On the Supersingular Divisors of Nilpotent Admissible Indigenous Bundles

Yuichiro Hoshi

March 2018

Abstract. — In the present paper, we give a characterization of the supersingular divisors [i.e., the zero loci of the Hasse invariants] of nilpotent admissible/ordinary indigenous bundles on hyperbolic curves. By applying the characterization, we also obtain lists of the nilpotent indigenous bundles on certain hyperbolic curves. Moreover, we prove the hyperbolic ordinariness of certain hyperbolic curves.

Contents

Introduction

Let p be an *odd* prime number, k an algebraically closed field of characteristic p, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and

$$(X, D)$$

a *hyperbolic curve* of type (g, r) over k — i.e., a pair consisting of a projective smooth curve X of genus g over k and a reduced closed subscheme $D \subseteq X$ of X of degree r [cf. (1.a), (1.b)]. Write (X^F, D^F) for the hyperbolic curve over k obtained by forming the pull-back of (X, D) via the absolute Frobenius morphism of k; $\Phi\colon X \to X^F$ for the relative Frobenius morphism of X/k; $\tau^{\log}$, $(\tau^{\log})^F$ for the logarithmic tangent sheaves of $(X, D)/k$, $(X^F, D^F)/k$, respectively [cf. (1.c)].

2010 Mathematics Subject Classification. — 14G17.
Key words and phrases. — p-adic Teichmüller theory, nilpotent admissible indigenous bundle, nilpotent ordinary indigenous bundle, supersingular divisor, hyperbolically ordinary.

First, let us recall the notion of an *indigenous bundle* and some properties on an indigenous bundle. We shall say that a pair

$$(\pi\colon P \to X, \nabla_P)$$

consisting of a $\mathbb{P}^1$-bundle $\pi\colon P \to X$ over X and a connection ∇_P on P relative to $(X, D)/k$ is an *indigenous bundle* on $(X, D)/k$ if the monodromy operator of ∇_P at each point on $D \subseteq X$ is nilpotent, and, moreover, there exists a [unique — cf. [7], Chapter I, Proposition 2.4] nonhorizontal section [i.e., the *Hodge section*] $\sigma\colon X \to P$ of π *of canonical height* [cf. the discussion preceding [7], Chapter I, Definition 2.2] $\deg \tau^{\log}/2$ [cf. [7], Chapter I, Definition 2.2]. The notion of an indigenous bundle was introduced and studied by *R. C. Gunning* [cf. [3], §2] and enables one to understand the theory of uniformization of Riemann surfaces in a somewhat more algebraic setting.

Let $(\pi\colon P \to X, \nabla_P)$ be an indigenous bundle on $(X, D)/k$. Then the connection ∇_P on P determines a horizontal homomorphism [i.e., the *p-curvature homomorphism*]

$$\mathcal{P}\colon\ \Phi^*(\tau^{\log})^F \ \longrightarrow\ \pi_*\tau_{P/X}.$$

We shall say that the indigenous bundle $(\pi\colon P \to X, \nabla_P)$ is *nilpotent* (respectively, *admissible*) if the square of $\mathcal{P}$ is zero (respectively, the dual of $\mathcal{P}$ is surjective) [cf. [7], Chapter II, Definition 2.4]. Moreover, we shall refer to the composite

$$\Phi^*(\tau^{\log})^F \ \xrightarrow{\mathcal{P}}\ \pi_*\tau_{P/X} \ \twoheadrightarrow\ \tau^{\log}$$

of the p-curvature homomorphism $\mathcal{P}$ and the surjection $\pi_*\tau_{P/X} \twoheadrightarrow \tau^{\log}$ determined by the Hodge section of $(\pi\colon P \to X, \nabla_P)$ as the *square Hasse invariant* of $(\pi\colon P \to X, \nabla_P)$ [cf. [7], Chapter II, Proposition 2.6, (1)]. Then, by applying "H^1" to the square Hasse invariant, one may obtain a k-linear homomorphism $H^1(X^F, (\tau^{\log})^F) \to H^1(X, \tau^{\log})$, i.e., the *Frobenius on* $H^1(X, \tau^{\log})$ *induced by* $(\pi\colon P \to X, \nabla_P)$ [cf. the discussion following [7], Chapter II, Lemma 2.11]. We shall say that the indigenous bundle $(\pi\colon P \to X, \nabla_P)$ is *ordinary* if the Frobenius on $H^1(X, \tau^{\log})$ induced by $(\pi\colon P \to X, \nabla_P)$ is an isomorphism [cf. [7], Chapter II, Definition 3.1].

Nilpotent admissible/ordinary indigenous bundles play some important roles in the theory of *hyperbolically ordinary curves* established by *S. Mochizuki* [cf. [7]]. Now let us recall that, in [2], *L. R. A. Finotti* studied nilpotent ordinary indigenous bundles on hyperbolic curves of type $(2, 0)$ [cf. also [5], Remark 6.1.2]. Moreover, in [1], *I. I. Bouw* and *S. Wewers* studied nilpotent ordinary indigenous bundles on hyperbolic curves of type $(0, 4)$ [cf. also Remark 4.7.1]. In the present paper, we study nilpotent admissible/ordinary indigenous bundles.

Let $(\pi\colon P \to X, \nabla_P)$ be a nilpotent admissible indigenous bundle on $(X, D)/k$. Then let us recall that, by the theory of hyperbolically ordinary curves, one may prove that there exists an effective divisor E on X such that $2E$ coincides with the zero locus of the square Hasse invariant of $(\pi\colon P \to X, \nabla_P)$ [cf. [7], Chapter II, Proposition 2.6, (3)]. We shall refer to this effective divisor E on X as the *supersingular divisor* of $(\pi\colon P \to X, \nabla_P)$ [cf. [7], Chapter II, Proposition 2.6, (3)]. The supersingular divisor is an important invariant of a nilpotent admissible indigenous bundle. For instance, the isomorphism class of a nilpotent admissible indigenous bundle is *completely determined* by the supersingular divisor [cf. [7], Chapter II, Proposition 2.6, (4)].

In [5], the author of the present paper gave a characterization of the supersingular divisors of nilpotent admissible/ordinary indigenous bundles in the case where $(r, p) =$

$(0, 3)$, i.e., on *projective* hyperbolic curves *of characteristic three*. The characterization of [5] asserts that if $(r, p) = (0, 3)$, then it holds that a given effective divisor on X coincides with the *supersingular divisor* of a nilpotent admissible indigenous bundle on X if and only if the divisor is *reduced* and may be obtained by forming the zero locus of a *Cartier eigenform* [cf. [5], Definition A.8, (ii)] associated to a square-trivialized invertible sheaf [cf. [5], Definition A.3] on X [cf. [5], Theorem B]. Moreover, in this case, it holds that the nilpotent admissible indigenous bundle on X is *ordinary* if and only if either

• the underlying invertible sheaf of the square-trivialized invertible sheaf is *trivial*, and the Jacobian variety of X is *ordinary*, or

• the underlying invertible sheaf of the square-trivialized invertible sheaf is *nontrivial* [i.e., *of order two*], and the Prym variety associated to the underlying invertible sheaf is *ordinary*

[cf. [5], Theorem B].

In the present paper, we give another characterization of the supersingular divisors of nilpotent admissible/ordinary indigenous bundles on hyperbolic curves [in the case where (r, p) is not necessarily equal to $(0, 3)$]. The main result of the present paper is as follows [cf. Theorem 3.9, Theorem 3.10].

THEOREM A. — *Let us apply the notational conventions introduced in* §1*. By abuse of notation, write*

$$C\colon\ \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

for the [*necessarily surjective*] *k-linear homomorphism obtained by applying "$\Gamma(X^F, -\otimes_{\mathcal{O}_{X^F}} (\omega^{\log})^F)$" to the Cartier operator associated to X/k and*

$$d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

for the k-linear homomorphism determined by the exterior differentiation operator. Let

$$E$$

be an effective divisor on X. Then the following hold.

(i) *It holds that the divisor E coincides with the* **supersingular divisor** *of a* **nilpotent admissible** *indigenous bundle on $(X, D)/k$ if and only if the following three conditions are satisfied.*

(1) *The divisor E is* **of degree** $p^* \deg \omega^{\log}$.

(2) *The composite*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \hookrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \overset{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is **surjective**.

(3) *The k-vector space $\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$ is* **not generated** *by the subspace*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

and the image of the k-linear homomorphism

$$d\colon\ \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)).$$

(ii) *It holds that the divisor E coincides with the* **supersingular divisor** *of a* **nilpotent ordinary** *indigenous bundle on $(X, D)/k$ if and only if the following three conditions are satisfied.*

(1) *The divisor E is* **of degree** $p^{\divideontimes} \deg \omega^{\log}$.

(2′) *The composite*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \hookrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \overset{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is **surjective**.

(3) *The k-vector space $\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$ is* **not generated** *by the subspace*

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

and the image of the k-linear homomorphism

$$d\colon \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)).$$

By applying Theorem A, we obtain the following result concerning nilpotent indigenous bundles on certain hyperbolic curves [cf. Proposition 4.6, Proposition 5.2, Proposition 5.5, Proposition 5.7].

THEOREM B. — *The following hold.*

(i) *Suppose that $(g, r, p) = (0, 4, 3)$. Then (X, D) has* **precisely three** *nilpotent indigenous bundles. Moreover, every nilpotent indigenous bundle on $(X, D)/k$ is* **ordinary**, *hence also* **admissible**. *The* **supersingular divisor** *of a nilpotent [necessarily admissible] indigenous bundle on $(X, D)/k$ coincides with the reduced effective divisor on X of degree two obtained by forming the* **fixed locus** *of one of the three nontrivial* **nonspecial** *[cf. Definition 4.5] automorphisms of (X, D) over k.*

(ii) *Suppose that $(g, r, p) = (1, 1, 3)$. Then (X, D) has* **precisely three** *nilpotent indigenous bundles. Moreover, every nilpotent indigenous bundle on $(X, D)/k$ is* **ordinary**, *hence also* **admissible**. *The* **supersingular divisor** *of a nilpotent [necessarily admissible] indigenous bundle on $(X, D)/k$ coincides with the reduced effective divisor on X of degree one determined by one of the three* **nontrivial 2-torsion points** *of the elliptic curve determined by (X, D).*

(iii) *Suppose that $(g, r, p) = (1, 1, 5)$. If the elliptic curve over k determined by (X, D) is* **ordinary** *(respectively,* **supersingular**), *then (X, D) has* **precisely five** *(respectively,* **four**) *nilpotent indigenous bundles. Moreover, every nilpotent indigenous bundle on $(X, D)/k$ is* **admissible**. *The* **supersingular divisor** *of a nilpotent [necessarily admissible] indigenous bundle on $(X, D)/k$ may be described explicitly [cf. Proposition 5.5, (iii)]. Finally, a nilpotent indigenous bundle on $(X, D)/k$ is* **ordinary** *if and only if one of the following two conditions is satisfied.*

(1) *The supersingular divisor of the nilpotent [necessarily admissible] indigenous bundle coincides with the reduced effective divisor on X of degree two determined by two of the three* **nontrivial 2-torsion points** *of the elliptic curve determined by (X, D).*

(2) *The elliptic curve determined by (X, D) is* **ordinary**.

(iv) *Suppose that* $(g, r, p) = (1, 1, 7)$*. Then* (X, D) *has* **at least one** *nilpotent* **ordinary** *indigenous bundle whose* **supersingular divisor** *coincides with the reduced effective divisor on* X *of degree three determined by the three* **nontrivial 2-torsion points** *of the elliptic curve determined by* (X, D)*.*

Here, let us recall the following basic question in *p-adic Teichmüller theory* discussed in [8], Introduction, §2.1 [cf. [8], Introduction, §2.1, (1)].

> Is every pointed stable curve *hyperbolically ordinary* [cf. [7], Chapter II, Definition 3.3]?

In the present paper, we prove the following result concerning the above basic question [cf. Corollary 4.7, Corollary 5.3, Corollary 5.8].

Theorem C. — *If*

$$(g, r, p) \ \in \ \{(0, 4, 3),\ (1, 1, 3),\ (1, 1, 7)\},$$

then every hyperbolic curve of type (g, r) *over a connected noetherian scheme of characteristic* p *is* **hyperbolically ordinary***.*

Now we have the following remarks concerning Theorem C.

- A similar result to Theorem C in the case where

$$(g, r) \ = \ (0, 3)$$

is a consequence of [7], Chapter II, Theorem 2.3 [cf. Proposition 4.2 of the present paper and the discussion at the beginning of §4, (4.a), of the present paper]. In §4, (4.a), of the present paper, we give an alternative verification of this result by means of the main result of the present paper.

- A similar result to Theorem C in the case where

$$(g, r, p) \ = \ (1, 1, 5)$$

has already been verified in [8] [cf. Remark 5.6.1 of the present paper]. In §5, (5.b), of the present paper, we give an alternative verification of this result by means of the main result of the present paper [cf. Corollary 5.6].

- A similar result to Theorem C in the case where

$$(g, r, p) \ = \ (2, 0, 3)$$

is the content of [5], Theorem D.

- Theorem C in the case where

$$(g, r, p) \ = \ (0, 4, 3)$$

"*follows*" from [1], Proposition 6.4. However, unfortunately, the proof of [1], Lemma 6.3 — which implies [1], Proposition 6.4 — contains an *error* [cf. Remark 4.7.1 of the present paper].

Finally, in §A, we discuss the relationship between the zero loci of *square Hasse invariants* [cf. [7], Chapter II, Proposition 2.6, (1)] and the zero loci of *canonical sections* discussed in [1], §3.

Acknowledgments

The author would like to thank *Irene I. Bouw* for discussions [by email, March 2015] concerning Remark 4.7.1. This research was supported by the Inamori Foundation and JSPS KAKENHI Grant Number 15K04780.

1. Notational Conventions

In the present §1, we introduce some notational conventions applied in the present paper.

(1.a). Throughout the present paper, let p be an *odd* prime number and k an algebraically closed field of characteristic p. We shall write

$$p^{*} \stackrel{\text{def}}{=} \frac{p-1}{2}.$$

If "$(-)$" is either a scheme over k, a sheaf of modules on a scheme over k, or a section of a sheaf of modules on a scheme over k, then we shall write "$(-)^F$" for the corresponding object over k obtained by forming the pull-back of "$(-)$" via the absolute Frobenius morphism of k.

(1.b). Throughout the present paper, let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$ and

$$(X, D)$$

a *hyperbolic curve* of type (g, r) over k, i.e., a pair consisting of a projective smooth curve X of genus g over k and a reduced closed subscheme $D \subseteq X$ of X of degree r. We shall write

$$\omega$$

for the *cotangent sheaf* of X/k,

$$\tau$$

for the *tangent sheaf* of X/k, and

$$\Phi\colon\ X \longrightarrow X^F$$

for the *relative Frobenius morphism* of X/k. In particular, the sheaves ω^F, τ^F [cf. (1.a)] may be naturally identified with the cotangent sheaf, tangent sheaf of X^F/k, respectively.

(1.c). It is immediate that the pair (X, D) of (1.b) naturally determines a *log smooth* [cf. [6], (3.3)] *fine log scheme* [cf. [6], (2.3)] over k [cf. [6], Example (2.5)]. We shall write

$$\omega^{\log}$$

for the *cotangent sheaf* of the resulting log scheme over k [cf. [6], (1.7)] and

$$\tau^{\log} \stackrel{\text{def}}{=} \mathcal{H}om_{\mathcal{O}_X}(\omega^{\log}, \mathcal{O}_X)$$

for the *tangent sheaf* of the resulting log scheme over k. Note that it follows immediately from the various definitions involved that the natural morphism from the resulting log scheme to X determines *isomorphisms* of $\mathcal{O}_X$-modules

$$\omega(D) \xrightarrow{\sim} \omega^{\log}, \quad \tau(-D) \xrightarrow{\sim} \tau^{\log}.$$

We shall write

$$d \colon \mathcal{O}_X \longrightarrow \omega$$

for the exterior differentiation operator. By abuse of notation, we shall write

$$d \colon \mathcal{O}_X \longrightarrow \omega^{\log}$$

for the exterior differentiation operator obtained by forming the composite of d and the natural inclusion $\omega \hookrightarrow \omega^{\log}$. Note that since (X, D) is *hyperbolic*, it holds that the invertible sheaf $\omega^{\log}$ on X is *ample*, i.e., that $\deg \omega^{\log}$ $(= 2g - 2 + r)$ is *positive*.

(1.d). If $\mathcal{L}$ is an invertible sheaf on X, then, by mapping the p-th power of each local section l of $\mathcal{L}$ to the pull-back, via Φ, of the local section l^F of $\mathcal{L}^F$ determined by the local section l of $\mathcal{L}$, we have an *isomorphism* of $\mathcal{O}_X$-modules

$$\mathcal{L}^{\otimes p} \xrightarrow{\sim} \Phi^* \mathcal{L}^F.$$

Let us always identify $\mathcal{L}^{\otimes p}$ with $\Phi^* \mathcal{L}^F$ by means of this isomorphism.

