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A Note on Dormant Opers of Rank $p - 1$ in Characteristic $p$

By

Yuichiro HOSHI

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY, Kyoto, Japan
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Abstract. — In the present paper, we prove that the set of equivalence classes of dormant opers of rank $p - 1$ over a projective smooth curve of genus $\geq 2$ over an algebraically closed field of characteristic $p > 0$ whose Jacobian variety is ordinary is of cardinality one.

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Introduction

In the present paper, we study dormant opers of rank $p - 1$ over projective smooth curves of characteristic $p > 0$. In particular, we prove that the set of equivalence classes of dormant opers of rank $p - 1$ over a suitable curve of characteristic $p > 0$ is of cardinality one.

The notion of oper was introduced in [1] [cf. also [2]]. Let $k$ be an algebraically closed field and $X$ a projective smooth curve of genus $\geq 2$ over $k$. Let us recall that an oper over $X/k$ is a suitable triple consisting of a locally free coherent $\mathcal{O}_X$-module, a connection on the module relative to $X/k$, and a filtration of the module. The study of opers in positive characteristic was initiated by, for instance, [4], [5], [9], [10]. Suppose that we are in the situation in which $k$ is of characteristic $p > 0$. Then we shall say that a given oper is dormant if the $p$-curvature of the connection of the oper is zero. We refer to Definition 1.1 [cf. also Definition 1.2] concerning the precise definition of the notion of “dormant oper” discussed in the present paper. Note that a dormant oper of rank two is essentially the same as a dormant indigenous bundle studied in the $p$-adic Teichmüller theory [cf. [7]].

The main result of the present paper, which is a generalization of the first portion of [3], Theorem A [cf. Remark 2.1.1], is as follows [cf. Proposition 1.3, Theorem 2.1]:

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Theorem A. — Let $k$ be an algebraically closed field of characteristic $p > 0$ and $X$ a projective smooth curve of genus $\geq 2$ over $k$. Then the following hold:

(i) There exists a dormant oper of rank $p - 1$ over $X/k$.

(ii) Suppose that the Jacobian variety of $X$ is ordinary. Then the set of equivalence classes of dormant opers of rank $p - 1$ over $X/k$ is of cardinality one.

In [5], Joshi posed a conjecture concerning the number of equivalence classes of dormant opers of rank $r$ over a projective smooth curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p > 0$ for $p > C(r, g) \stackrel{\text{def}}{=} r(r-1)(r-2)(g-1)$ [cf. [5], Conjecture 8.1]; moreover, Wakabayashi proved, in [10], this conjecture for a sufficiently general curve [cf. [10], Theorem 8.7.1]. Note that the triple $(r, g, p)$ does not satisfy the condition $p > C(r, g)$ unless $p \in \{2, 3\}$.

1. Construction of Dormant Opers of Rank $p - 1$

In the present §1, let $p$ be a prime number, $g \geq 2$ an integer, $k$ an algebraically closed field of characteristic $p$, and $X$ a projective smooth curve over $k$ [i.e., a scheme which is projective, smooth, geometrically connected, and of relative dimension one over $k$] of genus $g$. Write $X^F$ for the projective smooth curve over $k$ obtained by base-changing $X$ via the absolute Frobenius morphism of $k$, $\Phi: X \to X^F$ for the relative Frobenius morphism over $k$, $\mathcal{I} \subseteq \mathcal{O}_{X \times_k X}$ for the ideal of $\mathcal{O}_{X \times_k X}$ which defines the diagonal morphism with respect to $X/k$, and $X_{(n)} \subseteq X \times_k X$ for the closed subscheme of $X \times_k X$ defined by the ideal $\mathcal{I}^{n+1} \subseteq \mathcal{O}_{X \times_k X}$ [where $n$ is a nonnegative integer]. In particular, it follows that $\mathcal{I}/\mathcal{I}^2 = \omega_{X/k}$ (respectively, $\text{Hom}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \tau_{X/k}$), where we use the notation “$\omega$” (respectively, “$\tau$”) to denote the cotangent (respectively, tangent) sheaf. Finally, write $d: \mathcal{O}_X \to \omega_{X/k}$ for the exterior differentiation operator.

