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# Mono-anabelian Reconstruction of Number Fields

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Abstract. — The *Neukirch-Uchida theorem* asserts that every outer isomorphism between the absolute Galois groups of number fields arises from a uniquely determined isomorphism between the given number fields. We thus conclude that the isomorphism class of a number field is completely determined by the isomorphism class of the absolute Galois group of the number field. On the other hand, the Neukirch-Uchida theorem, as well as the proof of the theorem, does *not give* any "explicit reconstruction of the given number field". In other words, the Neukirch-Uchida theorem yields only a *bi-anabelian reconstruction* of number fields. In the present paper, we discuss a *mono-anabelian reconstruction* of number fields. In particular, we give a *functorial "group-theoretic" algorithm* for reconstructing, from [a suitable quotient of] the absolute Galois group of a number field, [the subfield of] the algebraic closure of the given number field — which determines the [quotient of the] absolute Galois group — equipped with the natural action of the [quotient of the] absolute Galois group. In our discussion, we construct a *global cyclotome* [i.e., a cyclotome constructed from a global Galois group] and the *local-global cyclotomic synchronization isomorphism* [i.e., a suitable isomorphism between a global cyclotome and a local cyclotome]. Moreover, we also prove a *compatibility* of our reconstruction algorithm with the reconstruction algorithm given by *S. Mochizuki* concerning the étale fundamental groups of hyperbolic orbicurves of strictly Belyi type over number fields. Finally, we discuss the *global mono-anabelian log-Frobenius compatibility* of the reconstruction algorithm obtained in the present paper.

## Contents



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## INTRODUCTION

The theme of the present paper is the following *naive question*:

> Can one reconstruct a number field [i.e., a finite extension of the field of rational numbers] from the absolute Galois group of the given number field?

Now let us recall the following result, i.e., the *Neukirch-Uchida theorem* [cf., e.g., [11], Theorem 12.2.1]:

> For $\square \in \{\circ, \bullet\}$, let $F_\square$ be a *number field* and $\overline{F}_\square$ an algebraic closure of $F_\square$. Write $G_\square \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}_\square/F_\square)$;
> $$\mathrm{Isom}(\overline{F}_\bullet/F_\bullet, \overline{F}_\circ/F_\circ)$$
> for the set of isomorphisms $\overline{F}_\bullet \xrightarrow{\sim} \overline{F}_\circ$ of fields which map $F_\bullet$ bijectively onto $F_\circ$;
> $$\mathrm{Isom}(G_\circ, G_\bullet)$$
> for the set of isomorphisms $G_\circ \xrightarrow{\sim} G_\bullet$ of profinite groups. Then the natural map
> $$\mathrm{Isom}(\overline{F}_\bullet/F_\bullet, \overline{F}_\circ/F_\circ) \longrightarrow \mathrm{Isom}(G_\circ, G_\bullet)$$
> is *bijective*.

That is to say, every outer isomorphism between the absolute Galois groups of number fields arises from a uniquely determined isomorphism between the given number fields. In other words, the functor of "*taking the absolute Galois group*" from the full subcategory consisting of number fields of the category of fields and field isomorphisms to the category of profinite groups and outer isomorphisms is *fully faithful*. It follows from the [surjectivity portion of the] Neukirch-Uchida theorem that the isomorphism class of a number field is *completely determined* by the isomorphism class of the absolute Galois group of the number field. From this point of view, one may consider that the Neukirch-Uchida theorem gives an *affirmative answer* to the above *naive question*.

On the other hand, let us observe that the Neukirch-Uchida theorem [as well as the proof of the theorem] does *not give* any "explicit reconstruction of the given number field". That is to say, although one may conclude from the Neukirch-Uchida theorem that the isomorphism class of a number field is completely determined by the isomorphism class of the associated absolute Galois group, the Neukirch-Uchida theorem does *not tell us how to reconstruct explicitly the given number field* from the associated absolute Galois group. In other words, the Neukirch-Uchida theorem yields only a *bi-anabelian reconstruction* — in the sense of [9], Introduction [cf. also [9], Remark 1.9.8] — of number fields.

In the present paper, we discuss a *mono-anabelian reconstruction* — in the sense of [9], Introduction [cf. also [9], Remark 1.9.8] — of number fields. In particular, we center around the task of establishing a "*group-theoretic software*" [i.e., "*group-theoretic* *algorithm*"] whose input data consists of a *single abstract profinite group* — which is isomorphic to [a suitable quotient of] the absolute Galois group of a number field — and whose output data consists of a *field* — which is isomorphic to [a suitable subfield of] an algebraic closure of a number field — *equipped with an action of the profinite group.*

We shall say that an algebraic extension of the field of rational numbers is *absolutely Galois* (respectively, *solvably closed*) if the extension field is Galois over the field of rational numbers (respectively, does not admit a nontrivial finite abelian extension) [cf. Definition 3.1]. We shall say that a profinite group $G$ is *of AGSC-type* if there exist a number field $F$, a Galois extension $\widetilde{F}$ of $F$ which is absolutely Galois and solvably closed, and an isomorphism of $G$ with $\mathrm{Gal}(\widetilde{F}/F)$ of profinite groups [cf. Definition 3.2]. [In particular, if a profinite group is isomorphic to the absolute Galois group of a number field, then the profinite group is *of AGSC-type*.] Then the main result of the present paper may be summarized as follows:

**THEOREM A.** — *There exists a* **functorial "group-theoretic" algorithm** [*cf.* [9], *Remark* 1.9.8*, for more on the meaning the terminology "group-theoretic"*]

$$G \quad \mapsto \quad (G \curvearrowright \widetilde{F}(G))$$

*for constructing, from a profinite group $G$* **of AGSC-type** [*cf. Definition* 3.2]*, a field $\widetilde{F}(G)$ which is* **absolutely Galois** *and* **solvably closed** *equipped with an action of $G$ such that the subfield $\widetilde{F}(G)^G$ of $\widetilde{F}(G)$ consisting of $G$-invariants is a* **number field***, and, moreover, the action of $G$ on $\widetilde{F}(G)$ determines an* **isomorphism of profinite groups**

$$G \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}(G)/\widetilde{F}(G)^G).$$

We thus conclude from Theorem A that every profinite group which is isomorphic to the absolute Galois group of a number field admits a *ring-theoretic basepoint* [i.e., a *"ring-theoretic interpretation"* or a *"ring-theoretic label"*] *group-theoretically* constructed from the given profinite group. Note that, in the proof of Theorem A, the Neukirch-Uchida theorem plays a crucial role; in particular, [the proof of] Theorem A does *not give* an alternative proof of the Neukirch-Uchida theorem.

In the present paper, we also verify a *compatibility* of the reconstruction algorithm of Theorem A with the reconstruction algorithm obtained in [9], Theorem 1.9, in the case where the "$k$" of [9], Theorem 1.9, is a number field. More precisely, we verify the following assertion [cf. Theorem 5.13]: Let $\Pi$ be a profinite group which is isomorphic to the étale fundamental group of a hyperbolic orbicurve of strictly Belyi type over a number field [cf. [8], Definition 3.5]. Write

$$\Pi \curvearrowright \overline{F}(\Pi)$$

for the algebraically closed field equipped with an action of $\Pi$ obtained by applying the *functorial "group-theoretic" algorithm* given in [9], Theorem 1.9, to $\Pi$ [i.e., the field "$\overline{k}^{\times}_{\mathrm{NF}} \cup \{0\}$" of [9], Theorem 1.9, (e)] and

$$\Pi \twoheadrightarrow Q$$

for the *arithmetic quotient* of $\Pi$, i.e., the quotient of $\Pi$ by the [uniquely determined — cf. [7], Theorem 2.6, (vi)] maximal topologically finitely generated normal closed subgroup of $\Pi$. [Thus, $Q$ is a profinite group *of AGSC-type* — cf. [7], Theorem 2.6, (vi) — which thus implies that one may apply Theorem A to $Q$ to construct a field $\widetilde{F}(Q)$ equipped with an action of $Q$.] Then the natural surjection $\Pi \twoheadrightarrow Q$ *group-theoretically determines an isomorphism of fields*

$$\widetilde{F}(Q) \xrightarrow{\sim} \overline{F}(\Pi)$$

which is *compatible* with the natural actions of $Q$ and $\Pi$ relative to the surjection $\Pi \twoheadrightarrow Q$.

Finally, we verify that the reconstruction algorithm of Theorem A also satisfies the *global mono-anabelian log-Frobenius compatibility* [cf. Theorem 6.10], i.e., a *compatibility* with the NF-log-Frobenius functor $\mathfrak{log}$ [cf. Definition 6.8].

The present paper is organized as follows: In §1, we review mono-anabelian reconstructions of various objects which arise from a mixed characteristic local field [cf. Theorem 1.4]. In §2, we discuss an *NF-monoid* [cf. Definition 2.3]. In particular, we obtain a mono-anabelian reconstruction of the "*additive structure*" on an NF-monoid [cf. Theorem 2.9]. Note that the main result of §2 was already essentially proved in [3]; however, the author discussed, in [3], the issue of reconstruction of additive structure in not a "*mono-anabelian" fashion* but a "*bi-anabelian" fashion*. In §3, we define a *cyclotome* [cf. Proposition 3.7, (4)] associated to a profinite group *of GSC-type* [cf. Definition 3.2]. Moreover, we discuss the *local-global cyclotomic synchronization isomorphism* [cf. Theorem 3.8, (ii)], i.e., a suitable isomorphism of a global cyclotome with a local cyclotome. By means of the local-global cyclotomic synchronization isomorphism, we construct the *Kummer containers* associated to a profinite group of GSC-type [cf. Proposition 3.11]. In §4, we discuss a *GSC-Galois pair* [cf. Definition 4.1]. In particular, by means of the main result of §2, we obtain a mono-anabelian reconstruction of the "*additive structure*" on a GSC-Galois pair [cf. Theorem 4.4]. In §5, we finish establishing the *functorial "group-theoretic" algorithm* of Theorem A and prove a compatibility of our reconstruction algorithm with the reconstruction algorithm obtained in [9], Theorem 1.9. In §6, we give an interpretation of the global reconstruction result obtained in the present paper in terms of a certain compatibility with the *NF-log-Frobenius functor* [cf. Theorem 6.10].

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## 0. Notations and Conventions

Numbers. — The notation $\mathbb{N}$ will be used to denote the monoid of nonnegative rational integers [with respect to the addition]. The notation $\mathbb{Z}$ will be used to denote the ring of rational integers. The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. If $n \in \mathbb{Z}$, then we shall write $\mathbb{Z}_{\geq n} \subseteq \mathbb{Z}$ for the subset of $\mathbb{Z}$ consisting of $m \in \mathbb{Z}$ such that $m \geq n$. If $p$ is a prime number, then we shall write $\mathbb{Q}_p$ for the field obtained by forming the $p$-adic completion of $\mathbb{Q}$ and $\mathbb{F}_p \stackrel{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}$ for the finite field of cardinality $p$.

Sets. — Let $S$ be a finite set. Then we shall write $\sharp S$ for the *cardinality* of $S$.

Let $G$ be a group and $S$ a $G$-set. Then we shall write $S^G \subseteq S$ for the subset of $S$ consisting of $G$-invariants.

Monoids. — In the present paper, a "monoid" always means a "commutative monoid". Let $M$ be a monoid. [The monoid operation of $M$ will be written *multiplicatively.*] Then we shall write $M^\times \subseteq M$ for the abelian group of invertible elements of $M$. We shall write $M^{\mathrm{gp}}$ for the *groupification* of $M$, i.e., the monoid [which is, in fact, an *abelian group*] given by the set of equivalence classes with respect to the relation "$\sim$" on $M \times M$ defined as follows: For $(a_1, b_1)$, $(a_2, b_2) \in M \times M$, it holds that $(a_1, b_1) \sim (a_2, b_2)$ if and only if there exists an element $c \in M$ such that $ca_1b_2 = ca_2b_1$. We shall write $M^{\mathrm{pf}}$ for the *perfection* of $M$, i.e., the monoid given by the induction limit of the inductive system $I_*$ of monoids

$$\cdots \longrightarrow M \longrightarrow M \longrightarrow \cdots$$

given by assigning to each element of $n \in \mathbb{Z}_{\geq 1}$ a copy of $M$, which we denote by $I_n$, and to every two elements $n$, $m \in \mathbb{Z}_{\geq 1}$ such that $n$ divides $m$ the morphism $I_n = M \to I_m = M$ given by multiplication by $m/n$. We shall write $M^{\circledast} \stackrel{\text{def}}{=} M \cup \{*_M\}$; we regard $M^{\circledast}$ as a *monoid* by $a \cdot *_M \stackrel{\text{def}}{=} *_M$, $*_M \cdot a \stackrel{\text{def}}{=} *_M$, $*_M \cdot *_M \stackrel{\text{def}}{=} *_M$ for every $a \in M$.

Modules. — Let $M$ be a module. If $n \in \mathbb{Z}$, then we shall write $M[n] \subseteq M$ for the submodule obtained by forming the kernel of the endomorphism of $M$ given by multiplication by $n$. We shall write $M_{\mathrm{tor}} \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{Z}_{\geq 1}} M[n] \subseteq M$ for the submodule of torsion elements of $M$,

$$M^\wedge \stackrel{\text{def}}{=} \varprojlim_n M/nM$$

— where the projective limit is taken over the positive integers $n$ — and $\widehat{\mathbb{Z}} \stackrel{\text{def}}{=} \mathbb{Z}^\wedge$. Thus, if $M$ is *finitely generated*, then $M^\wedge$ is naturally isomorphic to the *profinite completion* of $M$.

Groups. — Let $G$ be a group and $H \subseteq G$ a subgroup of $G$. Then we shall write $Z_G(H) \subseteq G$ for the *centralizer* of $H$ in $G$, i.e., the subgroup consisting of $g \in G$ such that $gh = hg$ for every $h \in H$. We shall write $N_G(H) \subseteq G$ for the *normalizer* of $H$ in $G$, i.e., the subgroup consisting of $g \in G$ such that $H = gHg^{-1}$. We shall write $C_G(H) \subseteq G$ for the *commensurator* of $H$ in $G$, i.e., the subgroup consisting of $g \in G$ such that $H \cap gHg^{-1}$ is of finite index in both $H$ and $gHg^{-1}$. We shall say that $H$ is *normally terminal* (respectively, *commensurably terminal*) in $G$ if $N_G(H) = H$ (respectively, $C_G(H) = H$).

TOPOLOGICAL GROUPS. — Let $G$ be a topological group. Then we shall write $G^{\mathrm{ab}}$ for the *abelianization* of $G$ [i.e., the quotient of $G$ by the closure of the commutator subgroup of $G$], $G^{\mathrm{ab/tor}}$ for the quotient of $G^{\mathrm{ab}}$ by the closure of $(G^{\mathrm{ab}})_{\mathrm{tor}} \subseteq G^{\mathrm{ab}}$, and $\mathrm{Aut}(G)$ for the group of [continuous] automorphisms of $G$.

Let $G$ be a profinite group and $p$ a prime number. Then we shall write $G^{(p)}$ for the *maximal pro-p quotient* of $G$ and $G^{(p')}$ for the *maximal pro-prime-to-p quotient* of $G$.

RINGS. — In the present paper, a "ring" always means a "unital associative commutative ring". Let $R$ be a ring. Then we shall write $R^{\times} \subseteq R$ for the abelian group of invertible elements of $R$ and $R^{\triangleright} \stackrel{\mathrm{def}}{=} R \setminus \{0\} \subseteq R$ for the monoid of nonzero elements of $R$ [with respect to the multiplication]. Thus, we have a natural inclusion $R^{\times} \subseteq R^{\triangleright}$ of monoids.

FIELDS. — Let $K$ be a field [i.e., a ring such that $K^{\times} = K^{\triangleright}$]. Then we shall write $\boldsymbol{\mu}(K) \stackrel{\mathrm{def}}{=} (K^{\times})_{\mathrm{tor}}$ for the group of roots of unity of $K$ and $K_{\times}$ for the monoid obtained by forgetting the additive structure of $K$. Thus, we have a natural isomorphism $(K^{\times})^{\circledast} \xrightarrow{\sim} K_{\times}$ of monoids. If, moreover, $K$ is *algebraically closed* and *of characteristic zero*, then we shall write

$$\Lambda(K) \stackrel{\mathrm{def}}{=} \varprojlim_{n} \boldsymbol{\mu}(K)[n] = \varprojlim_{n} K^{\times}[n]$$

— where the projective limits are taken over the positive integers $n$ — and refer to $\Lambda(K)$ as the *cyclotome* associated to $K$. Thus, the cyclotome is [noncanonically] isomorphic to $\widehat{\mathbb{Z}}$; moreover, we have a natural identification $\boldsymbol{\mu}(K)[n] = \Lambda(K)/n\Lambda(K)$.

We shall refer to a field which is isomorphic to a finite extension of $\mathbb{Q}$ as an *NF* [i.e., a *number field*]. We shall refer to a field which is isomorphic to a finite extension of $\mathbb{Q}_p$ for some prime number $p$ as an *MLF* [i.e., a *mixed characteristic local field*].

## 1. Review of the Local Theory

In the present §1, let us review *mono-anabelian reconstructions* of various objects which arise from an *MLF* [cf. Theorem 1.4 below].

In the present §1, let

$$k$$

be an *MLF*. We shall write

- $\mathcal{O}_k \subseteq k$ for the ring of integers of $k$,
- $\mathfrak{m}_k \subseteq \mathcal{O}_k$ for the maximal ideal of $\mathcal{O}_k$,
- $\underline{k} \stackrel{\mathrm{def}}{=} \mathcal{O}_k/\mathfrak{m}_k$ for the residue field of $\mathcal{O}_k$,
- $p_k \stackrel{\mathrm{def}}{=} \mathrm{char}(\underline{k})$ for the characteristic of $\underline{k}$,
- $d_k$ for the extension degree of $k$ over the subfield of $k$ obtained by forming the closure of the prime field contained in $k$ [i.e., "$[k : \mathbb{Q}_{p_k}]$"],
- $\mathrm{ord}_k \colon k^\times \twoheadrightarrow \mathbb{Z}$ for the [uniquely determined] surjective valuation on $k$,
- $e_k \stackrel{\mathrm{def}}{=} \mathrm{ord}_k(p_k)$ for the absolute ramification index of $k$, and
- $f_k$ for the extension degree of $\underline{k}$ over the prime field contained in $\underline{k}$ [i.e., "$[\underline{k} : \mathbb{F}_{p_k}]$"].

Let

$$\overline{k}$$

be an algebraic closure of $k$. We shall write

- $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of $k$ with respect to $\overline{k}/k$,
- $I_k \subseteq G_k$ for the inertia subgroup of $G_k$,
- $P_k \subseteq I_k$ for the wild inertia subgroup of $G_k$, and
- $\mathrm{Frob}_k \in G_k/I_k$ for the [$\sharp\underline{k}$-th power] Frobenius element of $G_k/I_k$.

**DEFINITION 1.1.** — Let $G$ be a group. Then we shall refer to a collection of data

$$(K,\ \overline{K},\ \alpha\colon \mathrm{Gal}(\overline{K}/K) \xrightarrow{\sim} G)$$

consisting of an MLF $K$, an algebraic closure $\overline{K}$ of $K$, and an isomorphism $\alpha\colon \mathrm{Gal}(\overline{K}/K) \xrightarrow{\sim} G$ of groups as an *MLF-envelope* for $G$. We shall say that the group $G$ is *of MLF-type* if there exists an MLF-envelope for $G$.

**PROPOSITION 1.2.** — *Let $G$ be a group* **of MLF-type**. *Then the following hold:*

(i) *The natural homomorphism*

$$G \longrightarrow \varprojlim_N G/N$$

*— where the projective limit is taken over the normal subgroups $N \subseteq G$ of $G$* **of finite index** *— is an* **isomorphism** *of groups. By means of this isomorphism, we always regard a group* **of MLF-type** *as a* **profinite group**.

(ii) *Let*

$$(k,\ \overline{k},\ \alpha\colon G_k \xrightarrow{\sim} G)$$

*be an* **MLF-envelope** *for* $G$. *Then the isomorphism* $\alpha$ *is an* **isomorphism of profinite groups**.

PROOF. — Assertion (i) follows from [12], Theorem 1.1, together with the fact that the absolute Galois group of an MLF is *topologically finitely generated* [cf., e.g., [11], Theorem 7.4.1]. Assertion (ii) follows from assertion (i). This completes the proof of Proposition 1.2. $\square$

**REMARK 1.2.1.** — One verifies immediately that every open subgroup of a profinite group *of MLF-type* is *of MLF-type*.

**LEMMA 1.3.** — *The following hold:*

(i) *The reciprocity homomorphism* $k^\times \to G_k^{\mathrm{ab}}$ *in local class field theory determines a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}_k^\times \arrow[r] \arrow[d, "\wr"'] & k^\times \arrow[r, "\mathrm{ord}_k"] \arrow[d, "\wr"'] & \mathbb{Z} \arrow[r] \arrow[d, "\wr"'] & 1 \\
1 \arrow[r] & \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}}) \arrow[r] \arrow[d, equal] & G_k^{\mathrm{ab}} \times_{G_k/I_k} \mathrm{Frob}_k^{\mathbb{Z}} \arrow[r] \arrow[d] & \mathrm{Frob}_k^{\mathbb{Z}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}}) \arrow[r] & G_k^{\mathrm{ab}} \arrow[r] & G_k/I_k \arrow[r] & 1
\end{tikzcd}$$

— *where the horizontal sequences are* **exact**, *the upper vertical arrows are* **isomorphisms**, *the lower vertical arrows are the natural inclusions, the upper right-hand vertical arrow maps* $1 \in \mathbb{Z}$ *to* $\mathrm{Frob}_k \in \mathrm{Frob}_k^{\mathbb{Z}}$, *and we write* $\mathrm{Frob}_k^{\mathbb{Z}} \subseteq G_k/I_k$ *for the* [*discrete*] *subgroup of* $G_k/I_k$ *generated by* $\mathrm{Frob}_k$.

(ii) *The prime number* $p_k$ *may be* **characterized** *as a unique prime number* $l$ *such that* $\log_l(\sharp(G_k^{\mathrm{ab/tor}}/l \cdot G_k^{\mathrm{ab/tor}})) \geq 2$.

(iii) *It holds that* $d_k = \log_{p_k}(\sharp(G_k^{\mathrm{ab/tor}}/p_k \cdot G_k^{\mathrm{ab/tor}})) - 1$.

(iv) *It holds that* $f_k = \log_{p_k}(1 + \sharp((G_k^{\mathrm{ab}})_{\mathrm{tor}})^{(p_k')})$.

(v) *It holds that* $e_k = d_k/f_k$.

(vi) *The closed subgroup* $I_k \subseteq G_k$ *may be* **characterized** *as the intersection of the normal open subgroups* $N \subseteq G_k$ *of* $G_k$ *such that* $e_k = e_{k_N}$, *where we write* $k_N$ *for the intermediate extension of* $\overline{k}/k$ *corresponding to* $N$.

(vii) *The closed subgroup* $P_k \subseteq G_k$ *may be* **characterized** *as the intersection of the normal open subgroups* $N \subseteq G_k$ *of* $G_k$ *such that the integer* $e_{k_N}/e_k$ *is prime to* $p_k$, *where we write* $k_N$ *for the intermediate extension of* $\overline{k}/k$ *corresponding to* $N$.

(viii) *The element* $\mathrm{Frob}_k \in G_k/I_k$ *may be* **characterized** *as a unique element of* $G_k/I_k$ *such that the action on* [*the abelian group*] $I_k/P_k$ *by conjugation is given by multiplication by* $p_k^{f_k}$.

(ix) *The upper left-hand vertical arrow of the diagram of* (i) *determines an* **isomorphism** $\underline{k}^\times \xrightarrow{\sim} \mathrm{Im}(I_k \hookrightarrow G_k \twoheadrightarrow G_k^{\mathrm{ab}})^{(p_k')}$ *of modules.*

(x) *The exact sequences of* $G_k$*-modules*

$$1 \longrightarrow \boldsymbol{\mu}(\overline{k})[n] \longrightarrow \overline{k}^\times \xrightarrow{\ n\ } \overline{k}^\times \longrightarrow 1$$

— *where* $n$ *ranges over the positive integers* — *determine an* **injection**

$$\mathrm{Kmm}_k\colon\ k^\times \hookrightarrow H^1(G_k, \Lambda(\overline{k})).$$

PROOF. — Assertion (i) follows from *local class field theory* [cf., e.g., [10], Chapter V, §1]. Assertions (ii), (iii), (iv), (ix) follow immediately from assertion (i), together with the well-known explicit description of the topological module $k^\times$ [cf., e.g., [10], Chapter II, Proposition 5.3; also [10], Chapter II, Proposition 5.7, (i)]. Assertion (v) follows from [10], Chapter II, Proposition 6.8. Assertions (vi), (vii) follow immediately from the definitions of $I_k$, $P_k$, respectively. Assertion (viii) follows immediately from [11], Proposition 7.5.2, together with the easily verified *faithfulness* of the action of "$\Gamma$" [in *loc. cit.*] on "$\widehat{\mathbb{Z}}^{(p')}(1)$" [in *loc. cit.*]. Assertion (x) follows immediately from the fact that there is *no nontrivial divisible element* in $k^\times$ [cf., e.g., [10], Chapter II, Proposition 5.7, (i)]. This completes the proof of Lemma 1.3. □

**THEOREM 1.4.** — *In the notation introduced at the beginning of* §1, *let* $G$ *be a profinite group* **of MLF-type** [*cf. Definition* 1.1*; Proposition* 1.2, (i)]. *We shall define various objects which arise from* $G$ *as follows:*

(1) *It follows from Lemma* 1.3, (ii), *that there exists a* **unique** *prime number* $l$ *such that* $\log_l(\sharp(G^{\mathrm{ab/tor}}/l \cdot G^{\mathrm{ab/tor}})) \geq 2$. *We shall write*

$$p(G)$$

*for this prime number.*

(2) *We shall write*

$$d(G) \stackrel{\mathrm{def}}{=} \log_{p(G)}(\sharp(G^{\mathrm{ab/tor}}/p(G) \cdot G^{\mathrm{ab/tor}})) - 1,$$

$$f(G) \stackrel{\mathrm{def}}{=} \log_{p(G)}(1 + \sharp((G^{\mathrm{ab}})_{\mathrm{tor}})^{(p(G)')}),$$

$$e(G) \stackrel{\mathrm{def}}{=} d(G)/f(G).$$

*Note that it follows from Lemma* 1.3, (ii), (iii), (iv), *that* $d(G)$, $f(G)$, $e(G)$ *are* **positive integers**.

