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A Note on Dormant Opers of Rank $p-1$ in Characteristic p

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Abstract. — In the present paper, we prove that the set of equivalence classes of dormant opers of rank $p-1$ over a projective smooth curve of genus ≥ 2 over an algebraically closed field of characteristic $p > 0$ whose Jacobian variety is ordinary is of cardinality one.

Contents

Introduction

In the present paper, we study *dormant opers of rank* $p-1$ over projective smooth curves of characteristic $p > 0$. In particular, we prove that the set of equivalence classes of dormant opers of rank $p-1$ over a suitable curve of characteristic $p > 0$ is *of cardinality one*.

The notion of *oper* was introduced in [1] [cf. also [2]]. Let k be an algebraically closed field and X a projective smooth curve of genus ≥ 2 over k. Let us recall that an *oper* over X/k is a suitable triple consisting of a locally free coherent $\mathcal{O}_X$-module, a connection on the module relative to X/k, and a filtration of the module. The study of *opers in positive characteristic* was initiated by, for instance, [4], [5], [9], [10]. Suppose that we are in the situation in which k is *of characteristic* $p > 0$. Then we shall say that a given oper is *dormant* if the p-curvature of the connection of the oper is zero. We refer to Definition 1.1 [cf. also Definition 1.2] concerning the precise definition of the notion of "dormant oper" discussed in the present paper. Note that a *dormant oper of rank two* is essentially the same as a *dormant indigenous bundle* studied in the *p-adic Teichmüller theory* [cf. [7]].

The main result of the present paper, which is a *generalization* of the first portion of [3], Theorem A [cf. Remark 2.1.1], is as follows [cf. Proposition 1.3, Theorem 2.1]:

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THEOREM A. — *Let k be an algebraically closed field of characteristic $p > 0$ and X a projective smooth curve of genus ≥ 2 over k. Then the following hold:*

(i) *There exists a* **dormant oper of rank $p-1$** *over X/k.*

(ii) *Suppose that the Jacobian variety of X is* **ordinary**. *Then the set of equivalence classes of* **dormant opers of rank $p-1$** *over X/k is* **of cardinality one**.

In [5], *Joshi* posed a conjecture concerning the number of equivalence classes of dormant opers of rank r over a projective smooth curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p > 0$ for $p > C(r,g) \stackrel{\text{def}}{=} r(r-1)(r-2)(g-1)$ [cf. [5], Conjecture 8.1]; moreover, *Wakabayashi* proved, in [10], this conjecture for a sufficiently general curve [cf. [10], Theorem 8.7.1]. Note that the triple $(r,g,p) \stackrel{\text{def}}{=} (p-1,g,p)$ [i.e., the triple in the case discussed in Theorem A] does *not satisfy* the condition "$p > C(r,g)$" unless $p \in \{2,3\}$.

1. CONSTRUCTION OF DORMANT OPERS OF RANK $p-1$

In the present §1, let p be a prime number, $g \geq 2$ an integer, k an algebraically closed field *of characteristic* p, and X a *projective smooth curve* over k [i.e., a scheme which is projective, smooth, geometrically connected, and of relative dimension one over k] of genus g. Write X^F for the projective smooth curve over k obtained by base-changing X via the absolute Frobenius morphism of k, $\Phi\colon X \to X^F$ for the relative Frobenius morphism over k, $\mathcal{I} \subseteq \mathcal{O}_{X\times_k X}$ for the ideal of $\mathcal{O}_{X\times_k X}$ which defines the diagonal morphism with respect to X/k, and $X_{(n)} \subseteq X \times_k X$ for the closed subscheme of $X \times_k X$ defined by the ideal $\mathcal{I}^{n+1} \subseteq \mathcal{O}_{X\times_k X}$ [where n is a nonnegative integer]. In particular, it follows that $\mathcal{I}/\mathcal{I}^2 = \omega_{X/k}$ (respectively, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \tau_{X/k}$), where we use the notation "ω" (respectively, "τ") to denote the cotangent (respectively, tangent) sheaf. Finally, write $d\colon \mathcal{O}_X \to \omega_{X/k}$ for the exterior differentiation operator.