Let us define the notion of dormant oper, as well as the notion of equivalence of dormant opers, discussed in the present paper as follows [cf., e.g., [5], §3; also [4], Definitions 3.1.1, 3.1.2 and (1)-(5) of pp. 51-52]:

Definition 1.1. — Let $r$ be a positive integer. Then we shall say that a collection of data

$$(\mathcal{E}, \nabla_{\mathcal{E}}, \{0\} = \mathcal{E}_r \subseteq \mathcal{E}_{r-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E})$$

consisting of a locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$, a connection $\nabla_{\mathcal{E}}$ on $\mathcal{E}$ relative to $X/k$, and a filtration $\{0\} = \mathcal{E}_r \subseteq \mathcal{E}_{r-1} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E}$ by $\mathcal{O}_X$-submodules of $\mathcal{E}$ is a dormant oper of rank $r$ over $X/k$ if the following five conditions are satisfied:

(1) For every $i \in \{1, \ldots, r\}$, the subquotient $\mathcal{E}_{i-1}/\mathcal{E}_i$ is an invertible sheaf on $X$.

(2) For every $i \in \{1, \ldots, r\}$, it holds that $\nabla_{\mathcal{E}}(\mathcal{E}_i) \subseteq \mathcal{E}_{i-1} \otimes_{\mathcal{O}_X} \omega_{X/k}$.

(3) For every $i \in \{1, \ldots, r - 1\}$, the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{E}_i \xrightarrow{\nabla_{\mathcal{E}_i}} \mathcal{E}_{i-1} \otimes_{\mathcal{O}_X} \omega_{X/k} \to (\mathcal{E}_{i-1}/\mathcal{E}_i) \otimes_{\mathcal{O}_X} \omega_{X/k}$$
[cf. (2)] determines an isomorphism $E_i/E_{i+1} \sim (E_{i-1}/E_i) \otimes_{O_X} \omega_{X/k}$ of invertible sheaves on $X$ [cf. (1)].

(4) The $p$-curvature of $\nabla_E$ is zero.

(5) There exists an isomorphism $(\det E, \det \nabla_E) \sim (O_X, d)$.

**Definition 1.2.** — Let $r$ be a positive integer and

$$E \overset{\text{def}}{=} (E, \nabla_E, \{0\} = E_r \subseteq E_{r-1} \subseteq \cdots \subseteq E_1 \subseteq E_0 = E),$$

$$\mathcal{F} \overset{\text{def}}{=} (\mathcal{F}, \nabla_{\mathcal{F}}, \{0\} = \mathcal{F}_r \subseteq \mathcal{F}_{r-1} \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0 = \mathcal{F})$$

dormant opers of rank $r$ over $X/k$. Then we shall say that $E$ is equivalent to $\mathcal{F}$ if the following condition is satisfied: There exists a triple $(\mathcal{L}, \nabla_{\mathcal{L}}, \phi)$ consisting of an invertible sheaf $\mathcal{L}$ on $X$, a connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$ relative to $X/k$, and a horizontal isomorphism $\phi$ of $(E, \nabla_E)$ with $(\mathcal{F}, \nabla_{\mathcal{F}}) \otimes_{O_X} (\mathcal{L}, \nabla_{\mathcal{L}})$ such that $\phi$ maps, for each $i \in \{0, \ldots, r\}$, the subsheaf $E_i \subseteq E$ isomorphically onto the subsheaf $\mathcal{F}_i \otimes_{O_X} \mathcal{L} \subseteq \mathcal{F} \otimes_{O_X} \mathcal{L}$.  

**Remark 1.2.1.** — Note that the notion of the “equivalence class of a dormant oper of rank $r$” of the present paper coincides with the notion of “isomorphism class of a dormant $\text{PGL}(r)$-oper” in the terminology given in [5], §3.

**Remark 1.2.2.** — One verifies immediately from the various definitions involved that the notion of dormant oper of rank two is essentially the same as the notion of dormant indigenous bundle studied in [7].

