Two Categorical Characterizations of Local Fields

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Abstract. — In the present paper, we discuss two categorical characterizations of local fields. We first prove that a certain full subcategory of the category of finite flat coverings of the spectrum of the ring of integers of a local field equipped with coherent modules completely determines the isomorphism class of the local field. Next, we also prove that a certain full subcategory of the category of irreducible schemes which are finite over the spectrum of the ring of integers of a local field completely determines the isomorphism class of the local field.

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Introduction

Let K be a *local field*, i.e., a field which is isomorphic to a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime number p. Write $\mathcal{O}_K$ for the ring of integers of K and

$$\mathcal{B}_K$$

for the category of irreducible normal schemes which are finite, flat, and generically étale over $\mathcal{O}_K$ [cf. Definition 1.2]. Then one may verify that the category $\mathcal{B}_K$ is, by the functor taking function fields, equivalent to the category of finite separable extensions of K [cf. Lemma 1.4, (ii)]. Thus,

> the category $\mathcal{B}_K$ *completely determines and is completely determined* by the absolute Galois group of K

[cf. Theorem 1.10]. In particular, one may conclude from [4], §2, Theorem, that

> the equivalence class of the category $\mathcal{B}_K$ does *not determine* the isomorphism class of the field K

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[cf. Corollary 1.12, (i)]. In the present paper, we introduce two categories which contain, as a full subcategory, the above category $\mathcal{B}_K$ and prove that these categories *completely determine* the isomorphism class of the field K.

First, let us write

$$\overline{\mathcal{C}}_K$$

for the category of pairs of objects of $\mathcal{B}_K$ and coherent modules on the objects [cf. Definition 2.1] and take a full subcategory

$$\mathcal{C}_K$$

of $\overline{\mathcal{C}}_K$ which satisfies the condition ($\mathfrak{C}$) [cf. Definition 2.3], i.e., such that, roughly speaking,

($\mathfrak{C}$-a) $\mathcal{C}_K$ is closed under the operation of taking *submodules*, and

($\mathfrak{C}$-b) $\mathcal{C}_K$ contains every object of $\overline{\mathcal{C}}_K$ whose module is *torsion and generated by a single element.*

Then, by the conditions ($\mathfrak{C}$-a) and ($\mathfrak{C}$-b), one may regard the category $\mathcal{B}_K$ as a full subcategory of $\mathcal{C}_K$ [cf. Lemma 2.4, (iii)].

Next, let us write

$$\overline{\mathcal{F}}_K$$

for the category of irreducible schemes which are finite over $\mathcal{O}_K$ [cf. Definition 3.1] and take a full subcategory

$$\mathcal{F}_K$$

of $\overline{\mathcal{F}}_K$ which satisfies the condition ($\mathfrak{F}$) [cf. Definition 3.4], i.e., such that, roughly speaking,

($\mathfrak{F}$-a) $\mathcal{F}_K$ contains the object $\mathrm{Spec}(\mathcal{O}_K)$,

($\mathfrak{F}$-b) $\mathcal{F}_K$ is closed under the operation of taking *normalizations* of objects which are the spectra of integral domains of dimension one,

($\mathfrak{F}$-c) $\mathcal{F}_K$ is closed under the operation of taking *finite separable extensions and subfields* of the function fields of objects which are the spectra of integral domains of dimension one, and

($\mathfrak{F}$-d) $\mathcal{F}_K$ is closed under the operation of taking *closed subschemes.*

Then, by the conditions ($\mathfrak{F}$-a), ($\mathfrak{F}$-b), and ($\mathfrak{F}$-c), one may regard the category $\mathcal{B}_K$ as a full subcategory of $\mathcal{F}_K$ [cf. Lemma 3.5, (v)].

The main result of the present paper is as follows [cf. Theorem 2.14; Theorem 3.20]:

THEOREM. — *Let $K_\circ$, $K_\bullet$ be local fields. Then the following hold:*

(i) *Let $\mathcal{C}_{K_\circ}$, $\mathcal{C}_{K_\bullet}$ be full subcategories of $\overline{\mathcal{C}}_{K_\circ}$, $\overline{\mathcal{C}}_{K_\bullet}$ as above, respectively. Suppose that the category $\mathcal{C}_{K_\circ}$ is* **equivalent** *to the category $\mathcal{C}_{K_\bullet}$. Then the field $K_\circ$ is* **isomorphic** *to the field $K_\bullet$.*

(ii) *Let $\mathcal{F}_{K_\circ}$, $\mathcal{F}_{K_\bullet}$ be full subcategories of $\overline{\mathcal{F}}_{K_\circ}$, $\overline{\mathcal{F}}_{K_\bullet}$ as above, respectively. Suppose that the category $\mathcal{F}_{K_\circ}$ is* **equivalent** *to the category $\mathcal{F}_{K_\bullet}$. Then the field $K_\circ$ is* **isomorphic** *to the field $K_\bullet$.*

In §1, we discuss the category $\mathcal{B}_K$. In §2, we prove Theorem, (i). In §3, we prove Theorem, (ii). In the proof of Theorem, the main result of [2] plays an important role. Here, let us recall that the main result of [2] was generalized in [1].

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1. Category of Finite Flat Coverings

In the present §1, let us discuss a category of certain finite flat coverings of the spectrum of the ring of integers of a local field [cf. Definition 1.2].

Definition 1.1. — If K is a *local field*, i.e., a field which is isomorphic to a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime number p, then we shall write

- $\mathcal{O}_K \subseteq K$ for the ring of integers of K,
- $\mathfrak{m}_K \subseteq \mathcal{O}_K$ for the maximal ideal of $\mathcal{O}_K$, and
- $\underline{K} \stackrel{\text{def}}{=} \mathcal{O}_K/\mathfrak{m}_K$ for the residue field of $\mathcal{O}_K$.

In the remainder of the present §1, let K be a *local field.*

Definition 1.2. — We shall write $\mathcal{B}_K$ for the category defined as follows:

- An object of $\mathcal{B}_K$ is a pair (S, ϕ) consisting of a(n) [nonempty] irreducible normal scheme S and a morphism $\phi\colon S \to \mathrm{Spec}(\mathcal{O}_K)$ of schemes which is finite, flat, and generically étale. To simplify the exposition, we shall often refer to S [i.e., just the domain of the morphism ϕ] as an "object of $\mathcal{B}_K$".
- Let S, T be objects of $\mathcal{B}_K$. Then a morphism $S \to T$ in $\mathcal{B}_K$ is defined as a morphism of schemes from S to T lying over $\mathcal{O}_K$.

Definition 1.3. — Let S be an object of $\mathcal{B}_K$. Then we shall write K_S for the function field of S.

Lemma 1.4. — *The following hold:*

(i) *A* **terminal** *object of $\mathcal{B}_K$ is given by the pair* $(\mathrm{Spec}(\mathcal{O}_K), \mathrm{id}_{\mathrm{Spec}(\mathcal{O}_K)})$.

(ii) *The assignment "$S \mapsto K_S$" determines an* **equivalence** *of categories of $\mathcal{B}_K$ with the category defined as follows:*

- *An object of the category is a finite separable extension of K.*
- *A morphism in the category is a homomorphism of fields over K.*

PROOF. — These assertions follow immediately from the definition of the category $\mathcal{B}_K$. $\square$

DEFINITION 1.5.

(i) We shall say that a morphism $f: S \to T$ in $\mathcal{B}_K$ is *Galois* if the finite separable extension K_S/K_T determined by f [cf. Lemma 1.4, (ii)] is Galois.

(ii) We shall say that an object S of $\mathcal{B}_K$ is *Galois* if there exists a Galois morphism from S to a terminal object of $\mathcal{B}_K$ [cf. Lemma 1.4, (i)].

(iii) We shall say that a projective system $(S_\lambda)_{\lambda\in\Lambda}$ consisting of objects and morphisms of $\mathcal{B}_K$ is a *basepoint* of $\mathcal{B}_K$ if S_λ is Galois for each $\lambda \in \Lambda$, and, moreover, for each object T of $\mathcal{B}_K$, there exist an element $\lambda_T \in \Lambda$ and a morphism $S_{\lambda_T} \to T$ in $\mathcal{B}_K$.

(iv) Let $\widetilde{S} = (S_\lambda)_{\lambda\in\Lambda}$ be a basepoint of $\mathcal{B}_K$. Then we shall write

$$K_{\widetilde{S}} \stackrel{\mathrm{def}}{=} \varinjlim_{\lambda\in\Lambda} K_{S_\lambda}$$

for the field obtained by forming the injective limit of the K_{S_λ}'s and

$$\Pi_{\widetilde{S}} \stackrel{\mathrm{def}}{=} \varprojlim_{\lambda\in\Lambda} \mathrm{Aut}(S_\lambda)$$

for the profinite [cf. Lemma 1.4, (ii)] group obtained by forming the projective limit of the $\mathrm{Aut}(S_\lambda)$'s.

LEMMA 1.6. — *The following hold:*

(i) *There exists a* **basepoint** *of* $\mathcal{B}_K$.

(ii) *Let* S *be a* **Galois** *object of* $\mathcal{B}_K$. *Then* $\mathrm{Aut}(S)$ *is isomorphic to* $\mathrm{Gal}(K_S/K)$.

(iii) *Let* $\widetilde{S}$ *be a basepoint of* $\mathcal{B}_K$. *Then the field* $K_{\widetilde{S}}$ *is a* **separable closure** *of* K. *Moreover, the profinite group* $\Pi_{\widetilde{S}}$ *is* **isomorphic** *to the absolute Galois group* $\mathrm{Gal}(K_{\widetilde{S}}/K)$ *of* K.

PROOF. — These assertions follow, in light of Lemma 1.4, (ii), from elementary field theory. $\square$

LEMMA 1.7. — *Let* S, T *be objects of* $\mathcal{B}_K$; $f: S \to T$ *a morphism in* $\mathcal{B}_K$. *Then it holds that* f *is* **Galois** *if and only if, for each two morphisms* $g_1, g_2: U \to S$ *in* $\mathcal{B}_K$ *such that* $f \circ g_1 = f \circ g_2$, *there exists an automorphism* h *of* S *over* T *such that* $g_2 = h \circ g_1$.

