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On the Pro-p Absolute Anabelian Geometry of Proper Hyperbolic Curves

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Abstract. — In the present paper, we study the geometry of the stable models of proper hyperbolic curves over p-adic local fields via the geometrically pro-p étale fundamental groups of the curves. In particular, we establish functorial "group-theoretic" algorithms for reconstructing various objects related to the geometry of stable models from the geometrically pro-p étale fundamental groups. As an application, we also give a pro-p "group-theoretic" criterion for good reduction of ordinary proper hyperbolic curves over p-adic local fields.

Contents

Introduction

Let p be a prime number, k a p-adic local field [i.e., a finite extension of $\mathbb{Q}_p$], $\overline{k}$ an algebraic closure of k, and X a *proper hyperbolic curve* over k [i.e., a proper smooth curve over k of arithmetic genus ≥ 2]. Write $\underline{\overline{k}}$ for the residue field of $\overline{k}$, $X_{\overline{k}} \stackrel{\mathrm{def}}{=} X \times_k \overline{k}$ for the proper hyperbolic curve over $\overline{k}$ obtained by base changing X from k to $\overline{k}$, and

$$\Pi_X$$

for the geometrically pro-p étale fundamental group of X [cf. Definition 2.2]. Then it is well-known [cf. Theorem 1.3] that the hyperbolic curve $X_{\overline{k}}$ has *stable reduction* over the ring of integers of $\overline{k}$. We shall write $X_{\underline{\overline{k}}}$ for the stable curve over $\underline{\overline{k}}$ obtained by forming the special fiber of the stable model of $X_{\overline{k}}$.

In the present paper, we study the geometry of the stable curve $X_{\underline{\overline{k}}}$ via the profinite group Π_X. In particular, we center around the task of establishing *functorial "group-theoretic" algorithms* whose input data consist of the abstract profinite group Π_X and

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whose output data consist of objects related to the geometry of the stable curve $X_{\overline{k}}$ [cf. the main result of the present paper, i.e., Theorem 3.7]. By applying the *functorial "group-theoretic" algorithms* of the present paper, one may reconstruct, from Π_X, for instance, the following objects:

- The set of irreducible components of $X_{\overline{k}}$ whose normalizations are *of positive p-rank* [cf. Theorem 3.7, (viii)], as well as the [necessarily positive] p-ranks of the normalizations of elements of this set [cf. Theorem 3.7, (x)].

- The first Betti number of the [topological space determined by the] dual graph of $X_{\overline{k}}$ [cf. Theorem 3.7, (vii)].

We shall say that the proper hyperbolic curve X is *ordinary* if the arithmetic genus of X is equal to the p-rank of $X_{\overline{k}}$ [cf. Definition 2.6, (i)]. Moreover, we shall say that a profinite group Π satisfies the condition ($\dagger$) if there exist a prime number l and an isomorphism of Π with the geometrically pro-l étale fundamental group of a proper hyperbolic curve over an l-adic local field [cf. Definition 3.6]. [So the profinite group Π_X satisfies the condition ($\dagger$).] Some of consequences of the *functorial "group-theoretic" algorithms* of the present paper may be summarized as follows [cf. Theorem 3.7, (xi), (xiii)]:

THEOREM. — *The following hold:*

(i) *There exists a purely* **"group-theoretic" condition** *for profinite groups which satisfy* ($\dagger$) *such that the profinite group* Π_X *satisfies the* **condition** *if and only if the hyperbolic curve* X *is* **ordinary**.

(ii) *There exists a purely* **"group-theoretic" condition** *for profinite groups which satisfy* ($\dagger$) *such that the profinite group* Π_X *satisfies the* **condition** *if and only if the hyperbolic curve* X *is* **ordinary** *and has* **good reduction** [*i.e., over the ring of integers of* k].

In particular, we obtain the following result [cf. Corollary 3.8, (iv), (vi)]:

COROLLARY. — *For* $\square \in \{\circ, \bullet\}$, *let* $p_\square$ *be a prime number,* $k_\square$ *a* $p_\square$*-adic local field, and* $X_\square$ *a* **proper hyperbolic curve** *over* $k_\square$. *Suppose that the geometrically pro-*$p_\circ$ *étale fundamental group of* $X_\circ$ *is* **isomorphic** *to the geometrically pro-*$p_\bullet$ *étale fundamental group of* $X_\bullet$. *Then the following hold:*

(i) *It holds that* $X_\circ$ *is* **ordinary** *if and only if* $X_\bullet$ *is* **ordinary**.

(ii) *Suppose, moreover, that either* $X_\circ$ *or* $X_\bullet$ *is* **ordinary**. *Then it holds that* $X_\circ$ *has* **good reduction** *if and only if* $X_\bullet$ *has* **good reduction**.

Note that Theorem, as well as Corollary, may be regarded as a *pro-p "group-theoretic" criterion for good reduction* of *ordinary* proper hyperbolic curves over p-adic local fields. Here, let us recall [cf. Remark 3.8.1] that, for a nonempty set Σ of prime numbers such that $p \notin \Sigma$, we have already a *pro-*Σ *"group-theoretic" criterion for good reduction* of [not necessarily ordinary] hyperbolic curves over p-adic local fields proved by *T. Oda* [cf. [18], Theorem 3.2], *A. Tamagawa* [cf. [20], Theorem 5.3], and *S. Mochizuki* [cf. [12], Corollary 2.8].

Finally, let us discuss [cf. Remark 3.8.2] the *p-adic criterion for good reduction* of curves proved by *F. Andreatta*, *A. Iovita*, and *M. Kim* in [1] from the point of view of the present paper. The p-adic criterion of [1] asserts, roughly speaking, that X has *good reduction* if and only if every member of a certain collection of finite-dimensional representations of $G_k \overset{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ over $\mathbb{Q}_p$ determined by the profinite group Π_X and a splitting of the natural surjection $\Pi_X \twoheadrightarrow G_k$ arising from a k-rational point of X is *crystalline* [cf. [1], Theorem 1.9]. Here, observe that this criterion [is interesting even in a certain point of view of anabelian geometry but] should be considered to be *not "group-theoretic"* [i.e., to be *not useful* in *pro-p absolute anabelian geometry*] by the following two reasons:

(1) The issue of whether or not a given finite-dimensional representation of G_k over $\mathbb{Q}_p$ is *crystalline* is *not "group-theoretic"*. Indeed, it follows immediately from the discussion of [7], Remark 3.3.1, that there exist a prime number l, an l-adic local field L, an automorphism α of the absolute Galois group G_L of L, and a *crystalline* representation $\rho\colon G_L \to \mathrm{GL}_n(\mathbb{Q}_l)$ such that the composite $G_L \overset{\alpha}{\xrightarrow{\sim}} G_L \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{Q}_l)$ is *not crystalline*.

(2) It is *not clear* that the issue of whether or not a given splitting of the natural surjection $\Pi_X \twoheadrightarrow G_k$ arises from a k-rational point of X is *"group-theoretic"*. Note that it follows from [5], Theorem A, that there exist a prime number l, an l-adic local field L, a proper hyperbolic curve C over L, and a splitting of the natural surjection from the geometrically pro-l étale fundamental group of C onto the absolute Galois group of L which does *not arise from an L-rational point of C*.

As a consequence of this discussion, one *cannot*, at least in the immediate literal sense, *drop* the ordinary hypothesis in the statement of Corollary, (ii), even if one applies the p-adic criterion of [1].

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1. STABLE MODELS

Throughout the present paper, let p be a prime number. In the present §1, we introduce some notational conventions related to the geometry of the stable models of proper hyperbolic curves over p-adic local fields. We also recall a theorem of *P. Deligne* and *D. Mumford* [cf. Theorem 1.3 below] and a theorem of *M. Raynaud* [cf. Theorem 1.6 below]

DEFINITION 1.1. — Let V be a proper variety over a field F. Then we shall write

$$g_V \overset{\text{def}}{=} (-1)^{\dim(V)} \cdot (\chi_{\mathrm{Zar}}(\mathcal{O}_V) - 1)$$

for the *arithmetic genus* of V. If, moreover, F is algebraically closed and of characteristic p, then we shall write

$$\gamma_V \overset{\text{def}}{=} \dim_{\mathbb{F}_p} H^1_{\text{ét}}(V, \mathbb{F}_p)$$

for the *p-rank* of V.

In the remainder of the present §1, let k be a p-adic local field [i.e., a finite extension of $\mathbb{Q}_p$], $\overline{k}$ an algebraic closure of k, and X a *proper hyperbolic curve* over k [i.e., a proper smooth curve over k such that $g_X \geq 2$]. Write $\underline{\overline{k}}$ for the residue field of $\overline{k}$ and $X_{\overline{k}} \stackrel{\text{def}}{=} X \times_k \overline{k}$ for the proper hyperbolic curve over $\overline{k}$ obtained by base changing X from k to $\overline{k}$.

DEFINITION 1.2. — Let K be a(n) [possibly infinite] algebraic extension of k. Then we shall say that the hyperbolic curve $X \times_k K$ over K has *stable reduction* (respectively, *good reduction*) if the structure morphism $X \times_k K \to \mathrm{Spec}(K)$ extends to a stable curve (respectively, smooth stable curve) over the ring of integers of K [cf. [3], Definition 1.1].

THEOREM 1.3 (Deligne-Mumford). — *In the notational conventions introduced in the discussion preceding Definition* 1.2*, there exists a finite extension K of k such that the hyperbolic curve $X \times_k K$ over K has* **stable reduction** [*cf. Definition* 1.2]. *In particular, the hyperbolic curve $X_{\overline{k}}$ over $\overline{k}$ has* **stable reduction**.

PROOF. — This follows from [3], Corollary 2.7. □

DEFINITION 1.4.

(i) We shall write

$$X_{\underline{\overline{k}}}$$

for the stable curve over $\underline{\overline{k}}$ [of arithmetic genus g_X] obtained by forming the special fiber of the stable model of $X_{\overline{k}}$ over the ring of integers of $\overline{k}$ [cf. Theorem 1.3].

(ii) We shall write

$$\mathbb{G}_X$$

for the *dual graph* of $X_{\underline{\overline{k}}}$,

$$\mathrm{Irr}(X)$$

for the set of irreducible components of $X_{\underline{\overline{k}}}$ — i.e., the set of vertices of $\mathbb{G}_X$ — and

$$b_1(X) \stackrel{\text{def}}{=} \dim_{\mathbb{Q}} H_1(\mathbb{G}_X, \mathbb{Q})$$

for the first Betti number of [the topological space determined by] $\mathbb{G}_X$.

