On the Grothendieck Conjecture for Affine Hyperbolic Curves over Kummer-faithful Fields

Yuichiro Hoshi

July 2014

Abstract. — In the present paper, we discuss the Grothendieck conjecture for hyperbolic curves over Kummer-faithful fields. In particular, we prove that every point-theoretic and Galois-preserving outer isomorphism between the étale/tame fundamental groups of affine hyperbolic curves over Kummer-faithful fields arises from a uniquely determined isomorphism between the original hyperbolic curves.

Contents

Introduction

In the present paper, we discuss the [*semi-absolute version* of the] *Grothendieck conjecture for hyperbolic curves over Kummer-faithful fields.* In Introduction, let the symbol "$\square$" stand for either "$\circ$" or "$\bullet$". Let $k_\square$ be a *perfect* field and $X_\square$ an *affine hyperbolic curve* over $k_\square$. Write $X_\square^{\mathrm{cpt}}$ for the smooth compactification of $X_\square$, $D_{X_\square} \subseteq X_\square^{\mathrm{cpt}}$ for the divisor at infinity of $X_\square$, and

$$\Pi_{X_\square} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{tame}}(X_\square^{\mathrm{cpt}}, D_{X_\square})$$

for the tame fundamental group of $(X_\square^{\mathrm{cpt}}, D_{X_\square})$.

Then it follows immediately from the *functoriality* of the tame fundamental group, together with the elementary theory of algebraic curves, that an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ of schemes gives rise to a(n) [continuous] outer isomorphism $\Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$ of profinite groups. In this situation, one of the main questions in the *anabelian geometry* may be stated as follows.

2010 Mathematics Subject Classification. — 14H30.
Key words and phrases. — Grothendieck conjecture, hyperbolic curve, Kummer-faithful field.
This research was supported by Grant-in-Aid for Scientific Research (C), No. 24540016, Japan Society for the Promotion of Science.

> What is a necessary and sufficient condition for an outer isomorphism $\Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$ of profinite groups to arise [in the above sense] from an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ of schemes?

Let

$$\alpha \colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an outer isomorphism of profinite groups. In [4], §2, S. Mochizuki proved the following assertion [cf. [4], Corollary 2.2], which may be regarded as an answer to the above question in the case where $k_\circ$ (respectively, $k_\bullet$) is either an *MLF* [i.e., a finite extension of $\mathbb{Q}_p$ for some prime number p] or an *FF* [i.e., a finite field].

> Suppose that $k_\circ$ (respectively, $k_\bullet$) is either an *MLF* or *FF*. Then it holds that α arises from an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ of schemes if and only if α is *point-theoretic* [i.e., satisfies the condition that
>
> $(*)_{\mathrm{P}}$: α induces a bijection between the set of decomposition subgroups of $\Pi_{X_\circ}$ associated to closed points of $X_\circ^{\mathrm{cpt}}$ and the set of decomposition subgroups of $\Pi_{X_\bullet}$ associated to closed points of $X_\bullet^{\mathrm{cpt}}$ — cf. Definition 3.1, (i); also [4], Definition 1.5, (ii)].
>
> Moreover, such an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ of schemes is *uniquely determined.*

In the present paper, by refining various arguments given in [4], §1, §2; [6], §1, we generalize the above result of Mochizuki to the case of affine hyperbolic curves over arbitrary *Kummer-faithful* fields.

We shall say that a perfect field k is *Kummer-faithful* if, for every finite extension K of k and every semi-abelian variety A over K, the Kummer map associated to A is injective, or, equivalently, it holds that

$$\bigcap_N N \cdot A(K) = \{0\}$$

— where N ranges over the positive integers [cf. Definition 1.2; also [6], Definition 1.5]. A typical example of a Kummer-faithful field of characteristic zero is a *sub-p-adic field* for some prime number p [i.e., a field which is isomorphic to a subfield of a finitely generated extension of an MLF — cf. [3], Definition 15.4, (i)] [cf. [6], Remark 1.5.4]; a typical example of a Kummer-faithful field of positive characteristic is an *FF*. In particular, a field that appears in the statement of the above result of Mochizuki [i.e., a field which is either an *MLF* or *FF*] is *Kummer-faithful.* Here, we note that, in [6], §1, Mochizuki also proved a result on the [*semi-absolute version* of the] *birational Grothendieck conjecture for curves over Kummer-faithful fields* [cf. [6], Corollary 1.11].

Let us observe that one verifies easily that if the outer isomorphism α arises from an isomorphism of schemes, then α satisfies the above condition $(*)_{\mathrm{P}}$. Next, let us introduce another necessary condition to arise from an isomorphism of schemes. We shall say that [under the assumption that $k_\circ$ and $k_\bullet$ are *Kummer-faithful*] the outer isomorphism α is *Galois-preserving* if

> $(*)_{\mathrm{G}}$: α is compatible with the natural quotients $\Pi_{X_\circ} \twoheadrightarrow \pi_1(\mathrm{Spec}(k_\circ))$, $\Pi_{X_\bullet} \twoheadrightarrow \pi_1(\mathrm{Spec}(k_\bullet))$ [cf. Definition 3.1, (ii)].

Here, we note that it follows from [5], Corollary 2.8, (ii), that if $k_\circ$ (respectively, $k_\bullet$) is either a finite extension of $\mathbb{Q}$, an MLF, or an FF, then every outer isomorphism $\Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$ [hence also every outer isomorphism discussed in the above result of Mochizuki] is

Galois-preserving. Now let us observe that it follows immediately from the elementary theory of algebraic curves that if α arises from an isomorphism of schemes, then α satisfies the above condition $(*)_{\mathrm{G}}$.

The main result of the present paper may be summarized as follows [cf. Theorem 3.4, (ii), in the case where we take "$\Pi_{X_\square}$" to be "π_1^{tame}"].

THEOREM A. — *Let the symbol "$\square$" stand for either "$\circ$" or "$\bullet$". Let $k_\square$ be a* **Kummer-faithful** *field [cf. Definition 1.2] and $X_\square$ an* **affine hyperbolic curve** *over $k_\square$. Write $X_\square^{\mathrm{cpt}}$ for the smooth compactification of $X_\square$, $D_{X_\square} \subseteq X_\square^{\mathrm{cpt}}$ for the divisor at infinity of $X_\square$, and $\Pi_{X_\square} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{tame}}(X_\square^{\mathrm{cpt}}, D_{X_\square})$ for the tame fundamental group of $(X_\square^{\mathrm{cpt}}, D_{X_\square})$. Let*

$$\alpha\colon\ \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an outer isomorphism of profinite groups. Then it holds that α arises from an isomorphism of schemes

$$X_\circ \xrightarrow{\sim} X_\bullet$$

if and only if α is **point-theoretic** *[cf. Definition 3.1, (i)] and* **Galois-preserving** *[cf. Definition 3.1, (ii)]. Moreover, such an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ of schemes is* **uniquely determined**.

The present paper is organized as follows. In §1, we discuss various objects related to the étale/tame fundamental group of a hyperbolic curve over a *Kummer-faithful* field. In particular, we consider the image, via the Kummer theory, of the multiplicative group of the function field of the hyperbolic curve in a certain injective limit of cohomology modules [cf. Lemma 1.10]. In §2, we discuss the *maximal cuspidally abelian quotient* of the étale fundamental group of the second configuration space of a proper hyperbolic curve. In particular, in order to study the maximal cuspidally abelian quotient, we consider a *fundamental extension* of an open subgroup of the fundamental group of the second configuration space by means of a certain projective system of cohomology modules [cf. Lemma 2.6]. In §3, we prove the main result of the present paper [cf. Theorem 3.4].

0. NOTATIONS AND CONVENTIONS

NUMBERS. — We shall write $\mathfrak{Primes}$ for the set of all prime numbers. Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset of $\mathfrak{Primes}$. Then we shall say that a positive integer is a Σ-*integer* if every prime divisor of the integer is contained in Σ. We shall refer to a finite extension of $\mathbb{Q}_p$ (respectively, $\mathbb{F}_p$) for some prime number p as an *MLF* [i.e., a *mixed-characteristic local field*] (respectively, *FF* [i.e., a *finite field*]).

PROFINITE GROUPS. — We shall say that a profinite group is *slim* if every open subgroup of the profinite group is center-free. One verifies immediately that an extension of *center-free* (respectively, *slim*) profinite groups is *center-free* (respectively, *slim*).

Let G be a profinite group and $H \subseteq G$ a closed subgroup. We shall say that H is *characteristic* if every [continuous] isomorphism of G preserves H. We shall write $Z_G(H)$ for the *centralizer* of H in G, $Z(G) \stackrel{\mathrm{def}}{=} Z_G(G)$ for the *center* of G, G^{ab} for the *abelianization* of G [i.e., the quotient of G by the closure of the commutator subgroup of G], $\mathrm{Aut}(G)$ for the group of [continuous] automorphisms of G, $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G, and $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$ for the group of outer automorphisms of G. Note that if G is *topologically finitely generated*, then it follows immediately that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on $\mathrm{Aut}(G)$, hence also $\mathrm{Out}(G)$.

Let G be a *center-free* and *topologically finitely generated* profinite group and $\rho\colon J \to \mathrm{Out}(G)$ a homomorphism of profinite groups. Thus, we have a natural *exact sequence of profinite groups*

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

Then, by pulling back this exact sequence by the homomorphism ρ, we obtain a *profinite group* $G \stackrel{\mathrm{out}}{\rtimes} J$, which fits into an exact sequence of profinite groups

$$1 \longrightarrow G \longrightarrow G \stackrel{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

CURVES. — Let S be a scheme and X a scheme over S. Then we shall say that X is a *smooth curve* over S if there exist a scheme X^{cpt} which is smooth, proper, geometrically connected, and of relative dimension one over S and a closed subscheme $D \subseteq X^{\mathrm{cpt}}$ of X^{cpt} which is finite and étale over S such that the complement $X^{\mathrm{cpt}} \setminus D$ of D in X^{cpt} is isomorphic to X over S. Note that if S is the spectrum of a field k, then it follows immediately from elementary algebraic geometry that the pair "(X^{cpt}, D)" is uniquely determined up to canonical isomorphism over k; we shall refer to X^{cpt} as the *smooth compactification* of X and to D as the *divisor at infinity* of X.

Let S be a scheme. Then we shall say that a smooth curve X over S is *hyperbolic* if there exist a pair (X^{cpt}, D) satisfying the condition in the above definition of the term "smooth curve" and a pair (g, r) of nonnegative integers such that $2g - 2 + r > 0$, the [necessarily locally free] $\mathcal{O}_S$-module $(X^{\mathrm{cpt}} \to S)_*(\Omega^1_{X^{\mathrm{cpt}}/S})$ is of rank g, and the finite étale covering $D \hookrightarrow X^{\mathrm{cpt}} \to S$ is of degree r.

1. FUNDAMENTAL GROUPS OF HYPERBOLIC CURVES OVER KUMMER-FAITHFUL FIELDS

In the present §1, we discuss various objects related to the étale/tame fundamental group of a hyperbolic curve over a *Kummer-faithful* field [cf. Definition 1.2 below]. In the present §1, let k be a *perfect* field, $\overline{k}$ an algebraic closure of k, and X a *hyperbolic curve* over k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$. Note that one verifies immediately from the various definitions involved that since k is *perfect*, every connected finite étale covering of X is a *hyperbolic curve* over the finite extension of k obtained by forming the algebraic closure of k in the function field of the covering.

DEFINITION 1.1. — We shall write

- $\mathfrak{Primes}^{\times/k} \subseteq \mathfrak{Primes}$ for the set of prime numbers which are invertible in k,
- $\widehat{\mathbb{Z}}_{\times/k}$ for the maximal pro-$\mathfrak{Primes}^{\times/k}$ quotient of $\widehat{\mathbb{Z}}$,

- X^{cpt} for the *smooth compactification* of X,
- $D_X \subseteq X^{\mathrm{cpt}}$ for the *divisor at infinity* of X,
- g_X for the genus of X^{cpt},
- $r_X \stackrel{\mathrm{def}}{=} \sharp D_X(\overline{k})$,
- K_X for the function field of X,
- $X^{\mathrm{cl}+}$ for the set of closed points of X^{cpt}, and
- $\mathrm{Div}(X)$ for the group of divisors on X^{cpt}.

If $x \in X^{\mathrm{cpt}}(k)$ is a k-rational point of X^{cpt}, then we shall write

- $\mathrm{ord}_x \colon K_X^\times \twoheadrightarrow \mathbb{Z}$ for the [uniquely determined] surjective valuation associated to $x \in X^{\mathrm{cpt}}(k)$.

In the following, let

$$\Pi_X$$

be either the étale fundamental group $\pi_1(X)$ of X or the tame fundamental group $\pi_1^{\mathrm{tame}}(X^{\mathrm{cpt}}, D_X)$ of (X^{cpt}, D_X). Write

$$\Delta_X \ \subseteq \ \Pi_X$$

for the quotient of the étale fundamental group $\pi_1(X \otimes_k \overline{k}) \subseteq \pi_1(X)$ of $X \otimes_k \overline{k}$ determined by Π_X. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1.$$

Now let us recall [cf., e.g., [7], Corollary 1.4; [7], Proposition 1.11] that Δ_X is *slim*.

DEFINITION 1.2. — We shall say that k is *Kummer-faithful* if, for every finite extension K of k and every semi-abelian variety A over K, it holds that

$$\bigcap_N N \cdot A(K) \ = \ \{0\}$$

— where N ranges over the positive integers [cf. Remark 1.2.1 below].

REMARK 1.2.1. — If k is of *characteristic zero*, then it is immediate that k is *Kummer-faithful* in the sense of Definition 1.2 if and only if k is *Kummer-faithful* in the sense of [6], Definition 1.5.

