out}^{\operatorname{FCI}}(\Pi_n)$, $\operatorname{Out}(\Pi^{\operatorname{tpd}})^{\mathrm{M}} = \operatorname{Out}^{\mathrm{I}}(\Pi^{\operatorname{tpd}}) \cap \operatorname{Out}^{\mathrm{C}}(\Pi^{\operatorname{tpd}})$ [cf. Theorem 3.17, (i), (ii)] — from the definition of *I*-admissibility, together with [in the case where $\Sigma = \operatorname{\mathfrak{Primcs}}$] Corollary 2.13, (iii). Next, we verify assertion (ii). The equalities

$$\operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\Delta +} = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\Delta +}, \quad \operatorname{Out}^{\mathrm{F}}(\Pi_{n})^{\mathrm{M}\Delta +} = \operatorname{Out}^{\mathrm{FC}}(\Pi_{n})^{\mathrm{M}\Delta +}$$

follow immediately from [CbTpII], Theorem A, (ii), together with the various definitions involved. Next, let us observe that, to verify the

bijectivity of the various homomorphisms in question, it suffices to verify the *bijectivity* of the natural homomorphism

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_{n+1})^{\Delta_+} \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\Delta_+}.$$

On the other hand, this *bijectivity* follows immediately, in light of the various definitions involved, from [CmbCsp], Corollary 4.2, (i), (ii). This completes the proof of assertion (ii), hence also of Theorem 3.18. \Box

Remark 3.18.1. In the notation of Theorem 3.18, suppose that $n \ge 2$. Then in [CmbCsp], Definition 1.11, (ii), a definition was given for the notation "Out^{FC}(Π_n)^{$\Delta+$}", in the case of *arbitrary* (g, r), that *differs* somewhat from the definition given for this notation in Theorem 3.18, (ii), when (g, r) = (0, 3). On the other hand, one verifies easily, by applying the theory of [CbTpII], §3, that, when (g, r) = (0, 3), these two definitions are in fact *equivalent*. Indeed, when n = 2 (respectively, $n \ge 3$), this follows immediately from [CbTpII], Lemma 3.15, (ii) (respectively, [CbTpII], Theorems 3.16, (v); 3.18, (ii)).

Theorem 3.19 (Metric-, graph-admissible outomorphisms and tempered fundamental groups). In the notation of Theorem 3.17, write \overline{K}^{\wedge} for the p-adic completion of \overline{K} ;

$$\pi_1^{\text{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})$$

for the **tempered fundamental group** [cf. [André], §4, as well as the discussion of Definition 3.1, (ii), of the present paper] of $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$;

$$\Pi_n^{\text{tp}} \stackrel{\text{def}}{=} \varprojlim_N \pi_1^{\text{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})/N$$

for the Σ -tempered fundamental group of $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$ [cf. [CmbGC], Corollary 2.10, (iii)], i.e., the inverse limit given by allowing N to vary over the open normal subgroups of $\pi_1^{\text{temp}}((X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge})$ such that the quotient by N corresponds to a **topological covering** [cf. [André], §4.2, as well as the discussion of Definition 3.1, (ii), of the present paper] of some finite log étale Galois covering of $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$ of degree a product of primes $\in \Sigma$. [Here, we recall that, when n = 1, such a "topological covering" corresponds to a "combinatorial covering", i.e., a covering determined by a covering of the dual semi-graph of the special fiber of the stable model of some finite log étale covering of $(X_{\overline{K}})_n^{\log} \times_{\overline{K}} \overline{K}^{\wedge}$.] Then the following hold: (i) Let $l \in \Sigma$ be such that $l \neq p$. Then the natural inclusion $\operatorname{Out}^{\{l\}\operatorname{-G}}(\Pi_n) \hookrightarrow \operatorname{Out}(\Pi_n)$

[cf. Definition 3.13, (iv)] factors as a composite of homomorphisms

$$\operatorname{Out}^{\{l\}}(\Pi_n) \longrightarrow \operatorname{Out}(\Pi_n^{\operatorname{tp}}) \longrightarrow \operatorname{Out}(\Pi_n)$$