Let us define the notion of *dormant oper*, as well as the notion of *equivalence of dormant opers*, discussed in the present paper as follows [cf., e.g., [5], §3; also [4], Definitions 3.1.1, 3.1.2 and (1)-(5) of pp. 51-52]:

DEFINITION 1.1. — Let r be a positive integer. Then we shall say that a collection of data

$$(\mathcal{E},\ \nabla_{\mathcal{E}},\ \{0\} = \mathcal{E}_r \subseteq \mathcal{E}_{r-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E})$$

consisting of a locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$, a connection $\nabla_{\mathcal{E}}$ on $\mathcal{E}$ relative to X/k, and a filtration $\{0\} = \mathcal{E}_r \subseteq \mathcal{E}_{r-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E}$ by $\mathcal{O}_X$-submodules of $\mathcal{E}$ is a *dormant oper* of rank r over X/k if the following five conditions are satisfied:

(1) For every $i \in \{1,\ldots,r\}$, the subquotient $\mathcal{E}_{i-1}/\mathcal{E}_i$ is an invertible sheaf on X.

(2) For every $i \in \{1,\ldots,r\}$, it holds that $\nabla_{\mathcal{E}}(\mathcal{E}_i) \subseteq \mathcal{E}_{i-1} \otimes_{\mathcal{O}_X} \omega_{X/k}$.

(3) For every $i \in \{1,\ldots,r-1\}$, the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{E}_i \stackrel{\nabla_{\mathcal{E}}}{\to} \mathcal{E}_{i-1} \otimes_{\mathcal{O}_X} \omega_{X/k} \twoheadrightarrow (\mathcal{E}_{i-1}/\mathcal{E}_i) \otimes_{\mathcal{O}_X} \omega_{X/k}$$

[cf. (2)] determines an isomorphism $\mathcal{E}_i/\mathcal{E}_{i+1} \xrightarrow{\sim} (\mathcal{E}_{i-1}/\mathcal{E}_i) \otimes_{\mathcal{O}_X} \omega_{X/k}$ of invertible sheaves on X [cf. (1)].

(4) The p-curvature of $\nabla_{\mathcal{E}}$ is zero.

(5) There exists an isomorphism $(\det \mathcal{E}, \det \nabla_{\mathcal{E}}) \xrightarrow{\sim} (\mathcal{O}_X, d)$.

DEFINITION 1.2. — Let r be a positive integer and

$$\mathfrak{E} \stackrel{\text{def}}{=} (\mathcal{E},\, \nabla_{\mathcal{E}},\, \{0\} = \mathcal{E}_r \subseteq \mathcal{E}_{r-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E}),$$

$$\mathfrak{F} \stackrel{\text{def}}{=} (\mathcal{F},\, \nabla_{\mathcal{F}},\, \{0\} = \mathcal{F}_r \subseteq \mathcal{F}_{r-1} \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0 = \mathcal{F})$$

dormant opers of rank r over X/k. Then we shall say that $\mathfrak{E}$ is *equivalent* to $\mathfrak{F}$ if the following condition is satisfied: There exists a triple $(\mathcal{L}, \nabla_{\mathcal{L}}, \phi)$ consisting of an invertible sheaf $\mathcal{L}$ on X, a connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$ relative to X/k, and a horizontal isomorphism ϕ of $(\mathcal{E}, \nabla_{\mathcal{E}})$ with $(\mathcal{F}, \nabla_{\mathcal{F}}) \otimes_{\mathcal{O}_X} (\mathcal{L}, \nabla_{\mathcal{L}})$ such that ϕ maps, for each $i \in \{0, \ldots, r\}$, the subsheaf $\mathcal{E}_i \subseteq \mathcal{E}$ isomorphically onto the subsheaf $\mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{L} \subseteq \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$.

