RIMS-1811

Nilpotent Admissible Indigenous Bundles via Cartier Operators in Characteristic Three

By

Yuichiro HOSHI

October 2014

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Nilpotent Admissible Indigenous Bundles via Cartier Operators in Characteristic Three

Yuichiro Hoshi

October 2014

Abstract. — In the present paper, we study the *p-adic Teichmüller theory* in the case where $p = 3$. In particular, we discuss *nilpotent admissible/ordinary* indigenous bundles over a projective smooth curve in characteristic three. The main result of the present paper is a characterization of the *supersingular divisors* of nilpotent admissible/ordinary indigenous bundles in characteristic three by means of various *Cartier operators*. By means of this characterization, we prove that, for every nilpotent *ordinary* indigenous bundle over a projective smooth curve in characteristic three, there exists a connected finite étale covering of the curve on which the indigenous bundle is *not ordinary*. We also prove that every projective smooth curve *of genus two* in characteristic three is *hyperbolically ordinary*. These two applications yield *negative*, *positive partial* answers to *basic questions* in the *p-adic Teichmüller theory*, respectively.

Contents

2010 Mathematics Subject Classification. — 14G17.
Key words and phrases. — p-adic Teichmüller theory, nilpotent admissible indigenous bundle, nilpotent ordinary indigenous bundle, Cartier operator.
This research was supported by Grant-in-Aid for Scientific Research (C), No. 24540016, Japan Society for the Promotion of Science.

INTRODUCTION

In the present paper, we study the *p-adic Teichmüller theory* established by S. Mochizuki [cf. [5], [6]] in the case where $p = 3$. In particular, we discuss *nilpotent admissible/ordinary* indigenous bundles over a projective smooth curve in characteristic three. In the Introduction, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbb{F}_p$], and $f\colon X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g over S. Write $f^F\colon X^F \to S$ for the projective smooth curve over S obtained by base-changing $X \to S$ via the absolute Frobenius morphism of S and $\Phi\colon X \to X^F$ for the relative Frobenius morphism over S. We use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

First, let us recall the notion of an *indigenous bundle* and some properties on an indigenous bundle. We shall say that a pair

$$(\pi\colon P \to X, \nabla_P)$$

consisting of a $\mathbb{P}^1$-bundle $\pi\colon P \to X$ over X and a connection ∇_P on P relative to X/S is an *indigenous bundle* over X/S if there exists a [uniquely determined — cf. [5], Chapter I, Proposition 2.4] section [i.e., the *Hodge section*] $\sigma\colon X \to P$ of $\pi\colon P \to X$ such that the Kodaira-Spencer homomorphism $\sigma^*\omega_{P/X} \to \omega_{X/S}$ at σ relative to ∇_P [i.e., the homomorphism obtained by differentiating σ by means of ∇_P] is an isomorphism [cf. [5], Chapter I, Definition 2.2]. The notion of an indigenous bundle was introduced and studied by R. C. Gunning [cf. [2], §2] and enables one to understand the theory of uniformization of [algebraic] Riemann surfaces in a somewhat more algebraic setting.

Let $(\pi\colon P \to X, \nabla_P)$ be an indigenous bundle over X/S. Then the connection ∇_P on P determines a *horizontal* homomorphism [i.e., the *p-curvature*]

$$\mathcal{P}\colon\ \Phi^*\tau_{X^F/S} \longrightarrow \mathcal{A}d(P) \overset{\text{def}}{=} \pi_*\tau_{P/X}.$$

We shall say that the indigenous bundle $(\pi\colon P \to X, \nabla_P)$ is *nilpotent* (respectively, *admissible*; *dormant*) if the square of $\mathcal{P}$ is 0 (respectively, the zero locus of $\mathcal{P}$ is empty; $\mathcal{P} = 0$) [cf. [5], Chapter II, Definition 2.4 (respectively, [5], Chapter II, Definition 2.4; [6], Chapter II, Definition 1.1)]. Moreover, we shall refer to the composite of the p-curvature $\mathcal{P}$ and the surjection $\mathcal{A}d(P) \twoheadrightarrow \tau_{X/S}$ determined by the Hodge section of $(\pi\colon P \to X, \nabla_P)$ as the *square Hasse invariant* of $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter II, Proposition 2.6, (1)]. Then, by means of this *square Hasse invariant*, one may define the *Frobenius on* $\mathbb{R}^1 f_*\tau_{X/S}$ *induced by* $(\pi\colon P \to X, \nabla_P)$ [cf. the discussion following [5], Chapter II, Lemma 2.11]. We shall say that the indigenous bundle $(\pi\colon P \to X, \nabla_P)$ is *ordinary* if the Frobenius on $\mathbb{R}^1 f_*\tau_{X/S}$ induced by $(\pi\colon P \to X, \nabla_P)$ is an isomorphism [cf. [5], Chapter II, Definition 3.1]. A *nilpotent admissible/ordinary* indigenous bundle plays a central role in the "*classical*" *p-adic Teichmüller theory*, i.e., the *p-adic Teichmüller theory* discussed in [not [6] but] [5].

The first main result of the present paper is the following *uniqueness* of a *dormant* indigenous bundle in characteristic three [cf. Theorem 2.1, Corollary 2.6]:

THEOREM A. — *In the notation introduced at the beginning of the Introduction, suppose that $p = 3$. Then there exists a* **unique dormant** *indigenous bundle over X/S. In particular, there exists a natural* **bijection** *between*

- $H^0(S, f_*\omega_{X/S}^{\otimes 2}) = H^0(X, \omega_{X/S}^{\otimes 2})$ *and*
- *the set of isomorphism classes of indigenous bundles over* X/S

such that, for $\theta \in H^0(S, f_*\omega_{X/S}^{\otimes 2})$, *the* **dormant locus** *in* S *of the indigenous bundle over* X/S *corresponding to* θ **coincides** *with the zero locus in* S *of* θ.

If an indigenous bundle $(\pi\colon P \to X, \nabla_P)$ over X/S is *nilpotent admissible*, then there exist an invertible sheaf $\mathcal{H}$ on X and a global section χ of $\mathcal{H}$ such that $\mathcal{H}^{\otimes 2} \cong \mathcal{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S})$, and, moreover, the square of χ coincides with the square Hasse invariant of $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter II, Proposition 2.6, (3)]. We shall refer to χ as the *Hasse invariant* of $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter II, Proposition 2.6, (3)] and to the zero locus of the Hasse invariant as the *supersingular divisor* of $(\pi\colon P \to X, \nabla_P)$ [cf. [5], Chapter II, Proposition 2.6, (3)]. The *supersingular divisor* is an important invariant of a nilpotent admissible indigenous bundle; for instance, if S is *reduced*, then the isomorphism class of a nilpotent admissible indigenous bundle over X/S is *completely determined* by the *supersingular divisor* [cf. [5], Chapter II, Proposition 2.6, (4)]. The main result of the present paper is a characterization of the *supersingular divisors* of nilpotent admissible/ordinary indigenous bundles in characteristic three by means of various *Cartier operators*.

In order to present the main result of the present paper, let us recall some notions related to the *Cartier operator*. Let $(\mathcal{L}, \tau)$ be a square-trivialized invertible sheaf on X, i.e., a pair consisting of an invertible sheaf $\mathcal{L}$ on X and a trivialization τ of the square of $\mathcal{L}$ [cf. Definition A.3]. Then the [usual] Cartier operator $\Phi_*\omega_{X/S} \to \omega_{X^F/S}$, together with the trivialization τ, determines a homomorphism of $\mathcal{O}_S$-modules

$$C_{(\mathcal{L},\tau)}\colon\ f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

— where we write $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathcal{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S. We shall refer to this homomorphism as the *Cartier operator associated to* $(\mathcal{L}, \tau)$ [cf. Definition A.4]. On the other hand, the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S determines a *Frobenius-semi-linear* homomorphism

$$f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

For a global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$, we shall write u^F for the global section of $\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}$ obtained by forming the image of u via this Frobenius-semi-linear homomorphism. We shall say that a global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *normalized Cartier eigenform* associated to $(\mathcal{L}, \tau)$ if u defines a relative effective Cartier divisor of X/S, and, moreover, $C_{(\mathcal{L},\tau)}(u) = -u^F$ [cf. Definition A.8, (i)].

A part of the main result of the present paper is as follows [cf. Theorem 5.2, (ii)]:

Theorem B. — *In the notation introduced at the beginning of the Introduction, suppose that* $p = 3$. *Let* D *be a relative effective Cartier divisor of* X/S. *Then it holds that* D *is the* **supersingular divisor** *of a* **nilpotent admissible** (*respectively,* **nilpotent ordinary**) *indigenous bundle over* X/S *if and only if* D *is* **of CE-type** (*respectively,* **of CEO-type**) [*cf. Definition* 5.1, (iii)], *i.e., there exist an invertible sheaf* $\mathcal{L}$ *on* X, *a trivialization* τ *of the square of* $\mathcal{L}$, *and a global section* χ *of* $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ *such that* *the following two (respectively, three) conditions* (1), (2) (*respectively,* (1), (2), (3)) *are satisfied:*

(1) *The divisor D is* **étale** *over S and coincides with the zero locus of $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$.*

(2) *The global section $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a* **normalized Cartier eigenform** *associated to $(\mathcal{L}, \tau)$.*

(3) *The invertible sheaf $\mathcal{L}$ is* **parabolically ordinary** [*cf. Definition* A.7]*, i.e., the Cartier operator associated to $(\mathcal{L}, \tau)$ is* **injective***, or, equivalently* [*cf. Proposition* A.6]*, one of the following two conditions is satisfied:*

- $\mathcal{L}$ *is* **of relative order one** [*cf. Definition* A.2]*, and, moreover, X is* **parabolically ordinary** [*cf. Definition* A.5, (i)].
- $\mathcal{L}$ *is* **of relative order two** [*cf. Definition* A.2]*, and, moreover, the connected finite étale double covering of X which trivializes $\mathcal{L}$* [*determined by τ*] *is* **parabolically new-ordinary** [*cf. Definition* A.5, (ii)].

Here, let us recall the following two *basic questions* in the *p-adic Teichmüller theory* discussed in [6], Introduction, §2.1 [cf. [6], Introduction, §2.1, (1), (2)]:

(1) Is every pointed stable curve [of type (g, r), where $2g-2+r>0$] *hyperbolically ordinary*? That is to say, does every pointed stable curve [of type (g, r), where $2g-2+r>0$] admit, étale locally on S, a *nilpotent ordinary* indigenous bundle?

(2) Let P be a *nilpotent ordinary* indigenous bundle over a pointed stable curve X [of type (g, r), where $2g-2+r>0$] and $Y \to X$ a connected finite [log] étale covering of X. Then is the pull-back of P to Y still *ordinary*?

As a corollary of Theorem B, we obtain the following theorem, which yields a *negative answer* to the above basic question (2) [cf. Corollary 5.4]:

THEOREM C. — *Let X be a projective smooth curve of genus ≥ 2 over an algebraically closed field k of characteristic* 3*. Then, for every* **nilpotent ordinary** *indigenous bundle P over X/k, there exists a connected finite étale covering $Y \to X$ of X such that the* [*necessarily* **nilpotent admissible**] *indigenous bundle $(Y \to X)^*P$ over Y/k is* **not ordinary**.

In §6, we give, by applying the results obtained in the present paper, a complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over a projective smooth curve *of genus two* over an algebraically closed field of characteristic three [cf. Theorem 6.1]. Moreover, we prove the following theorem, which yields a *positive partial answer* to the above basic question (1) [cf. Corollary 6.6, Remark 6.6.1]:

THEOREM D. — *Every projective smooth curve of genus two over a connected noetherian scheme of characteristic three is* **hyperbolically ordinary** [*cf.* [5]*, Chapter II, Definition 3.3*].

1. CONSTRUCTION OF A DORMANT INDIGENOUS BUNDLE

In the present §1, we construct a *dormant indigenous bundle* over a projective smooth curve of genus ≥ 2 of characteristic 3 [cf. Proposition 1.1 below]. In the present §1, let $g \geq 2$ be an integer, S a connected noetherian scheme *of characteristic* 3 [i.e., over $\mathbb{F}_3$], and $f \colon X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g over S. Write $f^F \colon X^F \to S$ for the projective smooth curve over S obtained by base-changing $X \to S$ via the absolute Frobenius morphism of S, $\Phi \colon X \to X^F$ for the relative Frobenius morphism over S, $\mathcal{I} \subseteq \mathcal{O}_{X \times_S X}$ for the ideal of $\mathcal{O}_{X \times_S X}$ which defines the diagonal morphism with respect to X/S, and $X_{(n)} \subseteq X \times_S X$ for the closed subscheme of $X \times_S X$ defined by the ideal $\mathcal{I}^{n+1} \subseteq \mathcal{O}_{X \times_S X}$ [where n is a nonnegative integer]. In particular, it follows that $\mathcal{I}/\mathcal{I}^2 = \omega_{X/S}$ (respectively, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \tau_{X/S}$), where we use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

We shall write

$$\mathcal{B}_\circ \stackrel{\text{def}}{=} \operatorname{Coker}(\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X)$$

for the $\mathcal{O}_{X^F}$-module obtained by forming the cokernel of the natural homomorphism $\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X$ and

$$\mathcal{E}_\circ \stackrel{\text{def}}{=} \Phi^* \mathcal{B}_\circ.$$

Since the homomorphism $\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X$ admits a natural splitting after pulling back via Φ, which thus determines a natural isomorphism of $\mathcal{O}_X$-modules

$$\Phi^* \Phi_* \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X \oplus \mathcal{E}_\circ,$$

and Φ is *finite flat of degree* 3, it follows that $\mathcal{B}_\circ$, hence also $\mathcal{E}_\circ$, is *locally free of rank* 2. We shall write

$$\pi_\circ \colon P_\circ \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{E}_\circ) \longrightarrow X$$

for the $\mathbb{P}^1$*-bundle over* X *associated to* $\mathcal{E}_\circ$.

Next, let us observe that one verifies immediately that the natural morphism

$$X \times_{X^F} X \longrightarrow X \times_S X$$

determines an isomorphism

$$X \times_{X^F} X \xrightarrow{\sim} X_{(2)}.$$

In particular, the closed immersion $X_{(1)} \hookrightarrow X \times_S X$ determines a closed immersion $X_{(1)} \hookrightarrow X \times_{X^F} X$. Thus, it follows that the $\mathcal{O}_X$-module $\mathcal{E}_\circ$, hence also the $\mathbb{P}^1$-bundle $P_\circ$, on X admits a *natural connection* relative to X/S. We shall write

$$\nabla_{\mathcal{E}_\circ}, \quad \nabla_{P_\circ}$$

for the respective natural connections on $\mathcal{E}_\circ$, $P_\circ$. [So one verifies immediately that the connection $\nabla_{\mathcal{E}_\circ}$ *coincides* with the connection on $\mathcal{E}_\circ = \Phi^* \mathcal{B}_\circ$ determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$.] Moreover, the above isomorphism $X \times_{X^F} X \xrightarrow{\sim} X_{(2)}$, together with the cartesian diagram

$$\begin{array}{ccc} X \times_{X^F} X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \Phi} \\ X & \xrightarrow[\Phi]{} & X^F, \end{array}$$

determines *isomorphisms* of $\mathcal{O}_X$-modules

$$\Phi^*\Phi_*\mathcal{O}_X \xrightarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X\times_{X^F}X} \xleftarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X_{(2)}},$$

which are *compatible* with the respective natural surjections onto $\mathcal{O}_X$ [arising from the diagonal morphism with respect to X/X^F] from each of these three modules. In particular, by forming the kernels of the respective natural surjections onto $\mathcal{O}_X$, we obtain *isomorphisms* of $\mathcal{O}_X$-modules

$$\mathcal{E}_\circ \xrightarrow{\sim} \mathrm{Ker}(\mathrm{pr}_{1*}\mathcal{O}_{X\times_{X^F}X} \twoheadrightarrow \mathcal{O}_X) \xleftarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3).$$

We shall write

$$\sigma_\circ\colon X \longrightarrow P_\circ$$

for the section of $\pi_\circ\colon P_\circ \to X$ determined by the composite $\mathcal{E}_\circ \twoheadrightarrow \omega_{X/S}$ of the above isomorphism $\mathcal{E}_\circ \xrightarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3)$ and the natural surjection $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \twoheadrightarrow \mathcal{I}/\mathcal{I}^2 = \omega_{X/S}$. Then one verifies easily that the *Kodaira-Spencer homomorphism* $\sigma_\circ^*\omega_{P_\circ/X} \to \omega_{X/S}$ at $\sigma_\circ$ relative to $\nabla_{P_\circ}$ [i.e., the homomorphism obtained by differentiating $\sigma_\circ$ by means of $\nabla_{P_\circ}$] is an *isomorphism*. Thus, it follows immediately from our construction that the following proposition holds:

PROPOSITION 1.1. — *The pair* $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ *is an* **indigenous bundle** [*cf.* [5], *Chapter I, Definition 2.2*] *over* X/S *whose* **Hodge section** [*cf.* [5], *Chapter I, Proposition 2.4*] *is given by* $\sigma_\circ$. *Moreover, the indigenous bundle* $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ *is* **dormant** [*cf.* [6], *Chapter II, Definition 1.1*].