In the remainder of the present §1, let us construct a dormant oper of rank $p - 1$ over $X/k$ [cf. Proposition 1.3 below]. We shall write

$$B^\circ \overset{\text{def}}{=} \text{Coker}(O_{X_F} \rightarrow \Phi_*O_X)$$

for the $O_{X_F}$-module obtained by forming the cokernel of the natural homomorphism $O_{X_F} \rightarrow \Phi_*O_X$. Since [one verifies immediately that] the homomorphism $O_{X_F} \rightarrow \Phi_*O_X$ admits a natural splitting after pulling back via $\Phi$ — which thus determines a natural isomorphism of $O_X$-modules

$$\Phi^*\Phi_*O_X \overset{\sim}{\rightarrow} O_X \oplus \Phi^*B^\circ$$

— and $\Phi$ is finite flat of degree $p$, it follows that $B^\circ$, hence also $\Phi^*B^\circ$, is locally free of rank $p - 1$.

Next, let us observe that one verifies immediately that the natural morphism over $k$

$$X \times_{X_F} X \rightarrow X \times_k X$$

determines an isomorphism over $k$

$$X \times_{X_F} X \overset{\sim}{\rightarrow} X_{(p-1)}.$$
Moreover, this isomorphism $X \times_{X^F} X \sim X_{(p-1)}$, together with the cartesian diagram
\[
\begin{array}{ccc}
X \times_{X^F} X & \xrightarrow{pr_2} & X \\
pr_1 \downarrow & & \downarrow \Phi \\
X & \xrightarrow{\Phi} & X^F,
\end{array}
\]
determines isomorphisms of $\mathcal{O}_X$-modules
\[
\Phi^* \Phi_* \mathcal{O}_X \sim \text{pr}_{1*} \mathcal{O}_{X \times_{X^F} X} \sim \text{pr}_{1*} \mathcal{O}_{X_{(p-1)}},
\]
which are compatible with the respective natural surjections onto $\mathcal{O}_X$ [arising from the diagonal morphism with respect to $X/X^F$] from each of these three modules. In particular, by forming the kernels of the respective natural surjections onto $\mathcal{O}_X$, we obtain isomorphisms of $\mathcal{O}_X$-modules
\[
\Phi^* \mathcal{B}^o \sim \text{Ker}(\text{pr}_{1*} \mathcal{O}_{X \times_{X^F} X} \to \mathcal{O}_X) \sim \text{pr}_{1*} (\mathcal{I}/\mathcal{I}^p).
\]
Thus, by considering the filtration of $\text{pr}_{1*} (\mathcal{I}/\mathcal{I}^p)$
\[
\{0\} = \text{pr}_{1*} (\mathcal{I}^p/\mathcal{I}^p) \subseteq \text{pr}_{1*} (\mathcal{I}^{p-1}/\mathcal{I}^p) \subseteq \cdots \subseteq \text{pr}_{1*} (\mathcal{I}^2/\mathcal{I}^p) \subseteq \text{pr}_{1*} (\mathcal{I}/\mathcal{I}^p),
\]
we conclude that $\Phi^* \det \mathcal{B}^o = \det(\Phi^* \mathcal{B}^o)$ is isomorphic to $\omega_{X/k}^{1/p}$, which thus implies that $\deg \mathcal{B}^o = (p-1)(g-1)$. In particular, one verifies immediately that there exists an invertible sheaf $\mathcal{L}^o$ on $X^F$ such that $\det(\mathcal{L}^o \otimes \mathcal{O}_{X^F} \mathcal{B}^o) (= (\mathcal{L}^o)^{\otimes p-1} \otimes \mathcal{O}_{X^F} \det \mathcal{B}^o)$ is isomorphic to $\mathcal{O}_{X^F}$ [cf. Remark 1.3.1 below]. We shall write
\[
\mathcal{E}^o \overset{\text{def}}{=} \Phi^* (\mathcal{L}^o \otimes \mathcal{O}_{X^F} \mathcal{B}^o).
\]
Then since the closed immersion $X_{(1)} \hookrightarrow X \times_k X$ factors through the closed subscheme $X \times_{X^F} X \subseteq X \times_k X$ [as verified above], it follows that the $\mathcal{O}_X$-module $\mathcal{E}^o$ admits a natural connection relative to $X/k$. We shall write
\[
\nabla_{\mathcal{E}^o}
\]
for the connection on $\mathcal{E}^o$. Moreover, the above filtration of $\text{pr}_{1*} (\mathcal{I}/\mathcal{I}^p)$ [together with the above isomorphism $\Phi^* \mathcal{B}^o \sim \text{pr}_{1*} (\mathcal{I}/\mathcal{I}^p)$] determines a filtration of $\mathcal{E}^o$
\[
\{0\} = \mathcal{E}^o_{p-1} \subseteq \mathcal{E}^o_{p-2} \subseteq \cdots \subseteq \mathcal{E}^o_1 \subseteq \mathcal{E}^o_0 = \mathcal{E}^o.
\]
[Thus, one verifies immediately that $\mathcal{E}^o/\mathcal{E}^o_1 \cong \Phi^* \mathcal{L}^o \otimes \mathcal{O}_X \text{pr}_{1*} (\mathcal{I}/\mathcal{I}^2) = \Phi^* \mathcal{L}^o \otimes \mathcal{O}_X \omega_{X/k}$.]