REMARK 1.2.1. — Note that the notion of the "*equivalence class of a dormant oper of rank* r" of the present paper *coincides* with the notion of "*isomorphism class of a dormant* $\mathrm{PGL}(r)$*-oper*" in the terminology given in [5], §3.

REMARK 1.2.2. — One verifies immediately from the various definitions involved that the notion of *dormant oper of rank two* is essentially the same as the notion of *dormant indigenous bundle* studied in [7].

In the remainder of the present §1, let us construct a *dormant oper of rank* $p-1$ over X/k [cf. Proposition 1.3 below]. We shall write

$$\mathcal{B}^\circ \stackrel{\text{def}}{=} \mathrm{Coker}(\mathcal{O}_{X^F} \ \to \ \Phi_* \mathcal{O}_X)$$

for the $\mathcal{O}_{X^F}$-module obtained by forming the cokernel of the natural homomorphism $\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X$. Since [one verifies immediately that] the homomorphism $\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X$ admits a natural splitting after pulling back via Φ — which thus determines a natural isomorphism of $\mathcal{O}_X$-modules

$$\Phi^* \Phi_* \mathcal{O}_X \ \xrightarrow{\ \sim\ } \ \mathcal{O}_X \oplus \Phi^* \mathcal{B}^\circ$$

— and Φ is *finite flat of degree* p, it follows that $\mathcal{B}^\circ$, hence also $\Phi^* \mathcal{B}^\circ$, is *locally free of rank* $p-1$.

Next, let us observe that one verifies immediately that the natural morphism over k

$$X \times_{X^F} X \ \longrightarrow \ X \times_k X$$

determines an isomorphism over k

$$X \times_{X^F} X \ \xrightarrow{\ \sim\ } \ X_{(p-1)}.$$

Moreover, this isomorphism $X \times_{X^F} X \xrightarrow{\sim} X_{(p-1)}$, together with the cartesian diagram

$$\begin{array}{ccc} X \times_{X^F} X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \Phi} \\ X & \xrightarrow[\Phi]{} & X^F, \end{array}$$

determines *isomorphisms* of $\mathcal{O}_X$-modules

$$\Phi^*\Phi_*\mathcal{O}_X \xrightarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X\times_{X^F}X} \xleftarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X_{(p-1)}},$$

which are *compatible* with the respective natural surjections onto $\mathcal{O}_X$ [arising from the diagonal morphism with respect to X/X^F] from each of these three modules. In particular, by forming the kernels of the respective natural surjections onto $\mathcal{O}_X$, we obtain *isomorphisms* of $\mathcal{O}_X$-modules

$$\Phi^*\mathcal{B}^\circ \xrightarrow{\sim} \mathrm{Ker}(\mathrm{pr}_{1*}\mathcal{O}_{X\times_{X^F}X} \twoheadrightarrow \mathcal{O}_X) \xleftarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p).$$

Thus, by considering the filtration of $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p)$

$$\{0\} = \mathrm{pr}_{1*}(\mathcal{I}^p/\mathcal{I}^p) \subseteq \mathrm{pr}_{1*}(\mathcal{I}^{p-1}/\mathcal{I}^p) \subseteq \cdots \subseteq \mathrm{pr}_{1*}(\mathcal{I}^2/\mathcal{I}^p) \subseteq \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p),$$

we conclude that $\Phi^*\det\mathcal{B}^\circ = \det(\Phi^*\mathcal{B}^\circ)$ is *isomorphic* to $\omega_{X/k}^{\frac{p(p-1)}{2}}$, which thus implies that $\deg\mathcal{B}^\circ = (p-1)(g-1)$. In particular, one verifies immediately that there exists an invertible sheaf $\mathcal{L}^\circ$ on X^F such that $\det(\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ)$ $(= (\mathcal{L}^\circ)^{\otimes p-1} \otimes_{\mathcal{O}_{X^F}} \det\mathcal{B}^\circ)$ is *isomorphic* to $\mathcal{O}_{X^F}$ [cf. Remark 1.3.1 below].