— where the second arrow is the natural homomorphism [cf. Proposition 3.3, (i)]. In particular, the image of the natural homomorphism $\operatorname{Out}(\Pi_n^{\operatorname{tp}}) \to \operatorname{Out}(\Pi_n)$ contains the subgroup $\operatorname{Out}^{\{l\}-G}(\Pi_n) \subseteq \operatorname{Out}(\Pi_n)$, hence also the subgroup $\operatorname{Out}^{G}(\Pi_n) \subseteq$ $\operatorname{Out}(\Pi_n)$ [cf. Definition 3.13, (iv)].

(ii) Write

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}} \subseteq \operatorname{Out}(\Pi_n^{\operatorname{tp}})$$

for the inverse image of $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} \subseteq \operatorname{Out}(\Pi_n)$ [cf. Definition 3.7, (iii)] via the natural homomorphism $\operatorname{Out}(\Pi_n^{\operatorname{tp}}) \to \operatorname{Out}(\Pi_n)$ [cf. (i)]. Then the resulting natural homomorphism

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}} \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}}$$

is split surjective, i.e., there exists a homomorphism

 $\Phi \colon \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}}$

such that the composite

$$\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}} \xrightarrow{\Phi} \operatorname{Out}^{\operatorname{FC}}(\Pi_n^{\operatorname{tp}})^{\operatorname{M}} \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\operatorname{M}}$$

is the identity automorphism of $Out^{FC}(\Pi_n)^M$.

Proof. Assertion (i) follows immediately from Proposition 3.16, (ii). Assertion (ii) follows immediately from assertion (i), together with the fact that $\operatorname{Out}^{\operatorname{FC}}(\Pi_n)^{\mathrm{M}} \subseteq \operatorname{Out}^{\{l\}-\mathrm{G}}(\Pi_n)$ [cf. Theorem 3.17, (ii)]. This completes the proof of Theorem 3.19.

Remark 3.19.1. In the fourth line of the proof of [André], Proposition 8.6.2, it is *asserted* that one has an *injection*

$$\operatorname{Aut}^{\flat}(\Gamma_{0,r+1}^{\operatorname{alg}}) \hookrightarrow \operatorname{Aut}^{\flat}(\Gamma_{0,r}^{\operatorname{alg}})$$

In the notation of the present series of papers [cf. [CmbCsp], Proposition 1.3, (vi), (vii)], this homomorphism corresponds to the natural homomorphism

$$\operatorname{Aut}^{\operatorname{FC}}(\Pi_{n+1})^{\operatorname{cusp}} \longrightarrow \operatorname{Aut}^{\operatorname{FC}}(\Pi_n)^{\operatorname{cusp}}$$

in the case where $(g, r, \Sigma) = (0, 3, \mathfrak{Primes})$, and we observe that Π_n for $n \geq 1$ corresponds to " $\Gamma_{0,r}^{\text{alg}}$ for r-3" in the notation of [André], Proposition 8.6.2. However, this assertion is *false*. Indeed, since $\Gamma_{0,r+1}^{\text{alg}}$ and $\Gamma_{0,r}^{\text{alg}}$ are *center-free* [cf., e.g., [MzTa], Proposition 2.2, (ii)], it follows that

the respective subgroups of inner automorphisms determine compatible injections $\Gamma_{0,r+1}^{\text{alg}} \hookrightarrow \text{Aut}^{\flat}(\Gamma_{0,r+1}^{\text{alg}})$, $\Gamma_{0,r}^{\text{alg}} \hookrightarrow \text{Aut}^{\flat}(\Gamma_{0,r}^{\text{alg}})$. On the other hand, since the natural surjection $\Gamma_{0,r+1}^{\text{alg}} \twoheadrightarrow \Gamma_{0,r}^{\text{alg}}$ is far from injective, it thus follows that the natural homomorphism $\text{Aut}^{\flat}(\Gamma_{0,r+1}^{\text{alg}}) \to \text{Aut}^{\flat}(\Gamma_{0,r}^{\text{alg}})$ also fails to be injective. In particular, the proof given in [André] of the injectivity of the first displayed homomorphism