**PROPOSITION 1.3.** — The collection of data
\[
\mathcal{E}^o \overset{\text{def}}{=} (\mathcal{E}^o, \nabla_{\mathcal{E}^o}, \{0\} = \mathcal{E}^o_{p-1} \subseteq \mathcal{E}^o_{p-2} \subseteq \cdots \subseteq \mathcal{E}^o_1 \subseteq \mathcal{E}^o_0 = \mathcal{E}^o)
\]
forms a dormant oper of rank $p-1$ over $X/k$.

**PROOF.** — The assertion that $\mathcal{E}^o$ satisfies condition (1) of Definition 1.1 follows from the definition of the $\mathcal{O}_X$-submodules $\mathcal{E}^o_i \subseteq \mathcal{E}^o$. The assertion that $\mathcal{E}^o$ satisfies conditions (2) and (3) of Definition 1.1 follows immediately from a local explicit calculation [cf. also [4], Theorem 3.1.6]. The assertion that $\mathcal{E}^o$ satisfies condition (4) of Definition 1.1 follows from the definition of the connection $\nabla_{\mathcal{E}^o}$. The assertion that $\mathcal{E}^o$ satisfies condition (5) of Definition 1.1 follows from the fact that $\det(\mathcal{L}^o \otimes \mathcal{O}_{X^F} \mathcal{B}^o)$ is isomorphic to $\mathcal{O}_{X^F}$, together with the definition of the connection $\nabla_{\mathcal{E}^o}$. This completes the proof of Proposition 1.3. \(\square\)
Remark 1.3.1. — Let us observe that the choice of “$L^0$” in the discussion preceding Proposition 1.3 is not unique. More precisely, if we write $(\text{Pic} X)[p-1] \subseteq \text{Pic} X$ for the subgroup of $\text{Pic} X$ obtained by forming the kernel of the endomorphism of $\text{Pic} X$ given by multiplication by $p-1$, then one verifies immediately that the set consisting of isomorphism classes of “possible $L^0$’s” forms a $(\text{Pic} X)[p-1]$-torsor. On the other hand, one also verifies immediately from the various definitions involved that the adoption of another possible “$L^0$” does not affect the equivalence class of the dormant oper of Proposition 1.3.

2. Uniqueness of Dormant Opers of Rank $p - 1$

In the present §2, we maintain the notation of the preceding §1. In particular, we have a projective smooth curve $X$ over $k$. In the present §2, we prove the following theorem:

Theorem 2.1. — If the Jacobian variety of $X$ is ordinary, then every dormant oper of rank $p - 1$ over $X/k$ is equivalent [cf. Definition 1.2] to the dormant oper of Proposition 1.3 [cf. also Remark 1.3.1].

Remark 2.1.1. — One verifies immediately from the various definitions involved, together with Remark 1.2.2, that if $p = 3$, then Theorem 2.1 implies and is implied by the first portion of [3], Theorem A [cf. the first paragraph of the proof of [3], Theorem 2.1]. Thus, Theorem 2.1 is a generalization of the first portion of [3], Theorem A.