(1.e). If $\mathcal{E}$ is a locally free coherent $\mathcal{O}_{X^F}$-module, then it is immediate that the k-linear homomorphism

$$\Phi^* \mathcal{E} = \mathcal{O}_X \otimes_{\Phi^{-1}\mathcal{O}_{X^F}} \Phi^{-1}\mathcal{E} \xrightarrow{d \otimes \mathrm{id}_{\Phi^{-1}\mathcal{E}}} \omega^{\log} \otimes_{\Phi^{-1}\mathcal{O}_{X^F}} \Phi^{-1}\mathcal{E} = \omega^{\log} \otimes_{\mathcal{O}_X} \Phi^* \mathcal{E}$$

is a *connection* on $\Phi^* \mathcal{E}$ [relative to $(X, D)/k$]. We shall write

$$d_{\mathcal{E}}$$

for this connection on $\Phi^* \mathcal{E}$.

(1.f). By applying [6], Theorem (4.12), to the log smooth fine log scheme over k determined by the pair (X, D) (respectively, the scheme X), we obtain an *exact* sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_* \mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_* \omega^{\log} \xrightarrow{C^{\log}} (\omega^{\log})^F \longrightarrow 0$$

$$(\text{respectively, } 0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_* \mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_* \omega \xrightarrow{C} \omega^F \longrightarrow 0).$$

We shall refer to the fourth arrow

$$C^{\log} \colon \Phi_* \omega^{\log} \longrightarrow (\omega^{\log})^F \quad (\text{respectively, } C \colon \Phi_* \omega \longrightarrow \omega^F)$$

as the *Cartier operator* associated to $(X, D)/k$ (respectively, X/k).

(1.g). We shall write

$$\mathcal{T} \stackrel{\mathrm{def}}{=} \Phi^*(\tau^{\log})^F.$$

Thus, if $\mathcal{E}$ is a locally free coherent $\mathcal{O}_X$-module, then the sheaf $(\tau^{\log})^F \otimes_{\mathcal{O}_{X^F}} \Phi_*\mathcal{E}$ may be naturally identified with $\Phi_*(\mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{E})$. Moreover, we have a *connection* on $\mathcal{T}$ [cf. (1.e)]

$$\nabla_{\mathcal{T}} \stackrel{\mathrm{def}}{=} d_{(\tau^{\log})^F} \colon \mathcal{T} \longrightarrow \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}.$$

(1.h). We shall write

$$\mathcal{M}_{g,[r]}$$

for the moduli stack of hyperbolic curves of type (g, r) over k;

$$(\mathcal{X}_{g,[r]}, \mathcal{D}_{g,[r]})$$

for the universal hyperbolic curve over $\mathcal{M}_{g,[r]}$;

$$\mathcal{N}_{g,[r]}$$

for the moduli stack of *smooth nilcurves* [cf. the discussion preceding [8], Introduction, Theorem 0.1] of type (g, r) over k, i.e., the moduli stack of hyperbolic curves of type (g, r) over k equipped with nilpotent [cf. [7], Chapter II, Definition 2.4] indigenous bundles [cf. [7], Chapter I, Definition 2.2];

$$\mathcal{N}^{\mathrm{adm}}_{g,[r]} \subseteq \mathcal{N}_{g,[r]}$$

for the *admissible locus* of $\mathcal{N}_{g,[r]}$, i.e., the [necessarily open] substack which parametrizes hyperbolic curves of type (g, r) over k equipped with nilpotent *admissible* [cf. [7], Chapter II, Definition 2.4] indigenous bundles;

$$\mathcal{N}^{\mathrm{ord}}_{g,[r]} \subseteq \mathcal{N}^{\mathrm{adm}}_{g,[r]}$$

for the *ordinary locus* of $\mathcal{N}_{g,[r]}$, i.e., the [necessarily open] substack which parametrizes hyperbolic curves of type (g, r) over k equipped with nilpotent *ordinary* [cf. [7], Chapter II, Definition 3.1] indigenous bundles;

$$\mathcal{M}_{g,r} \longrightarrow \mathcal{M}_{g,[r]}$$

for the connected finite étale Galois covering [whose Galois group is isomorphic to $\mathfrak{S}_r$] which trivializes the étale local system on $\mathcal{M}_{g,[r]}$ obtained by considering *"ordering on the r marked points"*;

$$(\mathcal{X}_{g,r}, \mathcal{D}_{g,r}) \stackrel{\mathrm{def}}{=} (\mathcal{X}_{g,[r]}, \mathcal{D}_{g,[r]}) \times_{\mathcal{M}_{g,[r]}} \mathcal{M}_{g,r};$$

$$\mathcal{N}^{\mathrm{ord}}_{g,r} \stackrel{\mathrm{def}}{=} \mathcal{N}^{\mathrm{ord}}_{g,[r]} \times_{\mathcal{M}_{g,[r]}} \mathcal{M}_{g,r} \subseteq \mathcal{N}^{\mathrm{adm}}_{g,r} \stackrel{\mathrm{def}}{=} \mathcal{N}^{\mathrm{adm}}_{g,[r]} \times_{\mathcal{M}_{g,[r]}} \mathcal{M}_{g,r} \subseteq \mathcal{N}_{g,r} \stackrel{\mathrm{def}}{=} \mathcal{N}_{g,[r]} \times_{\mathcal{M}_{g,[r]}} \mathcal{M}_{g,r}.$$

Then the following three facts were proved in [7], Chapter II.

(i) The forgetful morphism of stacks

$$\mathcal{N}_{g,[r]} \longrightarrow \mathcal{M}_{g,[r]}$$

is *finite flat of degree* p^{3g-3+r} [cf. [7], Chapter II, Theorem 2.3].

(ii) The open substack

$$\mathcal{N}^{\mathrm{adm}}_{g,[r]} \subseteq \mathcal{N}_{g,[r]}$$

coincides with the *smooth locus* of the structure morphism $\mathcal{N}_{g,[r]} \to \mathrm{Spec}(k)$ [cf. [7], Chapter II, Corollary 2.16].

(iii) The open substack

$$\mathcal{N}^{\mathrm{ord}}_{g,[r]} \subseteq \mathcal{N}_{g,[r]}$$

coincides with the *étale locus* of the forgetful morphism of stacks

$$\mathcal{N}_{g,[r]} \longrightarrow \mathcal{M}_{g,[r]}$$

[cf. [7], Chapter II, Proposition 2.12; [7], Chapter II, Theorem 2.13].

2. Review of FL-bundles

In [7], Chapter II, §1, *S. Mochizuki* studied the notion of an *FL-bundle* [cf. [7], Chapter II, Definition 1.3; Definition 2.2 of the present paper], which defines a section of the torsor [i.e., under $H^1(X^F, (\tau^{\log})^F)$] of "mod p^2 liftings" of (X^F, D^F). In the present §2, let us review some portions of the theory of *FL-bundles* of [7], Chapter II, §1, from the point of view of the present paper.

Let us start our discussion with the exact sequence of $\mathcal{O}_{X^F}$-modules of §1, (1.f),

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega^{\log} \xrightarrow{C^{\log}} (\omega^{\log})^F \longrightarrow 0.$$

Thus, by applying "$H^1(X^F, - \otimes_{\mathcal{O}_{X^F}} (\tau^{\log})^F)$, we obtain a sequence of k-vector spaces

$$H^1(X^F, (\tau^{\log})^F) \longrightarrow H^1(X, \mathcal{T}) \xrightarrow{\nabla_{\mathcal{T}}} H^1(X, \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}).$$

The following lemma discusses the first de Rham cohomology

$$H^1_{\mathrm{DR}}(X, \mathcal{T}) \stackrel{\mathrm{def}}{=} H^1_{\mathrm{DR}}(X, (\mathcal{T}, \nabla_{\mathcal{T}}))$$

of $(\mathcal{T}, \nabla_{\mathcal{T}})$, i.e., the first hypercohomology of the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{T} \xrightarrow{\nabla_{\mathcal{T}}} \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T} \longrightarrow 0 \longrightarrow \cdots.$$

Lemma 2.1. — *In the above sequence*

$$H^1(X^F, (\tau^{\log})^F) \longrightarrow H^1(X, \mathcal{T}) \xrightarrow{\nabla_{\mathcal{T}}} H^1(X, \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}),$$

the following hold.

(i) *It holds that*

$$\Gamma(X, \mathcal{T}) = \Gamma(X, \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}) = \{0\}.$$

(ii) *The image of the composite of the two arrows is* **zero**.

(iii) *The first arrow is* **injective**.

(iv) *The kernel of the second arrow is naturally* **isomorphic** *to* $H^1_{\mathrm{DR}}(X, \mathcal{T})$*, i.e.,*

$$H^1_{\mathrm{DR}}(X, \mathcal{T}) \xrightarrow{\sim} \mathrm{Ker}\big(H^1(X, \mathcal{T}) \to H^1(X, \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T})\big).$$

(v) *The sequence under consideration determines a sequence of* **injections**

$$H^1(X^F, (\tau^{\log})^F) \hookrightarrow H^1_{\mathrm{DR}}(X, \mathcal{T}) \hookrightarrow H^1(X, \mathcal{T}).$$

(vi) *The cokernel of the first arrow of* (v) *is naturally* **isomorphic** *to* $k = \Gamma(X^F, \mathcal{O}_{X^F})$*, hence also* **of dimension one**.

PROOF. — Assertions (i), (ii) are immediate. Next, we verify assertion (iii). It is immediate that the kernel of the first arrow may be identified with a subspace of $\Gamma(X, \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T})$. Thus, assertion (iii) follows from assertion (i). This completes the proof of assertion (iii). Next, let us observe that, by considering the spectral sequence that arises from the "stupid filtration" of the complex $\nabla_{\mathcal{T}} \colon \mathcal{T} \to \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}$, one may conclude that assertion (iv) follows from assertion (i). Assertion (v) follows formally from assertions (ii), (iii), (iv). Finally, we verify assertion (vi). It follows immediately from assertion (iv) that the cokernel under consideration may be identified with the cokernel of the homomorphism $\Gamma(X, \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}) \to \Gamma(X^F, \mathcal{O}_{X^F})$ induced by the Cartier operator $C^{\log}$. Thus, assertion (vi) follows from assertion (i). This completes the proof of assertion (vi), hence also of Lemma 2.1. $\square$

DEFINITION 2.2. — Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ be a pair consisting of a coherent $\mathcal{O}_X$-module $\mathcal{E}$ and a connection $\nabla_{\mathcal{E}}$ on $\mathcal{E}$ relative to $(X, D)/k$. Then we shall say that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is an *FL-bundle* on $(X, D)/k$ [cf. [7], Chapter II, Definition 1.3] if $(\mathcal{E}, \nabla_{\mathcal{E}})$ admits a structure of extension

$$0 \longrightarrow (\mathcal{T}, \nabla_{\mathcal{T}}) \longrightarrow (\mathcal{E}, \nabla_{\mathcal{E}}) \longrightarrow (\mathcal{O}_X, d) \longrightarrow 0$$

whose extension class $\in H^1_{\mathrm{DR}}(X, \mathcal{T})$ is not contained in the subspace $H^1(X^F, (\tau^{\log})^F) \subseteq H^1_{\mathrm{DR}}(X, \mathcal{T})$ [cf. Lemma 2.1, (v)].

DEFINITION 2.3. — We shall say that an FL-bundle is *indigenous* if the projectivization of the FL-bundle is an indigenous bundle on $(X, D)/k$ [cf. [7], Chapter I, Definition 2.2].

The following proposition follows immediately from [7], Chapter II, Corollary 1.6.

PROPOSITION 2.4. — *Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ be an* **FL-bundle** *on $(X, D)/k$. Then the horizontal invertible subsheaf "$(\mathcal{T}, \nabla_{\mathcal{T}})$" of $(\mathcal{E}, \nabla_{\mathcal{E}})$ in the extension of Definition* 2.2 *is the* **unique maximal horizontal invertible subsheaf** *of $(\mathcal{E}, \nabla_{\mathcal{E}})$.*

DEFINITION 2.5. — Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ be an FL-bundle on $(X, D)/k$. Then we shall refer to the unique maximal horizontal invertible subsheaf of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [cf. Proposition 2.4] as the *conjugate filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$.

LEMMA 2.6. — *Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ be an* **FL-bundle** *on $(X, D)/k$. Then the* **monodromy** *operator of $\nabla_{\mathcal{E}}$ at each point on $D \subseteq X$ [cf., e.g., the discussion at the beginning of* [1], §2.2] *is* **nilpotent**.

PROOF. — This follows from the existence of a structure of extension as in Definition 2.2, together with the [easily verified] fact that the monodromy operator of the connection $\nabla_{\mathcal{T}}$ (respectively, d) on $\mathcal{T}$ (respectively, $\mathcal{O}_X$) at each point on $D \subseteq X$ is *zero*. $\square$

LEMMA 2.7. — *Let $(Y, D_Y) \to (X, D)$ be a finite flat tamely ramified covering between hyperbolic curves over k and $(\mathcal{E}, \nabla_{\mathcal{E}})$ an* **FL-bundle** *on $(X, D)/k$. Then it holds that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is* **indigenous** *if and only if the FL-bundle $(Y \to X)^*(\mathcal{E}, \nabla_{\mathcal{E}})$ on $(Y, D_Y)/k$ obtained by pulling back $(\mathcal{E}, \nabla_{\mathcal{E}})$ via $Y \to X$ is* **indigenous**.

PROOF. — Write (P, ∇_P) and (Q, ∇_Q) for the respective projectivizations of $(\mathcal{E}, \nabla_{\mathcal{E}})$ and $(Y \to X)^*(\mathcal{E}, \nabla_{\mathcal{E}})$. The *necessity* follows from [7], Chapter I, Proposition 2.3. To verify the *sufficiency*, suppose that (Q, ∇_Q) is *indigenous*. Then it follows immediately from the *uniqueness* discussed in [7], Chapter I, Proposition 2.4, that the *Hodge section* [cf. [7], Chapter I, Proposition 2.4] of the indigenous bundle (Q, ∇_Q) *descends* to a section of $P \to X$. Moreover, since the covering $(Y, D_Y) \to (X, D)$ is *tamely ramified*, it follows immediately from the various definitions involved that the resulting section of $P \to X$ is *of canonical height* $-\deg \omega^{\log}/2$ [cf. the discussion preceding [7], Chapter I, Definition 2.2]. Thus, in light of Lemma 2.6, we conclude that (P, ∇_P) is an *indigenous bundle* on $(X, D)/k$, as desired. □

LEMMA 2.8. — *Let $(Y, D_Y) \to (X, D)$ be a finite flat tamely ramified covering between hyperbolic curves over k and (P, ∇_P) an indigenous bundle on $(X, D)/k$. Then it holds that (P, ∇_P) is* **nilpotent** *(respectively,* **admissible***) [cf. [7], Chapter II, Definition 2.4] if and only if the indigenous bundle $(Y \to X)^*(P, \nabla_P)$ on $(Y, D_Y)/k$ obtained by pulling back (P, ∇_P) via $Y \to X$ is* **nilpotent** *(respectively,* **admissible***).*

PROOF. — This follows immediately from the [easily verified] fact that the p-curvature homomorphisms of indigenous bundles are *compatible* with the pull-back via a finite flat tamely ramified covering between hyperbolic curves. □

One of the main results of the theory of *FL-bundles* is as follows [cf. [7], Chapter II, Proposition 2.5].

THEOREM 2.9. — *The following hold.*

(i) *Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ be an* **FL-bundle** *on $(X, D)/k$. Suppose that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is* **indigenous**. *Then the indigenous bundle on $(X, D)/k$ obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$ is* **nilpotent** *and* **admissible**.

(ii) *Let $(\pi : P \to X, \nabla_P)$ be a* **nilpotent admissible indigenous bundle** *on $(X, D)/k$. Write $\tau_{P/X}$ for the relative tangent sheaf of P/X and $\mathcal{P} : \mathcal{T} \to \pi_* \tau_{P/X}$ for the p-curvature homomorphism of (P, ∇_P). Then the pair consisting of the kernel $\mathrm{Ker}(\mathcal{P}^\vee)$ of the dual $\mathcal{P}^\vee$ of $\mathcal{P}$ and the connection on $\mathrm{Ker}(\mathcal{P}^\vee)$ induced by ∇_P is an* **indigenous FL-bundle** *on $(X, D)/k$.*

(iii) *The constructions of* (i) *and* (ii) *determine a* **bijection** *between the set of isomorphism classes of* **indigenous FL-bundles** *on $(X, D)/k$ and the set of isomorphism classes of* **nilpotent admissible indigenous bundles** *on $(X, D)/k$.*

PROOF. — Let us first recall that if r is *even* [cf. the remark at the beginning of the discussion entitled “The Definition of the Verschiebung” in [7], Chapter II, §2], then these assertions follow immediately from [7], Chapter II, Proposition 2.5 [cf. also the proof of

[7], Chapter II, Proposition 2.5]. Next, let us observe that it is immediate that there exists a finite flat tamely ramified Galois covering $(Y, D_Y) \to (X, D)$ between hyperbolic curves over k such that "r" for (Y, D_Y) [i.e., the degree of the reduced closed subscheme $D_Y \subseteq Y$] is *even*, which thus implies that Theorem 2.9 for (Y, D_Y) holds.