In the remainder of the present §1, let us consider the invertible sheaves

$$\det(\mathcal{E}_\circ), \quad \det(\mathcal{B}_\circ), \quad \det(\Phi_*\omega_{X/S}).$$

Write $\mathcal{M} \overset{\mathrm{def}}{=} \mathcal{H}om_{\mathcal{O}_{X^F}}(\det(\mathcal{B}_\circ), \omega_{X^F/S})$. First, let us observe that since the $\mathcal{O}_X$-module $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \cong \mathcal{E}_\circ = \Phi^*\mathcal{B}_\circ$ fits into an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \longrightarrow \omega_{X/S} \longrightarrow 0,$$

it follows that

$$\det(\mathcal{E}_\circ) \cong \omega_{X/S}^{\otimes 3},$$

hence also

$$\Phi^*\mathcal{M} \cong \mathcal{O}_X.$$

Next, let us recall from the discussion preceding [8], Théorème 4.1.1, that the map

$$\begin{array}{ccc} \Phi_*\mathcal{O}_X \times \Phi_*\mathcal{O}_X & \longrightarrow & \omega_{X^F/S} \\ (f,g) & \mapsto & c(f\cdot\Phi_*d(g)) \end{array}$$

— where we write $d\colon \mathcal{O}_X \to \omega_{X/S}$ for the exterior differentiation operator and $c\colon \Phi_*\omega_{X/S} \to \omega_{X^F/S}$ for the *Cartier operator* — determines an *isomorphism* of $\mathcal{O}_{X^F}$-modules

$$\mathcal{B}_\circ \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{B}_\circ, \omega_{X^F/S}),$$

which thus implies that

$$\mathcal{M}^{\otimes 2} \cong \mathcal{O}_{X^F}.$$

Thus, we obtain:

LEMMA 1.2. — *It holds that*

$$\det(\mathcal{E}_\circ) \cong \omega_{X/S}^{\otimes 3}, \quad \det(\mathcal{B}_\circ) \cong \omega_{X^F/S}, \quad \det(\Phi_* \omega_{X/S}) \cong \omega_{X^F/S}^{\otimes 2}.$$

PROOF. — The first "$\cong$" has already been verified. Since the homomorphism between the relative Jacobian varieties of X^F/S, X/S induced by Φ is *finite flat of degree* 3^g, it follows from the fact that $\Phi^*\mathcal{M} \cong \mathcal{O}_X$, $\mathcal{M}^{\otimes 2} \cong \mathcal{O}_{X^F}$ verified above that $\mathcal{M}$ lies in $(f^F)^*\mathrm{Pic}(S)$. Thus, again by the fact that $\Phi^*\mathcal{M} \cong \mathcal{O}_X$, the second "$\cong$" follows. The third "$\cong$" follows from the second "$\cong$", together with the *well-known* exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

[cf., e.g., [4], Theorem 7.2]. □

2. THE DORMANT TRIVIALIZATION OF THE SCHWARZ TORSOR

In the present §2, we maintain the notation of the preceding §1. In particular, we have a projective smooth curve $f\colon X \to S$ and a *dormant indigenous bundle* $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ over X/S [cf. Proposition 1.1]. In the present §2, we prove the following theorem:

THEOREM 2.1. — *Every* **dormant** *indigenous bundle over X/S is* **isomorphic** *to the dormant indigenous bundle $(\pi_\circ\colon P_\circ \to X, \nabla_{P_\circ})$ of Proposition* 1.1.

REMARK 2.1.1. — It follows from [10], Corollary 5.4, that if $p > 2g-2$, then the number of isomorphism classes of *dormant indigenous bundles* over a "*sufficiently general*" projective smooth curve of genus g over an algebraically closed field of characteristic p is equal to

$$\frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{i=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi \cdot i}{p})} = \frac{(-p)^{g-1}}{2} \cdot \sum_{\zeta^p=1,\, \zeta\neq 1} \frac{\zeta^{g-1}}{(\zeta-1)^{2g-2}}.$$

On the other hand, one verifies easily that the above quantity in the case where $p = 3$ is always *equal to* 1. Thus, it follows from Theorem 2.1 that the formula of [10], Corollary 5.4, is *valid* for $p = 3$ *without the assumption that* $p > 2g-2$.

To verify Theorem 2.1, let us first recall some facts on the *p-adic Teichmüller theory* [cf. [5], [6]]. Write

$$\mathcal{M}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic 3 and

$$\mathcal{N}_g[\infty]$$

for the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *dormant indigenous bundles*. Then the natural (1-)morphism

$$\mathcal{N}_g[\infty] \longrightarrow \mathcal{M}_g$$

is *finite* and *faithfully flat*; moreover, there exists a *dense open* substack of $\mathcal{M}_g$ on which this (1-)morphism is *étale* [cf. the final portion of [6], Chapter II, Theorem 2.8]. Moreover, as is well-known, there exists a *dense open* substack of $\mathcal{M}_g$ on which the associated relative Jacobian variety is *ordinary*. Thus, to complete the verification of Theorem 2.1, we may assume without loss of generality, by considering a geometric point on the intersection of these two *dense open* substacks [i.e., a geometric point on the — necessarily *dense open* — substack of $\mathcal{M}_g$ on which the above (1-)morphism is *étale*, and, moreover, the associated relative Jacobian variety is *ordinary*], that

- S is the spectrum of an algebraically closed field [of characteristic 3], and that
- the Jacobian variety of X is *ordinary* [i.e., that X is *parabolically ordinary* — cf. Definition A.5, (i)].

To complete the verification of Theorem 2.1, let

$$(\pi\colon P \to X, \nabla_P)$$

be a *dormant indigenous bundle* over X/S. Let us first observe that it follows from [5], Chapter I, Proposition 2.5, that there exists an isomorphism over X

$$P \ \cong\ P_\circ \ =\ \mathbb{P}(\mathcal{E}_\circ).$$

By means of such an isomorphism, let us identify P with $P_\circ = \mathbb{P}(\mathcal{E}_\circ)$. Next, let us observe that since $(\pi\colon P \to X, \nabla_P)$ is *dormant*, one verifies immediately [cf. also [4], Theorem 5.1] that, by considering *horizontal* local sections of $\pi\colon P \to X$ with respect to ∇_P, we obtain a $\mathbb{P}^1$-bundle

$$\pi_Q\colon\ Q \ \longrightarrow\ X^F$$

over X^F and an isomorphism $P \cong \Phi^*Q$ over X relative to which a local section of π is *horizontal* [with respect to ∇_P] if and only if the local section arises from a local section of π_Q.

Next, let us observe that since S is the spectrum of an algebraically closed field, there exist invertible sheaves $\mathcal{L}_Q$, $\mathcal{L}_{X^F}$ on Q, X^F such that $\mathcal{L}_Q^{\otimes 2} \cong \tau_{Q/X^F}$, $\mathcal{L}_{X^F}^{\otimes 2} \cong \omega_{X^F/S}$, respectively. Thus, one verifies easily that Q is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(\pi_{Q*}\mathcal{L}_Q)$, hence also the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{L}_{X^F} \otimes_{\mathcal{O}_{X^F}} \pi_{Q*}\mathcal{L}_Q)$, over X^F. In particular, since $P = P_\circ$ is isomorphic to Φ^*Q, it follows that there exists an invertible sheaf $\mathcal{M}_X$ on X such that $\mathcal{E}_\circ \otimes_{\mathcal{O}_X} \mathcal{M}_X \cong \Phi^*(\mathcal{L}_{X^F} \otimes_{\mathcal{O}_{X^F}} \pi_{Q*}\mathcal{L}_Q)$. Next, let us observe that, by considering the following *well-known* exact sequence of $\mathcal{O}_Q$-modules

$$0 \ \longrightarrow\ \omega_{Q/X^F} \ \longrightarrow\ \mathcal{H}om_{\mathcal{O}_Q}(\mathcal{L}_Q, \pi_Q^*\pi_{Q*}\mathcal{L}_Q) \ \longrightarrow\ \mathcal{O}_Q \ \longrightarrow\ 0,$$

we obtain that $\pi_Q^* \det(\pi_{Q*}\mathcal{L}_Q) \cong \mathcal{O}_Q$, which thus implies that $\det(\pi_{Q*}\mathcal{L}_Q) \cong \mathcal{O}_{X^F}$. Thus, by considering the determinant of $\mathcal{E}_\circ \otimes_{\mathcal{O}_X} \mathcal{M}_X \cong \Phi^*(\mathcal{L}_{X^F} \otimes_{\mathcal{O}_{X^F}} \pi_{Q*}\mathcal{L}_Q)$, we obtain from Lemma 1.2 that $\mathcal{M}_X^{\otimes 2} \cong \mathcal{O}_X$. Thus, in summary, we obtain:

LEMMA 2.2. — *There exists a locally free coherent $\mathcal{O}_{X^F}$-module $\mathcal{B}_Q$ of rank 2 on X^F which satisfies the following conditions:*

(0) *There exists an isomorphism of Q with the $\mathbb{P}^1$-bundle associated to $\mathcal{B}_Q$ over X^F; moreover, the connection ∇_P coincides with the connection on $P \cong \Phi^*Q$ induced by the connection on $\Phi^*\mathcal{B}_Q$ determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$.*

(1) *There exists a(n) [not necessarily horizontal] isomorphism $\mathcal{E}_\circ \cong \Phi^*\mathcal{B}_Q$ of $\mathcal{O}_X$-modules.*

(2) $\det(\mathcal{B}_Q) \cong \omega_{X^F/S}$.

PROOF. — Since $\mathcal{M}_X^{\otimes 2} \cong \mathcal{O}_X$, S is the spectrum of an algebraically closed field, and the homomorphism between the Jacobian varieties of X^F, X induced by Φ is *finite flat of degree* 3^g, one verifies easily that there exists an invertible sheaf $\mathcal{M}_{X^F}$ on X^F such that $\mathcal{M}_{X^F}^{\otimes 2} \cong \mathcal{O}_{X^F}$ and $\Phi^*\mathcal{M}_{X^F} \cong \mathcal{M}_X$. Then it follows immediately from the above discussion that the $\mathcal{O}_{X^F}$-module

$$\mathcal{B}_Q \stackrel{\text{def}}{=} \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{M}_{X^F}, \mathcal{L}_{X^F} \otimes_{\mathcal{O}_{X^F}} \pi_{Q*}\mathcal{L}_Q)$$

satisfies the conditions in the statement of Lemma 2.2. $\square$

Next, let us observe that the surjection

$$\Phi^*\mathcal{B}_Q \;\cong\; \mathcal{E}_\circ \;\cong\; \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \;\twoheadrightarrow\; \omega_{X/S}$$

[cf. Lemma 2.2, (1)] determines a homomorphism of $\mathcal{O}_{X^F}$-modules

$$\alpha\colon\; \mathcal{B}_Q \;\longrightarrow\; \Phi_*\omega_{X/S},$$

hence, by pulling back via Φ, also a homomorphism of $\mathcal{O}_X$-modules

$$\Phi^*\alpha\colon\; \Phi^*\mathcal{B}_Q \;\longrightarrow\; \Phi^*\Phi_*\omega_{X/S}$$

which is *horizontal* [with respect to the connections determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$].

LEMMA 2.3. — *The homomorphism $\Phi^*\alpha$, hence also the homomorphism α, is a* **locally split injection**.

PROOF. — Let us first recall that we have a cartesian diagram

$$\begin{array}{ccc} X \times_{X^F} X \;\cong\; X_{(2)} & \xrightarrow{\mathrm{pr}_2} & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \Phi} \\ X & \xrightarrow[\Phi]{} & X^F \end{array}$$

[cf. the discussion preceding Proposition 1.1]. Thus, the closed subschemes $X = X_{(0)} \subseteq X_{(1)} \subseteq X_{(2)}$ determine a filtration

$$\{0\} \;=\; \mathcal{F}_0 \;\subseteq\; \mathcal{F}_1 \;\subseteq\; \mathcal{F}_2 \;\subseteq\; \mathcal{F}_3 \;=\; \Phi^*\Phi_*\omega_{X/S}$$

such that, for $i \in \{1, 2, 3\}$,

$$\mathcal{F}_i/\mathcal{F}_{i-1} \;\cong\; \omega_{X/S}^{\otimes 4-i}.$$

Moreover, one verifies immediately that the connection on $\Phi^*\Phi_*\omega_{X/S}$ determines an *isomorphism* of invertible sheaves on X

$$\mathcal{F}_2/\mathcal{F}_1 \;\xrightarrow{\sim}\; (\mathcal{F}_3/\mathcal{F}_2) \otimes_{\mathcal{O}_X} \omega_{X/S}.$$

Thus, since

$$\det(\Phi^*\mathcal{B}_Q) \;\cong\; \det(\mathcal{F}_3/\mathcal{F}_1) \;\cong\; \omega_{X/S}^{\otimes 3}$$

[cf. Lemma 2.2, (2)], to complete the verification of Lemma 2.3, it suffices to verify that the composite

$$\Phi^*\mathcal{B}_Q \;\stackrel{\Phi^*\alpha}{\to}\; \Phi^*\Phi_*\omega_{X/S} \;=\; \mathcal{F}_3 \;\twoheadrightarrow\; \mathcal{F}_3/\mathcal{F}_1$$

is *injective*.

Next, let us observe that it follows from the definition of $\Phi^*\alpha$ that the composite of $\Phi^*\alpha$ and the natural homomorphism $\Phi^*\Phi_*\omega_{X/S} \to \omega_{X/S}$ [i.e., the surjection $\Phi^*\Phi_*\omega_{X/S} = \mathcal{F}_3 \twoheadrightarrow \mathcal{F}_3/\mathcal{F}_2 \cong \omega_{X/S}$] determines an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \Phi^*\mathcal{B}_Q \quad (\cong \mathcal{E}_\circ) \longrightarrow \omega_{X/S} \longrightarrow 0,$$

which gives rise to the *Hodge section* $\sigma_\circ$ of $(\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ})$ [cf. Proposition 1.1], hence also of the indigenous bundle $(\pi \colon P \to X, \nabla_P)$ [cf. [5], Chapter I, Proposition 2.4]. Thus, to complete the verification of Lemma 2.3, it suffices to verify the *injectivity* of the homomorphism

$$\omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{F}_2/\mathcal{F}_1$$

induced by the composite $\Phi^*\mathcal{B}_Q \to \mathcal{F}_3/\mathcal{F}_1$ under consideration, or, equivalently, the *injectivity* of the composite

$$\omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{F}_2/\mathcal{F}_1 \xrightarrow{\sim} (\mathcal{F}_3/\mathcal{F}_2) \otimes_{\mathcal{O}_X} \omega_{X/S}$$

with the isomorphism discussed above. On the other hand, since $\Phi^*\alpha$ is *horizontal*, one verifies immediately that this composite *coincides* with the composite

$$\omega_{X/S}^{\otimes 2} \hookrightarrow \Phi^*\mathcal{B}_Q \to \Phi^*\mathcal{B}_Q \otimes_{\mathcal{O}_X} \omega_{X/S} \twoheadrightarrow \omega_{X/S} \otimes_{\mathcal{O}_X} \omega_{X/S} \cong (\mathcal{F}_3/\mathcal{F}_2) \otimes_{\mathcal{O}_X} \omega_{X/S}$$

— where the second arrow is the connection on $\Phi^*\mathcal{B}_Q$. In particular, if the composite $\omega_{X/S}^{\otimes 2} \to (\mathcal{F}_3/\mathcal{F}_2) \otimes_{\mathcal{O}_X} \omega_{X/S}$ under consideration is *not injective*, then it follows that the invertible subsheaf $\omega_{X/S}^{\otimes 2} \subseteq \Phi^*\mathcal{B}_Q$ is *preserved* by the connection on $\Phi^*\mathcal{B}_Q$, which thus implies [cf. Lemma 2.2, (0)] that the *Hodge section* $\sigma_\circ$ of the indigenous bundle $(\pi \colon P \to X, \nabla_P)$ is *horizontal* with respect to ∇_P; thus, we obtain a *contradiction*. This completes the proof of Lemma 2.3. $\square$

We conclude from Lemma 2.3 that the cokernel $\mathrm{Coker}(\alpha)$ of α is an *invertible sheaf* on X^F which is isomorphic to

$$\mathcal{H}om_{\mathcal{O}_{X^F}}(\det(\mathcal{B}_Q), \det(\Phi_*\omega_{X/S})).$$

In particular, it follows from Lemma 1.2; Lemma 2.2, (2), that we have an exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{B}_Q \xrightarrow{\alpha} \Phi_*\omega_{X/S} \longrightarrow \omega_{X^F/S} \longrightarrow 0.$$

Let us prove Theorem 2.1.

