FROBENIUS-PROJECTIVE STRUCTURES ON CURVES IN POSITIVE CHARACTERISTIC

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Abstract. — In the present paper, we study Frobenius-projective structures on projective smooth curves in positive characteristic. The notion of Frobenius-projective structures may be regarded as an analogue, in positive characteristic, of the notion of complex projective structures in the classical theory of Riemann surfaces. By means of the notion of Frobenius-projective structures, we obtain a relationship between a certain rational function, i.e., a pseudo-coordinate, and a certain collection of data which may be regarded as an analogue, in positive characteristic, of the notion of indigenous bundles in the classical theory of Riemann surfaces, i.e., a Frobenius-indigenous structure. As an application of this relationship, we also prove the existence of certain Frobenius-destabilized locally free coherent sheaves of rank two.

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Introduction

In the present paper, we study Frobenius-projective structures on projective smooth curves in positive characteristic. Let \( p \) be a prime number, \( k \) an algebraically closed field of characteristic \( p \), \( g \) a nonnegative integer, and

\[ X \]

a projective smooth curve over \( k \) of genus \( g \). Throughout the present paper, let us fix a positive integer \( N \).

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Write $X^F$ for the base-change of $X$ via the [not $p$-th if $N \neq 1$ but] $p^N$-th power Frobenius endomorphism of $k$ and $\Phi: X \to X^F$ for the relative $p^N$-th power Frobenius morphism over $k$. Thus, by pulling back the “$\text{PGL}_2$” on $X^F$ via $\Phi$, we obtain a sheaf $G$ of groups on $X$. Then a Frobenius-projective structure of level $N$ on $X$ is defined to be a subsheaf of the sheaf on $X$ of étale morphisms to the projective line $\mathbb{P}_k^1$ over $k$ which forms a $G$-torsor with respect to the natural action of $G$ on the sheaf on $X$ of morphisms over $k$ to $\mathbb{P}_k^1$ [cf. Definition 2.1]. One may find easily that the notion of Frobenius-projective structures may be regarded as an analogue, in positive characteristic, of the notion of complex projective structures [cf., e.g., [2], §2] in the classical theory of Riemann surfaces.

The main result of the present paper is as follows [cf. Theorem 3.13]. The main motivation of the study of the present paper is in fact to understand, in more conceptual terms, the notion of pseudo-tame rational function [cf. [9], Corollary 3.6]. The main motivation of the study of the present paper is in fact to understand, in more conceptual terms, the notion of pseudo-tame rational function in the sense of [9] if and only if $p = 2$, then, for a morphism to $\mathbb{P}_k^1$ over $k$, it holds that the morphism is a pseudo-coordinate in characteristic two of Belyi’s theorem. In the proof of their main theorem, i.e., Belyi’s theorem in characteristic two, they proved that [if $p = 2$, then] $X$ always admits a pseudo-tame rational function [cf. [9], Corollary 3.6]. The main motivation of the study of the present paper is in fact to understand, in more conceptual terms, the notion of pseudo-tame defined in [9].

A Frobenius-indigenous structure of level $N$ on $X$ is defined to be a pair consisting of a $\mathbb{P}_k^1$-bundle $P \to X^F$ over $X^F$ and a section $\sigma$ of the pull-back $\Phi^*P \to X$ such that the Kodaira-Spencer section of the connection $\nabla_{\Phi^*P}$ on $\Phi^*P$ [cf. Definition 3.2, (i), (ii)] at $\sigma$ [i.e., the global section of the invertible sheaf $\omega_{X/k} \otimes_{\mathcal{O}_X} \sigma^*\tau_{\Phi^*P/X}$ on $X$ obtained by differentiating $\sigma$ via the connection $\nabla_{\Phi^*P}$ — cf. Definition 3.3] is nowhere vanishing [cf. Definition 3.4]. One may find easily that the notion of Frobenius-indigenous structures may be regarded as an analogue, in positive characteristic, of the notion of indigenous bundles [cf., e.g., [2], §2] in the classical theory of Riemann surfaces. If $p \neq 2$, and $g \geq 2$, then the notion of Frobenius-indigenous structures of level 1 is essentially the same as the notion of dormant indigenous bundles studied in $p$-adic Teichmüller theory; moreover, there is a certain direct relationship between Frobenius-indigenous structures of level $N$ and objects studied in $p$-adic Teichmüller theory even if $N \neq 1$ [cf. Remark 3.4.1].

The main result of the present paper is as follows [cf. Theorem 3.13].

**Theorem A.** — Suppose that $(p, N) \neq (2, 1)$. Then there exist bijections between the following three sets:

1. the set of $G_{\text{et}}$-orbits of pseudo-coordinates of level $N$ on $X$
2. the set of Frobenius-projective structures of level $N$ on $X$
(3) the set of isomorphism classes of Frobenius-indigenous structures of level \( N \) on \( X \)

Note that a result in the case where \((p, N) = (2, 1)\) similar to Theorem A is discussed in Corollary 5.7.

Theorem A has some applications. For instance, by applying Theorem A and a result in \( p \)-adic Teichmüller theory, one may verify that if \( g \geq 2 \), then \( X \) always admits a pseudo-coordinate of level 1 [cf. Corollary 1.9; Corollary 4.9, (i)]. Moreover, Theorem A yields a [fifth — cf. Remark 3.12.1] proof of the uniqueness of the isomorphism class of dormant indigenous bundles on a projective smooth curve of genus \( \geq 2 \) in characteristic three [cf. Remark 3.4.1, (ii); Corollary 3.12].

Another application of Theorem A is as follows. Write \( \text{Fr}_X : X \to X \) for the \( p \)-th power Frobenius endomorphism of \( X \). Then it is immediate [cf. Remark 4.2.3, Proposition 4.7] that if \( g \geq 2 \), then there exists a bijection between the set of (3) of Theorem A and the set of \( \mathbb{P} \)-equivalence [cf. Definition 4.1] classes of locally free coherent \( \mathcal{O}_X \)-modules \( E \) of rank two which satisfy the following condition: If, for a nonnegative integer \( i \), we write

\[
E_i \overset{\text{def}}{=} \underbrace{\text{Fr}_X^* \circ \cdots \circ \text{Fr}_X^*}_i \mathcal{E},
\]

then

- the locally free coherent \( \mathcal{O}_X \)-module \( E_{N-1} \), hence also \( E \), is stable, but
- there exist an invertible sheaf \( \mathcal{L} \) on \( X \) of degree \( \frac{p^N}{2} \cdot \deg(\mathcal{E}) + g - 1 = \frac{1}{2} \cdot \deg(\mathcal{E}_N) + g - 1 \) and a locally split injection \( \mathcal{L} \hookrightarrow E_N \) of \( \mathcal{O}_X \)-modules. [In particular, the locally free coherent \( \mathcal{O}_X \)-module \( E_N \) is not semistable.]

Thus, by applying Theorem A and the above existence of pseudo-tame rational functions proved in [9], we obtain the following application [cf. Remark 6.2.1].

**Theorem B.** — Suppose that \( p = 2 \), and that \( g \geq 2 \). Then there exists a locally free coherent \( \mathcal{O}_X \)-module \( E \) of rank two such that

- the locally free coherent \( \mathcal{O}_X \)-module \( \text{Fr}_X^* \mathcal{E} \), hence also \( E \), is stable, but
- the locally free coherent \( \mathcal{O}_X \)-module \( \text{Fr}_X^* \text{Fr}_X^* \mathcal{E} \) admits an invertible subsheaf \( \mathcal{L} \subseteq \text{Fr}_X^* \text{Fr}_X^* \mathcal{E} \) of degree \( \frac{1}{2} \deg(\text{Fr}_X^* \text{Fr}_X^* \mathcal{E}) + g - 1 \), which thus implies that \( \text{Fr}_X^* \text{Fr}_X^* \mathcal{E} \) is not semistable.

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1. Pseudo-coordinates

In the present §1, we introduce and discuss the notion of pseudo-coordinates on curves [cf. Definition 1.3 below], which may be regarded as a generalization of the notion of pseudo-tame rational functions studied in [9] [cf. Remark 1.3.2 below].

In the present §1, let \( p \) be a prime number, \( k \) an algebraically closed field of characteristic \( p \), \( g \) a nonnegative integer, and

\[
X
\]
a projective smooth curve over \( k \) of genus \( g \). We shall write \( K_X \) for the function field of \( X \). Throughout the present paper, let us fix a positive integer \( N \).

If \((-)\) is an object over \( k \), then we shall write \((-)^F\) for the object over \( k \) obtained by base-changing \((-)\) via the \([\text{not } p\text{-th if } N \neq 1 \text{ but }] p^N\)-th power Frobenius endomorphism of \( k \). We shall write

\[
W : X^F \longrightarrow X
\]

for the morphism obtained by base-changing the \( p^N\)-th power Frobenius endomorphism of \( \text{Spec}(k) \) via the structure morphism \( X \to \text{Spec}(k) \). Thus, the \( p^N\)-th power Frobenius endomorphism of \( X \) factors as the composite

\[
X \longrightarrow X^F \overset{W}{\longrightarrow} X.
\]

We shall write

\[
\Phi : X \longrightarrow X^F
\]

for the first arrow in this composite, i.e., the relative \( p^N\)-th power Frobenius morphism over \( k \). Note that \( X^F \) is a projective smooth curve over \( k \) of genus \( g \), and \( \Phi \) is a finite flat morphism over \( k \) of degree \( p^N \).

**Definition 1.1.**

(i) Let \( S \) be a scheme. Then we shall write

\[
P^1_S \longrightarrow S
\]

for the trivial \( \mathbb{P}^1 \)-bundle over \( S \). Thus, \( \mathbb{P}^1_k \overset{\text{def}}{=} \mathbb{P}^1_{\text{Spec}(k)} \) denotes the projective line over \( k \).

(ii) We shall write

\[
P
\]

for the sheaf of sets on \( X \) that assigns, to an open subscheme \( U \subseteq X \), the set of morphisms from \( U \) to \( \mathbb{P}^1_k \) over \( k \),

\[
P^{\text{\acute{e}t}} \subseteq P
\]

for the subsheaf of \( P \) that assigns, to an open subscheme \( U \subseteq X \), the set of generically \( \acute{e}tale \) morphisms from \( U \) to \( \mathbb{P}^1_k \) over \( k \), and

\[
P^{\acute{e}t} \subseteq P^{\text{\acute{e}t}}
\]

for the subsheaf of \( P^{\text{\acute{e}t}} \) that assigns, to an open subscheme \( U \subseteq X \), the set of \( \acute{e}tale \) morphisms from \( U \) to \( \mathbb{P}^1_k \) over \( k \).
**Remark 1.1.1.** — One verifies easily that both $\mathcal{P}$ and $\mathcal{P}^\text{ét}$ are isomorphic to constant sheaves.

**Remark 1.1.2.** — One verifies easily that $\mathcal{P}$ may be naturally identified with the sheaf of sets on $X$ that assigns, to an open subscheme $U \subseteq X$, the set of sections of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}_U^1 \to U$.

