Reconstruction of Profinite Graphs from Profinite Groups of PIPSC-type

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Abstract. — In the present paper, we study profinite groups of PIPSC-type, i.e., abstract profinite groups isomorphic to the extensions determined by outer representations of PIPSC-type. In particular, we establish a "group-theoretic" algorithm for constructing, from a profinite group of PIPSC-type that is noncuspidal, a certain profinite graph.

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Introduction

In the present paper, we study the *combinatorial anabelian geometry* of *semi-graphs of anabelioids of PSC-type*, i.e., roughly speaking, semi-graphs of anabelioids associated to pointed stable curves [cf., e.g., [7], [3], [4], [5], [6]]. The focus of the present paper is on a "group-theoretic" reconstruction, from a profinite group *of PIPSC-type*, of a certain profinite graph [cf. Theorem A below].

Let Σ be a nonempty set of prime numbers and $\mathcal{G}$ a semi-graph of anabelioids of pro-Σ PSC-type [cf. [7], Definition 1.1, (i)]. Let us fix a universal pro-Σ covering $\widetilde{\mathcal{G}} \to \mathcal{G}$ of $\mathcal{G}$. Write $\widetilde{\mathbb{G}}$ for the underlying profinite semi-graph of $\widetilde{\mathcal{G}}$ [i.e., the projective system consisting of the underlying semi-graphs of the connected finite étale subcoverings of $\widetilde{\mathcal{G}} \to \mathcal{G}$] and $\Pi_{\mathcal{G}}$ for the [pro-Σ] fundamental group of $\mathcal{G}$ determined by $\widetilde{\mathcal{G}} \to \mathcal{G}$. Let I be a profinite group and $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ an outer representation of pro-Σ PSC-type [cf. [3], Definition 2.1, (i)], which thus determines a homomorphism $I \to \mathrm{Out}(\Pi_{\mathcal{G}})$. Then

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since $\Pi_{\mathcal{G}}$ is topologically finitely generated and center-free [cf. [7], Remark 1.1.3], the outer representation ρ determines a profinite group $\Pi_\rho \overset{\text{def}}{=} \Pi_{\mathcal{G}} \overset{\text{out}}{\rtimes} I$ that fits into an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_\rho \longrightarrow I \longrightarrow 1$$

[cf. the discussion entitled "Profinite Groups" in §0].

Main objects of the present paper are outer representations *of PIPSC-type* [cf. [6], Definition 1.3] and profinite groups *of PIPSC-type* [cf. Definition 3.1]. Let us recall that, roughly speaking, an outer representation of PIPSC-type is defined to be an outer representation of PSC-type whose restriction to some open subgroup of the domain is isomorphic to the outer representation arising from a pointed stable curve over a log point. A profinite group of PIPSC-type is defined to be a profinite group isomorphic, as an abstract profinite group, to the profinite group "Π_ρ" as above for some outer representation "ρ" of PIPSC-type; moreover, we shall say that a profinite group of PIPSC-type is *noncuspidal* if one may take the "$\mathcal{G}$" as above to be a semi-graph of anabelioids of pro-Σ PSC-type that has no cusp [cf. Definition 3.2; Proposition 3.3]. An example of a profinite group of PIPSC-type is as follows: Let R be a strictly henselian discrete valuation ring of residue characteristic zero. Then the étale fundamental group of a hyperbolic curve over the field of fractions of R is an example of a profinite group of PIPSC-type [cf. Remark 3.1.2, (ii)]. Moreover, in this situation, the profinite semi-graph "$\widetilde{\mathbb{G}}$" as above may be naturally identified with the projective system consisting of the dual semi-graphs of the special fibers of the geometric stable models of the connected finite étale coverings of X [i.e., dominated by a fixed universal profinite covering of X].

The main result of the present paper may be summarized as follows [cf. Theorem 3.13]:

THEOREM A. — *There exists a* **"group-theoretic" algorithm**

$$\widetilde{\mathbb{G}}\colon\ \Pi \ \mapsto\ (\Pi \curvearrowright \widetilde{\mathbb{G}}(\Pi))$$

for constructing, from a profinite group Π **of PIPSC-type** *that is* **noncuspidal***, a profinite graph* $\widetilde{\mathbb{G}}(\Pi)$ *equipped with an action of* Π *such that if the above* ρ *is* **of PIPSC-type** [*which thus implies that the above profinite group* Π_ρ *is* **of PIPSC-type**]*, and* $\mathcal{G}$ *has* **no cusp***, then there exists a natural isomorphism of* $\widetilde{\mathbb{G}}$ *with* $\widetilde{\mathbb{G}}(\Pi_\rho)$.