To complete the verification of Theorem 2.1, let

$$\mathcal{E} \overset{\text{def}}{=} (\mathcal{E}, \nabla_{\mathcal{E}}, \{0\} = \mathcal{E}_{p-1} \subseteq \mathcal{E}_{p-2} \subseteq \cdots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E})$$

be a dormant oper of rank $p - 1$ over $X/k$. We shall write

$$\Xi^0 \overset{\text{def}}{=} \mathcal{E}^0/\mathcal{E}_1^0, \quad \Xi \overset{\text{def}}{=} \Xi^0/\mathcal{E}_1.$$

Then it follows from the definition of a dormant oper that both $(\Xi^0)^{\otimes p-1}$ and $\Xi^{\otimes p-1}$ are isomorphic to $\bigotimes_{X/k}^{r-1} \mathcal{O}_{X/k}$, which thus implies that $\text{Hom}_{\mathcal{O}_X}(\Xi, \Xi^0)$ is of degree zero. In particular, one verifies immediately that there exists an invertible sheaf on $X^F$ whose pull back via $\Phi$ is isomorphic to $\text{Hom}_{\mathcal{O}_X}(\Xi, \Xi)$. Thus, to complete the verification of Theorem 2.1, we may assume without loss of generality, by tensoring $\mathcal{E}$ with the above pull back via $\Phi$, that

$$\Xi^0 \cong \Xi,$$

By means of such an isomorphism, let us identify $\Xi^0$ with $\Xi$.

Next, let us observe that since the $p$-curvature of $\nabla_{\mathcal{E}}$ is zero, and $(\det \mathcal{E}, \det \nabla_{\mathcal{E}})$ is isomorphic to $(\mathcal{O}_X, d)$, it follows from a theorem of Cartier [cf., e.g., [6], Theorem 5.1] that if we write $\mathcal{F} \overset{\text{def}}{=} \mathcal{E}^{\nabla_{\mathcal{E}}}$ for the $\mathcal{O}_X^F$-module of horizontal sections of $\mathcal{E}$ with respect to $\nabla_{\mathcal{E}}$, then

(a) $\mathcal{E}$ is naturally isomorphic to $\Phi^* \mathcal{F}$.
(b) the connection $\nabla_\mathcal{E}$ arises from the isomorphism of (a) [together with the factorization $X_{(1)} \hookrightarrow X \times_{X^F} X \hookrightarrow X \times_k X$ of the closed immersion $X_{(1)} \hookrightarrow X \times_k X$ discussed in §1], and

(c) $\det \mathcal{F}$ is isomorphic to $\mathcal{O}_{X^F}$.

Thus, the isomorphism $\Phi^* \mathcal{F} \sim \mathcal{E}$ of (a) and the surjection $\mathcal{E} \twoheadrightarrow \Xi = \Xi^o$ determine a surjection of $O_X$-modules

$$\Phi^* \mathcal{F} \twoheadrightarrow \Xi^o,$$

hence also a homomorphism of $O_{X^F}$-modules

$$\mathcal{F} \longrightarrow \Phi_* \Xi^o.$$

**Lemma 2.2.** — *The above homomorphism $\mathcal{F} \rightarrow \Phi_1 \Xi^o$ is a locally split injection.*

**Proof.** — Let us first observe that it is immediate that, to verify Lemma 2.2, it suffices to verify that the homomorphism $\gamma : \mathcal{E} \sim \Phi^* \mathcal{F} \rightarrow \mathcal{G} \overset{\text{def}}{=} \Phi^* \Phi_* \Xi^o$ of $O_X$-modules obtained by pulling back, via $\Phi$, the homomorphism under consideration is a *split injection*. Now one verifies immediately that the composite of $\gamma$ and the natural homomorphism $\mathcal{G} = \Phi^* \Phi_* \Xi^o \rightarrow \Xi^o \overset{\text{def}}{=} \Xi^o$ coincides with the natural surjection $\mathcal{E} \twoheadrightarrow \Xi = \Xi^o$.