REMARK 1.2.2. — If k is *Kummer-faithful*, then the following assertion holds:

If K is a finite extension of k, then it holds that

$$\bigcap_N (K^\times)^N \ = \ \{1\}$$

— where N ranges over the $\mathfrak{Primes}^{\times/k}$*-integers*.

Indeed, this follows immediately, by considering the semi-abelian variety "$\mathbb{G}_m$", from the definition of the term "*Kummer-faithful*", together with our assumption that k, hence also K, is *perfect*.

REMARK 1.2.3. — A typical example of a *Kummer-faithful* field of characteristic zero is a *sub-p-adic* field for some prime number p [i.e., a field which is isomorphic to a subfield of a finitely generated extension of an MLF — cf. [3], Definition 15.4, (i)] [cf. [6], Remark 1.5.4]. A typical example of a *Kummer-faithful* field of positive characteristic is an *FF*.

DEFINITION 1.3.

(i) Let G be a profinite group. Then we shall write $\mathfrak{C}(G)$ for the set of closed subgroups of G. Note that G acts on $\mathfrak{C}(G)$ by conjugation.

(ii) We shall write

$$\mathfrak{S}_X\colon\ X^{\mathrm{cl}+} \longrightarrow \mathfrak{C}(\Pi_X)/\Pi_X$$

[cf. (i)] for the map given by mapping a closed point of X^{cpt} to the Π_X-conjugacy class of a decomposition subgroup associated to the closed point.

DEFINITION 1.4. — We shall write

$$\Lambda_X$$

for the *cyclotome* associated to the semi-graph of anabelioids of pro-$\mathfrak{Primes}^{\times/k}$ PSC-type [with no nodes] arising from the hyperbolic curve $X \otimes_k \overline{k}$ [cf. [2], Definition 3.8, (i)].

REMARK 1.4.1. — In the notation of Definition 1.4:

(i) The cyclotome Λ_X is isomorphic, as a G_k-module, to $\varprojlim_N \mu_N(\overline{k})$ — where the projective limit is taken over the $\mathfrak{Primes}^{\times/k}$-integers N, and we write $\mu_N(\overline{k})$ for the group of N-th roots of unity in $\overline{k}$.

(ii) If X is *proper* over k [i.e., $r_X = 0$], then

$$\Lambda_X \overset{\mathrm{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}_{\times/k}}\big(H^2(\Delta_X, \widehat{\mathbb{Z}}_{\times/k}), \widehat{\mathbb{Z}}_{\times/k}\big).$$

PROPOSITION 1.5. — *Suppose that k is* **Kummer-faithful**. *Then the following hold:*

(i) *Write*

$$\chi^{\times/k}_{\mathrm{cyc}}\colon\ G_k \longrightarrow \mathrm{Aut}(\varprojlim_N \mu_N(\overline{k})) = (\widehat{\mathbb{Z}}_{\times/k})^{\times}$$

— where the projective limit is taken over the $\mathfrak{Primes}^{\times/k}$-integers N, and we write $\mu_N(\overline{k})$ for the group of N-th roots of unity in $\overline{k}$ — for the $\mathfrak{Primes}^{\times/k}$-adic cyclotomic character of G_k. Then it holds that $Z(G_k) \cap \mathrm{Ker}(\chi^{\times/k}_{\mathrm{cyc}}) = \{1\}$.

(ii) *The profinite group Π_X is* **slim**.

PROOF. — First, we verify assertion (i). Assume that there exists a *nontrivial* element $\gamma \in Z(G_k) \cap \mathrm{Ker}(\chi_{\mathrm{cyc}}^{\times/k})$. Let K be a finite Galois extension of k contained in $\overline{k}$ such that the corresponding normal open subgroup $G_K \subseteq G_k$ does *not contain* $\gamma \in G_k$. Then since $\gamma \in Z(G_k) \cap \mathrm{Ker}(\chi_{\mathrm{cyc}}^{\times/k})$, the natural action of γ on $H^1(G_K, \varprojlim_N \mu_N(\overline{k}))$ is *trivial*. On the other hand, it follows from Remark 1.2.2, together with the *Kummer theory*, that this *triviality* implies the *triviality* of the action of γ on K. Thus, since $\gamma \notin G_K$, we obtain a *contradiction*. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that since a connected finite étale covering of X is a *hyperbolic curve over a Kummer-faithful field* [cf. our assumption that k is *perfect*], to verify assertion (ii), it suffices to verify that Π_X is *center-free*. Next, let us observe that since Δ_X is *center-free*, the composite $Z(\Pi_X) \hookrightarrow \Pi_X \twoheadrightarrow G_k$ is an *injection*, whose image is *contained* in the center $Z(G_k)$ of G_k. On the other hand, it follows immediately from the various definitions involved that the natural action of $Z(\Pi_X)$ on Λ_X is *trivial*, i.e., that the image of the composite $Z(\Pi_X) \hookrightarrow \Pi_X \twoheadrightarrow G_k$ is *contained* in $\mathrm{Ker}(\chi_{\mathrm{cyc}}^{\times/k})$ [cf. Remark 1.4.1, (i)]. Thus, it follows from assertion (i) that $Z(\Pi_X) = \{1\}$. This completes the proof of assertion (ii), hence also of Proposition 1.5. □

REMARK 1.5.1. — In the situation of Proposition 1.5, (i), in general, it does *not hold* that $Z(G_k) = \{1\}$. Indeed, although [one verifies easily that] an FF is *Kummer-faithful*, the absolute Galois group of an FF is *abelian* and *nontrivial*.

LEMMA 1.6. — *The following hold:*

(i) *The quotient of Π_X by the normal closed subgroup generated by the intersections $\Delta_X \cap H$ — where H ranges over the closed subgroups of Π_X whose images in $\mathfrak{C}(\Pi_X)/\Pi_X$ are contained in the image of the map $\mathfrak{S}_X$ of Definition* 1.3, (ii) — **coincides** *with the quotient $\Pi_X \twoheadrightarrow \pi_1(X^{\mathrm{cpt}})$.*

(ii) *The subset $\mathfrak{Primes}^{\times/k} \subseteq \mathfrak{Primes}$ is the [uniquely determined] maximal* **infinite** *subset on which the map $\mathfrak{Primes} \to \mathbb{Z}$ given by mapping $p \in \mathfrak{Primes}$ to $\dim_{\mathbb{Q}_p}\big(\pi_1(X^{\mathrm{cpt}} \otimes_k \overline{k})^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p\big)$ ($< \infty$ — cf.* [7], *Proposition 1.1) is* **constant**.

(iii) *For every $p \in \mathfrak{Primes}^{\times/k}$, it holds that* $\dim_{\mathbb{Q}_p}(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p) = 2g_X + \max\{0, r_X - 1\}$, $\dim_{\mathbb{Q}_p}\big(\pi_1(X^{\mathrm{cpt}} \otimes_k \overline{k})^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p\big) = 2g_X$.

(iv) *It holds that X is* **proper** *over k [i.e., $r_X = 0$] if and only if, for every $p \in \mathfrak{Primes}^{\times/k}$, the maximal pro-$p$ quotient of Δ_X is* **not free pro-p**.

(v) *It holds that Δ_X is* **not topologically finitely generated** *if and only if* $\mathrm{char}(k) \neq 0$, $r_X \neq 0$, *and* $\Pi_X = \pi_1(X)$. *In particular, it holds that Δ_X is* **topologically finitely generated** *if and only if $\Pi_X = \pi_1^{\mathrm{tame}}(X^{\mathrm{cpt}}, D_X)$.*

PROOF. — Assertion (i) follows immediately from the various definitions involved. Assertions (ii) and (iii) follow immediately from [7], Corollary 1.2. Assertions (iv) and (v) follow immediately from [7], Proposition 1.1. This completes the proof of Lemma 1.6. □

LEMMA 1.7. — *Suppose that k is* **Kummer-faithful**. *Then the following hold:*

(i) *The map $\mathfrak{S}_X \colon X^{\mathrm{cl}+} \to \mathfrak{C}(\Pi_X)/\Pi_X$ of Definition* 1.3, (ii), *is* **injective**.

(ii) *Suppose that X is* **proper** *over k [i.e., $r_X = 0$]. Let m be a positive integer and $x_1, \ldots, x_m \in X(k)$ distinct k-rational points. Thus, for each $i \in \{1, \ldots, m\}$, the k-rational point $x_i \in X(k)$ determines a splitting s_i [well-defined up to Δ_X^{ab}-conjugation] of the exact sequence of profinite groups*

$$1 \longrightarrow \Delta_X^{\mathrm{ab}} \longrightarrow \Pi_X/\mathrm{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\mathrm{ab}}) \longrightarrow G_k \longrightarrow 1.$$

Let $(n_1, \ldots, n_m) \in \mathbb{Z}^{\oplus m}$ be such that $\sum_{i=1}^m n_i = 0$. Thus, by considering the linear combination "$\sum_{i=1}^m n_i \cdot s_i$" of the s_i's, we obtain a cohomology class $[\sum_{i=1}^m n_i \cdot s_i] \in H^1(G_k, \Delta_X^{\mathrm{ab}})$. Then the divisor $\sum_{i=1}^m n_i \cdot x_i$ [of degree zero] on X is **principal** *if and only if $[\sum_{i=1}^m n_i \cdot s_i] = 0$ in $H^1(G_k, \Delta_X^{\mathrm{ab}})$.*

PROOF. — Assertion (i) follows immediately from a similar argument to the argument applied in the proof of [7], Proposition 2.8, (i). Assertion (ii) follows immediately from a similar argument to the argument applied in the proof of [4], Proposition 2.2, (i). This completes the proof of Lemma 1.7. □

LEMMA 1.8. — *Suppose that k is* **Kummer-faithful**, *and that $D_X(\overline{k}) = D_X(k)$. For $x \in D_X(k)$, let $\mathfrak{I}_x \subseteq \Delta_X$ be an inertia subgroup of Δ_X associated to x. Then the following hold:*

(i) *The inclusions $\mathfrak{I}_x \hookrightarrow \Pi_X$ — where x ranges over the elements of $D_X(k)$ — and the surjection $\Pi_X \twoheadrightarrow G_k$ determine an exact sequence*

$$0 \longrightarrow H^1(G_k, \Lambda_X) \longrightarrow H^1(\Pi_X, \Lambda_X) \longrightarrow \bigoplus_{x \in D_X(k)} \mathrm{Hom}_{\widehat{\mathbb{Z}}}(\mathfrak{I}_x, \Lambda_X).$$

Thus, by considering the isomorphism [well-defined up to a $(\widehat{\mathbb{Z}}_{\times/k})^\times$-multiple]

$$(k^\times)^{\times/k} \stackrel{\mathrm{def}}{=} \varprojlim_N k^\times/(k^\times)^N \stackrel{\sim}{\longrightarrow} H^1(G_k, \Lambda_X)$$

— where the projective limit is taken over the $\mathfrak{Primes}^{\times/k}$-integers N — obtained by the Kummer theory [cf. Remark 1.4.1, (i)] *and the identification*

$$\widehat{\mathbb{Z}}_{\times/k} = \mathrm{Hom}_{\widehat{\mathbb{Z}}_{\times/k}}(\Lambda_X, \Lambda_X) \stackrel{\sim}{\longrightarrow} \mathrm{Hom}_{\widehat{\mathbb{Z}}_{\times/k}}(\mathfrak{I}_x^{\times/k}, \Lambda_X) \stackrel{\sim}{\longrightarrow} \mathrm{Hom}_{\widehat{\mathbb{Z}}}(\mathfrak{I}_x, \Lambda_X)$$

— where we write $\mathfrak{I}_x^{\times/k}$ for the maximal pro-$\mathfrak{Primes}^{\times/k}$ quotient of $\mathfrak{I}_x$ — obtained by the synchronization of cyclotomes $\mathfrak{I}_x^{\times/k} \stackrel{\sim}{\to} \Lambda_X$ discussed in [2], *Corollary 3.9,* (v), *we obtain an exact sequence*

$$0 \longrightarrow (k^\times)^{\times/k} \longrightarrow H^1(\Pi_X, \Lambda_X) \longrightarrow \bigoplus_{x \in D_X(k)} \widehat{\mathbb{Z}}_{\times/k}.$$

(ii) *The exact sequence of the final display of* (i) *fits into the following commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & k^\times & \longrightarrow & \mathcal{O}_X^\times(X) & \xrightarrow{\bigoplus_{x \in D_X(k)} \mathrm{ord}_x} & \bigoplus_{x \in D_X(k)} \mathbb{Z} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (k^\times)^{\times/k} & \longrightarrow & H^1(\Pi_X, \Lambda_X) & \longrightarrow & \bigoplus_{x \in D_X(k)} \widehat{\mathbb{Z}}_{\times/k} \end{array}$$

— *where the horizontal sequences are* **exact**, *the vertical arrows are* **injective**, *the left-hand and middle vertical arrows are the homomorphisms obtained by the Kummer theory, and the right-hand vertical arrow is the homomorphism determined by the natural inclusion* $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}_{\times/k}$.