**Definition 1.2.**

(i) Let $S$ be a scheme. Then we shall write $\text{PGL}_{2,S}$ for the sheaf of groups on $S$ that assigns, to an open subscheme $T \subseteq S$, the group $\text{Aut}_T(\mathbb{P}_T^1)$ of automorphisms over $T$ of the trivial $\mathbb{P}_T^1$-bundle $\mathbb{P}_T^1 \to T$.

(ii) We shall write $\mathcal{G} \overset{\text{def}}{=} \Phi^{-1} \text{PGL}_{2,X^F}$ and $\mathcal{G}_{\text{rstn}}$ for the group obtained by forming the stalk of $\mathcal{G}$ at the generic point of $X$ [i.e., the “$\text{PGL}_{2}$” for the function field of $X^F$].

**Remark 1.2.1.**

(i) It follows immediately from Remark 1.1.2 that $\mathcal{G}$ naturally acts, via $\Phi$, on $\mathcal{P}$. Moreover, one verifies easily that the subsheaves $\mathcal{P}^\text{ét} \subseteq \mathcal{P}^\text{ét} \subseteq \mathcal{P}$ of $\mathcal{P}$ are preserved by this action of $\mathcal{G}$ on $\mathcal{P}$.

(ii) It follows from Remark 1.1.1 that the actions of $\mathcal{G}$ on $\mathcal{P}$, $\mathcal{P}^\text{ét}$ of (i) determine actions of $\mathcal{G}_{\text{rstn}}$ on $\mathcal{P}(X)$, $\mathcal{P}^\text{ét}(X)$, respectively.

**Definition 1.3.** — We shall say that a global section $f \in \mathcal{P}^\text{ét}(X)$ of $\mathcal{P}^\text{ét}$ is a pseudo-coordinate of level $N$ if, for each closed point $x \in X$ of $X$, there exist an open subscheme $U \subseteq X$ of $X$ and an element $g \in \mathcal{G}_{\text{rstn}}$ of $\mathcal{G}_{\text{rstn}}$ such that $x \in U$, and, moreover, the restriction $g(f)|_U \in \mathcal{P}^\text{ét}(U)$ to $U$ of the result $g(f) \in \mathcal{P}^\text{ét}(X)$ of the action of $g \in \mathcal{G}_{\text{rstn}}$ on $f \in \mathcal{P}^\text{ét}(X)$ [cf. Remark 1.2.1, (ii)] is contained in the subset $\mathcal{P}^\text{ét}(U) \subseteq \mathcal{P}^\text{ét}(U)$ of $\mathcal{P}^\text{ét}(U)$.

We shall write $\text{pcd}_N(X) \subseteq \mathcal{P}^\text{ét}(X)$ for the subset of pseudo-coordinates of level $N$.

**Remark 1.3.1.** — One verifies easily that if a global section of $\mathcal{P}^\text{ét}$ is a pseudo-coordinate of level $N$, then every element of the $\mathcal{G}_{\text{rstn}}$-orbit ($\subseteq \mathcal{P}^\text{ét}(X)$) of the global section is a pseudo-coordinate of level $N$. 
Remark 1.3.2. — Suppose that \((p, N) = (2, 2)\). Then one verifies easily that, for a global section \(f \in \mathcal{P}^\text{gét}(X)\) of \(\mathcal{P}^\text{gét}\), it holds that \(f \in \mathcal{P}^\text{gét}(X)\) is a pseudo-coordinate of level \(N\) in the sense of Definition 1.3 if and only if \(f \in \mathcal{P}^\text{gét}(X)\) [i.e., the generically étale morphism \(f : X \to \mathbb{P}^1_k\) over \(k\)] is pseudo-tame in the sense of [9], Definition 2.1 [cf. also [9], Remark 2.5].

Definition 1.4. — Let \(f \in \mathcal{P}^\text{gét}(X)\) be a global section of \(\mathcal{P}^\text{gét}\) and \(x \in X\) a closed point of \(X\). Let us identify \(A \overset{\text{def}}{=} k[[t]]\) with the completion \(\mathring{O}_{X,x}\) of \(\mathcal{O}_{X,x}\) by means of a fixed isomorphism \(A \sim \mathring{O}_{X,x}\) over \(k\). Write \(F \in \mathcal{O}_{X,x}\) for the image via \(f\) in \(\mathcal{O}_{X,x}\) of a fixed uniformizer of the discrete valuation ring \(\mathcal{O}_{\mathbb{P}^1_k,f(x)}\) and

\[
F = \sum_{i \geq 1} a_i t^i \in A
\]

for the expansion of \(F\) in \(A\). Thus, the positive integer

\[
\text{ind}_x(f) \overset{\text{def}}{=} \nu_A(F) = \min \{ i \in \mathbb{Z}_{\geq 1} \mid a_i \neq 0 \}
\]

— where \(\nu_A\) denotes the \(t\)-adic valuation on \(A = k[[t]]\) — coincides with the ramification index of the dominant morphism \(f : X \to \mathbb{P}^1_k\) at \(x \in X\). Then we shall write

\[
\text{ind}_x^{\mathbb{P}^N}(f) \overset{\text{def}}{=} \min \{ i \in \mathbb{Z}_{\geq 1} \mid a_i \neq 0 \text{ and } i \not\in p^N \mathbb{Z} \}
\]

and

\[
\text{ind}_x^{\mathbb{P}^N}(f)
\]

for the uniquely determined positive integer such that \(1 \leq \text{ind}_x^{\mathbb{P}^N}(f) \leq p^N - 1\), and, moreover, \(\text{ind}_x^{\mathbb{P}^N}(f) - \text{ind}_x^{\mathbb{P}^N}(f) \in p^N \mathbb{Z}\).

Note that one verifies easily that since \(f\) is a global section of \(\mathcal{P}^\text{gét}\), it holds that \(\text{ind}_x^{\mathbb{P}^N}(f) < \infty\). Moreover, one also verifies easily that both \(\text{ind}_x^{\mathbb{P}^N}(f)\) and \(\text{ind}_x^{\mathbb{P}^N}(f)\) do not depend on the choices of the fixed isomorphism \(A \sim \mathring{O}_{X,x}\) and the fixed uniformizer of \(\mathcal{O}_{\mathbb{P}^1_k,f(x)}\).

Lemma 1.5. — Let \(f \in \mathcal{P}^\text{gét}(X)\) be a global section of \(\mathcal{P}^\text{gét}\) and \(x \in X\) a closed point of \(X\). Then the following hold:

(i) There exists an element \(g \in \mathcal{G}_{\text{rtn}}\) of \(\mathcal{G}_{\text{rtn}}\) such that

\[
\text{ind}_x(g(f)) = \text{ind}_x^{\mathbb{P}^N}(f)
\]

[which thus implies that \(\text{ind}_x(g(f)) = \text{ind}_x^{\mathbb{P}^N}(g(f))\)].

(ii) Suppose that \(\text{ind}_x(f) = \text{ind}_x^{\mathbb{P}^N}(f)\). Then there exist elements \(g_+, g_- \in \mathcal{G}_{\text{rtn}}\) of \(\mathcal{G}_{\text{rtn}}\) such that

\[
\text{ind}_x(g_+(f)) = \text{ind}_x^{\mathbb{P}^N}(f), \quad \text{ind}_x(g_-(f)) = p^N - \text{ind}_x^{\mathbb{P}^N}(f)
\]

[which thus implies that \(\text{ind}_x(g_+(f)) = \text{ind}_x^{\mathbb{P}^N}(g_+(f)), \text{ind}_x(g_-(f)) = p^N - \text{ind}_x^{\mathbb{P}^N}(g_-(f))\)].
PROOF. — Let us identify the scheme \( \text{Proj}(k[u,v]) \) with \( \mathbb{P}^1_k \) by means of a fixed isomorphism \( \text{Proj}(k[u,v]) \xrightarrow{\sim} \mathbb{P}^1_k \) over \( k \). Thus, the global section \( f \in \mathcal{P}^{\text{\acute{e}t}}(X) \) determines and is determined by an element \( F \in K_X \setminus K'_X \) of \( K_X \setminus K'_X \) [i.e., the image of \( u/v \in k(u/v) \) in \( K_X \) via \( f \)]. To verify Lemma 1.5, we may assume without loss of generality, by replacing \( f \) by the composite of \( f \) and a suitable element of \( \text{Aut}_k(\mathbb{P}^1_k) \), that the image of \( x \in X \) via \( f \) is the point \( "(u,v) = (0, 1)" \), i.e., that \( F \in \mathcal{O}_{X,x} \setminus \mathcal{O}_{X,x}' \).

We verify assertion (i). Write \( r \) for the uniquely determined nonnegative integer such that \( rp^N < \text{ind}_{\mathbb{P}^N}(f) < (r + 1)p^N \). Then it is immediate that there exists an element \( a \in \mathcal{O}_{X,x} \) of \( \mathcal{O}_{X,x} \) such that \( F - a^p^N \in m_{X,x}^{rp^N+1} \). Thus, if we take \( g \in \mathcal{G}_{\text{rttn}} \) to be an element which maps \( F \) to \( F - a^p^N \), e.g.,

\[
g = \begin{pmatrix} 1 & 0 \\ -a^p^N & 1 \end{pmatrix} \in \mathcal{G}_{\text{rttn}},
\]

then \( g \) satisfies the condition of assertion (i). This completes the proof of assertion (i).

Next, we verify assertion (ii). Let \( \pi \in \mathcal{O}_{X,x} \) be a uniformizer of \( \mathcal{O}_{X,x} \). Write \( r_+ \) for the uniquely determined nonnegative integer such that \( r_+p^N = \text{ind}_{\mathbb{P}^N}(f) - \text{ind}_{\mathbb{P}^N}(f) \) and \( r_- \overset{\text{def}}{=} r_+ + 1 \). Then if we take \( g_+, g_- \in \mathcal{G}_{\text{rttn}} \) to be elements which map \( F \) to \( F/\pi^{r_+p^N}, \pi^{-p^N}/F \), e.g.,

\[
g_+ = \begin{pmatrix} 1 & 0 \\ 0 & \pi^{r_+p^N} \end{pmatrix} \in \mathcal{G}_{\text{rttn}}, \quad g_- = \begin{pmatrix} 0 & 1 \\ \pi^{-p^N} & 0 \end{pmatrix} \in \mathcal{G}_{\text{rttn}},
\]

then \( g_+, g_- \) satisfy the conditions of assertion (ii), respectively. This completes the proof of assertion (ii), hence also of Lemma 1.5. \( \square \)

**Lemma 1.6.** — Let \( f \in \mathcal{P}^{\text{\acute{e}t}}(X) \) be a global section of \( \mathcal{P}^{\text{\acute{e}t}} \) and \( x \in X \) a closed point of \( X \). Suppose that \( \text{ind}_{\mathbb{P}^N}(f) \notin \{1, p^N - 1\} \). Then, for each \( g \in \mathcal{G}_{\text{rttn}} \), the result \( g(f) \in \mathcal{P}^{\text{\acute{e}t}}(X) \) of the action of \( g \in \mathcal{G}_{\text{rttn}} \) on \( f \in \mathcal{P}^{\text{\acute{e}t}}(X) \) is not \( \text{\acute{e}tale} \) at \( x \).