(iii) Let $v \in \mathrm{Irr}(X)$. Then we shall write

$$I_v$$

for the proper smooth curve over $\underline{\overline{k}}$ obtained by forming the normalization of the irreducible component of $X_{\underline{\overline{k}}}$ corresponding to $v \in \mathrm{Irr}(X)$,

$$g_v \stackrel{\text{def}}{=} g_{I_v}$$

for the *arithmetic genus* of I_v, and

$$\gamma_v \stackrel{\text{def}}{=} \gamma_{I_v}$$

for the *p-rank* of I_v.

(iv) We shall write

$$\mathrm{Irr}(X)^{\gamma=0} \stackrel{\mathrm{def}}{=} \{ v \in \mathrm{Irr}(X) \mid \gamma_v = 0 \} \subseteq \mathrm{Irr}(X)$$

for the set of irreducible components of $X_{\underline{\overline{k}}}$ [whose normalizations are] of p-rank zero and

$$\mathrm{Irr}(X)^{\gamma>0} \stackrel{\mathrm{def}}{=} \mathrm{Irr}(X) \setminus \mathrm{Irr}(X)^{\gamma=0} = \{ v \in \mathrm{Irr}(X) \mid \gamma_v > 0 \} \subseteq \mathrm{Irr}(X)$$

for the set of irreducible components of $X_{\underline{\overline{k}}}$ [whose normalizations are] of positive p-rank.

REMARK 1.4.1.

(i) It is well-known that, for each $v \in \mathrm{Irr}(X)$, it holds that $g_v \geq \gamma_v \geq 0$.

(ii) One verifies easily that

$$g_X = g_{X_{\underline{\overline{k}}}} = b_1(X) + \sum_{v \in \mathrm{Irr}(X)} g_v,$$

$$\gamma_{X_{\underline{\overline{k}}}} = b_1(X) + \sum_{v \in \mathrm{Irr}(X)} \gamma_v = b_1(X) + \sum_{v \in \mathrm{Irr}(X)^{\gamma>0}} \gamma_v.$$

[One may also find these equalities concerning $\gamma_{X_{\underline{\overline{k}}}}$ in the final discussion of [16], §0.]

REMARK 1.4.2. — Let $Y \to X$ be a connected finite étale covering of X.

(i) One verifies easily that Y is a *proper hyperbolic curve* over a finite extension k_Y of k [i.e., the algebraic closure of k in the function field of Y]. Moreover, one also verifies easily that the covering $Y \to X$ determines a *connected finite étale covering* $Y_{\overline{k}} \stackrel{\mathrm{def}}{=} Y \times_{k_Y} \overline{k} \to X_{\overline{k}}$ over $\overline{k}$.

(ii) It follows, in light of Theorem 1.3, from [10], Lemma 8.3, that the covering $Y_{\overline{k}} \to X_{\overline{k}}$ of (i) extends to a *uniquely determined proper* [*not necessarily finite*] *surjection* from the stable model of $Y_{\overline{k}}$ over the ring of integers of $\overline{k}$ to the stable model of $X_{\overline{k}}$ over the ring of integers of $\overline{k}$. In particular, we obtain a *proper* [*not necessarily finite*] *surjection* $Y_{\underline{\overline{k}}} \to X_{\underline{\overline{k}}}$ over $\underline{\overline{k}}$.

(iii) One verifies immediately from the existence of the morphism $Y_{\underline{\overline{k}}} \to X_{\underline{\overline{k}}}$ of (ii) that the inequalities

$$b_1(Y) \geq b_1(X), \quad \sharp\mathrm{Irr}(Y)^{\gamma>0} \geq \sharp\mathrm{Irr}(X)^{\gamma>0}$$

hold.

DEFINITION 1.5. — Let $Y \to X$ be a connected finite étale covering of X. Then we shall say that the covering $Y \to X$ is a *geometrically-p-covering* if the Galois closure of the connected finite étale covering $Y_{\overline{k}} \to X_{\overline{k}}$ over $\overline{k}$ [cf. Remark 1.4.2, (i)] is of degree a power of p [cf. Remark 2.2.1 below].

REMARK 1.5.1. — One verifies easily that the composite of finitely many *geometrically-p-coverings* is a *geometrically-p-covering*. Moreover, one also verifies easily that the connected finite étale covering obtained by the "composition" [i.e., obtained by considering the composite field of the function fields] of finitely many *geometrically-p-coverings* is a *geometrically-p-covering*.

THEOREM 1.6 (Raynaud). — *In the notational conventions introduced in the discussion preceding Definition* 1.2*, suppose that $X_{\overline{k}}$ has* **good reduction** [*cf. Definition* 1.2]*. Then it holds that $b_1(Y) = 0$* [*cf. Definition* 1.4, (ii)] *for every geometrically-p-covering $Y \to X$* [*cf. Definition* 1.5] *of X.*

PROOF. — Let $Y \to X$ be a geometrically-p-covering of X. Then it follows from Remark 1.4.2, (iii), that, to verify that $b_1(Y) = 0$, we may assume without loss of generality, by replacing $Y \to X$ by the Galois closure, that the geometrically-p-covering $Y \to X$ is *Galois*. Then since the Galois group of the Galois covering $Y_{\overline{k}} \to X_{\overline{k}}$ [cf. Remark 1.4.2, (i)] is a *p-group*, the equality $b_1(Y) = 0$ follows from [15], Théorème 1, (ii). $\square$

2. QUOTIENTS OF PRO-p FUNDAMENTAL GROUPS

In the present §2, we discuss certain quotients [cf. Definition 2.3 and Definition 2.4 below] of the pro-p geometric étale fundamental groups [cf. Definition 2.2 below] of proper hyperbolic curves over p-adic local fields. In the present §2, we maintain the notational conventions introduced in the discussion preceding Definition 1.2. Write $\pi_1(X)$ for the étale fundamental group of X relative to some choice of basepoint such that the algebraic closure of k determined by the basepoint coincides with $\overline{k}$, $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\overline{k}$, and $I_k \subseteq G_k$ for the inertia subgroup of G_k.

DEFINITION 2.1. — We shall say that X is *split* if the natural action of G_k on the dual graph $\mathbb{G}_X$ is trivial.

REMARK 2.1.1. — Since the graph $\mathbb{G}_X$ is *finite*, it is immediate that there exists a finite extension K of k such that the hyperbolic curve $X \times_k K$ over K is *split*.

DEFINITION 2.2. — We shall write

$$\Delta_X$$

for the *pro-p geometric étale fundamental group* of X — i.e., the maximal pro-p quotient of the étale fundamental group $\pi_1(X_{\overline{k}})$ of $X_{\overline{k}}$ relative to the basepoint which defines $\pi_1(X)$ — and

$$\Pi_X$$

for the *geometrically pro-p étale fundamental group* of X — i.e., the quotient of $\pi_1(X)$ by the normal closed subgroup obtained by forming the kernel of the natural surjection from $\pi_1(X_{\overline{k}})$ ($\subseteq \pi_1(X)$) to Δ_X. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1,$$

which thus determines an outer action of G_k on Δ_X.

REMARK 2.2.1. — Let $Y \to X$ be a connected finite étale covering of X. Then one verifies easily that the covering $Y \to X$ [is isomorphic to the covering which] corresponds to an *open subgroup of* Π_X if and only if the covering $Y \to X$ is a *geometrically-p-covering*.

DEFINITION 2.3.

(i) We shall write

$$\Delta_X^{\text{ét}}$$

for the *pro-p étale fundamental group* of $X_{\overline{k}}$ — i.e., the maximal pro-p quotient of the étale fundamental group $\pi_1(X_{\overline{k}})$ of $X_{\overline{k}}$ relative to the basepoint determined by the basepoint which defines $\pi_1(X)$. Thus, the natural open immersion from $X_{\overline{k}}$ into the stable model of $X_{\overline{k}}$ over the ring of integers of $\overline{k}$ determines a *surjection*

$$\Delta_X \twoheadrightarrow \Delta_X^{\text{ét}}.$$

(ii) Let $v \in \text{Irr}(X)$. Then we shall write

$$\mathfrak{D}_v \subseteq \Delta_X^{\text{ét}}$$

for the decomposition subgroup of $\Delta_X^{\text{ét}}$ [well-defined up to conjugation] associated to the irreducible component of $X_{\overline{k}}$ corresponding to $v \in \text{Irr}(X)$.

(iii) We shall write

$$\Delta_X^{\text{cmb}}$$

for the quotient of $\Delta_X^{\text{ét}}$ by the normal closed subgroup topologically normally generated by the $\mathfrak{D}_v$'s, where v ranges over the elements of $\text{Irr}(X)$. Thus, we have a natural surjection

$$\Delta_X^{\text{ét}} \twoheadrightarrow \Delta_X^{\text{cmb}}.$$

DEFINITION 2.4. — We shall write

$$\Delta_X^{\text{ab}}, \quad \Delta_X^{\text{ab-ét}}, \quad \Delta_X^{\text{ab-cmb}}$$

for the respective abelianizations of Δ_X, $\Delta_X^{\text{ét}}$, Δ_X^{cmb}. Thus, Δ_X^{ab}, $\Delta_X^{\text{ab-ét}}$, $\Delta_X^{\text{ab-cmb}}$ have natural structures of $\mathbb{Z}_p$-modules, respectively.

REMARK 2.4.1.

(i) One verifies easily that if X has *stable reduction*, then the quotients $\Delta_X^{\text{ab}} \twoheadrightarrow \Delta_X^{\text{ab-ét}} \twoheadrightarrow \Delta_X^{\text{ab-cmb}}$ of Δ_X^{ab} are *G_k-stable*.