(3) *We shall write*

$$I(G) \subseteq G$$

*for the normal closed subgroup obtained by forming the intersection of the normal open subgroups* $N \subseteq G$ *of* $G$ *such that* $e(N) = e(G)$ *and*

$$P(G) \subseteq G$$

*for the normal closed subgroup obtained by forming the intersection of the normal open subgroups $N \subseteq G$ of $G$ such that the positive integer $e(N)/e(G)$ is prime to $p(G)$.*

(4) *It follows from Lemma* 1.3, (viii), *that there exists a* **unique** *element of $G/I(G)$ whose action on* [*the abelian group*] $I(G)/P(G)$ *by conjugation is given by multiplication by $p(G)^{f(G)}$. We shall write*

$$\mathrm{Frob}(G) \in G/I(G)$$

*for this element.*

(5) *We shall write*

$$\mathcal{O}^{\times}(G) \stackrel{\mathrm{def}}{=} \mathrm{Im}(I(G) \hookrightarrow G \twoheadrightarrow G^{\mathrm{ab}})$$

*for the image of $I(G)$ in $G^{\mathrm{ab}}$. By considering the topology induced by the topology of $I(G)$, we regard $\mathcal{O}^{\times}(G)$ as a* **profinite**, *hence also topological, module. We shall write*

$$\underline{k}^{\times}(G) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}(G)^{(p(G)')}$$

*for the module obtained by forming the maximal pro-prime-to-$p(G)$ quotient of $\mathcal{O}^{\times}(G)$.*

(6) *We shall write*

$$k^{\times}(G) \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{Z}}$$

*— where we write* $\mathrm{Frob}(G)^{\mathbb{Z}}$ *for the* [*discrete*] *subgroup of $G/I(G)$ generated by* $\mathrm{Frob}(G)$ *— and*

$$\mathcal{O}^{\triangleright}(G) \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{N}}$$

*— where we write* $\mathrm{Frob}(G)^{\mathbb{N}}$ *for the* [*discrete*] *submonoid of $G/I(G)$ generated by* $\mathrm{Frob}(G)$. *Note that the topology of $\mathcal{O}^{\times}(G)$ discussed in* (5) *naturally determines respective structures of* **topological** *module, monoid on $k^{\times}(G)$, $\mathcal{O}^{\triangleright}(G)$.*

(7) *We shall write*

$$\mathrm{ord}(G)\colon\ k^{\times}(G) \twoheadrightarrow \mathrm{Frob}(G)^{\mathbb{Z}}$$

*for the natural surjection. Thus, we have an exact sequence of topological modules*

$$1 \longrightarrow \mathcal{O}^{\times}(G) \longrightarrow k^{\times}(G) \xrightarrow{\mathrm{ord}(G)} \mathrm{Frob}(G)^{\mathbb{Z}} \longrightarrow 1.$$

(8) *We shall write*

$$k_{\times}(G) \stackrel{\mathrm{def}}{=} k^{\times}(G)^{\circledast}, \quad \underline{k}_{\times}(G) \stackrel{\mathrm{def}}{=} \underline{k}^{\times}(G)^{\circledast}.$$

(9) *We shall write*

$$\overline{k}^{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H} k^{\times}(H), \quad \overline{k}_{\times}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H} k_{\times}(H) = \overline{k}^{\times}(G)^{\circledast},$$

$$\boldsymbol{\mu}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{H} (H^{\mathrm{ab}})_{\mathrm{tor}} = \overline{k}^{\times}(G)_{\mathrm{tor}}$$

*— where the injective limits are taken over the open subgroups $H \subseteq G$ of $G$, and the transition morphisms in the limits are given by the homomorphisms determined by the transfer maps — and*

$$\Lambda(G) \stackrel{\mathrm{def}}{=} \varprojlim_{n} \boldsymbol{\mu}(G)[n]$$

*— where the projective limit is taken over the positive integers $n$. Note that $G$ acts on $\overline{k}^{\times}(G)$, $\overline{k}_{\times}(G)$, $\boldsymbol{\mu}(G)$, and $\Lambda(G)$ by conjugation. We shall refer to the $G$-module $\Lambda(G)$ as*

*the* **cyclotome** *associated to* $G$. *Note that one verifies immediately from our construction that the cyclotome has a natural structure of* **profinite** [*cf. also the above definition of* $\Lambda(G)$], *hence also topological,* $G$*-module; moreover, we have a natural identification* $\boldsymbol{\mu}(G)[n] = \Lambda(G)/n\Lambda(G)$.

(10) *It follows from Lemma* 1.3, (x), *that the exact sequences of* $G$*-modules*

$$1 \longrightarrow \Lambda(G)/n\Lambda(G) \longrightarrow \overline{k}^{\times}(G) \xrightarrow{n} \overline{k}^{\times}(G) \longrightarrow 1$$

— *where* $n$ *ranges over the positive integers — determine an* **injection**

$$k^{\times}(G) \hookrightarrow H^1(G, \Lambda(G)).$$

*We shall write*

$$\mathrm{Kmm}(G)$$

*for this injection.*

*Let*

$$(k,\ \overline{k},\ \alpha\colon G_k \xrightarrow{\sim} G)$$

*be an* **MLF-envelope** *for* $G$ [*cf. Definition* 1.1]. *Then the following hold:*

(i) *It holds that*

$$p_k = p(G), \quad d_k = d(G), \quad e_k = e(G), \quad f_k = f(G).$$

(ii) *The isomorphism* $\alpha$ *determines* **isomorphisms**

$$I_k \xrightarrow{\sim} I(G), \quad P_k \xrightarrow{\sim} P(G).$$

*Moreover, the resulting isomorphism* $G_k/I_k \xrightarrow{\sim} G/I(G)$ **maps** $\mathrm{Frob}_k$ *to* $\mathrm{Frob}(G)$.

(iii) *The isomorphism* $\alpha$, *together with the field structure of* $k$, *determines a* **commutative** *diagram of topological modules*

$$\begin{array}{ccccccc}
\underline{k}^{\times} & \longleftarrow & \mathcal{O}_k^{\times} & \longrightarrow & \mathcal{O}_k^{\triangleright} & \longrightarrow & k^{\times} \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\underline{k}^{\times}(G) & \longleftarrow & \mathcal{O}^{\times}(G) & \longrightarrow & \mathcal{O}^{\triangleright}(G) & \longrightarrow & k^{\times}(G)
\end{array}$$

— *where the horizontal arrows are natural homomorphisms, and the vertical arrows are* **isomorphisms**. *Thus, the left-hand, right-hand vertical arrows of this diagram determine* **isomorphisms** *of monoids*

$$\underline{k}_{\times} \xrightarrow{\sim} \underline{k}_{\times}(G), \quad k_{\times} \xrightarrow{\sim} k_{\times}(G),$$

*respectively.*

(iv) *The isomorphism* $\alpha$, *together with the field structures of the various fields involved, determines* **isomorphisms** *of modules*

$$\overline{k}^{\times} \xrightarrow{\sim} \overline{k}^{\times}(G), \quad \boldsymbol{\mu}(\overline{k}) \xrightarrow{\sim} \boldsymbol{\mu}(G), \quad \Lambda(\overline{k}) \xrightarrow{\sim} \Lambda(G)$$

*and an* **isomorphism** *of monoids*

$$\overline{k}_{\times} \xrightarrow{\sim} \overline{k}_{\times}(G)$$

*which are* **compatible** *with the natural actions of* $G_k$ *and* $G$ *relative to* $\alpha$.

(v) *The isomorphisms* $k^\times \xrightarrow{\sim} k^\times(G)$ *of* (iii) *and* $\Lambda(\overline{k}) \xrightarrow{\sim} \Lambda(G)$ *of* (iv) *fit into a* **commutative** *diagram*

$$\begin{array}{ccc} k^\times & \xrightarrow{\mathrm{Kmm}_k} & H^1(G_k, \Lambda(\overline{k})) \\ \wr\downarrow & & \wr\downarrow \\ k^\times(G) & \xrightarrow{\mathrm{Kmm}(G)} & H^1(G, \Lambda(G)). \end{array}$$

PROOF. — These assertions follow immediately from Lemma 1.3, together with the various definitions involved. ☐

**REMARK 1.4.1.**

(i) It is well-known [cf., e.g., [4], §2] that there exists an MLF $k_\Box$, where $\Box \in \{\circ, \bullet\}$, such that $k_\circ$ is *not isomorphic* to $k_\bullet$ but the absolute Galois group of $k_\circ$ [for some choice of an algebraic closure of $k_\circ$] is *isomorphic* to the absolute Galois group of $k_\bullet$ [for some choice of an algebraic closure of $k_\bullet$]. Moreover, it is known [cf., e.g., the final portion of [11], Chapter VII] that, for each MLF $k$ such that $p_k$ is *odd*, there exists an outer automorphism of the absolute Galois group of $k$ which does *not arise* from an automorphism of $k$.

(ii) It follows immediately from the discussion of (i) that

> there is *no functorial "group-theoretic" algorithm* [as discussed in Theorem 1.4] for reconstructing, from the *absolute Galois group of an MLF*, [the field structure of] the *MLF*.

(iii) On the other hand, there are some results concerning the *geometricity* of an outer homomorphism between the absolute Galois groups of MLF. For instance, in [5], S. Mochizuki proved that, for an outer isomorphism between the absolute Galois groups of MLF, it holds that the outer isomorphism is *geometric* [i.e., arises from a — necessarily unique — isomorphism of the MLF] if and only if the outer isomorphism preserves the [positively indexed] higher ramification filtrations in the upper numbering. Mochizuki also gave, in [7], §3 [cf. [7], Theorem 3.5; [7], Corollary 3.7], other necessary and sufficient conditions for an outer open homomorphism between the absolute Galois groups of MLF to be *geometric* [i.e., arise from a — necessarily unique — embedding of the MLF]. Moreover, in [2], the author proved that, for an outer open homomorphism between the absolute Galois groups of MLF, it holds that the outer open homomorphism is *geometric* if and only if the outer open homomorphism is *Hodge-Tate-preserving* [i.e., the pull-back, via the outer open homomorphism under consideration, of a Hodge-Tate representation is still Hodge-Tate].

**REMARK 1.4.2.**

(i) In the proof of the main result of [5] [cf. Remark 1.4.1, (iii)], Mochizuki essentially proved the following assertion:

> For $\Box \in \{\circ, \bullet\}$, let $k_\Box$ be an MLF. Write $G_\Box$ for the absolute Galois group of $k_\Box$ [which is well-defined up to conjugation]. Let $\alpha\colon G_\circ \xrightarrow{\sim} G_\bullet$ be an outer isomorphism of profinite groups. Then it holds that $\alpha$ is

> *geometric* if and only if, in the notation of Theorem 1.4, the following condition is satisfied: For every open subgroup $G_\circ^\dagger \subseteq G_\circ$ of $G_\circ$, if we write $G_\bullet^\dagger \subseteq G_\bullet$ for the open subgroup of $G_\bullet$ corresponding to $G_\circ^\dagger \subseteq G_\circ$ via $\alpha$, then the isomorphism $k^\times(G_\circ^\dagger) \xrightarrow{\sim} k^\times(G_\bullet^\dagger)$ induced by $\alpha$ maps, for each positive integer $n$, the submodule of $k^\times(G_\circ^\dagger)$ corresponding to "$1+\mathfrak{m}_k^n$" bijectively onto the submodule of $k^\times(G_\bullet^\dagger)$ corresponding to "$1+\mathfrak{m}_k^n$".

Now let us observe that, in the above notation, it follows from the *functorial "group-theoretic" algorithm* discussed in Theorem 1.4 that the induced isomorphism $k^\times(G_\circ^\dagger) \xrightarrow{\sim} k^\times(G_\bullet^\dagger)$ maps the submodule of $k^\times(G_\circ^\dagger)$ corresponding to "$1+\mathfrak{m}_k$" [i.e., the kernel of the natural surjection $\mathcal{O}^\times(G_\circ^\dagger) \twoheadrightarrow \underline{k}^\times(G_\circ^\dagger)$] bijectively onto the submodule of $k^\times(G_\bullet^\dagger)$ corresponding to "$1+\mathfrak{m}_k$" [i.e., the kernel of the natural surjection $\mathcal{O}^\times(G_\bullet^\dagger) \twoheadrightarrow \underline{k}^\times(G_\bullet^\dagger)$].

(ii) By the discussion of (i) and Remark 1.4.1, (ii), we obtain the following *observation*:

> There is *no functorial "group-theoretic" algorithm* [as discussed in Theorem 1.4] for reconstructing, from a group *G of MLF-type*, the family of submodules of the module $k^\times(G)$ of Theorem 1.4, (6), corresponding to the family of submodules "$\{1+\mathfrak{m}_k^n\}_{n\geq 1}$" of "$k^\times$".

**REMARK 1.4.3.**

(i) Write $k_+$, $(\mathcal{O}_k)_+$ for the modules obtained by forming the underlying *additive* modules of the rings $k$, $\mathcal{O}_k$, respectively. Then, by considering the $p_k$-adic logarithm on $k$, we obtain an *isomorphism* $(\mathcal{O}_k^\times)^{\mathrm{pf}} \xrightarrow{\sim} k_+$ of modules. Thus, by assigning $G \mapsto \mathcal{O}^\times(G)^{\mathrm{pf}}$, we obtain a *functorial "group-theoretic" algorithm* [as discussed in Theorem 1.4] for reconstructing, from a group *G of MLF-type*, the module corresponding to "$k_+$". Then another interpretation of the assertion of Remark 1.4.2, (i), is as follows:

> For $\Box \in \{\circ, \bullet\}$, let $k_\Box$ be an MLF. Write $G_\Box$ for the absolute Galois group of $k_\Box$ [which is well-defined up to conjugation]. Let $\alpha\colon G_\circ \xrightarrow{\sim} G_\bullet$ be an outer isomorphism of profinite groups. Then it holds that $\alpha$ is *geometric* if and only if, in the notation of Theorem 1.4, the following condition is satisfied: For every open subgroup $G_\circ^\dagger \subseteq G_\circ$ of $G_\circ$, if we write $G_\bullet^\dagger \subseteq G_\bullet$ for the open subgroup of $G_\bullet$ corresponding to $G_\circ^\dagger \subseteq G_\circ$ via $\alpha$, then the isomorphism $\mathcal{O}^\times(G_\circ^\dagger)^{\mathrm{pf}} \xrightarrow{\sim} \mathcal{O}^\times(G_\bullet^\dagger)^{\mathrm{pf}}$ induced by $\alpha$ maps the submodule of $\mathcal{O}^\times(G_\circ^\dagger)^{\mathrm{pf}}$ corresponding to "$(\mathcal{O}_k)_+ \subseteq k_+$" bijectively onto the submodule of $\mathcal{O}^\times(G_\bullet^\dagger)^{\mathrm{pf}}$ corresponding to "$(\mathcal{O}_k)_+ \subseteq k_+$".

(ii) By the discussion of (i) and Remark 1.4.1, (ii), we obtain the following *observation*:

> There is *no functorial "group-theoretic" algorithm* [as discussed in Theorem 1.4] for reconstructing, from a group *G of MLF-type*, the submodule of the module $\mathcal{O}^\times(G)^{\mathrm{pf}}$ corresponding to the submodule "$(\mathcal{O}_k)_+$" of "$k_+$".

**LEMMA 1.5.** — *The following hold:*

(i) *It holds that*

$$\mathcal{O}_k^\times \;=\; \mathrm{Ker}\big(k^\times \overset{\mathrm{Kmm}_k}{\hookrightarrow} H^1(G_k, \Lambda(\overline{k})) \;\to\; H^1(I_k, \Lambda(\overline{k})^{(p'_k)})\big).$$

(ii) *The homomorphism*

$$\mathcal{O}_k^\times \longrightarrow H^1(G_k/I_k, \Lambda(\overline{k})^{(p_k')})$$

*determined by* $\mathrm{Kmm}_k$ [*cf.* (i)] *induces an* **isomorphism**

$$\underline{k}^\times \xrightarrow{\sim} H^1(G_k/I_k, \Lambda(\overline{k})^{(p_k')}).$$

PROOF. — These assertions follow immediately from the well-known explicit description of the topological module $k^\times$ [cf., e.g., [10], Chapter II, Proposition 5.3; also [10], Chapter II, Proposition 5.7, (i)], together with the *Kummer theory* for $k$, $\underline{k}$. □

## 2. Reconstruction of the Additive Structure on an NF-monoid

In the present §2, we discuss an *NF-monoid* [cf. Definition 2.3 below]. In particular, we obtain a *mono-anabelian reconstruction* of the "*additive structure*" on an NF-monoid [cf. Theorem 2.9 below]. Note that the main result of the present §2 was already essentially proved in [3]; however, the author discussed, in [3], the issue of reconstruction of additive structure in not a "*mono-anabelian*" *fashion* but a "*bi-anabelian*" *fashion*.

In the present §2, let

$$F$$

be an *NF*. We shall write

- $\mathcal{O}_F \subseteq F$ for the ring of integers of $F$,
- $\mathcal{V}_F$ for the set of nonarchimedean primes of $F$, and
- $F_{\mathrm{prm}} \subseteq F$ for the prime field contained in $F$ [i.e., "$\mathbb{Q}$"].

If $v \in \mathcal{V}_F$, then we shall write

- $\mathrm{ord}_v \colon F^\times \twoheadrightarrow \mathbb{Z}$ for the [uniquely determined] surjective valuation associated to $v$,
- $\mathcal{O}_{(v)} \subseteq F$ for the subring of $F$ obtained by forming the localization of $\mathcal{O}_F$ at the maximal ideal corresponding to $v$,
- $\mathfrak{m}_{(v)} \subseteq \mathcal{O}_{(v)}$ for the maximal ideal of $\mathcal{O}_{(v)}$,
- $\kappa_v \stackrel{\mathrm{def}}{=} \mathcal{O}_{(v)}/\mathfrak{m}_{(v)}$ for the residue field of $\mathcal{O}_{(v)}$,
- $\mathrm{char}(v) \stackrel{\mathrm{def}}{=} \mathrm{char}(\kappa_v)$ for the characteristic of $\kappa_v$, and
- $\mathcal{O}_{(v)}^{\equiv 1} \stackrel{\mathrm{def}}{=} 1 + \mathfrak{m}_{(v)} \subseteq \mathcal{O}_{(v)}^\times$ for the kernel of the natural homomorphism $\mathcal{O}_{(v)}^\times \twoheadrightarrow \kappa_v^\times$.

Finally, for $a \in F^\times$, we shall write

- $\mathrm{Supp}(a) \stackrel{\mathrm{def}}{=} \{\, v \in \mathcal{V}_F \mid \mathrm{ord}_v(a) \neq 0 \,\} \subseteq \mathcal{V}_F$.

**Definition 2.1.** — We shall say that the NF $F$ is *prime* if $F = F_{\mathrm{prm}}$.

**Definition 2.2.** — We shall refer to the collection of data

$$(F_\times,\ \mathcal{O}_F^{\triangleright} \subseteq F_\times,\ \mathcal{V}_F,\ \{\mathcal{O}_{(v)}^{\equiv 1} \subseteq F_\times\}_{v \in \mathcal{V}_F})$$

[consisting of the monoid $F_\times$, the submonoid $\mathcal{O}_F^{\triangleright} \subseteq F_\times$ of $F_\times$, the set $\mathcal{V}_F$, and the submonoid $\mathcal{O}_{(v)}^{\equiv 1} \subseteq F_\times$ of $F_\times$ labeled by each $v \in \mathcal{V}_F$] as the *NF-monoid* associated to $F$.

**Definition 2.3.** — Let

$$\mathcal{M} \ = \ (M,\ O^{\triangleright} \subseteq M,\ S,\ \{O_s^{\equiv 1} \subseteq M\}_{s \in S})$$

be a collection of data consisting of a monoid $M$ [the monoid operation of $M$ will be written *multiplicatively*], a submonoid $O^{\triangleright} \subseteq M$ of $M$, a set $S$, and a submonoid $O_s^{\equiv 1} \subseteq M$ of $M$ labeled by each $s \in S$. Then we shall refer to an isomorphism of the NF-monoid [cf. Definition 2.2] associated to an NF (respectively, a prime NF — cf. Definition 2.1) with

$\mathcal{M}$ [in the evident sense, i.e., a suitable pair consisting of an isomorphism of "$F_\times$" with $M$ and a bijection of "$\mathcal{V}_F$" with $S$] as an *NF-envelope* (respectively, a *prime NF-envelope*) for $\mathcal{M}$. We shall say that $\mathcal{M}$ is an *NF-monoid* (respectively, a *prime NF-monoid*) if there exists an NF-envelope (respectively, a prime NF-envelope) for $\mathcal{M}$.

**LEMMA 2.4.** — *The following hold:*

(i) *It holds that the NF $F$ is* **prime** *if and only if, for all but finitely many $v \in \mathcal{V}_F$, it holds that $\sharp\kappa_v$ is a* **prime** *number.*

(ii) *The element $0 \in F_\times$ of $F_\times$ may be* **characterized** *as a unique element of $F_\times \setminus F^\times$.*

(iii) *The element $1 \in F_\times$ of $F_\times$ may be* **characterized** *as a unique element $a \in F_\times$ such that $ax = x$ for any $x \in F_\times$.*

(iv) *The element $-1 \in F_\times$ of $F_\times$ may be* **characterized** *as a unique element $a \in F_\times$ such that $a \neq 1$ but $a^2 = 1$.*

(v) *Let $v \in \mathcal{V}_F$. Then the natural injection $\mathcal{O}^\times_{(v)} \hookrightarrow F^\times$ determines an* **isomorphism** $\kappa_v^\times \xrightarrow{\sim} (F^\times/\mathcal{O}^{\equiv 1}_{(v)})_{\mathrm{tor}}$.

(vi) *Let $v \in \mathcal{V}_F$. Then the prime number* $\mathrm{char}(v)$ *may be* **characterized** *as a unique prime number which divides $\sharp\kappa_v$.*

(vii) *Let $v \in \mathcal{V}_F$. Then the $\{\pm 1\}$-orbit of the valuation* $\mathrm{ord}_v \colon F^\times \to \mathbb{Z}$ *may be* **characterized** *as the $\{\pm 1\}$-orbit of the homomorphism $F^\times \to \mathbb{Z}$ obtained by forming the composite*

$$F^\times \twoheadrightarrow F^\times/\mathcal{O}^{\equiv 1}_{(v)} \twoheadrightarrow (F^\times/\mathcal{O}^{\equiv 1}_{(v)})^{\mathrm{ab/tor}} \xrightarrow{\sim} \mathbb{Z}$$

*— where we regard $F^\times/\mathcal{O}^{\equiv 1}_{(v)}$ as a topological group by the discrete topology, and the "$\xrightarrow{\sim}$" is an isomorphism of group. Moreover, the valuation* $\mathrm{ord}_v \colon F^\times \to \mathbb{Z}$ *may be* **characterized** *as a unique element of this orbit which maps $\mathcal{O}^\rhd_F \subseteq F^\times$ to $\mathbb{Z}_{\geq 0} \subseteq \mathbb{Z}$.*

(viii) *Let $v \in \mathcal{V}_F$. Then it holds that $\mathcal{O}^\times_{(v)} = \mathrm{Ker}(\mathrm{ord}_v)$.*

PROOF. — Assertion (i) follows immediately from *Čebotarev's density theorem* [cf. also [10], Chapter VII, Corollary 13.7]. Assertions (ii), (iii), (iv), (vi), (viii) follow from the various definitions involved. Assertion (v) and the first assertion of assertion (vii) follow immediately from the fact that $F^\times/\mathcal{O}^\times_{(v)}$ is [noncanonically] isomorphic to $\mathbb{Z}$, hence also *torsion-free* [cf. also the proof of [3], Lemma 1.5, (i)]. The final assertion of assertion (vii) follows from the various definitions involved. This completes the proof of Lemma 2.4. $\square$

**PROPOSITION 2.5.** — *Let*

$$\mathcal{M} = (M,\ O^\rhd \subseteq M,\ S,\ \{O^{\equiv 1}_s \subseteq M\}_{s\in S})$$

*be an* **NF-monoid**. *We shall define various objects which arise from $\mathcal{M}$ as follows:*

(1) *It follows from Lemma 2.4, (ii), that there exists a* **unique** *element of $M \setminus M^\times$. We shall write*

$$0_\mathcal{M} \in M$$

*for this element.*

(2) *It follows from Lemma* 2.4, (iii), *that there exists a* **unique** *element* $a \in M$ *of* $M$ *such that* $ax = x$ *for any* $x \in M$. *We shall write*

$$1_{\mathcal{M}} \ \in \ M$$

*for this element.*

(3) *It follows from Lemma* 2.4, (iv), *that there exists a* **unique** *element* $a \in M$ *of* $M$ *such that* $a \neq 1_{\mathcal{M}}$ *but* $a^2 = 1_{\mathcal{M}}$. *We shall write*

$$-1_{\mathcal{M}} \ \in \ M$$

*for this element.*

(4) *Let* $s \in S$. *Then we shall write*

$$\underline{O}_s^{\times} \ \stackrel{\mathrm{def}}{=} \ (M^{\times}/O_s^{\equiv 1})_{\mathrm{tor}}, \quad (\underline{O}_s)_{\times} \ \stackrel{\mathrm{def}}{=} \ (\underline{O}_s^{\times})^{\circledast}.$$

(5) *Let* $s \in S$. *Then it follows from Lemma* 2.4, (v), (vi), *that there exists a* **unique** *prime number which divides* $\sharp(\underline{O}_s)_{\times}$. *We shall write*

$$\mathrm{char}(s)$$

*for this prime number.*

(6) *Let* $s \in S$. *Then we shall write*

$$Z_s \ \stackrel{\mathrm{def}}{=} \ (M^{\times}/O_s^{\equiv 1})^{\mathrm{ab/tor}}$$

— *where we regard* $M^{\times}/O_s^{\equiv 1}$ *as a topological group by the discrete topology* — *and*

$$\text{pre-ord}_s \colon \ M^{\times} \ \twoheadrightarrow \ Z_s$$

*for the natural surjection.*

(7) *Let* $s \in S$ *and* $a \in M^{\times}$. *Then we shall define an integer*

$$\mathrm{ord}_s(a) \ \in \ \mathbb{Z}$$

*as follows: Write* $\text{pre-ord}_s(a)^{\mathbb{N}} \subseteq \text{pre-ord}_s(a)^{\mathbb{Z}} \subseteq Z_s$ *for the submonoid, subgroup of* $Z_s$ *generated by* $\text{pre-ord}_s(a) \in Z_s$ *and* $i_{s,a} \stackrel{\mathrm{def}}{=} [Z_s : \text{pre-ord}_s(a)^{\mathbb{Z}}]$. *Then*

$$\mathrm{ord}_s(a) \ \stackrel{\mathrm{def}}{=} \ \begin{cases} 0 & \textit{if } i_{s,a} = \infty, \\ i_{s,a} & \textit{if } i_{s,a} < \infty \textit{ and } \sharp\bigl(\text{pre-ord}_s(a)^{\mathbb{N}} \cap \text{pre-ord}_s(O^{\rhd})\bigr) \neq 1, \\ -i_{s,a} & \textit{if } i_{s,a} < \infty \textit{ and } \sharp\bigl(\text{pre-ord}_s(a)^{\mathbb{N}} \cap \text{pre-ord}_s(O^{\rhd})\bigr) = 1. \end{cases}$$

(8) *Let* $a \in M^{\times}$. *Then we shall write*

$$\mathrm{Supp}(a) \ \stackrel{\mathrm{def}}{=} \ \{\, s \in S \mid \mathrm{ord}_s(a) \neq 0 \,\} \ \subseteq \ S.$$

(9) *Let* $s \in S$. *Then we shall write*

$$O_s^{\times} \ \stackrel{\mathrm{def}}{=} \ \mathrm{Ker}(\mathrm{ord}_s) \ \subseteq \ M^{\times}.$$

*Let*

$$(\phi \colon F_{\times} \xrightarrow{\sim} M, \ \tau \colon \mathcal{V}_F \xrightarrow{\sim} S)$$

*be an* **NF-envelope** *for* $\mathcal{M}$. *Then the following hold:*

(i) *It holds that* $\mathcal{M}$ *is* **prime** *if and only if, for all but finitely many* $s \in S$*, it holds that* $\sharp(\underline{O}_s)_\times$ *is a* **prime** *number.*

(ii) *The isomorphism* $\phi\colon F_\times \xrightarrow{\sim} M$ *of monoids maps* $0$, $1$, $-1$ *to* $0_\mathcal{M}$, $1_\mathcal{M}$, $-1_\mathcal{M}$, *respectively.*

(iii) *Let* $v \in \mathcal{V}_F$. *Write* $s \stackrel{\mathrm{def}}{=} \tau(v)$. *Then it holds that*

$$\mathrm{char}(v) \ = \ \mathrm{char}(s), \quad \mathrm{ord}_v \ = \ \mathrm{ord}_s \circ \phi.$$

*Moreover, the isomorphism* $\phi\colon F_\times \xrightarrow{\sim} M$ *of monoids determines* **isomorphisms** *of monoids*

$$\kappa_v^\times \ \xrightarrow{\sim} \ \underline{O}_s^\times, \quad (\kappa_v)_\times \ \xrightarrow{\sim} \ (\underline{O}_s)_\times, \quad \mathcal{O}_{(v)}^\times \ \xrightarrow{\sim} \ O_s^\times.$$

(iv) *Let* $a \in F^\times$. *Then the bijection* $\tau\colon \mathcal{V}_F \xrightarrow{\sim} S$ *determines a* **bijection**

$$\mathrm{Supp}(a) \ \xrightarrow{\sim} \ \mathrm{Supp}(\phi(a)).$$

(v) *Let* $s \in S$. *Then the composite* $O_s^\times \hookrightarrow M^\times \twoheadrightarrow M^\times/O_s^{\equiv 1}$ *determines a* **surjection**

$$O_s^\times \ \twoheadrightarrow \ \underline{O}_s^\times$$

*which fits into a* **commutative** *diagram*

$$\begin{array}{ccc} \mathcal{O}_{(v)}^\times & \longrightarrow & \kappa_v^\times \\ \wr\big\downarrow & & \wr\big\downarrow \\ O_s^\times & \longrightarrow & \underline{O}_s^\times \end{array}$$

*— where the upper horizontal arrow is the natural surjection, and the vertical arrows are the isomorphisms of* (iii).