$$\mathrm{GT}_p^{(r+1)} \longrightarrow \mathrm{GT}_p^{(r)}$$

of [André], Proposition 8.6.2, (1) — hence also of

- [André], Proposition 8.6.2, (2),
- [André], Corollary 8.6.4,
- the final portion of [André], Theorem 8.7.1, and
- the portion of [André], Corollary 8.7.2, concerning " $GT_n^{(r)}$ "

— must be considered *incomplete*. Moreover, although it is not directly related to the *injectivity* of the above discussion, we observe in passing [cf. Remark 3.19.4 below for more details] that the discussion of [André], §8, also contains another *misleading error*.

Remark 3.19.2. Recall that, relative to the notation of the present series of papers, the usual *Grothendieck-Teichmüller group* corresponds to the group

$$\operatorname{GT} \stackrel{\operatorname{def}}{=} \operatorname{Out}^{\mathrm{F}}(\Pi_n)^{\Delta +} = \operatorname{Out}^{\mathrm{FC}}(\Pi_n)^{\Delta +}$$

discussed in Theorem 3.18, (ii) [cf. also Remark 3.18.1], in the case where $(g, r, \Sigma) = (0, 3, \mathfrak{Primes})$ [cf. [CmbCsp], Remark 1.11.1]. Thus, from the point of view of the present paper, it seems that one natural candidate for the notion of a *local version of the Grothendieck-Teichmüller group* is the "metrized Grothendieck-Teichmüller group"

$$\mathrm{GT}^{\mathrm{M}} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)^{\mathrm{M}\Delta +} = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}\Delta +} \subseteq \mathrm{GT}$$

discussed in Theorem 3.18, (ii), again in the case where $(g, r, \Sigma) = (0, 3, \mathfrak{Primes})$. Here, we recall that each of these groups GT^{M} , GT admits a natural *profinite topology*, hence, in particular, is *compact* [cf. Theorem 3.17, (iv)], and, moreover, is *independent*, up to *canonical isomorphism*, of the choice of $n \geq 1$ [cf. Theorem 3.18, (ii)]. Finally, one verifies immediately from the existence of the *natural splitting* of the split surjection discussed in Theorem 3.19, (ii) [cf. also the discussion of the construction of this splitting in the proof of Proposition 3.16, (ii); Remark 3.19.3 below] that, for any positive integer n, one has a *natural inclusion*

$$\mathrm{GT}^{\mathrm{M}} \hookrightarrow \mathrm{GT}_{p}^{(n+3)}$$

[cf. [André], Notation 8.6.1], hence also a natural inclusion

 $\mathrm{GT}^{\mathrm{M}} \hookrightarrow \mathrm{GT}_{p}$

[cf. [André], Definition 8.6.3]. In particular, one obtains a *natural* outer action of GT^{M} on the "tower" of tempered fundamental groups " $(\Gamma_{0,r}^{\mathrm{temp}})_{r\geq 4}$ " discussed in [André], Corollary 8.6.4, i.e., in the notation of Theorem 3.19 of the present paper, on the system of tempered fundamental groups $\{\Pi_n^{\mathrm{tp}}\}_{n\geq 1}$ that is manifestly compatible with the quotients $\Pi_n^{\mathrm{tp}} \to \Gamma_{0,n+3}^{\mathrm{temp}}$ [cf. [André], §8.5].