PROOF. — This follows, in light of Lemma 1.4, (ii), from elementary field theory. $\square$

DEFINITION 1.8. — Let S be an object of $\mathcal{B}_K$ and $\widetilde{S} = (S_\lambda)_{\lambda\in\Lambda}$ a basepoint of $\mathcal{B}_K$. Then we shall write

$$S(\widetilde{S}) \stackrel{\mathrm{def}}{=} \varinjlim_{\lambda\in\Lambda} \mathrm{Hom}(S_\lambda, S).$$

Lemma 1.9. — *Let $\widetilde{S} = (S_\lambda)_{\lambda \in \Lambda}$ be a basepoint of $\mathcal{B}_K$. Then the assignment "$S \mapsto S(\widetilde{S})$" determines an* **equivalence** *of categories of $\mathcal{B}_K$ with the category defined as follows:*

- *An object of the category is a [nonempty] finite set equipped with a continuous transitive action of $\Pi_{\widetilde{S}}$.*

- *Let A, B be objects of the category. Then a morphism $A \to B$ in the category is defined as a $\Pi_{\widetilde{S}}$-equivariant map from A to B.*

Proof. — This follows from Lemma 1.4, (ii), and Lemma 1.6, (iii), together with elementary Galois theory. □

Theorem 1.10. — *Let $K_\circ$, $K_\bullet$ be local fields. Then it holds that the category $\mathcal{B}_{K_\circ}$ [cf. Definition 1.2] is* **equivalent** *to the category $\mathcal{B}_{K_\bullet}$ if and only if the absolute Galois group of the field $K_\circ$ is* **isomorphic**, *as a profinite group, to the absolute Galois group of the field $K_\bullet$.*

Proof. — The *necessity* follows, in light of Lemma 1.6, (i), from Lemma 1.6, (iii), and Lemma 1.7. The *sufficiency* follows, in light of Lemma 1.6, (i), (iii), from Lemma 1.9. □

Lemma 1.11. — *Let G be a profinite group which is isomorphic to the absolute Galois group of K. Then the following hold:*

(i) *It holds that K is* **of characteristic zero** *if and only if, for each prime number l, there exists an open subgroup of G such that l divides the cardinality of the [necessarily finite] module consisting of torsion elements of the abelianization of the open subgroup.*

(ii) *Suppose that K is* **of positive characteristic**. *Then it holds that $\sharp \underline{K} - 1$* **coincides** *with the cardinality of the [necessarily finite] module consisting of torsion elements of the abelianization of G.*

Proof. — Let us first recall from *local class field theory* [cf., e.g., [3], §2], together with the well-known structure of the multiplicative group $K^\times$, that the abelianization of G [i.e., as a profinite group] is *isomorphic* to the profinite module $\mathcal{O}_K^\times \times \widehat{\mathbb{Z}}$. Next, let us also recall that if K is *of positive characteristic*, then, again by the well-known structure of the multiplicative group $K^\times$, the composite $\boldsymbol{\mu}(K) \hookrightarrow \mathcal{O}_K^\times \twoheadrightarrow \underline{K}^\times$ — where we write $\boldsymbol{\mu}(K) \subseteq \mathcal{O}_K^\times$ for the group of roots of unity of K — is an *isomorphism*. Thus, assertions (i), (ii) follow immediately from the [easily verified] fact that $\widehat{\mathbb{Z}}$ is *torsion-free*. This completes the proof of Lemma 1.11. □

Corollary 1.12. — *The following hold:*

(i) *There exist local fields $K_\circ$ and $K_\bullet$ such that the category $\mathcal{B}_{K_\circ}$ is* **equivalent** *to the category $\mathcal{B}_{K_\bullet}$, but the field $K_\circ$ is* **not isomorphic** *to the field $K_\bullet$.*

(ii) *Let $K_\circ$, $K_\bullet$ be local fields. Suppose that the category $\mathcal{B}_{K_\circ}$ is* **equivalent** *to the category $\mathcal{B}_{K_\bullet}$, and that either $K_\circ$ or $K_\bullet$ is* **of positive characteristic**. *Then the field $K_\circ$ is* **isomorphic** *to the field $K_\bullet$.*

PROOF. — Assertion (i) follows from Theorem 1.10, together with [4], §2, Theorem. Finally, we verify assertion (ii). Suppose that $\mathcal{B}_{K_\circ}$ is *equivalent* to $\mathcal{B}_{K_\bullet}$, and that $K_\circ$ is *of positive characteristic*. Then it follows from Theorem 1.10 that the absolute Galois group of $K_\circ$ is *isomorphic* to the absolute Galois group of $K_\bullet$. Thus, it follows immediately from Lemma 1.11, (i), that $K_\bullet$ is *of positive characteristic*. Moreover, it follows from Lemma 1.11, (ii), that $\sharp\underline{K}_\circ = \sharp\underline{K}_\bullet$. Thus, since [one verifies easily that] the fields $K_\circ$, $K_\bullet$ are *isomorphic* to the local fields "$\mathbb{F}_{\sharp\underline{K}_\circ}((t))$", "$\mathbb{F}_{\sharp\underline{K}_\bullet}((t))$", respectively, we conclude that $K_\circ$ is *isomorphic* to $K_\bullet$, as desired. This completes the proof of assertion (ii). □

2. CATEGORY OF FINITE FLAT COVERINGS WITH COHERENT MODULES

In the present §2, let us discuss a certain full subcategory of the category of finite flat coverings of the spectrum of the ring of integers of a local field equipped with coherent modules [cf. Definition 2.1; Definition 2.3]. In the present §2, let K be a *local field*, i.e., a field which is isomorphic to a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime number p.

DEFINITION 2.1. — We shall write $\overline{\mathcal{C}}_K$ for the category defined as follows:

- An object of $\overline{\mathcal{C}}_K$ is a pair $X = (S_X, \mathcal{F}_X)$ consisting of an object S_X of $\mathcal{B}_K$ [cf. Definition 1.2] and a coherent $\mathcal{O}_{S_X}$-module $\mathcal{F}_X$.

- Let $X = (S_X, \mathcal{F}_X)$, $Y = (S_Y, \mathcal{F}_Y)$ be objects of $\overline{\mathcal{C}}_K$. Then a morphism $X \to Y$ in $\overline{\mathcal{C}}_K$ is defined as a pair $f = (f_S, f_{\mathcal{F}})$ consisting of a morphism $f_S\colon S_X \to S_Y$ in $\mathcal{B}_K$ and a homomorphism $f_{\mathcal{F}}\colon \mathcal{F}_X \to f_S^*\mathcal{F}_Y$ of $\mathcal{O}_{S_X}$-modules.

DEFINITION 2.2. — Let X, Y be objects of $\overline{\mathcal{C}}_K$; $f\colon X \to Y$ a morphism in $\overline{\mathcal{C}}_K$.

(i) We shall say that X is *scheme-like* if $\mathcal{F}_X(S_X) = \{0\}$.

(ii) We shall say that f is a *scheme-isomorphism* if f_S is an isomorphism of schemes. [Thus, a scheme-isomorphism is not necessarily an isomorphism in $\overline{\mathcal{C}}_K$.]

(iii) Suppose that $X = Y$, and that f is an automorphism. Then we shall say that f is a *scheme-identity* if f_S is the identity automorphism of S_X. We shall write

$$\mathrm{Aut}_{\mathrm{id}}(X)$$

for the group of scheme-identities of X.

(iv) We shall say that f is a *rigidification* [of Y] if X is scheme-like, and f is a scheme-isomorphism.

(v) We shall write K_X for the function field of S_X.

DEFINITION 2.3. — Let $\mathcal{C}_K$ be a full subcategory of $\overline{\mathcal{C}}_K$. Then we shall say that $\mathcal{C}_K$ satisfies the condition $(\mathfrak{C})$ if

(a) the full subcategory $\mathcal{C}_K$ is closed under the operation of taking *submodules*, i.e., if X is an object of $\mathcal{C}_K$, and $\mathcal{G} \subseteq \mathcal{F}_X$ is an $\mathcal{O}_{S_X}$-submodule of $\mathcal{F}_X$, then the object $(S_X, \mathcal{G})$ of $\overline{\mathcal{C}}_K$ is an object of $\mathcal{C}_K$, and

(b) the full subcategory $\mathcal{C}_K$ contains every object of $\overline{\mathcal{C}}_K$ whose module is *torsion and generated by a single element*, i.e., if an object X of $\overline{\mathcal{C}}_K$ satisfies the condition that the $\mathcal{O}_{K_X}$-module [cf. Definition 1.1; Definition 2.2, (v)] $\mathcal{F}_X(S_X)$ is torsion and generated by a single element, then X is an object of $\mathcal{C}_K$.

In the remainder of the present §2, let $\mathcal{C}_K$ be a full subcategory of $\overline{\mathcal{C}}_K$ which satisfies the condition ($\mathfrak{C}$).

Lemma 2.4. — *The following hold:*

(i) *Every* **scheme-like** *object of* $\overline{\mathcal{C}}_K$ *is an object of* $\mathcal{C}_K$.

(ii) *A* **terminal** *object of* $\mathcal{C}_K$ *is given by the pair* $(\mathrm{Spec}(\mathcal{O}_K), \{0\})$. *Moreover, every terminal object of* $\mathcal{C}_K$ *is* **scheme-like**.

(iii) *There exists a(n) [tautological]* **equivalence** *of categories of* $\mathcal{B}_K$ *with the full subcategory of* $\mathcal{C}_K$ *consisting of* **scheme-like** *objects of* $\mathcal{C}_K$.

Proof. — These assertions follow immediately from the definition of the category $\mathcal{C}_K$ [cf. Definition 2.3, (a), (b)]. □

Lemma 2.5. — *Let* X, Y *be objects of* $\mathcal{C}_K$; $f\colon X \to Y$ *a morphism in* $\mathcal{C}_K$. *Then the following hold:*

(i) *It holds that* f *is a* **monomorphism** [*i.e., in* $\mathcal{C}_K$] *if and only if* f *is a* **scheme-isomorphism**, *and, moreover, the homomorphism* $f_{\mathcal{F}}(S_X)\colon \mathcal{F}_X(S_X) \to f_S^*\mathcal{F}_Y(S_X)$ *of* $\mathcal{O}_{K_X}$*-modules is* **injective**.

(ii) *It holds that* f *is a* **rigidification** *if and only if* f *is a* **monomorphism**, *and, moreover,* f *is an* **initial** *object among monomorphisms whose codomains are* Y.

(iii) *It holds that* X *is* **scheme-like** *if and only if there exists a* **rigidification** *in* $\mathcal{C}_K$ *whose domain is* X.

(iv) *It holds that* f *is a* **scheme-isomorphism** *if and only if there exist* **rigidifications** $g\colon Z \to X$, $h\colon Z \to Y$ *in* $\mathcal{C}_K$ *such that* $f \circ g = h$.

(v) *Suppose that* $X = Y$, *and that* f *is an automorphism. Then it holds that* f *is a* **scheme-identity** *if and only if there exists a* **rigidification** $g\colon Z \to X$ *in* $\mathcal{C}_K$ *such that* $g = f \circ g$.

Proof. — First, we verify assertion (i). The *sufficiency* follows immediately from the [easily verified] *flatness* of a morphism in $\mathcal{B}_K$. In the remainder of the proof of assertion (i), we verify the *necessity*.