(ii) One also verifies easily from the various definitions involved that the following hold:

• If X has *stable reduction*, then the action of I_k on the G_k*-stable* [cf. (i)] quotient $\Delta_X^{\text{ab-ét}}$ is *trivial.*

• If X is *split*, then the action of G_k on the G_k*-stable* [cf. (i)] quotient $\Delta_X^{\text{ab-cmb}}$ is *trivial.*

PROPOSITION 2.5. — *The following hold:*

(i) *The profinite groups $\Delta_X^{\text{ét}}$, Δ_X^{cmb} are* **free pro-*p*** *of rank $\gamma_{X_{\overline{k}}}$, $b_1(X)$, respectively. In particular, the $\mathbb{Z}_p$-modules $\Delta_X^{\text{ab-ét}}$, $\Delta_X^{\text{ab-cmb}}$ are* **free** *of rank $\gamma_{X_{\overline{k}}}$, $b_1(X)$, respectively.*

(ii) *Let $v \in \mathrm{Irr}(X)$. Then the profinite group $\mathfrak{D}_v$ is* **free pro-*p*** *of rank γ_v. In particular, the abelianization $\mathfrak{D}_v^{\text{ab}}$ of $\mathfrak{D}_v$ is a* **free** *$\mathbb{Z}_p$-module of rank γ_v.*

(iii) *The natural inclusions $\mathfrak{D}_v \hookrightarrow \Delta_X^{\text{ét}}$ — where v ranges over the elements of $\mathrm{Irr}(X)$ — and the natural surjection $\Delta_X^{\text{ét}} \twoheadrightarrow \Delta_X^{\text{cmb}}$ determine an* **exact** *sequence of finitely generated free $\mathbb{Z}_p$-modules*

$$0 \longrightarrow \bigoplus_{v \in \mathrm{Irr}(X)} \mathfrak{D}_v^{\text{ab}} \longrightarrow \Delta_X^{\text{ab-ét}} \longrightarrow \Delta_X^{\text{ab-cmb}} \longrightarrow 0.$$

(iv) *Let v, $w \in \mathrm{Irr}(X)^{\gamma>0}$. Then the following conditions are equivalent:*

(1) *It holds that $v = w$.*

(2) *The conjugacy class of $\mathfrak{D}_v$* **coincides** *with the conjugacy class of $\mathfrak{D}_w$.*

(3) *The intersection $\mathfrak{D}_v \cap \mathfrak{D}_w$ is* **nontrivial** *for some choices of $\mathfrak{D}_v$ and $\mathfrak{D}_w$ [i.e., among their conjugates].*

(v) *Let $v \in \mathrm{Irr}(X)^{\gamma>0}$. Then the closed subgroup $\mathfrak{D}_v \subseteq \Delta_X^{\text{ét}}$ is* **commensurably terminal***, i.e., for $\delta \in \Delta_X^{\text{ét}}$, it holds that $\delta \in \mathfrak{D}_v$ if and only if the intersection $\mathfrak{D}_v \cap (\delta \mathfrak{D}_v \delta^{-1})$ is of finite index in both $\mathfrak{D}_v$ and $\delta \mathfrak{D}_v \delta^{-1}$.*

(vi) *Suppose that X has* **stable reduction** *[which thus implies that the quotients $\Delta_X^{\text{ab}} \twoheadrightarrow \Delta_X^{\text{ab-ét}} \twoheadrightarrow \Delta_X^{\text{ab-cmb}}$ of Δ_X^{ab} are* **G_k-stable** *— cf. Remark* 2.4.1, (i)*]. Then, for every open subgroup $J \subseteq G_k$ of G_k, there is* **no nontrivial torsion-free *J*-stable quotient** *of*

$$\mathrm{Ker}(\Delta_X^{\text{ab-ét}} \twoheadrightarrow \Delta_X^{\text{ab-cmb}})$$

on which J acts **trivially***.*

PROOF. — First, we verify assertion (i). Let us first observe that it follows immediately from the definition of Δ_X^{cmb} that Δ_X^{cmb} is naturally isomorphic to the pro-p completion of the topological fundamental group of the [topological space determined by the] graph $\mathbb{G}_X$. Next, let us recall the well-known fact that the topological fundamental group of the [topological space determined by the] graph $\mathbb{G}_X$ is *free of rank* $b_1(X)$. Thus, the profinite group Δ_X^{cmb} is *free pro-p* of rank $b_1(X)$, as desired.

Next, to verify the assertion for $\Delta_X^{\text{ét}}$ in assertion (i), let us recall the well-known fact that $H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}/p\mathbb{Z}) = \{0\}$ [cf., e.g., [9], Chapter VI, Remark 1.5, (b)]. Thus, it follows from [19], Corollary A.1.4, that $H^2(\Delta_X^{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) = \{0\}$. In particular, it follows from [17], Theorem 7.7.4, that $\Delta_X^{\text{ét}}$ is *free pro-p* [of rank $\gamma_{\overline{k}}$ — cf. Definition 1.1]. This completes the proof of assertion (i).

Next, we verify assertions (ii), (iii). For each $v \in \text{Irr}(X)$, write Δ_v for the maximal pro-p quotient of the étale fundamental group of the proper smooth curve I_v over $\overline{k}$. Then it follows from a similar argument to the argument applied in the proof of the assertion for $\Delta_X^{\text{ét}}$ in assertion (i) that

(a) the profinite group Δ_v is *free pro-p of rank* γ_v [which thus implies that the abelianization Δ_v^{ab} of Δ_v is a *free* $\mathbb{Z}_p$*-module of rank* γ_v].

Next, let us observe that since $\mathfrak{D}_v$ is a closed subgroup of a *free pro-p* [cf. assertion (i)] group $\Delta_X^{\text{ét}}$, it follows from [17], Corollary 7.7.5, that

(b) the profinite group $\mathfrak{D}_v$ is *free pro-p* [which thus implies that the $\mathbb{Z}_p$-module $\mathfrak{D}_v^{\text{ab}}$ is *free*].

Moreover, it follows from the definition of $\mathfrak{D}_v$ that

(c) the natural finite morphism $I_v \to X_{\overline{k}}$ over $\overline{k}$ determines a *surjection* $\Delta_v \twoheadrightarrow \mathfrak{D}_v$ [well-defined up to $N_{\Delta_X^{\text{ét}}}(\mathfrak{D}_v)$-conjugation — where we write $N_{\Delta_X^{\text{ét}}}(\mathfrak{D}_v)$ for the normalizer of $\mathfrak{D}_v$ in $\Delta_X^{\text{ét}}$].

Thus, it follows from (a), (b), (c) that, to verify assertion (ii), it suffices to verify the following assertion:

(A) The surjection $\Delta_v^{\text{ab}} \twoheadrightarrow \mathfrak{D}_v^{\text{ab}}$ determined by the surjection of (c) is *injective*.

Next, let us observe that one verifies easily that the various homomorphisms appearing in the statement of assertion (iii) determine an *exact* sequence of $\mathbb{Z}_p$-modules

$$\bigoplus_{v \in \text{Irr}(X)} \mathfrak{D}_v^{\text{ab}} \longrightarrow \Delta_X^{\text{ab-ét}} \longrightarrow \Delta_X^{\text{ab-cmb}} \longrightarrow 0.$$

In particular, to verify assertion (iii), it suffices to verify the following assertion:

(B) The natural homomorphism $\bigoplus_{v \in \text{Irr}(X)} \mathfrak{D}_v^{\text{ab}} \to \Delta_X^{\text{ab-ét}}$ is *injective*.

Thus, we conclude [cf. (A), (B)] that, to complete the verification of assertions (ii), (iii), it suffices to verify the following assertion:

(C) The homomorphism $\bigoplus_{v \in \text{Irr}(X)} \Delta_v^{\text{ab}} \to \Delta_X^{\text{ab-ét}}$ determined by the natural finite morphisms $I_v \to X_{\overline{k}}$ — where v ranges over the elements of $\text{Irr}(X)$ — is *injective*.

On the other hand, (C) follows immediately from a similar argument to the argument applied in the proof of [6], Lemma 1.4 [cf. also Remark 2.5.1, (ii), below]. This completes the proofs of assertions (ii), (iii).

Assertion (iv) follows immediately from assertions (ii), (iii), together with the fact that every nontrivial closed subgroup of a free pro-p group is *infinite* [cf. [17], Corollary 7.7.5]. Assertion (v) is a formal consequence of assertion (iv). Assertion (vi) follows immediately from assertion (iii) [cf. also (A)] and [20], Proposition 3.3, (ii). This completes the proof of Proposition 2.5. $\square$

REMARK 2.5.1.

(i) One can also verify the equalities concerning $\gamma_{X_{\overline{k}}}$ of Remark 1.4.1, (ii), from Proposition 2.5, (i), (ii), (iii).

(ii) The assertion (C) in the proof of Proposition 2.5 also follows, in light of the exact sequence in the discussion preceding the assertion (B), from the equalities concerning $\gamma_{X_{\overline{k}}}$ of Remark 1.4.1, (ii), together with Proposition 2.5, (i), and the assertions (a), (c) in the proof of Proposition 2.5.

DEFINITION 2.6.

(i) We shall say that X is *ordinary* if g_X [i.e., $g_{X_{\overline{k}}}$ — cf. Remark 1.4.1, (ii)] is equal to $\gamma_{\overline{k}}$.

(ii) We shall say that X is *rationally degenerate* if $g_v = 0$ for every $v \in \mathrm{Irr}(X)$.

LEMMA 2.7. — *The following hold:*

(i) *It holds that X is* **ordinary** *if and only if $g_v = \gamma_v$ for every $v \in \mathrm{Irr}(X)$.*

(ii) *It holds that X is* **rationally degenerate** *if and only if the following condition is satisfied: The hyperbolic curve X is* **ordinary**, *and* $\mathrm{Irr}(X)^{\gamma>0} = \emptyset$.

(iii) *If X is* **ordinary**, *then it holds that either $b_1(X) \neq 0$,* $\mathrm{Irr}(X)^{\gamma=0} = \emptyset$, *or* $\sharp\mathrm{Irr}(X)^{\gamma>0} \geq 3$.

PROOF. — Assertion (i) follows from Remark 1.4.1, (i), (ii). Assertion (ii) follows from assertion (i), together with Remark 1.4.1, (i). Assertion (iii) follows immediately from assertion (i), together with the definition of a stable curve. □

DEFINITION 2.8. — Let C be a hyperbolic curve over $\overline{k}$. Then we shall say that $X_{\overline{k}}$ is *p-isogenous* to C if there exist a hyperbolic curve Z over $\overline{k}$ and finite étale coverings $Z \to X_{\overline{k}}$, $Z \to C$ over $\overline{k}$ such that the respective Galois closures of $Z \to X_{\overline{k}}$, $Z \to C$ are of degree a power of p.

THEOREM 2.9. — *In the notational conventions introduced at the beginning of* §2, *consider the following conditions:*

(1) *The hyperbolic curve $X_{\overline{k}}$ has* **good reduction** [*cf. Definition* 1.2].

(2) *The hyperbolic curve $X_{\overline{k}}$ is* ***p*-isogenous** [*cf. Definition* 2.8] *to a hyperbolic curve over $\overline{k}$ which has* **good reduction**.

(3) *It holds that $b_1(Y) = 0$* [*cf. Definition* 1.4, (ii)] *for every geometrically-p-covering $Y \to X$* [*cf. Definition* 1.5] *of X.*

(4) *It holds that* $\sharp\mathrm{Irr}(Y)^{\gamma>0} \leq 1$ [*cf. Definition* 1.4, (iv)] *for every geometrically-p-covering $Y \to X$ of X.*

Then the following hold:

(i) *The implications*

$$(1) \implies (2) \implies (3) \implies (4)$$

hold.