PROOF. — Let us first observe that it follows immediately from Lemma 1.5, (i), (ii), that we may assume without loss of generality, by replacing \( f \) by the result of the action of a suitable element of \( \mathcal{G}_{\text{rttn}} \) on \( f \), that

\[(a) \quad \text{ind}_x(f) = d_0 \overset{\text{def}}{=} \text{ind}_{\mathbb{P}^N}(f) \notin \{1, p^N - 1\} \).
\]

Let us identify \( A \overset{\text{def}}{=} k[[t]] \) with the completion \( \hat{\mathcal{O}}_{X,x} \) of \( \mathcal{O}_{X,x} \) by means of a fixed isomorphism \( A \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x} \) over \( k \). Then it is immediate that, to verify Lemma 1.6, it suffices to verify that

\[(*)_1: \text{for each } g \in \mathcal{G}_{\text{rttn}}, \text{the composite of the natural morphism } \text{Spec}(A) \to X \text{ with } g(f): X \to \mathbb{P}^1_k \text{ is not formally \( \text{\acute{e}tale} \).}
\]

Let \( g \in \mathcal{G}_{\text{rttn}} \) be an element of \( \mathcal{G}_{\text{rttn}} \). Next, let us identify the scheme \( \text{Proj}(k[u,v]) \) with \( \mathbb{P}^1_k \) by means of a fixed isomorphism \( \text{Proj}(k[u,v]) \xrightarrow{\sim} \mathbb{P}^1_k \) over \( k \). Write \( K \) for the field of fractions of \( A \) and

\[
\text{Proj}(k[u,v]) \xleftarrow{\text{Spec}(A)} \quad (u,v) \mapsto (f_u, f_v)
\]

— where \( f_u, f_v \in A \) — for the composite of the natural morphism \( \text{Spec}(A) \to X \) with \( f: X \to \mathbb{P}^1_k \). Thus, there exist \( a_g, b_g, c_g, d_g \in k[[t^{p^N}]] = A^{p^N} \subseteq A \) [which thus implies
that \( \nu_A(a_g), \nu_A(b_g), \nu_A(c_g), \nu_A(d_g) \in p^N \mathbb{Z} \) such that \( a_g d_g - b_g c_g \neq 0 \), and, moreover, the composite of the natural morphism \( \text{Spec}(A) \to X \) with \( g(f) : X \to \mathbb{P}_k^1 \) coincides with the morphism determined by the composite

\[
\begin{align*}
\text{Proj}(K[u,v]) & \to \text{Proj}(K[u,v]) \\
(u,v) & \mapsto (u,v) \\
(\nu) & \mapsto (a_g u + b_g v, c_g u + d_g v) \\
(u,v) & \mapsto (f_u, f_v).
\end{align*}
\]

Next, let us observe that, to verify \((*)_1\), we may assume without loss of generality, by replacing \( f \) by the composite of \( f \) and a suitable element of \( \text{Aut}_k(\mathbb{P}_k^1) \), that the image of \( x \in X \) via \( f \) is the point \( \{(u,v) \mid (u,v) = (0,1)\} \), i.e., that \([\text{cf. (a)}]\]

(b) \( \nu_A(f_u) = d_0 \) and \( f_v = 1 \). [Recall that \( 2 \leq d_0 \leq p^N - 2 \) — cf. (a).]

Next, let us observe that, to verify \((*)_1\), we may assume without loss of generality, by replacing \( g \) by the product of \( g \) and a suitable element of \( \text{Aut}_k(\mathbb{P}_k^1) \), that the image of \( x \in X \) via \( g(f) \) is the point \( \{(u,v) \mid (u,v) = (0,1)\} \), i.e., that \([\text{cf. (b)}]\]

(c) if write

\[
F \overset{\text{def}}{=} \frac{a_g f_u + b_g}{c_g f_u + d_g} \in K,
\]

then \( F \in A \), and, moreover, \( \nu_A(F) \geq 1 \).

Thus, it is immediate that, to verify \((*)_1\), it suffices to verify that

\[(*)_2: \nu_A(F) \neq 1.\]

Next, let us observe that, to verify \((*)_2\), we may assume without loss of generality, by replacing \( (a_g, b_g, c_g, d_g) \) by \( t^{-\min(\nu_A(a_g), \nu_A(b_g), \nu_A(c_g), \nu_A(d_g))} \cdot (a_g, b_g, c_g, d_g) \), that

(d) \( 0 \in \{\nu_A(a_g), \nu_A(b_g), \nu_A(c_g), \nu_A(d_g)\} \).

Here, let us verify that

(e) \( \nu_A(b_g) \geq p^N \).

Indeed, if \( \nu_A(b_g) = 0 \), then it follows from (b) that \( \nu_A(a_g f_u + b_g) = 0 \), which thus implies that \( \nu_A(F) \leq 0 \) — in contradiction to (c). This completes the proof of (e).

Next, suppose that \( \nu_A(d_g) = 0 \). Then it follows from (b) that \( \nu_A(c_g f_u + d_g) = 0 \). In particular, it follows from (b) and (e) that \( \nu_A(F) = \nu_A(a_g f_u + b_g) \geq 2 \), as desired. Thus, to verify \((*)_2\), we may assume without loss of generality that

(f) \( \nu_A(d_g) \geq p^N \).

Next, let us verify that

(g) \( \nu_A(a_g) \geq p^N \).

Indeed, if \( \nu_A(a_g) = 0 \), then it follows from (b) and (e) that \( \nu_A(a_g f_u + b_g) = d_0 \). In particular, it follows from (c) that \( 1 \leq \nu_A(F) = d_0 - \nu_A(c_g f_u + d_g) \), which thus implies that \( \nu_A(c_g f_u + d_g) \leq d_0 - 1 \). Assume that \( \nu_A(c_g) = 0 \); then it follows from (b) and (f) that \( \nu_A(c_g f_u + d_g) = d_0 \) — in contradiction to the above inequality \( \nu_A(c_g f_u + d_g) \leq d_0 - 1 \). Assume that \( \nu_A(c_g) \geq p^N \); then it follows from (f) that \( \nu_A(c_g f_u + d_g) \geq p^N \) — in contradiction to the above inequality \( \nu_A(c_g f_u + d_g) \leq d_0 - 1 \). This completes the proof of (g).
It follows from (d), (e), (f), (g) that \( \nu_A(c_p) = 0 \). Thus, it follows from (b) and (f) that \( \nu_A(c_g f_u + d_g) = d_0 \leq p^N - 2 \). In particular, since \( \nu_A(a_g f_u + b_g) \geq p^N \) [cf. (e), (g)], it holds that \( \nu_A(F) \geq 2 \), as desired. This completes the proof of Lemma 1.6.

\[ \]

**Proposition 1.7.** — Let \( f \in \mathcal{P}^{\text{gel}}(X) \) be a global section of \( \mathcal{P}^{\text{gel}} \). Then it holds that \( f \) is a pseudo-coordinate of level \( N \) if and only if, for each closed point \( x \in X \) of \( X \), it holds that

\[
\text{ind}_{\mathcal{P}^{\text{gel}}}(f) \in \{1, p^N - 1\}.
\]

**Proof.** — The sufficiency follows immediately from Lemma 1.5, (i), (ii). The necessity follows immediately from Lemma 1.6.

\[ \]

**Proposition 1.8.** — Suppose that \( p \in \{2, 3\} \), and that \( N = 1 \). Then the following hold:

(i) It holds that \( \sharp(\mathcal{P}^{\text{gel}}(X)/\mathcal{G}_{\text{gel}}) = 1 \).

(ii) It holds that \( \mathcal{P}^{\text{gel}}(X) = \text{P}\mathcal{G}\mathcal{O}_N(X) \).

**Proof.** — First, we verify assertion (i). Let \( F \in K_X \) be an element of \( K_X \) such that \( \text{ord}_x(F) = 1 \) for some closed point \( x \in X \) of \( X \). [It is immediate that such an \( F \in K_X \) always exists.] Then one verifies easily that \( \{F^i\}_{i=0}^{p-1} \) forms a basis of the vector space \( K_X \) over \( K_X^p \). Thus, it is immediate that, to verify assertion (i), it suffices to verify the following assertion:

For each \( a_0, \ldots, a_{p-1} \in K_X^p \) such that \( (a_1, \ldots, a_{p-1}) \neq (0, \ldots, 0) \), there exist \( a, b, c, d \in K_X^p \) such that

\[
a_0 + a_1 F + \cdots + a_{p-1} F^{p-1} = \frac{aF + b}{cF + d}, \quad ad - bc \neq 0.
\]

On the other hand, since \( p \in \{2, 3\} \), this assertion may be easily verified. This completes the proof of assertion (i). Assertion (ii) follows from Proposition 1.7. This completes the proof of Proposition 1.8.

\[ \]

**Remark 1.8.1.** — Proposition 1.8 in the case where \( p \notin \{2, 3\} \) does not hold as follows:

(i) Let us discuss Proposition 1.8, (i), in the case where \( p \notin \{2, 3\} \). Suppose that we are in the situation of the first paragraph of the proof of Lemma 1.5. Suppose, moreover, that the element \( F \in K_X \) satisfies that \( \text{ord}_x(F) = 1 \). [It is immediate that such a pair \( (f, x) \) always exists.] Then it follows immediately from Lemma 1.6 that [if \( p \neq 3 \), then] \( F \in K_X \) is not contained in the \( \text{PGL}_2(K_X^p) \)-orbit of \( F^2 \in K_X^p \). One verifies easily from this observation that Proposition 1.8, (i), in the case where \( p \notin \{2, 3\} \) does not hold.

(ii) Let us discuss Proposition 1.8, (ii), in the case where \( p \notin \{2, 3\} \). Suppose that we are in the situation of the discussion of (i). Then it follows from Proposition 1.7 that [if \( p \notin \{2, 3\} \), then] \( F^2 \in K_X \setminus K_X^p \) determines a global section of \( \mathcal{P}^{\text{gel}} \) which is not a pseudo-coordinate of level \( N \). In particular, Proposition 1.8, (ii), in the case where \( p \notin \{2, 3\} \) does not hold.
Corollary 1.9. — Suppose that $p \in \{2, 3\}$, and that $N = 1$. Then $\sharp(p_{cdN}(X)/\mathcal{G}_{\text{rtu}}) = 1$.

Proof. — This assertion follows from Proposition 1.8, (i), (ii). □

2. Frobenius-projective Structures

In the present §2, we introduce and discuss the notion of Frobenius-projective structures on curves [cf. Definition 2.1 below]. Moreover, we also discuss a relationship between Frobenius-projective structures and pseudo-coordinates [cf. Proposition 2.7 below]. In the present §2, we maintain the notational conventions introduced at the beginning of §1.

Definition 2.1. — We shall say that a subsheaf $S \subseteq \mathcal{P}^\text{\acute{e}t}$ of $\mathcal{P}^\text{\acute{e}t}$ is a Frobenius-projective structure of level $N$ on $X$ if $S$ is preserved by the action of $\mathcal{G}$ on $\mathcal{P}^\text{\acute{e}t}$ [cf. Remark 1.2.1, (i)], and, moreover, the sheaf $S$ forms, by the resulting action of $\mathcal{G}$ on $S$, a $\mathcal{G}$-torsor on $X$.