First, suppose that f_S is *not an isomorphism*. Then since the finite extension K_X/K_Y determined by f is *nontrivial* and *separable* [cf. Lemma 1.4, (ii); Lemma 2.4, (iii)], it follows from elementary field theory that there exist a finite separable extension L of

K and two inclusions i_1, $i_2\colon K_X \hookrightarrow L$ such that $i_1 \neq i_2$ but $i_1|_{K_Y} = i_2|_{K_Y}$. Thus, by considering the two morphisms from the object $(\mathrm{Spec}(\mathcal{O}_L), \{0\})$ of $\mathcal{C}_K$ [cf. Lemma 2.4, (i)] to X determined by i_1, i_2, respectively, we conclude that f is *not a monomorphism*.

Next, suppose that f_S is an *isomorphism*, but that the homomorphism $f_{\mathcal{F}}(S_X)$ of $\mathcal{O}_{K_X}$-modules is *not injective*, i.e., that $\{0\} \neq \mathrm{Ker}(f_{\mathcal{F}}(S_X)) \subseteq \mathcal{F}_X(S_X)$. Then we have the natural inclusion $j_1\colon \mathrm{Ker}(f_{\mathcal{F}}(S_X)) \hookrightarrow \mathcal{F}_X(S_X)$ and the zero homomorphism $j_2\colon \mathrm{Ker}(f_{\mathcal{F}}(S_X)) \to (\{0\} \hookrightarrow)\ \mathcal{F}_X(S_X)$. Write Z for the object of $\mathcal{C}_K$ determined by the pair $(S_X, \mathrm{Ker}(f_{\mathcal{F}}(S_X)))$ [cf. Definition 2.3, (a)]. Then, by considering the natural two scheme-isomorphisms from Z to X determined by j_1, j_2, respectively, we conclude that f is *not a monomorphism*. This completes the proof of the *necessity*, hence also of assertion (i).

Assertion (ii) follows immediately, in light of Lemma 2.4, (i), from assertion (i). Assertions (iii), (iv), and (v) follow immediately, in light of Lemma 2.4, (i), from the various definitions involved. □

DEFINITION 2.6.

(i) Let X, Y be scheme-like objects of $\mathcal{C}_K$. Then we shall say that a morphism $f\colon X \to Y$ in $\mathcal{C}_K$ is *Galois* if the finite separable extension K_X/K_Y determined by f [cf. Lemma 1.4, (ii); Lemma 2.4, (iii)] is Galois.

(ii) Let X be a scheme-like object of $\mathcal{C}_K$. Then we shall say that X is *Galois* if there exists a Galois morphism from X to a terminal object of $\mathcal{C}_K$ [cf. Lemma 2.4, (ii)].

(iii) We shall say that a projective system $(X_\lambda)_{\lambda\in\Lambda}$ consisting of objects and morphisms of $\mathcal{C}_K$ is *basepoint* of $\mathcal{C}_K$ if X_λ is Galois [hence also scheme-like] for each $\lambda \in \Lambda$, and, moreover, for each scheme-like object Y of $\mathcal{C}_K$, there exist an element $\lambda_Y \in \Lambda$ and a morphism $X_{\lambda_Y} \to Y$ in $\mathcal{C}_K$.

(iv) Let $\widetilde{X} = (X_\lambda)_{\lambda\in\Lambda}$ be a basepoint of $\mathcal{C}_K$. Then we shall write

$$K_{\widetilde{X}} \stackrel{\mathrm{def}}{=} \varinjlim_{\lambda\in\Lambda} K_{X_\lambda}$$

for the field obtained by forming the injective limit of the K_{X_λ}'s and

$$\Pi_{\widetilde{X}} \stackrel{\mathrm{def}}{=} \varprojlim_{\lambda\in\Lambda} \mathrm{Aut}(X_\lambda)$$

for the profinite [cf. Lemma 1.4, (ii); Lemma 2.4, (iii)] group obtained by forming the projective limit of the $\mathrm{Aut}(X_\lambda)$'s.

LEMMA 2.7. — *The following hold:*

(i) *There exists a* **basepoint** *of* $\mathcal{C}_K$.

(ii) *Let X be a* **Galois** *object of* $\mathcal{C}_K$. *Then* $\mathrm{Aut}(X)$ *is isomorphic to* $\mathrm{Gal}(K_X/K)$.

(iii) *Let $\widetilde{X}$ be a basepoint of $\mathcal{C}_K$. Then the field $K_{\widetilde{X}}$ is a* **separable closure** *of K. Moreover, the profinite group $\Pi_{\widetilde{X}}$ is* **isomorphic** *to the absolute Galois group* $\mathrm{Gal}(K_{\widetilde{X}}/K)$ *of K.*

PROOF. — These assertions follow, in light of Lemma 2.4, (iii), from Lemma 1.6. □

LEMMA 2.8. — *Let X, Y be* **scheme-like** *objects of $\mathcal{C}_K$; $f\colon X \to Y$ a morphism in $\mathcal{C}_K$. Then it holds that f is* **Galois** *if and only if, for each* **scheme-like** *object Z in $\mathcal{C}_K$ and each two morphisms g_1, $g_2\colon Z \to X$ in $\mathcal{C}_K$ such that $f \circ g_1 = f \circ g_2$, there exists an automorphism h of X over Y such that $g_2 = h \circ g_1$.*

PROOF. — This follows, in light of Lemma 2.4, (iii), from Lemma 1.7. □

LEMMA 2.9. — *Let X, Y be objects of $\mathcal{C}_K$; $f\colon X \to Y$ a* **rigidification** *in $\mathcal{C}_K$. Then the following hold:*

(i) *For each automorphism g of Y, there exists a* **unique automorphism** *$\widetilde{g}$ of X such that $f \circ \widetilde{g} = g \circ f$.*

(ii) *The assignment "$g \mapsto \widetilde{g}$" of* (i) *determines an exact sequence of groups*

$$1 \longrightarrow \mathrm{Aut}_{\mathrm{id}}(Y) \longrightarrow \mathrm{Aut}(Y) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1.$$

PROOF. — First, we verify assertion (i). Since X is *scheme-like*, the automorphism of S_X given by $f_S^{-1} \circ g_S \circ f_S$ determines an automorphism $\widetilde{g}$ of X such that $f \circ \widetilde{g} = g \circ f$. Moreover, the uniqueness of such a "$\widetilde{g}$" follows from the fact that a rigidification is a *monomorphism* [cf. Lemma 2.5, (ii)]. This completes the proof of assertion (i).

Finally, we verify assertion (ii). One verifies easily that, to verify assertion (ii), it suffices to verify the following two assertions:

(1) For each $g \in \mathrm{Aut}(Y)$, it holds that $\widetilde{g}$ is the identity automorphism of X if and only if g is a *scheme-identity*.

(2) For each $h \in \mathrm{Aut}(X)$, there exists $g \in \mathrm{Aut}(Y)$ such that $h = \widetilde{g}$.

On the other hand, assertion (1) follows from the description of "$\widetilde{g}$" given in the proof of assertion (i); assertion (2) is immediate. This completes the proof of Lemma 2.9. □

DEFINITION 2.10.

(i) Let X, Y be objects of $\mathcal{C}_K$; $f\colon X \to Y$ a rigidification in $\mathcal{C}_K$. Then it follows from Lemma 2.9, (ii), that we have an exact sequence of groups

$$1 \longrightarrow \mathrm{Aut}_{\mathrm{id}}(Y) \longrightarrow \mathrm{Aut}(Y) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1,$$

which thus determines an outer action of $\mathrm{Aut}(X)$ on $\mathrm{Aut}_{\mathrm{id}}(Y)$:

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Out}(\mathrm{Aut}_{\mathrm{id}}(Y)).$$

We shall write

$$\mathrm{Aut}(X)_f \stackrel{\mathrm{def}}{=} \mathrm{Ker}\big(\mathrm{Aut}(X) \to \mathrm{Out}(\mathrm{Aut}_{\mathrm{id}}(Y))\big) \subseteq \mathrm{Aut}(X)$$

for the kernel of this action.

(ii) Let X be an object of $\mathcal{C}_K$ and n a nonnegative integer. Then we shall say that X is n*-simple* if the $\mathcal{O}_{K_X}$-module $\mathcal{F}_X(S_X)$ is isomorphic to $\mathcal{O}_{K_X}/\mathfrak{m}_{K_X}^n$ [cf. Definition 1.1; Definition 2.2, (v)].

LEMMA 2.11. — *Let X be a* **scheme-like** *object of $\mathcal{C}_K$ and n a nonnegative integer. Then there exists a* **rigidification** *of an* $\boldsymbol{n}$**-simple** *object whose domain is X.*

PROOF. — This is immediate [cf. Definition 2.3, (b)]. □

LEMMA 2.12. — *Let X be an object of $\mathcal{C}_K$. Then the following hold:*

(i) *It holds that X is* **0-simple** *if and only if X is* **scheme-like**.

(ii) *Let n be a positive integer. Then it holds that X is* $\boldsymbol{n}$**-simple** *if and only if there exists a morphism $f\colon Y \to X$ in $\mathcal{C}_K$ which satisfies the following conditions:*

(1) *The object Y is* $\boldsymbol{(n-1)}$**-simple**.

(2) *The morphism f is a* **monomorphism** *but* **not an isomorphism**.

(3) *Let $g\colon Y \to Z$, $h\colon Z \to X$ be morphisms in $\mathcal{C}_K$ such that $f = h \circ g$. If both g and h are* **monomorphisms**, *then either g or h is an* **isomorphism**.

(4) *The group* $\mathrm{Aut}_{\mathrm{id}}(X)$ *is* **abelian**.

PROOF. — Assertion (i) is immediate. In the remainder of the proof, we verify assertion (ii). The *necessity* follows immediately from Lemma 2.5, (i) [cf. Definition 2.3, (a)]. To verify the *sufficiency*, suppose that there exists a morphism $f\colon Y \to X$ in $\mathcal{C}_K$ which satisfies conditions (1), (2), (3), and (4). Then it follows immediately, in light of Lemma 2.5, (i), from conditions (1), (2), and (3) that the $\mathcal{O}_{K_X}$-module $\mathcal{F}_X(S_X)$ is isomorphic to either $\mathcal{O}_{K_X}/\mathfrak{m}^n_{K_X}$ or $(\mathcal{O}_{K_X}/\mathfrak{m}^{n-1}_{K_X}) \oplus (\mathcal{O}_{K_X}/\mathfrak{m}_{K_X})$. Thus, it follows from condition (4) that the $\mathcal{O}_{K_X}$-module $\mathcal{F}_X(S_X)$ is isomorphic to $\mathcal{O}_{K_X}/\mathfrak{m}^n_{K_X}$, as desired. This completes the proof of the *sufficiency*. □

LEMMA 2.13. — *Let X, Y be objects of $\mathcal{C}_K$; $f\colon X \to Y$ a morphism in $\mathcal{C}_K$; n a nonnegative integer. Suppose that X is* **Galois**, *that Y is* $\boldsymbol{n}$**-simple**, *and that f is a* **rigidification**. *Then the subgroup* $\mathrm{Aut}(X)_f \subseteq \mathrm{Aut}(X)$ *corresponds, relative to the natural isomorphism of* $\mathrm{Aut}(X)$ *with* $\mathrm{Gal}(K_X/K)$ [*cf. Lemma* 2.7, (ii)], *to the kernel*

$$\mathrm{Ker}\bigl(\mathrm{Gal}(K_X/K) \to \mathrm{Aut}(\mathcal{O}_{K_X}/\mathfrak{m}^n_{K_X})\bigr)$$

of the natural action of $\mathrm{Gal}(K_X/K)$ *on* $\mathcal{O}_{K_X}/\mathfrak{m}^n_{K_X}$.