PROOF OF THEOREM 2.1. — Let us recall the *well-known* exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{B}_\circ \longrightarrow \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

— cf. the exact sequence of $\mathcal{O}_{X^F}$-modules which appears in the proof of Lemma 1.2. If $\mathcal{B}_\circ = \mathrm{Im}(\alpha)$, then it follows from Lemma 2.2, (0), that the dormant indigenous bundle $(\pi \colon P \to X, \nabla_P)$ is *isomorphic* to the dormant indigenous bundle $(\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ})$ of Proposition 1.1. Thus, assume that $\mathcal{B}_\circ \neq \mathrm{Im}(\alpha)$. Then since $\det(\mathcal{B}_\circ) \cong \det(\mathcal{B}_Q) \cong \omega_{X^F/S}$ [cf. Lemma 1.2; Lemma 2.2, (2)], it holds that $\mathcal{B}_\circ \not\subseteq \mathrm{Im}(\alpha)$, which thus implies that the composite

$$\mathcal{B}_\circ = \mathrm{Ker}(c) \hookrightarrow \Phi_*\omega_{X/S} \twoheadrightarrow \Phi_*\omega_{X/S}/\mathrm{Im}(\alpha) \cong \omega_{X^F/S}$$

is *nonzero.* In particular, since we have an isomorphism $\mathcal{B}_\circ \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{B}_\circ, \omega_{X^F/S})$ of $\mathcal{O}_{X^F}$-modules [cf. the discussion preceding Lemma 1.2], we conclude that the $\mathcal{O}_S$-module $f^F_*\mathcal{B}_\circ$ admits a *nonzero section.* Thus, it follows, in light of the exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \longrightarrow \mathcal{B}_\circ \longrightarrow 0,$$

that the Jacobian variety of X is *not ordinary* — in *contradiction* to our assumption that X is *parabolically ordinary.* This completes the proof of Theorem 2.1. $\square$

It follows from Theorem 2.1 [together with the discussion following Remark 2.1.1] that the natural (1-)morphism

$$\mathcal{N}_g[\infty] \longrightarrow \mathcal{M}_g$$

is an *isomorphism*, hence also *étale.* Thus, by the final portion of [10], Theorem 3.3, we obtain:

COROLLARY 2.4. — *Every dormant indigenous bundle over X/S is* **dormant ordinary** [*cf.* [10], *Definition 3.2*].

We shall write

$$\mathcal{C}_g \longrightarrow \mathcal{M}_g$$

for the universal curve over $\mathcal{M}_g$ and

$$\mathcal{S}_g \longrightarrow \mathcal{M}_g$$

for the *Schwarz torsor* over $\mathcal{M}_g$ [cf. [6], Introduction, §0.4], i.e., the torsor over the locally free coherent $\mathcal{O}_{\mathcal{M}_g}$-module of rank $3g-3$

$$(\mathcal{C}_g \to \mathcal{M}_g)_*\omega^{\otimes 2}_{\mathcal{C}_g/\mathcal{M}_g}$$

obtained by forming the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *indigenous bundles* [cf. also [5], Chapter I, Corollary 2.9]. By considering the composite of the above natural isomorphism $\mathcal{M}_g \xleftarrow{\sim} \mathcal{N}_g[\infty]$ and the natural closed immersion $\mathcal{N}_g[\infty] \hookrightarrow \mathcal{S}_g$ of stacks, we obtain a trivialization

$$\mathcal{M}_g \longrightarrow \mathcal{S}_g$$

of the Schwarz torsor.

DEFINITION 2.5. — We shall refer to this trivialization $\mathcal{M}_g \to \mathcal{S}_g$ of the Schwarz torsor as the *dormant trivialization.*

By the dormant trivialization of Definition 2.5, we obtain an isomorphism of $\mathcal{S}_g$ with the geometric vector bundle over $\mathcal{M}_g$ associated to $(\mathcal{C}_g \to \mathcal{M}_g)_*\omega^{\otimes 2}_{\mathcal{C}_g/\mathcal{M}_g}$. Thus:

COROLLARY 2.6. — *There exists a natural* **bijection** *between the following two sets:*

- $\Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$.
- *The set of isomorphism classes of indigenous bundles over* X/S.

For $\theta \in \Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$, *the indigenous bundle over* X/S *corresponding to* θ *is given as follows: Let us recall the pair* $(\mathcal{E}_\circ, \nabla_{\mathcal{E}_\circ})$ *and the exact sequence of* $\mathcal{O}_X$*-modules*

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{E}_\circ \longrightarrow \omega_{X/S} \longrightarrow 0$$

discussed in §1. Write $\phi^\theta \colon \mathcal{E}_\circ \to \mathcal{E}_\circ \otimes_{\mathcal{O}_X} \omega_{X/S}$ *for the homomorphism of* $\mathcal{O}_X$*-modules obtained by forming the composite*

$$\mathcal{E}_\circ \twoheadrightarrow \omega_{X/S} \xrightarrow{\theta} \omega_{X/S}^{\otimes 3} = \omega_{X/S}^{\otimes 2} \otimes_{\mathcal{O}_X} \omega_{X/S} \hookrightarrow \mathcal{E}_\circ \otimes_{\mathcal{O}_X} \omega_{X/S}.$$

We shall write

$$\nabla_{P_\circ}^{\theta}$$

for the connection on $P_\circ$ *determined by the connection*

$$\nabla_{\mathcal{E}_\circ}^{\theta} \stackrel{\text{def}}{=} \nabla_{\mathcal{E}_\circ} + \phi^\theta$$

on $\mathcal{E}_\circ$. *Then the indigenous bundle over* X/S *corresponding to* θ *is given by*

$$P_\theta \stackrel{\text{def}}{=} (\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ}^{\theta}).$$

Moreover, for $\theta \in \Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$, *the* **dormant locus** *in* S *of* P_θ [*i.e., the maximal closed subscheme* $F \subseteq S$ *of* S *such that the restriction of* P_θ *to* $X \times_S F$ *is* **dormant**] **coincides** *with the zero locus in* S *of* θ [*i.e., the maximal closed subscheme* $F \subseteq S$ *of* S *such that the restriction of* θ *to* $X \times_S F$ *is identically zero*].

3. LOCAL CRITERIA

In the present §3, we prove *local criteria* for some properties on indigenous bundles [cf. Proposition 3.1; Proposition 3.8, (ii), below]. We maintain the notation introduced at the beginning of §1.

Let

$$\theta \in \Gamma(X, \omega_{X/S}^{\otimes 2})$$

be a global section of $\omega_{X/S}^{\otimes 2}$. Thus, it follows from Corollary 2.6 that we obtain a connection

$$\nabla_{P_\circ}^{\theta}$$

on the $\mathbb{P}^1$-bundle $P_\circ$ such that the pair

$$P_\theta \stackrel{\text{def}}{=} (\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ}^{\theta})$$

forms an *indigenous bundle* over X/S.

Let $x \in X$ be a point of X and $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x. Write $\phi_x = \phi \in \mathcal{O}_X$ for the local function on X at x which fits into the equality

$$\theta = \phi \cdot dt \otimes dt.$$

Then one verifies immediately that the local sections

$$e_1 \stackrel{\text{def}}{=} 1 \otimes t - t \otimes 1, \quad e_2 \stackrel{\text{def}}{=} e_1^2 \in \mathrm{pr}_{1*}\mathcal{O}_{X \times_{X^F} X} \xleftarrow{\sim} \Phi^*\Phi_*\mathcal{O}_X$$

[cf. the discussion preceding Proposition 1.1] are contained in the submodules

$$\mathrm{Ker}(\mathrm{pr}_{1*}\mathcal{O}_{X\times_{X^F}X} \twoheadrightarrow \mathcal{O}_X) \xleftarrow{\sim} \mathcal{E}_\circ,$$

and that, in the natural exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{E}_\circ \longrightarrow \omega_{X/S} \longrightarrow 0,$$

the local section e_2 determines a *local trivialization* of the invertible sheaf $\omega_{X/S}^{\otimes 2}$, and the local section e_1 determines a *local splitting* of the surjection $\mathcal{E}_\circ \twoheadrightarrow \omega_{X/S}$; in particular, $\{e_1, e_2\}$ forms a *local basis* of $\mathcal{E}_\circ$.

Next, let us observe that it follows immediately from the definition of $\nabla_{\mathcal{E}_\circ}$ that

$$\nabla_{\mathcal{E}_\circ}(e_1, e_2) \;=\; (e_1, e_2)\cdot\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\otimes dt.$$

Thus, one verifies immediately from the definition of $\nabla^\theta_{\mathcal{E}_\circ}$ [cf. Corollary 2.6] that

$$\nabla^\theta_{\mathcal{E}_\circ}(e_1, e_2) \;=\; (e_1, e_2)\cdot\begin{pmatrix}0 & 1\\ \phi & 0\end{pmatrix}\otimes dt.$$

In particular, it follows that the *p-curvature* $\mathcal{P}^\theta$ of the connection $\nabla^\theta_{\mathcal{E}_\circ}$ [cf., e.g., the discussion preceding [4], Theorem 5.1] is given by

$$\begin{array}{rccc}\mathcal{P}^\theta : & \Phi^*\tau_{X^F/S} & \longrightarrow & \mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ)\\ & \Phi^{-1}\delta_{t^F} & \mapsto & \left((e_1,e_2) \mapsto (e_1,e_2)\cdot\begin{pmatrix}-\phi' & \phi\\ \phi^2+\phi'' & \phi'\end{pmatrix}\right)\end{array}$$

— where we write $t^F \in \mathcal{O}_{X^F}$ for the local parameter of X^F/S determined by the local parameter $t \in \mathcal{O}_X$, δ_{t^F} (respectively, δ_t) for the local trivialization of $\tau_{X^F/S}$ (respectively, $\tau_{X/S}$) which maps dt^F (respectively, dt) to 1, ∂_t for the local derivation corresponding to δ_t, "$(-)'$" for "$\partial_t(-)$" [i.e., "$(-)'$" is the "derivative of $(-)$ with respect to t"], and

$$\mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ) \;\subseteq\; \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_\circ)$$

for the submodule of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_\circ)$ consisting of *trace zero* endomorphisms of locally free coherent $\mathcal{O}_X$-module $\mathcal{E}_\circ$. This local computation [cf. Remark 3.1.1 below] leads us to the following *local criteria* for some properties on indigenous bundles:

PROPOSITION 3.1. — *The following hold:*

(i) *The indigenous bundle P_θ is* **nilpotent** [*cf.* [5], *Chapter II, Definition 2.4*] *if and only if, for every point $x \in X$, the equality*

$$(\phi_x')^2 + \phi_x\cdot\phi_x'' + \phi_x^3 \;=\; 0$$

holds.

(ii) *Suppose that S is the spectrum of an algebraically closed field* [*of characteristic* 3]. *Then the indigenous bundle P_θ is* **admissible** [*cf.* [5], *Chapter II, Definition 2.4*] *if and only if, for every closed point $x \in X$, it holds that*

$$\mathrm{ord}_x(\phi_x) \;\leq\; 2.$$

PROOF. — Assertion (i) follows from the definition, together with the above local computation. To verify assertion (ii), let us observe that

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

forms a local basis of the locally free coherent $\mathcal{O}_X$-module $\mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ)$. Thus, assertion (ii) follows immediately from the definition, together with the above local computation. $\square$

REMARK 3.1.1. — We note that since $\det(\mathcal{E}_\circ) \cong \omega_{X/S}^{\otimes 3} \not\cong \mathcal{O}_X$ [cf. Lemma 1.2], the pair $(\mathcal{E}_\circ, \nabla_{\mathcal{E}_\circ})$, as well as the pair $(\mathcal{E}_\circ, \nabla_{\mathcal{E}_\circ}^{\theta})$ [cf. Corollary 2.6], is *not an indigenous vector bundle* [cf. [5], Chapter I, Definition 2.2; also the discussion preceding [5], Chapter I, Definition 2.2]. One verifies easily from the fact that $\det(\mathcal{B}_\circ) \cong \omega_{X^F/S}$ [cf. Lemma 1.2] that if $\mathcal{L}$ is an invertible sheaf on X^F such that $\mathcal{L}^{\otimes 2} \cong \tau_{X^F/S}$ [note that since 2 is *invertible* on S, such an invertible sheaf always exists after étale localizing S], then an *indigenous vector bundle* whose projectivization is isomorphic to $(\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ})$ is given by tensoring $(\mathcal{E}_\circ, \nabla_{\mathcal{E}_\circ})$ with the invertible sheaf $\Phi^*\mathcal{L}$ equipped with the connection determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$. On the other hand, one also verifies easily that the operation of taking tensor product with a *dormant invertible sheaf* [i.e., an invertible sheaf equipped with a connection whose p-curvature is identically zero] does *not affect* the *local computation of the p-curvature as above* [as well as the *validity* of the following results].

REMARK 3.1.2. — If $\mathcal{P}^\theta = 0$, then it follows from the above local computation that $\phi = 0$, hence also $\theta = 0$. By means of this observation, one can give an *alternative proof* of Theorem 2.1.

Next, let us observe that the natural exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \omega_{X/S}^{\otimes 2} \longrightarrow \mathcal{E}_\circ \longrightarrow \omega_{X/S} \longrightarrow 0$$

determines a homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ) \hookrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_\circ) \to \mathcal{H}om_{\mathcal{O}_X}(\omega_{X/S}^{\otimes 2}, \omega_{X/S}) \cong \tau_{X^F/S};$$

moreover, the *square Hasse invariant* [cf. [5], Chapter II, Proposition 2.6, (1)] of the indigenous bundle P_θ is defined as the composite of the p-curvature $\mathcal{P}^\theta$ and this homomorphism. Thus, by the above local computation, we obtain:

PROPOSITION 3.2. — *The* **square Hasse invariant** *of the indigenous bundle P_θ is, up to multiplication by a global section of $\mathcal{O}_S^\times$, given by*

$$\theta \in \Gamma(X, \omega_{X/S}^{\otimes 2}) \cong \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S})).$$

In particular, if, moreover, the indigenous bundle P_θ is **admissible**, *then the* **double supersingular divisor** [*cf.* [5], *Chapter II, Proposition 2.6,* (2)] *of P_θ* **coincides** *with the zero locus of θ.*

In particular, we obtain the following two corollaries:

COROLLARY 3.3. — *Suppose that the indigenous bundle* P_θ *is* **nilpotent** *and* **admissible**. *Then the* **supersingular divisor** [*cf.* [5], *Chapter II, Proposition 2.6,* (3)] *of* P_θ *is* **finite étale** *over* S.

PROOF. — Since [it follows from the definition that] the supersingular divisor of P_θ is *finite flat* over S [cf. also [5], Chapter II, Proposition 2.6, (2)], to complete the verification of Corollary 3.3, it suffices to verify the *unramifiedness*. Thus, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then the *unramifiedness* follows from Proposition 3.1, (ii); Proposition 3.2, together with the definition of the *supersingular divisor*. □

COROLLARY 3.4. — *Suppose that* S *is* **reduced**. *Then the isomorphism class of* **nilpotent** *indigenous bundle over* X/S *is* **completely determined** *by the zero locus of the square Hasse invariant.*

PROOF. — First, let us observe that since S is *reduced*, it follows from [6], Chapter I, Proposition 1.5, that, to verify Corollary 3.4, we may assume without loss of generality that S is the spectrum of an algebraically closed field k [of characteristic 3]. Next, let us observe that one verifies easily that if ϕ is *nonzero* and *satisfies* the equality "$(\phi')^2 + \phi \cdot \phi'' + \phi^3 = 0$" of Proposition 3.1, (i), then, for every $c \in k \setminus \{0, 1\}$, $c \cdot \phi$ does *not satisfy* the equality "$(\phi')^2 + \phi \cdot \phi'' + \phi^3 = 0$" of Proposition 3.1, (i). Thus, Corollary 3.4 follows from Proposition 3.1, (i); Proposition 3.2, together with Corollary 2.6. □

REMARK 3.4.1. — Observe that Corollary 3.4 is a *generalization* of [5], Chapter II, Proposition 2.6, (4), in the case where $p = 3$.