We shall write

$$\mathfrak{fps}_N(X)$$

for the set of Frobenius-projective structures of level $N$ on $X$.

Remark 2.1.1.

(i) One may find easily that the notion of Frobenius-projective structures may be regarded as an analogue, in positive characteristic, of the notion of complex projective structures [cf., e.g., [2], §2] in the classical theory of Riemann surfaces.

(ii) One may also find another algebraic analogue of the notion of complex projective structures in [6], i.e., the notion of Schwarz structures defined in [6], Chapter I, Definition 1.2.

(iii) As discussed in Proposition 3.11 below, a Frobenius-projective structure is related to a Frobenius-indigenous structure [cf. Definition 3.4 below]. On the other hand, a suitable Schwarz structure is related to an indigenous bundle [cf. [6], Chapter I, Corollary 2.9].

Lemma 2.2. — Let $S \subseteq \mathcal{P}^\text{\acute{e}t}$ be a Frobenius-projective structure of level $N$ on $X$. Then the following hold:

(i) Let $U, V \subseteq X$ be open subschemes of $X$, $f_U \in S(U)$, and $f_V \in S(V)$. Then the global section of $\mathcal{P}^\text{\acute{e}t}$ determined by $f_U \in S(U)$ [cf. Remark 1.1.1] is contained in the $\mathcal{G}_{\text{rtu}}$-orbit of the global section of $\mathcal{P}^\text{\acute{e}t}$ determined by $f_V \in S(V)$.

(ii) The global section of $\mathcal{P}^\text{\acute{e}t}$ determined by a local section of $S$ is a pseudo-coordinate of level $N$.

Proof. — Since $X$ is irreducible, assertion (i) follows from the fact that $S$ is a $\mathcal{G}$-torsor. Assertion (ii) follows from assertion (i), together with the fact that $S$ is contained in $\mathcal{P}^\text{\acute{e}t}$. □
Then it is immediate that, to verify the necessity, it suffices to verify that:

$$f$$

where

$$\ast$$

Replacing ($$a$$)

$$f \in P$$

$$\in G$$

Moreover, let us observe that since

$$\hat{g}$$

with the completion

$$N$$

coordinates of level

$$N$$.

Then it follows from Lemma 2

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that

$$m$$

composite determined by the composite

composite of the natural morphism

$$\ast$$

$$a$$

$$\in P$$

is the point "($$a$$)

$$\nu$$

(b)

Next, let us observe that, to verify ($$\nu$$)

(a) 0

$$\nu_A(a_g), \nu_A(b_g), \nu_A(c_g), \nu_A(d_g) \in p^n \mathbb{Z}$$

such that

$$a_g d_g - b_g c_g \neq 0$$.

and, moreover, the composite of the natural morphism

$$A$$

$$\sim \hat{O}_{X,x}$$

over

$$k$$.

Thus, we obtain a map

$$\hat{\text{ps}}_{N}(X) \rightarrow \text{pcd}_{N}(X)/G_{\text{rtu}}.$$
Moreover, let us observe that, to verify (*1), we may assume without loss of generality, by replacing $g$ by the product of $g$ and a suitable element of $\text{Aut}_k(\mathbb{P}^1_k)$, that the image of $x \in X$ via $g(f)$ is the point “$(u, v) = (0, 1)$”, i.e., that [cf. (c)]

\[(d) \quad \text{if write} \quad F \equiv \frac{a_g f_u + b_g}{c_g f_u + d_g} \in K,\]

then $F \in A$, and, moreover, $\nu_A(F) \geq 1$.

Thus, it is immediate that, to verify (*1), it suffices to verify that

\[(*2) : \nu_A(F) \neq 1.\]

Here, let us verify that

\[(e) \quad \nu_A(b_g) \geq p^N.\]

Indeed, if $\nu_A(b_g) = 0$, then it follows from (c) that $\nu_A(a_g f_u + b_g) = 0$, which thus implies that $\nu_A(F) \leq 0$ — in contradiction to (d). This completes the proof of (e).

Next, suppose that $\nu_A(d_g) = 0$. Then it follows from (b) and (e) that $\nu_A(a_g) \geq p^N$. On the other hand, since $\nu_A(d_g) = 0$, it follows from (c) that $\nu_A(c_g f_u + d_g) = 0$. In particular, it follows from (e) that $\nu_A(F) = \nu_A(a_g f_u + b_g) \geq p^N \geq 2$, as desired. Thus, to verify (*2), we may assume without loss of generality that

\[(f) \quad \nu_A(d_g) \geq p^N.\]

Next, let us verify that

\[(g) \quad \nu_A(a_g) \geq p^N.\]

Indeed, if $\nu_A(a_g) = 0$, then it follows from (c) and (e) that $\nu_A(a_g f_u + b_g) = 1$. In particular, it follows from (c) and (f) that $\nu_A(F) = 1 - \nu_A(c_g f_u + d_g) \leq 1 - 1 = 0$ — in contradiction to (d). This completes the proof of (g).

It follows from (a), (e), (f), (g) that $\nu_A(c_g) = 0$. Thus, it follows from (c) and (f) that $\nu_A(c_g f_u + d_g) = 1$. In particular, it follows from (e) and (g) that $\nu_A(F) = \nu_A(a_g f_u + b_g) - 1 \geq p^N - 1$. Now since [we have assumed that] $(p, N) \neq (2, 1)$, it holds that $p^N - 1 \geq 2$, as desired. This completes the proof of Lemma 2.4. \hfill \Box

**Remark 2.4.1.** — The necessity of Lemma 2.4 in the case where $(p, N) = (2, 1)$ does not hold. Indeed, suppose that we are in the situation of the first paragraph of the proof of Lemma 1.5. Suppose, moreover, that the element $F \in K_X$ satisfies that $\text{ord}_x(F) = 1$, which thus implies that $f \in \mathcal{P}^\text{et}(U)$ for some open subscheme $U \subseteq X$ of $X$ such that $x \in U$. [It is immediate that such a pair “$(f, x)$” always exists.] Let us consider the element $g \in G_{\text{tin}}$ of $G_{\text{tin}}$ determined by the matrix

$$\begin{pmatrix} F^2 & 1 \\ F^2 & F^2 \end{pmatrix} \in \text{GL}_2(K_X).$$

Then since the determinant of this matrix is $F^4 - F^2 \not\in \mathcal{O}^\times_{X,x}$, it holds that $g \in G_{\text{tin}}$ is not contained in $G(V) \subseteq G_{\text{tin}}$ for every open subscheme $V \subseteq X$ of $X$ such that $x \in V$. On the other hand, since

$$\frac{F \cdot F^2 + F^2}{F \cdot 1 + F^2} = F,$$
the result $g(f)$ of the action of $g$ on $f$ is given by $f$, which thus implies that $g(f) \in \mathcal{P}^{\text{ét}}(U)$.

**Lemma 2.5.** — Suppose that $(p, N) \neq (2, 1)$. Let $f \in \mathcal{P}^{\text{ét}}(X)$ be a pseudo-coordinate of level $N$. Then the following hold:

(i) Write $\mathcal{S}_f \subseteq \mathcal{P}^{\text{ét}}$ for the subsheaf of $\mathcal{P}^{\text{ét}}$ that assigns, to an open subscheme $U \subseteq X$, the subset of $\mathcal{P}^{\text{ét}}(U)$ obtained by forming the intersection of $\mathcal{P}^{\text{ét}}(U)$ and the $\mathcal{G}_{\text{rtn}}$-orbit ($\subseteq \mathcal{P}^{\text{ét}}(U)$) of $f|_U$ [cf. Remark 1.1.1; Remark 1.2.1, (i)]:

$$\mathcal{S}_f(U) \overset{\text{def}}{=} \mathcal{P}^{\text{ét}}(U) \cap (\mathcal{G}_{\text{rtn}} \cdot f|_U).$$

Then the subsheaf $\mathcal{S}_f$ is a Frobenius-projective structure of level $N$ on $X$.

(ii) Let $g \in \mathcal{P}^{\text{ét}}(X)$ be a global section of $\mathcal{P}^{\text{ét}}$ which is contained in the $\mathcal{G}_{\text{rtn}}$-orbit of $f \in \mathcal{P}^{\text{ét}}(X)$. [So $g$ is a pseudo-coordinate of level $N$ — cf. Remark 1.3.1.] Then $\mathcal{S}_f = \mathcal{S}_g$ [cf. (i)].

**Proof.** — Assertion (i) follows immediately from Lemma 2.4, together with the definition of a pseudo-coordinate of level $N$. Assertion (ii) follows immediately from the definition of “$\mathcal{S}_f$”. □

**Definition 2.6.** — Suppose that $(p, N) \neq (2, 1)$. Let $f \in \mathcal{P}^{\text{ét}}(X)$ be a pseudo-coordinate of level $N$. Then it follows from Lemma 2.5, (i), that $f$ determines a Frobenius-projective structure of level $N$ on $X$. We shall refer to this Frobenius-projective structure of level $N$ as the Frobenius-projective structure of level $N$ associated to $f$. Thus, we obtain a map

$$\text{pcd}_N(X)/\mathcal{G}_{\text{rtn}} \longrightarrow \mathfrak{Fps}_N(X)$$

[cf. Lemma 2.5, (ii)].

**Proposition 2.7.** — Suppose that $(p, N) \neq (2, 1)$. Then the assignments of Definition 2.3 and Definition 2.6 determine a bijection

$$\mathfrak{Fps}_N(X) \xrightarrow{\sim} \text{pcd}_N(X)/\mathcal{G}_{\text{rtn}}.$$ 

**Proof.** — This assertion follows immediately from the constructions of Lemma 2.2 and Lemma 2.5. □

**Remark 2.7.1.** — If $(p, N) = (2, 1)$, then, as discussed in Corollary 5.7, (ii), below, the map

$$\mathfrak{Fps}_N(X) \longrightarrow \text{pcd}_N(X)/\mathcal{G}_{\text{rtn}}$$

of Definition 2.3 is surjective but not injective.

**Corollary 2.8.** — Suppose that $(p, N) = (3, 1)$. Then $\sharp \mathfrak{Fps}_N(X) = 1$.

**Proof.** — This assertion follows from Proposition 2.7, together with Corollary 1.9. □
3. Frobenius-indigenous Structures

In the present §3, we introduce and discuss the notion of Frobenius-indigenous structures on curves [cf. Definition 3.4 below], which may be regarded as a generalization of the notion of dormant indigenous bundles studied in [7] [cf. Remark 3.4.1, (ii), below]. Moreover, we also discuss a relationship between Frobenius-indigenous structures and Frobenius-projective structures [cf. Proposition 3.11 below]. In the present §3, we maintain the notational conventions introduced at the beginning of §1. Moreover, if $U \subseteq X$ is an open subscheme of $X$, and $i \in \{1, 2\}$, then write $\mathcal{J}_U \subseteq \mathcal{O}_{U \times_k U}$ for the ideal of $\mathcal{O}_{U \times_k U}$ which defines the diagonal morphism with respect to $U/k$, $U^{(1)} \subseteq U \times_k U$ for the closed subscheme of $U \times_k U$ defined by the ideal $\mathcal{J}_U^2 \subseteq \mathcal{O}_{U \times_k U}$, and $\text{pr}_i : U^{(1)} \rightarrow U$ for the $i$-th projection morphism. We use the notation $\omega_{U/k}$ (respectively, $\tau_{U/k}$) to denote the relative cotangent (respectively, tangent) sheaf. Thus, it holds that $\omega_{U/k} = \mathcal{J}_U/\mathcal{J}_U^2$ and $\tau_{U/k} = \text{Hom}_{\mathcal{O}_U}(\mathcal{J}_U/\mathcal{J}_U^2, \mathcal{O}_U)$.