PROOF. — It follows immediately from the definition of an *n-simple* object that $\mathrm{Aut}_{\mathrm{id}}(Y)$ is naturally isomorphic to $(\mathcal{O}_{K_Y}/\mathfrak{m}^n_{K_Y})^\times$. Thus, the subgroup $\mathrm{Aut}(X)_f \subseteq \mathrm{Aut}(X)$ corresponds, relative to the natural isomorphism of $\mathrm{Aut}(X)$ with $\mathrm{Gal}(K_X/K)$, to the kernel

$$\mathrm{Ker}\bigl(\mathrm{Gal}(K_X/K) \to \mathrm{Aut}((\mathcal{O}_{K_X}/\mathfrak{m}^n_{K_X})^\times)\bigr).$$

In particular, Lemma 2.13 follows immediately from the [easily verified] fact that

$$\mathcal{O}_{K_X}/\mathfrak{m}^n_{K_X} \;=\; \mathfrak{m}_{K_X}/\mathfrak{m}^n_{K_X} \;\cup\; (\mathcal{O}_{K_X}/\mathfrak{m}^n_{K_X})^\times,$$

$$1 + (\mathfrak{m}_{K_X}/\mathfrak{m}^n_{K_X}) \;\subseteq\; (\mathcal{O}_{K_X}/\mathfrak{m}^n_{K_X})^\times.$$

This completes the proof of Lemma 2.13. □

Theorem 2.14. — *Let* $K_\circ$, $K_\bullet$ *be local fields;* $\mathcal{C}_{K_\circ}$, $\mathcal{C}_{K_\bullet}$ *full subcategories of* $\overline{\mathcal{C}}_{K_\circ}$, $\overline{\mathcal{C}}_{K_\bullet}$ [*cf. Definition* 2.1] *which satisfy the condition* ($\mathfrak{C}$) [*cf. Definition* 2.3], *respectively. Suppose that the category* $\mathcal{C}_{K_\circ}$ *is* **equivalent** *to the category* $\mathcal{C}_{K_\bullet}$. *Then the field* $K_\circ$ *is* **isomorphic** *to the field* $K_\bullet$.

Proof. — Suppose that there exists an equivalence of categories $\phi\colon \mathcal{C}_{K_\circ} \xrightarrow{\sim} \mathcal{C}_{K_\bullet}$. Let $X_\circ$, $Y_\circ$ be objects of $\mathcal{C}_{K_\circ}$; $f_\circ\colon X_\circ \to Y_\circ$ a morphism in $\mathcal{C}_{K_\circ}$. Write $X_\bullet$, $Y_\bullet$ for the objects of $\mathcal{C}_{K_\bullet}$ corresponding, via ϕ, to $X_\circ$, $Y_\circ$, respectively; $f_\bullet\colon X_\bullet \to Y_\bullet$ for the morphism in $\mathcal{C}_{K_\bullet}$ corresponding, via ϕ, to $f_\circ$. Then it follows from Lemma 2.5, (ii), that

(a) it holds that $f_\circ$ is a *rigidification* if and only if $f_\bullet$ is a *rigidification*.

Thus, it follows from Lemma 2.5, (iii), that

(b) it holds that $X_\circ$ is *scheme-like* if and only if $X_\bullet$ is *scheme-like*;

moreover, it follows from Lemma 2.5, (iv) (respectively, (v)), that

(c) it holds that $f_\circ$ is a *scheme-isomorphism* (respectively, *scheme-identity*) if and only if $f_\bullet$ is a *scheme-isomorphism* (respectively, *scheme-identity*).

In particular, it follows from Lemma 2.12 that, for each nonnegative integer n,

(d) it holds that $X_\circ$ is n*-simple* if and only if $X_\bullet$ is n*-simple*.

Next, let $\widetilde{X}_\circ = ((X_\circ)_\lambda)_{\lambda\in\Lambda}$ be a *basepoint* of $\mathcal{C}_{K_\circ}$ [cf. Lemma 2.7, (i)]. Then it follows from Lemma 2.8, together with (b), that the projective system $\widetilde{X}_\bullet = ((X_\bullet)_\lambda)_{\lambda\in\Lambda}$ consisting of objects and morphisms of $\mathcal{C}_{K_\bullet}$ corresponding, via ϕ, to $\widetilde{X}_\circ$ is a *basepoint* of $\mathcal{C}_{K_\bullet}$. Thus, the equivalence ϕ determines an isomorphism of profinite groups

$$\Pi_\phi\colon\ \Pi_{\widetilde{X}_\circ} = \varprojlim_{\lambda\in\Lambda} \mathrm{Aut}((X_\circ)_\lambda) \xrightarrow{\ \sim\ } \Pi_{\widetilde{X}_\bullet} = \varprojlim_{\lambda\in\Lambda} \mathrm{Aut}((X_\bullet)_\lambda).$$

In particular, if either $K_\circ$ or $K_\bullet$ is *of positive characteristic*, then it follows, in light of Lemma 2.7, (iii), from Theorem 1.10 and Corollary 1.12, (ii), that $K_\circ$ is *isomorphic* to $K_\bullet$, as desired. In the remainder of the proof,

suppose that both $K_\circ$ and $K_\bullet$ are *of characteristic zero*.

Next, let λ be an element of Λ, n a nonnegative integer, and $(f_\circ)_\lambda\colon (X_\circ)_\lambda \to (Y_\circ)_\lambda$ a *rigidification* of an n*-simple* object $(Y_\circ)_\lambda$ whose domain is the member $(X_\circ)_\lambda$ of $\widetilde{X}_\circ$ [cf. Lemma 2.11]. Write

$$\Pi_{\phi,\lambda}\colon\ \mathrm{Aut}((X_\circ)_\lambda) \xrightarrow{\ \sim\ } \mathrm{Aut}((X_\bullet)_\lambda)$$

for the isomorphism induced by Π_ϕ and $(f_\bullet)_\lambda\colon (X_\bullet)_\lambda \to (Y_\bullet)_\lambda$ for the *rigidification* [cf. (a)] of the n*-simple* object $(Y_\bullet)_\lambda$ [cf. (d)] corresponding, via ϕ, to $(f_\circ)_\lambda\colon (X_\circ)_\lambda \to (Y_\circ)_\lambda$. Then it follows from (c) that the isomorphism $\Pi_{\phi,\lambda}$ restricts to an isomorphism of subgroups

$$\mathrm{Aut}((X_\circ)_\lambda)_{(f_\circ)_\lambda} \xrightarrow{\ \sim\ } \mathrm{Aut}((X_\bullet)_\lambda)_{(f_\bullet)_\lambda}.$$

Thus, it follows from Lemma 2.13 that the isomorphism $\Pi_{\phi,\lambda}$ is *compatible* — relative to the natural identifications [cf. Lemma 2.7, (ii)] of $\mathrm{Aut}((X_\circ)_\lambda)$, $\mathrm{Aut}((X_\bullet)_\lambda)$ with $\mathrm{Gal}(K_{(X_\circ)_\lambda}/K_\circ)$, $\mathrm{Gal}(K_{(X_\bullet)_\lambda}/K_\bullet)$, respectively — with the respective filtrations of *higher ramification subgroups in the lower numbering*, hence also [cf., e.g., [3], §4.1] *in the upper numbering*. In particular, the isomorphism Π_ϕ is *compatible* — relative to the natural identifications [cf. Lemma 2.7, (iii)] of $\Pi_{\widetilde{X}_\circ}$, $\Pi_{\widetilde{X}_\bullet}$ with $\mathrm{Gal}(K_{\widetilde{X}_\circ}/K_\circ)$, $\mathrm{Gal}(K_{\widetilde{X}_\bullet}/K_\bullet)$, respectively — with the respective *filtrations of higher ramification subgroups in the upper numbering*. Thus, it follows from [2], Theorem, that $K_\circ$ is *isomorphic* to $K_\bullet$, as desired. This completes the proof of Theorem 2.14. $\square$

3. Category of Finite Schemes

In the present §3, let us discuss a certain full subcategory of the category of irreducible schemes which are finite over the spectrum of the ring of integers of a local field [cf. Definition 3.1; Definition 3.4]. In the present §3, let K be a *local field*, i.e., a field which is isomorphic to a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime number p.

Definition 3.1. — We shall write $\overline{\mathcal{F}}_K$ for the category defined as follows:

- An object of $\overline{\mathcal{F}}_K$ is a pair (S, ϕ) consisting of a(n) [nonempty] irreducible scheme S and a finite morphism $\phi\colon S \to \mathrm{Spec}(\mathcal{O}_K)$ of schemes. To simplify the exposition, we shall often refer to S [i.e., just the domain of the morphism ϕ] as an "object of $\overline{\mathcal{F}}_K$".
- Let S, T be objects of $\overline{\mathcal{F}}_K$. Then a morphism $S \to T$ in $\overline{\mathcal{F}}_K$ is defined as a morphism of schemes from S to T lying over $\mathcal{O}_K$.

Lemma 3.2. — *The following hold:*

(i) *Every object of $\overline{\mathcal{F}}_K$ is isomorphic to the spectrum of a* **noetherian complete local ring of dimension zero or one**.

(ii) *Every object of $\overline{\mathcal{F}}_K$ is* **of cardinality one or two**.

(iii) *Every morphism in $\overline{\mathcal{F}}_K$ is* **injective**. *In particular, if the domain* (*respectively, codomain*) *of a morphism in $\overline{\mathcal{F}}_K$ is of cardinality two* (*respectively, one*)*, then the morphism is* **bijective**.