Write $\nabla_\mathcal{G}$ for the connection on $\mathcal{G}$ determined by the factorization $X_{(1)} \hookrightarrow X \times_{X^F} X \hookrightarrow X \times_k X$ of the closed immersion $X_{(1)} \hookrightarrow X \times_k X$ discussed in §1. Thus, it follows immediately from the definitions of $\nabla_\mathcal{E}$ and $\nabla_\mathcal{G}$ that $\gamma$ is *horizontal* with respect to $\nabla_\mathcal{E}$ and $\nabla_\mathcal{G}$. Moreover, for $i \in \{1, \ldots, p\}$, let us define submodules $\mathcal{G}_i \subseteq \mathcal{G}$ inductively as follows: We shall write $\mathcal{G}_1 \overset{\text{def}}{=} \ker(\mathcal{G} = \Phi^* \Phi_* \Xi^o \rightarrow \Xi^o)$. If $i \geq 2$, then we shall write $\mathcal{G}_i \overset{\text{def}}{=} \ker(\mathcal{G}_{i-1} \hookrightarrow \mathcal{G} \overset{\nabla_\mathcal{G}}{\longrightarrow} \mathcal{G} \otimes_{O_X} \omega_{X/k} \rightarrow (\mathcal{G}/\mathcal{G}_{i-1}) \otimes_{O_X} \omega_{X/k})$. Then it follows that the submodule $\mathcal{G}_i \subseteq \mathcal{G}$ is an $O_X$-submodule, and $\mathcal{G}_p = \{0\}$; moreover, one verifies immediately from a local explicit calculation [cf. also [4], Theorem 3.1.6] that the collection of data

$$(\mathcal{G}, \nabla_\mathcal{G}, \{0\}) \subseteq \mathcal{G}_{p-1} \subseteq \cdots \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_0 \overset{\text{def}}{=} \mathcal{G}$$

satisfies conditions (1), (2), (3), and (4) of Definition 1.1.

Now I claim that the following assertion holds:

For each $i \in \{1, \ldots, p-1\}$, the composite $\mathcal{E}_{i-1} \hookrightarrow \mathcal{E} \overset{\gamma}{\twoheadrightarrow} \mathcal{G}$ determines an *isomorphism $\mathcal{E}_{i-1}/\mathcal{E}_i \sim \mathcal{G}_{i-1}/\mathcal{G}_i$ of $O_X$-modules.*

Let us verify this claim by induction on $i$. If $i = 1$, then the desired assertion has already been verified [in the first paragraph of this proof]. Let $i \in \{2, \ldots, p-1\}$. Suppose that $\gamma$ determines an *isomorphism $\mathcal{E}_{i-2}/\mathcal{E}_{i-1} \sim \mathcal{G}_{i-2}/\mathcal{G}_{i-1}$ of $O_X$-modules*, which thus implies that $\gamma(\mathcal{E}_{i-1}) \subseteq \mathcal{G}_{i-1}$. Thus, since $\gamma$ is *horizontal*, the diagram

$$\begin{array}{ccc}
\mathcal{E}_{i-1} & \overset{\epsilon}{\longrightarrow} & \mathcal{E} \\
\downarrow & \gamma \downarrow & \gamma \circ \text{id} \\
\mathcal{G}_{i-1} & \overset{\epsilon}{\longrightarrow} & \mathcal{G}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{E}_{i-1} & \overset{\nabla_\mathcal{E}}{\longrightarrow} & \mathcal{E} \otimes_{O_X} \omega_{X/k} \\
\downarrow & \gamma \otimes \text{id} \downarrow & \\
\mathcal{G}_{i-1} & \overset{\nabla_\mathcal{G}}{\longrightarrow} & \mathcal{G} \otimes_{O_X} \omega_{X/k}
\end{array}$$

commutes. In particular, it follows immediately from condition (3) of Definition 1.1, together with the induction hypothesis, that $\gamma$ determines an *isomorphism $\mathcal{E}_{i-1}/\mathcal{E}_i \sim \mathcal{G}_{i-1}/\mathcal{G}_i$ of $O_X$-modules*, as desired. This completes the proof of the above claim.
By the above claim, one verifies easily that the composite \( \mathcal{E} \xrightarrow{\gamma} \mathcal{G} \rightarrow \mathcal{G}/G_{p-1} \) is an isomorphism of \( \mathcal{O}_X \)-modules. In particular, \( \gamma \) is split injection. This completes the proof of Lemma 2.2. \( \square \)