(ii) *Suppose that there exists a geometrically-p-covering $Y \to X$ of X such that* $\mathrm{Irr}(Y)^{\gamma>0} \neq \emptyset$. *Then the equivalence*

$$(3) \iff (4)$$

holds.

(iii) *Suppose that X is* **ordinary** [*cf. Definition* 2.6, (i)]. *Then the equivalence*

$$(1) \iff (3)$$

holds.

PROOF. — First, we verify assertion (i). The implication (1) ⇒ (2) is immediate. The implication (2) ⇒ (3) follows, in light of Remark 1.4.2, (iii), and Remark 1.5.1, from Theorem 1.6. Finally, we verify the implication (3) ⇒ (4). Suppose that condition (4) is not satisfied, i.e., that there exist a geometrically-p-covering $Y \to X$ and distinct elements v_1, $v_2 \in \mathrm{Irr}(Y)^{\gamma>0}$. Then it follows from Proposition 2.5, (ii), (iii), that there exists a Galois geometrically-p-covering $Z \to Y$ of Y such that

- the surjection $\Delta_Y \twoheadrightarrow \Delta_Y/\Delta_Z$ [cf. Remark 2.2.1] *factors through* $\Delta_Y \twoheadrightarrow \Delta_Y^{\text{ét}}$,
- $\Delta_Y/\Delta_Z \cong \mathbb{Z}/p\mathbb{Z}$, and, moreover,
- for each $w \in \mathrm{Irr}(Y)$, it holds that the image of the composite $\mathfrak{D}_w \hookrightarrow \Delta_Y^{\text{ét}} \twoheadrightarrow \Delta_Y/\Delta_Z$ is *nontrivial* if and only if $w \in \{v_1, v_2\}$.

Then, by considering *liftings* in $\mathbb{G}_Z$ — relative to the finite étale covering $Z_{\overline{k}} \to Y_{\overline{k}}$ [cf. Remark 1.4.2, (ii)] — of a "simple path" in $\mathbb{G}_Y$ from v_1 to v_2, one verifies easily that $b_1(Z) \neq 0$, which thus implies [cf. Remark 1.5.1] that condition (3) is not satisfied. This completes the proof of the implication (3) ⇒ (4), hence also of assertion (i).

Next, we verify assertion (ii). Suppose that there exists a geometrically-p-covering $Y \to X$ of X such that $\mathrm{Irr}(Y)^{\gamma>0} \neq \emptyset$, and that condition (3) is not satisfied. Thus, it follows from Remark 1.4.2, (iii), and Remark 1.5.1 that there exists a geometrically-p-covering $Z \to Y$ of Y such that $b_1(Z) \neq 0$, which thus implies that $\Delta_Z^{\text{ab-cmb}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z} \neq \{0\}$ [cf. Proposition 2.5, (i)]. Let $W \to Z$ be a geometrically-p-covering of Z such that the open subgroup $\Delta_W \subseteq \Delta_Z$ [cf. Remark 2.2.1] *coincides* with the kernel of the natural surjection $\Delta_Z \twoheadrightarrow \Delta_Z^{\text{ab-cmb}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z}$. Then it is immediate that

$$0 < \sharp\mathrm{Irr}(Y)^{\gamma>0} \leq \sharp\mathrm{Irr}(Z)^{\gamma>0} < \sharp(\Delta_Z^{\text{ab-cmb}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z}) \cdot \sharp\mathrm{Irr}(Z)^{\gamma>0} = \sharp\mathrm{Irr}(W)^{\gamma>0}$$

[cf. Remark 1.4.2, (iii)]. Thus, condition (4) is not satisfied [cf. Remark 1.5.1]. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Suppose that X is *ordinary*, and that condition (3) is satisfied [which thus implies that condition (4) is satisfied — cf. assertion (i)]. Then it follows from Lemma 2.7, (iii), together with the fact that $b_1(X) = 0$ [cf. condition (3)], that it holds that either $\mathrm{Irr}(X)^{\gamma=0} = \emptyset$ or $\sharp\mathrm{Irr}(X)^{\gamma>0} \geq 3$. In particular, it follows from the fact that $\sharp\mathrm{Irr}(X)^{\gamma>0} \leq 1$ [cf. condition (4)] that $\mathrm{Irr}(X)^{\gamma=0} = \emptyset$. Thus, again by the fact that $\sharp\mathrm{Irr}(X)^{\gamma>0} \leq 1$ [cf. condition (4)], it follows that

$$1 \geq \sharp\mathrm{Irr}(X)^{\gamma>0} = \sharp\mathrm{Irr}(X) - \sharp\mathrm{Irr}(X)^{\gamma=0} = \sharp\mathrm{Irr}(X).$$

In particular, again by the fact that $b_1(X) = 0$ [cf. condition (3)], it follows that $X_{\overline{k}}$ is *smooth* over $\overline{k}$, as desired. This completes the proof of assertion (iii). $\square$

REMARK 2.9.1. — Suppose that we are in the situation of Theorem 2.9:

(i) In general, the implication (2) $\Rightarrow$ (1) does *not hold* as follows: Let us recall the well-known fact that the $\mathbb{Z}_p$-module Δ_X^{ab} is *free of rank* $2g_X$ $(= 2g_{X_{\overline{k}}} > \gamma_{X_{\overline{k}}})$. Thus, it follows from Proposition 2.5, (i), that the natural surjection $\Delta_X \twoheadrightarrow \Delta_X^{\text{ét}}$ is *not an isomorphism*. Now suppose that $X_{\overline{k}}$ has *good reduction*. Thus, it follows from [20], Lemma 5.5, that there exists a geometrically-p-covering $Y \to X$ of X such that $Y_{\overline{k}}$ does *not have good reduction*. Then the hyperbolic curve Y *violates* the implication (2) $\Rightarrow$ (1).

(ii) It follows from (i) that, in general, the implication (3) $\Rightarrow$ (1), hence also the implication (4) $\Rightarrow$ (1), does *not hold*.

COROLLARY 2.10. — *In the notational conventions introduced at the beginning of* §2*, let Y be an* **ordinary** [*cf. Definition* 2.6*,* (i)] *proper hyperbolic curve over k such that $Y_{\overline{k}}$ has* **good reduction** [*cf. Definition* 1.2]*. Consider the following conditions:*

(1) *The hyperbolic curve X is* **ordinary***.*

(2) *The hyperbolic curve $X_{\overline{k}}$ has* **good reduction***.*

Then the following hold:

(i) *If $X_{\overline{k}}$ is* **p-isogenous** [*cf. Definition* 2.8] *to $Y_{\overline{k}}$, then the implication*

$$(1) \Longrightarrow (2)$$

holds.

(ii) *If there exists a geometrically-p-covering $X \to Y$* [*cf. Definition* 1.5] *over k such that the connected finite étale covering $X_{\overline{k}} \to Y_{\overline{k}}$ over $\overline{k}$* [*cf. Remark* 1.4.2*,* (i)] *is* **Galois***, then the equivalence*

$$(1) \Longleftrightarrow (2)$$

holds.

PROOF. — First, we verify assertion (i). Suppose that X is *ordinary*, and that $X_{\overline{k}}$ is *p-isogenous* to $Y_{\overline{k}}$. Since X satisfies condition (2) of Theorem 2.9, it follows from Theorem 2.9, (i), that X satisfies condition (3) of Theorem 2.9. Thus, since [we have assumed that] X is *ordinary*, it follows from Theorem 2.9, (iii), that the hyperbolic curve $X_{\overline{k}}$ has *good reduction*, as desired. This completes the proof of assertion (i).

The implication (2) $\Rightarrow$ (1) in the case where there exists a geometrically-p-covering $X \to Y$ over k such that the connected finite étale covering $X_{\overline{k}} \to Y_{\overline{k}}$ over $\overline{k}$ is *Galois* follows immediately from [20], Lemma 5.5, together with the *Riemann-Roch formula* [for genus] and the *Deuring-Shafarevich formula* [for p-rank]. This completes the proof of Corollary 2.10. $\square$

REMARK 2.10.2. — Note that Corollary 2.10, (ii), may be regarded as a *special case* of [16], Proposition 3.

3. PRO-p GROUP-THEORETIC ALGORITHMS

In the present §3, we establish functorial "group-theoretic" algorithms for reconstructing various objects related to the geometry of the stable models of proper hyperbolic curves over p-adic local fields from the geometrically pro-p étale fundamental groups of the curves [cf. Theorem 3.7 below]. In the present §3, we maintain the notational conventions introduced at the beginning of §2.

DEFINITION 3.1. — We shall write

$$\Lambda_X \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{Z}_p}\big(H^2(\Delta_X, \mathbb{Z}_p), \mathbb{Z}_p\big)$$

for the *pro-p cyclotome* associated to X.

REMARK 3.1.1. — One verifies easily that the natural outer action of G_k on Δ_X determines an action of G_k on the cyclotome Λ_X. Moreover, one also verifies easily [cf., e.g., [9], Chapter V, Theorem 2.1, (a)] that the resulting G_k-module is isomorphic to the G_k-module "$\mathbb{Z}_p(1)$" obtained by forming the projective limit $\varprojlim_n \boldsymbol{\mu}_{p^n}(\overline{k})$ — where the projective limit is taken over the positive integers n — of the groups $\boldsymbol{\mu}_{p^n}(\overline{k}) \subseteq \overline{k}^{\times}$ of p^n-th roots of unity in $\overline{k}$.

Let us first recall the following well-known fact:

LEMMA 3.2. — *Suppose that X has* **stable reduction***. Then there exists a sequence of G_k-stable $\mathbb{Z}_p$-submodules of Δ_X^{ab}*

$$F_0 = \{0\} \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 = \Delta_X^{\mathrm{ab}}$$

which satisfies the following conditions:

(1) *For each $0 \leq i \leq 4$, the quotient F_{i+1}/F_i is a* **free** *$\mathbb{Z}_p$-module.*

(2) *The submodule F_3 (respectively, F_4) coincides with the* **kernel** *of the natural surjection $\Delta_X^{\mathrm{ab}} \twoheadrightarrow \Delta_X^{\text{ab-ét}}$ (respectively, $\Delta_X^{\mathrm{ab}} \twoheadrightarrow \Delta_X^{\text{ab-cmb}}$). In particular, we obtain* ***G_k*-equivariant** *isomorphisms*

$$F_5/F_3 \xrightarrow{\sim} \Delta_X^{\text{ab-ét}}, \quad F_5/F_4 \xrightarrow{\sim} \Delta_X^{\text{ab-cmb}}.$$

(3) *There exist* ***G_k*-equivariant** *isomorphisms*

$$F_1 \cong \mathrm{Hom}_{\mathbb{Z}_p}(\Delta_X^{\text{ab-cmb}}, \Lambda_X), \quad F_2 \cong \mathrm{Hom}_{\mathbb{Z}_p}(\Delta_X^{\text{ab-ét}}, \Lambda_X).$$

(4) *For every open subgroup $J \subseteq I_k$ of I_k, there is* **no nontrivial torsion-free *J*-stable quotient** *of F_3/F_2 on which J acts* **trivially**.