Remark 3.19.3. The construction of the *splitting* Φ given in the proof of Theorem 3.19, (ii), appears, at first glance, to depend on the choice of the prime l, as well as on the *ordering* of the n factors of the configuration spaces that give rise to Π_n , Π_n^{tp} . In fact, however, it is not difficult to verify — by

- observing that symmetries [e.g., that arise from permutations of the *n* factors] of finite étale coverings of the various configuration spaces over *fields* that appear always *extend* to symmetries of the corresponding *stable polycurves* [cf. the discussion of Remark 3.10.1, (i), (e); [ExtFam], Theorem A];
- applying the *functoriality* of the various constructions involved [cf. the discussion of *"functorial bijections"* in the proof of Proposition 3.6] to relate the *"decomposition groups"* of the various *strata* that appear in the proof of Proposition 3.16, (ii); and
- observing that these strata may be described in terms of "jumps" in the rank of the group-characteristic sheaf [cf. [MzTa], Definition 5.1, (i)] associated to the log structure of the stable polycurves that appear [cf. the discussion of Remark 3.10.1, (i), (e)], hence are independent of the ordering of the n factors of the configuration spaces that appear

— that Φ is *independent* of the choice of l, as well as of the ordering of the *n* factors of the configuration spaces that give rise to Π_n , Π_n^{tp} .

Remark 3.19.4. In passing, we observe, relative to the discussion of [André], §8, that the *second isomorphism*

 $\operatorname{Out}^{\sharp} \pi_{1}^{\operatorname{top}}((\mathbb{P}^{1}_{\mathbb{C}})^{\operatorname{an}} \setminus \{x_{1}, \dots, x_{r}\}) \cong \operatorname{Ker}[\operatorname{Out} F_{r-1} \to \operatorname{GL}_{r-1}(\mathbb{Z})]$

of the final display of [André], §8.2, is *false* as stated and should be replaced by an *inclusion arrow* " \rightarrow ". Indeed, let us first observe that the *first isomorphism* of the final display of [André], §8.2, is *correct* as stated — and indeed is a special case of the well-known theorem of *Dehn-Nielsen-Baer* — if one interprets the phrase "local monodromies"

in the definition of "Out[#]" as referring to generators γ_i of the inertia groups at the points x_i . Write $\prod \stackrel{\text{def}}{=} \pi_1^{\text{top}}((\mathbb{P}^1_{\mathbb{C}})^{\text{an}} \setminus \{x_1, \ldots, x_r\})$. Then the falsity of the second isomorphism — i.e., the non-surjectivity of the natural inclusion " \hookrightarrow " induced by an isomorphism $\Pi \xrightarrow{\sim} F_{r-1}$ may be verified as follows. First, we observe that if one assumes the surjectivity of this natural inclusion, then it follows that the subgroup $\operatorname{Out}^{\sharp}(\Pi) \subseteq \operatorname{Out}(\Pi)$ is normal. Thus, it suffices to obtain a contradiction under the assumption that the subgroup $\operatorname{Out}^{\sharp}(\Pi) \subset \operatorname{Out}(\Pi)$ is normal. Next, let us observe that the discrete free group Π on r-1 generators is generated by $\gamma_1, \ldots, \gamma_{r-1}$. In particular, for any element $\delta \in \Pi$ that appears as one of a collection of r-1 generators of Π , there exists an element $\phi \in \operatorname{Out}(\Pi)$ such that $\phi(\delta) = \gamma_1$. Thus, since $\operatorname{Out}^{\sharp}(\Pi) \subseteq$ $\operatorname{Out}(\Pi)$ is normal, it follows that any element of $\operatorname{Out}^{\sharp}(\Pi)$ preserves the conjugacy class of δ . Write $\delta_1 \stackrel{\text{def}}{=} \gamma_1 \cdot \gamma_2$; for $i = 2, \ldots, r-3$, $\delta_i \stackrel{\text{def}}{=} \delta_{i-1} \cdot \gamma_i$. Then, by taking " δ " to be $\delta_1, \ldots, \delta_{r-3}$, we conclude that any element of $\operatorname{Out}^{\sharp}(\Pi)$ preserves the conjugacy classes of each of $\delta_1, \ldots, \delta_{r-3}$. On the other hand, one verifies immediately that, for a standard choice of generators $\gamma_1, \ldots, \gamma_{r-1}$, the elements $\delta_1, \ldots, \delta_{r-3}$ may be regarded as generators of the *nodal inertia groups* associated to the nodes that appear in a totally degenerate pointed stable curve [over the field of complex numbers] that arises as a degeneration of the pointed stable curve corresponding to the given Riemann surface $(\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}}$. Thus, it follows from [CbTpIV], Corollary 2.19, (i); [CmbGC], Theorem 1.6, (ii); [CmbGC], Proposition 1.3, that any element of $Out^{\sharp}(\Pi)$ is graphic, i.e., in particular, preserves the conjugacy classes of the verticial and nodal subgroups of Π that arise from this totally degenerate structure. On the other hand, in light of [CbTpIV], Corollary 2.21, (iii); [CbTpIV], Theorem 2.24, (ii), this implies that $\operatorname{Out}^{\sharp}(\Pi)$ is an *extension* of a *finite* group by an abelian group, i.e., in contradiction to the fact that $\operatorname{Out}^{\sharp}(\Pi)$ admits a *surjection* to a [*highly nonabelian*!] discrete free group of rank > 2.