**Lemma 3.1.** — Let $U \subseteq X$ be an open subscheme of $X$. Then the natural morphism $U \times_{U^F} U \longrightarrow U \times_k U$ is a closed immersion whose image contains the closed subscheme $U^{(1)} \subseteq U \times_k U$.

**Proof.** — This assertion follows from a straightforward computation. □

**Definition 3.2.** — Let $U \subseteq X$ be an open subscheme of $X$.

(i) Let $S$ be an object over $U$. Then we shall refer to an isomorphism $\text{pr}_2^* S \overset{\sim}{\longrightarrow} \text{pr}_1^* S$ over $U^{(1)}$ that restricts, on the diagonal closed subscheme $U \hookrightarrow U^{(1)}$, to the identity automorphism of $S$ as a connection on $S$ [relative to $U/k$] [cf. Remark 3.2.1 below].

(ii) Let $S$ be an object over $U^F$. Then it follows from Lemma 3.1 that we have a natural identification of $\text{pr}_1^* \Phi^* S$ with $\text{pr}_2^* \Phi^* S$ over $U^{(1)}$. We shall write $\nabla_{\Phi^* S}$ for the connection on $\Phi^* S$ relative to $U/k$ obtained by forming this identification.

**Remark 3.2.1.** — In the situation of Definition 3.2, (i), suppose that $S$ is a quasi-coherent $\mathcal{O}_U$-module. Then it is well-known [cf., e.g., [1], Proposition 2.9] that a connection on $S$ relative to $U/k$ may be naturally regarded as a $k$-linear homomorphism

$$\nabla : S \longrightarrow \omega_{U/k} \otimes_{\mathcal{O}_U} S$$

that satisfies the equality $\nabla(fs) = f\nabla(s) + df \otimes s$ — where $f$, $s$ are local sections of $\mathcal{O}_U$, $S$, respectively.

**Definition 3.3.** — Let $U \subseteq X$ be an open subscheme of $X$, $P \rightarrow U$ a $\mathbb{P}^1$-bundle over $U$, $\nabla$ a connection on $P$ relative to $U/k$, and $\sigma$ a section of $P \rightarrow U$. Then, by considering the difference between the two deformations $U^{(1)} \overset{\text{pr}_1^* \sigma}{\longrightarrow} \text{pr}_1^* P$, $U^{(1)} \overset{\text{pr}_2^* \sigma}{\longrightarrow} \text{pr}_2^* P \overset{\nabla}{\longrightarrow} \text{pr}_1^* P$
of the section \( \sigma \), we have a global section of the invertible sheaf on \( U \)
\[ \omega_{U/k} \otimes_{\mathcal{O}_U} \sigma^* \tau_{P/U}. \]

[Note that let us recall from elementary algebraic geometry that the deformations \( U^{(1)} \to \text{pr}_1^* P \) of the section \( \sigma: U \to P \) form a torsor under \( \omega_{U/k} \otimes_{\mathcal{O}_U} \sigma^* \tau_{P/U} \).] We shall refer to this global section as the *Kodaira-Spencer section* of \( \nabla \) at \( \sigma \).

**Definition 3.4.** We shall say that a pair \((P \to X^F, \sigma)\) consisting of a \( \mathbb{P}^1 \)-bundle \( P \to X^F \) over \( X^F \) and a section \( \sigma \) of the pull-back \( \Phi^* P \to X \) is a *Frobenius-indigenous structure of level \( N \) on \( X \) if the Kodaira-Spencer section of the connection \( \nabla_{\Phi^* P} \) at \( \sigma \) is nowhere vanishing.

For two Frobenius-indigenous structures \( \mathcal{I}_1 = (P_1 \to X^F, \sigma_1) \), \( \mathcal{I}_2 = (P_2 \to X^F, \sigma_2) \) of level \( N \) on \( X \), we shall say that \( \mathcal{I}_1 \) is isomorphic to \( \mathcal{I}_2 \) if there exists an isomorphism \( P_1 \sim \to P_2 \) over \( X^F \) compatible with \( \sigma_1 \) and \( \sigma_2 \).

We shall write \( \mathfrak{F}_{isN}(X) \) for the set of isomorphism classes of Frobenius-indigenous structures of level \( N \) on \( X \).

**Remark 3.4.1.** Suppose that \( p \neq 2 \), and that \( g \geq 2 \).

(i) Write \( \Pi_{N+1} \) for the *VF-pattern of pure tone \( N + 1 \) [cf. \( \text{[7]}, \text{Chapter IV, Definition 2.6} \). Then it is immediate that there is a certain direct relationship between Frobenius-indigenous structures of level \( N \) and objects parametrized by the stack “\( X \)" defined in \( \text{[7]}, \text{Chapter III, \S 1.3} \), in the case where we take the VF-pattern “\( \Pi \)" to be \( \Pi_{N+1} \). In the notation of \( \text{[7]}, \text{Chapter III, \S 1} \), by taking the VF-pattern "\( \Pi \)" to be \( \Pi_{N+1} \) and the triple "\( (p, g, r) \)" to be \( (p, g, 0) \), we have natural functors
\[ X \to \mathcal{W} \to \mathcal{R}^{\Pi_{N+1}(0), \Pi_{N+1}(-1)}_{g, 0} \to \mathcal{M}_{g, 0}. \]

Then one verifies easily that if the classifying morphism \( \text{Spec}(k) \to \mathcal{M}_{g, 0} \) of the projective smooth curve \( X \) over \( k \) factors through the stack \( X \)[relative to the above functors], then there exists a Frobenius-indigenous structure of level \( N \) on \( X \).

(ii) Suppose, moreover, that \( N = 1 \). Then one verifies easily that the notion of Frobenius-indigenous structures of level \( N \) in the sense of Definition 3.4 is essentially the same as the notion of dormant indigenous bundles in the sense of \( \text{[6]}, \text{Chapter I, Definition 2.2; [7], Chapter II, Definition 1.1} \).

**Lemma 3.5.** Let \((P \to X^F, \sigma)\) be a Frobenius-indigenous structure of level \( N \) on \( X \), \( \mathcal{E} \) a locally free coherent \( \mathcal{O}_{X^F} \)-module of rank two whose projectivization \( \mathbb{P}(\mathcal{E}) \) is isomorphic to \( P \) over \( X^F \), and \( \Phi^* \mathcal{E} \to Q \) a surjection of \( \mathcal{O}_{X^F} \)-modules onto an invertible sheaf \( Q \) on \( X \) which defines, relative to an isomorphism of \( P \) with \( \mathbb{P}(\mathcal{E}) \) over \( X^F \), the section \( \sigma \). Write \( \mathcal{L} \) for the kernel of the surjection \( \Phi^* \mathcal{E} \to Q \). Then it holds that
\[ 2 \cdot \deg(Q) = p^N \cdot \deg(\mathcal{E}) - 2g + 2, \text{ hence also that } 2 \cdot \deg(\mathcal{L}) = p^N \cdot \deg(\mathcal{E}) + 2g - 2. \]
Proof. — This assertion follows immediately from our assumption that the Kodaira-
Spencer section of the connection $\nabla_{\Phi^*P}$ at $\sigma$ is nowhere vanishing, i.e., that the homo-
morphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$L \hookrightarrow \Phi^*E \xrightarrow{\nabla_{\Phi^*E}} \omega_{X/k} \otimes_{\mathcal{O}_X} \Phi^*E \twoheadrightarrow \omega_{X/k} \otimes_{\mathcal{O}_X} Q$$

is an isomorphism.

Lemma 3.6. — Suppose that $g \geq 2$. Let $P \to X^F$ be a $\mathbb{P}^1$-bundle over $X^F$ and $\sigma_1, \sigma_2$ sections of $\Phi^*P \to X$. Then if both $(P \to X^F, \sigma_1)$ and $(P \to X^F, \sigma_2)$ are Frobenius-
indigenous structures of level $N$ on $X$, then $\sigma_1 = \sigma_2$.

Proof. — Let $E$ be a locally free coherent $\mathcal{O}_{X^F}$-module of rank two whose projectivization $\mathbb{P}(E)$ is isomorphic to $P$ over $X^F$. Let us fix an isomorphism of $P$ with $\mathbb{P}(E)$ over $X^F$. For each $i \in \{1, 2\}$, let $\Phi^*E \to Q_i$ be a surjection of $\mathcal{O}_X$-modules onto an invertible sheaf $Q_i$ on $X$ which defines, relative to the fixed isomorphism $P \sim \mathbb{P}(E)$, the section $\sigma_i$. Write $L_i$ for the kernel of the surjection $\Phi^*E \to Q_i$. Then since [we have assumed that] $g \geq 2$, it follows from Lemma 3.5 that $\deg(Q_i) < \deg(L_j)$ for each $i, j \in \{1, 2\}$, which thus implies that $L_1 = L_2$, as desired. This completes the proof of Lemma 3.6.

Lemma 3.7. — Let $S \subseteq \mathcal{P}^{\text{et}}$ be a Frobenius-projective structure of level $N$ on $X$. Thus, the sheaf $\Phi_*S$ is a $\text{PGL}_{2, X^F}$-torsor on $X^F$. Write $P_S \to X^F$ for the $\mathbb{P}^1$-bundle associated to the $\text{PGL}_{2, X^F}$-torsor $\Phi_*S$ [i.e., the quotient of $\Phi_*S \times_{X^F} \mathbb{P}^{1}_{X^F}$ by the diagonal action of $\text{PGL}_{2, X^F}$]. For each local section $s$ of $\Phi_*S$, write $\sigma_s$ for the local section of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1_X \to X$ [cf. Remark 1.1.2]. Then the pair consisting of

- the $\mathbb{P}^1$-bundle $P_S \to X^F$ over $X^F$

and

- the section of $\Phi^*P_S \to X$ determined by the various pairs “$(s, \sigma_s)$” — where “$s$” ranges over the local sections of $\Phi_*S$ —

is a Frobenius-indigenous structure of level $N$ on $X$.

Proof. — This assertion follows immediately from the fact that $S$ is contained in $\mathcal{P}^{\text{et}}$.