By Lemma 2.2, together with the fact that \( \Xi^o \) is isomorphic to \( \Phi^*\mathcal{L}^o \otimes \mathcal{O}_X \omega_{X/k} \) [cf. the discussion preceding Proposition 1.3], we have an exact sequence of locally free \( \mathcal{O}_{X^F} \)-modules

\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{L}^o \otimes \mathcal{O}_{X^F} \Phi_*\omega_{X/k} \rightarrow \mathcal{Q} \rightarrow 0.
\]

**Lemma 2.3.** — The above \( \mathcal{O}_{X^F} \)-module \( \mathcal{Q} \) is isomorphic to \( \mathcal{L}^o \otimes \mathcal{O}_{X^F} \omega_{X^F/k} \).

**Proof.** — Let us first observe that \( \mathcal{F} \) is of rank \( p-1 \), and \( \mathcal{L}^o \otimes \mathcal{O}_{X^F} \Phi_*\omega_{X/k} \) is of rank \( p \), which thus implies that \( \mathcal{Q} \) is an invertible sheaf on \( X^F \). Thus, by the exact sequence of the discussion preceding Lemma 2.3, together with (c) of the discussion preceding Lemma 2.2, it holds that

\[
\mathcal{Q} \cong (\mathcal{L}^o)^{\otimes p} \otimes \mathcal{O}_{X^F} \det \Phi_*\omega_{X/k}.
\]

In particular, since [it follows from our choice of \( \mathcal{L}^o \) — cf. the discussion preceding Proposition 1.3 — that] \( (\mathcal{L}^o)^{\otimes p-1} \) is isomorphic to \( \mathcal{H}om_{\mathcal{O}_{X^F}}(\det \mathcal{B}^o, \mathcal{O}_{X^F}) \), we obtain that

\[
\mathcal{Q} \cong \mathcal{L}^o \otimes \mathcal{O}_{X^F} \mathcal{H}om_{\mathcal{O}_{X^F}}(\det \mathcal{B}^o, \det \Phi_*\omega_{X/k}).
\]

Next, let us recall [cf., e.g., [6], Theorem 7.2] the well-known exact sequence of \( \mathcal{O}_{X^F} \)-modules

\[
0 \rightarrow \mathcal{O}_{X^F} \rightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_*\phi} \Phi_*\omega_{X/k} \xrightarrow{c} \omega_{X^F/k} \rightarrow 0
\]

where we write \( c \) for the Cartier operator. Thus, it follows from the definition of \( \mathcal{B}^o \) that

\[
\det \Phi_*\omega_{X/k} \cong \omega_{X^F/k} \otimes \mathcal{O}_{X^F} \det \mathcal{B}^o.
\]

This completes the proof of Lemma 2.3. \( \square \)

By Lemma 2.3, we have an exact sequence of locally free \( \mathcal{O}_{X^F} \)-modules

\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{L}^o \otimes \mathcal{O}_{X^F} \Phi_*\omega_{X/k} \rightarrow \mathcal{L}^o \otimes \mathcal{O}_{X^F} \omega_{X^F/k} \rightarrow 0.
\]

On the other hand, we have an exact sequence of locally free \( \mathcal{O}_{X^F} \)-modules

\[
0 \rightarrow \mathcal{L}^o \otimes \mathcal{O}_{X^F} \mathcal{B}^o \rightarrow \mathcal{L}^o \otimes \mathcal{O}_{X^F} \Phi_*\omega_{X/k} \xrightarrow{\mathcal{L}^o \otimes \mathcal{O}_{X^F} \Phi_*\omega_{X/k}} \mathcal{L}^o \otimes \mathcal{O}_{X^F} \omega_{X^F/k} \rightarrow 0
\]

[cf. the well-known exact sequence that appears in the second paragraph of the proof of Lemma 2.3].