Proof. — First, we verify assertion (i). Let S be an object of $\overline{\mathcal{F}}_K$. Let us first observe that since S is *finite* over $\mathcal{O}_K$, the scheme S is isomorphic to the spectrum of a *finite*, hence also *noetherian*, $\mathcal{O}_K$-algebra A. Thus, since A is *finite* over the *complete* [hence also *henselian*] local ring $\mathcal{O}_K$, and S is *irreducible*, it holds that A is a *complete local ring*. Finally, since A is *finite* over the local ring $\mathcal{O}_K$ *of dimension one*, it holds that A is *of dimension zero or one*. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let S be an object of $\overline{\mathcal{F}}_K$. Then since the scheme S is *irreducible* and *finite* over the *complete* [hence also *henselian*] local ring $\mathcal{O}_K$, the fiber of the structure morphism $S \to \mathrm{Spec}(\mathcal{O}_K)$ at the [uniquely determined] closed (respectively, generic) point of $\mathrm{Spec}(\mathcal{O}_K)$ is *of cardinality one* (respectively, *of cardinality zero or one*). In particular, the scheme S is *of cardinality one or two*. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Let us first observe that it is immediate that, to verify assertion (iii), it suffices to verify that the structure morphism "ϕ" of each object "(S, ϕ)" of $\overline{\mathcal{F}}_K$ is *injective*. On the other hand, this *injectivity* follows from the proof of assertion (ii). This completes the proof of assertion (iii). $\square$

Definition 3.3. — Let S be an object of $\overline{\mathcal{F}}_K$.

(i) We shall say that S is *point-like* if S is of cardinality one, or, alternatively, is of dimension zero; we shall say that S is *non-point-like* if S is not of cardinality one [i.e., is of cardinality two, or, alternatively, is of dimension one — cf. Lemma 3.2, (i), (ii)].

(ii) We shall say that S is a *trait* (respectively, *quasi-trait*) if S is normal (respectively, integral) and non-point-like.

(iii) Suppose that S is a quasi-trait. Then we shall write K_S for the function field of S.

Remark 3.3.1. — One verifies easily from Lemma 3.2, (i), that it holds that an object of $\overline{\mathcal{F}}_K$ is a *trait* if and only if the object is isomorphic to the spectrum of a *complete discrete valuation ring.*

Definition 3.4. — Let $\mathcal{F}_K$ be a full subcategory of $\overline{\mathcal{F}}_K$. Then we shall say that $\mathcal{F}_K$ satisfies the condition ($\mathfrak{F}$) if

(a) the full subcategory $\mathcal{F}_K$ contains the object $(\mathrm{Spec}(\mathcal{O}_K), \mathrm{id}_{\mathrm{Spec}(\mathcal{O}_K)})$,

(b) the full subcategory $\mathcal{F}_K$ is closed under the operation of taking *normalizations* of quasi-traits, i.e., if S is a quasi-trait of $\mathcal{F}_K$, then the trait of $\overline{\mathcal{F}}_K$ obtained by forming the normalization of S is an object of $\mathcal{F}_K$,

(c) the full subcategory $\mathcal{F}_K$ is closed under the operation of taking *finite separable extensions and subfields* of the function fields of quasi-traits, i.e., if S is a quasi-trait of $\mathcal{F}_K$, and L is a finite separable extension of K_S (respectively, an intermediate extension of K_S/K), then there exist a quasi-trait T of $\mathcal{F}_K$ and a morphism $T \to S$ (respectively, $S \to T$) in $\mathcal{F}_K$ such that K_T is isomorphic, over K_S (respectively, as an intermediate extension of K_S/K), to L, and

(d) the full subcategory $\mathcal{F}_K$ is closed under the operation of taking *closed subschemes*, i.e., if S is an object of $\mathcal{F}_K$, then every closed immersion in $\overline{\mathcal{F}}_K$ whose codomain is S is a morphism in $\mathcal{F}_K$.

In the remainder of the present §3, let $\mathcal{F}_K$ be a full subcategory of $\overline{\mathcal{F}}_K$ which satisfies the condition ($\mathfrak{F}$).

Lemma 3.5. — *The following hold:*

(i) *A* **terminal** *object of $\mathcal{F}_K$ is given by the pair* $(\mathrm{Spec}(\mathcal{O}_K), \mathrm{id}_{\mathrm{Spec}(\mathcal{O}_K)})$. *Moreover, every terminal object of $\mathcal{F}_K$ is a* **trait**.

(ii) *The assignment "$S \mapsto K_S$" determines a* **faithful** *functor from the full subcategory of $\mathcal{F}_K$ consisting of* **quasi-traits** *of $\mathcal{F}_K$ to the category defined as follows:*

- *An object of the category is a finite extension of K.*
- *A morphism in the category is a homomorphism of fields over K.*

(iii) *Let S be a* **trait** *of $\mathcal{F}_K$, T a* **quasi-trait** *of $\mathcal{F}_K$, and $i\colon K_T \hookrightarrow K_S$ a homomorphism of fields over K. Then there exists a* **unique** *morphism $S \to T$ in $\mathcal{F}_K$ which induces, via the functor of* (ii)*, the homomorphism i.*

(iv) *The restriction of the functor of* (ii) *to the full subcategory of $\mathcal{F}_K$ consisting of* **traits** *of $\mathcal{F}_K$ is* **full***.*

(v) *There exists a(n)* [*tautological*] **equivalence** *of categories of $\mathcal{B}_K$* [*cf. Definition* 1.2] *with the full subcategory of $\mathcal{F}_K$ consisting of* **traits** *of $\mathcal{F}_K$ which are* **generically étale** *over $\mathcal{O}_K$.*

PROOF. — Assertions (i), (ii), and (iii) follow immediately from the definition of the category $\mathcal{F}_K$ [cf. Definition 3.4, (a)]. Assertion (iv) follows from assertion (iii). Assertion (v) follows from the definition of the category $\mathcal{F}_K$ [cf. Definition 3.4, (a), (b), (c)]. $\square$

DEFINITION 3.6. — We shall say that $\mathcal{F}_K$ is *separable* if the essential image of the functor of Lemma 3.5, (ii), consists of finite separable extensions of K.

DEFINITION 3.7.

(i) Let S, T be traits of $\mathcal{F}_K$. Then we shall say that a morphism $f\colon S \to T$ in $\mathcal{F}_K$ is *Galois* if the finite extension K_S/K_T determined by f [cf. Lemma 3.5, (ii)] is Galois.

(ii) Let S be a trait of $\mathcal{F}_K$. Then we shall say that S is *Galois* if there exists a Galois morphism from S to a terminal object of $\mathcal{F}_K$ [cf. Lemma 3.5, (i)].

(iii) We shall say that a projective system $(S_\lambda)_{\lambda\in\Lambda}$ consisting of objects and morphisms of $\mathcal{F}_K$ is *basepoint* of $\mathcal{F}_K$ if S_λ is Galois [hence also a trait which is generically étale over $\mathcal{O}_K$] for each $\lambda \in \Lambda$, and, moreover, for each trait T of $\mathcal{F}_K$ which is generically étale over $\mathcal{O}_K$, there exist an element $\lambda_T \in \Lambda$ and a morphism $S_{\lambda_T} \to T$ in $\mathcal{F}_K$.

(iv) Let $\widetilde{S} = (S_\lambda)_{\lambda\in\Lambda}$ be a basepoint of $\mathcal{F}_K$. Then we shall write

$$K_{\widetilde{S}} \overset{\mathrm{def}}{=} \varinjlim_{\lambda\in\Lambda} K_{S_\lambda}$$

for the field obtained by forming the injective limit of the K_{S_λ}'s and

$$\Pi_{\widetilde{S}} \overset{\mathrm{def}}{=} \varprojlim_{\lambda\in\Lambda} \mathrm{Aut}(S_\lambda)$$

for the profinite [cf. Lemma 1.4, (ii); Lemma 3.5, (v)] group obtained by forming the projective limit of the $\mathrm{Aut}(S_\lambda)$'s.

LEMMA 3.8. — *The following hold:*

(i) *There exists a* **basepoint** *of $\mathcal{F}_K$.*

(ii) *Let S be a* **Galois** *object of $\mathcal{F}_K$. Then* $\mathrm{Aut}(S)$ *is isomorphic to* $\mathrm{Gal}(K_S/K)$*.*

(iii) *Let $\widetilde{S}$ be a basepoint of $\mathcal{F}_K$. Then the field $K_{\widetilde{S}}$ is a* **separable closure** *of K. Moreover, the profinite group $\Pi_{\widetilde{S}}$ is* **isomorphic** *to the absolute Galois group* $\mathrm{Gal}(K_{\widetilde{S}}/K)$ *of K.*

Proof. — These assertions follow, in light of Lemma 3.5, (v), from Lemma 1.6. □

Lemma 3.9. — *Let S, T be objects of $\mathcal{F}_K$; $f\colon S \to T$ a morphism in $\mathcal{F}_K$. Then the following hold:*

(i) *It holds that f is a* **monomorphism** *[i.e., in $\mathcal{F}_K$] if and only if f is a* **closed immersion**.

(ii) *It holds that S is* **point-like** *if and only if there exists a morphism $S \to O$, where O is a* **terminal** *object of $\mathcal{F}_K$ [cf. Lemma 3.5, (i)], which satisfies the following condition: The morphism $S \to O$* **factors** *through a* **closed immersion** *$U \to O$ in $\mathcal{F}_K$ which is* **not an isomorphism**.

(iii) *It holds that S is* **non-point-like** *if and only if S is* **not point-like**.

(iv) *It holds that S is* **integral** *and* **point-like** *if and only if there exists a* **closed immersion** *$S \to U$ in $\mathcal{F}_K$ which is an* **initial** *object among closed immersions whose codomains are U.*

(v) *Suppose that S is* **non-point-like**. *Then it holds that S is a* **quasi-trait** *if and only if there exists a* **closed immersion** *$S \to U$ in $\mathcal{F}_K$ which is an* **initial** *object among closed immersions whose codomains are U and whose domains are* **non-point-like**.

(vi) *Suppose that S is a* **quasi-trait**. *Then it holds that S is a* **trait** *if and only if there exists a* **birational** *morphism $S \to U$ in $\mathcal{F}_K$ which is an* **initial** *object among birational morphisms whose codomains are U and whose domains are* **quasi-traits** *of $\mathcal{F}_K$.*

Proof. — First, we verify assertion (i). The *sufficiency* is immediate. To verify the *necessity*, suppose that f is a *monomorphism*. Write $A_S \stackrel{\text{def}}{=} \mathcal{O}_S(S)$ and $A_T \stackrel{\text{def}}{=} \mathcal{O}_T(T)$. Then since [one verifies easily that] the homomorphism $A_T \to A_S$ determined by f is *finite*, to verify that f is a *closed immersion*, we may assume without loss of generalities, by replacing A_T by the residue field [cf. Lemma 3.2, (i)], that A_T is a [necessarily *finite*, hence also *perfect*] *field* [cf. Definition 3.4, (d)].