Corollary 3.20 (Characterization of the local Galois groups in the global Galois image associated to a hyperbolic curve). Let F be a number field, i.e., a finite extension of the field of rational numbers; \mathfrak{p} a nonarchimedean prime of F; $\overline{F}_{\mathfrak{p}}$ an algebraic closure of the \mathfrak{p} -adic completion $F_{\mathfrak{p}}$ of F; $\overline{F} \subseteq \overline{F}_{\mathfrak{p}}$ the algebraic closure of F in $\overline{F}_{\mathfrak{p}}$; X_F^{\log} a smooth log curve over F. Write $\overline{F}_{\mathfrak{p}}^{\wedge}$ for the completion of $\overline{F}_{\mathfrak{p}}$; $G_{\mathfrak{p}} \stackrel{\text{def}}{=} \operatorname{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \subseteq G_F \stackrel{\text{def}}{=} \operatorname{Gal}(\overline{F}/F)$; $X_{\overline{F}}^{\log} \stackrel{\text{def}}{=} X_F^{\log} \times_F \overline{F}$; $\pi_1(X_{\overline{E}}^{\log})$

for the log fundamental group of $X_{\overline{F}}^{\log}$ [which, in the following, we identify with the log fundamental groups of $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}$, $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}$

- cf. the definition of \overline{F} !];

 $\pi_1^{\mathrm{temp}}(X_F^{\mathrm{log}}\times_F\overline{F}_{\mathfrak{p}}^\wedge)$

for the tempered fundamental group of $X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}$ [cf. [André], §4];

$$\rho_{X_F^{\log}} \colon G_F \longrightarrow \operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))$$

for the natural outer Galois action associated to X_F^{\log} ;

$$\rho_{X_F^{\log},\mathfrak{p}}^{\text{temp}} \colon G_{\mathfrak{p}} \longrightarrow \text{Out}(\pi_1^{\text{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge}))$$

for the natural outer Galois action associated to $X_F^{\log} \times_F F_{\mathfrak{p}}$ [cf. [André], Proposition 5.1.1];

$$\operatorname{Out}(\pi_1(X_{\overline{F}}^{\operatorname{log}}))^{\operatorname{M}} \subseteq (\operatorname{Out}(\pi_1^{\operatorname{temp}}(X_F^{\operatorname{log}} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})) \subseteq) \operatorname{Out}(\pi_1(X_{\overline{F}}^{\operatorname{log}}))$$

for the subgroup of **M-admissible** outcomorphisms of $\pi_1(X_{\overline{F}}^{\log})$ [cf. Definition 3.7, (i), (ii); Proposition 3.6, (i)]. Then the following hold:

- (i) The outer Galois action $\rho_{X_F^{\log},\mathfrak{p}}^{\text{temp}}$ factors through the subgroup $\operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{M}} \subseteq \operatorname{Out}(\pi_1^{\operatorname{temp}}(X_F^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})).$
- (ii) We have a natural commutative diagram

$$\begin{array}{cccc}
G_{\mathfrak{p}} & & \longrightarrow & \operatorname{Out}(\pi_{1}(X_{\overline{F}}^{\log}))^{\mathrm{M}} \\
\downarrow & & & \downarrow \\
G_{F} & & & \downarrow \\
\end{array}$$

$$\begin{array}{cccc}
G_{F} & \xrightarrow{\rho_{X_{F}^{\log}}} & \operatorname{Out}(\pi_{1}(X_{\overline{E}}^{\log})) \\
\end{array}$$

— where the vertical arrows are the natural inclusions, the upper horizontal arrow is the homomorphism arising from the factorization of (i), and all arrows are **injective**.

(iii) The diagram of (ii) is **cartesian**, i.e., if we regard the various groups involved as subgroups of $Out(\pi_1(X_{\overline{F}}^{log}))$, then we have an equality

$$G_{\mathfrak{p}} = G_F \cap \operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{M}}$$

Proof. Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from the *injectivity* of the lower horizontal arrow $\rho_{X_F^{\log}}$ [cf. [NodNon], Theorem C], together with the various definitions involved. Finally, we verify assertion (iii). First, let us observe that if the smooth log curve " X_F^{\log} " is the smooth log curve associated to $\mathbb{P}_F^1 \setminus \{0, 1, \infty\}$, then assertion (iii) follows immediately from [André], Theorem 7.2.1. Write $(X_{\overline{F}})_3^{\log}$ for the 3-rd log configuration space of $X_{\overline{F}}^{\log}$. Then it follows immediately from [NodNon], Theorem B, that the group $\operatorname{Out}^{\operatorname{FC}}(\pi_1((X_{\overline{F}})_3^{\log}))$ of FC-admissible outomorphisms of the log fundamental group $\pi_1((X_{\overline{F}})_3^{\log})$ of $(X_{\overline{F}})_3^{\log}$ [which, in the following, we *identify* with the log fundamental groups of $(X_{\overline{F}})_3^{\log} \times_{\overline{F}}$ $\overline{F}_{\mathfrak{p}}, \ (X_{\overline{F}})_3^{\log} \times_{\overline{F}} \overline{F}_{\mathfrak{p}}^{\wedge}$ — cf. the definition of \overline{F} !] may be regarded as a closed subgroup of $\operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))$. Moreover, it follows immediately from the various definitions involved that the respective images $\operatorname{Im}(\rho_{X_F^{\log}}), \operatorname{Im}(\rho_{X_F^{\log},\mathfrak{p}}^{\operatorname{temp}})$ of the natural outer Galois actions $\rho_{X_F^{\log}}, \rho_{X_F^{\log},\mathfrak{p}}^{\operatorname{temp}}$ associated to $X_F^{\log}, X_F^{\log} \times_F F_{\mathfrak{p}}$ are *contained* in this closed subgroup $\operatorname{Out}^{\operatorname{FC}}(\pi_1((X_{\overline{F}})_3^{\log})) \subseteq \operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))$. Thus, to verify assertion (iii), one verifies easily that it suffices to verify the equality

$$\mathrm{Im}(\rho_{X_F^{\mathrm{log}},\mathfrak{p}}^{\mathrm{temp}}) = \mathrm{Im}(\rho_{X_F^{\mathrm{log}}}) \cap \mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})_3^{\mathrm{log}}))^{\mathrm{M}}$$