Let us complete the proof of Theorem 2.1:

**Proof of Theorem 2.1.** — In the above two exact sequences, assume that \( \mathcal{L}^o \otimes \mathcal{O}_{X^F} \mathcal{B}^o \nsubseteq \mathcal{F} \) [in \( \mathcal{L}^o \otimes \mathcal{O}_{X^F} \Phi_*\omega_{X/k} \)]. Then it is immediate that the composite

\[
\mathcal{L}^o \otimes \mathcal{O}_{X^F} \mathcal{B}^o \hookrightarrow \mathcal{L}^o \otimes \mathcal{O}_{X^F} \Phi_*\omega_{X/k} \twoheadrightarrow (\mathcal{L}^o \otimes \mathcal{O}_{X^F} \Phi_*\omega_{X/k})/\mathcal{F} \cong \mathcal{L}^o \otimes \mathcal{O}_{X^F} \omega_{X^F/k}
\]

is nonzero. In particular, since we have an isomorphism \( \mathcal{B}^o \cong \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{B}^o, \omega_{X^F/k}) \) of \( \mathcal{O}_{X^F} \)-modules [cf. the discussion preceding [8], Théorème 4.1.1], we conclude that the
$\mathcal{O}_{XF}$-module $B^0$ admits a nonzero global section. Thus, it follows — in light of the exact sequence of $\mathcal{O}_{XF}$-modules

$$0 \to \mathcal{O}_{XF} \to \Phi_* \mathcal{O}_X \to B^0 \to 0$$

— that the Jacobian variety of $X$ is not ordinary — in contradiction to our assumption. Thus, we conclude that $L^0 \otimes \mathcal{O}_{XF} B^0 \subseteq \mathcal{F}$ [in $L^0 \otimes \mathcal{O}_{XF} \Phi_* \omega_X/k$].

Next, let us observe that since $\mathcal{O}_{XF} \cong \det(L^0 \otimes \mathcal{O}_{XF} B^0) \cong \det \mathcal{F}$, the inclusion $L^0 \otimes \mathcal{O}_{XF} B^0 \subseteq \mathcal{F}$ implies the equality $L^0 \otimes \mathcal{O}_{XF} B^0 = \mathcal{F}$ [in $L^0 \otimes \mathcal{O}_{XF} \Phi_* \omega_X/k$]. In particular, we have an isomorphism $\phi: \mathcal{E}^0 \sim \mathcal{E}$ [i.e., “$\Phi^* \text{id}$”]. Moreover, it follows immediately from the definition of $\nabla_{\mathcal{E}^0}$, together with (b) of the discussion preceding Lemma 2.2, that the isomorphism $\phi$ is horizontal with respect to $\nabla_{\mathcal{E}^0}, \nabla_{\mathcal{E}}$.

Finally, we verify that $\phi(\mathcal{E}^0_{p^{-1}}) = \mathcal{E}_{p^{-1}}$ for each $i \in \{1, \ldots, p-1\}$ by induction on $i$. The equality ($\{0\} = \phi(\mathcal{E}^0_{p^{-1}}) = \mathcal{E}_{p^{-1}} (= \{0\})$ is immediate. Let $i \in \{2, \ldots, p-1\}$. Suppose that $\phi(\mathcal{E}^0_{p^{-1+i}}) = \mathcal{E}_{p^{-1+i}}$ holds. Then, to verify that $\phi(\mathcal{E}^0_{p^{-i}}) = \mathcal{E}_{p^{-i}}$, it suffices to verify that the induced isomorphism $\mathcal{E}^0 / \mathcal{E}^0_{p^{-i+1}} \sim \mathcal{E} / \mathcal{E}_{p^{-i+1}}$ maps $\mathcal{E}^0_{p^{-i}} / \mathcal{E}^0_{p^{-i+1}}$ isomorphically onto $\mathcal{E}_{p^{-i}} / \mathcal{E}_{p^{-i+1}}$. On the other hand, since [one verifies easily from the definition of a dormant oper that]

$$\mathcal{E}^0_{j-1} / \mathcal{E}^0_j \cong \mathcal{E}_{j-1} / \mathcal{E}_j \cong \mathcal{E}^0 / \omega_X^{\otimes (j-1)}$$

for each $j \in \{1, \ldots, p-1\}$, the desired assertion follows immediately from the ampleness of $\omega_X/k$. This completes the proof of Theorem 2.1. \hfill $\square$

**References**


(Yuichiro Hoshi) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN

E-mail address: yuichiro@kurims.kyoto-u.ac.jp