We shall write

$$\mathcal{E}^\circ \stackrel{\mathrm{def}}{=} \Phi^*(\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ).$$

Then since the closed immersion $X_{(1)} \hookrightarrow X \times_k X$ *factors* through the closed subscheme $X \times_{X^F} X \subseteq X \times_k X$ [as verified above], it follows that the $\mathcal{O}_X$-module $\mathcal{E}^\circ$ admits a *natural connection* relative to X/k. We shall write

$$\nabla_{\mathcal{E}^\circ}$$

for the connection on $\mathcal{E}^\circ$. Moreover, the above filtration of $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p)$ [together with the above isomorphism $\Phi^*\mathcal{B}^\circ \xrightarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^p)$] determines a filtration of $\mathcal{E}^\circ$

$$\{0\} = \mathcal{E}^\circ_{p-1} \subseteq \mathcal{E}^\circ_{p-2} \subseteq \cdots \subseteq \mathcal{E}^\circ_1 \subseteq \mathcal{E}^\circ_0 = \mathcal{E}^\circ.$$

[Thus, one verifies immediately that $\mathcal{E}^\circ/\mathcal{E}^\circ_1 \cong \Phi^*\mathcal{L}^\circ \otimes_{\mathcal{O}_X} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^2) = \Phi^*\mathcal{L}^\circ \otimes_{\mathcal{O}_X} \omega_{X/k}$.]

PROPOSITION 1.3. — *The collection of data*

$$\mathfrak{E}^\circ \stackrel{\mathrm{def}}{=} (\mathcal{E}^\circ, \nabla_{\mathcal{E}^\circ}, \{0\} = \mathcal{E}^\circ_{p-1} \subseteq \mathcal{E}^\circ_{p-2} \subseteq \cdots \subseteq \mathcal{E}^\circ_1 \subseteq \mathcal{E}^\circ_0 = \mathcal{E}^\circ)$$

forms a **dormant oper of rank $p-1$** *over X/k.*

PROOF. — The assertion that $\mathfrak{E}^\circ$ satisfies condition (1) of Definition 1.1 follows from the definition of the $\mathcal{O}_X$-submodules $\mathcal{E}^\circ_i \subseteq \mathcal{E}^\circ$. The assertion that $\mathfrak{E}^\circ$ satisfies conditions (2) and (3) of Definition 1.1 follows immediately from a local explicit calculation [cf. also [4], Theorem 3.1.6]. The assertion that $\mathfrak{E}^\circ$ satisfies condition (4) of Definition 1.1 follows from the definition of the connection $\nabla_{\mathcal{E}^\circ}$. The assertion that $\mathfrak{E}^\circ$ satisfies condition (5) of Definition 1.1 follows from the fact that $\det(\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ)$ is *isomorphic* to $\mathcal{O}_{X^F}$, together with the definition of the connection $\nabla_{\mathcal{E}^\circ}$. This completes the proof of Proposition 1.3. $\square$

Remark 1.3.1. — Let us observe that the choice of "$\mathcal{L}^\circ$" in the discussion preceding Proposition 1.3 is *not unique*. More precisely, if we write $(\mathrm{Pic}\, X)[p-1] \subseteq \mathrm{Pic}\, X$ for the subgroup of $\mathrm{Pic}\, X$ obtained by forming the kernel of the endomorphism of $\mathrm{Pic}\, X$ given by multiplication by $p-1$, then one verifies immediately that the set consisting of isomorphism classes of "possible $\mathcal{L}^\circ$'s" forms a $(\mathrm{Pic}\, X)[p-1]$-*torsor*. On the other hand, one also verifies immediately from the various definitions involved that the adoption of another possible "$\mathcal{L}^\circ$" does *not affect* the equivalence class of the dormant oper of Proposition 1.3.