Assertion (i) follows from assertion (i) for (Y, D_Y), together with Lemma 2.8. Next, we verify assertion (ii). Let us first observe that it follows immediately from a similar argument to the argument applied in the proof of [7], Chapter II, Proposition 2.5, that the pair under consideration is an *FL-bundle*. Moreover, it follows from assertion (ii) for (Y, D_Y), together with Lemma 2.7, that the pair under consideration is also *indigenous*. This completes the proof of assertion (ii). Assertion (iii) follows immediately from the various definitions involved. This completes the proof of Theorem 2.9. $\square$

3. A Characterization of Supersingular Divisors

In the present §3, we give a *characterization of the supersingular divisors* of nilpotent admissible/ordinary indigenous bundles [cf. Theorem 3.9, Theorem 3.10, Corollary 3.12 below].

Definition 3.1. — We shall say that an effective divisor on X is *of NA-type* (respectively, *of NO-type*) relative to $(X, D)/k$ if there exists a nilpotent admissible (respectively, nilpotent ordinary — cf. [7], Chapter II, Definition 3.1) indigenous bundle on $(X, D)/k$ whose supersingular divisor [cf. [7], Chapter II, Proposition 2.6, (3)] coincides with the effective divisor.

The following fact is well-known [cf. [7], Chapter II, Proposition 2.6, (2), (3); Proposition A.4 of the present paper].

Proposition 3.2. — *Let E be an effective divisor on X* **of NA-type** *relative to $(X, D)/k$. Then the following hold.*

(i) *The divisor E is* **of degree** $p^* \deg \omega^{\log}$.

(ii) *The divisor E is* **reduced**.

(iii) *It holds that $E \cap D = \emptyset$.*

Since a nilpotent *ordinary* indigenous bundle is *admissible* [cf. [7], Chapter II, Proposition 3.2], the following proposition holds.

Proposition 3.3. — *If an effective divisor on X is* **of NO-type** *relative to $(X, D)/k$, then the divisor is* **of NA-type** *relative to $(X, D)/k$.*

Let

$$(\mathcal{E}, \nabla_{\mathcal{E}})$$

be an *FL-bundle* on $(X, D)/k$. Write

$$\mathcal{C} \subseteq \mathcal{E}$$

for the *conjugate filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [cf. Definition 2.5] and fix horizontal isomorphisms

$$\mathcal{T} \xrightarrow{\sim} \mathcal{C}, \quad \mathcal{O}_X \xrightarrow{\sim} \mathcal{E}/\mathcal{C}.$$

Let us identify $\mathcal{T}$, $\mathcal{O}_X$ with $\mathcal{C}$, $\mathcal{E}/\mathcal{C}$ by means of these horizontal isomorphisms, respectively.

Let E be an effective divisor on X of degree $< -\deg \mathcal{T} = p \deg \omega^{\log}$. Then the natural inclusion $\mathcal{O}_X(-E) \hookrightarrow \mathcal{O}_X$ determines an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}(E) \longrightarrow \mathcal{T}(E)|_E \longrightarrow 0,$$

which thus determines an *exact* sequence of k-vector spaces

$$0 \longrightarrow \Gamma(E, \mathcal{T}(E)|_E) \longrightarrow H^1(X, \mathcal{T}) \longrightarrow H^1(X, \mathcal{T}(E)) \longrightarrow 0.$$

Let us regard $\Gamma(E, \mathcal{T}(E)|_E)$ as a subspace of $H^1(X, \mathcal{T})$, i.e.,

$$\Gamma(E, \mathcal{T}(E)|_E) \subseteq H^1(X, \mathcal{T}),$$

by means of the second arrow of this sequence.

DEFINITION 3.4. — We shall say that E is *liftable* with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$ if the natural inclusion $\mathcal{O}_X(-E) \hookrightarrow \mathcal{O}_X$ lifts to a [necessarily injective] homomorphism $\mathcal{O}_X(-E) \hookrightarrow \mathcal{E}$ of $\mathcal{O}_X$-modules [relative to the natural surjection $\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{C} = \mathcal{O}_X$].

Thus, it is immediate from the definition of the term "*liftable*", together with the above exact sequence of k-vector spaces

$$0 \longrightarrow \Gamma(E, \mathcal{T}(E)|_E) \longrightarrow H^1(X, \mathcal{T}) \longrightarrow H^1(X, \mathcal{T}(E)) \longrightarrow 0,$$

that the following lemma holds.

LEMMA 3.5. — *The following two conditions are equivalent.*

(1) *The effective divisor E is* **liftable** *with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$.*

(2) *The FL-bundle $(\mathcal{E}, \nabla_{\mathcal{E}})$ has a structure of extension as in Definition* 2.2 *whose extension class $\in H^1_{\mathrm{DR}}(X, \mathcal{T})$ $(\subseteq H^1(X, \mathcal{T}))$ [cf. Lemma* 2.1, (v)] *is* **contained** *in the subspace $\Gamma(E, \mathcal{T}(E)|_E) \subseteq H^1(X, \mathcal{T})$.*

LEMMA 3.6. — *If E is* **liftable** *with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$, then it holds that $p^* \deg \omega^{\log} \leq \deg E$.*

PROOF. — Since E is *liftable* with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$, the natural inclusion $\mathcal{O}_X(-E) \hookrightarrow \mathcal{O}_X$ lifts to a homomorphism $\mathcal{O}_X(-E) \hookrightarrow \mathcal{E}$. Now we may assume without loss of generality, by replacing E by a suitable effective subdivisor of E, that the lifting $\mathcal{O}_X(-E) \hookrightarrow \mathcal{E}$ is *locally split*. Then since $\det \mathcal{E} \cong \mathcal{T}$, it holds that $\mathcal{E}/\mathcal{O}_X(-E) \cong \mathcal{T}(E)$.

Let us consider the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{O}_X(-E) \hookrightarrow \mathcal{E} \xrightarrow{\nabla_{\mathcal{E}}} \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{E} \twoheadrightarrow \omega^{\log} \otimes_{\mathcal{O}_X} (\mathcal{E}/\mathcal{O}_X(-E)) \cong \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}(E).$$

Then it follows immediately from Proposition 2.4 that this composite is *injective*. Thus, we obtain that

$$-\deg E \;=\; \deg \mathcal{O}_X(-E) \;\leq\; \deg(\omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}(E)) \;=\; (1-p)\deg \omega^{\log} + \deg E,$$

which thus implies the desired inequality. This completes the proof of Lemma 3.6. $\square$

Proposition 3.7. — *The following two conditions are equivalent.*

(1) *The FL-bundle* $(\mathcal{E}, \nabla_{\mathcal{E}})$ *is* **indigenous**.

(2) *There exists an effective divisor on* X *of degree* $p^{*} \deg \omega^{\log}$ *which is* **liftable** *with respect to* $(\mathcal{E}, \nabla_{\mathcal{E}})$.

Moreover, in this case, the effective divisor of (2) **coincides** *with the* **supersingular divisor** *of the* **nilpotent admissible indigenous bundle** *on* $(X, D)/k$ *obtained by forming the projectivization of* $(\mathcal{E}, \nabla_{\mathcal{E}})$ [*cf. Theorem* 2.9, (i)].

Proof. — First, we verify the implication (1) $\Rightarrow$ (2). Suppose that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is *indigenous*. Write $\mathcal{L} \subseteq \mathcal{E}$ for the *Hodge filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [i.e., the invertible subsheaf which defines the *Hodge section* of the indigenous bundle obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$]. Then it follows immediately from the definition of an indigenous bundle that the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{L} \;\hookrightarrow\; \mathcal{E} \;\stackrel{\nabla_{\mathcal{E}}}{\to}\; \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{E} \;\twoheadrightarrow\; \omega^{\log} \otimes_{\mathcal{O}_X} (\mathcal{E}/\mathcal{L})$$

is an *isomorphism*. In particular, since $(\mathcal{E}/\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L} \cong \det \mathcal{E} \cong \mathcal{T}$, it holds that $\deg \mathcal{L} = -p^{*} \deg \omega^{\log}$, and that the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{L} \;\hookrightarrow\; \mathcal{E} \;\twoheadrightarrow\; \mathcal{E}/\mathcal{C} \;=\; \mathcal{O}_X$$

is thus *injective* [cf. also Proposition 2.4]. Thus, there exists an effective divisor F on X of degree $-\deg \mathcal{L} = p^{*} \deg \omega^{\log}$ such that the injection $\mathcal{L} \hookrightarrow \mathcal{O}_X$ determines an isomorphism $\mathcal{L} \stackrel{\sim}{\to} \mathcal{O}_X(-F)$. In particular, condition (2) is satisfied. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that E is of degree $p^{*} \deg \omega^{\log}$ and *liftable* with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$. Since E is *liftable* with respect to $(\mathcal{E}, \nabla_{\mathcal{E}})$, the natural inclusion $\mathcal{O}_X(-E) \hookrightarrow \mathcal{O}_X$ lifts to a homomorphism $\mathcal{O}_X(-E) \hookrightarrow \mathcal{E}$. Let us observe that it follows immediately from Lemma 3.6 that this lifting $\mathcal{O}_X(-E) \hookrightarrow \mathcal{E}$ is *locally split*. Moreover, since $\det \mathcal{E} \cong \mathcal{T}$, it holds that $\mathcal{E}/\mathcal{O}_X(-E) \cong \mathcal{T}(E)$.

Consider the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{O}_X(-E) \;\hookrightarrow\; \mathcal{E} \;\stackrel{\nabla_{\mathcal{E}}}{\to}\; \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{E} \;\twoheadrightarrow\; \omega^{\log} \otimes_{\mathcal{O}_X} (\mathcal{E}/\mathcal{O}_X(-E)) \;\cong\; \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{T}(E).$$

Since E is *of degree* $p^{*} \deg \omega^{\log}$, and this composite is *injective* [cf. Proposition 2.4], this composite is in fact an *isomorphism*, which thus implies that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is *indigenous* [cf. also Lemma 2.6]. This completes the proof of the implication (2) $\Rightarrow$ (1).

The final assertion follows immediately from the proof of the implication (1) $\Rightarrow$ (2), together with a similar argument to the argument applied in the verification of [5], Proposition B.4 [cf. also Proposition A.3, (iv), and Lemma A.10, (i), of the present paper]. This completes the proof of Proposition 3.7. $\square$

PROPOSITION 3.8. — *It holds that E is* **of NA-type** *relative to $(X, D)/k$ if and only if the following three conditions are satisfied.*

(1) *It holds that* $\deg E = p^{*} \deg \omega^{\log}$.

(2) *It holds that* $H^1(X^F, (\tau^{\log})^F) \cap \Gamma(E, \mathcal{T}(E)|_E) = \{0\}$.

(3) *It holds that* $H^1_{\mathrm{DR}}(X, \mathcal{T}) \cap \Gamma(E, \mathcal{T}(E)|_E) \neq \{0\}$.

PROOF. — First, we verify the *sufficiency*. Take a *nonzero* element $c \in H^1_{\mathrm{DR}}(X, \mathcal{T}) \cap \Gamma(E, \mathcal{T}(E)|_E)$ [cf. condition (3)]. Then it follows from condition (2) that $c \notin H^1(X^F, (\tau^{\log})^F)$. In particular, the class c determines an *FL-bundle* on $(X, D)/k$. Thus, it follows, in light of Lemma 3.5, from the implication (2) $\Rightarrow$ (1) of Proposition 3.7, together with condition (1), that the projectivization of the FL-bundle is a(n) [necessarily *nilpotent admissible* — cf. Theorem 2.9, (i)] *indigenous bundle* on $(X, D)/k$. Moreover, it follows from the final assertion of Proposition 3.7 that the supersingular divisor of the nilpotent admissible indigenous bundle *coincides* with E. Thus, the divisor E is *of NA-type* relative to $(X, D)/k$. This completes the proof of the *sufficiency*.

Finally, we verify the *necessity*. Suppose that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is *indigenous*, and that E *coincides* with the supersingular divisor of the nilpotent admissible indigenous bundle on $(X, D)/k$ determined by $(\mathcal{E}, \nabla_{\mathcal{E}})$ [cf. Theorem 2.9, (i), (iii)]. Then it follows from Proposition 3.2, (i), that condition (1) is satisfied. Now let us observe that it follows from the definition of an FL-bundle that the conjugate filtration $\mathcal{C} \subseteq \mathcal{E}$ of $(\mathcal{E}, \nabla_{\mathcal{E}})$, together with the identifications $\mathcal{C} = \mathcal{T}$, $\mathcal{E}/\mathcal{C} = \mathcal{O}_X$, determines an extension class $c_{\mathcal{E}} \in H^1(X, \mathcal{T})$ such that $c_{\mathcal{E}} \notin H^1(X^F, (\tau^{\log})^F)$, $c_{\mathcal{E}} \in H^1_{\mathrm{DR}}(X, \mathcal{T})$. Moreover, let us observe that it follows, in light of Lemma 3.5, from the implication (1) $\Rightarrow$ (2) of Proposition 3.7 and the final assertion of Proposition 3.7 that $c_{\mathcal{E}} \in \Gamma(E, \mathcal{T}(E)|_E)$ [which thus implies that condition (3) is satisfied]. Thus, to complete the verification of the *necessity*, it suffices to verify condition (2), i.e., $H^1(X^F, (\tau^{\log})^F) \cap \Gamma(E, \mathcal{T}(E)|_E) = \{0\}$.

Assume that there exists a *nonzero* element $a \in H^1(X^F, (\tau^{\log})^F) \cap \Gamma(E, \mathcal{T}(E)|_E)$. Then it is immediate that $c_{\mathcal{E}} + a \notin H^1(X^F, (\tau^{\log})^F)$, $c_{\mathcal{E}} + a \in H^1_{\mathrm{DR}}(X, \mathcal{T})$, and $c_{\mathcal{E}} + a \in \Gamma(E, \mathcal{T}(E)|_E)$. Thus, it follows immediately, in light of Lemma 3.5, from the implication (2) $\Rightarrow$ (1) of Proposition 3.7 and the final assertion of Proposition 3.7 that the class $c_{\mathcal{E}} + a \in H^1(X, \mathcal{T})$ determines an *FL-bundle* $(\mathcal{E}', \nabla_{\mathcal{E}'})$ on $(X, D)/k$ such that the projectivization of $(\mathcal{E}', \nabla_{\mathcal{E}'})$ is a(n) [necessarily *nilpotent admissible* — cf. Theorem 2.9, (i)] *indigenous bundle* whose supersingular divisor *coincides* with E. In particular, it follow from [7], Chapter II, Proposition 2.6, (4), together with Theorem 2.9, (iii), that $(\mathcal{E}, \nabla_{\mathcal{E}})$ is *isomorphic* to $(\mathcal{E}', \nabla'_{\mathcal{E}})$. On the other hand, it follows immediately from Proposition 2.4 that this isomorphism restricts to an isomorphism between the respective *conjugate filtrations* of $\mathcal{E}$ and $\mathcal{E}'$, which thus implies that $c_{\mathcal{E}} + a \in H^1(X, \mathcal{T})$ is a *k-multiple* of $c_{\mathcal{E}}$ — in contradiction to the fact that $a \in H^1(X^F, (\tau^{\log})^F) \setminus \{0\}$ and $c_{\mathcal{E}} \notin H^1(X^F, (\tau^{\log})^F)$. This completes the proof of the *necessity*, hence also of Proposition 3.8. □

It follows from the definitions of the two subspaces

$$H^1_{\mathrm{DR}}(X, \mathcal{T}),\ \Gamma(E, \mathcal{T}(E)|_E) \ \subseteq\ H^1(X, \mathcal{T})$$

[cf. also Lemma 2.1, (iv)] that condition (2) (respectively, (3)) of the statement of Proposition 3.8 is *equivalent* to the condition that

$$\mathrm{Ker}\big(H^1(X^F, (\tau^{\log})^F) \ \hookrightarrow\ H^1(X, \mathcal{T}) \twoheadrightarrow\ H^1(X, \mathcal{T}(E))\big) =\ \{0\}$$

(respectively, $\mathrm{Ker}\big(H^1(X,\mathcal{T}) \to H^1(X,\omega^{\log}\otimes_{\mathcal{O}_X}\mathcal{T})\oplus H^1(X,\mathcal{T}(E))\big) \neq \{0\}$).
Thus, in light of Proposition 3.2 and Proposition 3.3, by applying the *Serre duality*, together with [7], Chapter II, Lemma 2.11, we obtain the following theorem, which is one of the main results of the present paper.

THEOREM 3.9. — *In the notational conventions introduced in* §1, *by abuse of notation, write*

$$C\colon\ \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)) \twoheadrightarrow \Gamma(X^F,((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

for the [*necessarily surjective*] *k-linear homomorphism obtained by applying* "$\Gamma(X^F,-\otimes_{\mathcal{O}_{X^F}}(\omega^{\log})^F)$" *to the Cartier operator associated to* X/k *and*

$$d\colon\ \Gamma(X,(\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D))$$

for the k-linear homomorphism determined by the exterior differentiation operator. Let

$$E$$

be an effective divisor on X. *Then it holds that the divisor* E *is* **of NA-type** *relative to* $(X,D)/k$ *if and only if the following three conditions are satisfied.*

(1) *The divisor* E *is* **of degree** $p^{\divideontimes}\deg\omega^{\log}$.

(2) *The composite*

$$\Gamma(X,(\omega^{\log})^{\otimes p+1}(-D-E)) \hookrightarrow \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)) \overset{C}{\twoheadrightarrow} \Gamma(X^F,((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is **surjective**.