PROOF. — These assertions follow immediately from Lemma 2.4, together with the various definitions involved. $\square$

**LEMMA 2.6.** — *Suppose that* $F$ *is* **prime**. *Write* $(\mathcal{O}_F)_+ \subseteq \mathcal{O}_F$ *for the subset of positive rational integers, i.e., the subset* $\mathbb{Z}_{\geq 1} \subseteq \mathbb{Z}$. *For a prime number* $p$*, write* $v_p \in \mathcal{V}_F$ *for the nonarchimedean prime of* $F$ *corresponding to the maximal ideal* $p\mathcal{O}_F \subseteq \mathcal{O}_F$ *of* $\mathcal{O}_F$. *Then the following hold:*

(i) *The nonarchimedean prime* $v_2$ (*respectively,* $v_3$*;* $v_5$) *of* $F$ *may be* **characterized** *as a unique nonarchimedean prime* $v$ *of* $F$ *such that* $\mathrm{char}(v) = 2$ (*respectively,* $3$*;* $5$).

(ii) *The element* $2 \in \mathcal{O}_F^\triangleright$ *of* $\mathcal{O}_F^\triangleright$ *may be* **characterized** *as a unique element* $a \in \mathcal{O}_F^\triangleright$ *such that* $\mathrm{Supp}(a) = \{v_2\}$, $\mathrm{ord}_{v_2}(a) = 1$, *and* $a \notin \mathcal{O}_{(v_3)}^{\equiv 1}$.

(iii) *The element* $3 \in \mathcal{O}_F^\triangleright$ *of* $\mathcal{O}_F^\triangleright$ *may be* **characterized** *as a unique element* $a \in \mathcal{O}_F^\triangleright$ *such that* $\mathrm{Supp}(a) = \{v_3\}$, $\mathrm{ord}_{v_3}(a) = 1$, *and* $2a \in \mathcal{O}_{(v_5)}^{\equiv 1}$.

(iv) *Let* $a \in \mathcal{O}_F^\triangleright$ *be such that* $a \notin \{-2, -1, 1, 2\}$. *Then it holds that*

$$\{a-1, a+1\} \ = \ \{\, b \in \mathcal{O}_F^\triangleright \mid \mathrm{Supp}(a) \cap \mathrm{Supp}(b) = \emptyset,\ a \cdot b^{-1} \notin \mathcal{O}_{(v)}^{\equiv 1} \text{ for all } v \in \mathcal{V}_F \,\}.$$

(v) *Let* $a \in \mathcal{O}_F^{\rhd}$ *be such that* $a \notin \{-2, -1, 1, 2\}$*, and, moreover,* $\mathrm{Supp}(a) \not\subseteq \{v_2\}$*. Then it holds that*

$$\{a+1\} \ = \ \{a-1, a+1\} \ \cap \bigcap_{v \in \mathrm{Supp}(a)} \mathcal{O}_{(v)}^{\equiv 1}.$$

(vi) *Let* $a \in \mathcal{O}_F^{\rhd}$ *be such that* $a \notin \{-2, -1, 1, 2\}$*, and, moreover,* $\mathrm{Supp}(a) \subseteq \{v_2\}$*. Then, for every* $b \in \{a-1, a+1\}$*, it holds that* $b \notin \{-2, -1, 1, 2\}$ *and* $\mathrm{Supp}(b) \not\subseteq \{v_2\}$*.*

(vii) *The map* $\mathcal{O}_F \to \mathcal{O}_F$ *given by mapping* $a$ *to* $a+1$ *is* **bijective***.*

(viii) *The subset* $(\mathcal{O}_F)_+ \subseteq \mathcal{O}_F$ *is the uniquely determined* **minimal** *subset of* $\mathcal{O}_F$ *which contains* $1 \in \mathcal{O}_F$ *and, moreover, is [nonbijectively] preserved by the bijection discussed in* (vii)*.*

(ix) *Let* $v \in \mathcal{V}_F$*. Then the composite* $(\mathcal{O}_F)_+ \cap \mathcal{O}_{(v)}^{\times} \hookrightarrow \mathcal{O}_{(v)}^{\times} \twoheadrightarrow \kappa_v^{\times}$ *is* **surjective***.*

PROOF. — These assertions follow from the various definitions involved. $\square$

**PROPOSITION 2.7.** — *Let*

$$\mathcal{M} \ = \ (M, \ O^{\rhd} \subseteq M, \ S, \ \{O_s^{\equiv 1} \subseteq M\}_{s \in S})$$

*be a* **prime NF-monoid***. We shall define various objects which arise from* $\mathcal{M}$ *as follows:*

(1) *It follows from Lemma* 2.6, (i)*, that there exists a* **unique** *element* $s \in S$ *such that* $\mathrm{char}(s) = 2$ *(respectively,* 3*;* 5*). We shall write*

$$(2)_{\mathcal{M}} \ (\textit{respectively}, \ (3)_{\mathcal{M}}; \ (5)_{\mathcal{M}}) \ \in \ S$$

*for this element.*

(2) *It follows from Lemma* 2.6, (ii)*, that there exists a* **unique** *element* $a \in O^{\rhd}$ *of* $O^{\rhd}$ *such that* $\mathrm{Supp}(a) = \{(2)_{\mathcal{M}}\}$*,* $\mathrm{ord}_{(2)_{\mathcal{M}}}(a) = 1$*, and* $a \notin O_{(3)_{\mathcal{M}}}^{\equiv 1}$*. We shall write*

$$2_{\mathcal{M}} \ \in \ O^{\rhd}$$

*for this element and*

$$-2_{\mathcal{M}} \ \overset{\mathrm{def}}{=} \ -1_{\mathcal{M}} \cdot 2_{\mathcal{M}} \ \in \ O^{\rhd}.$$

(3) *It follows from Lemma* 2.6, (iii)*, that there exists a* **unique** *element* $a \in O^{\rhd}$ *of* $O^{\rhd}$ *such that* $\mathrm{Supp}(a) = \{(3)_{\mathcal{M}}\}$*,* $\mathrm{ord}_{(3)_{\mathcal{M}}}(a) = 1$*, and* $2_{\mathcal{M}} \cdot a \in O_{(5)_{\mathcal{M}}}^{\equiv 1}$*. We shall write*

$$3_{\mathcal{M}} \ \in \ O^{\rhd}$$

*for this element.*

(4) *Let* $a \in O^{\rhd} \setminus \{-2_{\mathcal{M}}, -1_{\mathcal{M}}, 1_{\mathcal{M}}, 2_{\mathcal{M}}\}$*. Then we shall write*

$$N(a) \ \overset{\mathrm{def}}{=} \ \{\, b \in O^{\rhd} \mid \mathrm{Supp}(a) \cap \mathrm{Supp}(b) = \emptyset, \ a \cdot b^{-1} \notin O_s^{\equiv 1} \ \textit{for all } s \in S \,\} \ \subseteq \ O^{\rhd}.$$

(5) *Let* $a \in O^{\rhd} \setminus \{-2_{\mathcal{M}}, -1_{\mathcal{M}}, 1_{\mathcal{M}}, 2_{\mathcal{M}}\}$*. Suppose that* $\mathrm{Supp}(a) \not\subseteq \{(2)_{\mathcal{M}}\}$*. Then it follows from Lemma* 2.6, (iv), (v)*, that the intersection*

$$N(a) \ \cap \bigcap_{s \in \mathrm{Supp}(a)} O_s^{\equiv 1}$$

*is* **of cardinality one**. *We shall write*

$$\mathrm{next}_{\mathcal{M}}(a) \ \in \ O^{\rhd}$$

*for the unique element of this intersection.*

(6) *Let* $a \in O^{\rhd} \setminus \{-2_{\mathcal{M}}, -1_{\mathcal{M}}, 1_{\mathcal{M}}, 2_{\mathcal{M}}\}$. *Suppose that* $\mathrm{Supp}(a) \subseteq \{(2)_{\mathcal{M}}\}$. *Then it follows from Lemma* 2.6, (iv), (v), (vi), *that there exists a* **unique** *element* $b \in N(a)$ *of* $N(a)$ *such that* $b \in O^{\rhd} \setminus \{-2_{\mathcal{M}}, -1_{\mathcal{M}}, 1_{\mathcal{M}}, 2_{\mathcal{M}}\}$, $\mathrm{Supp}(b) \not\subseteq \{(2)_{\mathcal{M}}\}$, *and, moreover,* $a \neq \mathrm{next}_{\mathcal{M}}(b)$ [*cf.* (5)]. *We shall write*

$$\mathrm{next}_{\mathcal{M}}(a) \ \in \ O^{\rhd}$$

*for this element.*

(7) *We shall write*

$$\mathrm{next}_{\mathcal{M}}(-2_{\mathcal{M}}) \ \stackrel{\mathrm{def}}{=} \ -1_{\mathcal{M}}, \ \mathrm{next}_{\mathcal{M}}(-1_{\mathcal{M}}) \ \stackrel{\mathrm{def}}{=} \ 0_{\mathcal{M}}, \ \mathrm{next}_{\mathcal{M}}(0_{\mathcal{M}}) \ \stackrel{\mathrm{def}}{=} \ 1_{\mathcal{M}},$$

$$\mathrm{next}_{\mathcal{M}}(1_{\mathcal{M}}) \ \stackrel{\mathrm{def}}{=} \ 2_{\mathcal{M}}, \ \mathrm{next}_{\mathcal{M}}(2_{\mathcal{M}}) \ \stackrel{\mathrm{def}}{=} \ 3_{\mathcal{M}}.$$

*Then, by Lemma* 2.6, (vii), *together with our construction, we have a* **bijection**

$$\mathrm{next}_{\mathcal{M}} \colon \ O^{\rhd} \cup \{0_{\mathcal{M}}\} \ \xrightarrow{\sim} \ O^{\rhd} \cup \{0_{\mathcal{M}}\}.$$

(8) *It follows from Lemma* 2.6, (viii), *that there exists a* **unique** *minimal subset of* $O^{\rhd} \cup \{0_{\mathcal{M}}\}$ *which contains* $1_{\mathcal{M}}$ *and, moreover, is* [*nonbijectively*] *preserved by* $\mathrm{next}_{\mathcal{M}}$. *We shall write*

$$O_{+} \ \subseteq \ O^{\rhd} \cup \{0_{\mathcal{M}}\}$$

*for this subset.*

(9) *Let* $s \in S$*;* $a$*,* $b \in (\underline{O}_s)_{\times}$. *Then we shall define an element of* $(\underline{O}_s)_{\times}$

$$a \boxplus_s b \ \in \ (\underline{O}_s)_{\times}$$

*as follows: Write* $\underline{0}_s \in (\underline{O}_s)_{\times}$ *for the unique element of* $(\underline{O}_s)_{\times} \setminus \underline{O}_s^{\times}$. *If* $a = \underline{0}_s$, *then* $a \boxplus_s b \stackrel{\mathrm{def}}{=} b$. *If* $b = \underline{0}_s$, *then* $a \boxplus_s b \stackrel{\mathrm{def}}{=} a$. *In the following, suppose that* $a$*,* $b \in \underline{O}_s^{\times}$. *Then it follows from Lemma* 2.6, (ix), *that there exist respective liftings* $\widetilde{a}$*,* $\widetilde{b} \in O_{+} \cap O_s^{\times}$ *of* $a$*,* $b \in \underline{O}_s^{\times}$ [*relative to the surjection* $O_s^{\times} \twoheadrightarrow \underline{O}_s^{\times}$ *of Proposition* 2.5, (v)]. *Write* $n_{\widetilde{b}} \in \mathbb{Z}$ *for the positive integer defined by* $\prod_{s \in S} \mathrm{char}(s)^{\mathrm{ord}_s(\widetilde{b})}$ *and*

$$c \ \stackrel{\mathrm{def}}{=} \ \overbrace{\mathrm{next}_{\mathcal{M}} \circ \cdots \circ \mathrm{next}_{\mathcal{M}}}^{n_{\widetilde{b}}}(\widetilde{a}) \ \in \ O_{+}.$$

*Then*

$$a \boxplus_s b \ \stackrel{\mathrm{def}}{=} \ \begin{cases} \underline{0}_s & \textit{if } c \notin O_s^{\times}, \\ \textit{the image of } c \textit{ in } (\underline{O}_s)_{\times} & \textit{if } c \in O_s^{\times}. \end{cases}$$

*Note that one verifies immediately from our construction that "*$a \boxplus_s b$*" does* **not depend** *on the choice of the respective liftings* $\widetilde{a}$*,* $\widetilde{b} \in O_{+} \cap O_s^{\times}$ *of* $a$*,* $b \in \underline{O}_s^{\times}$.

(10) *Let* $s \in S$. *Then it follows immediately from our construction that the "*$\boxplus_s$*" of* (9), *together with the monoid structure of* $(\underline{O}_s)_{\times}$, *determines a* **structure of field** *on* $(\underline{O}_s)_{\times}$. *We shall write*

$$\underline{O}_s$$

*for the resulting field.*

*Let*

$$(\phi\colon F_\times \xrightarrow{\sim} M,\ \tau\colon \mathcal{V}_F \xrightarrow{\sim} S)$$

*be a(n) [necessarily* **prime** *— cf. Lemma* 2.4, (i)*; Proposition* 2.5, (i), (iii)] **NF-envelope** *for* $\mathcal{M}$ *and* $v \in \mathcal{V}_F$*. Write* $s \overset{\text{def}}{=} \tau(v)$*. Then the isomorphism of monoids*

$$(\kappa_v)_\times \ \xrightarrow{\ \sim\ }\ (\underline{O}_s)_\times$$

*of Proposition* 2.5, (iii)*, determines an* **isomorphism of fields**

$$\kappa_v \ \xrightarrow{\ \sim\ }\ \underline{O}_s.$$

PROOF. — This follows immediately from Lemma 2.6, together with the various definitions involved. $\square$

**LEMMA 2.8.** — *The following hold:*

(i) *For* $a \in F^\times$*, it holds that* $a \in F^\times_{\text{prm}}$ *if and only if, for all but finitely many* $v \in \mathcal{V}_F$*, it holds that* $a^{\text{char}(v)-1} \in \mathcal{O}^{\equiv 1}_{(v)}$*.*

(ii) *Let* $v \in \mathcal{V}_F$*. Then the intersection* $F^\times_{\text{prm}} \cap \mathcal{O}^{\rhd}_F$ *(respectively,* $F^\times_{\text{prm}} \cap \mathcal{O}^{\equiv 1}_{(v)}$*)* **coincides** *with "*$\mathcal{O}^{\rhd}_F$*" (respectively, "*$\mathcal{O}^{\equiv 1}_{(v)}$*") in the case where we take "*$(F, v)$*" to be* $(F_{\text{prm}}, v_{\text{char}(v)})$ *[cf. the notation introduced in Lemma* 2.6*].*

(iii) *Write* $\mathcal{V}^{f=1}_F \subseteq \mathcal{V}_F$ *for the subset of* $\mathcal{V}_F$ *consisting of* $v \in \mathcal{V}_F$ *such that* $\sharp\kappa_v = \text{char}(v)$*. Then* $\mathcal{V}^{f=1}_F$ *is* **infinite***.*

(iv) *Let* $a,\ b \in F^\times$ *be such that* $0 \notin \{a, b, a+b\}$*. Then the element* $a+b \in F^\times$ *may be* **characterized** *as a unique element* $c \in F^\times$ *which satisfies the following condition: For infinitely many* $v \in \mathcal{V}_F$ *such that* $\{a, b, c\} \subseteq \mathcal{O}^\times_{(v)}$*, if we write* $\overline{a},\ \overline{b},\ \overline{c} \in \kappa^\times_v$ *for the respective images of* $a,\ b,\ c \in \mathcal{O}^\times_{(v)}$*, then it holds that* $\overline{a} + \overline{b} = \overline{c}$*.*

PROOF. — Assertion (i) follows from [3], Lemma 2.3. Assertions (ii) and (iv) follow from the various definitions involved. Assertion (iii) follows from *Čebotarev's density theorem* [cf., e.g., [10], Chapter VII, Theorem 13.4]. This completes the proof of Lemma 2.8. $\square$

**THEOREM 2.9.** — *In the notation introduced at the beginning of* §2*, let*

$$\mathcal{M} \ = \ (M,\ O^{\rhd} \subseteq M,\ S,\ \{O^{\equiv 1}_s \subseteq M\}_{s \in S})$$

*be an* **NF-monoid** *[cf. Definition* 2.3*]. We shall define various objects which arise from* $\mathcal{M}$ *as follows:*

(1) *We shall write*

$$M^\times_{\text{prm}} \ \subseteq\ M^\times$$

*for the submodule consisting of* $a \in M^\times$ *such that, for all but finitely many* $s \in S$*, it holds that* $a^{\text{char}(s)-1} \in O^{\equiv 1}_s$*;*

$$M_{\text{prm}} \ \overset{\text{def}}{=}\ M^\times_{\text{prm}} \cup \{0_{\mathcal{M}}\} \ \subseteq\ M; \quad O^{\rhd}_{\text{prm}} \ \overset{\text{def}}{=}\ M_{\text{prm}} \cap O^{\rhd}.$$

(2) *We shall write*

$$S_{\text{prm}} \ \overset{\text{def}}{=}\ S/ \sim_{\text{prm}}$$

*for the set of equivalence classes with respect to the relation "$\sim_{\mathrm{prm}}$" on $S$ defined as follows: For $s_1,\ s_2 \in S$, it holds that $s_1 \sim_{\mathrm{prm}} s_2$ if and only if* $\mathrm{char}(s_1) = \mathrm{char}(s_2)$.

(3) *Let $\overline{s} \in S_{\mathrm{prm}}$. Then it follows from Lemma* 2.8, (i), (ii), *that the intersection $M_{\mathrm{prm}} \cap O_s^{\equiv 1}$ does* **not depend** *on the choice of a lifting $s \in S$ of $\overline{s}$. We shall write*

$$O_{\overline{s}}^{\equiv 1} \ \subseteq \ M_{\mathrm{prm}}$$

*for this intersection.*

(4) *It follows from Lemma* 2.8, (i), (ii), *that the collection of data*

$$\mathcal{M}_{\mathrm{prm}} \ \stackrel{\mathrm{def}}{=} \ (M_{\mathrm{prm}},\ O_{\mathrm{prm}}^{\triangleright} \subseteq M_{\mathrm{prm}},\ S_{\mathrm{prm}},\ \{O_{\overline{s}}^{\equiv 1} \subseteq M_{\mathrm{prm}}\}_{\overline{s} \in S_{\mathrm{prm}}})$$

*forms a* **prime NF-monoid** [*cf. Definition* 2.3].

(5) *We shall write*

$$S^{f=1} \ \stackrel{\mathrm{def}}{=} \ \{\, s \in S \mid \sharp(\underline{O}_s)_{\times} = \mathrm{char}(s) \,\}.$$

*Then it follows from Lemma* 2.8, (iii), *that $S^{f=1}$ is* **infinite**.

(6) *Let $s \in S^{f=1}$. Write $\overline{s} \in S_{\mathrm{prm}}$ for the element of $S_{\mathrm{prm}}$ determined by $s \in S^{f=1}$. Then one verifies immediately that the homomorphism $(\underline{O}_{\overline{s}})_{\times} \to (\underline{O}_s)_{\times}$ of monoids induced by the natural inclusion $M_{\mathrm{prm}} \hookrightarrow M$ is an* **isomorphism**. *Thus, it follows from Proposition* 2.7, (10), *that the "$\boxplus_s$" of Proposition* 2.7, (9), *in the case where we take the "$(\mathcal{M}, s)$" of Proposition* 2.7, (9), *to be $(\mathcal{M}_{\mathrm{prm}}, \overline{s})$, together with the monoid structure of $(\underline{O}_s)_{\times}$, determines a* **structure of field** *on $(\underline{O}_s)_{\times}$. We shall write*

$$\underline{O}_s$$

*for the resulting field.*

(7) *Let $a,\ b \in M$. Then we shall define an element of $M$*

$$a \boxplus_{\mathcal{M}} b \ \in \ M$$

*as follows: If $a = 0_{\mathcal{M}}$, then $a \boxplus_{\mathcal{M}} b \stackrel{\mathrm{def}}{=} b$. If $b = 0_{\mathcal{M}}$, then $a \boxplus_{\mathcal{M}} b \stackrel{\mathrm{def}}{=} a$. If $a = -1_{\mathcal{M}} \cdot b$, then $a \boxplus_{\mathcal{M}} b \stackrel{\mathrm{def}}{=} 0_{\mathcal{M}}$. Suppose that $a,\ b \in M^{\times}$, and that $a \neq -1_{\mathcal{M}} \cdot b$. Then $a \boxplus_{\mathcal{M}} b$ is defined to be a* **uniquely determined** [*cf. Lemma* 2.8, (iv)] *element $c \in M^{\times}$ of $M^{\times}$ which satisfies the following condition: For infinitely many $s \in S^{f=1}$ such that $\{a, b, c\} \subseteq O_s^{\times}$, if we write $\overline{a},\ \overline{b},\ \overline{c} \in \underline{O}_s^{\times}$ for the respective images of $a,\ b,\ c \in O_s^{\times}$, then it holds that $\overline{a} \boxplus_s \overline{b} = \overline{c}$, where we write $\boxplus_s$ for the addition of the field $\underline{O}_s$ defined in* (6).

(8) *It follows immediately from our construction that the "$\boxplus_{\mathcal{M}}$" of* (7), *together with the monoid structure of $M$, determines a* **structure of field** *on $M$. We shall write*

$$M^{\mathrm{fld}}$$

*for the resulting field.*

*Let*

$$(\phi\colon F_{\times} \xrightarrow{\sim} M,\ \tau\colon \mathcal{V}_F \xrightarrow{\sim} S)$$

*be an* **NF-envelope** *for $\mathcal{M}$* [*cf. Definition* 2.3]. *Then the isomorphism of monoids*

$$\phi\colon \ F_{\times} \ \xrightarrow{\ \sim\ } \ M$$

*determines an* **isomorphism of fields**

$$F \xrightarrow{\sim} M^{\mathrm{fld}}.$$

*In particular, the field* $M^{\mathrm{fld}}$ *of* (8) *is an* **NF**.

PROOF. — This follows immediately from Lemma 2.8, together with the various definitions involved. $\square$

## 3. Local-global Cyclotomic Synchronization

In the present §3, we define a *cyclotome* [cf. Proposition 3.7, (4), below] associated to a profinite group *of GSC-type* [cf. Definition 3.2 below]. Moreover, we discuss the *local-global cyclotomic synchronization isomorphism* [cf. Theorem 3.8, (ii), below], i.e., a suitable isomorphism of a *global cyclotome* with a *local cyclotome*. Finally, by means of the local-global cyclotomic synchronization isomorphism, we construct the *Kummer containers* associated to a profinite group *of GSC-type* [cf. Proposition 3.11 below].

In the present §3, we maintain the notation introduced at the beginning of §2. In particular, we have an *NF* $F$. Let

$$\overline{F}$$

be an algebraic closure of $F$. We shall write

- $d_F \stackrel{\text{def}}{=} [F : F_{\text{prm}}]$ for the extension degree of $F$ over $F_{\text{prm}}$,
- $\mathbb{I}_F^{\text{fin}}$ for the group of finite idèles of $F$, and
- $\mathbb{I}_F$ for the group of idèles of $F$.

If $v \in \mathcal{V}_F$, then we shall write

- $F_v$ for the MLF obtained by forming the completion of $F$ at $v$.

We shall write

- $\mathcal{V}_F^{d=1} \subseteq \mathcal{V}_F$ for the subset consisting of $v \in \mathcal{V}_F$ such that $d_{F_v} = 1$ [cf. the notation introduced at the beginning of §1].

**Definition 3.1.** — Let $E$ be a field of characteristic zero which is algebraic over the prime field contained in $E$ [i.e., "$\mathbb{Q}$"]. Then we shall say that $E$ is *absolutely Galois* if $E$ is Galois over the prime field contained in $E$ [i.e., "$\mathbb{Q}$"]. We shall say that $E$ is *solvably closed* if there is no nontrivial finite abelian extension of $E$.

**Definition 3.2.** — Let $G$ be a profinite group. Then we shall refer to a collection of data

$$(K,\ \widetilde{K},\ \alpha\colon \text{Gal}(\widetilde{K}/K) \xrightarrow{\sim} G)$$

consisting of an NF $K$, a Galois extension $\widetilde{K}$ of $K$ which is solvably closed (respectively, absolutely Galois and solvably closed; algebraically closed), and an isomorphism of profinite groups $\alpha\colon \text{Gal}(\widetilde{K}/K) \xrightarrow{\sim} G$ as a *GSC-envelope* (respectively, an *AGSC-envelope*; an *NF-envelope*) [where "GSC" (respectively, "AGSC"; "NF") is to be understood as an abbreviation for "Global Solvably Closed" (respectively, "Absolutely Galois and Solvably Closed"; "Number Field"] for $G$. We shall say that the profinite group $G$ is *of GSC-type* (respectively, *of AGSC-type*; *of NF-type*) if there exists a GSC-envelope (respectively, an AGSC-envelope; an NF-envelope) for $G$.

**REMARK 3.2.1.**

(i) One verifies immediately that every open subgroup of a profinite group *of GSC-type* (respectively, *of AGSC-type*; *of NF-type*) is *of GSC-type* (respectively, *of AGSC-type*; *of NF-type*).

(ii) It follows from the definitions that

$$\text{of NF-type} \quad \Longrightarrow \quad \text{of AGSC-type} \quad \Longrightarrow \quad \text{of GSC-type.}$$

(iii) A typical example of a field which is *absolutely Galois* and *solvably closed* is the solvable closure of an absolutely Galois NF.

Now let us recall the famous *Neukirch-Uchida theorem*:

**THEOREM 3.3** (Neukirch-Uchida). — *For $\Box \in \{\circ, \bullet\}$, let $F_\Box$ be an* **NF** *and $\widetilde{F}_\Box$ a Galois extension of $F_\Box$ which is* **solvably closed**. *Write $Q_\Box \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{F}_\Box/F_\Box)$;*

$$\mathrm{Isom}(\widetilde{F}_\bullet/F_\bullet, \widetilde{F}_\circ/F_\circ)$$

*for the set of isomorphisms $\widetilde{F}_\bullet \xrightarrow{\sim} \widetilde{F}_\circ$ of fields which map $F_\bullet$ bijectively onto $F_\circ$;*

$$\mathrm{Isom}(Q_\circ, Q_\bullet)$$

*for the set of isomorphisms $Q_\circ \xrightarrow{\sim} Q_\bullet$ of profinite groups. Then the natural map*

$$\mathrm{Isom}(\widetilde{F}_\bullet/F_\bullet, \widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Isom}(Q_\circ, Q_\bullet)$$

*is* **bijective**.

PROOF. — This follows from [13], Theorem. $\Box$

In the remainder of the present §3, let

$$\widetilde{F}$$

be a Galois extension of $F$ which is *solvably closed* and contained in $\overline{F}$. We shall write

- $\mathcal{V}_{\widetilde{F}}$ for the set of nonarchimedean primes of $\widetilde{F}$ and
- $Q_F \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{F}/F)$ for the Galois group of $\widetilde{F}/F$.

Note that, for $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$, if we write $v \in \mathcal{V}_F$ for the nonarchimedean prime of $F$ determined by $\widetilde{v}$, then since $\widetilde{F}$ is *solvably closed*, it follows immediately from [6], Proposition 2.3, (iii) [i.e., the *Grunwald-Wang Theorem* — cf., e.g., [11], Theorem 9.2.8], that the pair $(\widetilde{F}, \widetilde{v})$ determines an algebraic closure $\overline{F}_{\widetilde{v}}$ of $F_v$ equipped with an inclusion $\widetilde{F} \hookrightarrow \overline{F}_{\widetilde{v}}$ of fields.