2. Uniqueness of Dormant Opers of Rank $p-1$

In the present §2, we maintain the notation of the preceding §1. In particular, we have a projective smooth curve X over k. In the present §2, we prove the following theorem:

Theorem 2.1. — *If the Jacobian variety of X is* **ordinary**, *then every* **dormant oper of rank $p-1$** *over X/k is* **equivalent** [*cf. Definition* 1.2] *to the dormant oper of Proposition* 1.3 [*cf. also Remark* 1.3.1].

Remark 2.1.1. — One verifies immediately from the various definitions involved, together with Remark 1.2.2, that if $p = 3$, then Theorem 2.1 implies and is implied by the first portion of [3], Theorem A [cf. the first paragraph of the proof of [3], Theorem 2.1]. Thus, Theorem 2.1 is a *generalization* of the first portion of [3], Theorem A.

To complete the verification of Theorem 2.1, let

$$\mathfrak{E} \stackrel{\mathrm{def}}{=} (\mathcal{E},\, \nabla_{\mathcal{E}},\, \{0\} = \mathcal{E}_{p-1} \subseteq \mathcal{E}_{p-2} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E})$$

be a *dormant oper of rank* $p-1$ over X/k. We shall write

$$\Xi^\circ \stackrel{\mathrm{def}}{=} \mathcal{E}^\circ/\mathcal{E}_1^\circ, \quad \Xi \stackrel{\mathrm{def}}{=} \mathcal{E}/\mathcal{E}_1.$$

Then it follows from the definition of a dormant oper that both $(\Xi^\circ)^{\otimes p-1}$ and $\Xi^{\otimes p-1}$ are *isomorphic* to $\tau_{X/k}^{\otimes \frac{(p-1)(p-2)}{2}}$, which thus implies that $\mathcal{H}om_{\mathcal{O}_X}(\Xi, \Xi^\circ)$ is *of degree zero*. In particular, one verifies immediately that there exists an invertible sheaf on X^F whose pull back via Φ is *isomorphic* to $\mathcal{H}om_{\mathcal{O}_X}(\Xi, \Xi^\circ)$. Thus, to complete the verification of Theorem 2.1, we may assume without loss of generality, by tensoring $\mathcal{E}$ with the above pull back via Φ, that

$$\Xi^\circ \cong \Xi.$$

By means of such an isomorphism, let us identify Ξ° with Ξ.

Next, let us observe that since the p-curvature of $\nabla_{\mathcal{E}}$ is *zero*, and $(\det \mathcal{E}, \det \nabla_{\mathcal{E}})$ is *isomorphic* to $(\mathcal{O}_X, d)$, it follows from a theorem of *Cartier* [cf., e.g., [6], Theorem 5.1] that if we write $\mathcal{F} \stackrel{\mathrm{def}}{=} \mathcal{E}^{\nabla_{\mathcal{E}}}$ for the $\mathcal{O}_{X^F}$-module of horizontal sections of $\mathcal{E}$ with respect to $\nabla_{\mathcal{E}}$, then

(a) $\mathcal{E}$ is *naturally isomorphic* to $\Phi^*\mathcal{F}$,

(b) the connection $\nabla_{\mathcal{E}}$ *arises* from the isomorphism of (a) [together with the factorization $X_{(1)} \hookrightarrow X \times_{X^F} X \hookrightarrow X \times_k X$ of the closed immersion $X_{(1)} \hookrightarrow X \times_k X$ discussed in §1], and

(c) $\det \mathcal{F}$ is *isomorphic* to $\mathcal{O}_{X^F}$.

Thus, the isomorphism $\Phi^*\mathcal{F} \xrightarrow{\sim} \mathcal{E}$ of (a) and the surjection $\mathcal{E} \twoheadrightarrow \Xi = \Xi^\circ$ determine a surjection of $\mathcal{O}_X$-modules

$$\Phi^*\mathcal{F} \twoheadrightarrow \Xi^\circ,$$

hence also a homomorphism of $\mathcal{O}_{X^F}$-modules

$$\mathcal{F} \longrightarrow \Phi_*\Xi^\circ.$$

LEMMA 2.2. — *The above homomorphism $\mathcal{F} \to \Phi_*\Xi^\circ$ is a* **locally split injection**.