(3) *The k-vector space* $\Gamma(X,(\omega^{\log})^{\otimes p+1}(-D))$ *is* **not generated** *by the subspace*

$$\Gamma(X,(\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D))$$

and the image of the k-linear homomorphism

$$d\colon\ \Gamma(X,(\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)).$$

Moreover, we also obtain the following theorem, which is one of the main results of the present paper.

THEOREM 3.10. — *In the situation of Theorem* 3.9, *let*

$$E$$

be an effective divisor on X. *Then it holds that the divisor* E *is* **of NO-type** *relative to* $(X,D)/k$ *if and only if the following three conditions are satisfied.*

(1) *The divisor* E *is* **of degree** $p^{\divideontimes}\deg\omega^{\log}$.

(2′) *The composite*

$$\Gamma(X,(\omega^{\log})^{\otimes p+1}(-D-2E)) \hookrightarrow \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)) \overset{C}{\twoheadrightarrow} \Gamma(X^F,((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is **surjective** [*or, alternatively, an* **isomorphism** — *cf. Remark* 3.10.1, (i), (iii), *below*].

(3) *The k-vector space* $\Gamma(X,(\omega^{\log})^{\otimes p+1}(-D))$ *is* **not generated** *by the subspace*

$$\Gamma(X,(\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D))$$

and the image of the k-linear homomorphism

$$d\colon\ \Gamma(X,(\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)).$$

PROOF. — First, to verify the *sufficiency*, suppose that three conditions (1), (2′), (3) are satisfied. Then since [it is immediate that] condition (2′) implies condition (2) in the statement of Theorem 3.9, it follows from Theorem 3.9 that E is *of NA-type*. In particular, the divisor $2E$ coincides with the zero locus of the *square Hasse invariant* [cf. [7], Chapter II, Proposition 2.6, (1)] of a nilpotent admissible indigenous bundle on $(X,D)/k$. Thus, it follows from condition (2′), together with [7], Chapter II, Proposition 2.12, that the nilpotent admissible indigenous bundle is *ordinary*, which thus implies that E is *of NO-type*. This completes the proof of the *sufficiency*.

Finally, to verify the *necessity*, suppose that E is *of NO-type*. Let us observe that it follows from Proposition 3.3 and Theorem 3.9 that, to verify the *necessity*, it suffices to verify that condition (2′) is satisfied. Next, let us recall that since E is *of NO-type*, the divisor $2E$ coincides with the zero locus of the *square Hasse invariant* of a nilpotent *ordinary* indigenous bundle on $(X,D)/k$. Thus, it follows from [7], Chapter II, Proposition 2.12, that condition (2′) is satisfied. This completes the proof of the *necessity*, hence also of Theorem 3.10. □

REMARK 3.10.1. — In Theorem 3.9 and Theorem 3.10, we consider the two k-linear homomorphisms

$$C\colon\ \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)) \twoheadrightarrow \Gamma(X^F,((\omega^{\log})^F)^{\otimes 2}(-D^F)),$$
$$d\colon\ \Gamma(X,(\omega^{\log})^{\otimes p}(-D)) \rightarrow \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D))$$

and the two subspaces

$$\Gamma(X,(\omega^{\log})^{\otimes p+1}(-D-2E)) \subseteq \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)).$$

Let us first observe that it follows from the *Riemann-Roch formula* that

(i) the domain, codomain of the k-linear homomorphism

$$C\colon\ \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)) \twoheadrightarrow \Gamma(X^F,((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

are of dimension

$$1-g+(p+1)\deg\omega^{\log}-r = (2p+1)\cdot g-(2p+1)+pr,$$
$$\dim\mathcal{M}_{g,[r]} = 3g-3+r,$$

respectively,

(ii) the domain, codomain of the k-linear homomorphism

$$d\colon\ \Gamma(X,(\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D))$$

are of dimension

$$1-g+p\deg\omega^{\log}-r = (2p-1)\cdot g-(2p-1)+(p-1)\cdot r,$$
$$1-g+(p+1)\deg\omega^{\log}-r = (2p+1)\cdot g-(2p+1)+pr,$$

respectively, and

(iii) if condition (1) of the statement of Theorem 3.9 is satisfied, then the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

of $\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$ are of dimension

$$\dim \mathcal{M}_{g,[r]} = 3g-3+r,$$

$$1-g+(p^{*}+2)\deg\omega^{\log}-r = (2p^{*}+3)\cdot g-(2p^{*}+3)+(p^{*}+1)\cdot r,$$

respectively.

Next, let us recall that it follows immediately from the various definitions involved [cf. also the discussion preceding Lemma 2.1] that

(iv) the image of the composite

$$\Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \stackrel{d}{\to} \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \stackrel{C}{\twoheadrightarrow} \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is *zero*, and

(v) the kernel of the k-linear homomorphism

$$d\colon \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

is of dimension

$$\dim_k H^1(X^F, \mathcal{O}_{X^F}) = g.$$

Finally, let us observe that it follows from Lemma 2.1, (vi), that

(vi) the cokernel of the k-linear homomorphism

$$d\colon \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \longrightarrow \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

is of dimension

$$1+\dim_k \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F)) = 3g-2+r.$$

DEFINITION 3.11. — Suppose that we are in the situation of Theorem 3.9.

(i) We shall write

$$V_{(X,D)} \stackrel{\text{def}}{=} \operatorname{Coker}\bigl(d\colon \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \to \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))\bigr).$$

(ii) We shall write

$$V_{(X,D)}[2E] \subseteq V_{(X,D)}[E] \subseteq V_{(X,D)}$$

for the subspaces of $V_{(X,D)}$ determined by the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)),$$

respectively.

(iii) We shall write

$$C\colon V_{(X,D)} \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

for the surjective k-linear homomorphism determined by the homomorphism C in the statement of Theorem 3.9 [cf. Remark 3.10.1, (iv)].

It follows from Remark 3.10.1, (vi), that the kernel of the surjective k-linear homomorphism of Definition 3.11, (iii),

$$C\colon\ V_{(X,D)} \ \twoheadrightarrow\ \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is *of dimension one*. Thus, the following corollary follows immediately from Theorem 3.9 and Theorem 3.10, together with Remark 3.10.1, (i), (iii).

Corollary 3.12. — *In the situation of Theorem* 3.9, *let E be an effective divisor on X of degree $p^* \deg \omega^{\log}$. Then the following hold.*

(i) *It holds that E is* **of NA-type** *relative to $(X,D)/k$ if and only if the composite*

$$V_{(X,D)}[E] \ \hookrightarrow\ V_{(X,D)} \ \overset{C}{\twoheadrightarrow}\ \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

is an **isomorphism***, i.e., the subspace $V_{(X,D)}[E] \subseteq V_{(X,D)}$ determines a* **splitting** *of $C\colon V_{(X,D)} \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$.*

(ii) *It holds that E is* **of NO-type** *relative to $(X,D)/k$ if and only if the two composites*

$$V_{(X,D)}[E] \ \hookrightarrow\ V_{(X,D)} \ \overset{C}{\twoheadrightarrow}\ \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F)),$$

$$V_{(X,D)}[2E] \ \hookrightarrow\ V_{(X,D)} \ \overset{C}{\twoheadrightarrow}\ \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

are **isomorphisms***, i.e., the subspaces $V_{(X,D)}[E]$, $V_{(X,D)}[2E] \subseteq V_{(X,D)}$ determine* **splittings** *of $C\colon V_{(X,D)} \twoheadrightarrow \Gamma(X^F, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$, respectively.*

4. Explicit Computations in Cases of Genus Zero

In the present §4, we apply the characterization of Corollary 3.12 to some hyperbolic curves *of genus zero*.

In the present §4, suppose that

$$g \ = \ 0,$$

which thus implies that

$$\deg \omega^{\log} \ = \ r-2.$$

Thus, there exists a function $t \in \Gamma(X \setminus D, \mathcal{O}_X^\times)$ which determines an isomorphism over k

$$\mathrm{Spec}\Big(k\Big[t, \frac{1}{t}, \frac{1}{t-1}, \frac{1}{t-a_1}, \ldots, \frac{1}{t-a_{r-3}}\Big]\Big) \ \xrightarrow{\sim}\ X \setminus D$$

for some distinct $r-3$ elements $a_1, \ldots, a_{r-3} \in k \setminus \{0,1\}$ of $k \setminus \{0,1\}$. Let us identify the left-hand side with the right-hand side by means of this isomorphism. We shall write

$$f_0(t) \ \overset{\mathrm{def}}{=}\ t \cdot (t-1) \cdot (t-a_1) \cdots (t-a_{r-3}) \ \in\ \Gamma(X \setminus D, \mathcal{O}_X^\times)$$

and

$$\omega_0 \ \in\ \Gamma(X, \omega^{\log})$$

for the unique global section of $\omega^{\log}$ whose restriction to $X \setminus D$ is given by

$$\frac{dt}{f_0(t)} \ = \ \frac{dt}{t \cdot (t-1) \cdot (t-a_1) \cdots (t-a_{r-3})} \ \in\ \Gamma(X \setminus D, \omega^{\log}).$$

Write, moreover, for each integer d,

$$k[t]^{\leq d} \stackrel{\text{def}}{=} \{ f(t) \in k[t] \mid \deg f(t) \leq d \}.$$

Then it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes p}(-D)$, $(\omega^{\log})^{\otimes p+1}(-D)$, and $((\omega^{\log})^F)^{\otimes 2}(-D^F)$ and the homomorphisms d and C that there exist isomorphisms of k-vector spaces

$$\begin{array}{ccc} k[t]^{\leq p(r-2)-r} & \xrightarrow{\sim} & \Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \\ g(t) & \mapsto & g(t)dt \otimes \omega_0^{\otimes p-1}, \\ k[t]^{\leq p(r-2)-2} & \xrightarrow{\sim} & \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \\ f(t) & \mapsto & f(t)dt \otimes \omega_0^{\otimes p}, \\ k[t^F]^{\leq r-4} & \xrightarrow{\sim} & \Gamma(X, ((\omega^{\log})^F)^{\otimes 2}(-D^F)) \\ h(t^F) & \mapsto & h(t^F)dt^F \otimes \omega_0^F, \end{array}$$

and that the sequence of k-vector spaces

$$\Gamma(X, (\omega^{\log})^{\otimes p}(-D)) \xrightarrow{d} \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)) \xrightarrow{C} \Gamma(X, ((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

corresponds, relative to the above isomorphisms, to the sequence of k-vector spaces

$$\begin{array}{ccccc} k[t]^{\leq p(r-2)-r} & \longrightarrow & k[t]^{\leq p(r-2)-2} & \longrightarrow & k[t^F]^{\leq r-4} \\ g(t) & \mapsto & \dfrac{d}{dt}(g(t) \cdot f_0(t)) & & \\ & & f(t) & \mapsto & -\dfrac{d^{p-1}}{dt^{p-1}} f(t)\Big|_{t^p = t^F}. \end{array}$$

Next, let

$$e_1, \ldots, e_{p^*(r-2)} \in k \setminus \{0, 1, a_1, \ldots, a_{r-3}\}$$

be distinct $p^*(r-2)$ $(= p^* \deg \omega^{\log})$ elements of $k \setminus \{0, 1, a_1, \ldots, a_{r-3}\}$. Write $[e_j]$ for the principal divisor defined by the closed point of X corresponding to $e_j \in k$ [where $j \in \{1, \ldots, p^*(r-2)\}$],

$$E = \sum_{i=1}^{p^*(r-2)} [e_i]$$

for the [necessarily reduced effective] divisor on X *of degree* $p^*(r-2)$ $(= p^* \deg \omega^{\log})$ determined by the e_i's, and

$$f_E(t) \stackrel{\text{def}}{=} (t - e_1) \cdots (t - e_{p^*(r-2)}) \in \Gamma(X \setminus (D \cup E), \mathcal{O}_X^\times).$$

Then it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes p+1}(-D-2E)$, $(\omega^{\log})^{\otimes p+1}(-D-E)$, and $(\omega^{\log})^{\otimes p+1}(-D)$ that the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D))$$

correspond, relative to the above isomorphism

$$k[t]^{\leq p(r-2)-2} \xrightarrow{\sim} \Gamma(X, (\omega^{\log})^{\otimes p+1}(-D)),$$

to the subspaces

$$f_E(t)^2 \cdot k[t]^{\leq r-4} \stackrel{\text{def}}{=} \{ f(t) \cdot f_E(t)^2 \in k[t]^{\leq p(r-2)-2} \mid f(t) \in k[t]^{\leq r-4} \}$$

$$\subseteq f_E(t) \cdot k[t]^{\leq (p^*+1)(r-2)-2} \stackrel{\text{def}}{=} \{ f(t) \cdot f_E(t) \in k[t]^{\leq p(r-2)-2} \mid f(t) \in k[t]^{\leq (p^*+1)(r-2)-2} \}$$

$$\subseteq k[t]^{\leq p(r-2)-2},$$

respectively. Thus, by Corollary 3.12, we obtain the following proposition.

PROPOSITION 4.1. — *It holds that E is* **of NA-type** (*respectively,* **of NO-type**) *relative to $(X, D)/k$ if and only if the following two conditions* (1), (2) (*respectively,* (1), (2′)) *are satisfied.*

(1) *The k-linear homomorphism*

$$\begin{array}{ccc} f_E(t) \cdot k[t]^{\leq (p^*+1)(r-2)-2} & \longrightarrow & k[t^F]^{\leq r-4} \\ f_E(t) \cdot f(t) & \mapsto & -\frac{d^{p-1}}{dt^{p-1}}(f_E(t) \cdot f(t))\Big|_{t^p = t^F} \end{array}$$

is **surjective**.

(2) *The k-vector space $k[t]^{\leq p(r-2)-2}$ is* **not generated** *by the subspace*

$$f_E(t) \cdot k[t]^{\leq (p^*+1)(r-2)-2} \subseteq k[t]^{\leq p(r-2)-2}$$

and the image of the k-linear homomorphism

$$\begin{array}{ccc} k[t]^{\leq p(r-2)-r} & \longrightarrow & k[t]^{\leq p(r-2)-2} \\ g(t) & \mapsto & \frac{d}{dt}(g(t) \cdot f_0(t)). \end{array}$$

(2′) *The subspace*

$$f_E(t) \cdot k[t]^{\leq (p^*+1)(r-2)-2} \subseteq k[t]^{\leq p(r-2)-2}$$

is **contained** *in the subspace of $k[t]^{\leq p(r-2)-2}$ generated by the subspace*

$$f_E(t)^2 \cdot k[t]^{\leq r-4} \subseteq k[t]^{\leq p(r-2)-2}$$

and the image of the k-linear homomorphism

$$\begin{array}{ccc} k[t]^{\leq p(r-2)-r} & \longrightarrow & k[t]^{\leq p(r-2)-2} \\ g(t) & \mapsto & \frac{d}{dt}(g(t) \cdot f_0(t)). \end{array}$$

(4.a). In the present (4.a), suppose that

$$(g, r) = (0, 3),$$

which thus implies that

$$\deg \omega^{\log} = 1.$$

In this situation, it follows from §1, (1.h), (i), (iii), that

- the hyperbolic curve (X, D) over k has a *unique* nilpotent indigenous bundle, and
- the unique nilpotent indigenous bundle is *ordinary*.

Thus, since the projectivization of the relative first de Rham cohomology — equipped with the Gauss-Manin connection — of the Legendre family of elliptic curves over $X \setminus D$ forms a *nilpotent ordinary indigenous bundle* on $(X, D)/k$ [cf., e.g., the discussion preceding [7], Proposition 3.5], one may conclude that the supersingular divisor of the unique nilpotent ordinary indigenous bundle on $(X, D)/k$ coincides with the divisor determined by the *Hasse polynomial*

$$\chi_{\mathrm{Hss}}(t) \stackrel{\mathrm{def}}{=} \sum_{i=0}^{p^*} \binom{p^*}{i}^2 \cdot t^i.$$

In summary, in this situation, we already obtained the following assertion.

PROPOSITION 4.2. — *There exists a* **precisely one divisor of NA-type** — *relative to* $(X, D)/k$ — *on* X. *The divisor of NA-type is* **of NO-type** *relative to* $(X, D)/k$ *and obtained by forming the* **zero locus** *of the* **Hasse polynomial** $\chi_{\mathrm{Hss}}(t)$.

In the remainder of (4.a), let us verify the assertion that

the zero locus of $\chi_{\mathrm{Hss}}(t)$ satisfies conditions (1), (2′) of Proposition 4.1,

which thus gives an alternative verification of the assertion that

the zero locus of $\chi_{\mathrm{Hss}}(t)$ is *of NO-type* [hence also *of NA-type*] relative to $(X, D)/k$

by means of the characterization of Corollary 3.12.

To verify the assertion that the zero locus of $\chi_{\mathrm{Hss}}(t)$ satisfies conditions (1), (2′) of Proposition 4.1, let us first observe that since $r - 4 < 0$, it holds that

$$k[t^F]^{\leq r-4} = \{0\}, \quad f_E(t)^2 \cdot k[t]^{\leq r-4} = \{0\}.$$

In particular, condition (1) of Proposition 4.1 is always satisfied, and, moreover, condition (2′) of Proposition 4.1 is *equivalent* to the following assertion.