(iii) *Let* $y \in X(k)$ *be a* k*-rational point. Then the composite*

$$\mathcal{O}_X^\times(X) \ \longrightarrow \ H^1(\Pi_X, \Lambda_X) \ \longrightarrow \ H^1(G_k, \Lambda_X) \ \xleftarrow{\ \sim\ } \ (k^\times)^{\times/k}$$

— *where the first arrow is the middle vertical arrow of the diagram of* (ii), *and the second arrow is the homomorphism determined by the splitting* [*well-defined up to* Δ_X*-conjugation*] *of* $\Pi_X \twoheadrightarrow G_k$ *induced by* $y \in X(k)$, *i.e.,* "$\mathfrak{S}_X(y)$" — **coincides** [*up to a* $(\widehat{\mathbb{Z}}_{\times/k})^\times$*-multiple — cf. the isomorphism* $(k^\times)^{\times/k} \xrightarrow{\sim} H^1(G_k, \Lambda_X)$ *discussed in* (i)] *with the composite*

$$\begin{array}{ccccc} \mathcal{O}_X^\times(X) & \longrightarrow & k^\times & \longrightarrow & (k^\times)^{\times/k} \\ f & \mapsto & f(y) & & \end{array}$$

— *where the second arrow is the natural homomorphism.*

PROOF. — Since the G_k-*invariant* of the $\mathfrak{Primes}^{\times/k}$-adic Tate module of the Jacobian variety of X^{cpt} is *trivial* [by our assumption that k is *Kummer-faithful*], assertion (i) follows immediately from a similar argument to the argument applied in the proof of [4], Proposition 2.1, (ii). Assertions (ii) and (iii) follow immediately — in light of Remark 1.2.2 — from the functoriality of the Kummer class, together with the various definitions involved. This completes the proof of Lemma 1.8. ☐

DEFINITION 1.9. — Suppose that k is *Kummer-faithful*, and that X is *proper* [i.e., $r_X = 0$]. Let $S \subseteq X(k)$ be a finite subset and $x \in X(k) \setminus S$. Thus, since $X \setminus S$ is a *hyperbolic curve* over k, it follows from Lemma 1.8, (i), that we have an exact sequence

$$0 \ \longrightarrow \ (k^\times)^{\times/k} \ \longrightarrow \ H^1(\Pi_{X\setminus S}, \Lambda_X) \ \longrightarrow \ \bigoplus_{s \in S} \widehat{\mathbb{Z}}_{\times/k}$$

— where we write $\Pi_{X\setminus S} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{tame}}(X, S)$. We shall write

$$\mathcal{P}(X,S) \ \stackrel{\mathrm{def}}{=} \ \Big\{ (n_s)_{s\in S} \in \bigoplus_{s \in S} \mathbb{Z} \ \Big| \ \text{The divisor} \sum_{s \in S} n_s \cdot s \text{ is principal.} \Big\} \ \subseteq \ \bigoplus_{s\in S} \widehat{\mathbb{Z}}_{\times/k};$$

$$\mathcal{O}^\times(\Pi_X, S) \ \subseteq \ H^1(\Pi_{X\setminus S}, \Lambda_X)$$

for the submodule obtained by forming the inverse image of the submodule $\mathcal{P}(X,S) \subseteq \bigoplus_{s\in S} \widehat{\mathbb{Z}}_{\times/k}$ via the third arrow of the above exact sequence;

$$\mathrm{ev}_x(\Pi_X, S)\colon \ \mathcal{O}^\times(\Pi_X, S) \ \longrightarrow \ H^1(G_k, \Lambda_X)$$

for the restriction to $\mathcal{O}^\times(\Pi_X, S) \subseteq H^1(\Pi_{X\setminus S}, \Lambda_X)$ of the homomorphism $H^1(\Pi_{X\setminus S}, \Lambda_X) \to H^1(G_k, \Lambda_X)$ determined by the splitting [well-defined up to $\Delta_{X\setminus S}$-conjugation] of $\Pi_{X\setminus S} \twoheadrightarrow G_k$ induced by $x \in X(k) \setminus S$, i.e., "$\mathfrak{S}_{X\setminus S}(x)$";

$$\mathcal{K}^\times(\Pi_X) \stackrel{\text{def}}{=} \varinjlim_{K,T} \mathcal{O}^\times(\Pi_{X\otimes_k K}, T)$$

— where the injective limit is taken over the finite extensions K of k contained in $\overline{k}$ and the finite subsets $T \subseteq (X \otimes_k K)(K)$. Here, we note that the natural injection $\Pi_{X\otimes_k K} \hookrightarrow \Pi_X$ [well-defined up to Δ_X-conjugation] and the natural surjection $\Pi_{X\otimes_k K} \twoheadrightarrow G_K$ determine an *isomorphism* $\Pi_{X\otimes_k K} \xrightarrow{\sim} \Pi_X \times_{G_k} G_K$ [well-defined up to $(\Delta_X \times \{1\})$-conjugation].

LEMMA 1.10. — *Suppose that k is* **Kummer-faithful**, *and that X is* **proper** [*i.e.,* $r_X = 0$]. *Then the following hold:*

(i) *The middle vertical arrows of the diagram of Lemma* 1.8, (ii), *in the case where we take "X" of Lemma* 1.8, (ii), *to be $(X \otimes_k K) \setminus T$ — where K ranges over the finite extensions of k contained in $\overline{k}$ and T ranges over the finite subsets of $(X \otimes_k K)(K)$ — determine an* **injective** *homomorphism*

$$K^\times_{X\otimes_k \overline{k}} \hookrightarrow \mathcal{K}^\times(\Pi_X)$$

[*cf. Definition* 1.1].

(ii) *Let $S \subseteq X(k)$ be a finite subset. Then the diagram of Lemma* 1.8, (ii), *determines a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k^\times & \longrightarrow & \mathcal{O}_X^\times(X \setminus S) & \xrightarrow{\bigoplus_{x\in S} \mathrm{ord}_x} & \mathcal{P}(X, S) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^1(G_k, \Lambda_X) & \longrightarrow & \mathcal{O}^\times(\Pi_X, S) & \longrightarrow & \mathcal{P}(X, S) & \longrightarrow & 0. \end{array}$$

— *where the horizontal sequences are* **exact**, *and the vertical arrows are* **injective**.

(iii) *Let $S \subseteq X(k)$ be a finite subset and $x \in X(k) \setminus S$. Then the kernel of the homomorphism*

$$\mathrm{ev}_x(\Pi_X, S)\colon\ \mathcal{O}^\times(\Pi_X, S) \longrightarrow H^1(G_k, \Lambda_X)$$

coincides, *relative to the middle vertical* **injection** *of the diagram of* (ii), *with the subgroup*

$$\{ f \in \mathcal{O}_X^\times(X \setminus S) \mid f(x) = 1 \}$$

of $\mathcal{O}_X^\times(X \setminus S)$, i.e.,

$$\mathrm{Ker}\big(\mathrm{ev}_x(\Pi_X, S)\big) = \{ f \in \mathcal{O}_X^\times(X \setminus S) \mid f(x) = 1 \} \subseteq \mathcal{O}_X^\times(X \setminus S).$$

In particular, for every $y \in X(k) \setminus S$, relative to the left-hand vertical **injection** *of the diagram of* (ii), *it holds that*

$$\mathrm{ev}_y(\Pi_X, S)(\mathrm{Ker}\big(\mathrm{ev}_x(\Pi_X, S)\big)) \subseteq k^\times.$$

(iv) *Let x_1, $x_2 \in X(k)$ be such that $x_1 \neq x_2$. Then*

(a) *the subgroups*

$$\mathrm{Ker}\big(\mathrm{ev}_{(x_i)_K}(\Pi_{X\otimes_k K}, S_i)\big) \ \subseteq\ \mathcal{O}^\times(\Pi_{X\otimes_k K}, S_i) \ \subseteq\ \mathcal{K}^\times(\Pi_X)$$

— *where i ranges over the elements of $\{1,2\}$, K ranges over the finite extensions of k contained in $\overline{k}$, S_i ranges over the finite subsets of $(X \otimes_k K)(K)$ which do not contain $(x_i)_K$, and we write $(x_i)_K \in (X \otimes_k K)(K)$ for the K-rational point determined by x_i — and*

(b) *the subgroups*

$$\mathrm{ev}_{(x_2)_K}(\Pi_{X\otimes_k K}, S)(\mathrm{Ker}\big(\mathrm{ev}_{(x_1)_K}(\Pi_{X\otimes_k K}, S)\big)) \subseteq\ H^1(G_K, \Lambda_X) \ \subseteq\ \mathcal{K}^\times(\Pi_X)$$

— *where K ranges over the finite extensions of k contained in $\overline{k}$, S ranges over the finite subsets of $(X \otimes_k K)(K)$ which do not contain $(x_1)_K$ and $(x_2)_K$, and we write $(x_i)_K \in (X \otimes_k K)(K)$ for the K-rational point determined by x_i —*

generate *the image of the injection of* (i)*.*

PROOF. — Assertions (i), (ii) follow immediately from the various definitions involved, together with our assumption that k is *Kummer-faithful*. Next, we verify assertion (iii). Let us first observe that one verifies immediately from Lemma 1.8, (iii), together with the various definitions involved, that, to complete the verification of assertion (iii), it suffices to verify that $\mathrm{Ker}(\mathrm{ev}_x(\Pi_X, S)) \subseteq \mathcal{O}_X^\times(X \setminus S)$. Let $f \in \mathrm{Ker}(\mathrm{ev}_x(\Pi_X, S))$. Next, let us observe that it follows immediately from assertion (ii) that there exist $g \in \mathcal{O}_X^\times(X \setminus S)$ and $a \in H^1(G_k, \Lambda_X)$ such that $f = a \cdot g$. Thus, it holds that

$$1 \ =\ \mathrm{ev}_x(\Pi_X, S)(f) \ =\ a \cdot \mathrm{ev}_x(\Pi_X, S)(g),$$

which thus implies that $a = \mathrm{ev}_x(\Pi_X, S)(g)^{-1} \in k^\times$ [cf. Lemma 1.8, (iii)]. In particular, we conclude that $f = a \cdot g \in \mathcal{O}_X^\times(X \setminus S)$. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). Write $\mathcal{F} \subseteq \mathcal{K}^\times(\Pi_X)$ for the subgroup generated by the various subgroups (a), (b) appearing in the statement of assertion (iv) and regard $K^\times_{X\otimes_k\overline{k}}$ as a subgroup of $\mathcal{K}^\times(\Pi_X)$ by means of the injection of assertion (i). Then let us observe that it follows from assertion (iii) that $\mathcal{F} \subseteq K^\times_{X\otimes_k\overline{k}}$. Moreover, by considering the subgroups (b), one verifies immediately — in light of Lemma 1.11, (i), below — from assertion (iii), together with Lemma 1.8, (iii), that

$$\overline{k}^\times \ \subseteq\ \mathcal{F} \ \subseteq\ K^\times_{X\otimes_k\overline{k}}.$$

In particular, by considering the subgroups (a), we conclude from assertion (iii), together with Lemma 1.8, (iii), that, for a rational function $f \in K^\times_{X\otimes_k\overline{k}}$, if $f((x_i)_{\overline{k}}) \notin \{0, \infty\}$ for some $i \in \{1,2\}$ [where we write $(x_i)_{\overline{k}} \in (X \otimes_k \overline{k})(\overline{k})$ for the $\overline{k}$-valued point determined by x_i], then $f \in \mathcal{F}$. Thus, the equality $\mathcal{F} = K^\times_{X\otimes_k\overline{k}}$ follows immediately from Lemma 1.11, (ii), below. This completes the proof of assertion (iv). $\square$

LEMMA 1.11. — *Let Ω be an algebraically closed field; C a proper hyperbolic curve over Ω; x, $y \in C(\Omega)$ distinct Ω-valued points of C. Then the following hold:*

(i) *For every $\lambda \in \Omega \setminus \{0,1\}$, there exists a rational function $f \in K_C$ [cf. Definition* 1.1*] such that $f(x) = 1$ and $f(y) = \lambda$.*

(ii) *The multiplicative group $K_C^\times$ is* **generated** *by rational functions $f \in K_C^\times$ such that $f(\{x,y\}) \not\subseteq \{0,\infty\}$.*

PROOF. — Assertion (i) follows immediately by considering, for instance, a suitable *linear fractional transformation* $(ag+b)/(cg+d)$ [where a, b, c, $d \in \Omega$] of a rational function $g \in K_C^\times$ such that $g(x) \neq g(y)$. Next, we verify assertion (ii). Write $\mathcal{F} \subseteq K_C^\times$ for the subgroup of $K_C^\times$ generated by rational functions $f \in K_C^\times$ such that $f(\{x,y\}) \not\subseteq \{0,\infty\}$. To complete the verification of the equality $\mathcal{F} = K_C^\times$, let us take a rational function $g \in K_C^\times$ such that $g(\{x,y\}) \subseteq \{0,\infty\}$. Now, to verify $g \in \mathcal{F}$, we may assume without loss of generality, by replacing g by g^{-1} if necessary, that $g(x) = \infty$, i.e., that $\mathrm{ord}_x(g) < 0$. Then one verifies immediately from the Riemann-Roch theorem that there exists a rational function $h \in K_C^\times$ such that $\mathrm{ord}_x(g) = \mathrm{ord}_x(h)\,(= \mathrm{ord}_x(h+1))$ and $h(y) = 0$ [i.e., $(h+1)(y) = 1$]. Thus, since $g/(h+1)$, $h+1 \in \mathcal{F}$, we conclude that $g \in \mathcal{F}$. This completes the proof of assertion (ii). $\square$

DEFINITION 1.12. — We shall write

$$\Delta_X^{\text{c-ab}} \quad (\text{respectively, } \Delta_X^{\text{c-cn}})$$

for the maximal quotient of Δ_X such that the natural surjection $\Delta_X \twoheadrightarrow \pi_1(X^{\mathrm{cpt}} \otimes_k \overline{k})$ factors through the surjection $\Delta_X \twoheadrightarrow \Delta_X^{\text{c-ab}}$ (respectively, $\Delta_X^{\text{c-cn}}$), and, moreover, the kernel of the resulting surjection $\Delta_X^{\text{c-ab}}$ (respectively, $\Delta_X^{\text{c-cn}}$) $\twoheadrightarrow \pi_1(X^{\mathrm{cpt}} \otimes_k \overline{k})$ is pro-$\mathfrak{Primes}^{\times/k}$ and abelian (respectively, pro-$\mathfrak{Primes}^{\times/k}$ and contained in the center of $\Delta_X^{\text{c-cn}}$). We shall write

$$\Pi_X^{\text{c-ab}} \quad (\text{respectively, } \Pi_X^{\text{c-cn}})$$

for the quotient of Π_X by the kernel of $\Delta_X \twoheadrightarrow \Delta_X^{\text{c-ab}}$ (respectively, $\Delta_X^{\text{c-cn}}$). Thus, we have a commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \Delta_X \arrow[r] \arrow[d] & \Pi_X \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta_X^{\text{c-ab}} \arrow[r] \arrow[d] & \Pi_X^{\text{c-ab}} \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta_X^{\text{c-cn}} \arrow[r] & \Pi_X^{\text{c-cn}} \arrow[r] & G_k \arrow[r] & 1
\end{tikzcd}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*.