Write $\underline{A}_S$ for the residue field of A_S. Now assume that the composite $A_T \to A_S \twoheadrightarrow \underline{A}_S$ is *not an isomorphism*. Then it follows from elementary field theory that there exist a finite extension M of $\underline{A}_S$ and two inclusions $i_1, i_2 \colon \underline{A}_S \hookrightarrow M$ such that $i_1 \neq i_2$ but $i_1|_{A_T} = i_2|_{A_T}$. In particular, since f is a *monomorphism* [which thus implies that the morphism in $\mathcal{F}_K$ from the spectrum of $\underline{A}_S$ to T determined by the composite $A_T \to A_S \twoheadrightarrow \underline{A}_S$ is a *monomorphism*], we obtain a *contradiction* [cf. Definition 3.4, (c), (d)]. Thus, the composite $A_T \to A_S \twoheadrightarrow \underline{A}_S$ is an *isomorphism*. In particular, we conclude that the morphism $f\colon S \to T$ has a *splitting*, i.e., a morphism $s\colon T \to S$ such that $f \circ s = \mathrm{id}_T$.

Now we have the identity automorphism id_S of S and the composite $S \xrightarrow{f} T \xrightarrow{s} S$. Since f is a *monomorphism*, we conclude that $\mathrm{id}_S = s \circ f$, i.e., that f is a *closed immersion*. This completes the proof of the *necessity*, hence also of assertion (i).

Assertion (ii) follows immediately from Lemma 3.2, (iii); Lemma 3.5, (i) [cf. Definition 3.4, (d)]. Assertion (iii) is immediate.

Next, we verify assertion (iv). The *necessity* follows from the observation that if S is *integral* and *point-like*, then the identity automorphism of S satisfies the condition in the statement of assertion (iv). Next, to verify the *sufficiency*, suppose that there exists a closed immersion $S \to U$ in $\mathcal{F}_K$ that satisfies the condition in the statement of assertion (iv). Write $T \to U$ for the closed immersion determined by the residue field [cf. Lemma 3.2, (i)] of $\mathcal{O}_U(U)$ [cf. Definition 3.4, (d)]. Then it follows from our assumption that the closed immersion $S \to U$ *factors* through the closed immersion $T \to U$, which thus implies that we obtain a closed immersion $S \to T$. Now observe that since T is the spectrum of a field, the closed immersion $S \to T$ is an *isomorphism*, which thus implies that S is *integral* and *point-like*, as desired. This completes the proof of the *sufficiency*, hence also of assertion (iv).

Next, we verify assertion (v). The *necessity* follows from the observation that if S is a *quasi-trait*, then the identity automorphism of S satisfies the condition in the statement of assertion (v). Next, to verify the *sufficiency*, suppose that there exists a closed immersion $S \to U$ in $\mathcal{F}_K$ that satisfies the condition in the statement of assertion (v). Write $T \to U$ for the closed immersion defined by the ideal of $\mathcal{O}_U(U)$ of nilpotent elements [cf. Definition 3.4, (d)]. Note that since U is *non-point-like* [cf. Lemma 3.2, (iii)], and the closed immersion $T \to U$ is *bijective* [cf. Lemma 3.2, (iii)], it follows that T is *non-point-like*, hence also a *quasi-trait*. Thus, it follows from our assumption that the closed immersion $S \to U$ *factors* through the closed immersion $T \to U$, which thus implies that we obtain a closed immersion $S \to T$. Now observe that since T is a *quasi-trait*, the [necessarily *bijective* — cf. Lemma 3.2, (iii)] closed immersion $S \to T$ is an *isomorphism*, which thus implies that S is a *quasi-trait*, as desired. This completes the proof of the *sufficiency*, hence also of assertion (v).

Finally, we verify assertion (vi). The *necessity* follows from the observation that if S is a *trait*, then, by the *Zariski main theorem*, the identity automorphism of S satisfies the condition in the statement of assertion (vi). Next, to verify the *sufficiency*, suppose that there exists a birational morphism $S \to U$ in $\mathcal{F}_K$ that satisfies the condition in the statement of assertion (vi). Write $T \to U$ for the normalization of U [cf. Definition 3.4, (b)]. Then it follows from our assumption that the birational morphism $S \to U$ *factors* through the birational morphism $T \to U$, which thus implies that we obtain a birational morphism $S \to T$. Now observe that since T is a *trait*, it follows from the *Zariski main theorem* that the birational morphism $S \to T$ is an *isomorphism*, which thus implies that S is a *trait*, as desired. This completes the proof of the *sufficiency*, hence also of assertion (vi). $\square$

DEFINITION 3.10. — Let S, T be quasi-traits of $\mathcal{F}_K$; $f\colon S \to T$ a morphism in $\mathcal{F}_K$.

(i) We shall say that f is *purely inseparable* (respectively, *quasi-Galois*) if the finite extension K_S/K_T determined by f [cf. Lemma 3.5, (ii)] is purely inseparable (respectively, quasi-Galois, or, alternatively, normal, i.e., K_S is Galois over the purely inseparable closure of K_T in K_S).

(ii) Suppose that f is quasi-Galois. Then we shall write $\mathrm{qGal}(f) \stackrel{\mathrm{def}}{=} \mathrm{Gal}(K_S/L)$ $(= \mathrm{Aut}_{K_T}(K_S))$, where we write $L \subseteq K_S$ for the purely inseparable closure of K_T in K_S [which thus implies that the finite extension K_S/L is Galois].

Lemma 3.11. — *Let S, T be* **quasi-traits** *of $\mathcal{F}_K$; $f: S \to T$ a morphism in $\mathcal{F}_K$. Then the following hold:*

(i) *It holds that f is either* **birational** *or* **purely inseparable** *if and only if the following condition is satisfied: For each* **quasi-trait** *U of $\mathcal{F}_K$ and each two morphisms $g_1, g_2: U \to S$ in $\mathcal{F}_K$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.*

(ii) *Consider the following conditions:*

(1) *For each* **quasi-trait** *U of $\mathcal{F}_K$ and each two morphisms $g: S \to U$, $h: U \to T$ such that $f = h \circ g$, if every automorphism of S over T is an automorphism* **over $\boldsymbol{U}$** *[i.e., relative to g], then h is either* **birational** *or* **purely inseparable***.*

(2) *The morphism f is* **quasi-Galois***.*

Then (1) *implies* (2)*. If, moreover, S is a* **trait***, then* (1) *is equivalent to* (2)*.*

Proof. — First, we verify assertion (i). The *necessity* follows, in light of Lemma 3.5, (ii), from elementary field theory. Next, we verify the *sufficiency*. Suppose that f is *neither birational nor purely inseparable*. Then it follows from elementary field theory that there exist a finite separable extension L of K_S and two inclusions $i_1, i_2: K_S \hookrightarrow L$ such that $i_1 \neq i_2$ but $i_1|_{K_T} = i_2|_{K_T}$. Thus, by considering suitable two morphisms from a trait whose generic point is isomorphic to the spectrum of L [cf. Definition 3.4, (b), (c)] to S, we conclude from Lemma 3.5, (ii), that f does *not satisfy* the condition in the statement of assertion (i). This completes the proof of the *sufficiency*, hence also of assertion (i).

Finally, we verify assertion (ii). Let us first observe that if condition (1) is satisfied, then it follows immediately from Lemma 3.5, (ii), that the intermediate extension of K_S/K_T consisting of $\mathrm{Aut}_T(S)$-invariants in K_S is [either the trivial extension or] a *purely inseparable* extension of K_T. Thus, the implication (1) $\Rightarrow$ (2) follows from Lemma 3.5, (ii), together with elementary field theory. The implication (2) $\Rightarrow$ (1) in the case where S is a *trait* follows immediately, in light of Lemma 3.5, (ii), (iv), from elementary field theory. This completes the proof of assertion (ii). $\square$

Lemma 3.12. — *Let S, T be* **quasi-traits** *of $\mathcal{F}_K$; $f: S \to T$ a* **quasi-Galois** *morphism in $\mathcal{F}_K$. Then the following hold:*

(i) *For each* **quasi-trait** *U of $\mathcal{F}_K$ and each morphism $g: U \to S$ in $\mathcal{F}_K$ which is either* **birational** *or* **purely inseparable***, it holds that $\mathrm{Aut}_T(U)$ is* **isomorphic** *to a subgroup of $\mathrm{qGal}(f)$.*

(ii) *There exist a* **quasi-trait** *U of $\mathcal{F}_K$ and a* **birational** *morphism $g: U \to S$ in $\mathcal{F}_K$ such that $\mathrm{Aut}_T(U)$ is* **isomorphic** *to $\mathrm{qGal}(f)$.*

Proof. — Assertion (i) follows from Lemma 3.5, (ii), together with elementary field theory. Assertion (ii) follows from Lemma 3.5, (ii), (iv) [cf. Definition 3.4, (b)]. $\square$

Lemma 3.13. — *Let O be a* **terminal** *object of $\mathcal{F}_K$ [cf. Lemma 3.5, (i)]. Then the following hold:*

(i) *Consider the following conditions:*

(i-1) *The category* $\mathcal{F}_K$ *is* **separable**.

(i-2) *For each* **quasi-trait** S *of* $\mathcal{F}_K$*, there exists a morphism in* $\mathcal{F}_K$ *whose codomain is* S *and whose domain is* **Galois**.

(i-3) *For each* **quasi-trait** S *of* $\mathcal{F}_K$*, there exist a morphism from a* **quasi-trait** T *to* S *and a* **quasi-Galois** *morphism* $T \to O$ *in* $\mathcal{F}_K$.

(i-4) *For each* **quasi-trait** S *of* $\mathcal{F}_K$ *and each morphism* $f \colon S \to O$ *in* $\mathcal{F}_K$*, if* f *is either* **birational** *or* **purely inseparable***, then* f *is an* **isomorphism**.

Then the following equivalences hold:

$$\text{(i-1)} \iff \text{(i-2)} + \text{(i-4)} \iff \text{(i-3)} + \text{(i-4)}.$$

(ii) *Let* p *be a prime number. Then the following conditions are equivalent:*

(ii-1) *It holds that* $\underline{K}$ *is* **of characteristic** $\boldsymbol{p}$.

(ii-2) *There exists a finite subquotient of the absolute Galois group of* K *which is* **isomorphic** *to* $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(ii-3) *There exist* **traits** S, T *of* $\mathcal{F}_K$ *and a* **Galois** *morphism* $S \to T$ *in* $\mathcal{F}_K$ *such that* $\mathrm{Aut}_T(S)$ *is* **isomorphic** *to* $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(ii-4) *There exist* **quasi-traits** S, T *of* $\mathcal{F}_K$ *and a* **quasi-Galois** *morphism* $f \colon S \to T$ *in* $\mathcal{F}_K$ *such that* $\mathrm{qGal}(f)$ *is* **isomorphic** *to* $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(iii) *Let* $q_\times$ *be a positive integer. Then the following conditions are equivalent:*

(iii-1) *It holds that* $\sharp\underline{K}^\times = q_\times$*, i.e., that* $\sharp\underline{K} = q_\times + 1$.

(iii-2) *The positive integer* $q_\times$ *is the maximum positive integer such that* $q_\times$ *is* **not divisible** *by the characteristic of* $\underline{K}$*, and, moreover, there exists a finite quotient of the absolute Galois group which is* **isomorphic** *to* $\mathbb{Z}/q_\times\mathbb{Z} \times \mathbb{Z}/q_\times\mathbb{Z}$.