($\dagger_1$): The subspace

$$\chi_{\mathrm{Hss}}(t) \cdot k[t]^{\leq p^*-1} \subseteq k[t]^{\leq p-2}$$

is *contained* in the image of the k-linear homomorphism

$$\begin{array}{ccc} k[t]^{\leq p-3} & \longrightarrow & k[t]^{\leq p-2} \\ g(t) & \mapsto & \dfrac{d}{dt}(g(t) \cdot t \cdot (t-1)). \end{array}$$

Next, to verify the assertion ($\dagger_1$), for each $f(t) \in k[t]^{\leq p-2}$, let us write

$$\int f(t)dt \in k[t]^{\leq p-1}$$

for the unique element of $k[t]^{\leq p-1}$ such that

$$\frac{d}{dt} \int f(t)dt = f(t) \quad \text{and} \quad \int f(t)dt\Big|_{t=0} = 0,$$

i.e., the unique "*indefinite integral*" of degree $\leq p-1$ whose *constant of integration is zero*. Then it is immediate that, to verify the assertion ($\dagger_1$), it suffices to verify the following assertion.

($\dagger_2$): For each $0 \leq n \leq p^* - 1$, it holds that $\displaystyle\int t^n \cdot \chi_{\mathrm{Hss}}(t)dt\Big|_{t=1} = 0$.

Next, to verify the assertion ($\dagger_2$), for each $0 \leq n_1, n_2 \leq p^{*}-1$ such that $n_1+n_2 \leq p^{*}-1$, let us write

$$I(n_1, n_2) \ \stackrel{\mathrm{def}}{=} \ \int t^{n_1} \cdot \Big(\overbrace{\int \cdots \int}^{n_2} \chi_{\mathrm{Hss}}(t) \, \overbrace{dt \cdots dt}^{n_2} \Big) dt \Big|_{t=1}.$$

Thus, the assertion ($\dagger_2$) is *equivalent* to the assertion that $I(n, 0) = 0$ for each $0 \leq n \leq p^{*} - 1$. In particular, to verify the assertion ($\dagger_2$), it suffices to verify the following assertion.

($\dagger_3$): For each $0 \leq n_1, n_2 \leq p^{*} - 1$ such that $n_1 + n_2 \leq p^{*} - 1$, it holds that $I(n_1, n_2) = 0$.

Let us observe that, for each $0 \leq n \leq p^{*} - 1$, since

$$\overbrace{\int \cdots \int}^{n+1} \chi_{\mathrm{Hss}}(t) \, \overbrace{dt \cdots dt}^{n+1} \ = \ \sum_{i=0}^{p^{*}} \binom{p^{*}}{i}^2 \cdot \frac{1}{(i+1)\cdots(i+n+1)} \cdot t^{i+n+1}$$

$$= \ \frac{1}{(p^{*}+1)\cdots(p^{*}+n+1)} \cdot \sum_{i=0}^{p^{*}} \binom{p^{*}}{i} \cdot \binom{p^{*}+n+1}{i+n+1} \cdot t^{i+n+1},$$

it follows from "*Vandermonde's convolution*" that

$$\begin{aligned} I(0, n) \ &= \ \frac{1}{(p^{*}+1)\cdots(p^{*}+n+1)} \cdot \sum_{i=0}^{p^{*}} \binom{p^{*}}{i} \cdot \binom{p^{*}+n+1}{i+n+1} \\ &= \ \frac{1}{(p^{*}+1)\cdots(p^{*}+n+1)} \cdot \binom{p^{*}+p^{*}+n+1}{p^{*}} \\ &= \ \frac{1}{(p^{*}+1)\cdots(p^{*}+n+1)} \cdot \binom{p+n}{p^{*}} \ = \ 0. \end{aligned}$$

This completes the proof of the fact that $I(n_1, n_2) = 0$ if $n_1 = 0$. Thus, the assertion ($\dagger_3$) follows from *induction on* n_1, together with the equality

$$I(n_1, n_2) \ = \ t^{n_1}|_{t=1} \cdot I(0, n_2) - n_1 \cdot I(n_1 - 1, n_2 + 1)$$

obtained by "*partial integration*". This completes the proof of the assertion that the zero locus of $\chi_{\mathrm{Hss}}(t)$ satisfies conditions (1), (2$'$) of Proposition 4.1.

(4.b). In the present (4.b), suppose that

$$(g, r, p) \ = \ (0, 4, 3),$$

which thus implies that

$$p^{*} \ = \ 1, \quad \deg \omega^{\log} \ = \ 2.$$

Write

$$a \ \stackrel{\mathrm{def}}{=} \ a_1 \ \in \ k \setminus \{0, 1\}.$$

[So $f_0(t) = t \cdot (t-1) \cdot (t-a)$.] Then, by Proposition 4.1, we obtain the following lemma.

LEMMA 4.3. — *It holds that E is* **of NO-type** *relative to $(X, D)/k$ if and only if the following two conditions are satisfied.*

(1) *The k-linear homomorphism*

$$\begin{array}{ccc} f_E(t) \cdot k[t]^{\leq 2} & \longrightarrow & k[t^F]^{\leq 0} \\ f_E(t) \cdot f(t) & \mapsto & -\dfrac{d^2}{dt^2}(f_E(t) \cdot f(t)) \end{array}$$

is **surjective**.

(2) *The subspace*

$$f_E(t) \cdot k[t]^{\leq 2} \subseteq k[t]^{\leq 4}$$

is **contained** *in the subspace of $k[t]^{\leq 4}$ generated by the subspace*

$$f_E(t)^2 \cdot k[t]^{\leq 0} \subseteq k[t]^{\leq 4}$$

and the image of the k-linear homomorphism

$$\begin{array}{ccc} k[t]^{\leq 2} & \longrightarrow & k[t]^{\leq 4} \\ g(t) & \mapsto & \dfrac{d}{dt}(g(t) \cdot f_0(t)). \end{array}$$

Here, let us recall the following well-known [cf., e.g., [4], Proposition 3.2, and its proof] fact concerning automorphisms of (X, D) over k.

PROPOSITION 4.4. — *The following hold.*

(i) *The homomorphism of groups*

$$\mathrm{Aut}_{\mathcal{M}_{0,4}}(\mathcal{X}_{0,4}, \mathcal{D}_{0,4}) \longrightarrow \mathrm{Aut}_k(X, D)$$

obtained by considering restrictions, relative to some choice of an ordering on the 4 *marked points of (X, D), is* **injective**.

(ii) *The group $\mathrm{Aut}_{\mathcal{M}_{0,4}}(\mathcal{X}_{0,4}, \mathcal{D}_{0,4})$ is* **isomorphic** *to $\mathbb{Z}/2 \times \mathbb{Z}/2$.*

(iii) *The three* [*cf.* (ii)] *nontrivial automorphisms of (X, D) contained in the image of the injective* [*cf.* (i)] *homomorphism of* (i) *are the three automorphisms determined by the following three automorphisms of $X \setminus D$ over k.*

$$\sigma_0\colon\ t \ \mapsto\ \frac{t-a}{t-1}, \quad \sigma_1\colon\ t \ \mapsto\ \frac{a}{t}, \quad \sigma_\infty\colon\ t \ \mapsto\ a \cdot \frac{t-1}{t-a}.$$

In particular, the image of the injective homomorphism of (i) *does* **not depend** *on the choice of an ordering on the* 4 *marked points of (X, D).*

DEFINITION 4.5. — We shall refer to an automorphism of the hyperbolic curve (X, D) over k which is contained in the image of the homomorphism of Proposition 4.4, (i) [cf. also the final assertion of Proposition 4.4, (iii)], as a *nonspecial* automorphism of (X, D).

Let σ be a nontrivial *nonspecial* automorphism of (X, D). Now I *claim* that

> the reduced effective divisor on X of degree 2 ($= p^{\divideontimes} \deg \omega^{\log}$) obtained by forming the *fixed locus* of σ is *of NO-type* relative to $(X, D)/k$.

To verify this *claim*, let us take "E" of the discussion preceding Proposition 4.1 to be the reduced effective divisor on X obtained by forming the *fixed locus* of σ.

First, let us observe that it follows from Proposition 4.4, (iii), that we may assume without loss of generality, by applying a suitable change of coordinate, that the automorphism σ is the automorphism determined by σ_1 of Proposition 4.4, (iii). Thus, we obtain that

$$f_E(t) \;=\; t^2 - a.$$

Since

$$-\frac{d^2}{dt^2} f_E(t) \;=\; -\frac{d^2}{dt^2}(t^2 - a) \;=\; 1,$$

it holds that E satisfies condition (1) of Lemma 4.3. Next, to verify the assertion that E satisfies condition (2) of Lemma 4.3, let us observe that the following equalities hold.

$$f_E(t) \;=\; \frac{1}{a}\Big(f_E(t)^2 + \frac{d}{dt}\big((t+1)\cdot(t+a)\cdot f_0(t)\big)\Big),$$

$$t \cdot f_E(t) \;=\; \frac{d}{dt}(t \cdot f_0(t)),$$

$$t^2 \cdot f_E(t) \;=\; 2 \cdot f_E(t)^2 + \frac{d}{dt}\big((t+1)\cdot(t+a)\cdot f_0(t)\big).$$

Thus, we conclude that E satisfies condition (2) of Lemma 4.3. In particular, it follows from Lemma 4.3 that E is *of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above *claim*.

Next, let us recall that it follows immediately from §1, (1.h), (i), that the hyperbolic curve (X, D) over k has at most 3 ($= p^{3g-3+r}$) nilpotent indigenous bundles. Thus, the above *claim*, together with §1, (1.h), (i), (iii), leads us to the following list of the nilpotent indigenous bundles on $(X, D)/k$.

PROPOSITION 4.6. — *The following hold.*

(i) *The hyperbolic curve (X, D) over k has* **precisely three** *nilpotent indigenous bundles.*

(ii) *Every nilpotent indigenous bundle on $(X, D)/k$ is* **ordinary***, hence also* **admissible**.

(iii) *The* **supersingular divisor** *of a nilpotent [necessarily admissible — cf.* (ii)*] indigenous bundle on $(X, D)/k$ coincides with the reduced effective divisor obtained by forming the* **fixed locus** *of one of the three nontrivial* **nonspecial** *automorphisms of (X, D) over k.*

REMARK 4.6.1. — By Proposition 4.6, (i), (ii), the following assertion holds.

> Every sufficiently general hyperbolic curve of type $(0, 4)$ over k has *precisely three* nilpotent ordinary indigenous bundles.

On the other hand, this assertion has already been verified [cf. [8], Chapter V, Corollary 1.3, (3)].

The following corollary follows from Proposition 4.6, (i), (ii), together with [7], Chapter II, Proposition 3.4.

COROLLARY 4.7. — *Every hyperbolic curve of type* $(0,4)$ *over a connected noetherian scheme of characteristic* 3 *is* **hyperbolically ordinary** [*cf.* [7], *Chapter* II, *Definition* 3.3].

REMARK 4.7.1. — In the present Remark 4.7.1, let us discuss §6.2 of [1]. In the remainder of the present Remark 4.7.1, suppose that we are in the situation of §1, (1.h). [In particular, the field "k" is *not necessarily of characteristic three*.]

(i) [1], Lemma 6.3, asserts that the forgetful morphism $\mathcal{N}_{0,4} \to \mathcal{M}_{0,4}$ of stacks admits a *splitting*. Thus, since [it has already been verified that] $\mathcal{N}_{0,4}$ is *smooth* over k, it follows from §1, (1.h), (i), (iii), that the restriction of the morphism $\mathcal{N}_{0,4} \to \mathcal{M}_{0,4}$ of stacks to the ordinary locus $\mathcal{N}_{0,4}^{\mathrm{ord}} \subseteq \mathcal{N}_{0,4}$ is *surjective* [cf. [1], Proposition 6.4], and, moreover, the stack $\mathcal{N}_{0,4}$ is *not connected* [cf. [1], Corollary 6.5]. In particular, one may conclude that Corollary 4.7 holds [even if $p > 3$].

(ii) In the first and second paragraphs of the proof of [1], Lemma 6.3, the authors of [1] claimed that

> there exists a *nonzero* vector $(u_0, \ldots, u_{p-1})$ in the field $k_\lambda \stackrel{\mathrm{def}}{=} k(\lambda)$ of rational functions in λ over k such that the recursion (6.5) of [1], i.e.,
>
> $$\lambda \cdot (i+1)^2 \cdot u_{i+1} \;=\; (1+\lambda) \cdot (i^2+i+1) \cdot u_i - i^2 \cdot u_{i-1} \quad (i \in \{0, \ldots, p-1\})$$
>
> — where we write $u_{-1} \stackrel{\mathrm{def}}{=} u_p \stackrel{\mathrm{def}}{=} 0$ — holds.

However, this assertion is *false* in general. Indeed, if we are in the situation in which $p = 3$, then the above recursion is *equivalent* to the equality

$$\begin{pmatrix} 1+\lambda & -\lambda & 0 \\ 1 & 0 & \lambda \\ 0 & -1 & 1+\lambda \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

On the other hand, the determinant of the left-hand matrix is equal to $-\lambda \cdot (1+\lambda) \neq 0$. Thus, there is *no nonzero* vector (u_0, u_1, u_2) in k_λ which satisfies the recursion (6.5) of [1]. [Note that, in the fourth paragraph of the proof of [1], Lemma 6.3, it is asserted that the v_i's also satisfy the recursion (6.5) of [1]. However, the author of the present paper *cannot find* any reason which implies that the v_i's satisfy the recursion (6.5) of [1].]

(iii) As a consequence of the discussion of (ii), the proof given in [1] of [1], Lemma 6.3 — hence also of [1], Proposition 6.4; [1], Corollary 6.5 — must be considered *incomplete*.

(iv) On the other hand, by a straightforward computation of a similar recursion to the recursion (6.5) of [1] which arises from the differential operator $L_{\lambda,\beta}$ of (6.3) of [1], one can verify the *validity* of [1], Lemma 6.3, at least in the case where $p = 3$, which thus implies [cf. the discussion of (i)] [1], Proposition 6.4, in the case where $p = 3$ and [1], Corollary 6.5, in the case where $p = 3$. In particular, one may conclude that Corollary 4.7 of the present paper may also be deduced from the consideration of §6.2 of [1].

(v) However, after pointing out the error discussed in (ii) to the authors of [1], the author of the present paper was informed by *I. I. Bouw* [who is one of the authors of [1]] that she could verify that [1], Lemma 6.3, in the case where $p \in \{11, 13\}$ is in fact *false* by a straightforward computation of a similar recursion to the recursion (6.5) of [1] which arises from the differential operator $L_{\lambda,\beta}$ of (6.3) of [1].

5. Explicit Computations in Cases of Once-punctured Elliptic Curves

In the present §5, we apply the characterizations of Theorem 3.9 and Theorem 3.10 to some *once-punctured elliptic curves.*

In the present §5, suppose that

$$(g, r) \;=\; (1, 1),$$

which thus implies that

$$\deg \omega^{\log} \;=\; 1.$$

Thus, there exist functions s, $t \in \Gamma(X \setminus D, \mathcal{O}_X)$ which determine an isomorphism over k

$$\mathrm{Spec}(k[s,t]/(s^2 - t \cdot (t-1) \cdot (t-a)) \;\xrightarrow{\sim}\; X \setminus D$$

for some element $a \in k \setminus \{0,1\}$ of $k \setminus \{0,1\}$. Let us identify the left-hand side with the right-hand side by means of this isomorphism. We shall write

$$f_0(t) \;\stackrel{\mathrm{def}}{=}\; t \cdot (t-1) \cdot (t-a) \;\in\; \Gamma(X \setminus D, \mathcal{O}_X),$$

$$f_0'(t) \;\stackrel{\mathrm{def}}{=}\; \frac{d}{dt} f_0(t) \;=\; 3t^2 - 2(1+a)t + a,$$

$$U \;\stackrel{\mathrm{def}}{=}\; \mathrm{Spec}\Big(k\Big[s, \frac{1}{s}, t\Big]/(s^2 - f_0(t))\Big) \;\subseteq\; X \setminus D$$

for the largest open subscheme of $X \setminus D$ on which the function $s \in \Gamma(X \setminus D, \mathcal{O}_X)$ is invertible, and

$$\omega_0 \;\in\; \Gamma(X, \omega) \;=\; \Gamma(X, \omega^{\log})$$

for the unique global section of ω ($\subseteq \omega^{\log}$) whose restriction to $U \subseteq X$ is given by

$$\frac{dt}{s} \;\in\; \Gamma(U, \omega^{\log}).$$

Write, moreover, for each integer d,

$$k[s,t]^{\leq d} \;\stackrel{\mathrm{def}}{=}\; \Big\{\, f(s,t) = \sum_{i,j} c_{i,j} \cdot s^i \cdot t^j \in k[s,t] \;\Big|\; c_{i,j} = 0 \text{ if } 3i + 2j > d \,\Big\}$$

and

$$V^{\leq d} \;\subseteq\; k\Big[s, \frac{1}{s}, t\Big]/(s^2 - f_0(t))$$

for the subspace obtained by forming the image of $k[s,t]^{\leq d} \subseteq k[s,t]$. [Thus, the equality

$$\dim_k V^{\leq d} \;=\; \begin{cases} d & \text{if } d \geq 1 \\ 1 & \text{if } d = 0 \\ 0 & \text{if } d \leq -1 \end{cases}$$

holds.] Then it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes p}(-D)$, $(\omega^{\log})^{\otimes p+1}(-D)$, and $((\omega^{\log})^F)^{\otimes 2}(-D^F)$ and the homomorphisms d and C that there exist isomorphisms of k-vector spaces

$$\begin{array}{ccc} V^{\leq p-1} & \xrightarrow{\sim} & \Gamma(X,(\omega^{\log})^{\otimes p}(-D)) \\ g(s,t) & \mapsto & g(s,t)\cdot\omega_0^{\otimes p}, \end{array}$$

$$\begin{array}{ccc} V^{\leq p} & \xrightarrow{\sim} & \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)) \\ f(s,t) & \mapsto & f(s,t)\cdot\omega_0^{\otimes p+1}, \end{array}$$

$$\begin{array}{ccc} V^{\leq 0} & \xrightarrow{\sim} & \Gamma(X,((\omega^{\log})^F)^{\otimes 2}(-D^F)) \\ c & \mapsto & c\cdot(\omega_0^F)^{\otimes 2}, \end{array}$$

and that the sequence of k-vector spaces

$$\Gamma(X,(\omega^{\log})^{\otimes p}(-D)) \xrightarrow{d} \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)) \xrightarrow{C} \Gamma(X,((\omega^{\log})^F)^{\otimes 2}(-D^F))$$

corresponds, relative to the above isomorphisms, to the sequence of k-vector spaces

$$\begin{array}{ccccc} V^{\leq p-1} & \longrightarrow & V^{\leq p} & \longrightarrow & V^{\leq 0} \\ g(s,t) & \mapsto & s\cdot\frac{d}{dt}g(s,t)\ \Big(=\frac{1}{s^p}\cdot f_0(t)^{p^*+1}\cdot\frac{d}{dt}g(s,t)\Big) & & \\ & & f(s,t) & \mapsto & -\frac{d^{p-1}}{dt^{p-1}}(f(s,t)\cdot f_0(t)^{p^*}). \end{array}$$

Note that the first arrow of this sequence coincides with the homomorphism given by

$$\begin{array}{ccc} V^{\leq p-1} & \longrightarrow & V^{\leq p} \\ t^n & \mapsto & n\cdot t^{n-1}\cdot s \\ t^n\cdot s & \mapsto & G_n(t) \stackrel{\mathrm{def}}{=} n\cdot t^{n-1}\cdot f_0(t)+t^n\cdot\frac{f_0'(t)}{2}. \end{array}$$

Thus, we obtain the following lemma.