LEMMA 1.13. — *The following hold:*

(i) *The natural surjections*

$$\Pi_X \twoheadrightarrow \Pi_X^{\text{c-ab}} \twoheadrightarrow \Pi_X^{\text{c-cn}}$$

determine isomorphisms

$$H^1(\Pi_X^{\text{c-cn}}, \Lambda_X) \xrightarrow{\sim} H^1(\Pi_X^{\text{c-ab}}, \Lambda_X) \xrightarrow{\sim} H^1(\Pi_X, \Lambda_X).$$

(ii) *Suppose that X is* **proper** *[i.e., $r_X = 0$]. Let $x_1, \ldots, x_n \in X(k)$ be distinct k-rational points. For $i \in \{1, \ldots, n\}$, write $U_i \stackrel{\mathrm{def}}{=} X \setminus \{x_i\}$ and $U \stackrel{\mathrm{def}}{=} X \setminus \{x_1, \ldots, x_n\}$. Then* *the natural open immersions* $U \hookrightarrow U_i$ *— where* i *ranges over the elements of* $\{1, \dots, n\}$ *— determine an isomorphism of profinite groups* [*well-defined up to* $\Delta^{\text{c-cn}}_{U_1} \times_{\Delta_X} \cdots \times_{\Delta_X} \Delta^{\text{c-cn}}_{U_n}$*-conjugation*]

$$\Pi^{\text{c-cn}}_U \xrightarrow{\sim} \Pi^{\text{c-cn}}_{U_1} \times_{\Pi_X} \cdots \times_{\Pi_X} \Pi^{\text{c-cn}}_{U_n}.$$

PROOF. — Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from a similar argument to the argument applied in the proof of the final portion of [4], Proposition 1.6, (iii). □

2. MAXIMAL CUSPIDALLY ABELIAN QUOTIENTS

In the present §2, we discuss the *maximal cuspidally abelian quotients* "$\Pi^{\text{c-ab}}_{U_{X\times_k X}}$" [cf. Definition 2.1, (ii), below] of the étale fundamental groups of the second configuration spaces of proper hyperbolic curves. In the present §2, we maintain the notation of the preceding §1. Suppose, moreover, that X is *proper* over k [i.e., $r_X = 0$]. Write

$$\Pi_{X\times_k X} \stackrel{\text{def}}{=} \pi_1(X \times_k X) \quad \supseteq \quad \Delta_{X\times_k X} \stackrel{\text{def}}{=} \pi_1((X \times_k X) \otimes_k \overline{k})$$

for the respective étale fundamental groups of $X \times_k X$, $(X \times_k X) \otimes_k \overline{k}$. Then let us recall that the two projections $X \times_k X \to X$ determine an isomorphism of profinite groups

$$\Pi_{X\times_k X} \xrightarrow{\sim} \Pi_X \times_{G_k} \Pi_X,$$

which restricts to an isomorphism of profinite groups

$$\Delta_{X\times_k X} \xrightarrow{\sim} \Delta_X \times \Delta_X.$$

Let N be a $\mathfrak{Primes}^{\times/k}$-integer. Write

$$\Lambda_{X,N} \stackrel{\text{def}}{=} \Lambda_X / N\Lambda_X \;=\; \mathrm{Hom}_{\widehat{\mathbb{Z}}_{\times/k}}\big(H^2(\Delta_X, \widehat{\mathbb{Z}}_{\times/k}), \mathbb{Z}/N\mathbb{Z}\big)$$

[cf. Definition 1.4; Remark 1.4.1, (ii)].

DEFINITION 2.1.

(i) We shall write $U_{X\times_k X} \subseteq X \times_k X$ for the *second configuration space* of X, i.e., the open subscheme of $X \times_k X$ obtained by forming the complement of the diagonal divisor $X \subseteq X \times_k X$. Thus, the natural inclusion $U_{X\times_k X} \hookrightarrow X \times_k X$ determines a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1((U_{X\times_k X}) \otimes_k \overline{k}) & \longrightarrow & \pi_1(U_{X\times_k X}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_{X\times_k X} & \longrightarrow & \Pi_{X\times_k X} & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*.

(ii) We shall write

$$\Delta^{\text{c-}(\times/k)}_{U_{X\times_k X}} \quad \text{(respectively, } \Delta^{\text{c-ab}}_{U_{X\times_k X}}; \quad \Delta^{\text{c-cn}}_{U_{X\times_k X}})$$

for the maximal quotient of $\pi_1((U_{X\times_k X}) \otimes_k \overline{k})$ such that the left-hand vertical arrow of the diagram of (i) factors through the surjection $\pi_1((U_{X\times_k X}) \otimes_k \overline{k}) \twoheadrightarrow \Delta^{\text{c-}(\times/k)}_{U_{X\times_k X}}$ (respectively,

$\Delta^{\text{c-ab}}_{U_{X\times_k X}}$; $\Delta^{\text{c-cn}}_{U_{X\times_k X}}$), and, moreover, the kernel of the resulting surjection $\Delta^{\text{c-}(\times/k)}_{U_{X\times_k X}}$ (respectively, $\Delta^{\text{c-ab}}_{U_{X\times_k X}}$; $\Delta^{\text{c-cn}}_{U_{X\times_k X}}$) $\twoheadrightarrow \Delta_{X\times_k X}$ is pro-$\mathfrak{Primes}^{\times/k}$ (respectively, pro-$\mathfrak{Primes}^{\times/k}$ and abelian; pro-$\mathfrak{Primes}^{\times/k}$ and contained in the center of $\Delta^{\text{c-cn}}_{U_{X\times_k X}}$). We shall write

$$\Pi^{\text{c-}(\times/k)}_{U_{X\times_k X}} \quad (\text{respectively, } \Pi^{\text{c-ab}}_{U_{X\times_k X}}; \quad \Pi^{\text{c-cn}}_{U_{X\times_k X}})$$

for the quotient of $\pi_1(U_{X\times_k X})$ by the kernel of $\pi_1((U_{X\times_k X})\otimes_k \overline{k}) \twoheadrightarrow \Delta^{\text{c-}(\times/k)}_{U_{X\times_k X}}$ (respectively, $\Delta^{\text{c-ab}}_{U_{X\times_k X}}$; $\Delta^{\text{c-cn}}_{U_{X\times_k X}}$). Thus, the diagram of (i) determines a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^{\text{c-}(\times/k)}_{U_{X\times_k X}} & \longrightarrow & \Pi^{\text{c-}(\times/k)}_{U_{X\times_k X}} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^{\text{c-ab}}_{U_{X\times_k X}} & \longrightarrow & \Pi^{\text{c-ab}}_{U_{X\times_k X}} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^{\text{c-cn}}_{U_{X\times_k X}} & \longrightarrow & \Pi^{\text{c-cn}}_{U_{X\times_k X}} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_{X\times_k X} & \longrightarrow & \Pi_{X\times_k X} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*.

LEMMA 2.2. — *Let $x \in X(k)$ be a k-rational point. Write $U \stackrel{\text{def}}{=} X \setminus \{x\}$. Then the splitting [well-defined up to Δ_X-conjugation] $s\colon G_k \to \Pi_X$ induced by x, i.e., "$\mathfrak{S}_X(x)$", determines an isomorphism of profinite groups over G_k*

$$\Pi^{\text{c-cn}}_U \xrightarrow{\sim} \Pi^{\text{c-cn}}_{U_{X\times_k X}} \times_{\Pi_{X\times_k X}} (s(G_k) \times_{G_k} \Pi_X)$$

[*cf. Definition* 1.12].

PROOF. — This follows immediately from a similar argument to the argument applied in the proof of [4], Proposition 1.6, (iii), (iv). ☐

LEMMA 2.3. — *Suppose that Π_X is* **slim**, *and that k is* **p-cyclotomically full** *for every $p \in \mathfrak{Primes}^{\times/k}$ [i.e., the image of the p-adic cyclotomic character of G_k is* **open** *in $\mathbb{Z}_p^\times$ for every $p \in \mathfrak{Primes}^{\times/k}$]. Then an extension of an open subgroup of either Π_X or $\Pi_{X\times_k X}$ by a [possibly empty] finite product of copies of Λ_X is* **slim**.

PROOF. — Let us first observe that since the profinite group $\Pi_{X\times_k X} \xrightarrow{\sim} \Pi_X \times_{G_k} \Pi_X$ has a structure of extension of Π_X by Δ_X, and Δ_X is *slim*, it follows that $\Pi_{X\times_k X}$ is *slim*. In particular, since k is *p-cyclotomically full* for every $p \in \mathfrak{Primes}^{\times/k}$, it follows immediately from Remark 1.4.1, (i), that an extension of an open subgroup of either Π_X or $\Pi_{X\times_k X}$ by a finite product of copies of Λ_X is *slim*. This completes the proof of Lemma 2.3. ☐

REMARK 2.3.1. — It follows immediately from Proposition 1.5, (ii), together with a similar argument to the argument given in [6], Remark 1.5.1, that if k is *Kummer-faithful*, then the two assumptions in the statement of Lemma 2.3 are satisfied. In particular, in this situation, it follows from Lemma 2.3 that an extension of an open subgroup of either Π_X or $\Pi_{X\times_k X}$ by a [possibly empty] finite product of copies of Λ_X is *slim*.

DEFINITION 2.4. — Let $\Delta \subseteq \Delta_X$ be a *characteristic* open subgroup of Δ_X and $\Pi_Y \subseteq \Pi_X$ an open subgroup of Π_X such that $\Delta_Y = \Pi_Y \cap \Delta_X = \Delta$. Write $G_{k_Y} \subseteq G_k$ for the image of the composite $\Pi_Y \hookrightarrow \Pi_X \twoheadrightarrow G_k$. [Thus, the connected finite étale covering $Y \to X$ [corresponding to $\Pi_Y \subseteq \Pi_X$] is a *hyperbolic curve* over the finite extension k_Y of k [corresponding to $G_{k_Y} \subseteq G_k$].]

(i) By conjugation, we obtain an action $\Pi_X \to \mathrm{Aut}(\Delta)$, hence also a semi-direct product $\Delta \rtimes \Pi_X$, which fits into an exact sequence of profinite groups

$$1 \longrightarrow \Delta \rtimes \Delta_X \longrightarrow \Delta \rtimes \Pi_X \longrightarrow G_k \longrightarrow 1.$$

Observe that since Δ_X is *slim*, it follows that $\Delta \rtimes \Delta_X$ is *slim*.

(ii) By restricting the action $\Pi_X \to \mathrm{Aut}(\Delta)$ of (i) to $\Pi_Y \subseteq \Pi_X$, we obtain a semi-direct product $\Delta \rtimes \Pi_Y$. Then one verifies easily from the fact that Δ is *center-free* that the centralizer $Z_{\Delta\rtimes\Delta_Y}(\Delta) \subseteq \Delta \rtimes \Delta_Y$ determines a *splitting* of the exact sequence of profinite groups

$$1 \longrightarrow \Delta \longrightarrow \Delta \rtimes \Delta_Y \longrightarrow \Delta_Y \longrightarrow 1.$$

Thus, the natural surjections $\Delta \rtimes \Delta_Y \twoheadrightarrow (\Delta \rtimes \Delta_Y)/Z_{\Delta\rtimes\Delta_Y}(\Delta)$, Δ_Y determine isomorphisms of profinite groups

$$\Delta \rtimes \Delta_Y \xrightarrow{\sim} \big((\Delta \rtimes \Delta_Y)/Z_{\Delta\rtimes\Delta_Y}(\Delta)\big) \times \Delta_Y \xleftarrow{\sim} \Delta \times \Delta_Y,$$

which are *compatible* with the natural outer actions of G_{k_Y}. In particular, by considering "$(-) \overset{\mathrm{out}}{\rtimes} G_{k_Y}$" [cf. the *slimness* of $\Delta \rtimes \Delta_X$ discussed in (i)], we obtain an isomorphism of profinite groups

$$\Delta \rtimes \Pi_Y \xrightarrow{\sim} \Pi_Y \times_{G_{k_Y}} \Pi_Y \ (\xleftarrow{\sim} \Pi_{Y\times_{k_Y}Y}).$$

Observe that one verifies immediately from the various definitions involved that the natural splitting of the surjection $\Pi_{Y\times_{k_Y}Y} \xrightarrow{\sim} \Delta\rtimes\Pi_Y \twoheadrightarrow \Pi_Y$ arises from the *diagonal morphism* $Y \hookrightarrow Y \times_{k_Y} Y$.