Next, let us observe that it follows from the equality of Proposition 3.1, (i), that the following lemma holds:

LEMMA 3.5. — *Suppose that* S *is the spectrum of an algebraically closed field* [*of characteristic* 3], *and that the indigenous bundle* P_θ *is* **nilpotent**. *Then, for every closed point* $x \in X$, *it holds that* $\mathrm{ord}_x(\phi_x) \notin 3\mathbb{Z} + 1$.

PROOF. — Assume that $n \stackrel{\mathrm{def}}{=} \mathrm{ord}_x(\phi_x) \in 3\mathbb{Z} + 1$ for some closed point $x \in X$. Write

$$\phi_x \ = \ \sum_{i=0}^{\infty} a_i t_x^i$$

by regarding ϕ_x as an element of the completion $\mathcal{O}_{X,x}^{\wedge}$. Then, by considering the coefficient of the "t_x^{2n-2}" of the left-hand side of the equality "$(\phi')^2+\phi\cdot\phi''+\phi^3 = 0$" of Proposition 3.1, (i), we obtain that $a_n = 0$. Thus, we obtain a *contradiction*. □

By Lemma 3.5, we obtain:

COROLLARY 3.6. — *Suppose that $g = 2$. If a* **nilpotent** *indigenous bundle over X/S is* **active** [*cf.* [6]*, Chapter II, Definition 1.1*]*, then it is* **admissible***.*

PROOF. — Let us first observe that it follows from the definition of the *admissibility* that, to verify Corollary 3.6, we may assume without loss of generality that S is the spectrum of an algebraically closed field k [of characteristic 3]. On the other hand, in this case, since $\deg(\omega_{X/S}^{\otimes 2}) = 4$, it follows immediately from Proposition 3.1, (ii), together with Lemma 3.5, that every *nilpotent* and *active* indigenous bundle over X/S is *admissible.* $\square$

We shall write

$$\mathcal{N}_g$$

for the moduli stack of smooth *nilcurves* [cf. the discussion preceding [6], Introduction, Theorem 0.1] of genus g of characteristic 3, i.e., the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *nilpotent indigenous bundles.* Note that it follows from [5], Chapter II, Theorem 2.3 [cf. also the discussion following [5], Chapter II, Definition 2.4], that the natural (1-)morphism

$$\mathcal{N}_g \longrightarrow \mathcal{M}_g$$

is *finite flat of degree* 3^{3g-3}.

COROLLARY 3.7. — *Suppose that $g = 2$. Then the open substack of $\mathcal{N}_2$*

$$\mathcal{N}_2 \setminus \mathcal{N}_2[\infty]$$

is **smooth** *over $\mathbb{F}_3$.*

PROOF. — This follows from Corollary 3.6, together with [5], Chapter II, Corollary 2.16. $\square$

PROPOSITION 3.8. — *Suppose that S is the spectrum of an algebraically closed field k* [*of characteristic* 3]*, and that the indigenous bundle P_θ is* **nilpotent***. Then the following hold:*

(i) *We shall write*

$$T_\theta$$

for the relative tangent space of $\mathcal{N}_g/\mathcal{M}_g$ at the k-valued point of $\mathcal{N}_g$ corresponding to P_θ. Then T_θ is naturally isomorphic to the subspace of $\Gamma(X, \omega_{X/S}^{\otimes 2})$ consisting of global sections η of $\omega_{X/S}^{\otimes 2}$ such that if, for some closed point $x \in X$, we write

$$\eta = \psi_x \cdot dt_x \otimes dt_x,$$

then it holds that

$$(\phi_x \cdot \psi_x)'' = 0.$$

(ii) *It holds that the indigenous bundle P_θ is* **ordinary** [*cf.* [5]*, Chapter II, Definition 3.1*] *if and only if the following condition is satisfied: For every nonzero global section η of $\omega_{X/S}^{\otimes 2}$, if, for some closed point $x \in X$, we write*

$$\eta = \psi_x \cdot dt_x \otimes dt_x,$$

then it holds that

$$(\phi_x \cdot \psi_x)'' \neq 0.$$

PROOF. — Assertion (ii) follows immediately from assertion (i). Thus, to complete the verification of Proposition 3.8, it suffices to verify assertion (i). Write $A \stackrel{\text{def}}{=} k[\epsilon]/(\epsilon^2)$, where ϵ is an indeterminate. Then it follows from Proposition 3.1, (i), that T_θ is naturally isomorphic to the subspace of $\Gamma(X, \omega_{X/S}^{\otimes 2})$ consisting of global sections η of $\omega_{X/S}^{\otimes 2}$ such that if, for some closed point $x \in X$, we write

$$\eta = \psi_x \cdot dt_x \otimes dt_x,$$

then the equality

$$((\phi + \epsilon\psi)')^2 + (\phi + \epsilon\psi) \cdot (\phi + \epsilon\psi)'' + (\phi + \epsilon\psi)^3 = 0$$

— where write $\psi \stackrel{\text{def}}{=} \psi_x$ — in $A \otimes_k \Gamma(X, \omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2}) \oplus \epsilon \cdot \Gamma(X, \omega_{X/S}^{\otimes 2})$ holds. On the other hand, again by Proposition 3.1, (i), one verifies easily that it holds that this equality holds if and only if the equality

$$\phi'' \cdot \psi + \phi \cdot \psi'' - \phi' \cdot \psi' \ (= \ (\phi \cdot \psi)'') = 0$$

holds. This completes the proof of assertion (i). $\square$

REMARK 3.8.1. — Proposition 3.8, (ii), also follows immediately from Proposition 3.2; Lemma A.9, (i) [in the case where we take the pair "$(\mathcal{L}, \tau)$" of Lemma A.9, (i), to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$ — cf. Remark A.4.1], together with [5], Chapter II, Proposition 2.12.

Thus, we obtain:

COROLLARY 3.9. — *Suppose that S is the spectrum of an algebraically closed field k [of characteristic 3], and that the indigenous bundle P_θ is* **nilpotent**. *Then the following conditions are equivalent:*

(1) *The indigenous bundle P_θ is* **dormant**.

(2) *The vector space T_θ over k of Proposition 3.8, (i), is* **of dimension $3g - 3$**.

PROOF. — If P_θ is *dormant*, then $\theta = 0$ [cf. Corollary 2.6]. Thus, the implication (1) $\Rightarrow$ (2) follows from Proposition 3.8, (i). On the other hand, if condition (2) is satisfied, then it follows from Proposition 3.8, (i) [in the case where we take the "η" of Proposition 3.8, (i), to be θ], that $(\phi^2)'' = 0$. Thus, since $0 = (\phi^2)'' = -(\phi')^2 - \phi \cdot \phi'' = \phi^3$ [cf. Proposition 3.1, (i)], we conclude that $\phi = 0$, hence also $\theta = 0$, i.e., that condition (1) is satisfied [cf. Corollary 2.6]. This completes the proof of Corollary 3.9. $\square$

4. Indigenous Bundles Arising from Squares

In the present §4, we discuss some properties on an indigenous bundle which arises from the square of a "*twisted*" differential form, i.e., the square of a global section of a "*square root*" of the square of the relative cotangent sheaf [cf. Proposition 4.1, Proposition 4.2, Proposition 4.4 below]. In the present §4, we maintain the notation introduced at the beginning of §1.

Let

$$\mathcal{L} \ = \ (\mathcal{L}, \tau \colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X)$$

be a *square-trivialized invertible sheaf* on X [cf. Definition A.3] and

$$\chi \ \in \ \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$$

a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$. Let us recall [cf. the discussion following Definition A.3] that we have isomorphisms of invertible sheaves

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\sim} & \mathcal{L}^{\otimes 3} & \xrightarrow{\sim} & \Phi^* \mathcal{L}^F \\ \tau(l \otimes l) \cdot l & \mapsto & l \otimes l \otimes l & \mapsto & \Phi^{-1} l^F \end{array}$$

— where we write $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathcal{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S, l is a local section of $\mathcal{L}$, and l^F is the local section of $\mathcal{L}^F$ determined by l.

Let $x \in X$ be a point of X, $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x, and $l_x = l \in \mathcal{L}$ a local trivialization of $\mathcal{L}$ at x. Then the global trivialization τ and the local trivialization $l_x = l$ determine a local unit

$$\delta_x \ = \ \delta \ \stackrel{\mathrm{def}}{=} \ \tau(l \otimes l) \ \in \ \mathcal{O}_X^{\times}$$

at x. Moreover, the global section χ determines a local function $\phi_x = \phi \in \mathcal{O}_X$ on X at x which fits into the equality

$$\chi \ = \ \phi \cdot l \otimes dt$$

at x.

Next, let us observe that the trivialization τ determines an isomorphism

$$\tau \colon \ \Gamma(X, (\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \ \xrightarrow{\sim} \ \Gamma(X, \omega_{X/S}^{\otimes 2}).$$

Thus, by considering the image via this isomorphism of the *square*

$$\theta \ \stackrel{\mathrm{def}}{=} \ \chi \otimes \chi \ \in \ \Gamma(X, (\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$$

of χ, we obtain a global section

$$\tau(\theta) \ \in \ \Gamma(X, \omega_{X/S}^{\otimes 2})$$

of $\omega_{X/S}^{\otimes 2}$. On the other hand, it follows from Corollary 2.6 that this global section $\tau(\theta)$ gives rise to an *indigenous bundle* over X/S

$$P_{\tau(\theta)} \ \stackrel{\mathrm{def}}{=} \ (\pi_\circ \colon P_\circ \to X, \nabla_{P_\circ}^{\tau(\theta)}).$$

Proposition 4.1. — *Suppose that χ defines a relative effective Cartier divisor of X/S. Then the following conditions are equivalent:*

(1) *The indigenous bundle $P_{\tau(\theta)}$ is* **nilpotent** *and* **active**.

(2) *The global section* $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *is a* **normalized Cartier eigenform** *associated to* $\mathcal{L} = (\mathcal{L}, \tau)$ [*cf. Definition* A.8, (i)].

PROOF. — Let us first observe that it follows from the definitions of $\tau(\theta)$ that $\tau(\theta)$ fits into the equality

$$\tau(\theta) \;=\; \phi^2 \cdot \delta \cdot dt \otimes dt$$

at x. Thus, it follows from Proposition 3.1, (i), that it holds that $P_{\tau(\theta)}$ is *nilpotent* if and only if, for every point $x \in X$,

$$\begin{aligned} &((\phi^2 \cdot \delta)')^2 + (\phi^2 \cdot \delta) \cdot (\phi^2 \cdot \delta)'' + (\phi^2 \cdot \delta)^3 \\ =\;& \phi^4 \cdot (\delta')^2 - \phi^3 \cdot \phi' \cdot \delta \cdot \delta' - \phi^3 \cdot \phi'' \cdot \delta^2 + \phi^4 \cdot \delta \cdot \delta'' + \phi^6 \cdot \delta^3 \\ =\;& \phi^3 \cdot \delta^3 \cdot (-(\phi \cdot \delta^{-1})'' + \phi^3) \end{aligned}$$

is equal to zero. In particular, Proposition 4.1 follows from Lemma A.9, (ii), together with Corollary 2.6. ☐

PROPOSITION 4.2. — *Suppose that the indigenous bundle* $P_{\tau(\theta)}$ *is* **nilpotent** *and* **active**. *Then the following conditions are equivalent:*

(1) *The indigenous bundle* $P_{\tau(\theta)}$ *is* **nilpotent** *and* **admissible**.

(2) *The zero locus of the global section* $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *is* **finite étale** *over* S.

PROOF. — Since [one verifies immediately that] the locus [in S] on which condition (1) (respectively, (2)) is satisfied is *open*, to complete the verification of Proposition 4.2, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then the equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 3.1, (ii), together with the definition of $\tau(\theta)$. ☐

REMARK 4.2.1.

(i) Note that condition (2) of Proposition 4.1 does *not imply* condition (2) of Proposition 4.2. Such a counter-example is as follows: Let k be an algebraically closed field of characteristic 3. Let us consider the following polynomial:

$$f(t) \;=\; t^{12} + t^{10} + 1 \;\in\; k[t].$$

Then one verifies easily that $f(t)$ does *not have* any multiple root, which thus implies that the equation

$$s^2 \;=\; f(t)$$

determines a *hyperelliptic* projective smooth curve C of genus five over k.

Write $\omega \in \Gamma(C, \omega_{C/k})$ for the global section of $\omega_{C/k}$ whose restriction to the open subscheme of X on which f is invertible is of the form

$$\frac{\alpha \cdot t^4}{s} dt$$

— where $\alpha \in k$ satisfies that $\alpha^2 = 2$. Then one verifies easily from Lemma A.9, (ii), that ω is a *normalized Cartier eigenform* associated to $\mathcal{O}_C$ [equipped with the natural identification $\mathcal{O}_C \otimes_{\mathcal{O}_C} \mathcal{O}_C = \mathcal{O}_C$]. On the other hand, it is immediate that if we write $c \in C$ for the closed point corresponding to $(t, s) = (0, 1)$, then $\mathrm{ord}_c(\omega) = 4$.

(ii) It follows from Corollary 3.6 that a *nilpotent active* indigenous bundle over a projective smooth curve *of genus two* in characteristic three is *admissible*. On the other hand, it follows from the discussion of (i), together with Proposition 4.1 and Proposition 4.2, that there exists a *nilpotent active* indigenous bundle over a projective smooth curve in characteristic three which is *not admissible*.

PROPOSITION 4.3. — *Suppose that S is the spectrum of an algebraically closed field k [of characteristic 3], and that the indigenous bundle $P_{\tau(\theta)}$ is* **nilpotent** *and* **admissible**. *Write*

$$T_{\tau(\theta)}$$

for the relative tangent space of $\mathcal{N}_g/\mathcal{M}_g$ at the k-valued point of $\mathcal{N}_g$ corresponding to $P_{\tau(\theta)}$. Thus, it follows from Proposition 3.8, (i), that $T_{\tau(\theta)}$ may be regarded as a subspace of $\Gamma(X, \omega_{X/S}^{\otimes 2})$:

$$T_{\tau(\theta)} \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2}).$$

Then the map

$$\begin{array}{ccc} \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \otimes_k \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) & \longrightarrow & \Gamma(X, \omega_{X/S}^{\otimes 2}) \\ \alpha \otimes \beta & \mapsto & \tau(\alpha \otimes \beta) \end{array}$$

induces an **isomorphism** *of vector spaces over k*

$$\begin{array}{ccc} \mathrm{Ker}(C_{\mathcal{L}}) & \xrightarrow{\sim} & T_{\tau(\theta)} \\ \sigma & \mapsto & \tau(\sigma \otimes \chi) \end{array}$$

— *where we write $C_{\mathcal{L}}$ for the Cartier operator associated to $\mathcal{L} = (\mathcal{L}, \tau)$ [cf. Definition* A.4].

PROOF. — Let us first observe that [one verifies easily that] the homomorphism of vector spaces over k

$$\begin{array}{cccc} \Xi\colon & \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) & \longrightarrow & \Gamma(X, \omega_{X/S}^{\otimes 2}) \\ & \alpha & \mapsto & \tau(\alpha \otimes \chi) \end{array}$$

is *injective*. Thus, to verify Proposition 4.3, it suffices to verify the following two assertions:

(a) $\Xi(\mathrm{Ker}(C_{\mathcal{L}})) \subseteq T_{\tau(\theta)}$.

(b) The resulting [cf. (a)] homomorphism $\Xi\colon \mathrm{Ker}(C_{\mathcal{L}}) \to T_{\tau(\theta)}$ is *surjective*.