LEMMA 5.1. — *Let E be a reduced effective divisor on X of degree p^* $(=p^*\deg\omega^{\log})$. Then it holds that E satisfies condition* (3) *of Theorem* 3.9 [*i.e., condition* (3) *of Theorem* 3.10] *if and only if the k-vector space $V^{\leq p}$ is* **not generated** *by the subspace of $V^{\leq p}$ corresponding, relative to the above isomorphism*

$$V^{\leq p} \xrightarrow{\sim} \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D)),$$

to the subspace

$$\Gamma(X,(\omega^{\log})^{\otimes p+1}(-D-E)) \subseteq \Gamma(X,(\omega^{\log})^{\otimes p+1}(-D))$$

and

$$t^n\cdot s \quad (0\leq n\leq p^*-1), \quad G_m(t) \quad (0\leq m\leq p^*-2).$$

(5.a). In the present (5.a), suppose that

$$(g, r, p) = (1, 1, 3),$$

which thus implies that

$$p^{\circledast} = 1.$$

Let us first consider the principal divisor [i.e., the reduced effective divisor of degree $1 = p^{\circledast} \deg \omega^{\log}$] on X defined by a closed point of $X \setminus D$ which is *not a* 2*-torsion* point of the elliptic curve over k determined by (X, D). It is immediate that such a closed point of $X \setminus D$ is defined by the maximal ideal

$$(s - c_2, \, t - c_1) \subseteq k[s, t]/(s^2 - f_0(t))$$

for some pair (c_1, c_2) of elements of k such that $f_0(c_1) \neq 0$ and $c_2^2 = f_0(c_1)$. Write

$$E_{(c_1, c_2)} \subseteq X$$

for the principal divisor defined by this closed point. Then it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes 4}(-D - E_{(c_1,c_2)})$ and $(\omega^{\log})^{\otimes 4}(-D)$ that the subspace

$$\Gamma(X, (\omega^{\log})^{\otimes 4}(-D - E_{(c_1,c_2)})) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 4}(-D))$$

corresponds, relative to the isomorphism

$$V^{\leq 3} \xrightarrow{\sim} \Gamma(X, (\omega^{\log})^{\otimes 4}(-D))$$

discussed above, to the subspace

$$\langle t - c_1, \, s - c_2 \rangle \subseteq V^{\leq 3}.$$

Thus, since [it is immediate from the fact that $c_2 \neq 0$ that] the subspace $\langle t - c_1, \, s - c_2 \rangle$ and

$$s$$

generate $V^{\leq 3}$, it follows from Lemma 5.1 that $E_{(c_1,c_2)}$ does *not satisfy* condition (3) of Theorem 3.9. Thus, it follows from Theorem 3.9 that $E_{(c_1,c_2)}$ is *not of NA-type* relative to $(X, D)/k$.

Next, let us consider the principal divisor [i.e., the reduced effective divisor of degree $1 = p^{\circledast} \deg \omega^{\log}$] on X defined by a closed point of $X \setminus D$ which is a [necessarily nontrivial] 2*-torsion* point of the elliptic curve over k determined by (X, D). Let $c \in k$ be a solution of the equation "$f_0(t) = 0$", i.e., an element of $\{0, 1, a\}$. In the remainder of (5.a), write

$$E \subseteq X$$

for the principal divisor defined by the maximal ideal

$$(s, \, t - c) \subseteq k[s, t]/(s^2 - f_0(t)).$$

Now I *claim* that

> the reduced effective divisor E on X of degree 1 ($= p^{\circledast} \deg \omega^{\log}$) is *of NO-type* relative to $(X, D)/k$.

To verify this *claim*, let us first observe that we may assume without loss of generality, by applying a suitable change of coordinate, that $c = 0$. Then it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes 4}(-D - 2E)$, $(\omega^{\log})^{\otimes 4}(-D - E)$, and $(\omega^{\log})^{\otimes 4}(-D)$ that the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes 4}(-D - 2E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 4}(-D - E)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 4}(-D))$$

correspond, relative to the isomorphism

$$V^{\leq 3} \xrightarrow{\sim} \Gamma(X, (\omega^{\log})^{\otimes 4}(-D))$$

discussed above, to the subspaces

$$\langle t \rangle \subseteq \langle t,\, s \rangle \subseteq V^{\leq 3}.$$

Since

$$-\frac{d^2}{dt^2}(t \cdot f_0(t)) = -\frac{d^2}{dt^2}(t^2 \cdot (t-1) \cdot (t-a)) = a \neq 0,$$

it holds that E satisfies condition ($2'$) of Theorem 3.10. Moreover, since [it is immediate that] the subspace of $V^{\leq 3}$ generated by $\langle t,\, s \rangle$ and

$$s$$

is *of dimension* ≤ 2 (< 3), it follows from Lemma 5.1 that E satisfies condition (3) of Theorem 3.10. Thus, it follows from Theorem 3.10 that E is *of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above *claim*.

Next, let us recall that it follows immediately from §1, (1.h), (i), that the hyperbolic curve (X, D) over k has at most 3 ($= p^{3g-3+r}$) nilpotent indigenous bundles. Thus, the above *claim*, together with §1, (1.h), (i), (iii), leads us to the following list of the nilpotent indigenous bundles on $(X, D)/k$.

PROPOSITION 5.2. — *The following hold.*

(i) *The hyperbolic curve* (X, D) *over* k *has* **precisely three** *nilpotent indigenous bundles.*

(ii) *Every nilpotent indigenous bundle on* $(X, D)/k$ *is* **ordinary**, *hence also* **admissible**.

(iii) *The* **supersingular divisor** *of a nilpotent* [*necessarily admissible — cf.* (ii)] *indigenous bundle on* $(X, D)/k$ *coincides with the reduced effective divisor on* X *of degree one determined by one of the three* **nontrivial 2-torsion points** *of the elliptic curve determined by* (X, D).

REMARK 5.2.1. — By Proposition 5.2, (i), (ii), the following assertion holds.

> Every sufficiently general hyperbolic curve of type $(1, 1)$ over k has *precisely three* nilpotent ordinary indigenous bundles.

On the other hand, this assertion has already been verified [cf. [8], Chapter V, Corollary 1.3, (3)].

The following corollary follows from Proposition 5.2, (i), (ii), together with [7], Chapter II, Proposition 3.4.

COROLLARY 5.3. — *Every hyperbolic curve of type* $(1, 1)$ *over a connected noetherian scheme of characteristic* 3 *is* **hyperbolically ordinary**.

Let us observe that it follows from Proposition 5.2, (ii), that

$$\mathcal{N}_{1,[1]}^{\mathrm{ord}} = \mathcal{N}_{1,[1]}^{\mathrm{adm}} = \mathcal{N}_{1,[1]}.$$

Next, let us recall that the morphism of stacks

$$\mathcal{X}_{1,[1]} \longrightarrow \mathcal{M}_{1,[1]}$$

forms a *family of elliptic curves* over $\mathcal{M}_{1,[1]}$ whose identity section is given by $\mathcal{D}_{1,[1]} \subseteq \mathcal{X}_{1,[1]}$. For each positive integer n, we shall write

$$\mathcal{X}_{1,[1]}[n] \longrightarrow \mathcal{M}_{1,[1]}$$

for the kernel of the endomorphism of $\mathcal{X}_{1,[1]}$ over $\mathcal{M}_{1,[1]}$ obtained by multiplication by n. [So $\mathcal{X}_{1,[1]}[1] = \mathcal{D}_{1,[1]}$.] Then it follows from Proposition 5.2, (iii), that, by considering *supersingular divisors*, we obtain a *dominant* morphism of stacks

$$\mathcal{N}_{1,[1]}^{\mathrm{ord}} = \mathcal{N}_{1,[1]}^{\mathrm{adm}} = \mathcal{N}_{1,[1]} \longrightarrow \mathcal{X}_{1,[1]}[2] \setminus \mathcal{D}_{1,[1]}$$

over $\mathcal{M}_{1,[1]}$ [i.e., the "$(1,[1])$-version" of the *Hasse defect morphism* — cf. [5], Definition C.1]. Thus, both $\mathcal{N}_{1,[1]}^{\mathrm{ord}} = \mathcal{N}_{1,[1]}^{\mathrm{adm}} = \mathcal{N}_{1,[1]}$ and $\mathcal{X}_{1,[1]}[2] \setminus \mathcal{D}_{1,[1]}$ are *finite étale* and *of degree three* over $\mathcal{M}_{1,[1]}$ [cf. §1, (1.h), (i), (iii)], we obtain the following result.

COROLLARY 5.4. — *There exists a natural* **isomorphism** *of stacks*

$$\mathcal{N}_{1,[1]}^{\mathrm{ord}} = \mathcal{N}_{1,[1]}^{\mathrm{adm}} = \mathcal{N}_{1,[1]} \xrightarrow{\sim} \mathcal{X}_{1,[1]}[2] \setminus \mathcal{D}_{1,[1]}$$

over $\mathcal{M}_{1,[1]}$.

(5.b). In the present (5.b), suppose that

$$(g, r, p) = (1, 1, 5),$$

which thus implies that

$$p^{\circledast} = 2.$$

Let c_1, $c_2 \in k$ be two distinct solutions of the equation "$f_0(t) = 0$", i.e., two distinct elements of $\{0, 1, a\}$. Write

$$E_1 \subseteq X$$

for the reduced effective divisor of degree 2 ($= p^{\circledast} \deg \omega^{\log}$) defined by the ideal

$$(s, (t - c_1) \cdot (t - c_2)) \subseteq k[s,t]/(s^2 - f_0(t)).$$

Now I claim that the following assertion holds.

($\dagger_1$): The reduced effective divisor E_1 on X of degree 2 ($= p^{\circledast} \deg \omega^{\log}$) is *of NO-type* relative to $(X, D)/k$.

To verify the assertion ($\dagger_1$), let us first observe that we may assume without loss of generality, by applying a suitable change of coordinate, that $(c_1, c_2) = (0, 1)$. Then it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes 6}(-D-2E_1)$, $(\omega^{\log})^{\otimes 6}(-D-E_1)$, and $(\omega^{\log})^{\otimes 6}(-D)$ that the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes 6}(-D - 2E_1)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 6}(-D - E_1)) \subseteq \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

correspond, relative to the isomorphism

$$V^{\leq 5} \xrightarrow{\sim} \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

discussed above, to the subspaces

$$\langle t \cdot (t-1) \rangle \ \subseteq \ \langle s,\, t \cdot (t-1),\, t \cdot s \rangle \ \subseteq \ V^{\leq 5}.$$

Since

$$-\frac{d^4}{dt^4}(t \cdot (t-1) \cdot f_0(t)^2) \ = \ -\frac{d^4}{dt^4}(t^3 \cdot (t-1)^3 \cdot (t-a)^2) \ = \ 3 \cdot a \cdot (a-1) \ \neq \ 0,$$

it holds that E_1 satisfies condition ($2'$) of Theorem 3.10. Moreover, since [it is immediate that] the subspace of $V^{\leq 5}$ generated by $\langle s,\, t \cdot (t-1),\, t \cdot s \rangle$ and

$$s, \quad G_0(t) \ = \ 3 \cdot f_0'(t), \quad t \cdot s$$

is *of dimension* ≤ 4 (< 5), it follows from Lemma 5.1 that E_1 satisfies condition (3) of Theorem 3.10. Thus, it follows from Theorem 3.10 that E_1 is *of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above assertion ($\dagger_1$).

Next, let $c \in k$ be a solution of the equation "$f_0'(t) = 0$". [So the equality

$$c^2 + (1+a) \cdot c + 2a \ = \ 0$$

holds.] Write

$$E_2 \ \subseteq \ X$$

for the reduced effective divisor of degree 2 ($= p^* \deg \omega^{\log}$) defined by the ideal

$$(t-c) \ \subseteq \ k[s,t]/(s^2 - f_0(t)).$$

Now I claim that the following assertion holds.

($\dagger_2$): The reduced effective divisor E_2 on X of degree 2 ($= p^* \deg \omega^{\log}$) is *of NA-type* relative to $(X, D)/k$.

To verify the assertion ($\dagger_2$), let us first observe that it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes 6}(-D - E_2)$ and $(\omega^{\log})^{\otimes 6}(-D)$ that the subspace

$$\Gamma(X, (\omega^{\log})^{\otimes 6}(-D - E_2)) \ \subseteq \ \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

corresponds, relative to the isomorphism

$$V^{\leq 5} \ \xrightarrow{\sim} \ \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

discussed above, to the subspace

$$\langle t-c,\, (t-c)^2,\, (t-c) \cdot s \rangle \ \subseteq \ V^{\leq 5}.$$

Here, let us observe that it holds that

$$-\frac{d^4}{dt^4}((t-c) \cdot f_0(t)^2) \ = \ -\frac{d^4}{dt^4}((t-c) \cdot t^2 \cdot (t-1)^2 \cdot (t-a)^2)$$
$$= \ 3 \cdot a \cdot (1+a) - c \cdot (a^2 - a + 1).$$

If $a^2 - a + 1 = 0$ [which thus implies that a is a *primitive sixth root of unity*], then it is immediate that $3 \cdot a \cdot (1+a) - c \cdot (a^2 - a + 1) \neq 0$. Moreover, if $a^2 - a + 1 \neq 0$, and $3 \cdot a \cdot (1+a) - c \cdot (a^2 - a + 1) = 0$, then the equality $c^2 + (1+a) \cdot c + 2a = 0$ implies that

$$a^2 \cdot (a-1)^2 \ = \ 0$$

— in contradiction to the fact that $a \notin \{0, 1\}$. Thus, we conclude that

$$-\frac{d^4}{dt^4}((t-c) \cdot f_0(t)^2) \ \neq \ 0$$

— which thus implies that E_2 satisfies condition (2) of Theorem 3.9.

Next, let us observe that it is immediate that if $c' \in k$ is *not a solution* of the equation "$f_0'(t) = 0$", then $t - c' \in V^{\leq 5}$ is *not contained* in the subspace of $V^{\leq 5}$ generated by $\langle t - c,\ (t-c)^2,\ (t-c) \cdot s \rangle$ and

$$s, \quad G_0(t) \ = \ 3 \cdot f_0'(t), \quad t \cdot s.$$

In particular, it follows from Lemma 5.1 that E_2 satisfies condition (3) of Theorem 3.9. Thus, it follows from Theorem 3.9 that E_2 is *of NA-type* relative to $(X, D)/k$, as desired. This completes the proof of the above assertion $(\dagger_2)$.

Next, I claim that the following assertion holds.

> $(\dagger_3)$: If, moreover, the elliptic curve over k determined by (X, D) is *supersingular* [i.e., the equality $a^2 - a + 1 = 0$ holds — cf. the Hasse polynomial "$\chi_{\mathrm{Hss}}(t)$" discussed in §4, (4.a), in the case where $p = 5$], then the divisor E_2 is *not of NO-type* relative to $(X, D)/k$.