Here, let us recall that if we are in a situation in which the profinite group Π_ρ is equipped with the closed subgroup $\Pi_{\mathcal{G}} \subseteq \Pi_\rho$, then a similar reconstruction result to the reconstruction result of Theorem A was already essentially obtained by *S. Mochizuki* and the author of the present paper in [5], Theorem 1.9, (ii), without the noncuspidal assumption. That is to say, roughly speaking, we already have a "group-theoretic" algorithm

$$(\Pi_{\mathcal{G}} \subseteq \Pi_\rho) \ \mapsto\ (\Pi_{\mathcal{G}} \subseteq \Pi_\rho \curvearrowright \widetilde{\mathbb{G}})$$

for constructing, from the profinite group Π_ρ equipped with the closed subgroup $\Pi_{\mathcal{G}} \subseteq \Pi_\rho$, the profinite graph $\widetilde{\mathbb{G}}$ equipped with the natural action of Π_ρ. Thus, Theorem A may be regarded as a *refinement* of this reconstruction result of [5] in the noncuspidal case.

Next, let us observe that if one considers the maximal pro-Σ quotient "Π" of the étale fundamental group of a certain hyperbolic curve over the field of fractions of a strictly henselian discrete valuation ring of residue characteristic $\not\in \Sigma$ [cf. Remark 3.1.2], then one verifies easily that it holds that the hyperbolic curve has *potentially good reduction* if and only if the profinite graph "$\widetilde{\mathbb{G}}(\Pi)$" of Theorem A has *no node*. In particular, as discussed in the introductions of [7] and [3], this sort of result may be regarded as a *refinement* of the "group-theoretic" pro-l criterion by *Takayuki Oda* and *Akio Tamagawa* for such a hyperbolic curve to have *good reduction*.

Finally, in §4, we study analogues of the discussions of [2], §5, and [2], §7 [i.e., related to *mono-anabelian transport for MLF-pairs*], from the point of view of the present paper. A *PIPSC-pair* is defined to be a collection of data $\Pi \curvearrowright \mathbb{H}$ consisting of a profinite semi-graph $\mathbb{H}$, a profinite group Π, and a continuous action of Π on $\mathbb{H}$ which is isomorphic to the collection of data "$\Pi_\rho \curvearrowright \widetilde{\mathbb{G}}$" as above for some outer representation "ρ" of PIPSC-type [cf. Definition 4.2, (ii)]; moreover, we shall say that a PIPSC-pair is *noncuspidal* if one may take the "$\mathcal{G}$" as above to be a semi-graph of anabelioids of pro-Σ PSC-type that has no cusp [cf. Definition 4.2, (ii)]. As an application of Theorem A, we also prove the following result in §4 [cf. Theorem 4.5]:

THEOREM B. — *Let* $\Pi_\circ \curvearrowright \mathbb{H}_\circ$, $\Pi_\bullet \curvearrowright \mathbb{H}_\bullet$ *be* **noncuspidal PIPSC-pairs**. *Then the natural map*

$$\mathrm{Isom}(\Pi_\circ \curvearrowright \mathbb{H}_\circ, \Pi_\bullet \curvearrowright \mathbb{H}_\bullet) \longrightarrow \mathrm{Isom}(\Pi_\circ, \Pi_\bullet)$$

is **bijective**.

Here, observe that the bijectivity of the map of Theorem B may be regarded as an analogue of the bijectivity of the map of [2], Theorem 7.6, (iv), from the point of view of the present paper.

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0. NOTATIONS AND CONVENTIONS

PROFINITE GROUPS. — If G is a profinite group, then we shall write $\mathrm{Aut}(G)$ for the group of automorphisms of the profinite group G, $\mathrm{Out}(G)$ for the group of outer automorphisms of the profinite group G, G^{ab} for the *abelianization* of G [i.e., the maximal abelian quotient of G whose kernel is closed in G], and $G^{\text{ab-free}}$ for the maximal abelian torsion-free quotient of G whose kernel is closed in G.

If G is a profinite group, and $H \subseteq G$ is a closed subgroup of G, then we shall write $Z_G(H) \subseteq N_G(H) \subseteq C_G(H) \subseteq G$ for the *centralizer*, *normalizer*, and *commensurator* of H in G, respectively. We shall say that H is *characteristic* if every automorphism of the profinite group G restricts to an automorphism of H. We shall say that H is *commensurably terminal* if $H = C_G(H)$.