(iii) We shall write $Z_\Delta \to X \times_k X$ for the connected finite étale covering corresponding to the open subgroup

$$\Pi_{Z_\Delta} \overset{\mathrm{def}}{=} \Delta \rtimes \Pi_X \subseteq \Delta_X \rtimes \Pi_X \xrightarrow{\sim} \Pi_{X\times_k X}$$

— where the "$\xrightarrow{\sim}$" is the isomorphism obtained in (ii). Thus, the exact sequence of (i) determines an exact sequence of profinite groups

$$1 \longrightarrow \Delta_{Z_\Delta} \overset{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{Z_\Delta} \twoheadrightarrow G_k) \longrightarrow \Pi_{Z_\Delta} \longrightarrow G_k \longrightarrow 1.$$

Observe that one verifies immediately from the various definitions involved that the surjection $\Pi_{Z_\Delta} = \Delta\rtimes\Pi_X \twoheadrightarrow \Pi_X$ arises from an "*isotrivial*" [cf. (ii)] *hyperbolic curve* $Z_\Delta \to X$ over X, and the natural splitting of the surjection $\Pi_{Z_\Delta} = \Delta \rtimes \Pi_X \twoheadrightarrow \Pi_X$ arises from a section $\iota_\Delta\colon X \hookrightarrow Z_\Delta$ — that lies over the *diagonal morphism* $X \hookrightarrow X \times_k X$ [cf. (ii)] — of this hyperbolic curve $Z_\Delta \to X$.

REMARK 2.4.1. — One verifies easily from the various definitions involved that, in the notation of Definition 2.4, if k is either an MLF or FF, then the finite étale covering $Z_\Delta \to X \times_k X$ of Definition 2.4, (iii), is the *diagonal covering associated to the covering* $Y \to X$ in the sense of [4], Definition 1.2, (i).

LEMMA 2.5. — *In the notation of Definition* 2.4, *the following hold:*

(i) *Let $i \neq 0$ be an integer, $G \in \{\Delta, \Pi_{Z_\Delta}\}$, and A a finite G-module annihilated by a $\mathfrak{Primes}^{\times/k}$-integer. Then*

$$\varinjlim_H H^i(H, A) \;=\; \{0\}$$

— where the injective limit is taken over the open subgroups $H \subseteq G$, and the transition morphisms in the limit are given by the restriction maps.

(ii) *Let $i \neq 2$ be an integer and A a finite module equipped with the trivial action of Δ that is annihilated by a $\mathfrak{Primes}^{\times/k}$-integer. Then*

$$\varprojlim_H H^i(H, A) \;=\; \{0\}$$

— where the projective limit is taken over the open subgroups $H \subseteq \Delta$, and the transition morphisms in the limit are given by the corestriction maps.

(iii) *Let i be an integer and A a finite Π_{Z_Δ}-module annihilated by a $\mathfrak{Primes}^{\times/k}$-integer. Then the natural homomorphism*

$$H^i(\Pi_{Z_\Delta}, A) \;\longrightarrow\; H^i(Z_\Delta, A)$$

is an **isomorphism**.

PROOF. — Assertion (i) follows immediately from a similar argument to the argument applied in the proof of [1], Lemma 4.2, (iii). Next, we verify assertion (ii). Let us recall [cf., e.g., [4], Proposition 1.3, (ii)] that the homomorphism

$$H^i(H, A) \;\longrightarrow\; \mathrm{Hom}_{\widehat{\mathbb{Z}}_{\times/k}}\big(H^{2-i}(H, \Lambda_X), A\big)$$

determined by the cup product in group cohomology and the natural isomorphism of Λ_X with "Λ_X" with respect to H [cf., e.g., [4], Remark 1] is an *isomorphism*. Thus, assertion (ii) follows immediately from assertion (i). This completes the proof of assertion (ii). Assertion (iii) is a formal consequence of assertion (i) [cf., e.g., the proof of [1], Lemma 4.2, (iii)]. This completes the proof of Lemma 2.5. ☐

LEMMA 2.6. — *In the notation of Definition* 2.4, *write*

$$E_2^{i,j}(\Delta) \;=\; H^i(\Pi_X, H^j(\Delta, \Lambda_{X,N})) \;\Longrightarrow\; E^{i+j}(\Delta) \;=\; H^{i+j}(\Pi_{Z_\Delta}, \Lambda_{X,N})$$

for the spectral sequence associated to the exact sequence of profinite groups

$$1 \;\longrightarrow\; \Delta \;\longrightarrow\; \Pi_{Z_\Delta}\,(= \Delta \rtimes \Pi_X) \;\longrightarrow\; \Pi_X \;\longrightarrow\; 1.$$

Then the following hold:

(i) *The natural homomorphism*

$$\varprojlim_{\Delta^\dagger} E_2^{0,2}(\Delta^\dagger) \longrightarrow E_2^{0,2}(\Delta) \quad (= H^0(\Pi_X, H^2(\Delta, \Lambda_{X,N})) = \mathbb{Z}/N\mathbb{Z})$$

— where the projective limit is taken over the characteristic open subgroups $\Delta^\dagger \subseteq \Delta_X$ *contained in* Δ*, and the transition morphisms in the limit are given by the corestriction maps — is an* **isomorphism***.*

(ii) *The natural homomorphism*

$$\varprojlim_{\Delta^\dagger} E^2(\Delta^\dagger) \longrightarrow \varprojlim_{\Delta^\dagger} E_2^{0,2}(\Delta^\dagger)$$

— where the projective limits are taken over the characteristic open subgroups $\Delta^\dagger \subseteq \Delta_X$ *contained in* Δ*, and the transition morphisms in the limits are given by the corestriction maps — is an* **isomorphism***.*

(iii) *The image of* $1 \in \mathbb{Z}/N\mathbb{Z}$ *via the composite*

$$\mathbb{Z}/N\mathbb{Z} = H^0(\Pi_X, H^2(\Delta, \Lambda_{X,N})) = E_2^{0,2}(\Delta) \xleftarrow{\sim} \varprojlim_{\Delta^\dagger} E_2^{0,2}(\Delta^\dagger)$$

$$\xleftarrow{\sim} \varprojlim_{\Delta^\dagger} E^2(\Delta^\dagger) \longrightarrow E^2(\Delta) = H^2(\Pi_{Z_\Delta}, \Lambda_{X,N}) \xrightarrow{\sim} H^2(Z_\Delta, \Lambda_{X,N})$$

— where the first "$\xleftarrow{\sim}$*" is the isomorphism of* (i)*, the second "*$\xleftarrow{\sim}$*" is the isomorphism of* (ii)*, and the "*$\xrightarrow{\sim}$*" is the isomorphism of Lemma* 2.5, (iii) *—* **coincides** *with the first Chern class* $c_1(\iota_\Delta(X))$ *of the divisor* $\iota_\Delta(X) \subseteq Z_\Delta$ *obtained by forming the scheme-theoretic image of the section* $\iota_\Delta \colon X \hookrightarrow Z_\Delta$ *of the hyperbolic curve* $Z_\Delta \to X$ *[cf. Definition* 2.4, (iii)*].*

PROOF. — First, we verify assertion (i). Let us recall [cf., e.g., [4], Proposition 1.3, (ii)] that the homomorphism

$$(E_2^{0,2}(\Delta^\dagger) =) \quad H^2(\Delta^\dagger, \Lambda_{X,N}) \longrightarrow \mathrm{Hom}_{\widehat{\mathbb{Z}}_{\times/k}}\big(H^0(\Delta^\dagger, \Lambda_X), \Lambda_{X,N}\big) \quad (= \mathbb{Z}/N\mathbb{Z})$$

determined by the cup product in group cohomology and the natural isomorphism of Λ_X with "Λ_X" with respect to $\Delta^\dagger$ [cf., e.g., [4], Remark 1] is an *isomorphism*. Thus, assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from Lemma 2.5, (ii). Assertion (iii) follows immediately from the [easily verified] fact that the image of the compatible system

$$(c_1(\iota_{\Delta^\dagger}(X)))_{\Delta^\dagger} \in \varprojlim_{\Delta^\dagger} E^2(\Delta^\dagger)$$

[cf. Lemma 2.5, (iii)] via the composite of natural homomorphisms

$$\varprojlim_{\Delta^\dagger} E^2(\Delta^\dagger) \longrightarrow \varprojlim_{\Delta^\dagger} E_2^{0,2}(\Delta^\dagger) \longrightarrow E_2^{0,2}(\Delta) = \mathbb{Z}/N\mathbb{Z}$$

coincides with $1 \in \mathbb{Z}/N\mathbb{Z}$. This completes the proof of Lemma 2.6. $\square$

Definition 2.7. — In the notation of Lemma 2.6:

(i) We shall refer to an extension of Π_{Z_Δ} by $\Lambda_{X,N}$

$$1 \longrightarrow \Lambda_{X,N} \longrightarrow E_{\Delta,N} \longrightarrow \Pi_{Z_\Delta} \longrightarrow 1$$

whose associated extension class is given by the image of $1 \in \mathbb{Z}/N\mathbb{Z}$ via the composite

$$\mathbb{Z}/N\mathbb{Z} \;=\; H^0(\Pi_X, H^2(\Delta, \Lambda_{X,N})) \;=\; E_2^{0,2}(\Delta) \;\overset{\sim}{\longleftarrow}\; \varprojlim_{\Delta^\dagger} E_2^{0,2}(\Delta^\dagger)$$

$$\overset{\sim}{\longleftarrow}\; \varprojlim_{\Delta^\dagger} E^2(\Delta^\dagger) \;\longrightarrow\; E^2(\Delta) \;=\; H^2(\Pi_{Z_\Delta}, \Lambda_{X,N})$$

— where the first "$\overset{\sim}{\leftarrow}$" is the isomorphism of Lemma 2.6, (i), and the second "$\overset{\sim}{\leftarrow}$" is the isomorphism of Lemma 2.6, (ii) — as a *mod N fundamental extension* of Π_{Z_Δ}.

(ii) We shall refer to the extension of Π_{Z_Δ} by Λ_X

$$1 \longrightarrow \Lambda_X \longrightarrow E_\Delta \overset{\mathrm{def}}{=} \varprojlim_N E_{\Delta,N} \longrightarrow \Pi_{Z_\Delta} \longrightarrow 1$$

— where the projective limit is taken over the $\mathfrak{Primes}^{\times/k}$-integers N — obtained by forming the projective limit of a compatible system of mod N fundamental extensions $\{E_{\Delta,N}\}_N$ [cf. (i)] as a *fundamental extension* of Π_{Z_Δ}. [Here, let us observe that the natural surjection $\Pi_{Z_\Delta} \twoheadrightarrow G_k$, together with the *Kummer theory*, determines an exact sequence

$$1 \longrightarrow k^\times/(k^\times)^N \longrightarrow H^1(\Pi_{Z_\Delta}, \Lambda_{X,N}) \longrightarrow H^1(\Delta_{Z_\Delta}, \Lambda_{X,N});$$

moreover, one verifies easily that $H^1(\Delta_{Z_\Delta}, \Lambda_{X,N})$ is *finite* — cf. Lemma 1.6, (v). In particular, for every N, there exists a $\mathfrak{Primes}^{\times/k}$-integer N_0 which is divisible by N such that, if N' is a $\mathfrak{Primes}^{\times/k}$-integer which is divisible by N_0, then the image of $H^1(\Pi_{Z_\Delta}, \Lambda_{X,N'}) \to H^1(\Pi_{Z_\Delta}, \Lambda_{X,N})$ *coincides* with the image of $H^1(\Pi_{Z_\Delta}, \Lambda_{X,N_0}) \to H^1(\Pi_{Z_\Delta}, \Lambda_{X,N})$.]

Remark 2.7.1. — One verifies easily from Lemma 2.6, (iii), together with the various definitions involved, that, in the notation of Definition 2.7, if k is either an MLF or FF, then the notion of a *fundamental extension* of Π_{Z_Δ} defined in Definition 2.7, (ii), *coincides* with the notion of a *fundamental extension* of Π_{Z_Δ} defined in [4], Definition 1.2, (i) [cf. also Remark 2.4.1].

Definition 2.8. — Suppose that k is *Kummer-faithful*. Let

$$\Delta^\ddagger \;\subseteq\; \Delta^\dagger \;\subseteq\; \Delta \;\subseteq\; \Delta_X$$

be *characteristic* open subgroups of Δ_X;

$$1 \longrightarrow \Lambda_X \longrightarrow E_{\Delta^\dagger} \longrightarrow \Pi_{Z_{\Delta^\dagger}} \longrightarrow 1,$$

$$1 \longrightarrow \Lambda_X \longrightarrow E_{\Delta^\ddagger} \longrightarrow \Pi_{Z_{\Delta^\ddagger}} \longrightarrow 1$$

respective fundamental extensions of $\Pi_{Z_{\Delta^\dagger}}$, $\Pi_{Z_{\Delta^\ddagger}}$ [cf. Definition 2.7, (ii)]; $s\colon G_k \to \Pi_X$ a splitting of the natural surjection $\Pi_X \twoheadrightarrow G_k$. Write

$$Y^\ddagger \longrightarrow Y^\dagger \longrightarrow Y \longrightarrow X$$

for the connected finite étale coverings corresponding to the open subgroups

$$\Pi_{Y^\ddagger} \stackrel{\text{def}}{=} \Delta^\ddagger \cdot \mathrm{Im}(s) \subseteq \Pi_{Y^\dagger} \stackrel{\text{def}}{=} \Delta^\dagger \cdot \mathrm{Im}(s) \subseteq \Pi_Y \stackrel{\text{def}}{=} \Delta \cdot \mathrm{Im}(s) \subseteq \Pi_X.$$

(i) By similar procedures to the procedures given in the discussion following [4], Proposition 1.6, in the case where we take "$(X'' \to X' \to X^* \to X, \mathcal{D}'')$" in the discussion following [4], Proposition 1.6, to be $(Y^\ddagger \to Y^\dagger \to Y \to X, E_{\Delta^\ddagger})$ [cf. Remarks 2.4.1, 2.7.1], together with Remark 2.3.1, one may define extensions

$$1 \longrightarrow \prod_{\Delta/\Delta^\ddagger} \Lambda_X \longrightarrow \mathbb{S}_{Y^\ddagger/Y}(E_{\Delta^\ddagger}) \longrightarrow \Pi_{Z_\Delta} \longrightarrow 1,$$

$$1 \longrightarrow \prod_{\Delta/\Delta^\dagger} \Lambda_X \longrightarrow \mathrm{Tr}_{Y^\ddagger/Y^\dagger:Y}(E_{\Delta^\ddagger}) \longrightarrow \Pi_{Z_\Delta} \longrightarrow 1$$

— in which Π_{Z_Δ} is only determined up to $\Delta^\ddagger \times \{1\}$-inner automorphisms. We shall refer to $\mathbb{S}_{Y^\ddagger/Y}(E_{\Delta^\ddagger})$ as the $[Y^\ddagger/Y$-$]$*symmetrized fundamental extension* [cf. [4], Definition 1.3, (i)] and to $\mathrm{Tr}_{Y^\ddagger/Y^\dagger:Y}(E_{\Delta^\ddagger})$ as the $[Y^\ddagger/Y^\dagger : Y$-$]$*trace-symmetrized fundamental extension* [cf. [4], Definition 1.3, (i)].