To verify the assertion $(\dagger_3)$, suppose that the equality $a^2 - a + 1 = 0$ holds. Then it is immediate that the equation "$f_0'(t) = 0$" has a *multiple root*, which thus implies that

$$f_0'(t) \ = \ 3 \cdot (t - c)^2.$$

Next, let us observe that it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes 6}(-D - 2E_2)$ and $(\omega^{\log})^{\otimes 6}(-D)$ that the subspace

$$\Gamma(X, (\omega^{\log})^{\otimes 6}(-D - 2E_2)) \ \subseteq \ \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

corresponds, relative to the isomorphism

$$V^{\leq 5} \ \xrightarrow{\sim} \ \Gamma(X, (\omega^{\log})^{\otimes 6}(-D))$$

discussed above, to the subspace

$$\langle (t - c)^2 \rangle \ \subseteq \ V^{\leq 5}.$$

Thus, since $G_0(t)$ $(= 3 \cdot f_0'(t) = -(t - c)^2)$ is *contained* in the image of d, hence also the kernel of C [cf. Remark 3.10.1, (iv)], it holds that E_2 does *not satisfy* condition $(2')$ of Theorem 3.10. Thus, it follows from Theorem 3.10 that E_2 is *not of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above assertion $(\dagger_3)$.

By the assertions $(\dagger_1)$, $(\dagger_2)$, and $(\dagger_3)$, we obtain the following list of the nilpotent indigenous bundles on $(X, D)/k$.

PROPOSITION 5.5. — *Write A for the elliptic curve over k determined by (X, D). Then the following hold.*

(i) *If A is* **ordinary** (*respectively,* **supersingular**), *then the hyperbolic curve (X, D) over k has* **precisely five** (*respectively,* **four**) *nilpotent indigenous bundles.*

(ii) *Every nilpotent indigenous bundle on $(X, D)/k$ is* **admissible**.

(iii) *The* **supersingular divisor** *of a nilpotent [necessarily admissible — cf.* (ii)] *indigenous bundle on $(X, D)/k$ coincides with the reduced effective divisor on X of degree two determined by either*

(a) *two of the three* **nontrivial 2-torsion points** *of A or*

(b) *one of the solutions of the equation "*$f_0'(t) = 0$*".* [*Note that if A is* **ordinary** (*respectively,* **supersingular**)*, then the equation "*$f_0'(t) = 0$*" has exactly two* (*respectively, one*) *solution*(*s*)*.*]

(iv) *It holds that a nilpotent* [*necessarily admissible — cf.* (ii)] *indigenous bundle on* $(X, D)/k$ *is* **ordinary** *if and only if one of the following two conditions is satisfied.*

(1) *The supersingular divisor of the nilpotent indigenous bundle is given by* (a) *of* (iii)*.*

(2) *The elliptic curve A is* **ordinary***.*

(v) *If A is* **ordinary** (*respectively,* **supersingular**)*, then* (X, D) *has* **precisely five** (*respectively,* **three**) *nilpotent* **ordinary** *indigenous bundles.*

PROOF. — First, we verify Proposition 5.5 in the case where A is *ordinary.* Suppose that A is *ordinary.* Then it follows from the assertions ($\dagger_1$) and ($\dagger_2$) that the hyperbolic curve (X, D) has *at least five* nilpotent admissible indigenous bundles. Thus, it follows immediately from §1, (1.h), (i), that assertion (i) — hence also assertions (ii), (iii) [cf. the assertions ($\dagger_1$) and ($\dagger_2$)] — holds. Moreover, it follows immediately from §1, (1.h), (iii), that every nilpotent indigenous bundle is *ordinary*, which thus implies that assertions (iv), (v) hold. This completes the proof of Proposition 5.5 in the case where A is *ordinary.*

Next, we verify Proposition 5.5 in the case where A is *supersingular.* Suppose that A is *supersingular.* Then it follows from the assertions ($\dagger_1$) and ($\dagger_2$) that the hyperbolic curve (X, D) has *at least four* nilpotent admissible indigenous bundles. Moreover, it follows from the assertion ($\dagger_3$) that one of the four nilpotent admissible indigenous bundles is *not ordinary.* Thus, it follows immediately from §1, (1.h), (i), (iii), that assertion (i) — hence also assertions (ii), (iii) [cf. the assertions ($\dagger_1$) and ($\dagger_2$)] — holds. Moreover, it follows immediately from the assertions ($\dagger_1$) and ($\dagger_3$) that assertions (iv), (v) hold. This completes the proof of Proposition 5.5 in the case where A is *supersingular*, hence also of Proposition 5.5. ☐

REMARK 5.5.1. — By Proposition 5.5, (v), the following assertion holds.

> Every sufficiently general hyperbolic curve of type $(1, 1)$ over k has *precisely five* nilpotent ordinary indigenous bundles.

On the other hand, this assertion has already been verified [cf. [8], Chapter V, Corollary 1.3, (3)].

The following corollary follows from Proposition 5.5, (v), together with [7], Chapter II, Proposition 3.4 [cf. Remark 5.6.1 below].

COROLLARY 5.6. — *Every hyperbolic curve of type* $(1, 1)$ *over a connected noetherian scheme of characteristic* 5 *is* **hyperbolically ordinary***.*

REMARK 5.6.1. — Note that Corollary 5.6 has already been verified in the second remark of [8], Chapter IV, §1.3.

(5.c). In the present (5.c), suppose that

$$(g, r, p) \ = \ (1, 1, 7),$$

which thus implies that

$$p^{\circledast} \ = \ 3.$$

Write

$$E \ \subseteq \ X$$

for the reduced effective divisor of degree 3 ($= p^{\circledast} \deg \omega^{\log}$) defined by the ideal

$$(s) \ \subseteq \ k[s, t]/(s^2 - f_0(t)).$$

Now I *claim* that

> the reduced effective divisor E on X of degree 3 ($= p^{\circledast} \deg \omega^{\log}$) is *of NO-type* relative to $(X, D)/k$.

To verify this *claim*, let us first observe that it follows immediately from the definitions of the sheaves $(\omega^{\log})^{\otimes 8}(-D - 2E)$, $(\omega^{\log})^{\otimes 8}(-D - E)$, and $(\omega^{\log})^{\otimes 8}(-D)$ that the subspaces

$$\Gamma(X, (\omega^{\log})^{\otimes 8}(-D - 2E)) \ \subseteq \ \Gamma(X, (\omega^{\log})^{\otimes 8}(-D - E)) \ \subseteq \ \Gamma(X, (\omega^{\log})^{\otimes 8}(-D))$$

correspond, relative to the isomorphism

$$V^{\leq 7} \ \xrightarrow{\sim} \ \Gamma(X, (\omega^{\log})^{\otimes 8}(-D))$$

discussed above, to the subspaces

$$\langle f_0(t) \rangle \ \subseteq \ \langle s,\, t \cdot s,\, f_0(t),\, t^2 \cdot s \rangle \ \subseteq \ V^{\leq 7}.$$

Since

$$-\frac{d^6}{dt^6}(f_0(t) \cdot f_0(t)^3) \ = \ -\frac{d^6}{dt^6}(t^4 \cdot (t-1)^4 \cdot (t-a)^4) \ = \ -a^2 \cdot (a-1)^2 \ \neq \ 0,$$

it holds that E satisfies condition $(2')$ of Theorem 3.10. Moreover, since [it is immediate that] the subspace of $V^{\leq 7}$ generated by $\langle s,\, t \cdot s,\, f_0(t),\, t^2 \cdot s \rangle$ and

$$s, \quad G_0(t) \ = \ 4 \cdot f_0'(t), \quad t \cdot s, \quad G_1(t) \ = \ f_0(t) + 4 \cdot t \cdot f_0'(t), \quad t^2 \cdot s$$

is *of dimension* ≤ 6 (< 7), it follows from Lemma 5.1 that E satisfies condition (3) of Theorem 3.10. Thus, it follows from Theorem 3.10 that E is *of NO-type* relative to $(X, D)/k$, as desired. This completes the proof of the above *claim*.

By the above *claim*, we obtain the following proposition.

PROPOSITION 5.7. — *The hyperbolic curve (X, D) over k has a* **nilpotent ordinary indigenous bundle** *whose* **supersingular divisor** *coincides with the reduced effective divisor on X of degree three determined by the three* **nontrivial 2-torsion points** *of the elliptic curve determined by (X, D).*

The following corollary follows from Proposition 5.7, together with [7], Chapter II, Proposition 3.4.

COROLLARY 5.8. — *Every hyperbolic curve of type $(1, 1)$ over a connected noetherian scheme of characteristic 7 is* **hyperbolically ordinary**.

APPENDIX A. CANONICAL SECTIONS AND SQUARE HASSE INVARIANTS

In the present §A, we apply the notational conventions introduced in §1. In the present §A, we discuss the relationship between the zero loci of *square Hasse invariants* [cf. [7], Chapter II, Proposition 2.6, (1)] and the zero loci of *canonical sections* discussed in [1], §3. Moreover, we also verify that the application of the discussions of [1], §3, leads us to the relationship [cf. Proposition A.3 below] between the zero loci of *square Hasse invariants*, *generalized supersingular divisors* [cf. Definition A.2, (iii), below], and *spiked loci* [cf. Definition A.2, (iv), below].

DEFINITION A.1. — We shall say that an indigenous bundle on $(X, D)/k$ is *active* [cf. [8], Chapter II, Definition 1.1] if the p-curvature homomorphism of the indigenous bundle is a nonzero homomorphism.

DEFINITION A.2. — Let (P, ∇_P) be a nilpotent active indigenous bundle on $(X, D)/k$.

(i) We shall write

$$E_{\mathrm{sH}}$$

for the divisor on X obtained by forming the zero locus of the *square Hasse invariant* of (P, ∇_P).

(ii) Since (P, ∇_P) is nilpotent and active, there exists a unique horizontal section of $P \to X$. We shall refer to this unique horizontal section of $P \to X$ as the *conjugate section* of (P, ∇_P).

(iii) We shall write

$$E_{\mathrm{gss}}$$

for the divisor on X obtained by pulling back the conjugate section via the Hodge section of (P, ∇_P) [cf. [7], Chapter I, Proposition 2.4] and refer to E_{gss} as the *generalized supersingular divisor* of (P, ∇_P).

(iv) We shall write

$$E_{\mathrm{spk}}$$

for the divisor on X obtained by forming the zero locus of the p-curvature homomorphism of (P, ∇_P) and refer to E_{spk} as the *spiked locus* of (P, ∇_P) [cf. [8], Chapter II, Definition 3.1]. Thus, it follows from the various definitions involved that (P, ∇_P) is admissible if and only if $E_{\mathrm{spk}} = \emptyset$.

The purpose of the present §A is to verify the following proposition.

PROPOSITION A.3. — *Let* (P, ∇_P) *be a* **nilpotent active** *indigenous bundle on* $(X, D)/k$. *Then the following hold.*

(i) *The divisor* E_{gss} *is* **reduced**. *Moreover, it holds that* $E_{\mathrm{gss}} \cap D = \emptyset$.

(ii) *It holds that* $E_{\mathrm{spk}} \cap D = \emptyset$. *Moreover, there exists a divisor* $\underline{E}_{\mathrm{spk}}$ *on* X *such that* $E_{\mathrm{spk}} = p\underline{E}_{\mathrm{spk}}$.

(iii) *It holds that* $2E_{\mathrm{gss}} \leq E_{\mathrm{sH}}$, *that* $E_{\mathrm{spk}} \leq E_{\mathrm{sH}}$, *and that* $2 \deg E_{\mathrm{gss}} + \deg E_{\mathrm{spk}} = \deg E_{\mathrm{sH}}$.

(iv) *If* (P, ∇_P) *is* **admissible**, *then the divisor* E_{gss} *coincides with the* **supersingular divisor** *of* (P, ∇_P).

REMARK A.3.1. — Suppose that we are in the situation of Proposition A.3.

(i) Suppose that $E_{\mathrm{gss}} \cap E_{\mathrm{spk}} = \emptyset$. Then it follows immediately from Proposition A.3, (iii), that $E_{\mathrm{sH}} = 2E_{\mathrm{gss}} + E_{\mathrm{spk}}$. In particular, it follows from Proposition A.3, (i), (ii), that the order of E_{sH} at each closed point of X is $\in \{2\} \cup p\mathbb{Z}$.

(ii) Now let us recall that the content of [1], Proposition 3.6, (iv), is essentially the same as the equality $E_{\mathrm{gss}} \cap E_{\mathrm{spk}} = \emptyset$.

(iii) However, this equality $E_{\mathrm{gss}} \cap E_{\mathrm{spk}} = \emptyset$ does *not hold* in general. Indeed, let us first observe that, by (i), this equality implies that the order of E_{sH} at each closed point of X is $\in \{2\} \cup p\mathbb{Z}$. Now let us recall the global section "ω" discussed in the second paragraph of [5], Remark 4.2.1, (i). [Note that "(g, r, p)" for the situation of [5], Remark 4.2.1, (i), is given by $(5, 0, 3)$.] Then it follows from [5], Proposition 3.2, and [5], Proposition 4.1, that the zero locus of the square of this ω is the "E_{sH}" of a nilpotent active indigenous bundle. On the other hand, it is immediate that the zero locus of the square of this ω is *of order* $\in \{0, 8\}$ at each closed point. In particular, one may conclude that the equality $E_{\mathrm{gss}} \cap E_{\mathrm{spk}} = \emptyset$ does *not hold* in general. [Note that, at the time of writing, the author of the present paper was *not able* to follow the proof of [1], Proposition 3.6, (iv).]

The following proposition, which seem to be well-known to experts, follows immediately from Proposition A.3, (i), (iv).

PROPOSITION A.4. — *The* **supersingular divisor** *of a nilpotent admissible indigenous bundle on* $(X, D)/k$ *is* **reduced** *and does* **not intersect** *the closed subscheme* D.

Since [we have assumed that] $p \geq 3$, the following proposition follows immediately from Proposition A.3.

PROPOSITION A.5. — *Let* (P, ∇_P) *be a* **nilpotent active** *indigenous bundle on* $(X, D)/k$. *Then the following four conditions are equivalent.*

(1) *The indigenous bundle* (P, ∇_P) *is* **admissible**.

(2) *There exists a* **reduced** *divisor* $\underline{E}_{\mathrm{sH}}$ *on* X *such that* $E_{\mathrm{sH}} = 2\underline{E}_{\mathrm{sH}}$.

(3) *For each closed point* $x \in X$ *of* X, *the order of* E_{sH} *at* x *is* $\in \{0, 2\}$.

(4) *For each closed point* $x \in X$ *of* X, *the order of* E_{sH} *at* x *is* ≤ 2.

In the remainder of the present §A, let us prove Proposition A.3 by means of the discussions of [1], §3. In the present §A, for a closed point $x \in X$ of X and a positive integer d, we shall write

$$x_d$$

for the natural closed immersion from the [unique] closed subscheme of X of length d whose underlying set consists of x into X. [So x_1 is the natural closed immersion from the reduced closed subscheme of X determined by $x \in X$ into X.]

Let (P, ∇_P) be a *nilpotent active* indigenous bundle on $(X, D)/k$. Let us first observe that, to verify Proposition A.3, we may assume without loss of generality, by replacing (X, D) by a suitable connected finite flat tamely ramified covering of (X, D), that r is *even*. Then it follows from [7], Chapter I, Proposition 2.6, that there exists an *indigenous vector bundle* [cf. [7], Chapter I, Definition 2.2]

$$(\mathcal{E}, \nabla_{\mathcal{E}})$$

which determines the indigenous bundle (P, ∇_P). That is to say,

- $\mathcal{E}$ is a locally free coherent $\mathcal{O}_X$-module of rank two, and
- $\nabla_{\mathcal{E}}$ is a connection on $\mathcal{E}$ relative to $(X, D)/k$

such that

- the determinant of $(\mathcal{E}, \nabla_{\mathcal{E}})$ is isomorphic to $(\mathcal{O}_X, d)$ [cf. the discussion preceding [7], Chapter I, Definition 2.2], and, moreover,
- the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$ is isomorphic to (P, ∇_P).

Write

$$\mathcal{H} \subseteq \mathcal{E}$$

for the *Hodge filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$ [i.e., the invertible subsheaf which defines the Hodge section of (P, ∇_P)]. Thus, it follows from the definition of an indigenous bundle that the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{H} \hookrightarrow \mathcal{E} \xrightarrow{\nabla_{\mathcal{E}}} \omega^{\log} \otimes_{\mathcal{O}_X} \mathcal{E} \twoheadrightarrow \omega^{\log} \otimes_{\mathcal{O}_X} (\mathcal{E}/\mathcal{H})$$

is an *isomorphism*. In particular, since $\det \mathcal{E} \cong \mathcal{O}_X$, we have the following lemma.

LEMMA A.6. — *It holds that*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}, \mathcal{E}/\mathcal{H}) \cong \tau^{\log}, \quad \mathcal{H}^{\otimes 2} \cong \omega^{\log}.$$

Next, write

$$\mathcal{P} \colon \mathcal{T} \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$$

for the *p-curvature* homomorphism of $(\mathcal{E}, \nabla_{\mathcal{E}})$. Let us recall that, in this situation, the *square Hasse invariant* of (P, ∇_P) is defined as the composite

$$\mathcal{T} \xrightarrow{\mathcal{P}} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}, \mathcal{E}/\mathcal{H}) \cong \tau^{\log}$$

[cf. Lemma A.6]. Thus, by the definition, we have the following lemma.