PROOF. — This follows immediately, in light of Remark 3.1.1, from, for instance, the discussion preceding [10], Lemma 8.1, together with [10], Lemma 8.1. □

LEMMA 3.3. — *The following hold:*

(i) *Let V be a finite-dimensional representation of G_k over $\mathbb{Q}_p$. Suppose that the restriction of V to I_k is isomorphic to an extension of the direct product of finitely many copies of the trivial representation $\mathbb{Q}_p$ by the direct product of finitely many copies of the representation $\Lambda_X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then the representation V of G_k is* **semistable***.*

(ii) *Suppose that X is* **ordinary***. Then it holds that X has* **stable reduction** *if and only if the finite-dimensional representation $\Delta_X^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of I_k over $\mathbb{Q}_p$ is isomorphic to an extension of the direct product of g_X copies of the trivial representation $\mathbb{Q}_p$ by the direct product of g_X copies of the representation $\Lambda_X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

PROOF. — First, we verify assertion (i). Let us first observe that it follows from [4], Proposition of §5.1.5, that the representation V of G_k is *semistable* if and only if the restriction of V to I_k is *semistable*. Thus, to verify assertion (i), we may assume without loss of generality that the *representation V of G_k* is isomorphic to an extension of the direct product of finitely many copies of the trivial representation $\mathbb{Q}_p$ by the direct product of finitely many copies of the representation $\Lambda_X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then the assertion that the representation V of G_k is *semistable* follows immediately from the second comment following the table in the final discussion of [2], §16. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, we verify the *necessity*. Suppose that X has *stable reduction*. Then since [we have assumed that] X is *ordinary*, it follows from Proposition 2.5, (i), that the $\mathbb{Z}_p$-module $\Delta_X^{\text{ab-ét}}$ is *free of rank g_X*. Thus, since [it is well-known that] the $\mathbb{Z}_p$-module Δ_X^{ab} is *free of rank $2g_X$*, the *necessity* follows immediately, in light of Remark 2.4.1, (ii), from Lemma 3.2. This completes the proof of the *necessity*.

Finally, we verify the *sufficiency*. Suppose that the representation $\Delta_X^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of I_k is isomorphic to an extension of the direct product of g_X copies of the trivial representation $\mathbb{Q}_p$ by the direct product of g_X copies of the representation $\Lambda_X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then it follows from assertion (i) that the representation $\Delta_X^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of G_k is *semistable*. In particular, it follows from [2], Theorem 14.1, that the Jacobian variety of X has *semistable reduction* [i.e., over the ring of integers of k]. Thus, it follows from [3], Theorem 2.4, that X has *stable reduction*. This completes the proof of assertion (ii), hence also of Lemma 3.3. □

LEMMA 3.4. — *The following hold:*

(i) *The closed subgroup $\Delta_X \subseteq \Pi_X$ of Π_X may be* **characterized** *as the uniquely determined maximal* **nontrivial pro-l** *— for some prime number l —* **topologically finitely generated normal** *closed subgroup of Π_X.*

(ii) *The quotient $\Delta_X^{\mathrm{ab}} \twoheadrightarrow \Delta_X^{\text{ab-ét}}$ (respectively, $\Delta_X^{\mathrm{ab}} \twoheadrightarrow \Delta_X^{\text{ab-cmb}}$) of Δ_X^{ab} may be* **characterized** *as the uniquely determined maximal* **torsion-free** *quotient of Δ_X^{ab} which satisfies the following condition: There exists an open subgroup $J \subseteq G_k$ of G_k such that the quotient is* **J-stable***, and, moreover, the resulting action of $J \cap I_k$ (respectively, J) on the quotient is* **trivial***.*

PROOF. — First, we verify assertion (i) [cf. Remark 3.4.1 below]. Let l be a prime number and $N \subseteq \Pi_X$ a maximal *nontrivial pro-l topologically finitely generated normal* closed subgroup of Π_X. Then it is immediate that the image $\overline{N} \subseteq G_k$ of N in G_k is a *pro-l topologically finitely generated normal* closed subgroup of G_k. In particular, since G_k is *elastic* [cf. [13], Definition 1.1, (ii); [13], Theorem 1.7, (ii)], the closed subgroup $\overline{N}$ is either *trivial* or *open* in G_k. Thus, since [one verifies easily — by considering, for instance, the quotient determined by the maximal unramified extension — that] every open subgroup of G_k is *not pro-l*, we conclude that $\overline{N} = \{1\}$, i.e., that $N \subseteq \Delta_X$. Thus, since Δ_X is *pro-p*, and [we have assumed that] N is *nontrivial* and *pro-l*, it holds that $l = p$. Moreover, since Δ_X is a *nontrivial pro-p topologically finitely generated normal* closed subgroup of Π_X, it follows from the *maximality* of N that $N = \Delta_X$, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that, to verify assertion (ii), we may assume without loss of generality, by replaing k by a suitable finite extension of k contained in $\overline{k}$, that X has *stable reduction* [cf. Theorem 1.3] and is *split* [cf. Remark 2.1.1], which thus implies that [the quotients $\Delta_X^{\text{ab}} \twoheadrightarrow \Delta_X^{\text{ab-ét}} \twoheadrightarrow \Delta_X^{\text{ab-cmb}}$ are *G_k-stable* — cf. Remark 2.4.1, (i) — and, moreover]

(a) the action of I_k (respectively, G_k) on $\Delta_X^{\text{ab-ét}}$ (respectively, $\Delta_X^{\text{ab-cmb}}$) is *trivial* [cf. Remark 2.4.1, (ii)].

Thus, in light of Proposition 2.5, (i), to complete the verification of assertion (ii), it suffices to verify the following assertion:

> If $\Delta_X^{\text{ab}} \twoheadrightarrow Q$ is a *torsion-free G_k-stable* quotient of Δ_X^{ab} on which I_k (respectively, G_k) acts *trivially*, then the surjection $\Delta_X^{\text{ab}} \twoheadrightarrow Q$ *factors* through the surjection $\Delta_X^{\text{ab}} \twoheadrightarrow \Delta_X^{\text{ab-ét}}$ (respectively, $\Delta_X^{\text{ab}} \twoheadrightarrow \Delta_X^{\text{ab-cmb}}$).

To this end, let $\Delta_X^{\text{ab}} \twoheadrightarrow Q$ be a *torsion-free G_k-stable* quotient of Δ_X^{ab}. Now let us recall the sequence of G_k-stable $\mathbb{Z}_p$-submodules of Δ_X^{ab}

$$F_0 = \{0\} \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 = \Delta_X^{\text{ab}}$$

of Lemma 3.2.

To verify the non-resp'd portion of assertion (ii), suppose that the action of I_k on Q is *trivial.* Then it follows from (a), together with condition (3) of Lemma 3.2, that we have an *I_k-equivariant* isomorphism of F_2 with the direct product of finitely many copies of Λ_X. Thus, since [one verifies easily from Remark 3.1.1 that] the image of the character $I_k \to \mathbb{Z}_p^\times$ determined by the action of I_k on Λ_X is *open* in $\mathbb{Z}_p^\times$, the image of the composite $F_2 \hookrightarrow \Delta_X^{\text{ab}} \twoheadrightarrow Q$ is *zero.* Moreover, it follows from condition (4) of Lemma 3.2 that the image of $F_3/F_2 \subseteq \Delta_X^{\text{ab}}/F_2$ via the resulting surjection $\Delta_X^{\text{ab}}/F_2 \twoheadrightarrow Q$ is *zero.* Thus, the surjection $\Delta_X^{\text{ab}} \twoheadrightarrow Q$ *factors* through the surjection $\Delta_X^{\text{ab}} \twoheadrightarrow \Delta_X^{\text{ab}}/F_3 = \Delta_X^{\text{ab-ét}}$ [cf. condition (2) of Lemma 3.2]. This completes the proof of the non-resp'd portion of assertion (ii).

Next, to verify the resp'd portion of assertion (ii), suppose that the action of [not only I_k but also] G_k on the quotient Q is *trivial.* Thus, it follows from the above proof of the non-resp'd portion of assertion (ii) that, to verify the resp'd portion of assertion (ii), it suffices to verify that the image of F_4/F_3 via the resulting surjection $\Delta_X^{\text{ab}}/F_3 \twoheadrightarrow Q$ is *zero* [cf. condition (2) of Lemma 3.2]. On the other hand, this follows from Proposition 2.5, (vi), together with condition (2) of Lemma 3.2. This completes the proof of the resp'd portion of assertion (ii), hence also of assertion (ii). $\square$

REMARK 3.4.1. — Note that Lemma 3.4, (i), is a *special case* of [13], Theorem 2.6, (iv). Note, moreover, that the assertion for $\Delta_X^{\text{ab-ét}}$ in Lemma 3.4, (ii), may be considered to be *essentially the same* as [10], Lemma 8.2.