Next, let us recall from the proof of Proposition 4.1 that $\tau(\theta)$ fits into the equality

$$\tau(\theta) = \phi^2 \cdot \delta \cdot dt \otimes dt$$

at x. Thus, it follows from Proposition 3.8, (i), that the subspace $T_{\tau(\theta)} \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2})$ consists of global sections η of $\omega_{X/S}^{\otimes 2}$ such that if, for some closed point $x \in X$, we write

$$\eta = \psi \cdot dt \otimes dt,$$

then it holds that

$$(\phi^2 \cdot \delta \cdot \psi)'' = 0.$$

Now we verify the assertion (a). Let $\sigma \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ be such that $C_{\mathcal{L}}(\sigma) = 0$. Write

$$\sigma \;=\; \mu \cdot l \otimes dt$$

at x. Since $\sigma \in \mathrm{Ker}(C_{\mathcal{L}})$, it holds that $(\mu \cdot \delta^{-1})'' = 0$ [cf. Lemma A.9, (i)]. Thus, since

$$\tau(\sigma \otimes \chi) \;=\; \phi \cdot \mu \cdot \delta \cdot dt \otimes dt$$

at x, and

$$(\phi^3 \cdot \mu \cdot \delta^2)'' \;=\; (\phi^3 \cdot \delta^3)'' \cdot (\mu \cdot \delta^{-1}) - (\phi^3 \cdot \delta^3)' \cdot (\mu \cdot \delta^{-1})' + (\phi^3 \cdot \delta^3) \cdot (\mu \cdot \delta^{-1})'' \;=\; 0,$$

we conclude that $\Xi(\sigma) \in T_{\tau(\theta)}$. This completes the proof of the assertion (a).

Next, we verify the assertion (b). Let η be a global section of $\omega_{X/S}^{\otimes 2}$ which belongs to $T_{\tau(\theta)}$. Write

$$\eta \;=\; \psi \cdot dt \otimes dt$$

at x. Then since

$$0 \;=\; (\phi^2 \cdot \delta \cdot \psi)''$$

$$=\; (-(\phi')^2 - \phi \cdot \phi'') \cdot \delta \cdot \psi + \phi^2 \cdot \delta'' \cdot \psi + \phi^2 \cdot \delta \cdot \psi'' + \phi \cdot \phi' \cdot \delta' \cdot \psi - \phi^2 \cdot \delta' \cdot \psi' + \phi \cdot \phi' \cdot \delta \cdot \psi',$$

and ϕ is *of order* ≤ 1 [at x] by Proposition 4.2, it holds that $\mathrm{ord}_x(\phi) \geq 1$ implies $\mathrm{ord}_x(\psi) \geq 1$. Thus, it follows that $V(\chi) = V(\chi)_{\mathrm{red}} \subseteq V(\eta)_{\mathrm{red}} \subseteq V(\eta)$, where we write "$V(-)$" for the zero locus of "$(-)$", i.e., that $\eta \in \Gamma(X, \omega_{X/S}^{\otimes 2}(-V(\chi))) \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2})$. Now let us observe that since $(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2} \cong \omega_{X/S}^{\otimes 2}$, which thus implies that $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S} \cong \omega_{X/S}^{\otimes 2}(-V(\chi))$, we have an isomorphism

$$\begin{array}{ccc} \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) & \xrightarrow{\sim} & \Gamma(X, \omega_{X/S}^{\otimes 2}(-V(\chi))) \\ \sigma & \mapsto & \tau(\sigma \otimes \chi). \end{array}$$

Thus, we conclude that there exists a global section σ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ such that $\eta = \tau(\sigma \otimes \chi)$. Write

$$\sigma \;=\; \mu \cdot l \otimes dt$$

at x, which thus implies that

$$\psi \;=\; \mu \cdot \phi \cdot \delta$$

at x. Then since

$$\begin{aligned} 0 \;=\; (\phi^2 \cdot \delta \cdot \psi)'' \;&=\; (\mu \cdot \phi^3 \cdot \delta^2)'' \\ =\; (\phi^3 \cdot \delta^3)'' \cdot (\mu \cdot \delta^{-1}) &- (\phi^3 \cdot \delta^3)' \cdot (\mu \cdot \delta^{-1})' + (\phi^3 \cdot \delta^3) \cdot (\mu \cdot \delta^{-1})'' \\ &=\; \phi^3 \cdot \delta^3 \cdot (\mu \cdot \delta^{-1})'', \end{aligned}$$

it holds that $(\mu \cdot \delta^{-1})'' = 0$, i.e., that $\sigma \in \mathrm{Ker}(C_{\mathcal{L}})$ [cf. Lemma A.9, (i)]. This completes the proof of the assertion (b), hence also of Proposition 4.3. $\square$

PROPOSITION 4.4. — *Suppose that the indigenous bundle $P_{\tau(\theta)}$ is* **nilpotent** *and* **admissible**. *Then the following conditions are equivalent:*

(1) *The indigenous bundle $P_{\tau(\theta)}$ is* **nilpotent** *and* **ordinary**.

(2) *The invertible sheaf $\mathcal{L}$ is* **parabolically ordinary** [*cf. Definition* A.7].

PROOF. — Since [one verifies immediately that] the locus [in S] on which condition (1) (respectively, (2)) is satisfied is *open*, to complete the verification of Proposition 4.4, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then Proposition 4.4 follows from Proposition 4.3. $\square$

5. Nilpotent Admissible Indigenous Bundles via Cartier Operators

In the present §5, we prove the main result of the present paper [cf. Theorem 5.2 below], as well as some corollaries to the main result. In the present §5, we maintain the notation introduced at the beginning of §1.

DEFINITION 5.1.

(i) We shall say that a pair

$$(\mathcal{L}, \chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}))$$

consisting of an invertible sheaf $\mathcal{L}$ on X and a global section χ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is *of CE-type* [where "CE" stands for "Cartier Eigenform"] if

- $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$,
- $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a *Cartier eigenform* associated to $\mathcal{L}$ [cf. Definition A.8, (ii)], and
- the zero locus of χ is *étale* over S.

(ii) We shall say that a pair $(\mathcal{L}, \chi)$ of CE-type is *of CEO-type* [where "CEO" stands for "Cartier Eigenform and Ordinary"] if $\mathcal{L}$ is parabolically ordinary.

(iii) We shall say that a relative effective Cartier divisor D of X/S is *of CE-type* (respectively, *of CEO-type*) if there exists a pair $(\mathcal{L}, \chi)$ of CE-type (respectively, of CEO-type) such that D coincides with the zero locus of χ.

The main result of the present paper is as follows:

THEOREM 5.2. — *Let $g \geq 2$ be an integer, S a connected noetherian scheme of characteristic* 3 [*i.e., over* $\mathbb{F}_3$], *and $f: X \to S$ a projective smooth curve of genus g. Write $\omega_{X/S}$ for the relative cotangent bundle of X/S. Then the following hold:*

(i) *Let P be a* **nilpotent admissible** *indigenous bundle over X/S. Write $\mathcal{L}_P$ for the* **Hasse defect** *of P* [*cf. Definition* B.2] *and $\chi_P \in \Gamma(X, \mathcal{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})$ for the* **Hasse invariant** *of P* [*cf.* [5], *Chapter II, Proposition 2.6,* (3)*; also the final portion of Proposition* B.4]. *Then the pair*

$$(\mathcal{L}_P, \chi_P)$$

is **of CE-type** [*cf. Definition* 5.1, (i)]. *Moreover, it holds that P is* **nilpotent ordinary** *if and only if the pair $(\mathcal{L}_P, \chi_P)$ is* **of CEO-type** [*cf. Definition* 5.1, (ii)].

(ii) *Let D be a relative effective Cartier divisor of X/S. Then it holds that D is the* **supersingular divisor** [*cf.* [5], *Chapter II, Proposition 2.6,* (3)] *of a* **nilpotent**

admissible (*respectively,* **nilpotent ordinary**) *indigenous bundle over* X/S *if and only if* D *is* **of CE-type** (*respectively,* **of CEO-type**) [*cf. Definition* 5.1*,* (iii)]*.*

(iii) *Suppose that* S *is* **reduced***. Then, by considering the* **supersingular divisors***, we have a* **bijection** *between the following two sets:*

- *The set of isomorphism classes of* **nilpotent admissible** (*respectively,* **nilpotent ordinary**) *indigenous bundles over* X/S*.*
- *The set of relative effective Cartier divisors of* X/S **of CE-type** (*respectively,* **of CEO-type**)*.*

PROOF. — First, we verify the first assertion of assertion (i). Let us first observe that it follows from the final portion of Proposition B.3 that $\mathcal{L}_P^{\otimes 2} \cong \mathcal{O}_X$. Moreover, it follows from Corollary 3.3 that the zero locus of χ_P is *finite étale* over S. Thus, to complete the verification of the first assertion of assertion (i), it suffices to verify that there exists a trivialization $\tau\colon \mathcal{L}_P^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ such that χ_P is a *normalized Cartier eigenform* associated to $(\mathcal{L}_P, \tau)$.

Let us write $\theta_P \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ for the global section of $\omega_{X/S}^{\otimes 2}$ corresponding, via the bijection of Corollary 2.6, to the indigenous bundle P. Fix a trivialization $\tau\colon \mathcal{L}_P^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ of $\mathcal{L}_P^{\otimes 2}$ and write θ for the image via the isomorphism $\Gamma(X, (\mathcal{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \xrightarrow{\sim} \Gamma(X, \omega_{X/S}^{\otimes 2})$ induced by τ of the square $\chi_P \otimes \chi_P \in \Gamma(X, (\mathcal{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$ of χ_P. Then it follows from Proposition 3.2 that there exists a global unit $u \in \Gamma(S, \mathcal{O}_S^{\times})$ such that $\theta_P = u \cdot \theta$. Thus, we may assume without loss of generality, by replacing τ by $u^{-1} \cdot \tau$, that $\theta_P = \theta$. In particular, it follows from Proposition 4.1 that χ_P is a *normalized Cartier eigenform* associated to $(\mathcal{L}_P, \tau)$. This completes the proof of the first assertion of assertion (i). Moreover, the final assertion of assertion (i) follows from the first assertion of assertion (i), together with Proposition 4.4 [cf. also the equality "$\theta_P = \theta$" in the proof of the first assertion of assertion (i)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). The *necessity* follows from assertion (i). To verify the *sufficiency*, let D be a relative effective Cartier divisor of X/S *of CE-type* (respectively, *of CEO-type*). Thus, it follows from the definition that there exists a pair $(\mathcal{L}, \chi)$ *of CE-type* (respectively, *of CEO-type*) such that D is defined by χ. Now since $(\mathcal{L}, \chi)$ is *of CE-type*, the zero locus of χ is *étale* over S, and there exists a trivialization $\tau\colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ of $\mathcal{L}^{\otimes 2}$ such that χ is a *normalized Cartier eigenform* associated to $(\mathcal{L}, \tau)$. Thus, it follows from Proposition 4.1 and Proposition 4.2 that the indigenous bundle P over X/S corresponding, via the bijection of Corollary 2.6, to the image via the isomorphism $\Gamma(X, (\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \xrightarrow{\sim} \Gamma(X, \omega_{X/S}^{\otimes 2})$ induced by τ of the square $\chi \otimes \chi \in \Gamma(X, (\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$ of χ is *nilpotent* and *admissible*. Moreover, it follows from Proposition 4.4 that if $(\mathcal{L}, \chi)$ is *of CEO-type*, then the indigenous bundle P is *ordinary*. Write χ_P for the *Hasse invariant* of P. Then it follows from Proposition 3.2 that the zero locus of χ_P, i.e., the supersingular divisor of P, *coincides* with the zero locus of χ, i.e., D. This completes the proof of the *sufficiency*, hence also of assertion (ii).

The *injectivity* of the map of assertion (iii) follows from Corollary 3.4 [cf. also [5], Chapter II, Proposition 2.6, (4)]. The *surjectivity* of the map of assertion (iii) follows from assertion (ii). This completes the proof of Theorem 5.2. □

COROLLARY 5.3. — *Let X be a projective smooth curve of genus $g \geq 2$ over an algebraically closed field k of characteristic* 3 *and $\mathcal{L}$ an invertible sheaf on X such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$. Then the following hold:*

(i) *Suppose that $\mathcal{L} \cong \mathcal{O}_X$ (respectively, $\mathcal{L} \not\cong \mathcal{O}_X$). Then the number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over X/k whose Hasse defects are isomorphic to $\mathcal{L}$ is*

$$\leq \ \sharp\mathbb{P}^{g-1}(\mathbb{F}_3) = \ \frac{3^g - 1}{3-1} \quad \Big(\textit{respectively,} \quad \leq \ \sharp\mathbb{P}^{g-2}(\mathbb{F}_3) = \ \frac{3^{g-1}-1}{3-1}\Big).$$

(ii) *The number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over X/k is*

$$\leq \ \sharp\mathbb{P}^{g-1}(\mathbb{F}_3) + (\sharp((\mathbb{Z}/2\mathbb{Z})^{\oplus 2g}) - 1) \cdot \sharp\mathbb{P}^{g-2}(\mathbb{F}_3) \ = \ \frac{1}{2}\Big((3^g-1) + (2^{2g}-1)(3^{g-1}-1)\Big).$$

PROOF. — First, we verify assertion (i). Fix a trivialization $\tau \colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$. Let us first observe that it follows from Theorem 5.2, (iii) [cf. also Remark A.8.1], that, to verify assertion (i), it suffices to verify that the number of subspaces of $V_{\mathcal{L}} \overset{\mathrm{def}}{=} \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *of dimension* 1 which are *preserved* and *not annihilated* by the Cartier operator $V_{\mathcal{L}} \to \Gamma(X^F, \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})) \xleftarrow{\sim} V_{\mathcal{L}}$ associated to $(\mathcal{L}, \tau)$ is

$$\leq \ \sharp\mathbb{P}^{g-1}(\mathbb{F}_3) = \ \frac{3^g - 1}{3-1} \quad \Big(\text{respectively,} \quad \leq \ \sharp\mathbb{P}^{g-2}(\mathbb{F}_3) = \ \frac{3^{g-1}-1}{3-1}\Big)$$

if $\mathcal{L} \cong \mathcal{O}_X$ (respectively, $\mathcal{L} \not\cong \mathcal{O}_X$). To this end, let us observe that one verifies easily from the *Riemann-Roch theorem* that the vector space $V_{\mathcal{L}}$ over k is *of dimension g* (respectively, $g-1$) if $\mathcal{L} \cong \mathcal{O}_X$ (respectively, $\mathcal{L} \not\cong \mathcal{O}_X$). Thus, assertion (i) follows immediately from elementary linear algebra [cf. also [7], p.143, Corollary]. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i). This completes the proof of Corollary 5.3. □

COROLLARY 5.4. — *Let X be a projective smooth curve of genus ≥ 2 over an algebraically closed field k of characteristic* 3. *Then, for every* **nilpotent ordinary** *indigenous bundle P over X/k, there exists a connected finite étale covering $Y \to X$ of X such that the* [*necessarily* **nilpotent admissible**] *indigenous bundle $(Y \to X)^*P$ over Y/k is* **not ordinary**.

PROOF. — Write $\mathcal{L}_P$ for the *Hasse defect* of P and $Y_1 \to X$ for the connected finite étale covering of X which trivializes $\mathcal{L}_P$. [So if $\mathcal{L}_P \cong \mathcal{O}_X$ (respectively, $\mathcal{L}_P \not\cong \mathcal{O}_X$), then $Y_1 \to X$ is *of degree* 1 (respectively, 2).] Next, let $Y_2 \to X$ be a connected finite étale covering of X such that Y_2 is *not parabolically ordinary* [cf., e.g., [9], Théorème 2] and $Y \to X$ a connected finite étale covering of X which dominates $Y_1 \to X$ and $Y_2 \to X$.

Now let us observe that since $Y \to X$ *factors* through $Y_1 \to X$, one verifies immediately that the Hasse defect of the indigenous bundle $(Y \to X)^*P$ over Y/k is *trivial*. Thus, it follows from the final portion of Theorem 5.2, (i), that the indigenous bundle $(Y \to X)^*P$ over Y/k is *ordinary* if and only if Y is *parabolically ordinary*. On the other hand, since $Y \to X$ *factors* through $Y_2 \to X$, and Y_2 is *not parabolically ordinary*, it holds that Y

is *not parabolically ordinary*, which thus implies that the indigenous bundle $(Y \to X)^*P$ over Y/k is *not ordinary.* $\square$

REMARK 5.4.1. — Corollary 5.4 yields a *negative answer* to the basic question (2) of [6], Introduction, §2.1.