LEMMA A.7. — *The following hold.*

(i) *The divisor* E_{sH} [*cf. Definition* A.2, (i)] *is* **of degree** $(p-1)\deg\omega^{\log}$.

(ii) *For a closed point* $x \in X$ *of* X *and a positive integer* d*, the following two conditions are equivalent.*

(1) *It holds that* $\mathrm{ord}_x E_{\mathrm{sH}} \geq d$.

(2) *For each local section* ∂ *of* $\mathcal{T}$ *at* x*, the endomorphism* $x_d^*\mathcal{P}(\partial)$ *of* $x_d^*\mathcal{E}$ **preserves** *the submodule* $x_d^*\mathcal{H} \subseteq x_d^*\mathcal{E}$.

Next, write

$$\mathcal{C} \ \subseteq \ \mathcal{E}$$

for the *conjugate filtration* of $(\mathcal{E}, \nabla_\mathcal{E})$ [i.e., the invertible subsheaf which defines the conjugate section of (P, ∇_P) — cf. Definition A.2, (ii) — or, equivalently, the unique maximal horizontal invertible subsheaf of $(\mathcal{E}, \nabla_\mathcal{E})$] and

$$\nabla_\mathcal{C}, \quad \nabla_\mathcal{Q}$$

for the connections on

$$\mathcal{C}, \quad \mathcal{Q} \ \stackrel{\mathrm{def}}{=} \ \mathcal{E}/\mathcal{C}$$

induced by $\nabla_\mathcal{E}$, respectively.

LEMMA A.8. — *The following hold.*

(i) *The respective* ***p*-curvature** *homomorphisms of* $(\mathcal{C}, \nabla_\mathcal{C})$, $(\mathcal{Q}, \nabla_\mathcal{Q})$ *are* **zero**. *Moreover, the respective* **monodromy** *operators at each point on* $D \subseteq X$ *of* $(\mathcal{C}, \nabla_\mathcal{C})$, $(\mathcal{Q}, \nabla_\mathcal{Q})$ *are* **zero**.

(ii) *It holds that*

$$(\mathcal{C}, \nabla_\mathcal{C}) \otimes (\mathcal{Q}, \nabla_\mathcal{Q}) \ \cong \ (\mathcal{O}_X, d).$$

PROOF. — Assertion (i) follows from our assumption that the *p-curvature* homomorphism (respectively, *monodromy* operator at each point on $D \subseteq X$) of (P, ∇_P) is *nilpotent*. Assertion (ii) follows from the fact that $\det(\mathcal{E}, \nabla_\mathcal{E}) \cong (\mathcal{O}_X, d)$. ☐

In this situation, the *canonical section* of $(\mathcal{E}, \nabla_\mathcal{E})$ introduced in the discussion preceding [1], Lemma 3.4, may be defined as follows.

DEFINITION A.9. — We shall refer to the composite

$$\mathcal{H} \ \hookrightarrow \ \mathcal{E} \ \twoheadrightarrow \ \mathcal{Q}$$

as the *canonical section* of $(\mathcal{E}, \nabla_\mathcal{E})$ [cf. the discussion preceding [1], Lemma 3.4].

Thus, we have the following lemma.

LEMMA A.10. — *The following hold.*

(i) *The divisor on X obtained by forming the zero locus of the* **canonical section** *coincides with the* **generalized supersingular divisor** E_{gss} *of* (P, ∇_P) [*cf. Definition* A.2, (iii)].

(ii) *For a closed point $x \in X$ of X, the following two conditions are equivalent.*

(1) *It holds that* $\text{ord}_x E_{\text{gss}} \geq 1$.

(2) *It holds that* $x_1^* \mathcal{H} = x_1^* \mathcal{C}$.

PROOF. — Let us first observe that it follows immediately from the definition of E_{gss} that, for a closed point $x \in X$ of X and a positive integer d, the following three conditions are equivalent.

(a) It holds that $\text{ord}_x E_{\text{gss}} \geq d$.

(b) The pull-back of the Hodge section via x_d *coincides* with the pull-back of the conjugate section via x_d.

(c) It holds that $x_d^* \mathcal{H} = x_d^* \mathcal{C}$.

Next, let us observe that it follows from the definition of the canonical section that, for a closed point $x \in X$ of X and a positive integer d, the above condition (c) is equivalent to the following condition.

(d) The zero locus of the canonical section is *of order* $\geq d$ at x.

Finally, observe that the equivalence (a) $\Leftrightarrow$ (d) implies assertion (i), and that the equivalence (a) $\Leftrightarrow$ (c) implies assertion (ii). This completes the proof of Lemma A.10. $\square$

It follows from Lemma A.8, (i), that the p-curvature homomorphism $\mathcal{P}$ of $(\mathcal{E}, \nabla_{\mathcal{E}})$ factors through the subsheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{C}) \subseteq \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$, i.e.,

$$\mathcal{P} \colon \mathcal{T} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{C}) \cong \mathcal{C}^{\otimes 2}$$

[cf. Lemma A.8, (ii)]. In this situation, the *spiked locus* E_{spk} of (P, ∇_P) [cf. Definition A.2, (iv)] is defined as the zero locus of the resulting homomorphism $\mathcal{P} \colon \mathcal{T} \to \mathcal{C}^{\otimes 2}$. Thus, by definition, we have the following lemma.

LEMMA A.11. — *The following hold.*

(i) *The* ***p*-curvature** *homomorphism determines an* **isomorphism** *of $\mathcal{O}_X$-modules*

$$\mathcal{T}(E_{\text{spk}}) \xrightarrow{\sim} \mathcal{C}^{\otimes 2}.$$

(ii) *For a closed point $x \in X$ of X and a positive integer d, the following two conditions are equivalent.*

(1) *It holds that* $\text{ord}_x E_{\text{spk}} \geq d$.

(2) *For each local section ∂ of $\mathcal{T}$ at x, the image of the endomorphism $x_d^* \mathcal{P}(\partial)$ of $x_d^* \mathcal{E}$ is* **zero**.

By [1], §3, together with Lemma A.10, (i), we obtain the following lemma.

LEMMA A.12. — *The following hold.*

(i) *The divisor* E_{gss} *is* **of degree** $((p-1)\deg\omega^{\log} - \deg E_{\mathrm{spk}})/2$.

(ii) *The divisor* E_{gss} *is* **reduced**.

(iii) *It holds that* $E_{\mathrm{gss}} \cap D = \emptyset$.

(iv) *It holds that* $E_{\mathrm{spk}} \cap D = \emptyset$.

(v) *There exists a divisor* $\underline{E}_{\mathrm{spk}}$ *on* X *such that* $E_{\mathrm{spk}} = p\underline{E}_{\mathrm{spk}}$.

PROOF. — Assertion (i) follows from Lemma A.6; Lemma A.8, (ii); Lemma A.11, (i). Assertion (ii) is the content of the first assertion of [1], Lemma 3.4. Assertion (iii) is the content of the second assertion of [1], Lemma 3.4. Assertion (iv) is the content of [1], Proposition 3.6, (i). Assertion (v) is the content of [1], Proposition 3.6, (iii). □

In the remainder of the present §A, let us give a proof of Proposition A.3. Assertion (i) of Proposition A.3 follows from Lemma A.12, (ii), (iii). Assertion (ii) of Proposition A.3 follows from Lemma A.12, (iv), (v).

Next, we verify assertion (iii) of Proposition A.3.

LEMMA A.13. — *The following hold.*

(i) *It holds that* $E_{\mathrm{spk}} \leq E_{\mathrm{sH}}$.

(ii) *It holds that* $E_{\mathrm{gss}} \leq E_{\mathrm{sH}}$.

PROOF. — Assertion (i) follows from Lemma A.7, (ii), and Lemma A.11, (ii). Next, we verify assertion (ii). Let $x \in X$ be a closed point of X which is *contained* in the support of E_{gss}. Now let us observe that it follows from Lemma A.12, (ii), that, to verify assertion (ii), it suffices to verify that x is *contained* in the support of E_{sH}.

If x is *contained* in the support of E_{spk}, then it follows from assertion (i) that x is *contained* in the support of E_{sH}, as desired. Thus, we may assume without loss of generality that x is *not contained* in the support of the divisor E_{spk}, hence also [cf. Lemma A.12, (iii)] the divisor $E_{\mathrm{spk}} + D$. Then it follows from the definition of E_{spk}, together with Lemma A.8, (i), that, for each local generator ∂ of $\mathcal{T}$ at x, the *kernel* of the endomorphism $x_1^*\mathcal{P}(\partial)$ of $x_1^*\mathcal{E}$ *coincides* with the subspace $x_1^*\mathcal{C} \subseteq x_1^*\mathcal{E}$. In particular, it follows from Lemma A.10, (ii), that, for each local section ∂ of $\mathcal{T}$ at x, the image of the restriction of the endomorphism $x_1^*\mathcal{P}(\partial)$ of $x_1^*\mathcal{E}$ to the subspace $x_1^*\mathcal{H} \subseteq x_1^*\mathcal{E}$ is *zero*. Thus, it follows from Lemma A.7, (ii), that x is *contained* in the support of E_{sH}, as desired. This completes the proof of assertion (ii), hence also of Lemma A.13. □

It follows immediately, in light of Lemma A.7, (i), and Lemma A.12, (i), (iii), from Lemma A.13 that, to complete the verification of assertion (iii) of Proposition A.3, it suffices to verify that $2E_{\mathrm{gss}} \leq E_{\mathrm{sH}}$. To this end, let us take a closed point $x \in X$ of X which is *contained* in the support of E_{gss} [which thus implies that x is *not contained* in the support of the divisor D — cf. Lemma A.12, (iii)]. Thus, since $\mathrm{ord}_x E_{\mathrm{gss}} = 1$ [cf. Lemma A.12, (ii)], to verify that $2E_{\mathrm{gss}} \leq E_{\mathrm{sH}}$, it suffices to verify that $\mathrm{ord}_x E_{\mathrm{sH}} \geq 2$.

To verify that $\mathrm{ord}_x E_{\mathrm{sH}} \geq 2$, let us fix respective local generators $e_{\mathcal{H}}$ $e_{\mathcal{C}}$, ∂ of $\mathcal{H}$, $\mathcal{C}$, $\mathcal{T}$ at x. Now let us observe that since the p-curvature homomorphism of $\nabla_{\mathcal{Q}}$ is *zero* [cf.

Lemma A.8, (i)], there exists a local generator $e_{\mathcal{Q}}$ of $\mathcal{Q}$ at x such that $\nabla_{\mathcal{Q}}(e_{\mathcal{Q}}) = 0$. Write, moreover, $e_{\mathcal{E}} \stackrel{\text{def}}{=} \nabla_{\mathcal{E}}(\partial)(e_{\mathcal{H}})$ for the local section of $\mathcal{E}$ at x obtained by forming the image of $e_{\mathcal{H}}$ via $\nabla_{\mathcal{E}}(\partial)$. Thus, it follows from the definition of an indigenous bundle that the pair $(e_{\mathcal{H}}, e_{\mathcal{E}})$ forms a *basis* of $\mathcal{E}$ at x.

LEMMA A.14. — *Write*

- $f_{\mathcal{Q}}$ *for the local section of* $\mathcal{O}_X$ *at* x *such that the image of* $e_{\mathcal{H}}$ *via the natural surjection* $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ *coincides with* $f_{\mathcal{Q}} \cdot e_{\mathcal{Q}}$,
- $f_{\mathcal{C}}$ *for the local section of* $\mathcal{O}_X$ *at* x *such that the image* $\mathcal{P}(\partial)(e_{\mathcal{H}})$ *of* $e_{\mathcal{H}}$ *via* $\mathcal{P}(\partial)$ *coincides with* $f_{\mathcal{C}} \cdot e_{\mathcal{C}}$ [*cf. the discussion preceding Lemma* A.11], *and*
- $f_{\mathcal{H}}$, $f_{\mathcal{E}}$ *for the local sections of* $\mathcal{O}_X$ *at* x *such that*

$$\mathcal{P}(\partial)(e_{\mathcal{H}}) \ = \ f_{\mathcal{C}} \cdot e_{\mathcal{C}} \ = \ f_{\mathcal{H}} \cdot e_{\mathcal{H}} + f_{\mathcal{E}} \cdot e_{\mathcal{E}}.$$

[*Thus, it is immediate that the equalities*

$$\mathrm{ord}_x E_{\mathrm{gss}} \ = \ \mathrm{ord}_x f_{\mathcal{Q}}, \quad \mathrm{ord}_x E_{\mathrm{sH}} \ = \ \mathrm{ord}_x f_{\mathcal{E}}$$

hold.] *Then the following hold.*

(i) *It holds that* $\mathrm{ord}_x f_{\mathcal{Q}} = 1$.

(ii) *It holds that* $\mathrm{ord}_x f_{\mathcal{C}} \geq \mathrm{ord}_x f_{\mathcal{Q}}$.

(iii) *It holds that* $\mathrm{ord}_x f_{\mathcal{E}} \geq 1$.

(iv) *It holds that* $\mathrm{ord}_x f_{\mathcal{H}} \geq 1$.

(v) *It holds that* $\mathrm{ord}_x f_{\mathcal{E}} \geq 2$.

PROOF. — Assertion (i) follows from Lemma A.12, (ii). Assertion (ii) follows from the discussion preceding Lemma A.11. Assertion (iii) follows from Lemma A.13, (ii). Assertion (iv) follows from assertions (i), (ii), (iii), together with the equality $f_{\mathcal{C}} \cdot e_{\mathcal{C}} = f_{\mathcal{H}} \cdot e_{\mathcal{H}} + f_{\mathcal{E}} \cdot e_{\mathcal{E}}$ in the statement of Lemma A.14.

Finally, we verify assertion (v). Let us observe that since [we have assumed that] $\nabla_{\mathcal{Q}}(e_{\mathcal{Q}}) = 0$, the image of $e_{\mathcal{H}}$ in $\mathcal{Q}$ is given by $f_{\mathcal{Q}} \cdot e_{\mathcal{Q}}$, and the natural surjection $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ is *horizontal*, it holds that the image of $e_{\mathcal{E}} = \nabla_{\mathcal{E}}(\partial)(e_{\mathcal{H}})$ in $\mathcal{Q}$ *coincides* with $\partial f_{\mathcal{Q}} \cdot e_{\mathcal{Q}}$. Thus, by considering the image of "$f_{\mathcal{C}} \cdot e_{\mathcal{C}} = f_{\mathcal{H}} \cdot e_{\mathcal{H}} + f_{\mathcal{E}} \cdot e_{\mathcal{E}}$" in $\mathcal{Q}$, we obtain that

$$0 \ = \ f_{\mathcal{H}} \cdot f_{\mathcal{Q}} \cdot e_{\mathcal{Q}} + f_{\mathcal{E}} \cdot \partial f_{\mathcal{Q}} \cdot e_{\mathcal{Q}}.$$

Next, let us observe that since $\mathrm{ord}_x f_{\mathcal{Q}} = 1$ [cf. assertion (i)], it holds that $\mathrm{ord}_x \partial f_{\mathcal{Q}} = 0$. Thus, it follows from assertions (i), (iv), that

$$\mathrm{ord}_x f_{\mathcal{E}} \ = \ \mathrm{ord}_x (f_{\mathcal{H}} \cdot f_{\mathcal{Q}}) \ \geq \ 2,$$

as desired. This completes the proof of assertion (v), hence also of Lemma A.14. □

It follows from Lemma A.14, (v), that $\mathrm{ord}_x E_{\mathrm{sH}} \geq 2$, as desired. This completes the proof of assertion (iii) of Proposition A.3. Assertion (iv) of Proposition A.3 follows immediately from assertion (iii) of Proposition A.3 [cf. also [7], Chapter II, Proposition 2.6, (3)]. This completes the proof of Proposition A.3.

REFERENCES

[1] I. I. Bouw and S. Wewers, Indigenous bundles with nilpotent p-curvature, *Int. Math. Res. Not.* 2006, Art. ID 89254, 37.

[2] L. R. A. Finotti, Minimal degree liftings of hyperelliptic curves, *J. Math. Sci. Univ. Tokyo* **11** (2004), 1–47.

[3] R. C. Gunning, Special coordinate coverings of Riemann surfaces, *Math. Ann.* **170** (1967), 67–86.

[4] Y. Hoshi, Galois-theoretic characterization of isomorphism classes of monodromically full hyperbolic curves of genus zero, *Nagoya Math. J.* **203** (2011), 47–100.

[5] Y. Hoshi, Nilpotent admissible indigenous bundles via Cartier operators in characteristic three, *Kodai Math. J.* **38** (2015), no. **3**, 690–731.

[6] K. Kato, Logarithmic structures of Fontaine-Illusie, *Algebraic analysis, geometry, and number theory* (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[7] S. Mochizuki, A theory of ordinary p-adic curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), no. **6**, 957–1152.

[8] S. Mochizuki, *Foundations of p-adic Teichmüller theory*, AMS/IP Studies in Advanced Mathematics, **11**. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 1999.

(Yuichiro Hoshi) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: yuichiro@kurims.kyoto-u.ac.jp