**LEMMA 3.4.** — *The following hold:*

(i) *The map given by mapping $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$ to the decomposition subgroup of $Q_F$ associated to $\widetilde{v}$ determines a* **bijection** *of $\mathcal{V}_{\widetilde{F}}$ with the set of* **maximal** *closed subgroups of $Q_F$* **of MLF-type**. *Moreover, the natural map $\mathcal{V}_{\widetilde{F}} \to \mathcal{V}_F$ and the natural action of $Q_F$ on $\mathcal{V}_{\widetilde{F}}$ determines a* **bijection**

$$\mathcal{V}_{\widetilde{F}}/Q_F \xrightarrow{\sim} \mathcal{V}_F.$$

(ii) *Let $p$ be a prime number. Then it holds that*

$$d_F = \sum_{v \in \mathcal{V}_F;\ \mathrm{char}(v)=p} d_{F_v}.$$

PROOF. — Assertion (i) follows immediately, in light of [6], Proposition 2.3, (iii), (iv), from a similar argument to the argument applied in the proof of [11], Corollary 12.1.11. Assertion (ii) follows from [10], Chapter II, Corollary 8.4. This completes the proof of Lemma 3.4. □

**PROPOSITION 3.5.** — *Let $G$ be a profinite group* **of GSC-type**. *We shall define various objects which arise from $G$ as follows:*

(1) *We shall write*

$$\widetilde{\mathcal{V}}(G)$$

*for the set of* **maximal** *closed subgroups of $G$* **of MLF-type** *and*

$$\mathcal{V}(G) \stackrel{\mathrm{def}}{=} \widetilde{\mathcal{V}}(G)/G$$

*for the quotient of $\widetilde{\mathcal{V}}(G)$ by the action of $G$ by conjugation.*

(2) *Let $v \in \mathcal{V}(G)$. Then we shall write*

$$p(v) \stackrel{\mathrm{def}}{=} p(D), \quad d(v) \stackrel{\mathrm{def}}{=} d(D), \quad f(v) \stackrel{\mathrm{def}}{=} f(D), \quad e(v) \stackrel{\mathrm{def}}{=} e(D)$$

*for some $D \in v$. We shall write*

$$\mathcal{V}^{d=1}(G) \subseteq \mathcal{V}(G)$$

*for the subset of $\mathcal{V}(G)$ consisting of $v \in \mathcal{V}(G)$ such that $d(v) = 1$.*

(3) *It follows from Lemma* 3.4, (i), (ii)*, that, for $v_0 \in \mathcal{V}(G)$, the sum*

$$\sum_{v \in \mathcal{V}(G);\ p(v)=p(v_0)} d(v)$$

*does* **not depend** *on the choice of $v_0 \in \mathcal{V}(G)$. We shall write*

$$d(G)$$

*for this sum.*

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be a* **GSC-envelope** *for $G$. Then the following hold:*

(i) *The isomorphism $\alpha$ determines — relative to the first bijection of Lemma* 3.4, (i) *— a* **bijection**

$$\mathcal{V}_{\widetilde{F}} \xrightarrow{\sim} \widetilde{\mathcal{V}}(G).$$

*This bijection is* **compatible** *with the natural actions of $Q_F$ and $G$ relative to $\alpha$, which thus induces a* **bijection** [*cf. the second bijection of Lemma* 3.4, (i)]

$$\mathcal{V}_F \xrightarrow{\sim} \mathcal{V}(G).$$

*By means of these bijections, let us* **identify** *$\mathcal{V}_{\widetilde{F}}$, $\mathcal{V}_F$ with $\widetilde{\mathcal{V}}(G)$, $\mathcal{V}(G)$, respectively.*

(ii) *Let $v \in \mathcal{V}(G)$. Then it holds that*

$$p_{F_v} = p(v), \quad d_{F_v} = d(v), \quad f_{F_v} = f(v), \quad e_{F_v} = e(v).$$

(iii) *It holds that $d_F = d(G)$.*

(iv) *Let $H \subseteq G$ be an open subgroup of $G$. Then we have a* **bijection**

$$\begin{array}{ccc} \widetilde{\mathcal{V}}(G) & \xrightarrow{\sim} & \widetilde{\mathcal{V}}(H) \\ D & \mapsto & D \cap H \end{array}$$

*whose inverse is given by*

$$\begin{array}{ccc} \widetilde{\mathcal{V}}(H) & \xrightarrow{\sim} & \widetilde{\mathcal{V}}(G) \\ D & \mapsto & C_G(D). \end{array}$$

*Moreover, this inverse determines a* **surjection**

$$\mathcal{V}(H) \twoheadrightarrow \mathcal{V}(G).$$

PROOF. — Assertions (i), (ii), (iii) follow from Lemma 3.4, together with the various definitions involved. Assertion (iv) follows immediately from assertion (i), together with the *commensurable terminality* [cf. [6], Proposition 2.3, (v)] in $G$ of a closed subgroup of $G$ which is contained in $\widetilde{\mathcal{V}}(G)$. This completes the proof of Proposition 3.5. □

**LEMMA 3.6.** — *The following hold:*

(i) *By considering the "diagonal", we have an* **injection** *of groups*

$$F^\times \hookrightarrow \mathbb{I}_F^{\mathrm{fin}}.$$

*By means of this* **injection**, *we regard $F^\times$ as a subgroup of $\mathbb{I}_F^{\mathrm{fin}}$.*

(ii) *By considering the reciprocity homomorphism $\mathbb{I}_F \twoheadrightarrow (\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}} \xrightarrow{\sim})\ Q_F^{\mathrm{ab}}$ in global class field theory, together with the natural inclusion $\mathbb{I}_F^{\mathrm{fin}} \hookrightarrow \mathbb{I}_F$, we obtain [cf. also* (i)*] homomorphisms of groups*

$$F^\times \hookrightarrow \mathbb{I}_F^{\mathrm{fin}} \to Q_F^{\mathrm{ab}}.$$

[*Note that, in general, this composite is* **nontrivial**. *For instance, one verifies easily that if $F$ is* **prime**, *then the image of $-1 \in F^\times$ via this composite is* **nontrivial**.]

(iii) *It holds that, relative to the display of* (ii),

$$\mathrm{Ker}(\mathbb{I}_F^{\mathrm{fin}} \to Q_F^{\mathrm{ab}})_{\mathrm{tor}} \subseteq \boldsymbol{\mu}(F).$$

*If, moreover, $F$ is* **totally imaginary**, *then it holds that, relative to the display of* (ii),

$$\mathrm{Ker}(\mathbb{I}_F^{\mathrm{fin}} \to Q_F^{\mathrm{ab}})_{\mathrm{tor}} = \boldsymbol{\mu}(F).$$

(iv) *Let $n$ be a positive integer and $\zeta_n \in \overline{F}$ a primitive $n$-th root of unity. Then it holds that $\zeta_n \in \widetilde{F}$. Moreover, the subfield of $\widetilde{F}$ corresponding to the kernel of the natural action of $Q_F$ on*

$$\big(\varinjlim_{E} \mathrm{Ker}(\mathbb{I}_E^{\mathrm{fin}} \to Q_E^{\mathrm{ab}})_{\mathrm{tor}}\big)[n]$$

*— where the injective limit is taken over the finite extensions $E$ of $F$ contained in $\widetilde{F}$, and we write $Q_E \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}/E)$ —* **coincides** *with $F(\zeta_n)$.*

PROOF. — Assertions (i), (ii) follow from the various definitions involved. Next, we verify assertion (iii). Write $F^{\times}_{\mathrm{idl}} \subseteq \mathbb{I}_F$ for the image of the injection $F^{\times} \hookrightarrow \mathbb{I}_F$ obtained by considering the "diagonal" and $C^0 \subseteq \mathbb{I}_F/F^{\times}_{\mathrm{idl}}$ for the kernel of the *reciprocity homomorphism* $\mathbb{I}_F/F^{\times}_{\mathrm{idl}} \twoheadrightarrow Q^{\mathrm{ab}}_F$ in global class field theory, i.e., the connected component of $\mathbb{I}_F/F^{\times}_{\mathrm{idl}}$ containing the identity element [cf. [11], Corollary 8.2.2]. [Let us recall that the subgroup $F^{\times}_{\mathrm{idl}} \subseteq \mathbb{I}_F$ does *not coincide* with the image of the composite $F^{\times} \hookrightarrow \mathbb{I}^{\mathrm{fin}}_F \hookrightarrow \mathbb{I}_F$.] First, in order to verify the inclusion

$$\mathrm{Ker}(\mathbb{I}^{\mathrm{fin}}_F \to Q^{\mathrm{ab}}_F)_{\mathrm{tor}} \;\subseteq\; F^{\times},$$

take a *torsion* finite idèle $\alpha \in \mathbb{I}^{\mathrm{fin}}_F \subseteq \mathbb{I}_F$ whose image in $Q^{\mathrm{ab}}_F$ is *trivial*. Then one verifies immediately that the image of $\alpha$ via the composite

$$\mathbb{I}^{\mathrm{fin}}_F \;\hookrightarrow\; \mathbb{I}_F \;\twoheadrightarrow\; \mathbb{I}_F/F^{\times}_{\mathrm{idl}}$$

is *contained* in $C^0$ and *torsion*. In particular, it follows immediately from [11], Theorem 8.2.5, together with the fact that "$\widehat{\mathbb{Z}}/\mathbb{Z}$" and "$\mathbb{R}$" in *loc. cit.* are *torsion-free*, that there exists an *infinite idèle* $\beta \in \mathbb{I}_F$ such that the image of $\alpha$ in $\mathbb{I}_F/F^{\times}_{\mathrm{idl}}$ *coincides* with the image of $\beta$ in $\mathbb{I}_F/F^{\times}_{\mathrm{idl}}$, i.e., that $\alpha \cdot \beta^{-1} \in F^{\times}_{\mathrm{idl}}$ in $\mathbb{I}_F$. On the other hand, it follows immediately from the various definitions involved that this implies that $\alpha \in F^{\times}$ [not in $\mathbb{I}_F$ but] in $\mathbb{I}^{\mathrm{fin}}_F$. This completes the proof of the desired inclusion.

Next, in order to verify the inclusion

$$\boldsymbol{\mu}(F) \;\subseteq\; \mathrm{Ker}(\mathbb{I}^{\mathrm{fin}}_F \to Q^{\mathrm{ab}}_F)$$

under the assumption that $F$ is *totally imaginary*, take a *torsion* element $\alpha \in F^{\times}$. Then it follows immediately from the various definitions involved that, to complete the verification of the desired inclusion, it suffices to verify that the image of the *infinite idèle* determined by $\alpha \in F^{\times}$ in $\mathbb{I}_F/F^{\times}_{\mathrm{idl}}$ is *contained* in $C^0$. On the other hand, since $F$ is *totally imaginary*, this follows immediately from [11], Theorem 8.2.5. This completes the proof of the desired inclusion, hence also of assertion (iii). Assertion (iv) follows immediately from assertion (iii), together with our assumption that $\widetilde{F}$ is *solvably closed*. This completes the proof of Lemma 3.6. □

**PROPOSITION 3.7.** — *Let $G$ be a profinite group* **of GSC-type**. *We shall define various objects which arise from $G$ as follows:*

(1) *Let $v \in \mathcal{V}(G)$. Then one verifies immediately from the* **commensurable terminality** [*cf.* [6], *Proposition* 2.3, (v)] *in $G$ of a closed subgroup of $G$ which is contained in $\widetilde{\mathcal{V}}(G)$ that there exists a* **uniquely determined** *submodule* (*respectively, submonoid*)

$$k^{\times}(v) \;\subseteq\; \prod_{D\in v} k^{\times}(D) \quad (\subseteq \prod_{D\in v} D^{\mathrm{ab}})$$

$$(\textit{respectively, } \;\mathcal{O}^{\triangleright}(v) \;\subseteq\; \prod_{D\in v} \mathcal{O}^{\triangleright}(D) \quad (\subseteq \prod_{D\in v} D^{\mathrm{ab}}))$$

*which satisfies the following two conditions:*

(a) *The action of $G$ on $\prod_{D\in v} k^{\times}(D)$ (respectively, $\prod_{D\in v} \mathcal{O}^{\triangleright}(D)$) by conjugation [preserves and] induces the* **identity automorphism** *on the submodule $k^{\times}(v)$ (respectively, $\mathcal{O}^{\triangleright}(v)$).*

(b) *For every* $D_0 \in v$, *the composite*

$$k^\times(v) \hookrightarrow \prod_{D \in v} k^\times(D) \twoheadrightarrow k^\times(D_0)$$

$$(\textit{respectively, } \mathcal{O}^\triangleright(v) \hookrightarrow \prod_{D \in v} \mathcal{O}^\triangleright(D) \twoheadrightarrow \mathcal{O}^\triangleright(D_0))$$

*is an* **isomorphism** *of modules* (*respectively, monoids*).

*For each* $D \in v$, *let us* **identify** $k^\times(D)$, $\mathcal{O}^\triangleright(D)$ *with* $k^\times(v)$, $\mathcal{O}^\triangleright(v)$ *by means of the isomorphisms of* (b), *respectively. Moreover, by means of this identification, together with the topology on* $k^\times(D)$ (*respectively,* $\mathcal{O}^\triangleright(D)$), *we regard* $k^\times(v)$ (*respectively,* $\mathcal{O}^\triangleright(v)$) *as a* **topological** *module* (*respectively, monoid*). [*Note that one verifies immediately that this topology on* $k^\times(v)$ (*respectively,* $\mathcal{O}^\triangleright(v)$) *does* **not depend** *on the choice of* $D \in v$.]

(2) *We shall write*

$$k_\times(v) \stackrel{\text{def}}{=} k^\times(v)^\circledast \subseteq \prod_{D \in v} k_\times(D).$$

*Let us* **identify** $k_\times(D)$ *with* $k_\times(v)$ *by means of the identification of* $k^\times(D)$ *with* $k^\times(v)$ *of* (1).

(3) *Let* $\Sigma \subseteq \mathcal{V}(G)$ *be a finite subset of* $\mathcal{V}(G)$. *Then we shall write*

$$\mathbb{I}^{\text{fin}}_\Sigma(G) \stackrel{\text{def}}{=} \big(\prod_{v \in \Sigma} k^\times(v)\big) \times \big(\prod_{v \notin \Sigma} \mathcal{O}^\triangleright(v)^\times\big) \quad (\subseteq \prod_{D \in \widetilde{\mathcal{V}}(G)} D^{\text{ab}});$$

$$\mathbb{I}^{\text{fin}}(G) \stackrel{\text{def}}{=} \varinjlim_{\Sigma} \mathbb{I}^{\text{fin}}_\Sigma(G) \quad (\subseteq \prod_{D \in \widetilde{\mathcal{V}}(G)} D^{\text{ab}})$$

— *where the injective limit is taken over the finite subsets* $\Sigma \subseteq \mathcal{V}(G)$ *of* $\mathcal{V}(G)$.

(4) *It follows from our construction that the inclusions* $D \hookrightarrow G$, *where* $D$ *ranges over the elements of* $\widetilde{\mathcal{V}}(G)$, *determine a homomorphism of groups*

$$\mathbb{I}^{\text{fin}}(G) \longrightarrow G^{\text{ab}}.$$

*We shall write*

$$\boldsymbol{\mu}(G) \stackrel{\text{def}}{=} \varinjlim_{H} \text{Ker}(\mathbb{I}^{\text{fin}}(H) \to H^{\text{ab}})_{\text{tor}}$$

— *where the injective limit is taken over the open subgroups* $H \subseteq G$ *of* $G$, *and the transition morphisms in the limit are given by the homomorphisms determined by the transfer maps;*

$$\Lambda(G) \stackrel{\text{def}}{=} \varprojlim_{n} \boldsymbol{\mu}(G)[n]$$

— *where the projective limit is taken over the positive integers* $n$. *Note that* $G$ *acts on* $\boldsymbol{\mu}(G)$, $\Lambda(G)$ *by conjugation. We shall refer to the* $G$*-module* $\Lambda(G)$ *as the* **cyclotome** *associated to* $G$. *Note that one verifies immediately from our construction that the cyclotome has a natural structure of* **profinite** [*cf. also the above definition of* $\Lambda(G)$], *hence also topological,* $G$*-module; moreover, we have a natural identification* $\boldsymbol{\mu}(G)[n] = \Lambda(G)/n\Lambda(G)$ [*cf. Lemma* 3.6, (iii)].

(5) *Let $n$ be a positive integer. Then we shall write*

$$G^{(\boldsymbol{\mu}_n)} \subseteq G$$

*for the open subgroup of $G$ obtained by forming the kernel of the action*

$$G \longrightarrow \mathrm{Aut}(\Lambda(G)/n\Lambda(G)).$$

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be a* **GSC-envelope** *for $G$. Then the following hold:*

(i) *Let $v \in \mathcal{V}(G)$. Then the isomorphism $\alpha$, together with the field structure of $F$, determines a* **commutative** *diagram of topological monoids*

$$\begin{array}{ccc} \mathcal{O}^{\triangleright}_{F_v} & \longrightarrow & F_v^{\times} \\ \wr\downarrow & & \wr\downarrow \\ \mathcal{O}^{\triangleright}(v) & \longrightarrow & k^{\times}(v) \end{array}$$

*— where the horizontal arrows are the natural inclusions, and the vertical arrows are* **isomorphisms**. *Thus, the right-hand vertical arrow of this diagram determines an* **isomorphism** *of monoids*

$$(F_v)_{\times} \xrightarrow{\sim} k_{\times}(v).$$

(ii) *The diagram of groups*

$$\begin{array}{ccc} \mathbb{I}^{\mathrm{fin}}_F & \longrightarrow & Q_F^{\mathrm{ab}} \\ \wr\downarrow & & \wr\downarrow \\ \mathbb{I}^{\mathrm{fin}}(G) & \longrightarrow & G^{\mathrm{ab}} \end{array}$$

*— where the upper horizontal arrow is the homomorphism of Lemma* 3.6, (ii)*; the lower horizontal arrow is the homomorphism of* (4)*; the left-hand vertical arrow is the isomorphism induced by the various isomorphisms "$F_v^{\times} \xrightarrow{\sim} k^{\times}(v)$" of* (i)*; the right-hand vertical arrow is the isomorphism induced by $\alpha$ —* **commutes**.

(iii) *The commutative diagram of* (ii) *determines* **isomorphisms**

$$\boldsymbol{\mu}(\overline{F}) \xrightarrow{\sim} \boldsymbol{\mu}(G), \quad \Lambda(\overline{F}) \xrightarrow{\sim} \Lambda(G)$$

*which are* **compatible** *with the natural actions of $Q_F$ and $G$ relative to $\alpha$.*

(iv) *Let $n$ be a positive integer and $\zeta_n \in \overline{F}$ a primitive $n$-th root of unity. Then the isomorphism $\alpha$ determines an isomorphism of profinite groups*

$$\mathrm{Gal}(\widetilde{F}/F(\zeta_n)) \xrightarrow{\sim} G^{(\boldsymbol{\mu}_n)}$$

[*cf. the first assertion of Lemma* 3.6, (iv)].

PROOF. — These assertions follow immediately from Lemma 3.6, together with the various definitions involved. □

**THEOREM 3.8.** — *In the notation introduced at the beginning of* §3 *and the discussion following Theorem* 3.3*, let $G$ be a profinite group* **of GSC-type** [*cf. Definition* 3.2] *and $D \in \widetilde{\mathcal{V}}(G)$. Then the following hold:*

(i) *Let $H \subseteq G$ be an open subgroup of $G$. Then we have a natural* **identification**

$$\Lambda(G) \xrightarrow{\sim} \Lambda(H)$$

[*arising from the definition of "$\Lambda(-)$"*] *which is* ***H*-equivariant**.

(ii) *The natural homomorphism $\mathbb{I}^{\text{fin}}(G) \to k^{\times}(D)$* [*arising from the definition of $\mathbb{I}^{\text{fin}}(G)$*] *determines* ***D*-equivariant isomorphisms**

$$\boldsymbol{\mu}(G) \xrightarrow{\sim} \boldsymbol{\mu}(D), \quad \Lambda(G) \xrightarrow{\sim} \Lambda(D).$$

*We shall refer to the isomorphism $\Lambda(G) \xrightarrow{\sim} \Lambda(D)$ between the cyclotomes as the* **local-global cyclotomic synchronization isomorphism** *with respect to $D \in \widetilde{\mathcal{V}}(G)$.*

(iii) *Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be a* **GSC-envelope** *for $G$* [*cf. Definition* 3.2]*. Write $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$ for the nonarchimedean prime of $\widetilde{F}$ which corresponds to $D \in \widetilde{\mathcal{V}}(G)$* [*cf. Proposition* 3.5, (i)] *and $v \in \mathcal{V}_F$ for the nonarchimedean prime of $F$ determined by $\widetilde{v}$. Thus, by the discussion following Theorem* 3.3*, we have an algebraic closure $\overline{F}_{\widetilde{v}}$ of $F_v$ equipped with an inclusion $\widetilde{F} \hookrightarrow \overline{F}_{\widetilde{v}}$ of fields. Then the diagram*

$$\begin{array}{ccc}
\Lambda(\overline{F}) & \xrightarrow{\sim} & \Lambda(\overline{F}_{\widetilde{v}}) \\
\wr\downarrow & & \wr\downarrow \\
\Lambda(G) & \xrightarrow{\sim} & \Lambda(D)
\end{array}$$

*— where the upper horizontal arrow is the isomorphism induced by the inclusion $\widetilde{F} \hookrightarrow \overline{F}_{\widetilde{v}}$ of fields* [*cf. the first assertion of Lemma* 3.6, (iv)]*; the lower horizontal arrow is the local-global cyclotomic synchronization isomorphism; the left-hand vertical arrow is the isomorphism of Proposition* 3.7, (iii)*; the right-hand vertical arrow is the isomorphism of Theorem* 1.4, (iv) *—* **commutes**.

PROOF. — These assertions follow immediately from the various definitions involved. □

**DEFINITION 3.9.** — We shall write

$$\mathcal{H}^{\times}(F)$$

for the module obtained by forming the fiber product of the diagram of natural injections

$$\begin{array}{ccc}
 & & \prod_{v \in \mathcal{V}_F} F_v^{\times} \\
 & & \downarrow \\
(F^{\times})^{\wedge} & \longrightarrow & \prod_{v \in \mathcal{V}_F} (F_v^{\times})^{\wedge}
\end{array}$$

and

$$\mathcal{H}_{\times}(F) \stackrel{\text{def}}{=} \mathcal{H}^{\times}(F)^{\circledast}.$$

Thus, we have natural injections of monoids

$$F_{\times} \hookrightarrow \mathcal{H}_{\times}(F) \hookrightarrow \prod_{v \in \mathcal{V}_F} (F_v)_{\times}.$$

We shall refer to $\mathcal{H}^{\times}(F)$, $\mathcal{H}_{\times}(F)$ as the *Kummer containers* associated to $F$.

**LEMMA 3.10.** — *The following hold:*

(i) *We have a natural commutative diagram of modules*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}_F^{\times} & \longrightarrow & F^{\times} & \longrightarrow & F^{\times}/\mathcal{O}_F^{\times} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & (\mathcal{O}_F^{\times})^{\wedge} & \longrightarrow & \mathcal{H}^{\times}(F) & \longrightarrow & F^{\times}/\mathcal{O}_F^{\times} & \longrightarrow & 1 \end{array}$$

*— where the horizontal sequences are* **exact**, *and the vertical arrows are* **injective**.

(ii) *If, moreover,* $\mathcal{O}_F^{\times}$ *is* **finite** [*or, equivalently,* $F$ *is* **contained** *in an* **imaginary quadratic** *field — cf., e.g.,* [10], *Chapter* I, *Theorem* 7.4], *then the natural homomorphism* $F^{\times} \to \mathcal{H}^{\times}(F)$, *hence also* $F_{\times} \to \mathcal{H}_{\times}(F)$, *is an* **isomorphism** *of monoids.*

(iii) *The natural inclusion* $F^{\times} \hookrightarrow \mathcal{H}^{\times}(F)$ *determines an* **isomorphism of finite groups**

$$\boldsymbol{\mu}(F) \xrightarrow{\sim} \mathcal{H}^{\times}(F)_{\mathrm{tor}}.$$

(iv) *The module* $\mathcal{H}^{\times}(F)$ *is* **generated** *by* $(\mathcal{O}_F^{\times})^{\wedge}$ *and* $F^{\times}$.

(v) *The composite of natural homomorphisms*

$$\mathcal{H}_{\times}(F) \to \prod_{v \in \mathcal{V}_F} (F_v)_{\times} \twoheadrightarrow \prod_{v \in \mathcal{V}_F^{d=1}} (F_v)_{\times}$$

*is* **injective**.

(vi) *Let $n$ be a positive integer. Then the sequence of $Q_F$-modules*

$$1 \longrightarrow \boldsymbol{\mu}(\overline{F})[n] \longrightarrow \widetilde{F}^{\times} \xrightarrow{n} \widetilde{F}^{\times} \longrightarrow 1$$

[*cf. the first assertion of Lemma* 3.6, (iv)] *is* **exact**. *Moreover, these sequences — where $n$ ranges over the positive integers — determine an* **injection** *and an* **isomorphism**

$$\mathrm{Kmm}_{\widetilde{F}/F} \colon F^{\times} \hookrightarrow (F^{\times})^{\wedge} \xrightarrow{\sim} H^1(Q_F, \Lambda(\overline{F})).$$

PROOF. — First, we verify assertion (i). The [existence and] exactness of the lower horizontal sequence of the diagram of (i) follows immediately from [1], Lemma 3.3, (i), together with the various definitions involved. The injectivity of the left-hand, hence also middle, vertical arrow follows immediately from the fact that $\mathcal{O}_F^{\times}$ is a *finitely generated module* [cf., e.g., [10], Chapter I, Theorem 7.4]. This completes the proof of assertion (i). Assertions (ii), (iv) follow immediately from assertion (i). Assertion (iii) follows immediately from assertion (i), together with the [easily verified] fact that $F^{\times}/\mathcal{O}_F^{\times}$ is *torsion-free*. Next, we verify assertion (v). Let us first observe that the subset $\mathcal{V}_F^{d=1} \subseteq \mathcal{V}_F$

of $\mathcal{V}_F$ is *of density one* [cf., e.g., the discussion preceding [10], Chapter VII, Theorem 13.2]. Thus, it follows immediately from [11], Theorem 9.1.11, that the composite

$$(F^\times)^\wedge \longrightarrow \prod_{v\in\mathcal{V}_F} (F_v^\times)^\wedge \longrightarrow \prod_{v\in\mathcal{V}_F^{d=1}} (F_v^\times)^\wedge,$$

hence [cf. the easily verified *injectivity* of the natural homomorphism $F_v^\times \to (F_v^\times)^\wedge$] also the composite discussed in assertion (v), is *injective*. This completes the proof of assertion (v). Finally, we verify assertion (vi). The first assertion of assertion (vi) follows from our assumption that $\widetilde{F}$ is *solvably closed*. The final assertion of assertion (vi) follows from the *Kummer theory*, together with the fact that there is *no nontrivial divisible element* in $F^\times$ [cf., e.g., [10], Chapter I, Theorem 7.4, together with assertion (i)]. This completes the proof of assertion (vi), hence also of Lemma 3.10. ☐

**PROPOSITION 3.11.** — *Let $G$ be a profinite group* **of GSC-type**. *Then the inclusions $D \hookrightarrow G$ and the local-global cyclotomic synchronization isomorphisms $\Lambda(G) \xrightarrow{\sim} \Lambda(D)$, where $D$ ranges over the elements of $\widetilde{\mathcal{V}}(G)$, determine an homomorphism*

$$H^1(G,\Lambda(G)) \longrightarrow \prod_{D\in\widetilde{\mathcal{V}}(G)} H^1(D,\Lambda(D)).$$

*We shall write*

$$\mathcal{H}^\times(G) \subseteq \prod_{v\in\mathcal{V}(G)} k^\times(v)$$

*for the inverse image via the above homomorphism of the image of the composite of injections*

$$\prod_{v\in\mathcal{V}(G)} k^\times(v) \hookrightarrow \prod_{D\in\widetilde{\mathcal{V}}(G)} k^\times(D) \overset{\prod_{D\in\widetilde{\mathcal{V}}(G)}\mathrm{Kmm}(D)}{\hookrightarrow} \prod_{D\in\widetilde{\mathcal{V}}(G)} H^1(D,\Lambda(D));$$

$$\mathcal{H}_\times(G) \overset{\mathrm{def}}{=} \mathcal{H}^\times(G)^\circledast \subseteq \prod_{v\in\mathcal{V}(G)} k_\times(v).$$

*We shall refer to $\mathcal{H}^\times(G)$, $\mathcal{H}_\times(G)$ as the* **Kummer containers** *associated to $G$.*

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be a* **GSC-envelope** *for $G$. Then the following hold:*

(i) *The isomorphism $\alpha$, together with the field structure of $F$, determines a* **commutative** *diagram of monoids*

$$\begin{array}{ccc} \mathcal{H}_\times(F) & \longrightarrow & \prod_{v\in\mathcal{V}_F}(F_v)_\times \\ \wr\downarrow & & \wr\downarrow \\ \mathcal{H}_\times(G) & \longrightarrow & \prod_{v\in\mathcal{V}(G)} k_\times(v) \end{array}$$

*— where the horizontal arrows are the natural inclusions, and the right-hand vertical arrow is the isomorphism determined by the various isomorphisms of monoids of Proposition 3.7, (i).*

(ii) *The composite*

$$\mathcal{H}_\times(G) \hookrightarrow \prod_{v \in \mathcal{V}(G)} k_\times(v) \twoheadrightarrow \prod_{v \in \mathcal{V}^{d=1}(G)} k_\times(v)$$

*is* **injective**.