Definition 3.8. — Let $S \subseteq \mathcal{P}^{\text{et}}$ be a Frobenius-projective structure of level $N$ on $X$. Then it follows from Lemma 3.7 that $S$ determines a Frobenius-indigenous structure of level $N$ on $X$. We shall refer to this Frobenius-indigenous structure of level $N$ as the Frobenius-indigenous structure of level $N$ associated to $S$. Thus, we obtain a map

$$\mathcal{F}_{ps}^N(X) \to \mathcal{F}_{is}^N(X).$$

Lemma 3.9. — Let $(P \to X^F, \sigma)$ be a Frobenius-indigenous structure of level $N$ on $X$. Then the following hold:
(i) Let $U \subseteq X$ be an open subscheme of $X$ such that the restriction $P|_{U^F}$ is isomorphic to the trivial $\mathbb{P}^1$-bundle over $U^F$ and $\iota_U: P|_{U^F} \sim \mathbb{P}^1_k \times_k U^F$ an isomorphism over $U^F$. Write $f_{U, \iota_U} \in \mathcal{P}(U)$ for the section of $\mathcal{P}$ obtained by forming the composite

$$U \xrightarrow{\sigma|_U} (\Phi^*P)|_U \xrightarrow{\Phi^*\iota_U} \mathbb{P}^1_k \times_k U \xrightarrow{pr_1} \mathbb{P}^1.$$ 

Then $f_{U, \iota_U} \in \mathcal{P}^\text{ét}(U)$.

(ii) The collection of sections $f_{U, \iota_U} \in \mathcal{P}^\text{ét}(U)$ [cf. (i)] — where $(U, \iota_U)$ ranges over the pairs as in (i) — determines a Frobenius-projective structure of level $N$ on $X$.

**Proof.** — Assertion (i) follows from our assumption that the Kodaira-Spencer section of the connection $\nabla_{\Phi^*P}$ at $\sigma$ is nowhere vanishing. Assertion (ii) follows immediately from assertion (i). 

---

**Definition 3.10.** — Let $\mathcal{I}$ be a Frobenius-indigenous structure of level $N$ on $X$. Then it follows from Lemma 3.9, (ii), that $\mathcal{I}$ determines a Frobenius-projective structure of level $N$ on $X$. We shall refer to this Frobenius-projective structure of level $N$ as the Frobenius-projective structure of level $N$ associated to $\mathcal{I}$. Thus, we obtain a map

$$\mathfrak{fis}_N(X) \longrightarrow \mathfrak{fps}_N(X).$$

**Proposition 3.11.** — The assignments of Definition 3.8 and Definition 3.10 determine a bijection

$$\mathfrak{fps}_N(X) \sim \mathfrak{fis}_N(X).$$

**Proof.** — This assertion follows immediately from the constructions of Lemma 3.7 and Lemma 3.9. 

**Remark 3.11.1.** — Proposition 3.11 may be regarded as an analogue, in positive characteristic, of [2], Theorem 3.

**Corollary 3.12.** — Suppose that $(p, N) = (3, 1)$. Then $\# \mathfrak{fis}_N(X) = 1$.

**Proof.** — This assertion follows from Proposition 3.11, together with Corollary 2.8. 

**Remark 3.12.1.** — Suppose that $g \geq 2$. Then it follows from Remark 3.4.1, (ii), that the conclusion of Corollary 3.12 is equivalent to the following assertion:

\((\ast): \) If $p = 3$, then the set of isomorphism classes of dormant indigenous bundles on $X$ is of cardinality one.

Now let us recall that we already have four proofs of the assertion $(\ast)$ as follows:

1. the proof essentially obtained by the theory of molecules established by S. Mochizuki [cf. [3], Remark 2.1.1]
(2) the proof essentially obtained by the formula of the number of isomorphism classes of dormant indigenous bundles on a sufficiently general curve established by Y. Wakahayashi [cf. the proof of [3], Theorem 2.1]

(3) the proof obtained by an explicit local computation of the $p$-curvatures of indigenous bundles in characteristic three established by the author of the present paper [cf. [3], Remark 3.1.1]

(4) the proof obtained by the uniqueness of the isomorphism class of dormant opers of rank $p - 1$ established by the author of the present paper [cf. [4], Theorem A]

Thus, we conclude that the proof of the assertion (∗) given in the proof of Corollary 3.12, i.e.,

(5) the proof essentially obtained by the uniqueness of the $G_{etn}$-orbit of generically étale rational functions in the case where $(p, N) = (3, 1)$ [cf. Proposition 1.8, (i)], may be regarded as the fifth proof of the assertion (∗).

**Theorem 3.13.** — Suppose that $(p, N) \neq (2, 1)$. Then there exist bijections

$$\text{pcd}_N(X)/G_{etn} \xrightarrow{\sim} \text{fps}_N(X) \xrightarrow{\sim} \text{fis}_N(X).$$

**Proof.** — This assertion follows from Proposition 2.7 and Proposition 3.11. 

---

**4. Relationship Between Certain Frobenius-destabilized Bundles**

In the present §4, we discuss a relationship between Frobenius-indigenous structures and certain Frobenius-destabilized bundles over $X^F$ [cf. Proposition 4.7 below].

In the present §4, we maintain the notational conventions introduced at the beginnings of §1 and §3. Write, moreover,

$$X^f$$

for the “$X^F$” in the case where $N = 1$ and

$$\phi: X \rightarrow X^f$$

for the “$\Phi$” in the case where $N = 1$. Thus, the morphism $\Phi: X \rightarrow X^F$ factors as the composite

$$X \xrightarrow{\phi} X^f \rightarrow X^F.$$

We shall write

$$\Phi_{f \rightarrow F}: X^f \rightarrow X^F$$

for the second arrow in this composite [i.e., the “$\Phi$” in the case where we take the pair “$(X, N)$” to be $(X^f, N - 1)$].

In the present §4, suppose, moreover, that

$$g \geq 2.$$
**Definition 4.1.** — Let $S$ be a scheme and $E_1, E_2$ two $O_S$-modules. Then we shall say that $E_1$ is $P$-equivalent to $E_2$ if there exist an invertible sheaf $L$ on $S$ and an isomorphism $E_1 \otimes_{O_S} L \cong E_2$ of $O_S$-modules. We shall write $E_1 \sim_P E_2$ if $E_1$ is $P$-equivalent to $E_2$.

**Definition 4.2.** — Let $d$ be a positive integer and $E$ a locally free coherent $O_X$-module of rank two. Then we shall say that $E$ is $(N, d)$-Frobenius-destabilized if the following conditions are satisfied:

1. The locally free coherent $O_X$-module $\Phi^* f \to F E$ of rank two is stable. [In particular, the locally free coherent $O_X$-module $E$ of rank two is stable.]
2. There exist an invertible sheaf $L$ on $X$ of degree $p^N \cdot \deg(E) + d$ and a locally split injection $L \to \Phi^* E$ of $O_X$-modules. [In particular, the locally free coherent $O_X$-module $\Phi^* E$ of rank two is not semistable.] Note that one verifies easily that the quotient, which is an invertible sheaf on $X$, of $\Phi^* E$ by $L$ is of degree $p^N \cdot \deg(E) - d = \deg(L) - 2d$.

We shall write $\mathcal{F}_{ds}(X)$ for the set of $P$-equivalence classes [cf. Remark 4.2.1 below] of $(N, g - 1)$-Frobenius-destabilized locally free coherent $O_X$-modules of rank two.

**Remark 4.2.1.** — Let $E_1, E_2$ be locally free coherent $O_X$-modules of rank two. Suppose that $E_1 \sim_P E_2$. Then one verifies easily that it holds that $E_1$ is $(N, g - 1)$-Frobenius-destabilized if and only if $E_2$ is $(N, g - 1)$-Frobenius-destabilized.

**Remark 4.2.2.**

(i) Let $E$ be a locally free coherent $O_X$-module of rank two. Then one verifies easily that it holds that $E$ is $(N, g - 1)$-Frobenius-destabilized if and only if $\Phi^* f \to F E$ is $(1, g - 1)$-Frobenius-destabilized.

(ii) Let $P \to X^F$ be a $\mathbb{P}^1$-bundle over $X$ and $\sigma$ a section of $\Phi^* P \to X$. Then one verifies easily that it holds that $(P \to X^F, \sigma)$ is a Frobenius-indigenous structure of level $N$ on $X$ if and only if $(\Phi^* f \to F P \to X^F, \sigma)$ is a Frobenius-indigenous structure of level 1 on $X$.

**Remark 4.2.3.** — Write $\text{Fr}_X : X \to X$ for the $p$-th power Frobenius endomorphism of $X$. Then one verifies easily that the assignment “$E \mapsto W_* E$” determines a bijection of the set $\mathcal{F}_{ds}(X)$ with the set of $P$-equivalence classes of locally free coherent $O_X$-modules $F$ of rank two which satisfy the following condition: If, for a nonnegative integer $i$, we write

$$F_i \overset{\text{def}}{=} \text{Fr}^* X \circ \cdots \circ \text{Fr}^* X F,$$
then

- the locally free coherent $\mathcal{O}_X$-module $\mathcal{F}_{N-1}$, hence also $\mathcal{F}$, is stable, but

- there exist an invertible sheaf $\mathcal{L}$ on $X$ of degree $\frac{pN}{2} \cdot \deg(\mathcal{F}) + g - 1 = \frac{1}{2} \deg(\mathcal{F}_N) + g - 1$ and a locally split injection $\mathcal{L} \hookrightarrow \mathcal{F}_N$ of $\mathcal{O}_X$-modules. [In particular, the locally free coherent $\mathcal{O}_X$-module $\mathcal{F}_N$ is not semistable.]

**Lemma 4.3.** — Let $\mathcal{E}$ be an $(N, g - 1)$-Frobenius-destabilized locally free coherent $\mathcal{O}_{XF}$-module of rank two. Write $\mathbb{P}(\mathcal{E}) \to X^F$ for the projectivization of $\mathcal{E}$. Then there exists a [uniquely determined — cf. Lemma 3.6] section $\sigma$ of $\Phi^*\mathbb{P}(\mathcal{E}) \to X$ such that the pair $(\mathbb{P}(\mathcal{E}) \to X^F, \sigma)$ is a Frobenius-indigenous structure of level $N$ on $X$.