(ii) By a similar argument to the argument given in [4], Definition 1.3, (ii), together with similar results to the results given in [4], Proposition 1.7, (i), (ii), one may define the notion of a *morphism of trace type*

$$\mathbb{S}_{Y^\ddagger/X}(E_{\Delta^\ddagger}) \longrightarrow (\mathbb{S}_{Y^\dagger/X}(\mathrm{Tr}_{Y^\ddagger/Y^\dagger:Y^\dagger}(E_{\Delta^\ddagger})) \longrightarrow) \ \mathbb{S}_{Y^\dagger/X}(E_{\Delta^\dagger}).$$

Definition 2.9. — Suppose that k is *Kummer-faithful*. Let $s\colon G_k \to \Pi_X$ be a splitting of the natural surjection $\Pi_X \twoheadrightarrow G_k$ and

$$\cdots \subseteq \Delta_i \subseteq \cdots \subseteq \Delta_j \subseteq \cdots \subseteq \Delta_0 = \Delta_X$$

a system of *characteristic* open subgroups of Δ_X indexed by the nonnegative integers such that

$$\bigcap_{i\geq 0} \Delta_i = \{1\}.$$

[Note that since Δ_X is *topologically finitely generated* [by Lemma 1.6, (v)], such a system always exists.] Write

$$\cdots \to Y_i \to \cdots \to Y_j \to \cdots \to Y_0 = X$$

for the connected finite étale coverings corresponding to the open subgroups

$$\cdots \subseteq \Pi_{Y_i} \stackrel{\text{def}}{=} \Delta_i \cdot \mathrm{Im}(s) \subseteq \cdots \subseteq \Pi_{Y_j} \stackrel{\text{def}}{=} \Delta_j \cdot \mathrm{Im}(s) \subseteq \cdots \subseteq \Pi_X.$$

(i) We shall refer to a compatible system of morphisms of trace type [cf. Definition 2.8, (ii)], up to inner automorphisms of the appropriate type, between symmetrized fundamental extensions

$$\cdots \twoheadrightarrow \mathbb{S}_i \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{S}_j \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{S}_0 \ (\twoheadrightarrow \Pi_{X\times_k X})$$

— where $\mathbb{S}_i$ is the Y_i/X-symmetrized fundamental extension [cf. Definition 2.8, (i)] — as a *pro-symmetrized fundamental extension* of $\Pi_{X\times_k X}$ [cf. [4], Definition 1.3, (iii)]. In this situation, we shall refer to the profinite group

$$\mathbb{S}_\infty \stackrel{\mathrm{def}}{=} \varprojlim_{i\geq 0} \mathbb{S}_i \quad (\twoheadrightarrow \Pi_{X\times_k X})$$

as a *pro-fundamental extension* of $\Pi_{X\times_k X}$ [cf. [4], Definition 1.3, (iii)].

(ii) Let $\mathbb{S}_\infty$ be a pro-fundamental extension of $\Pi_{X\times_k X}$ [cf. (i)]. Then, by a similar argument to the argument given in [4], Definition 1.4, (ii), together with a similar result to the result given in [4], Proposition 1.8, (i), one may define the notion of a *pro-fundamental section*

$$\Pi_{U_{X\times_k X}} \longrightarrow \mathbb{S}_\infty.$$

PROPOSITION 2.10. — *Suppose that* k *is* **Kummer-faithful**, *and that the natural surjection* $\Pi_X \twoheadrightarrow G_k$ *has a splitting. Then the following hold:*

(i) *Let* $\mathbb{S}_\infty$ *be a* **pro-fundamental extension** *of* $\Pi_{X\times_k X}$ [*cf. Definition* 2.9, (i)]. *Then a* **pro-fundamental section** $\Pi_{U_{X\times_k X}} \to \mathbb{S}_\infty$ [*cf. Definition* 2.9, (ii)] *determines* **isomorphisms** *of profinite groups*

$$\Pi^{\text{c-ab}}_{U_{X\times_k X}} \xrightarrow{\sim} \mathbb{S}_\infty, \quad \Pi^{\text{c-cn}}_{U_{X\times_k X}} \xrightarrow{\sim} \mathbb{S}_0$$

— *where we write* $\mathbb{S}_0$ *for the* X/X*-symmetrized fundamental extension* [*cf. Definition* 2.8, (i)] [*i.e., a* **fundamental extension** *of* $\Pi_{Z_{\Delta_X}} = \Pi_{X\times_k X}$ — *cf. Definition* 2.7, (ii)] *appearing in the* **pro-symmetrized fundamental extension** *of* $\Pi_{X\times_k X}$ [*cf. Definition* 2.9, (i)] *that determines* $\mathbb{S}_\infty$.

(ii) *Let* $\mathfrak{D}_X \subseteq \Pi^{\text{c-ab}}_{U_{X\times_k X}}$ *be a decomposition subgroup associated to the diagonal divisor* $X \subseteq X \times_k X$ *such that the image of the composite* $\mathfrak{D}_X \hookrightarrow \Pi^{\text{c-ab}}_{U_{X\times_k X}} \twoheadrightarrow \Pi_{X\times_k X}$ *coincides with the image of the diagonal homomorphism* $\Pi_X \hookrightarrow \Pi_X \times_{G_k} \Pi_X \xleftarrow{\sim} \Pi_{X\times_k X}$. *Then if an automorphism* α *of* $\Pi^{\text{c-ab}}_{U_{X\times_k X}}$ *satisfies the following two conditions, then* α *is* $\mathrm{Ker}(\Delta^{\text{c-ab}}_{U_{X\times_k X}} \twoheadrightarrow \Delta_{X\times_k X})$*-inner:*

(a) α **preserves** *the quotient* $\Pi^{\text{c-ab}}_{U_{X\times_k X}} \twoheadrightarrow \Pi_{X\times_k X}$, *and, moreover, the resulting automorphism of* $\Pi_{X\times_k X}$ *is the* **identity automorphism**.

(b) α **preserves** *the inertia subgroup of* $\mathfrak{D}_X$.

(iii) *Let* $\{\mathbb{S}_i\}_i$ *be a* **pro-symmetrized fundamental extension** *of* $\Pi_{X\times_k X}$ *and* $x \in X(k)$ *a* k*-rational point. Write* $U \stackrel{\mathrm{def}}{=} X \setminus \{x\}$ *and* $s\colon G_k \to \Pi_X$ *for the splitting* [*well-defined up to* Δ_X*-conjugation*] *induced by* x, *i.e., "*$\mathfrak{S}_X(x)$*". Then the first isomorphism of* (i) *determines an* **isomorphism** *of profinite groups*

$$\Pi^{\text{c-ab}}_U \xrightarrow{\sim} \varprojlim_{i\geq 0} \big(\mathbb{S}_i \times_{\Pi_{X\times_k X}} (s(G_k) \times_{G_k} \Pi_X)\big)$$

[*cf. Definition* 1.12].

(iv) *In the notation of* (iii)*, let* $\mathfrak{D}_x \subseteq \Pi_U^{\text{c-ab}}$ *be a decomposition subgroup associated to* $x \in X(k)$*. Then if an automorphism* α *of* $\Pi_U^{\text{c-ab}}$ *satisfies the following two conditions, then* α *is* $\mathrm{Ker}(\Delta_U^{\text{c-ab}} \twoheadrightarrow \Delta_X)$*-inner:*

(a) α **preserves** *the quotient* $\Pi_U^{\text{c-ab}} \twoheadrightarrow \Pi_X$*, and, moreover, the resulting automorphism of* Π_X *is the* **identity automorphism***.*

(b) α **preserves** *the inertia subgroup of* $\mathfrak{D}_x$*.*

PROOF. — First, let us observe that if we regard Π_X as a closed subgroup of $\Pi_X \times_{G_k} \Pi_X$ by means of the diagonal homomorphism $\Pi_X \hookrightarrow \Pi_X \times_{G_k} \Pi_X$, then one verifies immediately that the set of $(\Pi_X \times_{G_k} \Pi_X)$-*conjugates* of Π_X *coincides* with the set of $(\Delta_X \times \{1\})$-*conjugates* of Π_X. Thus, assertion (i) (respectively, (iii)) follows immediately from a similar argument to the argument applied in the proof of [4], Proposition 1.9, (ii) (respectively, [4], Proposition 1.9, (i)). In particular, assertion (ii) (respectively, (iv)) follows immediately — in light of Lemma 2.11 below and a similar result to the result given as [4], Lemma 1.1 — from a similar argument to the argument applied in the proof of [4], Proposition 1.10, (i) (respectively, [4], Proposition 2.3, (i), together with Lemma 1.7, (i)). This completes the proof of Proposition 2.10. □

LEMMA 2.11. — *Suppose that* k *is* **Kummer-faithful***. Let* G *be an open subgroup of either* Π_X *or* $\Pi_{X\times_k X}$ *which* **surjects** *onto* G_k*. Then the natural surjection* $G \twoheadrightarrow G_k$ *determines an* **isomorphism**

$$H^1(G_k, \Lambda_X) \ \xrightarrow{\sim}\ H^1(G, \Lambda_X).$$

In particular, it holds that

$$\bigcap_n n \cdot H^1(G, \Lambda_X) \ = \ \{0\}$$

— where n *ranges over the* $\mathfrak{Primes}^{\times/k}$*-integers.*

PROOF. — The first portion of the statement follows immediately from a similar argument to the argument applied in the proof of Lemma 1.8, (i). The final portion of the statement follows immediately from the first portion of the statement, together with the *Kummer theory* [cf. Remark 1.4.1, (i)]. This completes the proof of Lemma 2.11. □

3. THE GROTHENDIECK CONJECTURE OVER KUMMER-FAITHFUL FIELDS

In the present §3, we discuss the [*semi-absolute version* of the] *Grothendieck conjecture for affine hyperbolic curves over Kummer-faithful fields.* In the present §3, let the symbol "$\square$" stand for either "$\circ$" or "$\bullet$". Let $k_\square$ be a *Kummer-faithful* field, $\overline{k}_\square$ an algebraic closure of $k_\square$, $X_\square$ a *hyperbolic curve* over $k_\square$, and

$$\Pi_{X_\square}$$

either the étale fundamental group $\pi_1(X_\square)$ of $X_\square$ or the tame fundamental group $\pi_1^{\mathrm{tame}}(X_\square^{\mathrm{cpt}}, D_{X_\square})$ of $(X_\square^{\mathrm{cpt}}, D_{X_\square})$. Write $G_{k_\square} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square)$ and

$$\Delta_{X_\square} \ \subseteq\ \Pi_{X_\square}$$

for the quotient of the étale fundamental group $\pi_1(X_\square \otimes_{k_\square} \overline{k}_\square) \subseteq \pi_1(X_\square)$ of $X_\square \otimes_{k_\square} \overline{k}_\square$ determined by $\Pi_{X_\square}$.

DEFINITION 3.1. — Let $\alpha\colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$ be an isomorphism of profinite groups.

(i) We shall say that α is *point-theoretic* if α determines a bijection between the set of decomposition subgroups of $\Pi_{X_\circ}$ associated to closed points of $X_\circ^{\mathrm{cpt}}$ and the set of decomposition subgroups of $\Pi_{X_\bullet}$ associated to closed points of $X_\bullet^{\mathrm{cpt}}$ [cf. Remark 3.1.1 below].

(ii) We shall say that α is *Galois-preserving* if α determines an isomorphism of profinite groups $\alpha_\Delta\colon \Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$. In particular, we obtain an isomorphism of profinite groups $\alpha_G\colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$.

We shall say that an outer isomorphism $\Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$ is *point-theoretic* (respectively, *Galois-preserving*) if it arises from a point-theoretic (respectively, Galois-preserving) isomorphism.

REMARK 3.1.1. — One verifies easily from [4], Proposition 2.2, (ii), that, in the notation of Definition 3.1, if k is either an MLF or FF, then α is *point-theoretic* in the sense of Definition 3.1, (i), if and only if α is *point-theoretic* in the sense of [4], Definition 1.5, (ii).

LEMMA 3.2. — *Let $\alpha\colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$ be a* **point-theoretic** *and* **Galois-preserving** *isomorphism of profinite groups. Then the following hold:*

(i) *The isomorphism α determines a bijection $\alpha^{\mathrm{cl}+}\colon X_\circ^{\mathrm{cl}+} \xrightarrow{\sim} X_\bullet^{\mathrm{cl}+}$ such that, for every $x_\circ \in X_\circ^{\mathrm{cl}+}$, if we write $x_\bullet \stackrel{\mathrm{def}}{=} \alpha^{\mathrm{cl}+}(x_\circ) \in X_\bullet^{\mathrm{cl}+}$, then the following conditions are satisfied:*

(a) *The diagram*

$$\begin{array}{ccc} X_\circ^{\mathrm{cl}+} & \xrightarrow{\alpha^{\mathrm{cl}+}} & X_\bullet^{\mathrm{cl}+} \\ {\scriptstyle \mathfrak{S}_{X_\circ}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{S}_{X_\bullet}} \\ \mathfrak{C}(\Pi_{X_\circ})/\Pi_{X_\circ} & \longrightarrow & \mathfrak{C}(\Pi_{X_\bullet})/\Pi_{X_\bullet} \end{array}$$

— *where the lower horizontal arrow is the bijection induced by α —* **commutes**.