LEMMA 3.5. — *The following hold:*

(i) *Let $N \subseteq \Delta_X^{\text{ét}}$ be a normal open subgroup of $\Delta_X^{\text{ét}}$. Write $Z \to X_{\underline{\overline{k}}}$ for the finite étale Galois covering corresponding to $N \subseteq \Delta_X^{\text{ét}}$ and $b_1(Z)$ for the first Betti number of the* [*topological space determined by the*] *dual graph of Z. Then the following conditions are equivalent:*

(1) *There exists an element $v \in \text{Irr}(X)$ such that $Z \times_{X_{\underline{\overline{k}}}} I_v$ is* **connected***, and, moreover, for each $w \in \text{Irr}(X) \setminus \{v\}$, the restriction of the covering $Z \to X_{\underline{\overline{k}}}$ to the generic point of the irreducible component corresponding to w is* **trivial***.*

(2) *It holds that*
$$b_1(Z) \;=\; [\Delta_X^{\text{ét}} : N] \cdot b_1(X).$$

(ii) *Consider the following set $\mathcal{I}_X$ and the following equivalence relation $\sim_{\mathcal{I}_X}$:*

- *The set $\mathcal{I}_X$ of minimal normal open subgroups $N \subseteq \Delta_X^{\text{ét}}$ of $\Delta_X^{\text{ét}}$ such that $\Delta_X^{\text{ét}}/N$ is* **abelian** *and* **annihilated by $\boldsymbol{p}$***, and, moreover, the subgroup N satisfies conditions* (1)*,* (2) *of* (i)*.*

- *For two elements N_1, N_2 of $\mathcal{I}_X$, we write $N_1 \sim_{\mathcal{I}_X} N_2$ if there exist two splittings $s_1, s_2 \colon \Delta_X^{\text{cmb}} \hookrightarrow \Delta_X^{\text{ét}}$ of the natural surjection $\Delta_X^{\text{ét}} \twoheadrightarrow \Delta_X^{\text{cmb}}$ such that, for each $i \in \{1, 2\}$, it holds that $N_i = (N_1 \cap N_2) \cdot \text{Im}(s_i)$.*

Then there exists a **bijection**
$$\text{Irr}(X)^{\gamma>0} \;\xrightarrow{\sim}\; \mathcal{I}_X / \sim_{\mathcal{I}_X}$$
which satisfies the following condition: Let N be an element of $\mathcal{I}_X$. Write $v \in \text{Irr}(X)$ for the element corresponding, via the bijection, to [*the class determined by*] *N. Then it holds that $\text{Ker}(\Delta_X^{\text{ét}} \twoheadrightarrow \Delta_X^{\text{cmb}}) \subseteq N \cdot \mathfrak{D}_v$.*

PROOF. — First, we verify assertion (i). Write $\text{Irr}(Z)$ for the set of irreducible components of Z. Write, moreover, $\text{Nd}(X)$, $\text{Nd}(Z)$ for the sets of nodes of the stable curves $X_{\underline{\overline{k}}}$, Z, respectively. Then let us first observe that since the covering $Z \to X_{\underline{\overline{k}}}$ is *Galois* and *of degree a power of* p, one verifies easily that condition (1) is *equivalent* to the following condition (1′):

(1′) The equality
$$\sharp\text{Irr}(Z) \;=\; [\Delta_X^{\text{ét}} : N] \cdot (\sharp\text{Irr}(X) - 1) + 1$$
holds.

Next, let us observe that it follows from a well-known fact concerning the first Betti numbers of [the topological spaces determined by] connected graphs that condition (2) is *equivalent* to the following condition (2′):

(2′) The equality
$$1 - \sharp\text{Irr}(Z) + \sharp\text{Nd}(Z) \;=\; [\Delta_X^{\text{ét}} : N] \cdot (1 - \sharp\text{Irr}(X) + \sharp\text{Nd}(X))$$

holds.

On the other hand, since the covering $Z \to X_{\overline{k}}$ is *finite étale*, it holds that

$$\sharp\mathrm{Nd}(Z) \ = \ [\Delta_X^{\text{ét}} : N] \cdot \sharp\mathrm{Nd}(X).$$

Thus, assertion (i) holds. This completes the proof of assertion (i).

Assertion (ii) follows immediately from assertion (i), together with Proposition 2.5, (i), (ii), (iii). This completes the proof of Lemma 3.5. □

REMARK 3.5.1. — Note that Lemma 3.5, (i), may be regarded as a "*pro-p variant*" of the discussion of [14], Remark 1.2.3, (iii), related to the term "verticially purely totally ramified". Note, moreover, that Lemma 3.5, (ii), may be regarded as a "*pro-p variant*" of the discussion of [14], Remark 1.2.3, (iv), related to the "functorial characterization of the set of vertices of $\mathcal{G}$".

DEFINITION 3.6. — We shall say that a profinite group Π *satisfies the condition* (†) if there exist a prime number l and an isomorphism of Π with the geometrically pro-l étale fundamental group of a proper hyperbolic curve over an l-adic local field.

REMARK 3.6.1. — One verifies easily [cf. Remark 1.4.2, (i)] that if a profinite group satisfies the condition (†), then every open subgroup of the profinite group satisfies the condition (†).

THEOREM 3.7. — *In the notational conventions introduced at the beginning of* §3, *let*

$$\Pi$$

be a profinite group which satisfies the condition (†) [*cf. Definition* 3.6]. *Suppose that* Π *is isomorphic to the geometrically pro-p étale fundamental group* Π_X *of* X [*cf. Definition* 2.2]. *Let*

$$\alpha \colon \ \Pi \ \xrightarrow{\sim} \ \Pi_X$$

be an isomorphism of profinite groups. Then the following hold:

(i) *We shall write*

$$\Delta_\Pi \ \subseteq \ \Pi$$

for the [*uniquely determined*] *maximal* **nontrivial pro-*l*** — *for some prime number* l — **topologically finitely generated normal** *closed subgroup of* Π. *Then* α *restricts to an isomorphism of profinite groups*

$$\alpha_\Delta \colon \ \Delta_\Pi \ \xrightarrow{\sim} \ \Delta_X$$

[*cf. Definition* 2.2].

(ii) *We shall write*

$$G_\Pi \ \stackrel{\text{def}}{=} \ \Pi/\Delta_\Pi$$

for the quotient of Π *by* Δ_Π*. Then* α *determines an isomorphism of profinite groups*

$$\alpha_G \colon G_\Pi \xrightarrow{\sim} G_k.$$

(iii) *The profinite group* G_Π *is* **of MLF-type** [*cf.* [8], *Definition* 1.1*; also* [8], *Proposition* 1.2, (i)]*. Thus, by applying the functorial "group-theoretic" algorithm of* [8], *Theorem* 1.4, (3)*, to* G_Π*, we obtain a normal closed subgroup*

$$I_\Pi \stackrel{\text{def}}{=} I(G_\Pi) \subseteq G_\Pi.$$

Then the isomorphism α_G *of* (ii) *restricts to an isomorphism of profinite groups*

$$\alpha_I \colon I_\Pi \xrightarrow{\sim} I_k.$$

(iv) *We shall write*

$$p_\Pi$$

for the [*uniquely determined*] *prime number such that* Δ_Π *is* **pro-p_Π**. *Then it holds that*

$$p_\Pi = p.$$

(v) *We shall write*

$$\Delta_\Pi^{\text{ét}} \stackrel{\text{def}}{=} \Delta_\Pi / J_\Pi^{\text{ét}} \quad (\textit{respectively, } \Delta_\Pi^{\text{cmb}} \stackrel{\text{def}}{=} \Delta_\Pi / J_\Pi^{\text{cmb}})$$

for the quotient of Δ_Π *by the normal closed subgroup*

$$J_\Pi^{\text{ét}} \subseteq \Delta_\Pi \quad (\textit{respectively, } J_\Pi^{\text{cmb}} \subseteq \Delta_\Pi)$$

obtained by forming the intersection of the normal open subgroups $N \subseteq \Delta_\Pi$ *of* Δ_Π *which satisfy the following condition: Let*

$$N_0 = N \subseteq N_1 \subseteq \cdots \subseteq N_{r-1} \subseteq N_r = \Delta_\Pi$$

be a finite sequence of normal open subgroups of Δ_Π *such that* N_{i+1}/N_i *is* **abelian** *for each* $0 \leq i \leq r-1$ [*note that since* Δ_Π *is* **pro-p_Π**, *one verifies easily that such a sequence always exists*] *and*

$$P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{r-1} \subseteq P_r = \Pi$$

a finite sequence of open subgroups of Π *such that* $P_i \cap \Delta_\Pi = N_i$ [*which thus implies that* P_i/N_i *may be regarded as an open subgroup of* G_Π] *for each* $0 \leq i \leq r$*. Then, for each* $0 \leq i \leq r-1$*, the surjection* $N_{i+1} \twoheadrightarrow N_{i+1}/N_i$ **factors** *through the surjection onto the* [*uniquely determined*] *maximal* **abelian torsion-free** *quotient of* N_{i+1} *which satisfies the following condition: There exists an open subgroup* $J_{i+1} \subseteq P_{i+1}/N_{i+1}$ *of* P_{i+1}/N_{i+1} *such that the quotient is* **J_{i+1}-stable**, *and, moreover, the resulting action of* $J_{i+1} \cap I_\Pi$ (*respectively,* J_{i+1}) *on the quotient is* **trivial**. *Then the isomorphism* α_Δ *of* (i) *determines a commutative diagram of profinite groups*

$$\begin{array}{ccccc}
\Delta_\Pi & \longrightarrow & \Delta_\Pi^{\text{ét}} & \longrightarrow & \Delta_\Pi^{\text{cmb}} \\
{\scriptstyle \alpha_\Delta}\downarrow\wr & & {\scriptstyle \alpha_\Delta^{\text{ét}}}\downarrow\wr & & {\scriptstyle \alpha_\Delta^{\text{cmb}}}\downarrow\wr \\
\Delta_X & \longrightarrow & \Delta_X^{\text{ét}} & \longrightarrow & \Delta_X^{\text{cmb}}
\end{array}$$

[*cf. Definition* 2.3, (i), (iii)] *— where the horizontal arrows are the natural surjections, and the vertical arrows are isomorphisms of profinite groups.*

(vi) *We shall write*

$$\Delta_\Pi^{\text{ab}} \twoheadrightarrow \Delta_\Pi^{\text{ab-ét}} \twoheadrightarrow \Delta_\Pi^{\text{ab-cmb}}$$

for the respective abelianizations of Δ_Π, $\Delta_\Pi^{\text{ét}}$, Δ_Π^{cmb}. *Then the diagram of* (v) *determines a commutative diagram of profinite groups*

$$\begin{array}{ccccc} \Delta_\Pi^{\text{ab}} & \longrightarrow & \Delta_\Pi^{\text{ab-ét}} & \longrightarrow & \Delta_\Pi^{\text{ab-cmb}} \\ {\scriptstyle \alpha_\Delta^{\text{ab}}}\downarrow\wr & & {\scriptstyle \alpha_\Delta^{\text{ab-ét}}}\downarrow\wr & & {\scriptstyle \alpha_\Delta^{\text{ab-cmb}}}\downarrow\wr \\ \Delta_X^{\text{ab}} & \longrightarrow & \Delta_X^{\text{ab-ét}} & \longrightarrow & \Delta_X^{\text{ab-cmb}} \end{array}$$

[*cf. Definition* 2.4] — *where the horizontal arrows are the natural surjections, and the vertical arrows are isomorphisms of profinite groups.*

(vii) *We shall write*

$$g_\Pi \stackrel{\text{def}}{=} \frac{1}{2}\cdot \text{rank}_{\mathbb{Z}_{p_\Pi}}(\Delta_\Pi^{\text{ab}}), \quad \gamma_\Pi \stackrel{\text{def}}{=} \text{rank}_{\mathbb{Z}_{p_\Pi}}(\Delta_\Pi^{\text{ab-ét}}), \quad b_1(\Pi) \stackrel{\text{def}}{=} \text{rank}_{\mathbb{Z}_{p_\Pi}}(\Delta_\Pi^{\text{ab-cmb}}).$$

Then it holds that

$$g_\Pi = g_X, \quad \gamma_\Pi = \gamma_{X_{\overline{k}}}, \quad b_1(\Pi) = b_1(X)$$

[*cf. Definition* 1.1*; Definition* 1.4, (ii)].