(iii) *Let $H \subseteq G$ be an open subgroup of $G$. Then the various restriction maps of cohomology groups involved determine a* **commutative** *diagram*

$$\begin{array}{ccc} \mathcal{H}_\times(G) & \longrightarrow & \prod_{v \in \mathcal{V}(G)} k_\times(v) \\ \downarrow & & \downarrow \\ \mathcal{H}_\times(H) & \longrightarrow & \prod_{w \in \mathcal{V}(H)} k_\times(w) \end{array}$$

— *where the arrows are* **injective**.

PROOF. — These assertions follow immediately from Lemma 3.10, (v), (vi), together with the various definitions involved. □

## 4. Reconstruction of the Additive Structure on a GSC-Galois Pair

In the present §4, we discuss a *GSC-Galois pair* [cf. Definition 4.1 below]. In particular, by means of the main result of §2, we obtain a *mono-anabelian reconstruction* of the "*additive structure*" on a GSC-Galois pair [cf. Theorem 4.4 below].

In the present §4, we maintain the notation introduced at the beginning of §3. Let

$$\widetilde{F}$$

be a Galois extension of $F$ which is *solvably closed* and contained in $\overline{F}$. We shall write

- $\mathcal{V}_{\widetilde{F}}$ for the set of nonarchimedean primes of $\widetilde{F}$,
- $\mathcal{O}_{\widetilde{F}} \subseteq \widetilde{F}$ for the ring of integers of $\widetilde{F}$, and
- $Q_F \overset{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}/F)$ for the Galois group of $\widetilde{F}/F$.

**Definition 4.1.** — Let

$$(G \curvearrowright M)$$

be a collection of data consisting of a group $G$ and a $G$-monoid $M$. Then we shall refer to a collection of data

$$(K,\ \widetilde{K},\ \alpha\colon \mathrm{Gal}(\widetilde{K}/K) \xrightarrow{\sim} G,\ \beta\colon \mathcal{O}^{\triangleright}_{\widetilde{K}} \xrightarrow{\sim} M)$$

consisting of an NF $K$, a Galois extension $\widetilde{K}$ of $K$ which is solvably closed (respectively, absolutely Galois and solvably closed; algebraically closed), an isomorphism of groups $\alpha\colon \mathrm{Gal}(\widetilde{K}/K) \xrightarrow{\sim} G$, and an isomorphism of monoids $\beta\colon \mathcal{O}^{\triangleright}_{\widetilde{K}} \xrightarrow{\sim} M$ which is compatible with the actions of $\mathrm{Gal}(\widetilde{K}/K)$ and $G$ relative to $\alpha$ as a *GSC-envelope* (respectively, an *AGSC-envelope*; an *NF-envelope*) for $(G \curvearrowright M)$. We shall say that the collection of data $(G \curvearrowright M)$ is a *GSC-Galois pair* (respectively, an *AGSC-Galois pair*; an *NF-Galois pair*) if there exists a GSC-envelope (respectively, an AGSC-envelope; an NF-envelope) for $(G \curvearrowright M)$.

**Lemma 4.2.** — *Let $H \subseteq Q_F$ be a subgroup of $Q_F$. Then $H$ is an* **open subgroup** *of $Q_F$ if and only if $H$ coincides with the* **stabilizer** *of an element of $\widetilde{F}^{\times}$ [with respect to the natural action of $Q_F$ on $\widetilde{F}^{\times}$].*

Proof. — This follows from elementary field theory. □

**Proposition 4.3.** — *Let $(G \curvearrowright M)$ be a* **GSC-Galois pair**. *Then the following hold:*

(i) *The natural homomorphism*

$$G \longrightarrow \varprojlim_{N} G/N$$

*— where the projective limit is taken over the normal subgroups $N \subseteq G$ of $G$ [necessarily* **of finite index** *— cf. Lemma 4.2] such that $N$ coincides with the* **stabilizer** *of an element of $M^{\mathrm{gp}}$ [with respect to the natural action of $G$ on $M^{\mathrm{gp}}$] — is an* **isomorphism** *of groups.*

*By means of this isomorphism, we always regard the group "$G$" of a* **GSC-Galois pair** *"$(G \curvearrowright M)$" as a* **profinite group**.

(ii) *Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G,\ \beta\colon \mathcal{O}^{\triangleright}_{\widetilde{F}} \xrightarrow{\sim} M)$$

*be a* **GSC-envelope** *for $(G \curvearrowright M)$. Then the isomorphism $\alpha$ is an* **isomorphism of profinite groups** [*cf.* (i)]. *In particular, the collection of data*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*forms a* **GSC-envelope** *for the profinite group $G$.*

(iii) *The profinite group $G$ is* **of GSC-type**. *If, moreover, the GSC-Galois pair $(G \curvearrowright M)$ is an* **AGSC-Galois pair** (*respectively,* **NF-Galois pair**), *then the profinite group $G$ is* **of AGSC-type** (*respectively,* **of NF-type**).

PROOF. — These assertions follow immediately from Lemma 4.2, together with the various definitions involved. □

**THEOREM 4.4.** — *In the notation introduced at the beginning of* §4, *let $(G \curvearrowright M)$ be a* **GSC-Galois pair** [*cf. Definition* 4.1]. *We shall define various objects which arise from $(G \curvearrowright M)$ as follows:*

(1) *We shall write*

$$\boldsymbol{\mu}(M) \ \stackrel{\mathrm{def}}{=}\ (M^{\times})_{\mathrm{tor}}$$

*and*

$$\Lambda(M) \ \stackrel{\mathrm{def}}{=}\ \varprojlim_{n} \boldsymbol{\mu}(M)[n]$$

*— where the projective limit is taken over the positive integers $n$. Note that $G$ acts on $\boldsymbol{\mu}(M)$ and $\Lambda(M)$. We shall refer to the $G$-module $\Lambda(M)$ as the* **cyclotome** *associated to $(G \curvearrowright M)$. Note that one verifies immediately from our construction that the cyclotome has a natural structure of* **profinite** [*cf. also the above definition of* $\Lambda(M)$], *hence also topological, $G$-module; moreover, we have a natural identification $\boldsymbol{\mu}(M)[n] = \Lambda(M)/n\Lambda(M)$.*

(2) *By Lemma* 3.10, (vi), *we have exact sequences of $G$-modules*

$$1 \longrightarrow \Lambda(M)/n\Lambda(M) \longrightarrow M^{\mathrm{gp}} \stackrel{n}{\longrightarrow} M^{\mathrm{gp}} \longrightarrow 1$$

*— where $n$ ranges over the positive integers — which determine an* **injection**

$$(M^{\mathrm{gp}})^G \hookrightarrow H^1(G, \Lambda(M)).$$

(3) *Let $D \in v \in \mathcal{V}(G)$* [*cf. Proposition* 4.3, (iii)]. *Then it follows immediately from Lemma* 1.5, (i), *that the kernel of the composite*

$$(M^{\mathrm{gp}})^G \hookrightarrow H^1(G, \Lambda(M)) \to H^1(I(D), \Lambda(M)^{(p(D)')})$$

**depends** *only on $v$* [*i.e., does* **not depend** *on the choice of $D \in v$*]. *We shall write*

$$(M^{\mathrm{gp}})^G|^{\times}_v \subseteq (M^{\mathrm{gp}})^G$$

*for this kernel. Now observe that it follows from the definition of* $(M^{\mathrm{gp}})^G|_v^{\times} \subseteq (M^{\mathrm{gp}})^G$ *that the composite*

$$(M^{\mathrm{gp}})^G \hookrightarrow H^1(G, \Lambda(M)) \to H^1(D, \Lambda(M)^{(p(D)')})$$

*determines a homomorphism*

$$(M^{\mathrm{gp}})^G|_v^{\times} \longrightarrow H^1(D/I(D), \Lambda(M)^{(p(D)')});$$

*moreover, it follows immediately from Lemma* 1.5, (ii)*, that the kernel of this homomorphism* **depends** *only on* $v$ [*i.e., does* **not depend** *on the choice of* $D \in v$]*. We shall write*

$$(M^{\mathrm{gp}})^G|_v^{\equiv 1} \subseteq (M^{\mathrm{gp}})^G|_v^{\times}$$

*for this kernel.*

(4) *It follows from our construction, together with Lemma* 1.5*, that the collection of data*

$$\mathcal{M}(G \curvearrowright M) \stackrel{\mathrm{def}}{=} (((M^{\mathrm{gp}})^G)^{\circledast},\ M^G \subseteq ((M^{\mathrm{gp}})^G)^{\circledast},\ \mathcal{V}(G),\ \{(M^{\mathrm{gp}})^G|_v^{\equiv 1} \subseteq ((M^{\mathrm{gp}})^G)^{\circledast}\}_{v \in \mathcal{V}(G)})$$

*forms an* **NF-monoid** [*cf. Definition* 2.3]*. Thus, by Theorem* 2.9*,* (7)*,* (8)*, we have a map*

$$\boxplus_{F(G \curvearrowright M)} \stackrel{\mathrm{def}}{=} \boxplus_{\mathcal{M}(G \curvearrowright M)} \colon ((M^{\mathrm{gp}})^G)^{\circledast} \times ((M^{\mathrm{gp}})^G)^{\circledast} \longrightarrow ((M^{\mathrm{gp}})^G)^{\circledast}$$

*such that the map* $\boxplus_{F(G \curvearrowright M)}$*, together with the monoid structure of* $((M^{\mathrm{gp}})^G)^{\circledast}$*, determines a* **structure of field** *on* $((M^{\mathrm{gp}})^G)^{\circledast}$*. We shall write*

$$F(G \curvearrowright M)$$

*for the resulting field.*

(5) *For an open subgroup* $H \subseteq G$ *of* $G$*, we shall write* $(H \curvearrowright M)$ *for the* **GSC-Galois pair** *obtained by forming the collection of data consisting of* $H$*,* $M$*, and the action of* $H$ *on* $M$ *induced by the action of* $G$ *on* $M$*. Then we shall write*

$$\widetilde{F}(G \curvearrowright M) \stackrel{\mathrm{def}}{=} \varinjlim_{H} F(H \curvearrowright M)$$

*— where the injective limit is taken over the open subgroups* $H \subseteq G$ *of* $G$*. Thus,* $G$ *acts on* $\widetilde{F}(G \curvearrowright M)$*.*

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G,\ \beta\colon \mathcal{O}^{\triangleright}_{\widetilde{F}} \xrightarrow{\sim} M)$$

*be a* **GSC-envelope** *for* $(G \curvearrowright M)$ [*cf. Definition* 4.1]*. Then the isomorphism* $\beta$ *determines a* **commutative diagram of fields**

$$\begin{array}{ccc} F & \longrightarrow & \widetilde{F} \\ \wr\downarrow & & \wr\downarrow \\ F(G \curvearrowright M) & \longrightarrow & \widetilde{F}(G \curvearrowright M) \end{array}$$

*— where the horizontal arrows are the natural inclusions, the vertical arrows are* **isomorphisms***, and the right-hand vertical arrow is* **compatible** *with the natural actions of* $Q_F$ *and* $G$ *relative to* $\alpha$*.*

Proof. — This follows immediately from the various definitions involved. $\square$

**REMARK 4.4.1.** — One verifies immediately from the various definitions involved that we have a natural identification

$$\widetilde{F}(G \curvearrowright M)_{\times} \;=\; (M^{\mathrm{gp}})^{\circledast}.$$

**COROLLARY 4.5.** — *Let* $(G \curvearrowright M)$ *be a* **GSC-Galois pair** [*cf. Definition* 4.1]. *Write*

$$\mathrm{Aut}(M)$$

*for the group of automorphisms of the monoid* $M$ *and*

$$\mathrm{Aut}^{\mathrm{fld}}(M) \;\subseteq\; \mathrm{Aut}(M)$$

*for the subgroup of* $\mathrm{Aut}(M)$ *consisting of automorphisms* $\alpha$ *of* $M$ *such that the automorphism of* $(M^{\mathrm{gp}})^{\circledast}$ *induced by* $\alpha$ *is* **compatible** *with the field structure of* $\widetilde{F}(G \curvearrowright M)$ [*cf. Remark* 4.4.1]. [*Thus, the image of the faithful action*

$$G \;\hookrightarrow\; \mathrm{Aut}(M)$$

*is* **contained** *in* $\mathrm{Aut}^{\mathrm{fld}}(M) \subseteq \mathrm{Aut}(M)$.] *Then it holds that*

$$N_{\mathrm{Aut}(M)}(G) \;\subseteq\; \mathrm{Aut}^{\mathrm{fld}}(M).$$

PROOF. — This follows immediately from Theorem 4.4. □

**COROLLARY 4.6.** — *Let* $\overline{\mathbb{Q}}$ *be an algebraic closure of* $\mathbb{Q}$. *Write* $\mathcal{O}_{\overline{\mathbb{Q}}} \subseteq \overline{\mathbb{Q}}$ *for the ring of integers of* $\overline{\mathbb{Q}}$ *and* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$ *for the group of automorphisms of the monoid* $\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}}$. *Thus, we have a natural injection*

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \;\hookrightarrow\; \mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}}).$$

*By means of this injection, we regard* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *as a subgroup of* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$. *Then the following hold:*

(i) *The subgroup* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *is* **normally terminal** *in* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$, *i.e., it holds that*

$$N_{\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \;=\; \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

(ii) *The centralizer of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *in* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$ *is* **trivial**, *i.e., it holds that*

$$Z_{\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \;=\; \{1\}.$$

(iii) *The group* $\mathrm{Aut}(\mathcal{O}^{\triangleright}_{\overline{\mathbb{Q}}})$ *is* **center-free**.

PROOF. — Assertion (i) follows from Corollary 4.5. Assertion (ii) follows from assertion (i), together with the well-known fact that $Z_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = \{1\}$ [cf., e.g., [11], Corollary 12.1.6]. Assertion (iii) follows from assertion (ii). This completes the proof of Corollary 4.6. □

## 5. MONO-ANABELIAN RECONSTRUCTION OF NUMBER FIELDS

In the present §5, we finish establishing a *functorial "group-theoretic" algorithm* for reconstructing, from [a suitable quotient of] the absolute Galois group of an NF, the given NF [cf. Theorem 5.11 below].

In the present §5, we maintain the notation introduced at the beginning §4. Suppose that $\widetilde{F}$ is *absolutely Galois*. We shall write

- $Q_{F_{\mathrm{prm}}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{F}/F_{\mathrm{prm}})$ for the Galois group of $\widetilde{F}/F_{\mathrm{prm}}$.

**LEMMA 5.1.** — *The following hold:*

(i) *It holds that $F$ is* **absolutely Galois** *if and only if the following condition is satisfied: For every $v$, $w \in \mathcal{V}_F$, if $p_{F_v} = p_{F_w}$, and $f_{F_v} = 1$, then $f_{F_w} = 1$.*

(ii) *There exists a uniquely determined minimal intermediate extension of $\widetilde{F}/F$ which is* **absolutely Galois** *and* **finite** *over $F$.*

(iii) *Suppose that $F$ is* **absolutely Galois**. *Then the action*

$$Q_{F_{\mathrm{prm}}} \longrightarrow \mathrm{Aut}(Q_F)$$

*by conjugation is an* **isomorphism** *of groups.*

PROOF. — Assertion (i) follows from [10], Chapter VII, Corollary 13.8. Assertion (ii) follows immediately from our assumption that $\widetilde{F}$ is *absolutely Galois*, together with elementary field theory. Assertion (iii) follows from Theorem 3.3. This completes the proof of Lemma 5.1. □

**PROPOSITION 5.2.** — *Let $G$ be a profinite group* **of AGSC-type** *[cf. Definition 3.2]. We shall define various objects which arise from $G$ as follows:*

(1) *We shall say that $G$ is* **absolutely Galois** *if the following condition is satisfied: For every $v$, $w \in \mathcal{V}(G)$, if $p(v) = p(w)$, and $f(v) = 1$, then $f(w) = 1$.*

(2) *It follows from Lemma 5.1, (i), (ii), that there exists a uniquely determined maximal open subgroup of $G$ which is* **absolutely Galois**. *We shall refer to this open subgroup as the* **Galois closure-subgroup** *of $G$.*

(3) *We shall write*

$$G_{\mathfrak{c}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(H)$$

*for the group obtained by forming the group of automorphisms of the Galois closure-subgroup $H \subseteq G$ of $G$. Thus, since $H$ is* **normal** *in $G$ [cf. Lemma 5.1, (i)], by considering the action of $G$ on $H$ by conjugation, we obtain a homomorphism of groups*

$$G \longrightarrow G_{\mathfrak{c}}.$$

(4) *It follows from Lemma 5.1, (iii), that the homomorphism $G \to G_{\mathfrak{c}}$ of (3) is* **injective** *whose image is* **of finite index**. *Thus, the structure of profinite group of $G$ determines a structure of profinite group of $G_{\mathfrak{c}}$. By means of this, we always regard $G_{\mathfrak{c}}$ as a* **profinite group**.

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be an* **AGSC-envelope** *for* $G$ [*cf. Definition* 3.2]. *Then the following hold:*

(i) *It holds that* $F$ *is* **absolutely Galois** *if and only if* $G$ *is* **absolutely Galois**.

(ii) *The isomorphism* $\alpha$ *determines a* **commutative** *diagram of profinite groups*

$$\begin{tikzcd} Q_F \arrow[r] \arrow[d, "\alpha", "\wr"'] & Q_{F_{\mathrm{prm}}} \arrow[d, "\wr"'] \\ G \arrow[r] & G_{\mathfrak{C}} \end{tikzcd}$$

— *where the horizontal arrows are the natural open injections, and the vertical arrows are* **isomorphisms**.

PROOF. — These assertions follow immediately from Lemma 5.1, together with the various definitions involved. □

**PROPOSITION 5.3.** — *Let* $G$ *be a profinite group* **of AGSC-type**. *We shall define various objects which arise from* $G$ *as follows:*

(1) *We shall write*

$$^{\dagger}F^{\times}(G_{\mathfrak{C}}) \ \stackrel{\mathrm{def}}{=}\ \mathcal{H}^{\times}(G_{\mathfrak{C}}) \ \subseteq\ ^{\dagger}F_{\times}(G_{\mathfrak{C}}) \ \stackrel{\mathrm{def}}{=}\ \mathcal{H}_{\times}(G_{\mathfrak{C}}) \ \subseteq\ \prod_{v\in\mathcal{V}(G_{\mathfrak{C}})} k_{\times}(v).$$

(2) *We shall write*

$$^{\dagger}\mathcal{O}^{\rhd}(G_{\mathfrak{C}}) \ \subseteq\ ^{\dagger}F_{\times}(G_{\mathfrak{C}})$$

*for the submonoid of* $^{\dagger}F_{\times}(G_{\mathfrak{C}})$ *consisting of* $a \in {}^{\dagger}F_{\times}(G_{\mathfrak{C}})$ *such that, for every* $v \in \mathcal{V}(G_{\mathfrak{C}})$, *the image in* $k_{\times}(v)$ *is contained in* $\mathcal{O}^{\rhd}(v) \subseteq k_{\times}(v)$.

(3) *For* $v \in \mathcal{V}(G_{\mathfrak{C}})$, *we shall write*

$$^{\dagger}F^{\times}(G_{\mathfrak{C}})|_v^{\times} \ \stackrel{\mathrm{def}}{=}\ \mathrm{Ker}\big(^{\dagger}F^{\times}(G_{\mathfrak{C}}) \to k^{\times}(v) \twoheadrightarrow k^{\times}(v)/\mathcal{O}^{\times}(v)\big),$$

$$^{\dagger}F^{\times}(G_{\mathfrak{C}})|_v^{\equiv 1} \ \stackrel{\mathrm{def}}{=}\ \mathrm{Ker}\big(^{\dagger}F^{\times}(G_{\mathfrak{C}})|_v^{\times} \to \mathcal{O}^{\times}(v) \twoheadrightarrow \underline{k}^{\times}(v)\big).$$

(4) *It follows from Lemma* 3.10, (ii), *together with our construction, that the collection of data*

$$\mathcal{M}(G_{\mathfrak{C}}) \ \stackrel{\mathrm{def}}{=}\ \big(^{\dagger}F_{\times}(G_{\mathfrak{C}}),\ ^{\dagger}\mathcal{O}^{\rhd}(G_{\mathfrak{C}}) \subseteq {}^{\dagger}F_{\times}(G_{\mathfrak{C}}),\ \mathcal{V}(G_{\mathfrak{C}}),\ \{^{\dagger}F^{\times}(G_{\mathfrak{C}})|_v^{\equiv 1} \subseteq {}^{\dagger}F_{\times}(G_{\mathfrak{C}})\}_{v\in\mathcal{V}(G_{\mathfrak{C}})}\big)$$

*forms a* **prime NF-monoid** [*cf. Definition* 2.3]. *Thus, by Theorem* 2.9, (7), (8), *we have a map*

$$\boxplus_{^{\dagger}F(G_{\mathfrak{C}})} \ \stackrel{\mathrm{def}}{=}\ \boxplus_{\mathcal{M}(G_{\mathfrak{C}})}\colon\ ^{\dagger}F_{\times}(G_{\mathfrak{C}}) \times {}^{\dagger}F_{\times}(G_{\mathfrak{C}}) \ \longrightarrow\ ^{\dagger}F_{\times}(G_{\mathfrak{C}})$$

*such that the map* $\boxplus_{^{\dagger}F(G_{\mathfrak{C}})}$, *together with the monoid structure of* $^{\dagger}F_{\times}(G_{\mathfrak{C}})$, *determines a* **structure of field** *on* $^{\dagger}F_{\times}(G_{\mathfrak{C}})$. *We shall write*

$$^{\dagger}F(G_{\mathfrak{C}})$$

*for the resulting field.*

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be an* **AGSC-envelope** *for* $G$. *Then the isomorphism* $\alpha$, *together with the field structure of* $F$, *determines an* **isomorphism of fields**

$$F_{\mathrm{prm}} \xrightarrow{\sim} {}^{\dagger}F(G_{\mathfrak{C}}).$$

PROOF. — This follows immediately from the various definitions involved. □

**LEMMA 5.4.** — *Let* $v \in \mathcal{V}_F$. *Write*

$$(F_v^{\times} \times F_v^{\times})^{\neq 0} \ \subseteq\ F_v^{\times} \times F_v^{\times}$$

*for the subset of* $F_v^{\times} \times F_v^{\times}$ *consisting of* $(a,b) \in F_v^{\times} \times F_v^{\times}$ *such that* $a+b \neq 0$. *Then the subset*

$$(F^{\times} \times F^{\times})\ \cap\ (F_v^{\times} \times F_v^{\times})^{\neq 0} \ \subseteq\ (F_v^{\times} \times F_v^{\times})^{\neq 0}$$

*is* **dense** *in* $(F_v^{\times} \times F_v^{\times})^{\neq 0}$.

PROOF. — This follows immediately from the various definitions involved. □

**PROPOSITION 5.5.** — *Let* $G$ *be a profinite group* **of AGSC-type**. *We shall define various objects which arise from* $G$ *as follows:*

(1) *Let* $v \in \mathcal{V}(G)$. *Then we shall write*

$$(k^{\times}(v) \times k^{\times}(v))^{=0} \ \subseteq\ k^{\times}(v) \times k^{\times}(v)$$

*for the subset of the topological space* $k^{\times}(v) \times k^{\times}(v)$ *consisting of* $(a,b) \in k^{\times}(v) \times k^{\times}(v)$ *such that* $ab^{-1} \neq 1$ *but* $(ab^{-1})^2 = 1$;

$$(k^{\times}(v) \times k^{\times}(v))^{\neq 0} \ \stackrel{\mathrm{def}}{=}\ \big(k^{\times}(v) \times k^{\times}(v)\big) \setminus \big((k^{\times}(v) \times k^{\times}(v))^{=0}\big).$$

(2) *Let* $v \in \mathcal{V}(G_{\mathfrak{C}})$. *Since [it follows from our construction that] the natural homomorphism* ${}^{\dagger}F^{\times}(G_{\mathfrak{C}}) \to k^{\times}(v)$ *is* **injective**, *let us regard* ${}^{\dagger}F^{\times}(G_{\mathfrak{C}})$ *as a submodule of* $k^{\times}(v)$. *Write* $* \in k_{\times}(v)$ *for the unique element of the set* $k_{\times}(v) \setminus k^{\times}(v)$. *Then we shall define a map*

$$\boxplus_{k(v)}\colon\ k_{\times}(v) \times k_{\times}(v) \ \longrightarrow\ k_{\times}(v)$$

*as follows:*

(a) *It holds that* $\boxplus_{k(v)}(*, a) = \boxplus_{k(v)}(a, *) = a$ *for every* $a \in k_{\times}(v)$.

(b) *The image of* $(k^{\times}(v) \times k^{\times}(v))^{=0}$ *via* $\boxplus_{k(v)}$ *is* $\{*\}$.

(c) *Let* $(a,b) \in (k^{\times}(v) \times k^{\times}(v))^{\neq 0}$. *Now it follows from Lemma* 5.4 *that there exists a sequence* $(a_i, b_i)_{i\geq 1}$ *consisting of elements of* $({}^{\dagger}F^{\times}(G_{\mathfrak{C}}) \times {}^{\dagger}F^{\times}(G_{\mathfrak{C}})) \cap (k^{\times}(v) \times k^{\times}(v))^{\neq 0}$ *such that* $\lim_{i\to\infty}(a_i, b_i) = (a,b)$ *[with respect to the topology of the topological module* $k^{\times}(v) \times k^{\times}(v)$*]. Then write* $\boxplus_{k(v)}(a.b) \stackrel{\mathrm{def}}{=} \lim_{i\to\infty} \boxplus_{{}^{\dagger}F(G_{\mathfrak{C}})}(a_i, b_i)$. *Note that it follows from our construction that this "*$\boxplus_{k(v)}(a.b)$*" does* **not depend** *on the choice of the sequence* $(a_i, b_i)_{i\geq 1}$.