(b) *It holds that $x_\circ$ lies on $X_\circ$ if and only if $x_\bullet$ lies on $X_\bullet$.*

(c) *If we write $\kappa(x_\circ)$, $\kappa(x_\bullet)$ for the residue fields at $x_\circ$, $x_\bullet$, respectively, then it holds that $[\kappa(x_\circ) : k_\circ] = [\kappa(x_\bullet) : k_\bullet]$.*

We shall write

$$\mathrm{Div}(\alpha)\colon\ \mathrm{Div}(X_\circ) \xrightarrow{\ \sim\ } \mathrm{Div}(X_\bullet)$$

[cf. Definition 1.1] for the isomorphism of groups determined by $\alpha^{\mathrm{cl}+}$.

(ii) *The isomorphism α determines an isomorphism of profinite groups*

$$\alpha^{\mathrm{cpt}}\colon \pi_1(X_\circ^{\mathrm{cpt}}) \xrightarrow{\sim} \pi_1(X_\bullet^{\mathrm{cpt}}),$$

which restricts to an isomorphism of profinite groups

$$\alpha_{\Delta}^{\mathrm{cpt}}\colon \pi_1(X_{\circ}^{\mathrm{cpt}} \otimes_{k_\circ} \overline{k}_\circ) \xrightarrow{\sim} \pi_1(X_{\bullet}^{\mathrm{cpt}} \otimes_{k_\bullet} \overline{k}_\bullet).$$

(iii) *It holds that* $(g_{X_\circ}, r_{X_\circ}, \mathrm{char}(k_\circ)) = (g_{X_\bullet}, r_{X_\bullet}, \mathrm{char}(k_\bullet))$.

(iv) *The isomorphism* α *determines an isomorphism* $\Lambda_\alpha\colon \Lambda_{X_\circ} \xrightarrow{\sim} \Lambda_{X_\bullet}$.

Proof. — First, we verify assertion (i). One verifies immediately from Lemma 1.7, (i), that there exists a [uniquely determined] bijection $\alpha^{\mathrm{cl+}}\colon X_\circ^{\mathrm{cl+}} \xrightarrow{\sim} X_\bullet^{\mathrm{cl+}}$ that satisfies condition (a). The assertion that $\alpha^{\mathrm{cl+}}$ satisfies condition (b) follows from the easily verified fact that, for $x_\Box \in X_\Box^{\mathrm{cl+}}$, it holds that $x_\Box$ lies on $X_\Box$ if and only if $\Delta_{X_\Box} \cap \mathfrak{S}_{X_\Box}(x_\Box) = \{1\}$, together with condition (a). The assertion that $\alpha^{\mathrm{cl+}}$ satisfies condition (c) follows from the easily verified fact that, for $x_\Box \in X_\Box^{\mathrm{cl+}}$, the index of the image of $\mathfrak{S}_{X_\Box}(x_\Box)$ in $G_{k_\Box}$ *coincides* with $[\kappa(x_\Box) : k_\Box]$, together with condition (a). This completes the proof of assertion (i).

Assertion (ii) follows immediately from Lemma 1.6, (i). Assertion (iii) follows immediately — in light of assertion (ii) — from Lemma 1.6, (ii), (iii), (iv). Assertion (iv) follows immediately — in light of the equality $\mathrm{char}(k_\circ) = \mathrm{char}(k_\bullet)$ in assertion (iii) — from conditions (a) and (b) of assertion (i), together with the definition of the cyclotome "$\Lambda_{(-)}$". This completes the proof of Lemma 3.2. □

Lemma 3.3. — *In the situation of Lemma* 3.2, *suppose, moreover, that the natural surjection* $\Pi_{X_\circ} \twoheadrightarrow G_{k_\circ}$ *has a splitting, and that* $X_\circ$ *is* **proper** *over* $k_\circ$. [*Thus, it follows from Lemma* 3.2, (iii), *that* $X_\bullet$ *is* **proper** *over* $k_\bullet$.] *Let* $S_\circ \subseteq X_\circ(k_\circ)$ *be a finite subset. Write* $S_\bullet \stackrel{\mathrm{def}}{=} \alpha^{\mathrm{cl+}}(S_\circ) \subseteq X_\bullet(k_\bullet)$ [*cf. condition* (c) *of Lemma* 3.2, (i)], $U_\circ \stackrel{\mathrm{def}}{=} X_\circ \setminus S_\circ$, *and* $U_\bullet \stackrel{\mathrm{def}}{=} X_\bullet \setminus S_\bullet$. *Then the following hold:*

(i) *Let* m *be a positive integer;* $x_\circ^1, \dots, x_\circ^m \in X_\circ(k_\circ)$; $n_1, \dots, n_m \in \mathbb{Z}$. *Then the divisor* $D_\circ \stackrel{\mathrm{def}}{=} \sum_{i=1}^m n_i \cdot x_\circ^i \in \mathrm{Div}(X_\circ)$ *is* **principal** *if and only if the divisor* $\mathrm{Div}(\alpha)(D_\circ) \in \mathrm{Div}(X_\bullet)$ [*cf. Lemma* 3.2, (i)] *is* **principal**.

(ii) *The isomorphism* α *determines an isomorphism* $\alpha^{\text{c-ab}}\colon \Pi^{\text{c-ab}}_{U_{X_\circ \times_{k_\circ} X_\circ}} \xrightarrow{\sim} \Pi^{\text{c-ab}}_{U_{X_\bullet \times_{k_\bullet} X_\bullet}}$ [*cf. Definition* 2.1, (ii)] [*well-defined up to* $\mathrm{Ker}(\Pi^{\text{c-ab}}_{U_{X_\bullet \times_{k_\bullet} X_\bullet}} \twoheadrightarrow \Pi_{X_\bullet \times_{k_\bullet} X_\bullet})$*-inner automorphisms*] *such that the diagram*

$$\begin{array}{ccc} \Pi^{\text{c-ab}}_{U_{X_\circ \times_{k_\circ} X_\circ}} & \xrightarrow{\alpha^{\text{c-ab}}} & \Pi^{\text{c-ab}}_{U_{X_\bullet \times_{k_\bullet} X_\bullet}} \\ \downarrow & & \downarrow \\ \Pi_{X_\circ \times_{k_\circ} X_\circ} & \longrightarrow & \Pi_{X_\bullet \times_{k_\bullet} X_\bullet} \end{array}$$

— *where the lower horizontal arrow is an isomorphism induced by* α *and the natural isomorphism* $\Pi_{X_\Box \times_{k_\Box} X_\Box} \xrightarrow{\sim} \Pi_{X_\Box} \times_{G_{k_\Box}} \Pi_{X_\Box}$ — **commutes**, *and, moreover,* $\alpha^{\text{c-ab}}$ *maps a decomposition subgroup of* $\Pi^{\text{c-ab}}_{U_{X_\circ \times_{k_\circ} X_\circ}}$ *associated to the diagonal divisor of* $X_\circ \times_{k_\circ} X_\circ$ *to a decomposition subgroup of* $\Pi^{\text{c-ab}}_{U_{X_\bullet \times_{k_\bullet} X_\bullet}}$ *associated to the diagonal divisor of* $X_\bullet \times_{k_\bullet} X_\bullet$.

(iii) *The isomorphism* α *determines an isomorphism* $\alpha^{\text{c-cn}}_{S_\circ}\colon \Pi^{\text{c-cn}}_{U_\circ} \xrightarrow{\sim} \Pi^{\text{c-cn}}_{U_\bullet}$ [*cf. Definition* 1.12] [*well-defined up to* $\mathrm{Ker}(\Pi^{\text{c-cn}}_{U_\bullet} \twoheadrightarrow \Pi_{X_\bullet})$*-inner automorphisms*] *such that the* *diagram*

$$\begin{array}{ccc} \Pi_{U_\circ}^{\text{c-cn}} & \xrightarrow{\alpha_{S_\circ}^{\text{c-cn}}} & \Pi_{U_\bullet}^{\text{c-cn}} \\ \downarrow & & \downarrow \\ \Pi_{X_\circ} & \xrightarrow{\alpha} & \Pi_{X_\bullet} \end{array}$$

commutes, *and, moreover,* $\alpha_{S_\circ}^{\text{c-cn}}$ *maps the [uniquely determined] inertia subgroup of* $\Pi_{U_\circ}^{\text{c-cn}}$ *associated to* $x_\circ \in S_\circ$ *to the [uniquely determined] inertia subgroup of* $\Pi_{U_\bullet}^{\text{c-cn}}$ *associated to* $\alpha^{\text{cl}+}(x_\circ) \in S_\bullet$.

(iv) *The isomorphism* $H^1(\Pi_{U_\circ}^{\text{c-cn}}, \Lambda_{X_\circ}) \xrightarrow{\sim} H^1(\Pi_{U_\bullet}^{\text{c-cn}}, \Lambda_{X_\bullet})$ *determined by the isomorphisms* $\alpha_{S_\circ}^{\text{c-cn}}$ *of* (iii) *and* Λ_α *of Lemma* 3.2, (iv), *determines — relative to the isomorphism of Lemma* 1.13, (i) *— an isomorphism*

$$\mathcal{O}^\times(\alpha, S_\circ) \colon \ \mathcal{O}^\times(\Pi_{X_\circ}, S_\circ) \xrightarrow{\sim} \mathcal{O}^\times(\Pi_{X_\bullet}, S_\bullet)$$

[*cf. Definition* 1.9] *such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(G_{k_\circ}, \Lambda_{X_\circ}) & \longrightarrow & \mathcal{O}^\times(\Pi_{X_\circ}, S_\circ) & \longrightarrow & \mathcal{P}(X_\circ, S_\circ) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathcal{O}^\times(\alpha, S_\circ)} & & \downarrow & & \\ 0 & \longrightarrow & H^1(G_{k_\bullet}, \Lambda_{X_\bullet}) & \longrightarrow & \mathcal{O}^\times(\Pi_{X_\bullet}, S_\bullet) & \longrightarrow & \mathcal{P}(X_\bullet, S_\bullet) & \longrightarrow & 0 \end{array}$$

— where the horizontal sequences are the lower exact sequence of the diagram of Lemma 1.10, (ii) [*in the case where we take "X" in Lemma* 1.10 *to be* $X_\circ$, $X_\bullet$]*; the left-hand vertical arrow is the isomorphism induced by the isomorphisms* α_G *of Definition* 3.1, (ii), *and* Λ_α *of Lemma* 3.2, (iv)*; the right-hand vertical arrow is the isomorphism determined by the isomorphism* $\text{Div}(\alpha)$ *of Lemma* 3.2, (i) *—* **commutes**.

(v) *The various isomorphisms "*$\mathcal{O}^\times(\alpha, S_\circ)$*" of* (iv) *determine an isomorphism of abelian groups*

$$\mathcal{K}^\times(\alpha) \colon \ \mathcal{K}^\times(\Pi_{X_\circ}) \xrightarrow{\sim} \mathcal{K}^\times(\Pi_{X_\bullet})$$

[*cf. Definition* 1.9].

PROOF. — Assertion (i) follows immediately from Lemma 1.7, (ii), together with conditions (a) and (c) of Lemma 3.2, (i). Assertion (ii) follows immediately — in light of Proposition 2.10, (i); Lemma 2.11 — from a similar argument to the argument applied in the proof of [4], Theorem 1.1, (iii), together with similar results to the results given as [4], Lemma 1.1; [4], Proposition 1.8. Assertion (iii) follows immediately — in light of condition (a) of Lemma 3.2, (i) — from assertion (ii), together with Lemma 1.13, (ii); Lemma 2.2. Assertion (iv) follows immediately from assertion (i), together with the various definitions involved. Assertion (v) follows immediately from the various definitions involved. This completes the proof of Lemma 3.3. □

THEOREM 3.4. — *Let the symbol "*$\square$*" stand for either "*$\circ$*" or "*$\bullet$*". Let* $k_\square$ *be a* **Kummer-faithful** *field [cf. Definition* 1.2] *and* $X_\square$ *a* **hyperbolic curve** *over* $k_\square$. *Write* $X_\square^{\text{cpt}}$ *for the smooth compactification of* $X_\square$ *and* $D_{X_\square} \subseteq X_\square^{\text{cpt}}$ *for the divisor at infinity of* $X_\square$. *Let*

$$\Pi_{X_\square}$$

be either the étale fundamental group $\pi_1(X_\Box)$ *of* $X_\Box$ *or the tame fundamental group* $\pi_1^{\mathrm{tame}}(X_\Box^{\mathrm{cpt}}, D_{X_\Box})$ *of* $(X_\Box^{\mathrm{cpt}}, D_{X_\Box})$. *Write*

$$\mathrm{Isom}(\Pi_{X_\circ}, \Pi_{X_\bullet})$$

for the set of isomorphisms of profinite groups $\Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$,

$$\mathrm{Isom}_{\mathrm{PG}}(\Pi_{X_\circ}, \Pi_{X_\bullet}) \subseteq \mathrm{Isom}(\Pi_{X_\circ}, \Pi_{X_\bullet})$$

for the subset of **point-theoretic** [*cf. Definition* 3.1, (i)] *and* **Galois-preserving** [*cf. Definition* 3.1, (ii)] *isomorphisms of profinite groups* $\Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$, *and*

$$\mathrm{Isom}(X_\circ, X_\bullet)$$

for the set of isomorphisms of schemes $X_\circ \xrightarrow{\sim} X_\bullet$. *Then the following hold:*

(i) *Suppose that* $\mathrm{Isom}_{\mathrm{PG}}(\Pi_{X_\circ}, \Pi_{X_\bullet})$ *is* **nonempty**. *Then it holds that* $\Pi_{X_\circ} = \pi_1(X_\circ)$ *if and only if* $\Pi_{X_\bullet} = \pi_1(X_\bullet)$.