(viii) *We shall write*

$$\mathcal{I}_\Pi$$

for the set of minimal normal open subgroups $N \subseteq \Delta_\Pi$ *of* Δ_Π *such that* N **contains** $J_\Pi^{\text{ét}}$, Δ_X/N *is* **abelian** *and* **annihilated by** $\boldsymbol{p_\Pi}$, *and, moreover, there exists an open subgroup* $P \subseteq \Pi$ *of* Π *such that* $P \cap \Delta_\Pi = N$ *and* $b_1(P) = [\Delta_\Pi : N]\cdot b_1(\Pi)$, *where we write* $b_1(P)$ *for the integer obtained by applying the "group-theoretic" algorithm "*$b_1(-)$*" of* (vii) *to the profinite group* P [*which satisfies the condition* (†) — *cf. Remark* 3.6.1];

$$\sim_{\mathcal{I}_\Pi}$$

for the equivalence relation on the set $\mathcal{I}_\Pi$ *defined as follows: For two elements* N_1, N_2 *of* $\mathcal{I}_\Pi$, *we write* $N_1 \sim_{\mathcal{I}_\Pi} N_2$ *if there exist two splittings* $s_1, s_2 \colon \Delta_\Pi^{\text{cmb}} \hookrightarrow \Delta_\Pi$ *of the natural surjection* $\Delta_\Pi \twoheadrightarrow \Delta_\Pi^{\text{cmb}}$ *such that, for each* $i \in \{1,2\}$, *it holds that* $N_i = (N_1 \cap N_2)\cdot \text{Im}(s_i)$;

$$\text{Irr}(\Pi)^{\gamma>0} \stackrel{\text{def}}{=} \mathcal{I}_\Pi/\sim_{\mathcal{I}_\Pi}.$$

Then the isomorphism $\alpha_\Delta^{\text{ét}}$ *of* (v) *determines — relative to the bijection of Lemma* 3.5, (ii) — *a bijection*

$$\alpha^{\text{Irr}} \colon \text{Irr}(\Pi)^{\gamma>0} \xrightarrow{\sim} \text{Irr}(X)^{\gamma>0}$$

[*cf. Definition* 1.4, (iv)].

(ix) *Let* $v_\Pi \in \text{Irr}(\Pi)^{\gamma>0}$. *Then we shall write*

$$\mathfrak{D}_{v_\Pi} \subseteq \Delta_\Pi^{\text{ét}}$$

for the [*uniquely determined, up to conjugation*] *maximal closed subgroup of* $\Delta_\Pi^{\text{ét}}$ *such that*

- *for each normal open subgroup* $P \subseteq \Pi$ *of* Π *such that* $J_\Pi^{\text{ét}} \subseteq P$, *the closed subgroup* $\mathfrak{D}_{v_\Pi} \subseteq \Delta_\Pi^{\text{ét}}$ *is contained in the* **stabilizer** [*with respect to the action induced by the action by conjugation*] *of an element of the set* $\text{Irr}(P)^{\gamma>0}$ *obtained by applying the "group-theoretic" algorithm "*$\text{Irr}(-)^{\gamma>0}$*" of* (viii) *to the profinite group* P [*which satisfies the condition* (†) — *cf. Remark* 3.6.1], *and, moreover,*

• *if* $N \subseteq \Delta_\Pi$ *is an element of* $\mathcal{I}_\Pi$ *which determines the class* $v_\Pi \in \mathrm{Irr}(\Pi)^{\gamma>0}$*, then it holds that* $\mathrm{Ker}(\Delta_\Pi^{\text{ét}} \twoheadrightarrow \Delta_\Pi^{\text{cmb}}) \subseteq \mathrm{Im}(N \hookrightarrow \Delta_\Pi \twoheadrightarrow \Delta_\Pi^{\text{ét}}) \cdot \mathfrak{D}_{v_\Pi}$.

Then the isomorphism $\alpha_\Delta^{\text{ét}}$ *of* (v) *determines a bijection between the set of conjugates of* $\mathfrak{D}_{v_\Pi} \subseteq \Delta_\Pi^{\text{ét}}$ *and the set of conjugates of* $\mathfrak{D}_{\alpha^{\mathrm{Irr}}(v_\Pi)} \subseteq \Delta_X^{\text{ét}}$ [*cf. Definition* 2.3*,* (ii)].

(x) *Let* $v_\Pi \in \mathrm{Irr}(\Pi)^{\gamma>0}$. *Then we shall write*

$$\gamma_{v_\Pi} \stackrel{\text{def}}{=} \mathrm{rank}_{\mathbb{Z}_{p_\Pi}}(\mathfrak{D}_{v_\Pi}^{\mathrm{ab}})$$

— where we write $\mathfrak{D}_{v_\Pi}^{\mathrm{ab}}$ *for the abelianization of* $\mathfrak{D}_{v_\Pi}$. *Then it holds that*

$$\gamma_{v_\Pi} = \gamma_{\alpha^{\mathrm{Irr}}(v_\Pi)}$$

[*cf. Definition* 1.4*,* (iii)].

(xi) *We shall say that the profinite group* Π *is* **ordinary** *if the equality* $g_\Pi = \gamma_\Pi$ *holds. We shall say that the profinite group* Π *is* **rationally degenerate** *if* Π *is ordinary, and, moreover,* $\mathrm{Irr}(\Pi)^{\gamma>0} = \emptyset$. *Then it holds that* Π *is* **ordinary** (*respectively,* **rationally degenerate**) *if and only if* X *is* **ordinary** [*cf. Definition* 2.6*,* (i)] (*respectively,* **rationally degenerate** [*cf. Definition* 2.6*,* (ii)]).

(xii) *Suppose that* Π *is* **ordinary** [*which thus implies that* X *is* **ordinary** *— cf.* (xi)]. *Then we shall say that the profinite group* Π *has* **stable reduction** *if the representation* $\Delta_\Pi^{\mathrm{ab}} \otimes_{\mathbb{Z}_{p_\Pi}} \mathbb{Q}_{p_\Pi}$ *of* I_Π *is isomorphic to an extension of the direct product of* g_Π *copies of the trivial representation* $\mathbb{Q}_{p_\Pi}$ *by the direct product of* g_Π *copies of the representation*

$$\mathrm{Hom}_{\mathbb{Z}_{p_\Pi}}\big(H^2(\Delta_\Pi, \mathbb{Z}_{p_\Pi}), \mathbb{Q}_{p_\Pi}\big).$$

Then it holds that Π *has* **stable reduction** *if and only if* X *has* **stable reduction** [*cf. Definition* 1.2].

(xiii) *Suppose that* Π *is* **ordinary** [*which thus implies that* X *is* **ordinary** *— cf.* (xi)]. *Then we shall say that the profinite group* Π *has* **good reduction** *if* Π *has stable reduction, and, moreover,* $b_1(P) = 0$ *for every open subgroup* $P \subseteq \Pi$ *of* Π*, where we write* $b_1(P)$ *for the integer obtained by applying the "group-theoretic" algorithm "*$b_1(-)$*" of* (vii) *to the profinite group* P [*which satisfies the condition* (†) *— cf. Remark* 3.6.1]. *Then it holds that* Π *has* **good reduction** *if and only if* X *has* **good reduction** [*cf. Definition* 1.2].

PROOF. — Assertions (i), (ii) follow from Lemma 3.4, (i). Assertion (iii) follows from [8], Theorem 1.4, (ii), together with assertion (ii). Assertion (iv) follows from assertion (i). Assertions (v), (vi) follow from Lemma 3.4, (ii), together with assertions (i), (ii), (iii). The assertion for g_Π in assertion (vii) follows from assertions (iv), (vi), together with the well-known fact that the $\mathbb{Z}_p$-module Δ_X^{ab} is *free of rank* $2g_X$. The assertion for γ_Π and $b_1(\Pi)$ in assertion (vii) follows from Proposition 2.5, (i), together with assertion (iv), (vi). Assertions (viii), (ix) follow, in light of the *finiteness* of $\mathrm{Irr}(-)^{\gamma>0}$, from Lemma 3.5, (ii), together with assertions (i), (iv), (v), (vii). Assertion (x) follows from Proposition 2.5, (ii), together with assertions (iv), (ix). Assertion (xi) follows from Lemma 2.7, (ii), together with assertions (vii), (viii). Assertion (xii) follows, in light of Definition 3.1, from Lemma 3.3, (ii), together with assertions (i), (iii), (iv), (vi), (vii), (xi). Assertion (xiii) follows from Theorem 2.9, (iii), together with assertions (vii), (xi), (xii). This completes the proof of Theorem 3.7. □

COROLLARY 3.8. — *For* $\square \in \{\circ, \bullet\}$, *let* $p_\square$ *be a prime number,* $k_\square$ *a* $p_\square$*-adic local field, and* $X_\square$ *a* **proper hyperbolic curve** *over* $k_\square$; *write* $\Pi_{X_\square}$ *for the geometrically pro-*$p_\square$ *étale fundamental group of* $X_\square$ [*cf. Definition* 2.2]. *Let*

$$\alpha\colon\ \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Then the following hold:

(i) *It holds that* $p_\circ = p_\bullet$, $g_{X_\circ} = g_{X_\bullet}$ [*cf. Definition* 1.1], *and* $b_1(X_\circ) = b_1(X_\bullet)$ [*cf. Definition* 1.4, (ii)].