Finally, we discuss the various *moduli stacks* related to the main result of the present paper. We shall apply the notational conventions for the various stacks established in the Appendix C [in the case where we take the "p" of the Appendix C to be 3]. The following corollary follows immediately from the final portion of Theorem 5.2, (i) [cf. also Proposition C.5]:

COROLLARY 5.5. — *We have a* **cartesian** *diagram of stacks*

$$\begin{array}{ccc} \mathcal{N}_g^{\mathrm{ord}} & \longrightarrow & \mathcal{J}_g[2]^{\text{pb-ord}} \\ \downarrow & & \downarrow \\ \mathcal{N}_g^{\mathrm{adm}} & \longrightarrow & \mathcal{J}_g[2] \end{array}$$

— *where the vertical arrows are the natural open immersions of stacks* [*cf. Definition* C.4]*, and the lower horizontal arrow is the* **Hasse defect morphism** [*cf. Definition* C.1]*.*

Next, for a nonnegative integer r, write

$$\mathcal{M}_{g,[r]}$$

for the moduli stack of *hyperbolic curves of type* (g, r) of characteristic 3, i.e., the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with relative effective *étale* Cartier divisors of relative degree r.

DEFINITION 5.6. — It follows from Corollary 3.3, together with Proposition B.3, that the supersingular divisor of the universal nilpotent admissible indigenous bundle over $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{N}_g^{\mathrm{adm}} \to \mathcal{N}_g^{\mathrm{adm}}$ determines a (1-)morphism over $\mathcal{M}_g$

$$\mathcal{N}_g^{\mathrm{adm}} \longrightarrow \mathcal{M}_{g,[2g-2]}.$$

We shall refer to this (1-)morphism as the *Hasse morphism.*

The following corollary follows immediately from Theorem 5.2, (ii):

COROLLARY 5.7. — *Let* (X, D) *be a hyperbolic curve of type* $(g, 2g-2)$ *over a connected noetherian scheme* S *of characteristic* 3*. Then the following conditions are equivalent:*

(1) *The classifying* (1-)*morphism* $S \to \mathcal{M}_{g,[2g-2]}$ *of* (X, D) **factors** *through the* **Hasse morphism** $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{M}_{g,[2g-2]}$.

(2) *The relative effective Cartier divisor* D *of* X/S *is* **of CE-type***.*

6. The Case of Genus Two

In the present §6, we give, by applying the results obtained in the present paper, a complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over a projective smooth curve *of genus two* over an algebraically closed field of characteristic three [cf. Theorem 6.1 below]. Moreover, we also prove that every projective smooth curve of genus two over a connected noetherian scheme of characteristic three is *hyperbolically ordinary* [cf. Corollary 6.6 below]. In the present §6, we maintain the notation introduced at the beginning of §1. Suppose, moreover, that $g = 2$ [i.e., that X is *of genus* 2], and that S is the spectrum of an algebraically closed field k [of characteristic 3].

Since X is of genus 2, X admits a *uniquely determined hyperelliptic involution* ι, which determines a *double* covering

$$\xi\colon\ X \longrightarrow Q$$

— where we write Q for the [scheme-theoretic] quotient of X by the action of ι. [Thus, Q is isomorphic to the projective line $\mathbb{P}^1_k$ over k.] We shall write

$$\mathrm{WP} \subseteq X$$

for the ramification locus of ξ, i.e., the zero locus of the global section of the invertible sheaf on X [of degree 6]

$$\mathcal{H}om_{\mathcal{O}_X}(\xi^*\omega_{Q/S}, \omega_{X/S})$$

determined by ξ. [Thus, $\mathrm{WP} \subseteq X$ is the set of *Weierstrass points* of X.] Then, as is well-known, the closed subscheme $\mathrm{WP} \subseteq X$ is *reduced*; moreover, we have a bijection [of finite sets of cardinality 15] between

- the set of subsets of WP *of cardinality* 2

and

- the set of isomorphism classes of invertible sheaves on X *of order* 2

given by mapping $D = \{x_1, x_2\} \subseteq \mathrm{WP}$ to $\mathcal{L}_D \stackrel{\mathrm{def}}{=} \mathcal{O}_X(x_1 - x_2)$. Finally, one verifies immediately that, for a subset $D \subseteq \mathrm{WP}$ of cardinality 2, if we write

$$\xi_D\colon\ X_D \longrightarrow X$$

for the connected finite étale double covering which trivializes $\mathcal{L}_D$ and

$$E_D$$

for the elliptic curve over k obtained by considering the double covering of Q ($\cong \mathbb{P}^1_k$) whose branch locus coincides with $\xi(\mathrm{WP} \setminus D)$, then the "*new part*" of ξ_D [i.e., the abelian variety obtained by forming the quotient of the Jacobian variety of X_D by the image — via the homomorphism induced by ξ_D — of the Jacobian variety of X] is *isogenous* to E_D.

By this observation, together with the results obtained in the present paper, we give the following complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over X/S:

Theorem 6.1. — *The following hold:*

(i) *Every* **nilpotent active** [*i.e.,* **nilpotent nondormant**] *indigenous bundle over X/S is* **admissible**.

(ii) *Let* $D \subseteq \mathrm{WP}$ *be a subset* **of cardinality 2** *and* $\theta_D \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ *a* [*uniquely determined, up to multiplication by an element of* $k^\times$] *global section of* $\omega_{X/S}^{\otimes 2}$ *such that the zero locus of* θ_D *coincides with* $2D$ [*if we naturally regard* D *as a reduced divisor of degree* 2], *and, moreover, the elliptic curve* E_D *is* **ordinary**. *Then a* [*uniquely determined — cf. Proposition* 3.2, *Corollary* 3.4] $k^\times$*-multiple of* θ_D *corresponds, via the bijection of Corollary* 2.6, *to a* **nilpotent** [*necessarily* **admissible** — *cf.* (i)] *indigenous bundle over* X/S.

(iii) *Let* $\omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S})$ *be a* **Cartier eigenform** *associated to* $\mathcal{O}_X$. *Then a* [*uniquely determined — cf. Proposition* 3.2, *Corollary* 3.4] $k^\times$*-multiple of* $\omega_{\mathrm{CE}} \otimes \omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ *corresponds, via the bijection Corollary* 2.6, *to a* **nilpotent** [*necessarily* **admissible** — *cf.* (i)] *indigenous bundle over* X/S.

(iv) *Every* **nilpotent active** [*i.e.,* **admissible** — *cf.* (i)] *indigenous bundle over* X/S *is obtained as the result of either* (ii) *or* (iii).

(v) *It holds that a* **nilpotent** *indigenous bundle over* X/S *is* **ordinary** *if and only if one of the following two conditions is satisfied:*

(1) *The indigenous bundle is obtained as the result of* (ii).

(2) *The indigenous bundle is obtained as the result of* (iii), *and, moreover,* X *is* **parabolically ordinary**.

PROOF. — Assertion (i) follows from Corollary 3.6. Next, we verify assertion (ii). Let $D \subseteq \mathrm{WP}$ be as in assertion (ii). Then it is immediate that $\mathcal{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S} \cong \mathcal{O}_X(D)$ [if we naturally regard D as a reduced divisor of degree 2]. In particular, it follows that $\Gamma(X, \mathcal{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S})$ is *of dimension* 1, which thus implies that the zero locus of every nonzero global section of $\mathcal{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S}$ *coincides* with [the reduced closed subscheme of X whose underlying subset is] D. Moreover, since E_D is *ordinary*, it follows from Proposition A.6 [cf. also Remark A.8.1] that every nonzero global section of $\mathcal{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *Cartier eigenform* associated to $\mathcal{L}_D$. Thus, it follows immediately from Proposition 4.1 that assertion (ii) holds. This completes the proof of assertion (ii).

Assertion (iii) follows from Proposition 4.1. Next, we verify assertion (iv). Let P be a *nilpotent admissible* indigenous bundle over X/S. If the Hasse defect of P is *trivial*, then it follows from Theorem 5.2, (iii), that P is obtained as the result of (iii). If the Hasse defect of P is *nontrivial*, then it follows from Theorem 5.2, (iii), together with Proposition A.6, that P is obtained as the result of (ii) [cf. also the proof of assertion (ii)]. This completes the proof of assertion (iv). Assertion (v) follows from the final portion of Theorem 5.2, (i), together with assertion (iv). This completes the proof of Theorem 6.1. □

REMARK 6.1.1. — It follows immediately from Proposition 3.2, together with the various definitions involved, that the *Hasse invariants* and the *supersingular divisors* of *nilpotent admissible* indigenous bundles over X/S are given as follows:

• Write P for the *nilpotent admissible* indigenous bundle over X/S obtained as the result of Theorem 6.1, (ii), with respect to a subset $D \subseteq \mathrm{WP}$ as in Theorem 6.1, (ii). Then the *supersingular divisor* of P is [the reduced closed subscheme of X whose underlying subset is] D. Next, let us observe that $\mathcal{O}_X(D)^{\otimes 2} \cong \omega_{X/S}^{\otimes 2}$, and, moreover, the vector space $\Gamma(X, \mathcal{O}_X(D))$ over k is *of dimension* 1. Let s be a nonzero global section of $\mathcal{O}_X(D)$. Then the *Hasse invariant* of the indigenous bundle P is a $k^\times$-multiple of

$$s \ \in \Gamma(X, \mathcal{O}_X(D)).$$

• Write P for the *nilpotent admissible* indigenous bundle over X/S obtained as the result of Theorem 6.1, (iii), with respect to a global section $\omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S})$ as in Theorem 6.1, (iii). Then the *Hasse invariant* of P is a $k^\times$-multiple of

$$\omega_{\mathrm{CE}} \ \in \Gamma(X, \omega_{X/S}).$$

The *supersingular divisor* of this indigenous bundle is the zero locus of ω_{CE}.

REMARK 6.1.2. — One verifies immediately that an indigenous bundle [implicitly] discussed in [1], §11, is a nilpotent admissible indigenous bundle obtained as the result of Theorem 6.1, (iii) [cf. the discussion in Remark 6.1.1 concerning *supersingular divisors*; also condition (2) of Theorem 6.1, (v), and the equivalence (a) $\Leftrightarrow$ (b) of [1], Theorem 2.8, (3)].

The following corollary follows immediately from Theorem 2.1 and Theorem 6.1 [cf. also Remark 6.1.1], together with elementary linear algebra [cf. also [7], p.143, Corollary]:

COROLLARY 6.2. — *Write n_{WP} for the number of subsets D of* WP *of cardinality* 2 *such that the elliptic curve E_D is* **ordinary**. *Write, moreover, γ_X ($\in \{0, 1, 2\}$) for the p-rank of the Jacobian variety of X. Then the following hold:*

(i) *The number of isomorphism classes of* **nilpotent** *indigenous bundles over X/S is given by*

$$1 + n_{\mathrm{WP}} + \sharp\mathbb{P}^{\gamma_X - 1}(\mathbb{F}_3)$$

— where we write $\sharp\mathbb{P}^{-1}(\mathbb{F}_3) \stackrel{\mathrm{def}}{=} 0$.

(ii) *The number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over X/S is given by*

$$n_{\mathrm{WP}} + \sharp\mathbb{P}^{\gamma_X - 1}(\mathbb{F}_3)$$

— where we write $\sharp\mathbb{P}^{-1}(\mathbb{F}_3) \stackrel{\mathrm{def}}{=} 0$.

(iii) *If X is* **parabolically ordinary** (*respectively,* **not parabolically ordinary**) [*i.e., $\gamma_X = 2$* (*respectively, $\gamma_X \neq 2$*)], *then the number of isomorphism classes of* **nilpotent ordinary** *indigenous bundles over X/S is given by*

$$n_{\mathrm{WP}} + 4 \quad (\textit{respectively, } \ n_{\mathrm{WP}}).$$

Next, let us recall the following *well-known* lemma on the p-rank of the Jacobian variety of a projective smooth curve of genus ≤ 2 over an algebraically closed field of characteristic three. The following well-known lemma follows immediately from, for instance, the characterization of the *Cartier operator* [cf., e.g., [4], Theorem 7.2], together with a *well-known* explicit description of the global differential forms on a *hyperelliptic* projective smooth curve:

LEMMA 6.3. — *The following hold:*

(i) *Suppose that X admits a dense open subscheme which is isomorphic to the affine scheme over k*

$$\mathrm{Spec}\big(k[s,t]/(s^2 - f(t))\big)$$

— where s and t are indeterminates, and

$$f(t) \;=\; t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \;\in\; k[t].$$

Then it holds that X is **parabolically ordinary** *if and only if $a_1 \neq a_2 \cdot a_4$.*

(ii) *If an elliptic curve E over k admits a dense open subscheme which is isomorphic to the affine scheme over k*

$$\mathrm{Spec}\big(k[s,t]/(s^2 - f(t))\big)$$

— where s and t are indeterminates, and

$$f(t) \;=\; t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \;\in\; k[t]$$
$$(\textit{respectively,} \quad f(t) \;=\; t^3 + a_2t^2 + a_1t + a_0 \;\in\; k[t])$$

— then it holds that E is **ordinary** *if and only if $a_2 \neq 0$.*

COROLLARY 6.4. — *There exists a* **dense open** *substack of $\mathcal{M}_2$ such that every projective smooth curve parametrized by a geometric point on this open substack admits* **exactly 19** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles.*

PROOF. — It follows from Theorem 6.1 [cf. also Corollary 6.2, (iii)] that, to verify Corollary 6.4, it suffices to verify the following assertion: There exists a *dense open* substack $U_{\mathrm{ord}} \subseteq \mathcal{M}_2$ (respectively, $U_{\mathrm{WP}} \subseteq \mathcal{M}_2$) of $\mathcal{M}_2$ such that every projective smooth curve parametrized by a geometric point on U_{ord} (respectively, U_{WP}) is *parabolically ordinary* (respectively, satisfies the condition that, for every subset $D \subseteq \mathrm{WP}$ of *cardinality* 2, the elliptic curve E_D is *ordinary*). On the other hand, the existence of "U_{ord}" (respectively, "U_{WP}") as above follows immediately from Lemma 6.3, (i) (respectively, Lemma 6.3, (ii)), together with a straightforward calculation. This completes the proof of Corollary 6.4. □

REMARK 6.4.1. — Let us observe that the number "19" in the statement of Corollary 6.4 *coincides* with the result of the formula of [6], Chapter V, Corollary 1.3, (3), i.e., the formula

$$n_{2,0}^{\mathrm{ord}} \;=\; \frac{p}{3}(2p^2 + 1).$$

Finally, we prove the *existence* of a *nilpotent ordinary* indigenous bundle over X/S:

PROPOSITION 6.5. — *The following hold:*

(i) *There exist* **at least 12** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles whose Hasse defects are* **nontrivial** *over X/S.*

(ii) *There exist* **at least 13** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles over X/S.*

PROOF. — Let us identify Q with $\mathbb{P}^1_k$ by an isomorphism over k. Moreover, let us naturally identify the set of closed points of $\mathbb{P}^1_k = Q$ with the set $k \cup \{\infty\}$.

First, I claim that the following assertion holds:

> Claim 6.5.A: If the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12, then, by considering a suitable automorphism of $\mathbb{P}^1_k = Q$, one may take the subset "WP" of $\mathbb{P}^1_k = Q$ to be
> $$\{0, 1, \infty, -1, \alpha, -\alpha\} \ \subseteq\ k \cup \{\infty\}$$
> — where $\alpha \in k$ satisfies that $\alpha^2 = 2$ — i.e., X admits a dense open subscheme which is isomorphic to the affine scheme over k
> $$\mathrm{Spec}\big(k[s,t]/(s^2 - f(t))\big)$$
> — where s and t are indeterminates, and
> $$f(t) \ = \ x(x-1)(x+1)(x-\alpha)(x+\alpha) \ = \ x^5 + 2x \ \in \ k[t].$$

Indeed, suppose that there exist 3 distinct subsets D_1, D_2, $D_3 \subseteq \mathrm{WP}$ of *cardinality* 2 such that the elliptic curves E_{D_1}, E_{D_2}, and E_{D_3} are *not ordinary*.

First, we consider the case where $\mathrm{WP} = D_1 \cup D_2 \cup D_3$. Then let us observe that we may assume without loss of generality, by considering a suitable automorphism of $\mathbb{P}^1_k = Q$, that $\xi(D_1) = \{0, \infty\}$ and $1 \in \xi(D_2)$. Then since E_{D_2} and E_{D_3} are *not ordinary*, it follows from Lemma 6.3, (ii), that there exists an element $\alpha \in k \setminus \{0, 1, -1\}$ such that
$$\xi(D_2) \ = \ \{1, -1\}, \quad \xi(D_3) \ = \ \{\alpha, -\alpha\}.$$
On the other hand, since
$$\alpha \cdot (-\alpha) + \alpha \cdot 1 + \alpha \cdot (-1) + (-\alpha) \cdot 1 + (-\alpha) \cdot (-1) + 1 \cdot (-1) \ = \ -\alpha^2 - 1,$$
and E_{D_1} is *not ordinary*, it follows from Lemma 6.3, (ii), that $\alpha^2 = 2$. Thus, one may take the subset "WP" to be as in Claim 6.5.A.