PROOF. — Let us first observe that it is immediate that, to verify Lemma 2.2, it suffices to verify that the homomorphism $\gamma\colon \mathcal{E} \xleftarrow{\sim} \Phi^*\mathcal{F} \to \mathcal{G} \stackrel{\text{def}}{=} \Phi^*\Phi_*\Xi^\circ$ of $\mathcal{O}_X$-modules obtained by pulling back, via Φ, the homomorphism under consideration is a *split injection*. Now one verifies immediately that the composite of γ and the natural homomorphism $\mathcal{G} = \Phi^*\Phi_*\Xi^\circ \to \Xi^\circ$ *coincides* with the natural surjection $\mathcal{E} \twoheadrightarrow \Xi = \Xi^\circ$.

Write $\nabla_{\mathcal{G}}$ for the connection on $\mathcal{G}$ determined by the factorization $X_{(1)} \hookrightarrow X \times_{X^F} X \hookrightarrow X \times_k X$ of the closed immersion $X_{(1)} \hookrightarrow X \times_k X$ discussed in §1. Thus, it follows immediately from the definitions of $\nabla_{\mathcal{E}}$ and $\nabla_{\mathcal{G}}$ that γ is *horizontal* with respect to $\nabla_{\mathcal{E}}$ and $\nabla_{\mathcal{G}}$. Moreover, for $i \in \{1, \ldots, p\}$, let us define submodules $\mathcal{G}_i \subseteq \mathcal{G}$ *inductively* as follows: We shall write $\mathcal{G}_1 \stackrel{\text{def}}{=} \operatorname{Ker}(\mathcal{G} = \Phi^*\Phi_*\Xi^\circ \twoheadrightarrow \Xi^\circ)$. If $i \geq 2$, then we shall write $\mathcal{G}_i \stackrel{\text{def}}{=} \operatorname{Ker}\bigl(\mathcal{G}_{i-1} \hookrightarrow \mathcal{G} \xrightarrow{\nabla_{\mathcal{G}}} \mathcal{G} \otimes_{\mathcal{O}_X} \omega_{X/k} \twoheadrightarrow (\mathcal{G}/\mathcal{G}_{i-1}) \otimes_{\mathcal{O}_X} \omega_{X/k}\bigr)$. Then it follows that the submodule $\mathcal{G}_i \subseteq \mathcal{G}$ is an $\mathcal{O}_X$*-submodule*, and $\mathcal{G}_p = \{0\}$; moreover, one verifies immediately from a local explicit calculation [cf. also [4], Theorem 3.1.6] that the collection of data

$$(\mathcal{G},\ \nabla_{\mathcal{G}},\ \{0\} = \mathcal{G}_p \subseteq \mathcal{G}_{p-1} \subseteq \cdots \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_0 \stackrel{\text{def}}{=} \mathcal{G})$$

satisfies conditions (1), (2), (3), and (4) of Definition 1.1.

Now I *claim* that the following assertion holds:

> For each $i \in \{1, \ldots, p-1\}$, the composite $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E} \xrightarrow{\gamma} \mathcal{G}$ determines an *isomorphism* $\mathcal{E}_{i-1}/\mathcal{E}_i \xrightarrow{\sim} \mathcal{G}_{i-1}/\mathcal{G}_i$ of $\mathcal{O}_X$-modules.