**Proof.** — Let us first observe that it follows from Remark 4.2.2, (i), (ii), that we may assume without loss of generality, by replacing $\mathcal{E}$ by $\Phi_{f_{-f}}^*\mathcal{E}$, that $N = 1$. Let $\mathcal{L} \subseteq \Phi^*\mathcal{E}$ be as in condition (2) of Definition 4.2. Then it is immediate that, to verify Lemma 4.3, it suffices to verify that the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{L} \hookrightarrow \Phi^*\mathcal{E} \overset{\nabla_{\Phi^*\mathcal{E}}}{\to} \omega_{X/k} \otimes_{\mathcal{O}_X} \Phi^*\mathcal{E} \rightarrow \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{Q}$$

— where we write $\mathcal{Q}$ for the invertible sheaf on $X$ obtained by forming the quotient of $\Phi^*\mathcal{E}$ by $\mathcal{L}$ — is an isomorphism. Thus, since $\deg(\mathcal{L}) = \deg(\omega_{X/k}) + \deg(\mathcal{Q})$ [cf. condition (2) of Definition 4.2], one verifies easily that, to verify Lemma 4.3, it suffices to verify that the image of the above composite is nonzero, i.e., that the submodule $\mathcal{L} \subseteq \Phi^*\mathcal{E}$ is not horizontal [with respect to the connection $\nabla_{\Phi^*\mathcal{E}}$]. On the other hand, if the submodule $\mathcal{L} \subseteq \Phi^*\mathcal{E}$ is horizontal, then since $N = 1$, we obtain [cf., e.g., [5], Theorem 5.1] an invertible sheaf $\mathcal{F}$ on $X^F$ of degree $\frac{1}{p} \cdot \deg(\mathcal{L})$ and a locally split injection $\mathcal{F} \hookrightarrow \mathcal{E}$ of $\mathcal{O}_{XF}$-modules; thus, since $2 \cdot \deg(\mathcal{F}) = \frac{2}{p} \cdot \deg(\mathcal{L}) = \deg(\mathcal{E}) + \frac{2p - 2}{p} > \deg(\mathcal{E})$ [cf. condition (2) of Definition 4.2], and $\mathcal{E}$ is stable [cf. condition (1) of Definition 4.2], we obtain a contradiction. This completes the proof Lemma 4.3. □

**Definition 4.4.** — Let $\mathcal{E}$ be an $(N, g - 1)$-Frobenius-destabilized locally free coherent $\mathcal{O}_{XF}$-module of rank two. Then it follows from Lemma 4.3 that $\mathcal{E}$ determines a Frobenius-indigenous structure of level $N$ on $X$. We shall refer to this Frobenius-indigenous structure of level $N$ as the Frobenius-indigenous structure of level $N$ associated to $\mathcal{E}$. Thus, we obtain a map

$$\mathfrak{F} \mathfrak{s}_N(X) \rightarrow \mathfrak{F} \mathfrak{s}_N(X).$$

**Lemma 4.5.** — Let $(P \to X^F, \sigma)$ be a Frobenius-indigenous structure of level $N$ on $X$ and $\mathcal{E}$ a locally free coherent $\mathcal{O}_{XF}$-module of rank two whose projectivization is isomorphic to $P$ over $X^F$. Then $\mathcal{E}$ is $(N, g - 1)$-Frobenius-destabilized.

**Proof.** — Let us first observe that it follows from Remark 4.2.2, (i), (ii), that we may assume without loss of generality, by replacing $P$ by $\Phi_{f_{-f}}^*P$, that $N = 1$. Let $\Phi^*\mathcal{E} \to \mathcal{Q}$ be a surjection of $\mathcal{O}_X$-modules onto an invertible sheaf $\mathcal{Q}$ on $X$ which defines, relative to an isomorphism of $P$ with the projectivization of $\mathcal{E}$ over $X^F$, the section $\sigma$. Write $\mathcal{L}$ for
the kernel of the surjection $\Phi^*E \to \mathcal{Q}$. Then since $2 \cdot \deg(L) = p \cdot \deg(E) + 2g - 2$ [cf. Lemma 3.5], we conclude that condition (2) of Definition 4.2 holds.

Assume that condition (1) of Definition 4.2 does not hold, i.e., that there exist an invertible sheaf $F$ on $X^F$ and a locally split injection $F \to E$ of $O_{X^F}$-modules such that

$$2 \cdot \deg(F) \geq \deg(E),$$

which thus implies that $2 \cdot \deg(\Phi^*E/\Phi^*F) \leq \deg(\Phi^*E)$. Then since $2 \cdot \deg(\Phi^*E/\Phi^*F) \leq \deg(\Phi^*E) = p \cdot \deg(E) < 2 \cdot \deg(L)$ [cf. Lemma 3.5], we obtain that $L \subseteq \Phi^*F$, which thus implies that $L = \Phi^*F$. In particular, the invertible subsheaf $L \subseteq \Phi^*E$ of $\Phi^*E$ is horizontal. On the other hand, since [we have assumed that] the Kodaira-Spencer section of the connection $\nabla_{\Phi^*F}$ at $\sigma$ is nowhere vanishing, we obtain a contradiction. This completes the proof of Lemma 4.5. □

**Remark 4.5.1.** — The contents of Lemma 4.3 and Lemma 4.5 may be considered to be essentially contained in [8], Proposition 4.2.

**Definition 4.6.** — Let $I$ be a Frobenius-indigenous structure of level $N$ on $X$. Then it follows from Lemma 4.5 that $I$ determines a $\mathbb{P}$-equivalence class of $(N, g - 1)$-Frobenius-destabilized locally free coherent $O_{X^F}$-modules of rank two. We shall refer to this $\mathbb{P}$-equivalence class as the $(N, g - 1)$-Frobenius-destabilized class associated to $I$. Thus, we obtain a map

$$\mathfrak{Fis}_N(X) \longrightarrow \mathfrak{Fds}_N(X).$$

**Proposition 4.7.** — The assignments of Definition 4.4 and Definition 4.6 determine a bijection

$$\mathfrak{Fds}_N(X) \sim \mathfrak{Fis}_N(X).$$

**Proof.** — This assertion follows immediately from the constructions of Lemma 4.3 and Lemma 4.5. □

**Corollary 4.8.** — Suppose that $(p, N) \neq (2, 1)$, and that $g \geq 2$. Then there exist bijections

$$\mathfrak{pcd}_N(X)/\mathcal{G}_{\text{rtu}} \sim \mathfrak{fps}_N(X) \sim \mathfrak{Fis}_N(X) \sim \mathfrak{Fds}_N(X).$$

**Proof.** — This assertion follows from Theorem 3.13 and Proposition 4.7. □

**Corollary 4.9.** — Suppose that $p \neq 2$, that $g \geq 2$, and that $N = 1$. Then the following hold:

(i) The four sets

$$\mathfrak{pcd}_N(X)/\mathcal{G}_{\text{rtu}}, \mathfrak{fps}_N(X), \mathfrak{Fis}_N(X), \mathfrak{Fds}_N(X)$$

are nonempty.
(ii) There exists a nonempty open subscheme of the coarse moduli space of projective smooth curves over $k$ of genus $g$ such that if the curve $X$ is parametrized by the open subscheme, then it holds that

$$\sharp(\text{pcd}_N(X)/\mathcal{G}_{\text{etm}}) = \sharp\mathfrak{ps}_N(X) = \sharp\mathfrak{is}_N(X) = \sharp\mathfrak{ds}_N(X)$$

$$= \frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{\theta=1}^{p-1} \sin \left( \frac{\pi \cdot \theta}{p} \right)^{2-2g} = \frac{(-p)^{g-1}}{2} \cdot \sum_{\zeta^{p}=1, \zeta \neq 1} \zeta^{g-1} \cdot (1 - \zeta)^{2-2g}.$$

**Proof.** — First, we verify assertion (i). Let us first observe that it follows from Corollary 4.8 that, to verify assertion (i), it suffices to verify that $\mathfrak{is}_N(X) \neq \emptyset$. On the other hand, this follows immediately, in light of Remark 3.4.1, (ii), from [10], Theorem 3.3. This completes the proof of assertion (i).

Assertion (ii) follows immediately, in light of Remark 3.4.1, (ii), and Corollary 4.8, from [10], Theorem A [cf. also the discussion given in the second paragraph of the proof of [3], Theorem 2.1]. This completes the proof of assertion (ii), hence also of Corollary 4.9. 

**Remark 4.9.1.** — Note that a result in the case where $p = 2$ similar to Corollary 4.9 will be discussed in Corollary 5.7, (iii), below.

**Corollary 4.10.** — Suppose that $p \neq 2$, and that $g \geq 2$. Then there exists a nonempty open subscheme of the coarse moduli space of projective smooth curves over $k$ of genus $g$ such that if the curve $X$ is parametrized by the open subscheme, then the four sets $\text{pcd}_N(X)/\mathcal{G}_{\text{etm}}, \mathfrak{ps}_N(X), \mathfrak{is}_N(X), \mathfrak{ds}_N(X)$ are nonempty.

**Proof.** — Let us first observe that it follows immediately, in light of Corollary 4.8, from Remark 3.4.1, (i), that, to verify Corollary 4.10, it suffices to verify that the image of the composite “$X \to \overline{\mathcal{M}}_{g,0}$” of the displayed functors of Remark 3.4.1, (i), contains a nonempty open substack of $\overline{\mathcal{M}}_{g,0}$. In particular, it follows immediately from the various definitions involved that, to verify Corollary 4.10, it suffices to verify that the image of the natural functor from the VF-stack $\overline{\mathcal{N}}_{g,0}^{\mathfrak{H}}$ of pure tone $N+1$ [cf. [7], Chapter IV, Definition 2.6] to $\overline{\mathcal{M}}_{g,0}$ contains a nonempty open substack of $\overline{\mathcal{M}}_{g,0}$. On the other hand, this follows immediately from [7], Chapter IV, Remark following Theorem 2.9, together with [7], Chapter III, Lemma 1.8. This completes the proof of Corollary 4.10. 

5. Frobenius-indigenous Structures of Level One in Characteristic Two

In the present §5, we discuss Frobenius-indigenous structures of level one in characteristic two [cf. Proposition 5.6, Corollary 5.7 below]. In the present §5, we maintain the notational conventions introduced at the beginnings of §1 and §3. Suppose, moreover, that

$$(p, N) = (2, 1).$$
**Definition 5.1.** — Let \( \mathcal{L} \) be an invertible sheaf on \( X \). Then we shall write
\[
\pi_\mathcal{L} \colon \Phi^* \Phi_* \mathcal{L} \twoheadrightarrow \mathcal{L}
\]
for the [necessarily surjective] homomorphism of \( \mathcal{O}_X \)-modules obtained by considering restrictions.

**Lemma 5.2.** — Let \( \mathcal{L} \) be an invertible sheaf on \( X \). Write \( P \to X^F \) for the \( \mathbb{P}^1 \)-bundle over \( X^F \) determined by the locally free coherent \( \mathcal{O}_{X^F} \)-module \( \Phi_* \mathcal{L} \) of rank two and \( \sigma \) for the section of \( \Phi^* P \to X \) determined by \( \pi_\mathcal{L} \). Then the pair \( (P \to X^F, \sigma) \) is a Frobenius-indigenous structure of level \( N \) on \( X \).

**Proof.** — This assertion follows immediately from an explicit local computation of the Kodaira-Spencer section of \( \nabla_{\Phi^* P} \) at \( \sigma \).

**Definition 5.3.** — Let \( \mathcal{L} \) be an invertible sheaf on \( X \). Then it follows from Lemma 5.2 that \( \mathcal{L} \) determines a Frobenius-indigenous structure of level \( N \) on \( X \). We shall write
\[
\mathcal{I}(\mathcal{L})
\]
for this Frobenius-indigenous structure of level \( N \).

**Lemma 5.4.** — Let \( \mathcal{L} \) be an invertible sheaf on \( X \). Then the following hold:

(i) It holds that \( \deg(\Phi^* \Phi_* \mathcal{L}) = 2 \cdot \deg(\mathcal{L}) + 2g - 2 \).

(ii) It holds that \( \deg(\Phi_* \mathcal{L}) = \deg(\mathcal{L}) + g - 1 \).