[cf. Definition 3.7, (iii)]. On the other hand, since the " $\rho_{X_F^{\log}}$ " that occurs in the case where we take " X_F^{\log} " to be the smooth log curve associated to $\mathbb{P}_F^1 \setminus \{0, 1, \infty\}$ is *injective* [cf. assertion (ii)], this equality follows immediately — by considering the images of the subgroups

$$\mathrm{Im}(\rho_{X_F^{\mathrm{log}},\mathfrak{p}}^{\mathrm{temp}}) \subseteq \mathrm{Im}(\rho_{X_F^{\mathrm{log}}}) \cap \mathrm{Out}^{\mathrm{FC}}(\pi_1((X_{\overline{F}})_3^{\mathrm{log}}))^{\mathrm{M}}$$

of $\operatorname{Out}^{\operatorname{FC}}(\pi_1((X_{\overline{F}})_3^{\log}))^{\operatorname{M}}$ via the tripod homomorphism associated to $\operatorname{Out}^{\operatorname{FC}}(\pi_1((X_{\overline{F}})_3^{\log}))$ [cf. [CbTpII], Definition 3.19] — from Theorem 3.18, (i), together with assertion (iii) in the case where we take " X_F^{\log} " to be the smooth log curve associated to $\mathbb{P}_F^1 \setminus \{0, 1, \infty\}$ [which was verified above]. This completes the proof of assertion (iii), hence also of Corollary 3.20.

Remark 3.20.1. Corollary 3.20, (iii), may be regarded as a generalization of [André], Theorems 7.2.1, 7.2.3, obtained at the cost of replacing, in effect, $\operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{G}}$ by the possibly smaller group $\operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{M}} \subseteq \operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{G}}$. Here, we note that unlike the subgroups $G_{\mathfrak{p}} \subseteq G_F$ [cf., e.g., [AbsHyp], Theorem 1.1.1, (i)] and $\operatorname{Out}(\pi_1^{\operatorname{temp}}(X_{\overline{F}}^{\log} \times_F \overline{F}_{\mathfrak{p}}^{\wedge})) \xrightarrow{\sim} \operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))^{\mathrm{G}} \subseteq \operatorname{Out}(\pi_1(X_{\overline{F}}^{\log}))$ [cf. Definition 3.7, (i); Proposition 3.6, (i); Remark 3.13.1, (i); Theorem 3.17, (v)], which are commensurably terminal, the subgroup $\operatorname{Out}(-)^{\mathrm{M}} \subseteq \operatorname{Out}(-)$ fails, in general [at least in the pro-l case], even to be normally terminal [cf. Remark 3.17.1].

Remark 3.20.2. Let us recall that, in the proof of [NodNon], Theorem C, the authors applied

• the *injectivity portion* of the theory of combinatorial cuspidalization, together with

- the *injectivity* of the outer Galois representation associated to a **tripod**, to prove
- the *injectivity* of the outer Galois representation associated to an **arbitrary hyperbolic curve**.

On the other hand, in the proof of Corollary 3.20, the authors applied

- the [almost pro-*l*] *injectivity portion* of the theory of combinatorial cuspidalization [in the form of Theorem 3.18, (i)], together with
- the characterization of the local Galois groups in the global Galois image for tripods, to prove
- an analogous *characterization* of the *local Galois groups* in the *global Galois image* for **arbitrary hyperbolic curves**.

The formal similarity of these two proofs suggests that it is perhaps natural to think of the *injectivity portion* of the theory of combinatorial cuspidalization as a sort of *tool for reducing* certain problems concerning **arbitrary hyperbolic curves** to the case of **tripods**.

Remark 3.20.3. By comparison to André's original characterization of the local Galois groups in the global Galois image [cf. [André], Theorems 7.2.1, 7.2.3], from the point of view of a researcher who is interested only in *tripods* [i.e., not in *arbitrary hyperbolic curves*], the motivation for the theory developed in the present paper concerning $Out(-)^{M}$ may at first glance appear insufficient. In fact, however, as discussed in Remarks 3.19.1, 3.19.2, even if one is interested only in tripods, it is necessary to apply the extensive theory developed in the present paper concerning $Out(-)^{M}$ in order to repair the mistake in [André] and realize the original goal of the present paper, i.e., of defining a suitable *local analogue of the Grothendieck-Teichmüller group*.

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