(iii-3) *The positive integer* $q_\times$ *is the maximum positive integer such that* $q_\times$ *is* **not divisible** *by the characteristic of* $\underline{K}$*, and, moreover, there exists a* **Galois** *object of* $\mathcal{F}_K$ *whose automorphism group is* **isomorphic** *to* $\mathbb{Z}/q_\times\mathbb{Z} \times \mathbb{Z}/q_\times\mathbb{Z}$.

(iii-4) *The positive integer* $q_\times$ *is the maximum positive integer such that* $q_\times$ *is* **not divisible** *by the characteristic of* $\underline{K}$*, and, moreover, there exist a* **quasi-trait** S *of* $\mathcal{F}_K$ *and a* **quasi-Galois** *morphism* $f \colon S \to O$ *in* $\mathcal{F}_K$ *such that* $\mathrm{qGal}(f)$ *is* **isomorphic** *to* $\mathbb{Z}/q_\times\mathbb{Z} \times \mathbb{Z}/q_\times\mathbb{Z}$.

PROOF. — First, we verify assertion (i). The implication (i-1) $\Rightarrow$ (i-2) follows immediately from the definition of the category $\mathcal{F}_K$ [cf. Definition 3.4, (b), (c)]. The implication (i-2) $\Rightarrow$ (i-3) is immediate. Next, we verify the implication (i-1) $\Rightarrow$ (i-4). Let us first observe that it follows from Lemma 3.5, (i), that O is a *trait*. Thus, it follows from Lemma 3.9, (vi), that the identity automorphism of O is an *initial* object among birational morphisms whose codomains are O and whose domains are *quasi-traits* of $\mathcal{F}_K$. On the other hand, it follows from (i-1) that f is *birational*. Thus, the morphism f is an *isomorphism*, as desired. This completes the proof of the implication (i-1) $\Rightarrow$ (i-4).

Thus, to complete the verification of assertion (i), it suffices to verify that if $\mathcal{F}_K$ *satisfies* condition (i-3) but does *not satisfy* condition (i-1), then $\mathcal{F}_K$ does *not satisfy* condition (i-4). On the other hand, this follows immediately from the definition of the category

$\mathcal{F}_K$, together with elementary field theory [cf. Definition 3.4, (c)]. This completes the proof of assertion (i).

Next, we verify assertions (ii), (iii). First, we verify the equivalences (ii-1) $\Leftrightarrow$ (ii-2) and (iii-1) $\Leftrightarrow$ (iii-2). Write G for the absolute Galois group of K and $p_{\underline{K}}$ for the characteristic of $\underline{K}$. Then it follows from *local class field theory* [cf., e.g., [3], §2], together with the well-known structure of the multiplicative group $K^\times$, that there exist a *cyclic* $p_{\underline{K}}$-group M_{cyc} and a *free* $\mathbb{Z}_{p_{\underline{K}}}$-module M_{free} of rank $[K : \mathbb{Q}_{p_{\underline{K}}}]$ (respectively, of infinite rank) if K is of characteristic zero (respectively, of positive characteristic) such that the abelianization of G [i.e., as a profinite group] is *isomorphic* to the profinite module $\underline{K}^\times \times M_{\mathrm{cyc}} \times M_{\mathrm{free}} \times \widehat{\mathbb{Z}}$. Thus, the equivalences (ii-1) $\Leftrightarrow$ (ii-2) and (iii-1) $\Leftrightarrow$ (iii-2) hold, as desired.

Moreover, the equivalences (ii-2) $\Leftrightarrow$ (ii-3) $\Leftrightarrow$ (ii-4) and (iii-2) $\Leftrightarrow$ (iii-3) $\Leftrightarrow$ (iii-4) follow immediately from Lemma 3.5, (iv), together with elementary field theory [cf. Definition 3.4, (b), (c)]. This completes the proofs of assertions (ii), (iii). ☐

Definition 3.14. — Let S, T be objects of $\mathcal{F}_K$; $f\colon S \to T$ a morphism in $\mathcal{F}_K$; n a positive integer. Then we shall say that f is *n-simple* if T is Galois [hence also a trait which is generically étale over $\mathcal{O}_K$], f is a closed immersion, and, moreover, the object $(T, f_*\mathcal{O}_S)$ of $\overline{\mathcal{C}}_K$ [cf. Definition 2.1; Lemma 3.5, (v)] is n-simple in the sense of Definition 2.10, (ii), i.e., and, moreover, the $\mathcal{O}_{K_T}$-module $\mathcal{O}_S(S)$ is isomorphic to $\mathcal{O}_{K_T}/\mathfrak{m}_{K_T}^n$ [cf. Definition 1.1; Definition 3.3, (iii)].

Lemma 3.15. — *Let S, T be objects of $\mathcal{F}_K$; $f\colon S \to T$ a morphism in $\mathcal{F}_K$; n a positive integer. Suppose that f is* **n-simple**. *Then, for each automorphism g of T, there exists a* **unique automorphism** *$\widetilde{g}$ of S such that $f \circ \widetilde{g} = g \circ f$. Moreover, the assignment "$g \mapsto \widetilde{g}$" determines a homomorphism of groups*

$$\mathrm{Aut}(T) \longrightarrow \mathrm{Aut}(S).$$

Proof. — The existence of such a "$\widetilde{g}$" is immediate from the definition of an n-simple morphism. Moreover, the uniqueness of such a "$\widetilde{g}$" follows from the fact that an n-simple morphism is a *monomorphism* [cf. Lemma 3.9, (i)]. Finally, the final assertion is immediate. This completes the proof of Lemma 3.15. ☐

Definition 3.16. — Let S, T be objects of $\mathcal{F}_K$; $f\colon S \to T$ a morphism in $\mathcal{F}_K$; n a positive integer. Suppose that f is *n-simple*. Then it follows from Lemma 3.15 that we have a homomorphism of groups

$$\mathrm{Aut}(T) \longrightarrow \mathrm{Aut}(S).$$

We shall write

$$\mathrm{Aut}(T)_f \stackrel{\mathrm{def}}{=} \mathrm{Ker}\big(\mathrm{Aut}(T) \to \mathrm{Aut}(S)\big) \subseteq \mathrm{Aut}(T)$$

for the kernel of this homomorphism.

Lemma 3.17. — *Let S be a* **Galois** *object of $\mathcal{F}_K$ and n a positive integer. Then there exists an* **n-simple** *morphism in $\mathcal{F}_K$ whose codomain is S.*

PROOF. — This is immediate [cf. Definition 3.4, (d)]. □

LEMMA 3.18. — *Let S, T be objects of $\mathcal{F}_K$; $f\colon S \to T$ a morphism in $\mathcal{F}_K$. Suppose that T is* **Galois**, *and that f is a* **closed immersion**. *Then the following hold:*

(i) *It holds that f is* **1-simple** *if and only if S is* **integral** *and* **point-like**.

(ii) *Let $n \geq 2$ be an integer. Then it holds that f is* **n-simple** *if and only if there exists a closed immersion $g\colon U \to S$ in $\mathcal{F}_K$ which satisfies the following conditions:*

(1) *The composite $f \circ g\colon U \to T$ is* **$(n-1)$-simple**.

(2) *The morphism g is* **not an isomorphism**.

(3) *Let $h\colon U \to V$, $i\colon V \to S$ be morphisms in $\mathcal{F}_K$ such that $g = i \circ h$. If both h and i are* **closed immersions**, *then either h or i is an* **isomorphism**.

PROOF. — This is immediate [cf. Definition 3.4, (d)]. □

LEMMA 3.19. — *Let S, T be objects of $\mathcal{F}_K$; $f\colon S \to T$ a morphism in $\mathcal{F}_K$; n a positive integer. Suppose that f is* **n-simple**. *Then the subgroup* $\mathrm{Aut}(T)_f \subseteq \mathrm{Aut}(T)$ *corresponds, relative to the natural isomorphism of* $\mathrm{Aut}(T)$ *with* $\mathrm{Gal}(K_T/K)$ [*cf. Lemma* 3.8, (ii)], *to the kernel*

$$\mathrm{Ker}\big(\mathrm{Gal}(K_T/K) \to \mathrm{Aut}(\mathcal{O}_{K_T}/\mathfrak{m}_{K_T}^n)\big)$$

of the natural action of $\mathrm{Gal}(K_T/K)$ *on* $\mathcal{O}_{K_T}/\mathfrak{m}_{K_T}^n$.

PROOF. — This is immediate. □

THEOREM 3.20. — *Let $K_\circ$, $K_\bullet$ be local fields; $\mathcal{F}_{K_\circ}$, $\mathcal{F}_{K_\bullet}$ full subcategories of $\overline{\mathcal{F}}_{K_\circ}$, $\overline{\mathcal{F}}_{K_\bullet}$* [*cf. Definition* 3.1] *which satisfy the condition* $(\mathfrak{F})$ [*cf. Definition* 3.4], *respectively. Suppose that the category $\mathcal{F}_{K_\circ}$ is* **equivalent** *to the category $\mathcal{F}_{K_\bullet}$. Then the field $K_\circ$ is* **isomorphic** *to the field $K_\bullet$.*

PROOF. — Suppose that there exists an equivalence of categories $\phi\colon \mathcal{F}_{K_\circ} \xrightarrow{\sim} \mathcal{F}_{K_\bullet}$. Let $S_\circ$, $T_\circ$ be objects of $\mathcal{F}_{K_\circ}$; $f_\circ\colon S_\circ \to T_\circ$ a morphism in $\mathcal{F}_{K_\circ}$. Write $S_\bullet$, $T_\bullet$ for the objects of $\mathcal{F}_{K_\bullet}$ corresponding, via ϕ, to $S_\circ$, $T_\circ$, respectively; $f_\bullet\colon S_\bullet \to T_\bullet$ for the morphism in $\mathcal{F}_{K_\bullet}$ corresponding, via ϕ, to $f_\circ$. Then it follows from Lemma 3.9, (i), (ii), (iii), (v), that

(a) it holds that $S_\circ$ is a *quasi-trait* if and only if $S_\bullet$ is a *quasi-trait*.

In particular, it follows from Lemma 3.11, (i), that

(b) if both $S_\circ$ and $T_\circ$ [hence also both $S_\bullet$ and $T_\bullet$ — cf. (a)] are *quasi-trait*, then it holds that $f_\circ$ is either *birational* or *purely inseparable* if and only if $f_\bullet$ is either *birational* or *purely inseparable*.

Now I claim that

(c) if both $S_\circ$ and $T_\circ$ are *traits* [which thus implies that both $S_\bullet$ and $T_\bullet$ are *quasi-trait* — cf. (a)], and $f_\circ$ is *Galois*, then $f_\bullet$ is *quasi-Galois*.