(ii) *The isomorphism* α *determines a commutative diagram of profinite groups*

$$\begin{array}{ccccc}
\Delta_{X_\circ} & \longrightarrow & \Delta^{\text{ét}}_{X_\circ} & \longrightarrow & \Delta^{\text{cmb}}_{X_\circ} \\
{\scriptstyle\alpha_\Delta}\downarrow\wr & & {\scriptstyle\alpha^{\text{ét}}_\Delta}\downarrow\wr & & {\scriptstyle\alpha^{\text{cmb}}_\Delta}\downarrow\wr \\
\Delta_{X_\bullet} & \longrightarrow & \Delta^{\text{ét}}_{X_\bullet} & \longrightarrow & \Delta^{\text{cmb}}_{X_\bullet}
\end{array}$$

[*cf. Definition* 2.3, (i), (iii)] — *where the horizontal arrows are the natural surjections, and the vertical arrows are isomorphisms of profinite groups.*

(iii) *There exists a bijection*

$$\alpha^{\mathrm{Irr}}\colon\ \mathrm{Irr}(X_\circ)^{\gamma>0} \xrightarrow{\sim} \mathrm{Irr}(X_\bullet)^{\gamma>0}$$

[*cf. Definition* 1.4, (iv)] *such that, for each* $v \in \mathrm{Irr}(X_\circ)^{\gamma>0}$,

(1) *the isomorphism* $\alpha^{\text{ét}}_\Delta$ [*cf.* (ii)] *determines a bijection between the set of conjugates of* $\mathfrak{D}_v \subseteq \Delta^{\text{ét}}_{X_\circ}$ [*cf. Definition* 2.3, (ii)] *and the set of conjugates of* $\mathfrak{D}_{\alpha^{\mathrm{Irr}}(v)} \subseteq \Delta^{\text{ét}}_{X_\bullet}$, *and*

(2) *it holds that* $\gamma_v = \gamma_{\alpha^{\mathrm{Irr}}(v)}$ [*cf. Definition* 1.4, (iii)].

(iv) *It holds that* $X_\circ$ *is* **ordinary** [*cf. Definition* 2.6, (i)] (*respectively,* **rationally degenerate** [*cf. Definition* 2.6, (ii)]) *if and only if* $X_\bullet$ *is* **ordinary** (*respectively,* **rationally degenerate**).

(v) *Suppose, moreover, that either* $X_\circ$ *or* $X_\bullet$ *is* **ordinary**. *Then it holds that* $X_\circ$ *has* **stable reduction** [*cf. Definition* 1.2] *if and only if* $X_\bullet$ *has* **stable reduction**.

(vi) *Suppose, moreover, that either* $X_\circ$ *or* $X_\bullet$ *is* **ordinary**. *Then it holds that* $X_\circ$ *has* **good reduction** [*cf. Definition* 1.2] *if and only if* $X_\bullet$ *has* **good reduction**.

PROOF. — Assertion (i) follows from Theorem 3.7, (iv), (vii). Assertion (ii) follows from Theorem 3.7, (v). Assertion (iii) follows from Theorem 3.7, (viii), (ix), (x). Assertion (iv) follows from Theorem 3.7, (xi). Assertion (v) follows, in light of assertion (iv), from Theorem 3.7, (xii). Assertion (vi) follows, in light of assertion (iv), from Theorem 3.7, (xiii). This completes the proof of Corollary 3.8. $\square$

REMARK 3.8.1.

(i) Note that Theorem 3.7, (xiii), may be regarded as a *pro-p "group-theoretic" criterion for good reduction* of *ordinary* proper hyperbolic curves over p-adic local fields. As a consequence of the *"group-theoreticity"*, Theorem 3.7, (xiii), implies in fact Corollary 3.8, (vi).

(ii) Let Σ be a nonempty set of prime numbers such that $p \notin \Sigma$. Then we have a *pro-Σ "group-theoretic" criterion for good reduction* of [not necessarily ordinary] hyperbolic curves over p-adic local fields in the following sense: Let C be a [not necessarily proper] hyperbolic curve over k and Π a profinite group which is isomorphic to the geometrically pro-Σ étale fundamental group of C [i.e., the quotient of the étale fundamental group of C obtained by replacing "pro-p" in the definition of the "geometrically pro-p étale fundamental group Π_X" in Definition 2.2 by pro-Σ]. Then it follows from [13], Theorem 2.6, (iv), that one may define a normal closed subgroup $\Delta_\Pi \subseteq \Pi$ of Π which corresponds to the pro-Σ geometric étale fundamental group of C [i.e., the quotient of the étale fundamental group of $C \times_k \overline{k}$ obtained by replacing "pro-p" in the definition of the "pro-p geometric étale fundamental group Δ_X" in Definition 2.2 by pro-Σ]. Thus, one may also define a normal closed subgroup $I_\Pi \subseteq \Pi/\Delta_\Pi$ of Π/Δ_Π which corresponds to the inertia subgroup I_k of G_k [cf., e.g., Theorem 3.7, (iii)]. Then [18], Theorem 3.2, and [20], Theorem 5.3, assert that

> it holds that C has *good reduction* [cf. [20], Definition 5.1] if and only if the image of the restriction of the action $\Pi \to \mathrm{Aut}(\Delta_\Pi)$ by conjugation to the closed subgroup $\Pi \times_{\Pi/\Delta_\Pi} I_\Pi \subseteq \Pi$ is contained in the *subgroup of inner automorphisms of* Δ_Π.

(iii) Note that, by [the proof of] [12], Corollary 2.8, in the situation of (ii), one may establish a functorial "group-theoretic" algorithm for reconstructing, from Π, the dual semi-graph of the special fiber of the stable model of $C \times_k \overline{k}$ over the ring of integers of $\overline{k}$.

REMARK 3.8.2. — Let us discuss the *p-adic criterion for good reduction* of curves proved by *F. Andreatta*, *A. Iovita*, and *M. Kim* in [1] from the point of view of the present paper:

(i) In [1], *F. Andreatta*, *A. Iovita*, and *M. Kim* proved a *p-adic criterion for good reduction* of curves. Here, let us recall [1], Theorem 1.9, briefly from the point of view of the present paper:

> In the notational conventions introduced at the beginning of §3 of the present paper, by considering [neutral tannakian] categories of certain finite-dimensional unipotent representations of the profinite group Δ_X over $\mathbb{Q}_p$, one may define, for each positive integer n, a finite-dimensional representation $\mathcal{E}_n^{\text{ét}}$ of Π_X over $\mathbb{Q}_p$. Let $b \in X(k)$ be a k-rational point of X. Then, by restricting the representation $\mathcal{E}_n^{\text{ét}}$ to the splitting [well-defined up to Δ_X-conjugation] of the natural surjection $\Pi_X \twoheadrightarrow G_k$ induced by b, one obtains, for each positive integer n, a finite-dimensional representation $\mathcal{E}_{n,b}^{\text{ét}}$ of G_k over $\mathbb{Q}_p$. Then [1], Theorem 1.9, asserts that X has *good reduction* if and only if the representation $\mathcal{E}_{n,b}^{\text{ét}}$ of G_k is *crystalline* for every positive integer n.

(ii) The p-adic criterion of (i) [is interesting even in a certain point of view of anabelian geometry but] should be considered to be *not "group-theoretic"* [i.e., to be *not useful* in *pro-p absolute anabelian geometry*] by the following two reasons:

(1) The issue of whether or not a given finite-dimensional representation of G_k over $\mathbb{Q}_p$ is *crystalline* is *not "group-theoretic"*. Indeed, it follows immediately from the discussion of [7], Remark 3.3.1, that there exist a prime number l, an l-adic local field L, an automorphism α of the absolute Galois group G_L of L, and a *crystalline* representation $\rho\colon G_L \to \mathrm{GL}_n(\mathbb{Q}_l)$ such that the composite $G_L \overset{\alpha}{\xrightarrow{\sim}} G_L \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{Q}_l)$ is *not crystalline*.

(2) It is *not clear* that the issue of whether or not a given splitting of the natural surjection $\Pi_X \twoheadrightarrow G_k$ arises from a k-rational point of X is *"group-theoretic"*. Note that it follows from [5], Theorem A, that there exist a prime number l, an l-adic local field L, a proper hyperbolic curve C over L, and a splitting of the natural surjection from the geometrically pro-l étale fundamental group of C onto the absolute Galois group of L which does *not arise from an L-rational point of C*.

(iii) As a consequence of the discussion of (ii), the p-adic criterion of (i) does *not*, at least in the immediate literal sense, *imply* the following assertion:

(3) In the situation of Corollary 3.8, it holds that $X_\circ$ has *good reduction* if and only if $X_\bullet$ has *good reduction*.

Note that it is not clear to the author at the time of writing whether or not the above assertion (3) is valid [without ordinary assumption].

(iv) In an attempt to apply the p-adic criterion of (i) to the study of assertion (3), in order to avoid the problem arising from the fact that the issue of whether or not a given finite-dimensional representation of G_k over $\mathbb{Q}_p$ is *crystalline* is *not "group-theoretic"* [i.e., (1) of the discussion of (ii)], one may consider the following assumption:

(4) In the situation of Corollary 3.8, if we write $p \overset{\mathrm{def}}{=} p_\circ = p_\bullet$ [cf. Corollary 3.8, (i)] and $\alpha_G\colon \mathrm{Gal}(\overline{k}_\circ/k_\circ) \xrightarrow{\sim} \mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$ — where $\overline{k}_\circ$, $\overline{k}_\bullet$ are respective appropriate algebraic closures of $k_\circ$, $k_\bullet$ — for the isomorphism induced by α [cf. Theorem 3.7, (ii)], then, for every finite extension $k'_\bullet$ of $k_\bullet$ in $\overline{k}_\bullet$ and every *crystalline* representation $\rho\colon \mathrm{Gal}(\overline{k}_\bullet/k'_\bullet) \to \mathrm{GL}_n(\mathbb{Q}_p)$ of $\mathrm{Gal}(\overline{k}_\bullet/k'_\bullet)$, the composite $\mathrm{Gal}(\overline{k}_\circ/k'_\circ) \overset{\alpha_G}{\xrightarrow{\sim}} \mathrm{Gal}(\overline{k}_\bullet/k'_\bullet) \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{Q}_p)$ — where we write $k'_\circ$ for the finite extension of $k_\circ$ in $\overline{k}_\circ$ corresponding, via α_G, to $k'_\bullet$ — is a *crystalline* representation of $\mathrm{Gal}(\overline{k}_\circ/k'_\circ)$.

On the other hand, it follows immediately from a similar argument to the argument applied in the proof of [7], Theorem, that assumption (4) implies that the isomorphism α_G arises from an isomorphism of fields $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ which restricts to an isomorphism of fields $k_\bullet \xrightarrow{\sim} k_\circ$. In particular, it follows immediately from [11], Theorem A, that α arises from an isomorphism of schemes $X_\circ \xrightarrow{\sim} X_\bullet$, which thus implies the equivalence discussed in assertion (3). That is to say, assertion (3) under assumption (4) may be verified *without the p-adic criterion of (i)*.

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