*If* $D \in v$, *then we shall write*

$$\boxplus_{k(D)}\colon\ k_{\times}(D) \times k_{\times}(D) \ \longrightarrow\ k_{\times}(D)$$

*for the map determined by* $\boxplus_{k(v)}$ *and the identification of* $k_\times(v)$ *with* $k_\times(D)$ *of Proposition* 3.7, (2). *Then it follows from our construction that the map* $\boxplus_{k(v)}$ (*respectively,* $\boxplus_{k(D)}$), *together with the monoid structure of* $k_\times(v)$ (*respectively,* $k_\times(D)$), *determines a* **structure of field** *on* $k_\times(v)$ (*respectively,* $k_\times(D)$). *We shall write*

$$k(v) \quad (\textit{respectively, } k(D))$$

*for the resulting field.*

(3) *Let* $v \in \mathcal{V}^{d=1}(G)$ *and* $D \in v$. *Write* $v_{\mathfrak{C}} \in \mathcal{V}(G_{\mathfrak{C}})$ *for the element determined by* $v$, *i.e., the* $G_{\mathfrak{C}}$*-conjugacy class of* $C_{G_{\mathfrak{C}}}(D) \subseteq G_{\mathfrak{C}}$ [*cf. Proposition* 3.5, (iv)]. *Then since* $d(D) = 1$, *it follows immediately from the various definitions involved that the natural inclusion* $D \hookrightarrow C_{G_{\mathfrak{C}}}(D)$ *determines an* **isomorphism** *of monoids*

$$(k_\times(v_{\mathfrak{C}}) \xrightarrow{\sim})\quad k_\times(C_{G_{\mathfrak{C}}}(D)) \xrightarrow{\sim} k_\times(D) \quad (\xleftarrow{\sim} k_\times(v)).$$

*We shall write*

$$\boxplus_{k(v)}\colon\ k_\times(v) \times k_\times(v) \longrightarrow k_\times(v), \quad \boxplus_{k(D)}\colon\ k_\times(D) \times k_\times(D) \longrightarrow k_\times(D)$$

*for the maps determined by* $\boxplus_{k(v_{\mathfrak{C}})}$. *Then it follows from our construction that the map* $\boxplus_{k(v)}$ (*respectively,* $\boxplus_{k(D)}$), *together with the monoid structure of* $k_\times(v)$ (*respectively,* $k_\times(D)$), *determines a* **structure of field** *on* $k_\times(v)$ (*respectively,* $k_\times(D)$). *We shall write*

$$k(v) \quad (\textit{respectively, } k(D))$$

*for the resulting field.*

(4) *It follows from Proposition* 3.11, (ii), *that the composite of homomorphisms of monoids*

$$\mathcal{H}_\times(G) \hookrightarrow \prod_{v \in \mathcal{V}(G)} k_\times(v) \twoheadrightarrow \prod_{v \in \mathcal{V}^{d=1}(G)} k_\times(v)$$

*is* **injective**. *Observe that the structures of fields on the* $k_\times(v)$*'s of* (3), *where* $v$ *ranges over the elements of* $\mathcal{V}^{d=1}(G)$, *determine a* **structure of ring** *on*

$$\prod_{v \in \mathcal{V}^{d=1}(G)} k_\times(v).$$

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be an* **AGSC-envelope** *for* $G$ *and* $v \in \mathcal{V}^{d=1}(G)$. *Then the isomorphism of monoids* $(F_v)_\times \xrightarrow{\sim} k_\times(v)$ *of Proposition* 3.7, (i), *determines an* **isomorphism of fields**

$$F_v \xrightarrow{\sim} k(v).$$

PROOF. — This follows immediately from the various definitions involved. □

**LEMMA 5.6.** — *Let* $E$ *be a finite Galois extension of* $F$ *contained in* $\widetilde{F}$ *and* $d$ *a positive integer. Thus, we have a commutative diagram of monoids*

$$\begin{tikzcd} F_\times \arrow[r] \arrow[d] & \mathcal{H}_\times(F) \arrow[d] & \\ E_\times \arrow[r] & \mathcal{H}_\times(E) \arrow[r] & \prod_{w \in \mathcal{V}^{d=1}(E)} (E_w)_\times \end{tikzcd}$$

— *where the arrows are* **injective** [*cf. Lemma* 3.10, (i), (v)]. *For a positive integer* $r$*, we use the notation* $\zeta_r \in \overline{F}$ *to denote a primitive* $r$*-th root of unity in* $\overline{F}$*. Then the following hold:*

(i) *Suppose that* $E = F(\zeta_d)$*. Let* $\zeta \in \mathcal{H}^\times(E)$ *be a* **torsion** *element of* $\mathcal{H}^\times(E)$ **of order** $\boldsymbol{d}$*. Then the submonoid* $E_\times \subseteq \mathcal{H}_\times(E)$ **coincides** *with the underlying* [*multiplicative*] *submonoid of the* **subring** *of* $\prod_{w \in \mathcal{V}^{d=1}(E)} E_w$ *generated by* $F$ *and* $\zeta$*.*

(ii) *Suppose that* $d$ *is a* **prime** *number, that* $\zeta_d \in F$*, and that* $\mathrm{Gal}(E/F)$ *is* **of order** $\boldsymbol{d}$*. Then there exists an element* $x \in \mathcal{H}_\times(E)$ *such that* $x \notin F$ *but* $x^d \in F$*. Moreover, the submonoid* $E_\times \subseteq \mathcal{H}_\times(E)$ **coincides** *with the underlying* [*multiplicative*] *submonoid of the* **subring** *of* $\prod_{w \in \mathcal{V}^{d=1}(E)} E_w$ *generated by* $F$ *and* $x$*.*

(iii) *Suppose that* $E$ *is* **contained** *in a finite* **solvable** *extension of* $F_{\mathrm{prm}}$*. Then, after possibly replacing* $E$ *by a finite extension of* $E$ *which is contained in a finite solvable extension of* $F_{\mathrm{prm}}$*, there exists a finite sequence of finite extensions of* $F_{\mathrm{prm}}$ *contained in* $E$

$$F_{\mathrm{prm}} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n \stackrel{\mathrm{def}}{=} E$$

*such that, for each* $i \in \{1, \ldots, n\}$*, the extension* $F_i/F_{i-1}$ *is* **Galois***, and, moreover, one of the following two conditions is satisfied:*

(a) *It holds that* $F_i = F_{i-1}(\zeta_r)$ *for some positive integer* $r$*.*

(b) *It holds that* $d_i \stackrel{\mathrm{def}}{=} \sharp\mathrm{Gal}(F_i/F_{i-1})$ *is a* **prime** *number, and, moreover,* $\zeta_{d_i} \in F_{i-1}$*.*

PROOF. — Assertion (i) follows immediately from Lemma 3.10, (iii), together with the various definitions involved. Next, we verify assertion (ii). The existence of such an $x$ follows immediately from the *Kummer theory*, together with our assumption that $\zeta_d \in F$ and $\mathrm{Gal}(E/F)$ is necessarily *cyclic of order* $d$. In order to verify the final assertion of assertion (ii), let $u \in (\mathcal{O}_E^\times)^\wedge$, $a \in E^\times$ be such that $x = u \cdot a$ [cf. Lemma 3.10, (iv)]. Since $x^d \in F^\times$, it follows from Lemma 3.10, (i), that $u^d \in \mathcal{O}_E^\times$. Thus, since [one verifies immediately that] the cokernel of the natural homomorphism $\mathcal{O}_E^\times \hookrightarrow (\mathcal{O}_E^\times)^\wedge$ is *torsion-free* [cf. also [1], Lemma 3.3, (ii)], it holds that $u \in \mathcal{O}_E^\times$, hence that $x \in E^\times$. In particular, the *subring* of $\prod_{w \in \mathcal{V}^{d=1}(E)} E_w$ generated by $F$ and $x$ determines an intermediate extension of the finite extension $E/F$. On the other hand,, since [we have assumed that] $d$ is a *prime* number, and that $\mathrm{Gal}(E/F)$ is of order $d$, the assumption that $x \notin F$ implies that this intermediate extension *coincides* with $E$. This completes the proof of assertion (ii). Assertion (iii) follows immediately from elementary field theory. This completes the proof of Lemma 5.6. □

**PROPOSITION 5.7.** — *Let* $G$ *be a profinite group* **of AGSC-type***. We shall define various objects which arise from* $G$ *as follows:*

(1) *Let* $H \subseteq G_{\mathfrak{C}}$ *be an open subgroup of* $G_{\mathfrak{C}}$*. Suppose that we are given a finite sequence of open subgroups of* $G_{\mathfrak{C}}$

$$H \stackrel{\mathrm{def}}{=} G_n \subseteq G_{n-1} \subseteq \cdots \subseteq G_1 \subseteq G_0 \stackrel{\mathrm{def}}{=} G_{\mathfrak{C}}$$

*such that, for each* $i \in \{1, \ldots, n\}$*,* $G_i$ *is* **normal** *in* $G_{i-1}$*, and, moreover, one of the following two conditions is satisfied:*

(a) *It holds that* $G_i = G_{i-1}^{(\boldsymbol{\mu}_r)}$ *for some positive integer* $r$.

(b) *It holds that* $d_i \stackrel{\mathrm{def}}{=} \sharp(G_{i-1}/G_i)$ *is a* **prime** *number, and, moreover,* $(G_{\mathfrak{C}})^{(\boldsymbol{\mu}_{d_i})} \subseteq G_{i-1}$.

*Then we shall inductively define submonoids* ${}^{\dagger}F_{\times}(G_i)$*'s of the* $\mathcal{H}_{\times}(G_i)$*'s*

$$\begin{array}{ccccccccccccc} {}^{\dagger}F_{\times}(G_0) & = & {}^{\dagger}F_{\times}(G_{\mathfrak{C}}) & \subseteq & {}^{\dagger}F_{\times}(G_1) & \subseteq & \cdots & \subseteq & {}^{\dagger}F_{\times}(G_{n-1}) & \subseteq & {}^{\dagger}F_{\times}(G_n) & = & {}^{\dagger}F_{\times}(H) \\ \cap & & & & \cap & & & & \cap & & \cap & & \\ \mathcal{H}_{\times}(G_0) & = & \mathcal{H}_{\times}(G_{\mathfrak{C}}) & \subseteq & \mathcal{H}_{\times}(G_1) & \subseteq & \cdots & \subseteq & \mathcal{H}_{\times}(G_{n-1}) & \subseteq & \mathcal{H}_{\times}(G_n) & = & \mathcal{H}_{\times}(H) \end{array}$$

*as follows: Let* $i \in \{1, \ldots, n\}$. *Suppose that we are given a submonoid* ${}^{\dagger}F_{\times}(G_{i-1}) \subseteq \mathcal{H}_{\times}(G_{i-1})$. [*Note that the submonoid* ${}^{\dagger}F_{\times}(G_0) = {}^{\dagger}F_{\times}(G_{\mathfrak{C}})$ *of* $\mathcal{H}_{\times}(G_0) = \mathcal{H}_{\times}(G_{\mathfrak{C}})$ *was already defined in Proposition* 5.3, (1).]

- *Suppose that* $G_i = G_{i-1}^{(\boldsymbol{\mu}_r)}$ *for some positive integer* $r$ [*cf. condition* (a)]. *Let* $\zeta \in \mathcal{H}_{\times}(G_i)$ *be a* **torsion** *element* **of order** $\boldsymbol{r}$. *Then we shall write*

$$ {}^{\dagger}F_{\times}(G_i) \subseteq \prod_{v \in \mathcal{V}^{d=1}(G_i)} k_{\times}(v)$$

*for the underlying* [*multiplicative*] *monoid of the* **subring** *of the ring* $\prod_{v \in \mathcal{V}^{d=1}(G_i)} k_{\times}(v)$ [*cf. Proposition* 5.5, (4)] *generated by the images of* ${}^{\dagger}F_{\times}(G_{i-1})$ *and* $\zeta$. *Then it follows from our construction* [*cf. also Lemma* 5.6, (i)] *that* ${}^{\dagger}F_{\times}(G_i)$ *is* **contained** *in* $\mathcal{H}_{\times}(G_i)$ [*relative to the injection discussed in Proposition* 5.5, (4)] *and, moreover,* **independent** *of the choice of* $\zeta$.

- *Suppose that* $d_i \stackrel{\mathrm{def}}{=} \sharp(G_{i-1}/G_i)$ *is a* **prime** *number, and, moreover,* $(G_{\mathfrak{C}})^{(\boldsymbol{\mu}_{d_i})} \subseteq G_{i-1}$ [*cf. condition* (b)]. *Then it follows from Lemma* 5.6, (ii), *that there exists an element* $x \in \mathcal{H}_{\times}(G_i)$ *such that* $x \notin {}^{\dagger}F_{\times}(G_{i-1})$ *but* $x^{d_i} \in {}^{\dagger}F_{\times}(G_{i-1})$. *We shall write*

$$ {}^{\dagger}F_{\times}(G_i) \subseteq \prod_{v \in \mathcal{V}^{d=1}(G_i)} k_{\times}(v)$$

*for the underlying* [*multiplicative*] *monoid of the* **subring** *of the ring* $\prod_{v \in \mathcal{V}^{d=1}(G_i)} k_{\times}(v)$ [*cf. Proposition* 5.5, (4)] *generated by the images of* ${}^{\dagger}F_{\times}(G_{i-1})$ *and* $x$. *Then it follows from our construction* [*cf. also Lemma* 5.6, (ii)] *that* ${}^{\dagger}F_{\times}(G_i)$ *is* **contained** *in* $\mathcal{H}_{\times}(G_i)$ [*relative to the injection discussed in Proposition* 5.5, (4)] *and, moreover,* **independent** *of the choice of* $x$.

*Next, let us observe that it follows immediately from our construction that the maps* $\boxplus_{k(v)}$, *where* $v$ *ranges over the elements of* $\mathcal{V}^{d=1}(H)$, *determine* [*cf. the injection discussed in Proposition* 5.5, (4)] *a map*

$$\boxplus_{{}^{\dagger}F(H)} \colon {}^{\dagger}F_{\times}(H) \times {}^{\dagger}F_{\times}(H) \longrightarrow {}^{\dagger}F_{\times}(H).$$

*Moreover, it follows from our construction that the map* $\boxplus_{{}^{\dagger}F(H)}$, *together with the monoid structure of* ${}^{\dagger}F_{\times}(H)$, *determines a* **structure of field** *on* ${}^{\dagger}F_{\times}(H)$. *We shall write*

$$ {}^{\dagger}F(H)$$

*for the resulting field. Note that it follows from our construction that the submonoid* ${}^{\dagger}F_{\times}(H) \subseteq \mathcal{H}_{\times}(H)$ *and the map* $\boxplus_{{}^{\dagger}F(H)}$*, hence also the field structure of* ${}^{\dagger}F(H)$*, do* **not depend** *on the choice of the sequence*

$$H \stackrel{\mathrm{def}}{=} G_n \subseteq G_{n-1} \subseteq \cdots \subseteq G_1 \subseteq G_0 \stackrel{\mathrm{def}}{=} G_{\mathfrak{C}}.$$

(2) *Write* $G_{\mathfrak{C}} \twoheadrightarrow G_{\mathfrak{C}}^{\mathrm{slv}}$ *for the maximal prosolvable quotient of* $G_{\mathfrak{C}}$*. Then it follows from Lemma* 5.6, (iii)*, that every open subgroup of* $G_{\mathfrak{C}}$ *which arises from an open subgroup of* $G_{\mathfrak{C}}^{\mathrm{slv}}$ **contains** *an open subgroup of* $G_{\mathfrak{C}}$ *which satisfies the condition imposed on "$H$" of* (1)*, i.e., the existence of a suitable sequence of an open subgroups of* $G_{\mathfrak{C}}$*. Thus, we have a submonoid*

$${}^{\dagger}F_{\times}^{\mathrm{slv}}(G_{\mathfrak{C}}) \stackrel{\mathrm{def}}{=} \varinjlim_{H} {}^{\dagger}F_{\times}(H) \subseteq \varinjlim_{H} \mathcal{H}_{\times}(H)$$

*— where the injective limits are taken over the open subgroups* $H \subseteq G_{\mathfrak{C}}$ *of* $G_{\mathfrak{C}}$ *which satisfy the condition imposed on "$H$" of* (1) *— equipped with the map* [*determined by the various* $\boxplus_{{}^{\dagger}F(H)}$*'s — where* $H$ *ranges over the open subgroups of* $G_{\mathfrak{C}}$ *which satisfy the condition imposed on "$H$" of* (1)]

$$\boxplus_{{}^{\dagger}F^{\mathrm{slv}}(G_{\mathfrak{C}})} \colon {}^{\dagger}F_{\times}^{\mathrm{slv}}(G_{\mathfrak{C}}) \times {}^{\dagger}F_{\times}^{\mathrm{slv}}(G_{\mathfrak{C}}) \longrightarrow {}^{\dagger}F_{\times}^{\mathrm{slv}}(G_{\mathfrak{C}}).$$

*Moreover, it follows from our construction that the map* $\boxplus_{{}^{\dagger}F^{\mathrm{slv}}(G_{\mathfrak{C}})}$*, together with the monoid structure of* ${}^{\dagger}F_{\times}^{\mathrm{slv}}(G_{\mathfrak{C}})$*, determines a* **structure of field** *on* ${}^{\dagger}F_{\times}^{\mathrm{slv}}(G_{\mathfrak{C}})$*. We shall write*

$${}^{\dagger}F^{\mathrm{slv}}(G_{\mathfrak{C}})$$

*for the resulting field.*

*Let*

$$(F,\ \widetilde{F},\ \alpha \colon Q_F \xrightarrow{\sim} G)$$

*be an* **AGSC-envelope** *for* $G$*. Write* $F_{\mathrm{prm}}^{\mathrm{slv}}$ *for the solvable closure of* $F_{\mathrm{prm}}$ *in* $\widetilde{F}$*. Then the isomorphism* $\alpha$*, together with the field structures of the various fields involved, determines an* **isomorphism of fields**

$$F_{\mathrm{prm}}^{\mathrm{slv}} \xrightarrow{\sim} {}^{\dagger}F^{\mathrm{slv}}(G_{\mathfrak{C}}).$$

PROOF. — This follows from Lemma 5.6, together with the various definitions involved. □

**PROPOSITION 5.8.** — *Let* $G$ *be a profinite group* **of AGSC-type***. We shall define various objects which arise from* $G$ *as follows:*

(1) *Let* $D \in \widetilde{\mathcal{V}}(G_{\mathfrak{C}})$*. Write* $G_{\mathfrak{C}} \twoheadrightarrow G_{\mathfrak{C}}^{\mathrm{slv}}$ *for the maximal prosolvable quotient of* $G_{\mathfrak{C}}$*. Then since the composite* $D \hookrightarrow G_{\mathfrak{C}} \twoheadrightarrow G_{\mathfrak{C}}^{\mathrm{slv}}$ *is* **injective** [*cf.* [6], *Proposition* 2.3, (iii)]*, there exists a sequence of normal open subgroups of* $G_{\mathfrak{C}}$ *which* **arise** *from open subgroups of* $G_{\mathfrak{C}}^{\mathrm{slv}}$

$$\cdots \subseteq G_{n+1} \subseteq G_n \subseteq \cdots \subseteq G_1 \subseteq G_0 = G_{\mathfrak{C}}$$

*such that if, for each* $n$*, we write* $D_n \stackrel{\mathrm{def}}{=} G_n \cap D$*, then*

$$\bigcap_{n \geq 0} D_n = \{1\}.$$

*Write* ${}^{\dagger}F^{\times}(G_n) \stackrel{\text{def}}{=} {}^{\dagger}F^{\text{slv}}(G_{\mathfrak{C}})^{G_n}$. *Then since [one verifies immediately that] the natural homomorphism* ${}^{\dagger}F^{\times}(G_n) \to k^{\times}(D_n)$ *is* **injective**, *let us regard* ${}^{\dagger}F^{\times}(G_n)$ *as a submodule of* $k^{\times}(D_n)$. *Write* $* \in k_{\times}(D_n)$ *for the unique element of the set* $k_{\times}(D_n) \setminus k^{\times}(D_n)$. *Then we shall define a map*

$$\boxplus_{k(D_n)}\colon\ k_{\times}(D_n) \times k_{\times}(D_n) \ \longrightarrow\ k_{\times}(D_n)$$

*as follows:*

(a) *It holds that* $\boxplus_{k(D_n)}(*, a) = \boxplus_{k(D_n)}(a, *) = a$ *for every* $a \in k_{\times}(D_n)$.

(b) *The image of* $(k^{\times}(D_n) \times k^{\times}(D_n))^{=0}$ [*cf. Proposition* 5.5, (1)] *via* $\boxplus_{k(D_n)}$ *is* $\{*\}$.

(c) *Let* $(a, b) \in (k^{\times}(D_n) \times k^{\times}(D_n))^{\neq 0}$ [*cf. Proposition* 5.5, (1)]. *Now it follows from Lemma* 5.4 *that there exists a sequence* $(a_i, b_i)_{i\geq 1}$ *consisting of elements of* $({}^{\dagger}F^{\times}(G_n)\times{}^{\dagger}F^{\times}(G_n))\cap(k^{\times}(D_n)\times k^{\times}(D_n))^{\neq 0}$ *such that* $\lim_{i\to\infty}(a_i, b_i) = (a, b)$ [*with respect to the topology of the topological module* $k^{\times}(D_n) \times k^{\times}(D_n)$]. *Then write* $\boxplus_{k(D_n)}(a.b) \stackrel{\text{def}}{=} \lim_{i\to\infty} \boxplus_{{}^{\dagger}F^{\text{slv}}(G_{\mathfrak{C}})}(a_i, b_i)$. *Note that it follows from our construction that this "*$\boxplus_{k(D_n)}(a.b)$*" does* **not depend** *on the choice of the sequence* $(a_i, b_i)_{i\geq 1}$.

*It follows from our construction that the map* $\boxplus_{k(D_n)}$, *together with the monoid structure of* $k_{\times}(D_n)$, *determines a* **structure of field** *on* $k_{\times}(D_n)$. *We shall write*

$$k(D_n)$$

*for the resulting field.*

(2) *In the notation of* (1), *since it holds that*

$$\overline{k}_{\times}(D) \ = \ \varinjlim_{n} k_{\times}(D_n)$$

[*cf. Theorem* 1.4, (9)], *it follows immediately from our construction that the various maps* $\boxplus_{k(D_n)}$, *where* $n$ *ranges over the nonnegative integers, determine a map*

$$\boxplus_{\overline{k}(D)}\colon\ \overline{k}_{\times}(D) \times \overline{k}_{\times}(D) \ \longrightarrow\ \overline{k}_{\times}(D)$$

*such that the map* $\boxplus_{\overline{k}(D)}$, *together with the monoid structure of* $\overline{k}_{\times}(D)$, *determines a* **structure of field** *on* $\overline{k}_{\times}(D)$. *We shall write*

$$\overline{k}(D)$$

*for the resulting field. Note that it follows from our construction that the map* $\boxplus_{\overline{k}(D)}$, *hence also the field structure of* $\overline{k}(D)$, *does* **not depend** *on the choice of the sequence*

$$\cdots \ \subseteq\ G_{n+1} \ \subseteq\ G_n \ \subseteq\ \cdots \ \subseteq\ G_1 \ \subseteq\ G_0 \ \stackrel{\text{def}}{=}\ G_{\mathfrak{C}}.$$

(3) *Let* $D \in \widetilde{\mathcal{V}}(G)$. *Thus, it holds that* $C_{G_{\mathfrak{C}}}(D) \in \widetilde{\mathcal{V}}(G_{\mathfrak{C}})$ [*cf. Proposition* 3.5, (iv)]. *Moreover, it follows immediately from the various definitions involved that* $\overline{k}_{\times}(D) = \overline{k}_{\times}(C_{G_{\mathfrak{C}}}(D))$. *We shall write*

$$\boxplus_{\overline{k}(D)}\colon\ \overline{k}_{\times}(D) \times \overline{k}_{\times}(D) \ \longrightarrow\ \overline{k}_{\times}(D)$$

*for the map corresponding to* $\boxplus_{\overline{k}(C_{G_{\mathfrak{C}}}(D))}$, *which thus determines a* **structure of field** *on* $\overline{k}_{\times}(D)$. *We shall write*

$$\overline{k}(D)$$

*for the resulting field.*

(4) *Let $D \in v \in \mathcal{V}(G)$. Then it follows immediately from the various definitions involved that $k_\times(D) = \overline{k}_\times(D)^D$. We shall write*

$$\boxplus_{k(D)}\colon\ k_\times(D) \times k_\times(D) \longrightarrow k_\times(D)$$

*for the map determined by $\boxplus_{\overline{k}(D)}$ and*

$$\boxplus_{k(v)}\colon\ k_\times(v) \times k_\times(v) \longrightarrow k_\times(v)$$

*for the map determined by $\boxplus_{k(D)}$ and the natural identification of $k_\times(v)$ with $k_\times(D)$ of Proposition* 3.7, (2)*. Then it follows from our construction that the map $\boxplus_{k(D)}$ (respectively, $\boxplus_{k(v)}$), together with the monoid structure of $k_\times(D)$ (respectively, $k_\times(v)$), determines a* **structure of field** *on $k_\times(D)$ (respectively, $k_\times(v)$). We shall write*

$$k(D) \quad (\textit{respectively, } k(v))$$

*for the resulting field.*

(5) *Observe that the structures of fields on the $k_\times(v)$'s of* (4)*, where $v$ ranges over the elements of $\mathcal{V}(G)$, determine a* **structure of ring** *on*

$$\prod_{v \in \mathcal{V}(G)} k_\times(v).$$

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be an* **AGSC-envelope** *for $G$ and $D \in v \in \mathcal{V}(G)$. Write $\widetilde{v} \in \mathcal{V}_{\widetilde{F}}$ for the element of $\mathcal{V}_{\widetilde{F}}$ corresponding, via $\alpha$, to $D \in \widetilde{\mathcal{V}}(G)$. Then the commutative diagram of monoids*

$$\begin{array}{ccc} (F_v)_\times & \longrightarrow & (\overline{F}_{\widetilde{v}})_\times \\ \wr\downarrow & & \wr\downarrow \\ k_\times(v) & \longrightarrow & \overline{k}_\times(D) \end{array}$$

*— where the horizontal arrows are natural inclusion, the left-hand vertical arrow is the isomorphism of monoids of Theorem* 1.4, (iii)*, and the right-hand vertical arrow is the isomorphism of monoids of Theorem* 1.4, (iv) *— determines a* **commutative diagram of fields**

$$\begin{array}{ccc} F_v & \longrightarrow & \overline{F}_{\widetilde{v}} \\ \wr\downarrow & & \wr\downarrow \\ k(v) & \longrightarrow & \overline{k}(D). \end{array}$$

PROOF. — This follows from the various definitions involved. □

**LEMMA 5.9.** — *Suppose that $F$ is* **absolutely Galois***. Then the following hold:*

(i) *There exists a* **subfield**

$$E \subseteq \prod_{v \in \mathcal{V}_F} F_v$$

*such that if we write* $E^{\mathrm{prm}} \subseteq E$ *for the* **prime** *subfield of* $E$*, then*

$$\sharp \mathrm{Aut}^{\mathrm{fld}}(E) \ = \ [E : E^{\mathrm{prm}}] \ = \ [F : F_{\mathrm{prm}}]$$

*— where we write* $\mathrm{Aut}^{\mathrm{fld}}(E)$ *for the group of* **field automorphisms** *of* $E$*.*

(ii) *In the situation of* (i)*, the field* $E$ *is, as an abstract field,* **isomorphic** *to the field* $F$*.*

(iii) *In the situation of* (i)*, let* $v_0 \in \mathcal{V}_F$ *and* $\overline{F}_{v_0}$ *an algebraic closure of* $F_{v_0}$*. Then the image of the composite*

$$E \ \hookrightarrow \ \prod_{v \in \mathcal{V}_F} F_v \ \twoheadrightarrow \ F_{v_0} \ \hookrightarrow \ \overline{F}_{v_0}$$

**coincides** *with the natural image of* $F$ *in* $\overline{F}_{v_0}$*. In particular, this image does* **not depend** *on the choice of "*$E$*" as in* (i)*.*

(iv) *Let* $M$ *be an intermediate extension of* $\overline{F}/F$*. Suppose that* $M$ *is* **Galois** *over* $F$ *and* **solvably closed***, and that* $\mathrm{Gal}(M/F)$ *is* **isomorphic** *to* $Q_F$*. Then it holds that* $M = \widetilde{F}$ *in* $\overline{F}$*.*

PROOF. — Assertion (i) follows from the fact that the various inclusions $F \hookrightarrow F_v$, where $v$ ranges over the elements of $\mathcal{V}_F$, determine a *subfield of the desired type*. Next, we verify assertion (ii). Let us first observe that it follows from the condition given in the statement of assertion (i) that $E$ is an *absolutely Galois NF*. Thus, since $E$ is *contained* in $\prod_{v \in \mathcal{V}_F} F_v$, one verifies immediately that, for a prime number $p$, if $p$ *splits completely* in $F$, then $p$ also *splits completely* in $E$. In particular, assertion (ii) follows immediately from [10], Chapter VII, Proposition 13.9, together with the equality $[E : E^{\mathrm{prm}}] = [F : F_{\mathrm{prm}}]$ in the statement of assertion (i). This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii), together with our assumption that $F$ is an *absolutely Galois NF*. Finally, we verify assertion (iv). Since $M$ is *solvably closed*, and $\mathrm{Gal}(M/F)$ is *isomorphic* to $Q_F$, it follows from Theorem 3.3 that $M$ is *isomorphic* to $\widetilde{F}$. Thus, since $\widetilde{F}$ is *absolutely Galois*, we conclude that $M = \widetilde{F}$ in $\overline{F}$. This completes the proof of assertion (iv), hence also of Lemma 5.9. □

**PROPOSITION 5.10.** — *Let* $G$ *be a profinite group* **of AGSC-type***. Suppose that* $G$ *is* **absolutely Galois***. We shall define various objects which arise from* $G$ *as follows:*

(1) *Let* $D \in \widetilde{\mathcal{V}}(G)$*. Now it follows from Lemma* 5.9, (i)*, that there exists a* [*not necessarily unique*] **subring**

$$R_{\mathrm{ncn}} \ \subseteq \prod_{v \in \mathcal{V}(G)} k_{\times}(v)$$

[*cf. Proposition* 5.8, (5)] *which satisfies the following two conditions* [*where "ncn" is to be understood as an abbreviation for "noncanonical"*]*:*

- *The ring* $R_{\mathrm{ncn}}$ *is a* **field***.*
- *If we write* $R_{\mathrm{ncn}}^{\mathrm{prm}} \subseteq R_{\mathrm{ncn}}$ *for the* **prime** *subfield of* $R_{\mathrm{ncn}}$*, then it holds that*

$$\sharp \mathrm{Aut}^{\mathrm{fld}}(R_{\mathrm{ncn}}) \ = \ [R_{\mathrm{ncn}} : R_{\mathrm{ncn}}^{\mathrm{prm}}] \ = \ d(G)$$

*— where we write* $\mathrm{Aut}^{\mathrm{fld}}(R_{\mathrm{ncn}})$ *for the group of* **field automorphisms** *of* $R_{\mathrm{ncn}}$*.*

*Then it follows from Lemma* 5.9, (iii), *that the image of the composite*

$$R_{\mathrm{ncn}} \hookrightarrow \prod_{v \in \mathcal{V}(G)} k_{\times}(v) \twoheadrightarrow k_{\times}(D) \hookrightarrow \overline{k}_{\times}(D)$$

*does* **not depend** *on the choice of* $R_{\mathrm{ncn}}$. *We shall write*

$$F(D) \subseteq \overline{k}(D)$$

*for the subfield obtained by forming this image and*

$$\overline{F}(D) \subseteq \overline{k}(D)$$

*for the* **algebraic closure** *of* $F(D)$ *in* $\overline{k}(D)$.