To this end, let us first observe that since $S_\circ$ is a *trait*, it follows from Lemma 3.11, (ii), that $f_\circ$ satisfies condition (1) of Lemma 3.11, (ii). Thus, it follows from (a), (b) that $f_\bullet$ satisfies condition (1) of Lemma 3.11, (ii). In particular, it follows from Lemma 3.11, (ii), that the morphism $f_\bullet$ is *quasi-Galois*, as desired. This completes the proof of (c).

Next, I claim that

(d) in the situation of (c), the four finite groups $\mathrm{Aut}_{T_\circ}(S_\circ)$, $\mathrm{qGal}(f_\circ)$, $\mathrm{Aut}_{T_\bullet}(S_\bullet)$, and $\mathrm{qGal}(f_\bullet)$ are *isomorphic*.

To this end, let us first observe that since $S_\circ$ is a *trait*, it is immediate [cf. Lemma 3.5, (iv)] that the three finite groups $\mathrm{Aut}_{T_\circ}(S_\circ)$, $\mathrm{qGal}(f_\circ)$, and $\mathrm{Aut}_{T_\bullet}(S_\bullet)$ are *isomorphic*. In particular, it follows from Lemma 3.12, (i), that, for each *quasi-trait* $U_\circ$ of $\mathcal{F}_{K_\circ}$ and each morphism $g_\circ\colon U_\circ \to S_\circ$ in $\mathcal{F}_{K_\circ}$ which is either *birational* or *purely inseparable*, it holds that $\mathrm{Aut}_{T_\circ}(U_\circ)$ is *isomorphic* to a subgroup of $\mathrm{Aut}_{T_\circ}(S_\circ)$. Thus, it follows from (a), (b) that, for each *quasi-trait* $U_\bullet$ of $\mathcal{F}_{K_\bullet}$ and each morphism $g_\bullet\colon U_\bullet \to S_\bullet$ in $\mathcal{F}_{K_\bullet}$ which is either *birational* or *purely inseparable*, it holds that $\mathrm{Aut}_{T_\bullet}(U_\bullet)$ is *isomorphic* to a subgroup of $\mathrm{Aut}_{T_\bullet}(S_\bullet)$. In particular, it follows from Lemma 3.12, (ii), that $\mathrm{qGal}(f_\bullet)$ is *isomorphic* to a subgroup of $\mathrm{Aut}_{T_\bullet}(S_\bullet)$, which thus implies [cf. Lemma 3.5, (ii)] that $\mathrm{qGal}(f_\bullet)$ is *isomorphic* to $\mathrm{Aut}_{T_\bullet}(S_\bullet)$. This completes the proof of (d).

Next, I claim that

(e) it holds that $(\mathrm{char}(\underline{K}_\circ), \sharp\underline{K}_\circ^\times) = (\mathrm{char}(\underline{K}_\bullet), \sharp\underline{K}_\bullet^\times)$.

To verify the equality $\mathrm{char}(\underline{K}_\circ) = \mathrm{char}(\underline{K}_\bullet)$, let us first observe that it follows from the implication (ii-1) $\Rightarrow$ (ii-3) of Lemma 3.13, (ii), that there exists a *Galois* morphism between *traits* of $\mathcal{F}_{K_\circ}$ whose automorphism group is isomorphic to the direct product of three copies of $\mathbb{Z}/\mathrm{char}(\underline{K}_\circ)\mathbb{Z}$. Thus, it follows from the implication (ii-4) $\Rightarrow$ (ii-1) of Lemma 3.13, (ii), together with (d), that the equality $\mathrm{char}(\underline{K}_\circ) = \mathrm{char}(\underline{K}_\bullet)$ holds. Next, to verify the equality $\sharp\underline{K}_\circ^\times = \sharp\underline{K}_\bullet^\times$, observe that it follows from the implication (iii-1) $\Rightarrow$ (iii-3) of Lemma 3.13, (iii), that there exists a *Galois* object of $\mathcal{F}_{K_\circ}$ whose automorphism group is isomorphic to the direct product of two copies of $\mathbb{Z}/\sharp\underline{K}_\circ^\times\mathbb{Z}$. Thus, it follows, in light of the equality $\mathrm{char}(\underline{K}_\circ) = \mathrm{char}(\underline{K}_\bullet)$, from the implication (iii-4) $\Rightarrow$ (iii-1) of Lemma 3.13, (iii), together with (d), that the inequality $\sharp\underline{K}_\circ^\times \leq \sharp\underline{K}_\bullet^\times$ holds. Thus, by applying, to ϕ^{-1}, a similar argument to the argument applied in the proof of the inequality $\sharp\underline{K}_\circ^\times \leq \sharp\underline{K}_\bullet^\times$, we conclude that $\sharp\underline{K}_\circ^\times = \sharp\underline{K}_\bullet^\times$. This completes the proof of (e).

Next, I claim that

(f) it holds that $\mathcal{F}_{K_\circ}$ is *separable* if and only if $\mathcal{F}_{K_\bullet}$ is *separable*.

To this end, suppose that $\mathcal{F}_{K_\circ}$ is *separable*. Then it follows from the implication (i-1) $\Rightarrow$ (i-4) of Lemma 3.13, (i), together with (a), (b), that $\mathcal{F}_{K_\bullet}$ satisfies condition (i-4) of Lemma 3.13, (i). Moreover, it follows from the implication (i-1) $\Rightarrow$ (i-2) of Lemma 3.13, (i), that $\mathcal{F}_{K_\circ}$ satisfies condition (i-2) of Lemma 3.13, (i). Thus, it follows from (a), (c) that $\mathcal{F}_{K_\bullet}$ satisfies condition (i-3) of Lemma 3.13, (i). In particular, it follows from Lemma 3.13, (i), that $\mathcal{F}_{K_\bullet}$ satisfies condition (i-1) of Lemma 3.13, (i), i.e., that $\mathcal{F}_{K_\bullet}$ is *separable*, as desired. This completes the proof of (f).

Now suppose that either $\mathcal{F}_{K_\circ}$ or $\mathcal{F}_{K_\bullet}$ is *not separable*. Then it follows from (f) that both $K_\circ$ and $K_\bullet$ are *of positive characteristic*. Thus, it follows immediately from (e) that $K_\circ$ is *isomorphic* to $K_\bullet$, as desired. In the remainder of the proof,

suppose that both $\mathcal{F}_{K_\circ}$ and $\mathcal{F}_{K_\bullet}$ are *separable*.

Then it follows from Lemma 3.9, (vi), together with (a), (b), that

(g) it holds that $S_\circ$ is a *trait* if and only if $S_\bullet$ is a *trait*.

Thus, it follows, in light of (c), from Lemma 3.18, together with Lemma 3.9, (i), (iv), that, for each positive integer n,

(h) it holds that $f_\circ$ is n-*simple* if and only if $f_\bullet$ is n-*simple*.

Next, let $\widetilde{S}_\circ = ((S_\circ)_\lambda)_{\lambda\in\Lambda}$ be a *basepoint* of $\mathcal{F}_{K_\circ}$ [cf. Lemma 3.8, (i)]. Then it follows from (c), (g), that the projective system $\widetilde{S}_\bullet = ((S_\bullet)_\lambda)_{\lambda\in\Lambda}$ consisting of objects and morphisms of $\mathcal{F}_{K_\bullet}$ corresponding, via ϕ, to $\widetilde{S}_\circ$ is a *basepoint* of $\mathcal{F}_{K_\bullet}$. Thus, the equivalence ϕ determines an isomorphism of profinite groups

$$\Pi_\phi \colon\ \Pi_{\widetilde{S}_\circ} \ =\ \varprojlim_{\lambda\in\Lambda} \mathrm{Aut}((S_\circ)_\lambda) \ \xrightarrow{\sim}\ \Pi_{\widetilde{S}_\bullet} \ =\ \varprojlim_{\lambda\in\Lambda} \mathrm{Aut}((S_\bullet)_\lambda).$$

In particular, if either $K_\circ$ or $K_\bullet$ is *of positive characteristic*, then it follows, in light of Lemma 3.8, (iii), from Theorem 1.10 and Corollary 1.12, (ii), that $K_\circ$ is *isomorphic* to $K_\bullet$, as desired. In the remainder of the proof,

suppose that both $K_\circ$ and $K_\bullet$ are *of characteristic zero*.

Let λ be an element of Λ, n a positive integer, and $(f_\circ)_\lambda \colon (T_\circ)_\lambda \to (S_\circ)_\lambda$ an n-*simple* morphism whose codomain is the member $(S_\circ)_\lambda$ of $\widetilde{S}_\circ$ [cf. Lemma 3.17]. Write

$$\Pi_{\phi,\lambda} \colon\ \mathrm{Aut}((S_\circ)_\lambda) \ \xrightarrow{\sim}\ \mathrm{Aut}((S_\bullet)_\lambda)$$

for the isomorphism induced by Π_ϕ and $(f_\bullet)_\lambda \colon (T_\bullet)_\lambda \to (S_\bullet)_\lambda$ for the n-*simple* [cf. (h)] morphism corresponding, via ϕ, to $(f_\circ)_\lambda \colon (T_\circ)_\lambda \to (S_\circ)_\lambda$. Then one verifies easily that the isomorphism $\Pi_{\phi,\lambda}$ restricts to an isomorphism of subgroups

$$\mathrm{Aut}((S_\circ)_\lambda)_{(f_\circ)_\lambda} \ \xrightarrow{\sim}\ \mathrm{Aut}((S_\bullet)_\lambda)_{(f_\bullet)_\lambda}.$$

Thus, it follows from Lemma 3.19 that the isomorphism $\Pi_{\phi,\lambda}$ is *compatible* — relative to the natural identifications [cf. Lemma 3.8, (ii)] of $\mathrm{Aut}((S_\circ)_\lambda)$, $\mathrm{Aut}((S_\bullet)_\lambda)$ with $\mathrm{Gal}(K_{(S_\circ)_\lambda}/K_\circ)$, $\mathrm{Gal}(K_{(S_\bullet)_\lambda}/K_\bullet)$, respectively — with the respective filtrations of *higher ramification subgroups in the lower numbering*, hence also [cf., e.g., [3], §4.1] *in the upper numbering*. In particular, the isomorphism Π_ϕ is *compatible* — relative to the natural identifications [cf. Lemma 3.8, (iii)] of $\Pi_{\widetilde{S}_\circ}$, $\Pi_{\widetilde{S}_\bullet}$ with $\mathrm{Gal}(K_{\widetilde{S}_\circ}/K_\circ)$, $\mathrm{Gal}(K_{\widetilde{S}_\bullet}/K_\bullet)$, respectively — with the respective filtrations of *higher ramification subgroups in the upper numbering*. Thus, it follows from [2], Theorem, that $K_\circ$ is *isomorphic* to $K_\bullet$, as desired. This completes the proof of Theorem 3.20. □

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