Next, we consider the case where $\mathrm{WP} \neq D_1 \cup D_2 \cup D_3$. Then let us observe that we may assume without loss of generality, by considering a suitable automorphism of $\mathbb{P}^1_k = Q$, that $\infty \in \xi(\mathrm{WP} \setminus (D_1 \cup D_2 \cup D_3))$, that $0 \in \xi(\mathrm{WP} \setminus (D_1 \cup D_2))$, and that $1 \in \xi(D_1 \setminus (D_1 \cap D_2))$. Then since E_{D_1} and E_{D_2} are *not ordinary*, it follows from Lemma 6.3, (ii), that there exists an element $\alpha \in k \setminus \{0, 1, -1\}$ such that
$$\xi(\mathrm{WP} \setminus D_1) \ = \ \{0, \infty, \alpha, -\alpha\}, \quad \xi(\mathrm{WP} \setminus D_2) \ = \ \{0, \infty, 1, -1\},$$
which thus implies that
$$\xi(\mathrm{WP}) \ = \ \{0, 1, -1, \alpha, -\alpha, \infty\}.$$
Thus, since $\infty \notin \xi(D_3)$, one verifies easily from Lemma 6.3, (ii), that E_{D_3} is *ordinary* — in *contradiction* to our assumption that E_{D_3} is *not ordinary*. This completes the proof of Claim 6.5.A.

Now we verify assertion (i). Suppose that the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12. Then it follows from Claim 6.5.A that one may take the subset "WP" to be as in Claim 6.5.A. In particular, it follows from Lemma 6.3, (ii), together with a straightforward calculation, that the number "n_{WP}" defined in the statement of Corollary 6.2 is *equal to* 12. Thus, assertion (i) follows from Theorem 6.1. This completes the proof of assertion (i).

Next, we verify assertion (ii). Assume that the set of isomorphism classes of *nilpotent ordinary* indigenous bundles over X/S is *of cardinality* ≤ 12. Then it follows from Corollary 6.2, (iii), that the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12. Thus, it follows from Claim 6.5.A that one may take the subset "WP" to be as in Claim 6.5.A. Then it follows from Lemma 6.3, (ii), together with a straightforward calculation, that the number "n_{WP}" defined in the statement of Corollary 6.2 is *equal to* 12. Moreover, since [it follows from Lemma 6.3, (i), that] X is *parabolically ordinary*, it follows from Corollary 6.2, (iii), that X/S admits *exactly* 16 $(= 12+4) > 12$ isomorphism classes of *nilpotent ordinary* indigenous bundles — in *contradiction* to our assumption that the set of isomorphism classes of *nilpotent ordinary* indigenous bundles over X/S is *of cardinality* ≤ 12. This completes the proof of assertion (ii). □

It follows from Proposition 6.5, (ii), together with [5], Chapter II, Proposition 3.4, that the following corollary holds:

COROLLARY 6.6. — *Every projective smooth curve of genus* 2 *over a connected noetherian scheme of characteristic* 3 *is* **hyperbolically ordinary** [*cf.* [5]*, Chapter II, Definition 3.3*]*.*

REMARK 6.6.1. — Corollary 6.6 yields a *positive partial answer* to the basic question (1) of [6], Introduction, §2.1. By Corollary 6.6, we conclude that the image of the natural (1-)morphism discussed in the basic question (1) of [6], Introduction, §2.1, in the case where $(g, r, p) = (2, 0, 3)$ *contains* the open substack $(\mathcal{M}_{2,0})_{\mathbb{F}_3} \subseteq (\overline{\mathcal{M}}_{2,0})_{\mathbb{F}_3}$.

REMARK 6.6.2. — Corollary 6.6 also yields an example of a *projective* smooth curve of positive characteristic which is *not parabolically ordinary* but *hyperbolically ordinary* [cf. also Lemma 6.3, (i)].

APPENDIX A. CARTIER OPERATOR ASSOCIATED TO A SQUARE-TRIVIALIZED INVERTIBLE SHEAF

In the Appendix A, let us recall the *Cartier operator* associated to a *square-trivialized* invertible sheaf on a projective smooth curve in positive characteristic. It seems to the author that the content of the Appendix A is *well-known*; however, since a suitable literature could not be found, the author has decided to discuss it in the Appendix A.

In the Appendix A, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbb{F}_p$], and $f \colon X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g. Write $f^F \colon X^F \to S$ for the projective smooth curve obtained by base-changing f via the absolute Frobenius morphism of S and $\Phi \colon X \to X^F$ for the relative Frobenius morphism over S. We shall use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

PROPOSITION A.1. — *Let $\mathcal{L}$ be an invertible sheaf on X such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$. Then the following conditions are equivalent:*

(1) *The restriction of $\mathcal{L}$ to every fiber of f is* **of order two** [*i.e., is* **nontrivial**].

(2) *There exists a point $s \in S$ of S such that the restriction of $\mathcal{L}$ to the fiber of f at s is* **of order two** [*i.e., is* **nontrivial**].

(3) *The invertible sheaf $\mathcal{L}$ does* **not arise** *from an invertible sheaf on S.*

(4) *The image of the classifying morphism of $\mathcal{L}$ [from S to the relative Jacobian variety of X/S] does* **not intersect** *the image of the identity section.*

PROOF. — The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are immediate. The implication (3) $\Rightarrow$ (4) follows immediately from our assumption that S is *connected*, together with the [well-known] fact that the endomorphism of the relative Jacobian variety of X/S obtained by multiplication by 2 is *finite étale*. This completes the proof of Proposition A.1. $\square$

DEFINITION A.2. — Let $\mathcal{L}$ be an invertible sheaf on X such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that $\mathcal{L}$ is *of relative order two* (respectively, *one*) if $\mathcal{L}$ satisfies (respectively, does not satisfy) the four conditions in the statement of Proposition A.1.

DEFINITION A.3. — We shall refer to a pair

$$(\mathcal{L}, \tau \colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X)$$

consisting of an invertible sheaf $\mathcal{L}$ on X and a global trivialization τ of the square $\mathcal{L}^{\otimes 2}$ of $\mathcal{L}$ as a *square-trivialized invertible sheaf* on X.

Let

$$\mathcal{L} \ = \ (\mathcal{L}, \tau \colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X)$$

be a *square-trivialized invertible sheaf* on X. Thus, the trivialization τ determines isomorphisms of invertible sheaves on X

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\sim} & \mathcal{L}^{\otimes p} & \xrightarrow{\sim} & \Phi^*\mathcal{L}^F \\ \tau(l \otimes l)^{\frac{p-1}{2}} \cdot l & \mapsto & l^{\otimes p} & \mapsto & \Phi^{-1} l^F \end{array}$$

— where we write $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathcal{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S, l is a local section of $\mathcal{L}$, and l^F is the local section of $\mathcal{L}^F$ determined by l.

Let us recall [cf., e.g., [4], Theorem 7.2] that we have an exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

— where we write d for the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$, and c for the Cartier operator. We shall write

$$\mathcal{B}_\circ \stackrel{\text{def}}{=} \mathrm{Coker}(\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X) \ = \ \mathrm{Ker}(c \colon \Phi_*\omega_{X/S} \to \omega_{X^F/S})$$

for the *locally free coherent $\mathcal{O}_{X^F}$-module of rank* $p-1$ obtained by forming the cokernel of the natural homomorphism $\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X$, or, alternatively, the kernel of the Cartier operator $c\colon \Phi_*\omega_{X/S} \to \omega_{X^F/S}$. Then, by tensoring with $\mathcal{L}^F$ and applying the above isomorphism $\mathcal{L} \xrightarrow{\sim} \Phi^*\mathcal{L}^F$ determined by τ, we obtain an exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ \longrightarrow \Phi_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S} \longrightarrow 0,$$

which thus determines an exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ) \longrightarrow f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

DEFINITION A.4. — We shall write

$$C_{\mathcal{L}}$$

for the third arrow of the above exact sequence of $\mathcal{O}_S$-modules and refer to

$$C_{\mathcal{L}}\colon\ f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

as the *Cartier operator* associated to $\mathcal{L} = (\mathcal{L}, \tau)$.

REMARK A.4.1. — If we take the pair "$(\mathcal{L}, \tau)$" to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$, then the Cartier operator $f_*\omega_{X/S} \to f^F_*\omega_{X^F/S}$ associated to $(\mathcal{L}, \tau)$ *coincides* with the [homomorphism induced by the] *usual* Cartier operator.

REMARK A.4.2. — One verifies easily that since the formation of

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

commutes with arbitrary change of base "$S' \to S$", the formation of

$$0 \longrightarrow f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ) \longrightarrow f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \xrightarrow{C_{\mathcal{L}}} f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

commutes with arbitrary change of base "$S' \to S$".

If [the underlying invertible sheaf of] $\mathcal{L}$ is *of relative order two*, then we shall write

$$\xi_{\mathcal{L}}\colon\ X_{\mathcal{L}} \longrightarrow X$$

for the connected finite étale *double* covering of X over S which *trivializes* the invertible sheaf $\mathcal{L}$ [determined by τ] and

$$\xi^F_{\mathcal{L}}\colon\ X^F_{\mathcal{L}} \longrightarrow X^F$$

for the connected finite étale covering of X^F over S obtained by base-changing $\xi_{\mathcal{L}}$ via the absolute Frobenius morphism of S. Thus, a trivialization of $\xi^*_{\mathcal{L}}\mathcal{L}$ determines respective isomorphisms of $\mathcal{O}_X$-, $\mathcal{O}_{X^F}$-modules

$$(\xi_{\mathcal{L}})_*\mathcal{O}_{X_{\mathcal{L}}} \cong \mathcal{O}_X \oplus \mathcal{L}, \quad (\xi^F_{\mathcal{L}})_*\mathcal{O}_{X^F_{\mathcal{L}}} \cong \mathcal{O}_{X^F} \oplus \mathcal{L}^F.$$

Moreover, one verifies immediately that the natural homomorphism of $\mathcal{O}_S$-modules

$$\mathbb{R}^1(f^F \circ \xi^F_{\mathcal{L}})_*\mathcal{O}_{X^F_{\mathcal{L}}} \longrightarrow \mathbb{R}^1(f \circ \xi_{\mathcal{L}})_*\mathcal{O}_{X_{\mathcal{L}}}$$

determined by the relative Frobenius morphism $X_{\mathcal{L}} \to X^F_{\mathcal{L}}$ over S is *decomposed* into the direct sum of the natural homomorphisms of $\mathcal{O}_S$-modules

$$\mathbb{R}^1 f^F_*\mathcal{O}_{X^F} \longrightarrow \mathbb{R}^1 f_*\mathcal{O}_X, \quad \mathbb{R}^1 f^F_*\mathcal{L}^F \longrightarrow \mathbb{R}^1 f_*\Phi^*\mathcal{L}^F \xleftarrow{\sim} \mathbb{R}^1 f_*\mathcal{L}$$

[cf. the isomorphism given in the discussion following Definition A.3].

DEFINITION A.5.

(i) We shall say that $f\colon X \to S$ is *parabolically ordinary* [cf. the discussion following [5], Chapter II, Definition 3.3] if the Jacobian variety of every fiber of f is ordinary.

(ii) Let $Y \to S$ be a projective smooth curve over S and $\xi\colon Y \to X$ a finite étale covering over S. Then we shall say that $\xi\colon Y \to X$ is *parabolically new-ordinary* if, for every point $s \in S$ of S, the "new part" of ξ at s [i.e., the abelian variety over s obtained by forming the quotient of the Jacobian variety of $Y \times_S s$ by the image — via the homomorphism induced by ξ — of the Jacobian variety of $X \times_S s$] is ordinary.

Thus, we obtain:

PROPOSITION A.6. — *It holds that the Cartier operator $C_{\mathcal{L}}\colon f_*(\mathcal{L}\otimes_{\mathcal{O}_X}\omega_{X/S}) \to f^F_*(\mathcal{L}^F\otimes_{\mathcal{O}_{X^F}}\omega_{X^F/S})$ associated to $\mathcal{L} = (\mathcal{L}, \tau)$ is* **injective** *if and only if one of the following conditions is satisfied:*

(1) *$\mathcal{L}$ is* **of relative order one***, and X/S is* **parabolically ordinary***.*

(2) *$\mathcal{L}$ is* **of relative order two***, and the connected finite étale double covering $\xi_{\mathcal{L}}\colon X_{\mathcal{L}} \to X$ is* **parabolically new-ordinary***.*

PROOF. — Let us first observe that it follows from Remark A.4.2 that, to verify Proposition A.6, we may assume without loss of generality that S is the spectrum of an algebraically closed field. Next, let us recall from the discussion preceding Definition A.4 that we have an isomorphism of $\mathcal{O}_S$-modules

$$f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ) \xrightarrow{\sim} \mathrm{Ker}(C_{\mathcal{L}}).$$

In particular, the exact sequence of $\mathcal{O}_{X^F}$-modules in the discussion preceding Definition A.4

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0,$$

together with the isomorphism $\mathcal{L} \xrightarrow{\sim} \Phi^*\mathcal{L}^F$ given in the discussion following Definition A.3, determines an exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \mathrm{Ker}(C_{\mathcal{L}}) \longrightarrow \mathbb{R}^1 f^F_*\mathcal{L}^F \longrightarrow \mathbb{R}^1 f_*\mathcal{L}.$$

Thus, it follows from the discussion preceding Definition A.5 that Proposition A.6 holds. This completes the proof of Proposition A.6. □

DEFINITION A.7. — Let $\mathcal{M}$ be an invertible sheaf on X such that $\mathcal{M}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that $\mathcal{M}$ is *parabolically ordinary* if, for some [or, equivalently, every] trivialization $\tau_{\mathcal{M}}\colon \mathcal{M}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$, the square-trivialized invertible sheaf $(\mathcal{M}, \tau_{\mathcal{M}})$ satisfies either (1) or (2) in the statement of Proposition A.6 [i.e., the Cartier operator associated to $(\mathcal{M}, \tau_{\mathcal{M}})$ is injective — cf. Proposition A.6].

Next, let us observe that the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S determines a *Frobenius-semi-linear* homomorphism

$$f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \longrightarrow f^F_*(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

For a global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$, we shall write u^F for the global section of $\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}$ obtained by forming the image of u via this Frobenius-semi-linear homomorphism.

Definition A.8.

(i) We shall say that a global section u of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *normalized Cartier eigenform* associated to $\mathcal{L} = (\mathcal{L}, \tau)$ if u defines a relative effective Cartier divisor of X/S, and, moreover, $C_{\mathcal{L}}(u) = -u^F$.

(ii) Let $\mathcal{M}$ be an invertible sheaf on X such that $\mathcal{M}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that a global section of $\mathcal{M} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *Cartier eigenform* associated to $\mathcal{M}$ if the global section is a normalized Cartier eigenform associated to the square-trivialized invertible sheaf obtained by equipping $\mathcal{M}$ with a trivialization of the square of $\mathcal{M}$.

Remark A.8.1. — One verifies immediately that if S is the spectrum of an algebraically closed field k [of characteristic p], then it holds that a global section $u \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a *Cartier eigenform* associated to [the underlying invertible sheaf of] $\mathcal{L}$ if and only if $u \neq 0$, and, moreover, $C_{\mathcal{L}}(u)$ is a $k^\times$-multiple of u^F; moreover, in this case, the subset of $k^\times$ consisting of $c \in k^\times$ such that cu is a *normalized Cartier eigenform* associated to $\mathcal{L} = (\mathcal{L}, \tau)$ forms an $\mathbb{F}_p^\times$*-torsor*, which thus implies that this subset is *of cardinality* $p-1$.

Remark A.8.2. — If we take the pair "$(\mathcal{L}, \tau)$" to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$ [i.e., if we are in the situation of Remark A.4.1], then the property of being a [*normalized*] *Cartier eigenform* is closely related to the property of being *locally logarithmic* [cf., e.g., [3], Théorème 2.1.17].