**Proof.** — Assertion (i) follows from Lemma 3.5 and Lemma 5.2. Assertion (ii) follows from assertion (i).

**Lemma 5.5.** — Let \( \mathcal{L}_1, \mathcal{L}_2 \) be invertible sheaves on \( X \). Then the following conditions are equivalent:

1. It holds that \( \Phi_* \mathcal{L}_1 \sim_{\mathbb{P}^1} \Phi_* \mathcal{L}_2 \).
2. It holds that \( \deg(\mathcal{L}_1) - \deg(\mathcal{L}_2) \) is even.

**Proof.** — The implication \( (1) \Rightarrow (2) \) follows immediately from Lemma 5.4, (ii). Next, we verify the implication \( (2) \Rightarrow (1) \). Suppose that condition (2) is satisfied. Then one verifies easily that there exists an invertible sheaf \( \mathcal{F} \) on \( X^F \) such that \( \mathcal{L}_1 \cong \mathcal{L}_2 \otimes_{\mathcal{O}_{X^F}} \Phi^* \mathcal{F} \), which thus implies that \( \Phi_* \mathcal{L}_1 \cong \Phi_* \mathcal{L}_2 \otimes_{\mathcal{O}_{X^F}} \mathcal{F} \). In particular, condition (1) is satisfied. This completes the proof of the implication \( (2) \Rightarrow (1) \), hence also of Lemma 5.5.

**Proposition 5.6.** — The following hold:

(i) Let \( \mathcal{L}_1, \mathcal{L}_2 \) be invertible sheaves on \( X \). Then if \( \mathcal{I}(\mathcal{L}_1) \) is isomorphic to \( \mathcal{I}(\mathcal{L}_2) \), then \( \deg(\mathcal{L}_1) - \deg(\mathcal{L}_2) \) is even.
(ii) Suppose that \( g \geq 2 \). Then, in the situation of (i), if \( \deg(L_1) - \deg(L_2) \) is even, then \( I(L_1) \) is isomorphic to \( I(L_2) \).

(iii) Suppose that \( g \geq 2 \). Let \( L_{\mathrm{odd}}, L_{\mathrm{even}} \) be invertible sheaves on \( X \) of odd, even degree, respectively. Write \( [I(L_{\mathrm{odd}})], [I(L_{\mathrm{even}})] \) for the isomorphism classes of the Frobenius-indigenous structures \( I(L_{\mathrm{odd}}), I(L_{\mathrm{even}}) \) of level \( N \) on \( X \), respectively. [So \( [I(L_{\mathrm{odd}})] \neq [I(L_{\mathrm{even}})] \) by (i).] Then it holds that
\[
\mathfrak{Fis}_N(X) = \{ [I(L_{\mathrm{odd}})], [I(L_{\mathrm{even}})] \}.
\]

**Proof.** — Assertion (i) follows from Lemma 5.5. Assertion (ii) follows immediately from Lemma 5.5, together with Lemma 3.6.

Finally, we verify assertion (iii). Let us first observe that it follows from assertion (ii) that, to verify assertion (iii), it suffices to verify the following assertion:

\( (*) \): Let \( I = (P \to X^F, \sigma) \) be a Frobenius-indigenous structure of level \( N \) on \( X \). Then there exists an invertible sheaf \( L \) on \( X \) such that \( I \) is isomorphic to \( I(L) \).

To this end, take a locally free coherent \( \mathcal{O}_{X^F} \)-module \( E \) of rank two whose projectivization \( \mathbb{P}(E) \) is isomorphic to \( P \) over \( X^F \). Write \( \Phi^*E \twoheadrightarrow L \) for the quotient which defines, relative to an isomorphism of \( P \) with \( \mathbb{P}(E) \) over \( X^F \), the section \( \sigma \). Then it follows from a similar argument to the argument applied in the proof of [4], Lemma 2.2, that the homomorphism \( E \to \Phi^*L \) of \( \mathcal{O}_{X^F} \)-modules determined by the surjection \( \Phi^*E \twoheadrightarrow L \) is an isomorphism. Thus, it follows immediately from Lemma 3.6 that \( I \) is isomorphic to \( I(L) \).

This completes the proof of assertion (iii), hence also of Proposition 5.6.

**Corollary 5.7.** — Suppose that \( (p, N) = (2, 1) \). Then the following hold:

(i) It holds that
\[
1 = \sharp(\text{pcd}_N(X)/\mathcal{G}_{\text{rtn}}) < 2 \leq \sharp\mathfrak{ps}_N(X) = \sharp\mathfrak{Fis}_N(X).
\]

(ii) The map
\[
\mathfrak{ps}_N(X) \to \text{pcd}_N(X)/\mathcal{G}_{\text{rtn}}
\]
of Definition 2.3 is surjective but not injective.

(iii) Suppose that \( g \geq 2 \). Then it holds that
\[
1 = \sharp(\text{pcd}_N(X)/\mathcal{G}_{\text{rtn}}) < 2 = \sharp\mathfrak{ps}_N(X) = \sharp\mathfrak{Fis}_N(X) = \sharp\mathfrak{Fds}_N(X).
\]

**Proof.** — Assertion (i) follows from Corollary 1.9, Proposition 3.11, and Proposition 5.6, (i). Assertion (ii) follows from assertion (i). Assertion (iii) follows from Proposition 4.7 and Proposition 5.6, (iii), together with assertion (i).

6. Applications of a Result of Sugiyama and Yasuda

In the present §6, we discuss applications of a result of Y. Sugiyama and S. Yasuda obtained in [9]. In the present §6, let us apply the notational conventions introduced at the beginnings of §1 and §4.

Let us recall that one important result obtained in [9] is as follows.
THEOREM 6.1. — Suppose that \((p, N) = (2, 2)\). Then there exists a pseudo-coordinate of level \(N\) on \(X\).

PROOF. — This assertion is the content of [9], Corollary 3.6 [cf. also Remark 1.3.2 of the present paper].

REMARK 6.1.1. — Suppose that \((p, N) = (2, 2)\). Let us recall that, for each \(f, g \in K_X \setminus K^p_X\), the “mysterious” element “\(a(f, g)\)” was defined in [9], Definition 2.6, and played an important role in the proof of [9], Corollary 3.6. This “mysterious” element satisfies the following three conditions:

- If \(f\) is contained in the \(G_{\text{rt}}\)-orbit of \(g\), then \(a(f, h) = a(g, h)\) [cf. [9], Proposition 2.8, (4)].
- It holds that \(a(f, g) = 0\) if and only if \(f\) is contained in the \(G_{\text{rt}}\)-orbit of \(g\) [cf. [9], Proposition 2.9].
- It holds that \(a(f, g) = a(f, h) + a(h, g)\) [cf. [9], Proposition 2.7; [9], Proposition 2.8, (1)].

These three properties are reminiscent of the following three properties of the Schwarzian derivative

\[
\theta_2(f, g) \overset{\text{def}}{=} \frac{2 \cdot f(1) \cdot f(3) - 3 \cdot (f(2))^2}{2 \cdot (f(1))^2} \quad \text{(where } f^{(n)} \overset{\text{def}}{=} \frac{d^n f}{dg^n})
\]

[cf., e.g., [2], §4] for local projective coordinates \(f, g\) on a Riemann surface in the classical theory of Riemann surfaces:

- If \(f\) is contained in the \(\text{PGL}_2(\mathbb{C})\)-orbit of \(g\), then \(\theta_2(f, h) = \theta_2(g, h)\).
- It holds that \(\theta_2(f, g) = 0\) if and only if \(f\) is contained in the \(\text{PGL}_2(\mathbb{C})\)-orbit of \(g\).
- It holds that \(\theta_2(f, g) \cdot (dg)^{\otimes 2} = \theta_2(f, h) \cdot (dh)^{\otimes 2} + \theta_2(h, g) \cdot (dg)^{\otimes 2}\).

Thus, the author of the present paper considers that, in the context of the present paper, this “mysterious” element “\(a(f, g)\)” defined in [9] should be regarded as a certain analogue in characteristic two of the Schwarzian derivative in the classical theory of Riemann surfaces. This observation was in fact a starting point of the study of the present paper.

COROLLARY 6.2. — Suppose that \((p, N) = (2, 2)\). Then there exist

- a Frobenius-projective structure of level \(N\) on \(X\)

and

- a Frobenius-indigenous structure of level \(N\) on \(X\).

If, moreover, \(g \geq 2\), then there exists

- an \((N, g - 1)\)-Frobenius-destabilized locally free coherent \(O_X\)-module of rank two.

PROOF. — The first assertion follows from Theorem 6.1, together with Theorem 3.13. The final assertion follows from Theorem 6.1, together with Corollary 4.8.
Remark 6.2.1. — Suppose that $p = 2$, and that $g \geq 2$. Then it follows, in light of Remark 4.2.3, from Corollary 6.2 that there exists a locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$ of rank two such that
\begin{itemize}
  \item the locally free coherent $\mathcal{O}_X$-module $\text{Fr}_X^* \mathcal{E}$, hence also $\mathcal{E}$, is stable, but
  \item the locally free coherent $\mathcal{O}_X$-module $\text{Fr}_X^* \text{Fr}_X^* \mathcal{E}$ admits an invertible subsheaf $\mathcal{L} \subseteq \text{Fr}_X^* \text{Fr}_X^* \mathcal{E}$ of degree $\frac{1}{2} \deg(\text{Fr}_X^* \text{Fr}_X^* \mathcal{E}) + g - 1$, which thus implies that $\text{Fr}_X^* \text{Fr}_X^* \mathcal{E}$ is not semistable.
\end{itemize}

Corollary 6.3. — Suppose that $(p, N) = (2, 2)$, and that $g \geq 2$. Let $\mathcal{L}$ be an invertible sheaf on $X$. Suppose, moreover, that $\mathcal{L}$ is of odd (respectively, even) degree if $g$ is even (respectively, odd). Then there exists a stable locally free coherent $\mathcal{O}_{X_F}$-module $\mathcal{E}$ of rank two such that $\phi_* \mathcal{L} \sim \Phi_{f \to F}^* \mathcal{E}$.

Proof. — Let $\mathcal{E}$ be an $(N, g - 1)$-Frobenius-destabilized locally free coherent $\mathcal{O}_{X_F}$-module of rank two [cf. Corollary 6.2]. Then it follows from Lemma 4.3, together with Remark 4.2.2, (ii), that there exists a [uniquely determined — cf. Lemma 3.6] section $\sigma$ of the $\mathbb{P}^1$-bundle $\Phi^* \mathbb{P}(\mathcal{E}) = \Phi^* \mathbb{P}(\Phi_{f \to F}^* \mathcal{E}) \to X$ such that the pair $\mathcal{I} \overset{\text{def}}{=} (\mathbb{P}(\Phi_{f \to F}^* \mathcal{E}) \to X\,' , \sigma)$ is a Frobenius-indigenous structure of level 1 on $X$. Here, let us observe that one verifies easily that $\deg(\Phi_{f \to F}^* \mathcal{E})$ is even. Thus, it follows from Proposition 5.6, (iii), together with Lemma 5.4, (ii), that $\mathcal{I}$ is isomorphic to $\mathcal{I}(\mathcal{L})$, which thus implies that $\Phi_{f \to F}^* \mathcal{E} \sim \Phi_{f \to F}^* \mathcal{E}$, as desired. This completes the proof of Corollary 6.3. □

References


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