(2) *Let* $D \in \widetilde{\mathcal{V}}(G)$. *Then it follows from Lemma* 5.9, (iv), *that there exists a* **uniquely determined** *intermediate extension of* $\overline{F}(D)/F(D)$

$$\widetilde{F}(D)$$

*such that* $\widetilde{F}(D)$ *is* **Galois** *over* $F(D)$ *and* **solvably closed**, *and, moreover, the profinite group* $\mathrm{Gal}(\widetilde{F}(D)/F(D))$ *is isomorphic to* $G$. *Let*

$$\alpha_{\mathrm{ncn}} \colon G \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}(D)/F(D))$$

*be an isomorphism of profinite groups* [*where "ncn" is to be understood as an abbreviation for "noncanonical"*]. *In particular, the collection of data*

$$(F(D),\ \widetilde{F}(D),\ \alpha_{\mathrm{ncn}}^{-1})$$

*forms an* **AGSC-envelope** *for* $G$ [*cf. also Proposition* 5.2, (i)]. *Thus, it follows from Theorem* 3.8, (iii), *that* $\alpha_{\mathrm{ncn}}$ *determines an isomorphism*

$$\Lambda(\overline{F}(D)) \xrightarrow{\sim} \Lambda(G)$$

*which are* **compatible** *with the natural actions of* $\mathrm{Gal}(\widetilde{F}(D)/F(D))$ *and* $G$ *relative to* $\alpha_{\mathrm{ncn}}$. *In particular, the isomorphism* $\alpha_{\mathrm{ncn}}$ *determines an* **isomorphism**

$$H^1(\mathrm{Gal}(\widetilde{F}(D)/F(D)), \Lambda(\overline{F}(D))) \xrightarrow{\sim} H^1(G, \Lambda(G)).$$

(3) *Let* $D \in \widetilde{\mathcal{V}}(G)$. *Then the exact sequences of* $\mathrm{Gal}(\widetilde{F}(D)/F(D))$*-modules*

$$1 \longrightarrow \Lambda(\overline{F}(D))/n\Lambda(\overline{F}(D)) \longrightarrow \widetilde{F}(D)^{\times} \xrightarrow{n} \widetilde{F}(D)^{\times} \longrightarrow 1$$

— *where* $n$ *ranges over the positive integers* — *determine an* **injection**

$$F(D)^{\times} \hookrightarrow H^1(\mathrm{Gal}(\widetilde{F}(D)/F(D)), \Lambda(\overline{F}(D)))$$

[*cf. Lemma* 3.10, (vi)]. *Moreover, it follows immediately from Theorem* 3.3 *that the image of the composite*

$$F(D)^{\times} \hookrightarrow H^1(\mathrm{Gal}(\widetilde{F}(D)/F(D)), \Lambda(\overline{F}(D))) \xrightarrow{\sim} H^1(G, \Lambda(G))$$

— *where the "$\xrightarrow{\sim}$" is the isomorphism discussed in* (2) — *does* **not depend** *on the choices of* $D$ *and* $\alpha_{\mathrm{ncn}}$ *and, moreover, is* **contained** *in* $\mathcal{H}^{\times}(G) \subseteq H^1(G, \Lambda(G))$. *We shall write*

$$F^{\times}(G) \subseteq \mathcal{H}^{\times}(G)$$

*for the submodule obtained by forming the image of the above composite and*

$$F_{\times}(G) \stackrel{\mathrm{def}}{=} F^{\times}(G)^{\circledast} \subseteq \mathcal{H}_{\times}(G).$$

(4) *We shall write*

$$\mathcal{O}^{\triangleright}(G) \ \subseteq \ F_{\times}(G)$$

*for the submonoid of* $F_{\times}(G)$ *consisting of* $a \in F_{\times}(G)$ *such that, for every* $v \in \mathcal{V}(G)$, *the image in* $k_{\times}(v)$ *is contained in* $\mathcal{O}^{\triangleright}(v) \subseteq k_{\times}(v)$.

*Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be an* **AGSC-envelope** *for* $G$. *Then the isomorphism* $\alpha$, *together with the field structure of* $F$, *determines an* **isomorphism** *of monoids*

$$\mathcal{O}_F^{\triangleright} \ \xrightarrow{\sim} \ \mathcal{O}^{\triangleright}(G).$$

PROOF. — This follows from Lemma 5.9 together with the various definitions involved. $\square$

**THEOREM 5.11.** — *In the notation introduced at the beginning of* §5, *let* $G$ *be a profinite group* **of AGSC-type** [*cf. Definition* 3.2]. *We shall write*

$$\widetilde{\mathcal{O}}^{\triangleright}(G) \ \stackrel{\text{def}}{=} \ \varinjlim_{H} \mathcal{O}^{\triangleright}(H)$$

— *where the injective limit is taken over the open subgroups* $H \subseteq G$ *of* $G$ *which are* **absolutely Galois** — *for the monoid obtained by forming the injective limit of the various monoids* $\mathcal{O}^{\triangleright}(H)$*'s of Proposition* 5.10, (4). *Then* $G$ *acts on* $\widetilde{\mathcal{O}}^{\triangleright}(G)$ *by conjugation; moreover, it follows from Proposition* 5.10 *that the collection of data*

$$(G \curvearrowright \widetilde{\mathcal{O}}^{\triangleright}(G))$$

*consisting of the profinite group* $G$ *and a* $G$*-monoid* $\widetilde{\mathcal{O}}^{\triangleright}(G)$ *forms an* **AGSC-Galois pair** [*cf. Definition* 4.1]. *We shall write*

$$\widetilde{F}(G) \ \stackrel{\text{def}}{=} \ \widetilde{F}(G \curvearrowright \widetilde{\mathcal{O}}^{\triangleright}(G)), \quad F(G) \ \stackrel{\text{def}}{=} \ F(G \curvearrowright \widetilde{\mathcal{O}}^{\triangleright}(G))$$

*for the fields obtained by applying Theorem* 4.4 *to the AGSC-Galois pair* $(G \curvearrowright \widetilde{\mathcal{O}}^{\triangleright}(G))$. *Then the following hold:*

(i) *Let* $D \in \widetilde{\mathcal{V}}(G)$. *Then the natural inclusion* $D \hookrightarrow G$ *determines a* **commutative diagram of fields**

$$\begin{array}{ccc} F(G) & \longrightarrow & \widetilde{F}(G) \\ \downarrow & & \downarrow \\ k(D) & \longrightarrow & \overline{k}(D) \end{array}$$

— *where the right-hand vertical arrow is* ***D*-equivariant**.

(ii) *Let*

$$(F,\ \widetilde{F},\ \alpha\colon Q_F \xrightarrow{\sim} G)$$

*be an* **AGSC-envelope** *for* $G$ [*cf. Definition* 3.2]. *Then the isomorphism* $\alpha$, *together with the field structures of the various fields involved, determines a* **commutative diagram**

**of fields**

$$\begin{array}{ccc} F & \longrightarrow & \widetilde{F} \\ \wr\big\downarrow & & \wr\big\downarrow \\ F(G) & \longrightarrow & \widetilde{F}(G) \end{array}$$

*— where the horizontal arrows are the natural inclusions, the vertical arrows are* **isomorphisms***, and the right-hand vertical arrow is* **compatible** *with the natural actions of $Q_F$ and $G$ relative to $\alpha$.*

PROOF. — These assertions follow from the various definitions involved. □

**REMARK 5.11.1.** — One verifies immediately from Remark 4.4.1, together with our construction, that we have a natural identification

$$\widetilde{F}(G)_{\times} \;=\; (\widetilde{\mathcal{O}}^{\triangleright}(G)^{\mathrm{gp}})^{\circledast}.$$

In particular, we have natural inclusions

$$\widetilde{F}(G)^{\times} \;\subseteq\; \varinjlim_{H} \mathcal{H}^{\times}(H) \;\subseteq\; \varinjlim_{H} H^1(H, \Lambda(H))$$

— where the injective limits are taken over the open subgroups $H \subseteq G$ of $G$ [cf. Proposition 5.2, (2)].

**REMARK 5.11.2.** — Recall that, in Proposition 5.10, (1), we *have reconstructed* an algebraically closed field $\overline{F}(D)$ corresponding to "$\overline{F}$". On the other hand, this algebraically closed field $\overline{F}(D)$ essentially *depends* on the choice of $D$. As a consequence of this dependence, one cannot *define directly* [i.e., without the discussions given in Proposition 5.10, (2), (3)] an *action*, of the desired type, of $G$ on a suitable subfield of this field $\overline{F}(D)$, i.e., the subfield of $\overline{F}(D)$ corresponding to "$\widetilde{F}$".

**REMARK 5.11.3.** — Note that, in the establishment of our global reconstruction result, the Neukirch-Uchida theorem [i.e., Theorem 3.3] plays a crucial role; in particular, the [proof/establishment of the] global reconstruction result does *not give* an alternative proof of the Neukirch-Uchida theorem.

**REMARK 5.11.4.** — We thus conclude from the global reconstruction result obtained in the present paper that every profinite group *of NF-type* admits a *ring-theoretic basepoint* [i.e., a *"ring-theoretic interpretation"* or a *"ring-theoretic label"*] *group-theoretically* constructed from the given profinite group.

**THEOREM 5.12.** — *Let $(G \curvearrowright M)$ be an* **AGSC-Galois pair** [*cf. Definition* 4.1]. *Recall that we have an* **injections**

$$M^G \;\hookrightarrow\; M^{\mathrm{gp}} \;\hookrightarrow\; \varinjlim_{H}\; H^1(H, \Lambda(M))$$

— *where the injective limit is taken over the open subgroups* $H \subseteq G$ *of* $G$ [*cf. Theorem* 4.4, (2)]. *Moreover, let us also recall that it follows from our construction that*

$$\mathcal{O}^{\triangleright}(G) \;\subseteq\; \widetilde{F}(G)^{\times} \;\subseteq\; \varinjlim_{H} H^1(H, \Lambda(H))$$

— *where the injective limit is taken over the open subgroups* $H \subseteq G$ *of* $G$ [*cf. Remark* 5.11.1]. *Then there exists a* **uniquely determined $G$-equivariant isomorphism**

$$\Lambda(M) \xrightarrow{\sim} \Lambda(G)$$

*such that the induced isomorphism* [*cf. also Theorem* 3.8, (i)]

$$\varinjlim_{H} H^1(H, \Lambda(M)) \xrightarrow{\sim} \varinjlim_{H} H^1(H, \Lambda(H))$$

— *where the injective limits are taken over the open subgroups* $H \subseteq G$ *of* $G$ — *maps* $M^G$ *bijectively onto* $\mathcal{O}^{\triangleright}(G)$. *Moreover, this induced isomorphism*

$$\varinjlim_{H} H^1(H, \Lambda(M)) \xrightarrow{\sim} \varinjlim_{H} H^1(H, \Lambda(H))$$

*also maps* $M^{\mathrm{gp}}$ *bijectively onto* $\widetilde{F}(G)^{\times}$. *We shall refer to this uniquely determined isomorphism* $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$ *as the* **cyclotomic synchronization isomorphism** *for* $(G \curvearrowright M)$.

PROOF. — The existence of such an isomorphism is immediate. The uniqueness of such an isomorphism follows immediately from the following elementary observation: Let $a \in \widehat{\mathbb{Z}}^{\times}$. Then it holds that the automorphism of $\widehat{\mathbb{Z}}$ obtained by multiplication by $a$ bijectively preserves the submonoid $\mathbb{N} \subseteq (\mathbb{Z} \subseteq)\, \widehat{\mathbb{Z}}$ if and only if $a = 1$. The final assertion follows immediately from the [existence and the] *uniqueness* of such an isomorphism. This completes the proof of Theorem 5.12. □

Finally, we prove a *compatibility* between the *functorial "group-theoretic" algorithm* obtained in the present paper and the *functorial "group-theoretic" algorithm* obtained in [9], Theorem 1.9:

**THEOREM 5.13.** — *Let* $\Pi$ *be a profinite group which is isomorphic to the étale fundamental group of a hyperbolic orbicurve of strictly Belyi type over an NF* [*cf.* [8], *Definition* 3.5]. *Write* $\Pi \twoheadrightarrow Q$ *for the* **arithmetic quotient** *of* $\Pi$, *i.e., the quotient of* $\Pi$ *by the* [*uniquely determined* — *cf.* [7], *Theorem* 2.6, (vi)] *maximal topologically finitely generated normal closed subgroup of* $\Pi$. *Thus,* $Q$ *is a profinite group* **of NF-type** [*cf.* [7], *Theorem* 2.6, (vi)]. *Write*

$$\Pi \curvearrowright \overline{F}(\Pi)$$

*for the algebraically closed field equipped with an action of* $\Pi$ *obtained by applying the* **functorial "group-theoretic" algorithm** *given in* [9], *Theorem* 1.9 *to* $\Pi$ [*i.e., the field* "$\overline{k}^{\times}_{\mathrm{NF}} \cup \{0\}$" *of* [9], *Theorem* 1.9, (e)]. *Thus, by the construction of* $\overline{F}(\Pi)$, *we have an inclusion*

$$\overline{F}(\Pi)^{\times} \hookrightarrow \varinjlim_{V} H^1(\Pi_V, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V))$$

— *where we refer to the notation in the statement of* [9]*, Theorem* 1.9*, concerning the notation* "$\varinjlim_V H^1(\Pi_V, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V))$"*. Then the natural homomorphism*

$$\varinjlim_{H} H^1(H, \Lambda(H)) \longrightarrow \varinjlim_{V} H^1(\Pi_V, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V))$$

— *where the first injective limit is taken over the open subgroups* $H \subseteq Q$ *of* $Q$ — *induced by the various natural surjections from the* "$\Pi_V$" *to the* "$H$" [*obtained by the fact that an* "$H$" *arises as the arithmetic quotient of a* "$\Pi_V$"]*, together with the isomorphisms of the* $\Lambda(H)$*'s with the* $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V)$*'s discussed in Lemma* 5.14 *below, determines* [*cf. Remark* 5.11.1] *a* **Π-equivariant isomorphism of fields**

$$\widetilde{F}(Q) \xrightarrow{\sim} \overline{F}(\Pi).$$

PROOF. — This follows immediately from the fact that, in the situation where the profinite groups involved are not just "some abstract profinite groups", but rather arises from familiar objects of scheme theory, the homomorphism

$$\varinjlim_{H} H^1(H, \Lambda(H)) \longrightarrow \varinjlim_{V} H^1(\Pi_V, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_V))$$

under consideration *coincides with the conventional homomorphism* between the injective limits of cohomology groups involved that arise from *conventional scheme theory.* This completes the proof of Theorem 5.13. □

**LEMMA 5.14.** — *Let* $\Pi$ *be a profinite group which is isomorphic to the étale fundamental group of a hyperbolic orbicurve over an NF. Write* $\Pi \twoheadrightarrow Q$ *for the* **arithmetic quotient** *of* $\Pi$*, i.e., the quotient of* $\Pi$ *by the* [*uniquely determined* — *cf.* [7]*, Theorem* 2.6*,* (vi)] *maximal topologically finitely generated normal closed subgroup of* $\Pi$*. Thus,* $Q$ *is a profinite group* **of NF-type** [*cf.* [7]*, Theorem* 2.6*,* (vi)]*. Let* $D \in \widetilde{\mathcal{V}}(Q)$*. Then the composite*

$$\Lambda(Q) \xrightarrow{\sim} \Lambda(D) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi \times_Q D) = \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi)$$

— *where we refer to the notation of* [9]*, Theorem* 1.9*,* (b)*, concerning the notation* "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(-)$"*; the first arrow is the* **local-global cyclotomic synchronization isomorphism** [*cf. Theorem* 3.8*,* (ii)]*; the second arrow is the natural isomorphism obtained in* [9]*, Corollary* 1.10*,* (ii)*,* (c) [*note that one verifies easily that the D-module* $\Lambda(D)$ coincides *with* "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$" *defined in* [9]*, Corollary* 1.10*,* (i)*,* (a)*, in the case where we take the* "$G$" *of loc. cit. to be* $D$]*; the* "$=$" *is the natural identification which arises from the definitions of* $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi \times_Q D)$ *and* $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi)$ — *is* **Π-equivariant** *and does* **not depend** *on the choice of* $D \in \widetilde{\mathcal{V}}(Q)$*.*

PROOF. — This follows immediately from the fact that, in the situation where the profinite groups involved are not just "some abstract profinite groups", but rather arises from familiar objects of scheme theory, the composite under consideration *coincides with the conventional identification* between the cyclotomes involved that arise from *conventional scheme theory.* This completes the proof of Lemma 5.14. □

## 6. Global Mono-anabelian Log-Frobenius Compatibility

In the present §6, we give an interpretation of the global reconstruction result obtained in the present paper in terms of a certain *compatibility* with the *NF-log-Frobenius functor* [cf. Theorem 6.10 below].

**Definition 6.1.** — Let $D$ be a profinite group *of MLF-type* [cf. Definition 1.1, Proposition 1.2, (i)]. Then we shall refer to a collection of data

$$(G,\ D \hookrightarrow G)$$

consisting of a profinite group $G$ of NF-type [cf. Definition 3.2] and an injection $D \hookrightarrow G$ of profinite groups as an *NF-holomorphic structure* on $D$.

**Definition 6.2.** — Let $D$ be a profinite group *of MLF-type* and $\mathfrak{hol} \stackrel{\mathrm{def}}{=} (G,\ D \hookrightarrow G)$ an *NF-holomorphic structure* on $D$. Then it follows immediately from [11], Theorem 12.1.9; [11], Lemma 12.1.10, that the injection $D \hookrightarrow G$ in $\mathfrak{hol}$ determines an *open injection* $D \hookrightarrow C_G(\mathrm{Im}(D))$ — where we write $\mathrm{Im}(D)$ for the image of the injection $D \hookrightarrow G$; moreover, it holds that $C_G(\mathrm{Im}(D)) \in \widetilde{\mathcal{V}}(G)$. Thus, we have an isomorphism of monoids

$$\overline{k}_\times(C_G(\mathrm{Im}(D))) \ \stackrel{\sim}{\longrightarrow}\ \overline{k}_\times(D)$$

which is *compatible* with the natural actions of $C_G(\mathrm{Im}(D))$ and $D$ relative to the open injection $D \hookrightarrow C_G(\mathrm{Im}(D))$. In particular, by means of the structure of field on $\overline{k}_\times(C_G(\mathrm{Im}(D)))$ constructed in Proposition 5.8, (3), we obtain a *structure of field* on $\overline{k}_\times(D)$. We shall write

$$\overline{k}(D, \mathfrak{hol})$$

for the resulting field.

**Remark 6.2.1.** — One verifies immediately from our construction that we have a natural identification

$$\overline{k}(D, \mathfrak{hol})_\times \ =\ \overline{k}_\times(D).$$

**Definition 6.3.** — Let $(D \curvearrowright M)$ be an *MLF-Galois* $\mathbb{TM}$*-pair of mono-analytic type* [cf. [9], Definition 3.1, (ii)]. Thus, $D$ is a profinite group *of MLF-type*. We shall refer to an NF-holomorphic structure on $D$ as an *NF-holomorphic structure* on $(D \curvearrowright M)$.

**Definition 6.4.** — Let $(D \curvearrowright M)$ be an MLF-Galois $\mathbb{TM}$-pair of mono-analytic type.

(i) We shall write

$$\boldsymbol{\mu}(M) \ \stackrel{\mathrm{def}}{=}\ (M^\times)_{\mathrm{tor}}$$

and

$$\Lambda(M) \ \stackrel{\mathrm{def}}{=}\ \varprojlim_n \boldsymbol{\mu}(M)[n]$$

— where the projective limit is taken over the positive integers $n$. Note that $D$ acts on $\boldsymbol{\mu}(M)$ and $\Lambda(M)$. We shall refer to the $D$-module $\Lambda(M)$ as the *cyclotome* associated to $(D \curvearrowright M)$. Note that one verifies immediately from our construction that the cyclotome $\Lambda(M)$ has a natural structure of *profinite* [cf. also the above definition of $\Lambda(M)$], hence also topological, $G$-module; moreover, we have a natural identification $\boldsymbol{\mu}(M)[n] = \Lambda(M)/n\Lambda(M)$. [Let us observe that one verifies easily that the $D$-modules $\boldsymbol{\mu}(M)$, $\Lambda(M)$ *coincide* with "$\boldsymbol{\mu}_{\mathbb{Q}/\mathbb{Z}}(M)$", "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M)$" defined in [9], Definition 3.1, (v), respectively.]

(ii) It follows from Lemma 1.3, (x), that the exact sequences of $D$-modules

$$1 \longrightarrow \Lambda(M)/n\Lambda(M) \longrightarrow M^{\mathrm{gp}} \xrightarrow{n} M^{\mathrm{gp}} \longrightarrow 1$$

— where $n$ ranges over the positive integers — determine an *injection*

$$(M^{\mathrm{gp}})^D \hookrightarrow H^1(D, \Lambda(M)).$$

(iii) By [9], Remark 3.2.1, we have a *functorial "group-theoretic" algorithm* for constructing, from $(D \curvearrowright M)$, a $D$-equivariant isomorphism

$$\Lambda(M) \xrightarrow{\sim} \Lambda(D)$$

[note that one verifies easily that the $D$-module $\Lambda(D)$ *coincides* with "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$" defined in [9], Corollary 1.10, (i), (a), in the case where we take the "$G$" of *loc. cit.* to be $D$] such that the induced isomorphism

$$H^1(D, \Lambda(M)) \xrightarrow{\sim} H^1(D, \Lambda(D))$$

determines, relative to the injection $\mathrm{Kmm}(D)$ and the injection of (ii), an *isomorphism* of modules

$$(M^{\mathrm{gp}})^D \xrightarrow{\sim} k^{\times}(D).$$

(iv) By applying the discussion of (iii) to the various open subgroups of $D$, we obtain an *isomorphism* of modules

$$M^{\mathrm{gp}} \xrightarrow{\sim} \overline{k}^{\times}(D),$$

hence also an *isomorphism* of monoids

$$(M^{\mathrm{gp}})^{\circledast} \xrightarrow{\sim} \overline{k}_{\times}(D).$$

Let us observe that it follows immediately from our construction that these isomorphisms of monoids are *$D$-equivariant*.

(v) Let $\mathfrak{hol}$ be an NF-holomorphic structure on $(D \curvearrowright M)$. Then the $D$-equivariant isomorphism of monoids of (iv), together with the field structure of $\overline{k}(D, \mathfrak{hol})$ obtained in Definition 6.2 [cf. also Remark 6.2.1], determines a *structure of field* on $(M^{\mathrm{gp}})^{\circledast}$. We shall write

$$\overline{k}(D \curvearrowright M, \mathfrak{hol})$$

for the resulting field.

**REMARK 6.4.1.** — One verifies immediately from our construction that we have a natural identification

$$\overline{k}(D \curvearrowright M, \mathfrak{hol})_{\times} = (M^{\mathrm{gp}})^{\circledast}.$$

**DEFINITION 6.5.** — Let $(D \curvearrowright M)$ be an MLF-Galois $\mathbb{TM}$-pair of mono-analytic type and $\mathfrak{hol}$ an NF-holomorphic structure on $(D \curvearrowright M)$.

(i) Write

$$(M^{\mathrm{gp}})^{\circledast}_{+}$$

for the module [whose underlying set is $(M^{\mathrm{gp}})^{\circledast}$] obtained by forming the underlying *additive* module of the field $\overline{k}(D \curvearrowright M, \mathfrak{hol})$ [cf. Remark 6.4.1]. Then the $p(D)$-adic logarithm on $\overline{k}(D \curvearrowright M, \mathfrak{hol})$ determines a *$D$-equivariant isomorphism* of modules

$$(M^{\times})^{\mathrm{pf}} \xrightarrow{\sim} (M^{\mathrm{gp}})^{\circledast}_{+}.$$

Thus, the field structure on $(M^{\mathrm{gp}})^{\circledast}_{+}$ [i.e., the field structure of $\overline{k}(D \curvearrowright M, \mathfrak{hol})$] determines a *structure of field* on $(M^{\times})^{\mathrm{pf}}$. We shall write

$$\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$$

for the resulting field.

(ii) We shall write

$$\mathcal{O}_{\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})} \subseteq \mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$$

for the ring of integers of $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$ and

$$\mathfrak{log}(D \curvearrowright M, \mathfrak{hol}) \stackrel{\mathrm{def}}{=} (\mathcal{O}_{\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})})^{\rhd}.$$

(iii) One verifies immediately from our construction that the action of $D$ on $M$ determines an action of $D$ on $\mathfrak{log}(D \curvearrowright M, \mathfrak{hol})$; moreover, the collection of data

$$(D \curvearrowright \mathfrak{log}(D \curvearrowright M, \mathfrak{hol}))$$

consisting of the profinite group $D$ and the topological [cf. Remark 6.5.2 below] $D$-monoid $\mathfrak{log}(D \curvearrowright M, \mathfrak{hol})$ forms an *MLF-Galois $\mathbb{TM}$-pair of mono-analytic type.*

**REMARK 6.5.1.** — One verifies immediately from our construction that if we write

$$\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})_{+}$$

for the underlying additive module of the field $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$, then we have a natural identification

$$\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})_{+} = (M^{\times})^{\mathrm{pf}}.$$

**REMARK 6.5.2.** — One verifies immediately that the field structure of $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$ naturally determines a *topology* on $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$, i.e., the "*$p$-adic topology*" of $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$. By means of this topology, one may regard $\mathfrak{log}(D \curvearrowright M, \mathfrak{hol})$ as a *topological monoid.*

**Definition 6.6.**

(i) We shall say that a collection of data

$$((G \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D \in \widetilde{\mathcal{V}}(G)},\ \{\rho_D \colon M \hookrightarrow M_D\}_{D \in \widetilde{\mathcal{V}}(G)})$$

consisting of an NF-Galois pair $(G \curvearrowright M)$ [cf. Definition 4.1], an MLF-Galois $\mathbb{TM}$-pair $(D \curvearrowright M_D)$ of mono-analytic type for each $D \in \widetilde{\mathcal{V}}(G)$, and a $D$-equivariant injection $\rho_D \colon M \hookrightarrow M_D$ of monoids for each $D \in \widetilde{\mathcal{V}}(G)$ is an *NF-Galois theater* if, for each $D \in \widetilde{\mathcal{V}}(G)$, the diagram

$$\begin{array}{ccc} M^{\mathrm{gp}} & \xrightarrow{\sim} & \widetilde{F}(G)^{\times} \\ \downarrow & & \downarrow \\ M_D^{\mathrm{gp}} & \xrightarrow{\sim} & \overline{k}^{\times}(D) \end{array}$$

— where the upper horizontal arrow is the isomorphism of the final assertion of Theorem 5.12; the lower horizontal arrow is the isomorphism of Definition 6.4, (iv); the left-hand vertical arrow is the homomorphism determined by $\rho_D$; the right-hand vertical arrow is the homomorphism induced by the right-hand vertical arrow of the diagram of Theorem 5.11, (i) — commutes.