(ii) *Suppose, moreover, that either* $X_\circ$ *or* $X_\bullet$ *is* **affine**. *Then the natural map*

$$\mathrm{Isom}(X_\circ, X_\bullet) \longrightarrow \mathrm{Isom}(\Pi_{X_\circ}, \Pi_{X_\bullet})/\Pi_{X_\bullet}$$

[*cf.* (i)] *determines a* **bijection**

$$\mathrm{Isom}(X_\circ, X_\bullet) \xrightarrow{\sim} \mathrm{Isom}_{\mathrm{PG}}(\Pi_{X_\circ}, \Pi_{X_\bullet})/\Pi_{X_\bullet}.$$

PROOF. — Assertion (i) follows immediately from Lemma 1.6, (v). Next, we verify assertion (ii). First, let us observe that it follows immediately from [7], Lemma 4.2, that the natural map

$$\mathrm{Isom}(X_\circ, X_\bullet) \longrightarrow \mathrm{Isom}(\Pi_{X_\circ}, \Pi_{X_\bullet})/\Pi_{X_\bullet}$$

factors through the subset $\mathrm{Isom}_{\mathrm{PG}}(\Pi_{X_\circ}, \Pi_{X_\bullet})/\Pi_{X_\bullet} \subseteq \mathrm{Isom}(\Pi_{X_\circ}, \Pi_{X_\bullet})/\Pi_{X_\bullet}$. Next, let us observe that the *injectivity* of the map under consideration follows immediately from Lemma 3.2, (i), together with Lemma 3.5 below. Thus, to complete the verification of assertion (ii), it suffices to verify the *surjectivity* of the map under consideration. To this end, let $\alpha\colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$ be a *point-theoretic* and *Galois-preserving* isomorphism of profinite groups.

Next, let us observe that it follows immediately from the *injectivity* of the map under consideration that we may assume without loss of generality, by replacing $\Pi_{X_\circ}$ by a suitable open subgroup of $\Pi_{X_\circ}$ if necessary, that $g_{X_\circ} \geq 2$, hence also $g_{X_\bullet} \geq 2$ [cf. Lemma 3.2, (iii)], and, moreover, $r_{X_\circ} \geq 3$, hence also $r_{X_\bullet} \geq 3$ [cf. Lemma 3.2, (iii)]. Next, again by the *injectivity* of the map under consideration, we may assume without loss of generality, by replacing $G_{k_\circ}$ by a suitable open subgroup of $G_{k_\circ}$ if necessary, that $D_{X_\circ}(\overline{k}_\circ) = D_{X_\circ}(k_\circ)$, hence also $D_{X_\bullet}(\overline{k}_\bullet) = D_{X_\bullet}(k_\bullet)$ [cf. conditions (b), (c) of Lemma 3.2, (i)], and that the natural surjection $\Pi_{X_\circ} \twoheadrightarrow G_{k_\circ}$, hence also the natural surjection $\Pi_{X_\bullet} \twoheadrightarrow G_{k_\bullet}$, has a splitting.

Let $S_\circ \subseteq X_\circ^{\mathrm{cpt}}(k_\circ)$ be a finite subset such that $D_{X_\circ}(k_\circ) \not\subseteq S_\circ$ and $x_\circ \in D_{X_\circ}(k_\circ) \setminus (D_{X_\circ}(k_\circ) \cap S_\circ)$. Write $S_\bullet \stackrel{\mathrm{def}}{=} \alpha^{\mathrm{cl}+}(S_\circ) \subseteq X_\bullet^{\mathrm{cpt}}(k_\bullet)$ [cf. condition (c) of Lemma 3.2, (i)], $x_\bullet \stackrel{\mathrm{def}}{=} \alpha^{\mathrm{cl}+}(x_\circ) \in D_{X_\bullet}(k_\bullet) \setminus (D_{X_\bullet}(k_\bullet) \cap S_\bullet)$ [cf. conditions (b), (c) of Lemma 3.2, (i)], $U_\circ \stackrel{\mathrm{def}}{=} X_\circ^{\mathrm{cpt}} \setminus S_\circ$, $U_\bullet \stackrel{\mathrm{def}}{=} X_\bullet^{\mathrm{cpt}} \setminus S_\bullet$, and

$$\alpha^{\mathrm{cpt}}\colon\ \Pi_{X_\circ^{\mathrm{cpt}}} \xrightarrow{\sim} \Pi_{X_\bullet^{\mathrm{cpt}}}$$

for the isomorphism of profinite groups obtained in Lemma 3.2, (ii). Let us observe that one verifies immediately from the various definitions involved that α^{cpt} is *point-theoretic* and *Galois-preserving*. Now I claim that the following assertion holds:

Claim 3.4.A: The diagram

$$\begin{array}{ccc} \mathcal{O}^{\times}(\Pi_{X_\circ^{\mathrm{cpt}}}, S_\circ) & \xrightarrow{\mathcal{O}^{\times}(\alpha^{\mathrm{cpt}}, S_\circ)} & \mathcal{O}^{\times}(\Pi_{X_\bullet^{\mathrm{cpt}}}, S_\bullet) \\ {\scriptstyle \mathrm{ev}_{x_\circ}(\Pi_{X_\circ^{\mathrm{cpt}}}, S_\circ)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_{x_\bullet}(\Pi_{X_\bullet^{\mathrm{cpt}}}, S_\bullet)} \\ H^1(G_{k_\circ}, \Lambda_{X_\circ}) & \longrightarrow & H^1(G_{k_\bullet}, \Lambda_{X_\bullet}) \end{array}$$

— where the upper horizontal arrow is the isomorphism obtained in Lemma 3.3, (iv); the lower horizontal arrow is the isomorphism induced by the isomorphisms α_G of Definition 3.1, (ii), and Λ_α of Lemma 3.2, (iv) — *commutes*.

Indeed, since [we have assumed that] $x_\circ$, hence also $x_\bullet$, is a *cusp* of the given hyperbolic curve, this follows immediately — in light of Proposition 2.10, (ii), (iv) — from a similar argument to the argument applied in [4], Remarks 15, 21. This completes the proof of Claim 3.4.A.

Since [we have assumed that] $\sharp D_{X_\circ}(k_\circ) = r_{X_\circ}$, $\sharp D_{X_\bullet}(k_\bullet) = r_{X_\bullet} \geq 3 \geq 2$, by applying Claim 3.4.A to the various isomorphisms "$\mathcal{O}^{\times}(\alpha, S_\circ)$" of Lemma 3.3, (iv), we conclude from Lemma 1.10, (iv), that the isomorphism $\mathcal{K}^{\times}(\alpha^{\mathrm{cpt}})\colon \mathcal{K}^{\times}(\Pi_{X_\circ^{\mathrm{cpt}}}) \xrightarrow{\sim} \mathcal{K}^{\times}(\Pi_{X_\bullet^{\mathrm{cpt}}})$ of Lemma 3.3, (v), determines an isomorphism of abelian groups

$$K_\alpha^{\times}\colon K^{\times}_{X_\circ \otimes_{k_\circ} \overline{k}_\circ} \xrightarrow{\sim} K^{\times}_{X_\bullet \otimes_{k_\bullet} \overline{k}_\bullet}$$

[relative to the injections $K^{\times}_{X_\circ \otimes_{k_\circ} \overline{k}_\circ} \hookrightarrow \mathcal{K}^{\times}(\Pi_{X_\circ^{\mathrm{cpt}}})$, $K^{\times}_{X_\bullet \otimes_{k_\bullet} \overline{k}_\bullet} \hookrightarrow \mathcal{K}^{\times}(\Pi_{X_\bullet^{\mathrm{cpt}}})$ of Lemma 1.10, (i), in the case where we take "X" in Lemma 1.10 to be $X_\circ^{\mathrm{cpt}}$, $X_\bullet^{\mathrm{cpt}}$], which restricts to an isomorphism of abelian groups

$$\overline{k}_\circ^{\times} \xrightarrow{\sim} \overline{k}_\bullet^{\times}.$$

Moreover, since [we have assumed that] $\sharp D_{X_\circ}(k_\circ) = r_{X_\circ}$, $\sharp D_{X_\bullet}(k_\bullet) = r_{X_\bullet} \geq 3$, it follows immediately — in light of Claim 3.4.A and the *commutativity* of the right-hand square of the diagram of Lemma 3.3, (iv) — from [7], Lemma 4.7, that the bijection

$$K_{X_\circ \otimes_{k_\circ} \overline{k}_\circ} = K^{\times}_{X_\circ \otimes_{k_\circ} \overline{k}_\circ} \cup \{0\} \xrightarrow{\sim} K_{X_\bullet \otimes_{k_\bullet} \overline{k}_\bullet} = K^{\times}_{X_\bullet \otimes_{k_\bullet} \overline{k}_\bullet} \cup \{0\}$$

induced by $K_\alpha^{\times}$ is an *isomorphism of fields*, which restricts to an *isomorphism of fields*

$$\overline{k}_\circ = \overline{k}_\circ^{\times} \cup \{0\} \xrightarrow{\sim} \overline{k}_\bullet = \overline{k}_\bullet^{\times} \cup \{0\}.$$

Thus, by considering the $\Pi_{X_\circ}$-, $\Pi_{X_\bullet}$-invariants, we obtain a commutative diagram of schemes

$$\begin{array}{ccc} X_\circ^{\mathrm{cpt}} & \xrightarrow{\sim} & X_\bullet^{\mathrm{cpt}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(k_\circ) & \xrightarrow{\sim} & \mathrm{Spec}(k_\bullet) \end{array}$$

— where the horizontal arrows are *isomorphisms*. Now let us observe that it follows immediately from our construction of the above diagram that the bijection $X_\circ^{\mathrm{cl}+} \xrightarrow{\sim} X_\bullet^{\mathrm{cl}+}$

induced by the upper horizontal arrow *coincides* with the bijection $\alpha^{\mathrm{cl}+}$ of Lemma 3.2, (i). Thus, it follows from condition (b) of Lemma 3.2, (i), that the upper horizontal arrow of the above diagram determines an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$. The assertion that the outer isomorphism $\Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$ induced by this isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ *coincides* with the outer isomorphism determined by α follows immediately from a similar argument to the argument given in the discussion preceding [3], Theorem 14.1. This completes the proof of assertion (ii), hence also of Theorem 3.4. □

It seems to the author that the following result is likely to be well known. However, the result could not be found in the literature.

LEMMA 3.5. — *Let C be a proper smooth curve over a field F and ϕ an automorphism of an abstract scheme C [i.e., not necessarily over F]. Suppose that ϕ induces the* **identity automorphism** *on the underlying set of C. Then ϕ is the* **identity automorphism** *of C.*

PROOF. — By abuse of notation, we shall write ϕ for the automorphism of the function field K of C induced by ϕ. Let $x \in C$ be a closed point of C and $f \in K^\times$. Then it follows from our assumption that ϕ preserves the local ring $\mathcal{O}_x \stackrel{\mathrm{def}}{=} \mathcal{O}_{C,x} \subseteq K$ at x, as well as the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_x$ of $\mathcal{O}_x$. Thus, one verifies immediately that ϕ induces the *identity automorphism* on the quotient $K^\times/\mathcal{O}_x^\times$ ($\simeq \mathbb{Z}$) of the group $K^\times$. In particular, it follows that there exists an element $\delta_f \in F^\times$ such that $\phi(f) = \delta_f \cdot f$.

Suppose that $f - 1 \in \mathfrak{m}_x$. Thus, it holds that $\delta_f \cdot f - 1 = \phi(f) - 1 = \phi(f-1) \in \mathfrak{m}_x$. Moreover, since $F \cdot \mathfrak{m}_x \subseteq \mathfrak{m}_x$, it holds that $\delta_f \cdot (f-1) \in \mathfrak{m}_x$. In particular, it follows that $\delta_f - 1 = (\delta_f \cdot f - 1) - \delta_f \cdot (f-1) \in \mathfrak{m}_x \cap F = \{0\}$, which thus implies that ϕ induces the *identity automorphism* on $\mathfrak{m}_x$. Thus, since [one verifies easily that] the field K is *generated* by the $\mathfrak{m}_x$'s, where x ranges over the closed points of C, we conclude that ϕ is the *identity automorphism* of C. This completes the proof of Lemma 3.5. □

REFERENCES

[1] Y. Hoshi, On the fundamental groups of log configuration schemes, *Math. J. Okayama Univ.* **51** (2009), 1–26.

[2] Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: inertia groups and profinite Dehn twists, *Galois-Teichmüller theory and arithmetic geometry*, 659–811, Adv. Stud. Pure Math., **63**, Math. Soc. Japan, Tokyo, 2012.

[3] S. Mochizuki, The local pro-p anabelian geometry of curves, *Invent. Math.* **138** (1999), no. **2**, 319–423.

[4] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), no. **3**, 451–539.

[5] S. Mochizuki, Topics in absolute anabelian geometry I: generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), no. **2**, 139–242.

[6] S. Mochizuki, *Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms*, RIMS Preprint **1626** (March 2008).

[7] A. Tamagawa, The Grothendieck conjecture for affine curves, *Compositio Math.* **109** (1997), no. **2**, 135–194.

(Yuichiro Hoshi) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: yuichiro@kurims.kyoto-u.ac.jp