Finally, we consider a local criterion for a *normalized Cartier eigenform.* Let $x \in X$ be a point of X, $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x, $l_x = l \in \mathcal{L}$ a local trivialization of $\mathcal{L}$ at x, and

$$\chi \ \in \ \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$$

a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$. Then the global trivialization τ and the local trivialization $l_x = l$ determine a local unit

$$\delta_x \ = \ \delta \ \stackrel{\text{def}}{=} \ \tau(l \otimes l) \ \in \ \mathcal{O}_X^\times$$

at x. Moreover, the global section χ determines a local function $\phi_x = \phi \in \mathcal{O}_X$ on X at x which fits into the equality

$$\chi \ = \ \phi \cdot l \otimes dt$$

at x. Then it follows immediately from the characterization of the *Cartier operator* [cf., e.g., [4], Theorem 7.2; also the discussion given in [3], §2.1 — especially, the equality (2.1.13) in [3], §2.1], together with the explicit description of the isomorphism $\mathcal{L} \overset{\sim}{\to} \Phi^* \mathcal{L}^F$ given in the discussion following Definition A.3, that the following lemma holds:

LEMMA A.9. — *Write ∂_{t_x} for the local derivation corresponding to the local trivialization of $\tau_{X/S}$ which maps dt_x to* 1 *[i.e., "$\partial_{t_x}(-)$" is the "derivative of $(-)$ with respect to t_x"]. Then the following hold:*

(i) *It holds that the global section χ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is* **annihilated** *by the Cartier operator $C_{\mathcal{L}}$ associated to $\mathcal{L} = (\mathcal{L}, \tau)$ if and only if, for every point $x \in X$, the equality*

$$(\overbrace{\partial_{t_x} \circ \cdots \circ \partial_{t_x}}^{p-1})(\delta_x^{-\frac{p-1}{2}} \cdot \phi_x) \;=\; 0$$

holds.

(ii) *It holds that the global section χ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a* **normalized Cartier eigenform** *associated to $\mathcal{L} = (\mathcal{L}, \tau)$ if and only if, for every point $x \in X$, the local function ϕ_x is* **not a zero-divisor**, *and, moreover, the equality*

$$\phi_x^p \;=\; (\overbrace{\partial_{t_x} \circ \cdots \circ \partial_{t_x}}^{p-1})(\delta_x^{-\frac{p-1}{2}} \cdot \phi_x)$$

holds.

APPENDIX B. THE HASSE BUNDLE OF A NILPOTENT ADMISSIBLE INDIGENOUS BUNDLE

In the Appendix B, we discuss the *Hasse bundle* of a *nilpotent admissible* indigenous bundle. In the Appendix B, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbb{F}_p$], and $f\colon X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g. Write $f^F\colon X^F \to S$ for the projective smooth curve obtained by base-changing f via the absolute Frobenius morphism of S and $\Phi\colon X \to X^F$ for the relative Frobenius morphism over S. We shall use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

Let

$$P \;=\; (\pi\colon P \to X, \nabla_P)$$

be a *nilpotent admissible* [cf. [5], Chapter II, Definition 2.4] indigenous bundle [cf. [5], Chapter I, Definition 2.2] over X/S. Write

$$\sigma_{\mathrm{Hdg}}\colon\; X \;\longrightarrow\; P$$

for the *Hodge section* of P [cf. [5], Chapter I, Proposition 2.4] and

$$\mathcal{I}_{\mathrm{Hdg}} \;\subseteq\; \mathcal{O}_P$$

for the ideal of $\mathcal{O}_P$ which defines the section σ_{Hdg}. Thus, it follows from the definition of an indigenous bundle that the Kodaira-Spencer homomorphism at σ_{Hdg} relative to ∇_P [i.e., the homomorphism obtained by differentiating σ_{Hdg} by means of ∇_P]

$$\sigma_{\mathrm{Hdg}}^*\omega_{P/X} \;\longrightarrow\; \omega_{X/S}$$

is an *isomorphism.*

PROPOSITION B.1. — *There exists a* **unique section** $\sigma_{\mathrm{con}}\colon X \to P$ *of* $\pi\colon P \to X$ *which satisfies the following conditions:*

(1) *The section* σ_{con} *is* **horizontal** *with respect to* ∇_P. *In particular, the connection* ∇_P *induces a connection on the invertible sheaf* $\sigma_{\mathrm{con}}^*\omega_{P/X}$ *on* X.

(2) *There exists a* **horizontal** *isomorphism* $\sigma_{\mathrm{con}}^*\omega_{P/X} \cong \Phi^*\tau_{X^F/S}$, *where we regard* $\Phi^*\tau_{X^F/S}$ *as an invertible sheaf equipped with a connection by equipping* $\Phi^*\tau_{X^F/S}$ *with the connection arising from the exterior differentiation operator* $\mathcal{O}_X \to \omega_{X/S}$.

We shall refer to this section σ_{con} *as the* **conjugate section** *of the indigenous bundle* P.

PROOF. — This follows from the second paragraph of [the statement of] [5], Chapter II, Proposition 2.5. $\square$

Write

$$\sigma_{\mathrm{con}}\colon\ X \ \longrightarrow\ P$$

for the *conjugate section* of P and

$$\mathcal{I}_{\mathrm{con}} \ \subseteq\ \mathcal{O}_P$$

for the ideal of $\mathcal{O}_P$ which defines the section σ_{con}.

DEFINITION B.2. — We shall refer to the invertible sheaf on X

$$\sigma_{\mathrm{Hdg}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P) \ = \ \mathcal{H}om_{\mathcal{O}_X}(\sigma_{\mathrm{Hdg}}^*\mathcal{I}_{\mathrm{con}}, \mathcal{O}_X)$$

obtained by pulling back $\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P)$ via σ_{Hdg} as the *Hasse bundle* of P. We shall refer to the invertible sheaf on X

$$\sigma_{\mathrm{Hdg}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P) \otimes_{\mathcal{O}_X} \tau_{X/S}^{\otimes\frac{p-1}{2}} \ = \ \mathcal{H}om_{\mathcal{O}_X}(\sigma_{\mathrm{Hdg}}^*\mathcal{I}_{\mathrm{con}}, \tau_{X/S}^{\otimes\frac{p-1}{2}})$$

obtained by tensoring the Hasse bundle with $\tau_{X/S}^{\otimes\frac{p-1}{2}}$ as the *Hasse defect* of P.

Write

$$\mathcal{H}_P \ \stackrel{\mathrm{def}}{=}\ \sigma_{\mathrm{Hdg}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P)$$

for the *Hasse bundle* of P. Then let us observe that since $\pi\colon P \to X$ is *of genus zero*, and the invertible sheaf on P

$$\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}})$$

is *of relative degree* 0 over X, it follows immediately that the natural homomorphisms

$$\pi_*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}}) \ \longrightarrow\ \sigma_{\mathrm{Hdg}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}}) \ \cong\ \mathcal{H}_P \otimes_{\mathcal{O}_X} \omega_{X/S},$$

$$\pi_*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}}) \ \longrightarrow\ \sigma_{\mathrm{con}}^*\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}}) \ \cong\ \sigma_{\mathrm{con}}^*\mathcal{I}_{\mathrm{Hdg}} \otimes_{\mathcal{O}_X} \Phi^*\omega_{X^F/S}$$

[cf. the discussion preceding Proposition B.1; Proposition B.1, (2)] are *isomorphisms*. Thus, by means of the natural identification $\sigma_{\mathrm{Hdg}}^*\mathcal{I}_{\mathrm{con}} = \sigma_{\mathrm{con}}^*\mathcal{I}_{\mathrm{Hdg}}$, we obtain:

PROPOSITION B.3. — *There exist isomorphisms of invertible sheaves on* X

$$\mathcal{H}_P^{\otimes 2} \ \cong\ \mathcal{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S}) \ \cong\ \omega_{X/S}^{\otimes p-1}.$$

In particular, the **square of the Hasse defect** *is* **trivial**.

Moreover, we obtain:

PROPOSITION B.4. — *The global section of*

$$\mathcal{H}om_{\mathcal{O}_X}(\Phi^* \tau_{X^F/S}, \tau_{X/S})$$

obtained, relative to the isomorphism of Proposition B.3*, by forming the square of the global section of $\mathcal{H}_P$ determined by the natural inclusion $\mathcal{I}_{\mathrm{con}} \hookrightarrow \mathcal{O}_P$ coincides, up to multiplication by a global section of $\mathcal{O}_S^{\times}$, with the* **square Hasse invariant** *of P* [*cf.* [5], *Chapter II, Proposition 2.6,* (1)]*. In particular, the global section of $\mathcal{H}_P$ determined by the natural inclusion $\mathcal{I}_{\mathrm{con}} \hookrightarrow \mathcal{O}_P$ coincides, up to multiplication by a global section of $\mathcal{O}_S^{\times}$, with the* **Hasse invariant** *of P* [*cf.* [5], *Chapter II, Proposition 2.6,* (3)]*.*

PROOF. — This follows from the discussion in the proof of [5], Chapter II, Proposition 2.6, (3). □

APPENDIX C. VARIOUS MODULI STACKS

In the Appendix C, we consider various moduli stacks related to the notions discussed in the present paper. In the Appendix C, let p be an *odd* prime number and $g \geq 2$ an integer.

We shall write

$$\mathcal{M}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic p;

$$\mathcal{C}_g \longrightarrow \mathcal{M}_g$$

for the universal curve over $\mathcal{M}_g$;

$$\mathcal{J}_g \longrightarrow \mathcal{M}_g$$

for the relative Jacobian variety of $\mathcal{C}_g \to \mathcal{M}_g$;

$$\mathcal{J}_g[n] \subseteq \mathcal{J}_g$$

for the kernel of the endomorphism of $\mathcal{J}_g$ over $\mathcal{M}_g$ obtained by multiplication by n [where n is a nonnegative integer]. Moreover, we shall write

$$\mathcal{N}_g$$

for the moduli stack of smooth *nilcurves* [cf. the discussion preceding [6], Introduction, Theorem 0.1] of genus g of characteristic p, i.e., the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent indigenous bundles*;

$$\mathcal{N}_g^{\mathrm{adm}} \subseteq \mathcal{N}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent admissible indigenous bundles*;

$$\mathcal{N}_g^{\mathrm{ord}} \subseteq \mathcal{N}_g^{\mathrm{adm}}$$

for the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent ordinary indigenous bundles*.

Definition C.1. — It follows from the final portion of Proposition B.3 that the *Hasse defect* of the universal nilpotent admissible indigenous bundle over $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{N}_g^{\mathrm{adm}} \to \mathcal{N}_g^{\mathrm{adm}}$ determines a (1-)morphism over $\mathcal{M}_g$

$$\mathcal{N}_g^{\mathrm{adm}} \longrightarrow \mathcal{J}_g[2].$$

We shall refer to this (1-)morphism as the *Hasse defect morphism.*

Proposition C.2. — *The following three open substacks of $\mathcal{N}_g^{\mathrm{adm}}$* **coincide***:*

(1) *The open substack $\mathcal{N}_g^{\mathrm{ord}} \subseteq \mathcal{N}_g^{\mathrm{adm}}$.*

(2) *The* **étale locus** *of the natural* (1-)*morphism $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{M}_g$.*

(3) *The* **étale locus** *of the Hasse defect morphism $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{J}_g[2]$.*

Proof. — The assertion that the open substack given in (1) *coincides* with the open substack given in (2) follows from the definition. On the other hand, since the (1-)morphism $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{M}_g$ is *flat* [cf. [5], Chapter II, Theorem 2.3], the assertion that the open substack given in (2) *coincides* with the open substack given in (3) follows from the well-known fact that the natural (1-)morphism $\mathcal{J}_g[2] \to \mathcal{M}_g$ is a *finite étale surjection.* $\square$

Now let us observe that since, as is well-known, $\mathcal{J}_g[2]$ is *finite étale* over $\mathcal{M}_g$, the identity section of $\mathcal{J}_g \to \mathcal{M}_g$ determines an isomorphism of stacks over $\mathcal{M}_g$

$$\mathcal{M}_g \sqcup (\mathcal{J}_g[2] \setminus \mathcal{J}_g[1]) \xrightarrow{\sim} \mathcal{J}_g[2].$$

Thus, by considering the Hasse defect morphism, we obtain:

Proposition C.3. — *Let $U \to \mathcal{N}_g^{\mathrm{adm}}$ be a scheme over $\mathcal{N}_g^{\mathrm{adm}}$. Suppose that there exist two geometric points s_1, s_2 of U such that the Hasse defect of the nilpotent admissible indigenous bundle corresponding to s_1* (*respectively, s_2*) *is* **of relative order one** (*respectively,* **two**)*. Then U is* **not connected**.

Next, we shall write

$$\mathcal{R}_g$$

for the moduli stack of "*nontrivial*" smooth *Prym curves* of genus g of characteristic p, i.e., the moduli stack of projective smooth curves of genus g of characteristic p equipped with square-trivialized invertible sheaves whose underlying invertible sheaves are of relative order two;

$$\underline{\mathcal{R}}_g \stackrel{\mathrm{def}}{=} \mathcal{J}_g[2] \setminus \mathcal{J}_g[1].$$

Thus, we have a natural (1-)morphism $\mathcal{R}_g \to \underline{\mathcal{R}}_g$ over $\mathcal{M}_g$. For a nonnegative integer d, write

$$\mathcal{A}_d$$

for the moduli stack of principally polarized abelian varieties of dimension d of characteristic p and

$$\mathcal{A}_d^{\mathrm{ord}} \subseteq \mathcal{A}_d$$

for the moduli stack of principally polarized *ordinary* abelian varieties of dimension d of characteristic p.

DEFINITION C.4. — Since $\mathcal{M}_g \sqcup \underline{\mathcal{R}}_g$ is naturally isomorphic to $\mathcal{J}_g[2]$ over $\mathcal{M}_g$, the inverse image of $\mathcal{A}_g^{\mathrm{ord}} \subseteq \mathcal{A}_g$ via the *Torelli morphism* $\mathcal{M}_g \to \mathcal{A}_g$ and the image in $\underline{\mathcal{R}}_g$ of the inverse image of $\mathcal{A}_{g-1}^{\mathrm{ord}} \subseteq \mathcal{A}_{g-1}$ via the *Prym morphism* $\mathcal{R}_g \to \mathcal{A}_{g-1}$ determine an open substack of $\mathcal{J}_g[2]$. We shall write

$$\mathcal{J}_g[2]^{\text{pb-ord}} \subseteq \mathcal{J}_g[2]$$

for this open substack.

Thus, it follows immediately from the various definitions involved that the following proposition holds:

PROPOSITION C.5. — *In the notation introduced at the beginning of the Appendix B, let P be a* **nilpotent admissible** *indigenous bundle over X/S. Then the following conditions are equivalent:*

(1) *The image of the composite*

$$S \longrightarrow \mathcal{N}_g^{\mathrm{ord}} \longrightarrow \mathcal{J}_g[2]$$

of the classifying (1-)*morphism $S \to \mathcal{N}_g^{\mathrm{ord}}$ of P and the Hasse defect morphism is* **contained** *in the open substack $\mathcal{J}_g[2]^{\text{pb-ord}} \subseteq \mathcal{J}_g[2]$.*

(2) *The Hasse defect of P is* **parabolically ordinary**.

REFERENCES

[1] L. R. A. Finotti, Minimal degree liftings of hyperelliptic curves, *J. Math. Sci. Univ. Tokyo* **11** (2004), no. **1**, 1–47.
[2] R. C. Gunning, Special coordinate coverings of Riemann surfaces, *Math. Ann.* **170** 1967 67–86.
[3] L. Illusie, Complexe de de Rham-Witt et cohomologie cristalline, *Ann. Sci. École Norm. Sup.* (**4**) **12** (1979), no. **4**, 501–661.
[4] N. M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin, *Inst. Hautes Études Sci. Publ. Math.* No. **39** (1970), 175–232.
[5] S. Mochizuki, A theory of ordinary p-adic curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), no. **6**, 957–1152.
[6] S. Mochizuki, *Foundations of p-adic Teichmüller theory*, AMS/IP Studies in Advanced Mathematics, **11**. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 1999.
[7] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. **5** Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London 1970.
[8] M. Raynaud, Sections des fibrés vectoriels sur une courbe, *Bull. Soc. Math. France* **110** (1982), no. **1**, 103–125.
[9] M. Raynaud, Revêtements des courbes en caractéristique $p > 0$ et ordinarité, *Compositio Math.* **123** (2000), no. **1**, 73–88.
[10] Y. Wakabayashi, An explicit formula for the generic number of dormant indigenous bundles, *Publ. Res. Inst. Math. Sci.* **50** (2014), no. **3**, 383–409.

(Yuichiro Hoshi) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
E-mail address: yuichiro@kurims.kyoto-u.ac.jp