Let us verify this *claim* by *induction on* i. If $i = 1$, then the desired assertion has already been verified [in the first paragraph of this proof]. Let $i \in \{2, \ldots, p-1\}$. Suppose that γ determines an *isomorphism* $\mathcal{E}_{i-2}/\mathcal{E}_{i-1} \xrightarrow{\sim} \mathcal{G}_{i-2}/\mathcal{G}_{i-1}$ of $\mathcal{O}_X$-modules, which thus implies that $\gamma(\mathcal{E}_{i-1}) \subseteq \mathcal{G}_{i-1}$. Thus, since γ is *horizontal*, the diagram

$$\begin{array}{ccccccc}
\mathcal{E}_{i-1} & \xrightarrow{\subseteq} & \mathcal{E} & \xrightarrow{\nabla_{\mathcal{E}}} & \mathcal{E} \otimes_{\mathcal{O}_X} \omega_{X/k} & \longrightarrow & (\mathcal{E}/\mathcal{E}_{i-1}) \otimes_{\mathcal{O}_X} \omega_{X/k} \\
\downarrow & & \gamma\downarrow & & \gamma\otimes\mathrm{id}\downarrow & & \downarrow \\
\mathcal{G}_{i-1} & \xrightarrow{\subseteq} & \mathcal{G} & \xrightarrow{\nabla_{\mathcal{G}}} & \mathcal{G} \otimes_{\mathcal{O}_X} \omega_{X/k} & \longrightarrow & (\mathcal{G}/\mathcal{G}_{i-1}) \otimes_{\mathcal{O}_X} \omega_{X/k}
\end{array}$$

commutes. In particular, it follows immediately from condition (3) of Definition 1.1, together with the induction hypothesis, that γ determines an *isomorphism* $\mathcal{E}_{i-1}/\mathcal{E}_i \xrightarrow{\sim} \mathcal{G}_{i-1}/\mathcal{G}_i$ of $\mathcal{O}_X$-modules, as desired. This completes the proof of the above *claim*.

By the above *claim*, one verifies easily that the composite $\mathcal{E} \xrightarrow{\gamma} \mathcal{G} \twoheadrightarrow \mathcal{G}/\mathcal{G}_{p-1}$ is an *isomorphism* of $\mathcal{O}_X$-modules. In particular, γ is *split injection*. This completes the proof of Lemma 2.2. $\square$

By Lemma 2.2, together with the fact that Ξ° is *isomorphic* to $\Phi^*\mathcal{L}^\circ \otimes_{\mathcal{O}_X} \omega_{X/k}$ [cf. the discussion preceding Proposition 1.3], we have an *exact sequence of locally free $\mathcal{O}_{X^F}$-modules*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

LEMMA 2.3. — *The above $\mathcal{O}_{X^F}$-module $\mathcal{Q}$ is* **isomorphic** *to $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/k}$.*

PROOF. — Let us first observe that $\mathcal{F}$ is *of rank* $p-1$, and $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k}$ is *of rank* p, which thus implies that $\mathcal{Q}$ is an *invertible sheaf* on X^F. Thus, by the exact sequence of the discussion preceding Lemma 2.3, together with (c) of the discussion preceding Lemma 2.2, it holds that

$$\mathcal{Q} \;\cong\; (\mathcal{L}^\circ)^{\otimes p} \otimes_{\mathcal{O}_{X^F}} \det \Phi_*\omega_{X/k}.$$

In particular, since [it follows from our choice of $\mathcal{L}^\circ$ — cf. the discussion preceding Proposition 1.3 — that] $(\mathcal{L}^\circ)^{\otimes p-1}$ is *isomorphic* to $\mathcal{H}om_{\mathcal{O}_{X^F}}(\det \mathcal{B}^\circ, \mathcal{O}_{X^F})$, we obtain that

$$\mathcal{Q} \;\cong\; \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{H}om_{\mathcal{O}_{X^F}}(\det \mathcal{B}^\circ, \det \Phi_*\omega_{X/k}).$$

Next, let us recall [cf., e.g., [6], Theorem 7.2] the *well-known* exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/k} \xrightarrow{c} \omega_{X^F/k} \longrightarrow 0$$

— where we write c for the *Cartier operator*. Thus, it follows from the definition of $\mathcal{B}^\circ$ that

$$\det \Phi_*\omega_{X/k} \;\cong\; \omega_{X^F/k} \otimes_{\mathcal{O}_{X^F}} \det \mathcal{B}^\circ.$$

This completes the proof of Lemma 2.3. $\square$

By Lemma 2.3, we have an *exact* sequence of locally free $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k} \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/k} \longrightarrow 0.$$

On the other hand, we have an *exact* sequence of locally free $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ \longrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k} \xrightarrow{c} \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/k} \longrightarrow 0$$

[cf. the *well-known* exact sequence that appears in the second paragraph of the proof of Lemma 2.3].