(ii) Let

$$\mathcal{T}_1 \overset{\mathrm{def}}{=} ((G_1 \curvearrowright M_1),\ \{(D_1 \curvearrowright M_{D_1})\}_{D_1 \in \widetilde{\mathcal{V}}(G_1)},\ \{\rho_{D_1} \colon M_1 \hookrightarrow M_{D_1}\}_{D_1 \in \widetilde{\mathcal{V}}(G_1)}),$$

$$\mathcal{T}_2 \overset{\mathrm{def}}{=} ((G_2 \curvearrowright M_2),\ \{(D_2 \curvearrowright M_{D_2})\}_{D_2 \in \widetilde{\mathcal{V}}(G_2)},\ \{\rho_{D_2} \colon M_2 \hookrightarrow M_{D_2}\}_{D_2 \in \widetilde{\mathcal{V}}(G_2)})$$

be *NF-Galois theaters.* Then we shall say that a collection of data

$$(\alpha \colon G_1 \hookrightarrow G_2,\ \tau \colon \widetilde{\mathcal{V}}(G_1) \xrightarrow{\sim} \widetilde{\mathcal{V}}(G_2),\ \beta \colon M_1 \xrightarrow{\sim} M_2,\ \{\beta_{D_1} \colon M_{D_1} \xrightarrow{\sim} M_{\tau(D_1)}\}_{D_1 \in \widetilde{\mathcal{V}}(G_1)})$$

consisting of an open injection $\alpha \colon G_1 \hookrightarrow G_2$ of profinite groups, a bijection $\tau \colon \widetilde{\mathcal{V}}(G_1) \xrightarrow{\sim} \widetilde{\mathcal{V}}(G_2)$, an isomorphism $\beta \colon M_1 \xrightarrow{\sim} M_2$ of monoids, and, for each $D_1 \in \widetilde{\mathcal{V}}(G_1)$, an isomorphism $\beta_{D_1} \colon M_{D_1} \xrightarrow{\sim} M_{\tau(D_1)}$ of topological monoids is a *morphism of NF-Galois theaters*

$$\mathcal{T}_1 \longrightarrow \mathcal{T}_2$$

if the following three conditions are satisfied:

(a) The isomorphism $\beta \colon M_1 \xrightarrow{\sim} M_2$ of monoids is *compatible* with the actions of $G_1$, $G_2$ relative to the open injection $\alpha$.

(b) For each $D_1 \in \widetilde{\mathcal{V}}(G_1)$, it holds that $\tau(D_1) = C_{G_2}(\alpha(D_1))$, which thus [cf. [11], Theorem 12.1.9; [11], Lemma 12.1.10] implies that $\alpha$ determines an open injection $D_1 \hookrightarrow \tau(D_1)$.

(c) For each $D_1 \in \widetilde{\mathcal{V}}(G_1)$, the isomorphism $\beta_{D_1} \colon M_{D_1} \xrightarrow{\sim} M_{\tau(D_1)}$ of topological monoids is *compatible* with the actions of $D_1$, $\tau(D_1)$ relative to the open injection $D_1 \hookrightarrow \tau(D_1)$ of (b).

(iii) We shall write

$$\mathfrak{Th}^{\mathrm{NF}}$$

for the category of *NF-Galois theaters* and *morphisms of NF-Galois theaters* defined in (i) and (ii).

(iv) We shall write

$$\mathbb{NF}$$

for the category whose objects are *profinite groups of NF-type* and morphisms are *open injections of profinite groups*. Thus, the assignment

$$((G \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D \in \widetilde{\mathcal{V}}(G)},\ \{\rho_D \colon M \hookrightarrow M_D\}_{D \in \widetilde{\mathcal{V}}(G)}) \ \mapsto\ G$$

determines a *functor*

$$\mathfrak{Th}^{\mathrm{NF}} \ \longrightarrow\ \mathbb{NF}.$$

**REMARK 6.6.1.** — In the notation of Definition 6.6, (i), if we write $\mathfrak{hol}_D$ for the *NF-holomorphic structure* on $D$ obtained by forming the natural inclusion $D \hookrightarrow G$, then it follows immediately from the definitions of $\widetilde{F}(G \curvearrowright M)$, $\overline{k}(D \curvearrowright M_D, \mathfrak{hol}_D)$ [cf. also Theorem 5.11, (i)] that the morphism of modules induced by $\rho_D$

$$M^{\mathrm{gp}} \ \longrightarrow\ M_D^{\mathrm{gp}}$$

determines an *inclusion of fields*

$$\widetilde{F}(G \curvearrowright M) \ \hookrightarrow\ \overline{k}(D \curvearrowright M_D, \mathfrak{hol}_D)$$

[cf. Remark 4.4.1, Remark 6.4.1].

**REMARK 6.6.2.** — In the notation of Definition 6.6, (ii), it follows immediately from [11], Lemma 12.1.10, that the bijection $\tau$ is *completely determined* by the open injection $\alpha$ as follows: $\tau(D_1)$ is a *unique* element of $\widetilde{\mathcal{V}}(G_2)$ which contains $\alpha(D_1)$.

**PROPOSITION 6.7.** — *We shall write*

$$\mathfrak{An}[\mathbb{NF}]$$

*for the category defined as follows: An object of* $\mathfrak{An}[\mathbb{NF}]$ *is a collection of data of the form*

$$\mathfrak{An}(G) \ \stackrel{\mathrm{def}}{=}\ ((G \curvearrowright \widetilde{\mathcal{O}}^{\rhd}(G)),\ \{(D \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}(D)})\}_{D \in \widetilde{\mathcal{V}}(G)},\ \{\widetilde{\mathcal{O}}^{\rhd}(G) \hookrightarrow \mathcal{O}^{\rhd}_{\overline{k}(D)}\}_{D \in \widetilde{\mathcal{V}}(G)})$$

*— where we write* $\mathcal{O}^{\rhd}_{\overline{k}(D)}$ *for the monoid of nonzero integers of* $\overline{k}(D)$*, and* $\widetilde{\mathcal{O}}^{\rhd}(G) \hookrightarrow \mathcal{O}^{\rhd}_{\overline{k}(D)}$ *is the natural inclusion — for some object* $G$ *of* $\mathbb{NF}$*. A morphism of* $\mathfrak{An}[\mathbb{NF}]$ *is the morphism induced by a morphism of* $\mathbb{NF}$*. Then the natural functors*

$$\mathbb{NF} \ \longrightarrow\ \mathfrak{An}[\mathbb{NF}] \ \longrightarrow\ \mathfrak{Th}^{\mathrm{NF}} \ \longrightarrow\ \mathbb{NF}$$

*— where the first arrow is the functor obtained by assigning* $G \mapsto \mathfrak{An}(G)$*, the second arrow is the functor obtained by forgetting the way in which the object* $\mathfrak{An}(G)$ *arose from* $G$*, and the third arrow is the functor of Definition* 6.6, (iv) *— all of which are* **equivalences of categories***. Moreover, the composite* $\mathbb{NF} \to \mathbb{NF}$ *of these arrows is naturally isomorphic to the* **identity** *functor.*

PROOF. — This follows immediately from Theorem 5.12; Remark 6.6.2; [9], Proposition 3.2, (iv), together with the various definitions involved. □

**DEFINITION 6.8.** — Let

$$\mathcal{T} \overset{\text{def}}{=} ((G \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D\in\widetilde{\mathcal{V}}(G)},\ \{\rho_D\colon M \hookrightarrow M_D\}_{D\in\widetilde{\mathcal{V}}(G)})$$

be an NF-Galois theater and $D \in \widetilde{\mathcal{V}}(G)$. Write $\mathfrak{hol}_D$ for the NF-holomorphic structure on $(D \curvearrowright M_D)$ obtained by forming the natural inclusion $D \hookrightarrow G$. Thus, by Definition 6.5, (iii), we have an MLF-Galois $\mathbb{TM}$-pair of mono-analytic type

$$(D \curvearrowright \mathfrak{log}(D \curvearrowright M_D, \mathfrak{hol}_D)).$$

Moreover, by the definition of the MLF-Galois $\mathbb{TM}$-pair $(D \curvearrowright \mathfrak{log}(D \curvearrowright M_D, \mathfrak{hol}_D))$ [i.e., the construction of $\mathfrak{log}(\overline{k})(D \curvearrowright M, \mathfrak{hol})$ via the $p(D)$-adic logarithm], we have a *natural $D$-equivariant isomorphism*

$$\iota_D\colon\ M_D \ \xrightarrow{\ \sim\ }\ \mathfrak{log}(D \curvearrowright M_D, \mathfrak{hol}_D).$$

Now one verifies immediately from the various definitions involved that the collection of data

$$\mathfrak{log}(\mathcal{T}) \overset{\text{def}}{=} ((G \curvearrowright M),\ \{(D \curvearrowright \mathfrak{log}(D \curvearrowright M_D, \mathfrak{hol}_D))\}_{D\in\widetilde{\mathcal{V}}(G)},\ \{\iota_D \circ \rho_D\}_{D\in\widetilde{\mathcal{V}}(G)})$$

forms an *NF-Galois theater*. Thus, we obtain a functor

$$\mathfrak{log}\colon\ \mathfrak{Th}^{\mathrm{NF}} \ \longrightarrow\ \mathfrak{Th}^{\mathrm{NF}}.$$

We shall refer to this functor $\mathfrak{log}$ as the *NF-log-Frobenius functor*.

**REMARK 6.8.1.** — One verifies immediately that the *NF-log-Frobenius functor* of Definition 6.8 is *naturally isomorphic to the identity functor*, hence, in particular, an *equivalence of category*.

**DEFINITION 6.9.** — Let $G$ be a profinite group *of NF-type*, i.e., an object of the category $\mathbb{NF}$, and $p$ a prime number.

(i) We shall write

$$\mathbb{NF}[G] \ \subseteq\ \mathbb{NF}$$

for the *full subcategory* of $\mathbb{NF}$ consisting of profinite groups which are isomorphic to $G$. [Thus, it follows from Theorem 3.3 that every morphism in this full subcategory is an *isomorphism*.] This full subcategory determines, in an evident fashion [cf. also the equivalences of categories of Proposition 6.7], *full subcategories*

$$\mathfrak{Th}^{\mathrm{NF}}[G] \ \subseteq\ \mathfrak{Th}^{\mathrm{NF}}, \quad \mathfrak{An}[\mathbb{NF}[G]] \ \subseteq\ \mathfrak{An}[\mathbb{NF}].$$

Then one verifies immediately that the *NF-log-Frobenius functor* $\mathfrak{log}\colon \mathfrak{Th}^{\mathrm{NF}} \to \mathfrak{Th}^{\mathrm{NF}}$ determines a *functor*

$$\mathfrak{log}\colon\ \mathfrak{Th}^{\mathrm{NF}}[G] \ \longrightarrow\ \mathfrak{Th}^{\mathrm{NF}}[G].$$

(ii) We shall write

$$\mathcal{N}_p \quad \text{(respectively, } \mathcal{N}_p^{\boxplus}\text{)}$$

for the category defined as follows: An object of the category $\mathcal{N}_p$ (respectively, $\mathcal{N}_p^{\boxplus}$) is a collection of data

$$((H \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D\in\widetilde{\mathcal{V}}(H)},\ \{\rho_D\colon M \hookrightarrow M_D\}_{D\in\widetilde{\mathcal{V}}(H)},\ \{(D \curvearrowright N_D)\}_{D\in\widetilde{\mathcal{V}}(H),\, p(D)=p})$$

— where

$$((H \curvearrowright M),\ \{(D \curvearrowright M_D)\}_{D \in \widetilde{\mathcal{V}}(H)},\ \{\rho_D \colon M \hookrightarrow M_D\}_{D \in \widetilde{\mathcal{V}}(H)})$$

is an object of $\mathfrak{Th}^{\mathrm{NF}}[G]$ [i.e., a certain NF-Galois theater], and, for each $D \in \widetilde{\mathcal{V}}(H)$ such that $p(D) = p$,

$$(D \curvearrowright N_D)$$

is an MLF-Galois $\mathbb{TS}$-pair (respectively, $\mathbb{TS}\boxplus$-pair) [necessarily of mono-analytic type] [cf. [9], Definition 3.1, (ii)]. A morphism in the category $\mathcal{N}_p$ (respectively, $\mathcal{N}_p^{\boxplus}$) is a morphism determined, in an evident fashion, by the notion of a morphism of NF-Galois theaters and the notion of a morphism of MLF-Galois $\mathbb{TS}$-pairs (respectively, $\mathbb{TS}\boxplus$-pairs) [cf. [9], Definition 3.1, (ii)]. Thus, we have *natural functors*

$$\mathcal{N}_p^{\boxplus} \longrightarrow \mathcal{N}_p \longrightarrow \mathbb{NF}[G].$$

(iii) Let $\nu$ be a vertex of the oriented graph "$\vec{\Gamma}^{\mathfrak{log}}_{\mathrm{non}}$" of [9], Definition 5.4, (iii). Then, by a similar procedure to the procedure of [9], Definition 5.4, (iv), one may define a *functor*

$$\lambda^{\boxplus}_{p,\nu} \colon \mathfrak{Th}^{\mathrm{NF}}[G] \longrightarrow \mathcal{N}_p^{\boxplus}.$$

(iv) Let $\epsilon$ be an edge of the oriented graph "$\vec{\Gamma}^{\ltimes}_{\mathrm{non}}$" of [9], Definition 5.4, (iii), running from a vertex $\nu_1$ to a vertex $\nu_2$. Then, by a similar procedure to the procedure of [9], Definition 5.4, (vii), one may define a *natural transformation*

$$\iota^{\boxplus}_{p,\epsilon} \colon \lambda^{\boxplus}_{p,\nu_1} \circ \Lambda_{\nu_1} \longrightarrow \lambda^{\boxplus}_{p,\nu_2}$$

— where, for each *pre-log* (respectively, *post-log*) vertex $\nu$ [cf. [9], Definition 5.4, (iii)] of the oriented graph "$\vec{\Gamma}^{\mathfrak{log}}_{\mathrm{non}}$" of [9], Definition 5.4, (iii), we take $\Lambda_\nu$ to be the *identity functor* on $\mathfrak{Th}^{\mathrm{NF}}[G]$ (respectively, *NF-log-Frobenius functor* $\mathfrak{log} \colon \mathfrak{Th}^{\mathrm{NF}}[G] \to \mathfrak{Th}^{\mathrm{NF}}[G]$ — cf. (i)).

Finally, we prove the following *global mono-anabelian log-Frobenius compatibility*:

**THEOREM 6.10.** — *Let $G$ be a profinite group* **of NF-type** [*cf. Definition* 3.2]. *Consider the diagram of categories $\mathcal{D}$* [*cf.* [9], *Definition* 3.5, (i)]

$$
\begin{array}{ccccccccc}
\dots & \xrightarrow{\mathfrak{log}} & \mathcal{X} & \xrightarrow{\mathfrak{log}} & \mathcal{X} & \xrightarrow{\mathfrak{log}} & \mathcal{X} & \xrightarrow{\mathfrak{log}} & \dots \\
\dots & & \mathrm{id}_{i-1} \searrow & & \downarrow \mathrm{id}_i & & \swarrow \mathrm{id}_{i+1} & & \dots \\
 & & & & \mathcal{X} & & & & \\
\dots & & \lambda^{\boxplus}_{p^\dagger} \swarrow \dots \swarrow & & \downarrow \dots \downarrow \lambda^{\boxplus}_p & & \searrow \dots \searrow \lambda^{\boxplus}_{p^\ddagger} & & \dots \\
\dots & & \mathcal{N}^{\boxplus}_{p^\dagger} & & \mathcal{N}^{\boxplus}_p & & \mathcal{N}^{\boxplus}_{p^\ddagger} & & \dots \\
\dots & & \downarrow & & \downarrow & & \downarrow & & \dots \\
\dots & & \mathcal{N}_{p^\dagger} & & \mathcal{N}_p & & \mathcal{N}_{p^\ddagger} & & \dots \\
\dots & & \searrow & & \downarrow & & \swarrow & & \dots \\
 & & & & \mathcal{E} & & & & \\
 & & & & \downarrow \kappa & & & & \\
 & & & & \mathfrak{An} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{E} & & & &
\end{array}
$$

— *where we write*

$$\mathcal{X} \stackrel{\mathrm{def}}{=} \mathfrak{Th}^{\mathrm{NF}}[G], \quad \mathcal{E} \stackrel{\mathrm{def}}{=} \mathbb{NF}[G], \quad \mathfrak{An} \stackrel{\mathrm{def}}{=} \mathfrak{An}[\mathbb{NF}[G]]$$

[*cf. Definition* 6.9, (i)]; *we write "$\mathcal{N}^{\boxplus}_p$", "$\mathcal{N}_p$" for the categories defined in Definition* 6.9, (ii); *the elements of $\mathbb{Z}$ correspond to vertices of the first row of $\mathcal{D}$; we write $\mathbb{Z}^{(\infty)} \stackrel{\mathrm{def}}{=} \mathbb{Z} \cup \{\infty\}$ for the ordered set obtained by appending to $\mathbb{Z}$ a formal symbol "$\infty$" — which we think of as corresponding to the unique vertex of the second row of $\mathcal{D}$ — such that $i < \infty$ for all $i \in \mathbb{Z}$; we write* $\mathrm{id}_i$ *for the identity functor at the vertex $i \in \mathbb{Z}$; for an element $n \in \{1, \dots, 7\}$, we write $\mathcal{D}_{\leq n}$ for the subdiagram of categories* [*cf.* [9], *Definition* 3.5, (i)] *of $\mathcal{D}$ determined by the first $n$* [*of the seven*] *rows of $\mathcal{D}$; the vertices of the third and fourth rows of $\mathcal{D}$ are indexed by the prime numbers $p$, $p^\dagger$, $p^\ddagger \dots$; the arrows from the second row to the category $\mathcal{N}^{\boxplus}_p$ in the third row are given by the collection of functors $\lambda^{\boxplus}_p \stackrel{\mathrm{def}}{=} \{\lambda^{\boxplus}_{p,\nu}\}_\nu$ of Definition* 6.9, (iii), *where $\nu$ ranges over the* **pre-log** *vertices of the oriented graph "$\vec{\Gamma}^{\mathfrak{log}}_{\mathrm{non}}$" of* [9], *Definition* 5.4, (iii) [*or, alternatively, over* **all** *the vertices of the oriented graph "$\vec{\Gamma}^{\mathfrak{log}}_{\mathrm{non}}$" of* [9], *Definition* 5.4, (iii), *subject to the proviso that we identify the functors associated to the* **space-link** *and* **post-log** *vertices*]; *the arrows from the third to fourth and from the fourth to fifth rows are the natural functors $\mathcal{N}^{\boxplus}_p \to \mathcal{N}_p \to \mathcal{E}$ of Definition* 6.9, (ii); *the arrows from the fifth to sixth and from the* *sixth to seventh rows are the* **natural equivalences of categories** $\mathcal{E} \to \mathfrak{An} \to \mathcal{E}$ *— the first of which we shall denote by* $\kappa$ *— of Proposition* 6.7 *restricted to "*$[G]$*"; we shall apply "*$[-]$*" to the names of arrows appearing in* $\mathcal{D}$ *to denote the* **path** [*cf. the discussion entitled "Combinatorics" in* [9], §0] *of length* 1 *associated to the arrow. Also, let us write*

$$\phi\colon\ \mathfrak{An} \longrightarrow \mathcal{X}$$

*for the equivalence of categories given by the "***forgetful functor***" of Proposition* 6.7 *restricted to "*$[G]$*";*

$$\pi\colon\ \mathcal{X} \longrightarrow \mathcal{E} \xrightarrow{\ \kappa\ } \mathfrak{An}$$

*for the quasi-inverse for* $\phi$ *given by the composite of the natural projection functor* $\mathcal{X} \to \mathcal{E}$ *with* $\kappa$*;*

$$\eta\colon\ \phi \circ \pi \xrightarrow{\ \sim\ } \mathrm{id}_{\mathcal{X}}$$

*for the isomorphism that exhibits* $\phi$, $\pi$ *as quasi-inverses to one another. Then the following hold:*

(i) *For* $n \in \{5, 6, 7\}$, $\mathcal{D}_{\leq n}$ *admits a natural structure of* **core** [*cf.* [9], *Definition* 3.5, (iii)] *on* $\mathcal{D}_{\leq n-1}$*. That is to say, loosely speaking,* $\mathcal{E}$, $\mathfrak{An}$ *"form cores" of the functors in* $\mathcal{D}$*.*

(ii) *The "***forgetful functor***"* $\phi$ *gives rise to a* **telecore structure** $\mathfrak{T}$ [*cf.* [9], *Definition* 3.5, (iv)] *on* $\mathcal{D}_{\leq 5}$ *— whose underlying diagram of categories we denote by* $\mathcal{D}_{\mathfrak{T}}$ *— by appending to* $\mathcal{D}_{\leq 6}$ **telecore edges** [*cf.* [9], *Definition* 3.5, (iv), (a)]

$$\begin{tikzcd}
 & & & \mathfrak{An} \arrow[dll, "\phi_{i-1}"'] \arrow[d, "\phi_i"] \arrow[drr, "\phi_{i+1}"] & & & \\
\cdots & & \mathcal{X} \arrow[r, "\mathfrak{log}"] & \mathcal{X} \arrow[r, "\mathfrak{log}"] & & \mathcal{X} \arrow[r, "\mathfrak{log}"] & \cdots \\
\cdots \arrow[urr, "\mathfrak{log}"] & & & & & & \cdots
\end{tikzcd}$$

$$\mathfrak{An} \xrightarrow{\ \phi_\infty\ } \mathcal{X}$$

*from the* **core** $\mathfrak{An}$ *to the various copies of* $\mathcal{X}$ *in* $\mathcal{D}_{\leq 2}$ *given by copies of* $\phi$ *— which we denote by* $\phi_i$ *— for* $i \in \mathbb{Z}^{(\infty)}$*. That is to say, loosely speaking,* $\phi$ *determines a telecore structure on* $\mathcal{D}_{\leq 5}$*. Finally, for each* $i \in \mathbb{Z}^{(\infty)}$*, let us write* $[\beta_i^0]$ *for the path on* $\vec{\Gamma}_{\mathcal{D}_{\mathfrak{T}}}$ *of length* 0 *at* $i$ *and* $[\beta_i^1]$ *for* **some** [*cf. the* **coricity** *of* (i)] *path on* $\vec{\Gamma}_{\mathcal{D}_{\mathfrak{T}}}$ *of length* $\in \{5, 6\}$ [*i.e., depending on whether or not* $i = \infty$] *that starts from* $i$*, descends via some path of length* $\in \{4, 5\}$ *to the core vertex "*$\mathfrak{An}$*", and returns to* $i$ *via the telecore edge* $\phi_i$*.*

*Then the collection of natural transformations*

$$\{\eta_{\infty, i},\ \eta_j,\ \eta_j^{-1}\}_{i \in \mathbb{Z},\, j \in \mathbb{Z}^{(\infty)}}$$

*— where we write*

$$\eta_{\infty, i}\colon\ \phi_\infty \xrightarrow{\ \sim\ } \mathrm{id}_i \circ \phi_i$$

*for the identity natural transformation and*

$$\eta_j\colon\ (\mathcal{D}_{\mathfrak{T}})_{[\beta_j^1]} \xrightarrow{\ \sim\ } (\mathcal{D}_{\mathfrak{T}})_{[\beta_j^0]}$$

[*cf.* [9], *Definition* 3.5, (i)] *for the isomorphism arising from* $\eta$ *— generate a* **contact structure** $\mathcal{H}$ [*cf.* [9], *Definition* 3.5, (iv)] *on the telecore* $\mathfrak{T}$*.*

(iii) *The natural transformations*

$$\iota^{\boxplus}_{p,\epsilon}:\ \lambda^{\boxplus}_{p,\nu_1} \circ \Lambda_{\nu_1} \longrightarrow \lambda^{\boxplus}_{p,\nu_2}$$

[*cf. Definition* 6.9, (iv)] — *where* $p$ *is a prime number;* $\epsilon$ *is an edge of the oriented graph* "$\vec{\Gamma}^{\ltimes}_{\mathrm{non}}$" *of* [9], *Definition* 5.4, (iii), *running from a vertex* $\nu_1$ *to a vertex* $\nu_2$*; if* $\nu_1$ *is a* **pre-log** *vertex, then we interpret the domain and codomain of* $\iota^{\boxplus}_{p,\epsilon}$ *as the arrows associated to the paths of length* 1 *from the second to third rows of* $\mathcal{D}$ *determined by* $p$ *and* $\nu_1$, $\nu_2$*; if* $\nu_1$ *is a* **post-log** *vertex, then we interpret the domain of* $\iota^{\boxplus}_{p,\epsilon}$ *as the arrow associated to the path of length* 3 *from the first to the third rows of* $\mathcal{D}$ *determined by* $p$, $\nu_1$, *and the condition that the initial length* 2 *portion of the path is a path of the form* $[\mathrm{id}_i] \circ [\mathfrak{log}]$ [*for* $i \in \mathbb{Z}$], *and we interpret the codomain of* $\iota^{\boxplus}_{p,\epsilon}$ *as the arrow associated to the path of length* 2 *from the first to the third rows of* $\mathcal{D}$ *determined by* $p$, $\nu_2$, *and the condition that the initial length* 1 *portion of the path is a path of the form* $[\mathrm{id}_{i+1}]$ [*for the* **same** $i \in \mathbb{Z}$] — *belong to a* **family of homotopies** [*cf.* [9], *Definition* 3.5, (ii)] *on* $\mathcal{D}_{\leq 3}$ *that determines on the portion of* $\mathcal{D}_{\leq 3}$ *indexed by* $p$ *a structure of* **observable** $\mathfrak{S}_{\mathfrak{log}}$ [*cf.* [9], *Definition* 3.5, (iii)] *on* $\mathcal{D}_{\leq 2}$. *Moreover, the family of homotopies that constitute* $\mathfrak{S}_{\mathfrak{log}}$ *is* **compatible** [*cf.* [9], *Definition* 3.5, (ii)] *with one another as well as with the families of homotopies that constitute the* **core** *and* **telecore** *structures of* (i), (ii).

(iv) *The diagram of categories* $\mathcal{D}_{\leq 2}$ *does* **not admit** *a structure of* **core** *on* $\mathcal{D}_{\leq 1}$ *which* [*i.e., whose constituent family of homotopies*] *is* **compatible** *with* [*the constituent family of homotopies of*] *the* **observable** $\mathfrak{S}_{\mathfrak{log}}$ *of* (iii). *Moreover, the* **telecore structure** $\mathfrak{T}$ *of* (ii), *the* **contact structure** $\mathcal{H}$ *of* (ii), *and the* **observable** $\mathfrak{S}_{\mathfrak{log}}$ *of* (iii) *are* **not simultaneously compatible**.

(v) *The unique vertex* $\infty$ *of the second row of* $\mathcal{D}$ *is a* **nexus** [*cf. the discussion entitled "Combinatorics" in* [9], §0] *of* $\vec{\Gamma}_{\mathcal{D}}$. *Moreover,* $\mathcal{D}$ *is* **totally** $\infty$**-rigid** [*cf.* [9], *Definition* 3.5, (vi)], *and the* **natural action** *of* $\mathbb{Z}$ *on the infinite linear oriented graph* $\vec{\Gamma}_{\mathcal{D}_{\leq 1}}$ **extends** *to an action of* $\mathbb{Z}$ *on* $\mathcal{D}$ *by* **nexus-classes of self-equivalences** *of* $\mathcal{D}$ [*cf.* [9], *Definition* 3.5, (vi)]. *Finally, the self-equivalences in these nexus-classes are* **compatible** *with the* **families of homotopies** *that constitute the* **cores** *and* **observable** *of* (i), (iii)*; these self-equivalences also extend naturally* [*cf. the technique of extension applied in* [9], *Definition* 3.5, (vi)] *to the diagram of categories* [*cf.* [9], *Definition* 3.5, (iv), (a)] *that constitutes the* **telecore** *of* (ii), *in a fashion that is* **compatible** *with both the* **family of homotopies** *that constitutes this telecore structure* [*cf.* [9], *Definition* 3.5, (iv), (b)] *and the contact structure* $\mathcal{H}$ *of* (ii).

PROOF. — This follows immediately from a similar argument to the argument applied in the proof of [9], Corollary 5.5. □

**REMARK 6.10.1.** — The "*general formal content*" of the remarks following [9], Corollaries 3.6, 3.7, applies to the situation discussed in Theorem 6.10, as well. We leave the routine details of translating these remarks into the language of the situation of Theorem 6.10 to the interested reader.

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