Let us complete the proof of Theorem 2.1:

PROOF OF THEOREM 2.1. — In the above two exact sequences, assume that $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ \not\subseteq \mathcal{F}$ [in $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k}$]. Then it is immediate that the composite

$$\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ \hookrightarrow \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k} \twoheadrightarrow (\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k})/\mathcal{F} \;\cong\; \mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/k}$$

is *nonzero*. In particular, since we have an isomorphism $\mathcal{B}^\circ \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{B}^\circ, \omega_{X^F/k})$ of $\mathcal{O}_{X^F}$-modules [cf. the discussion preceding [8], Théorème 4.1.1], we conclude that the

$\mathcal{O}_{X^F}$-module $\mathcal{B}^\circ$ admits a *nonzero global section*. Thus, it follows — in light of the exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \longrightarrow \mathcal{B}^\circ \longrightarrow 0$$

— that the Jacobian variety of X is *not ordinary* — in *contradiction* to our assumption. Thus, we conclude that $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ \subseteq \mathcal{F}$ [in $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k}$].

Next, let us observe that since $\mathcal{O}_{X^F} \cong \det(\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ) \cong \det \mathcal{F}$, the inclusion $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ \subseteq \mathcal{F}$ implies the equality $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \mathcal{B}^\circ = \mathcal{F}$ [in $\mathcal{L}^\circ \otimes_{\mathcal{O}_{X^F}} \Phi_*\omega_{X/k}$]. In particular, we have an isomorphism $\phi \colon \mathcal{E}^\circ \overset{\sim}{\to} \mathcal{E}$ [i.e., "Φ^*id"]. Moreover, it follows immediately from the definition of $\nabla_{\mathcal{E}^\circ}$, together with (b) of the discussion preceding Lemma 2.2, that the isomorphism ϕ is *horizontal* with respect to $\nabla_{\mathcal{E}^\circ}$, $\nabla_{\mathcal{E}}$.

Finally, we verify that $\phi(\mathcal{E}^\circ_{p-i}) = \mathcal{E}_{p-i}$ for each $i \in \{1, \ldots, p-1\}$ by *induction on* i. The equality $(\{0\} =)\ \phi(\mathcal{E}^\circ_{p-1}) = \mathcal{E}_{p-1}\ (= \{0\})$ is immediate. Let $i \in \{2, \ldots, p-1\}$. Suppose that $\phi(\mathcal{E}^\circ_{p-i+1}) = \mathcal{E}_{p-i+1}$ holds. Then, to verify that $\phi(\mathcal{E}^\circ_{p-i}) = \mathcal{E}_{p-i}$, it suffices to verify that the induced isomorphism $\mathcal{E}^\circ/\mathcal{E}^\circ_{p-i+1} \overset{\sim}{\to} \mathcal{E}/\mathcal{E}_{p-i+1}$ *maps* $\mathcal{E}^\circ_{p-i}/\mathcal{E}^\circ_{p-i+1}$ *isomorphically onto* $\mathcal{E}_{p-i}/\mathcal{E}_{p-i+1}$. On the other hand, since [one verifies easily from the definition of a dormant oper that]

$$\mathcal{E}^\circ_{j-1}/\mathcal{E}^\circ_j \cong \mathcal{E}_{j-1}/\mathcal{E}_j \cong \Xi^\circ \otimes_{\mathcal{O}_X} \omega_{X/k}^{\otimes j-1}$$

for each $j \in \{1, \ldots, p-1\}$, the desired assertion follows immediately from the *ampleness* of $\omega_{X/k}$. This completes the proof of Theorem 2.1. □

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