If G is a profinite group, then we shall refer to the inductive limit of the respective centralizers, in G, of the open subgroups of G as the *local center* of G. Thus, the local center of G contains the *center* $Z_G(G)$ of G. We shall say that G is *slim* if the local center of G is trivial.

If G is a *topologically finitely generated* profinite group, then one verifies easily that G admits a basis of *characteristic* open subgroups, which thus induces a *profinite topology* on $\mathrm{Aut}(G)$, hence also on $\mathrm{Out}(G)$, with respect to which the natural exact sequence of groups $G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$ — where the first arrow is given by the action by conjugation — determines an exact sequence of *profinite groups*. Now suppose, moreover, that G is *center-free* [which thus implies that the above exact sequence of profinite groups determines an exact sequence $1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$], and that we are given a profinite group J and a homomorphism $\rho\colon J \to \mathrm{Out}(G)$ of profinite groups. Then we shall write

$$G \overset{\mathrm{out}}{\rtimes} J \overset{\mathrm{def}}{=} \mathrm{Aut}(G) \times_{\mathrm{Out}(G)} J.$$

Thus, the profinite group $G \overset{\mathrm{out}}{\rtimes} J$ fits into an exact sequence of profinite groups

$$1 \longrightarrow G \longrightarrow G \overset{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

SEMI-GRAPHS. — In the present paper, we shall refer to a collection of data

$$\mathbb{G} = (\mathrm{Vert}(\mathbb{G}),\ \mathrm{Cusp}(\mathbb{G}),\ \mathrm{Node}(\mathbb{G}),\ \{\zeta_e\}_{e \in \mathrm{Cusp}(\mathbb{G}) \sqcup \mathrm{Node}(\mathbb{G})})$$

consisting of

- a nonempty set $\mathrm{Vert}(\mathbb{G})$,
- a set $\mathrm{Cusp}(\mathbb{G})$ of sets of cardinality one,
- a set $\mathrm{Node}(\mathbb{G})$ of sets of cardinality two, and,
- for each $e \in \mathrm{Cusp}(\mathbb{G}) \sqcup \mathrm{Node}(\mathbb{G})$, a map $\zeta_e\colon e \to \mathrm{Vert}(\mathbb{G})$ of sets

such that,

- for each e, $e' \in \mathrm{Cusp}(\mathbb{G}) \sqcup \mathrm{Node}(\mathbb{G})$, if $e \neq e'$, then $e \cap e' = \emptyset$

as a *semi-graph*. For two semi-graphs $\mathbb{G} = (\mathrm{Vert}(\mathbb{G}), \mathrm{Cusp}(\mathbb{G}), \mathrm{Node}(\mathbb{G}), \{\zeta_e\}_e)$ and $\mathbb{G}' = (\mathrm{Vert}(\mathbb{G}'), \mathrm{Cusp}(\mathbb{G}'), \mathrm{Node}(\mathbb{G}'), \{\zeta'_{e'}\}_{e'})$, we shall refer to a collection of data

$$\phi = (\phi_{\mathrm{Vert}},\ \phi_{\mathrm{Edge}},\ \{\phi_e\}_{e \in \mathrm{Cusp}(\mathbb{G}) \sqcup \mathrm{Node}(\mathbb{G})})$$

consisting of

- maps $\phi_{\mathrm{Vert}}\colon \mathrm{Vert}(\mathbb{G}) \to \mathrm{Vert}(\mathbb{G}')$, $\phi_{\mathrm{Edge}}\colon \mathrm{Cusp}(\mathbb{G}) \sqcup \mathrm{Node}(\mathbb{G}) \to \mathrm{Cusp}(\mathbb{G}') \sqcup \mathrm{Node}(\mathbb{G}')$ of sets and,
- for each $e \in \mathrm{Cusp}(\mathbb{G}) \sqcup \mathrm{Node}(\mathbb{G})$, a bijection $\phi_e\colon e \xrightarrow{\sim} \phi_{\mathrm{Edge}}(e)$ of sets

such that,

- for each $e \in \mathrm{Cusp}(\mathbb{G}) \sqcup \mathrm{Node}(\mathbb{G})$, the diagram

$$\begin{array}{ccc} e & \xrightarrow{\zeta_e} & \mathrm{Vert}(\mathbb{G}) \\ {\scriptstyle \phi_e}\downarrow & & \downarrow{\scriptstyle \phi_{\mathrm{Vert}}} \\ \phi_{\mathrm{Edge}}(e) & \xrightarrow[\zeta'_{\phi_{\mathrm{Edge}}(e)}]{} & \mathrm{Vert}(\mathbb{G}') \end{array}$$

commutes

as a *morphism* $\mathbb{G} \to \mathbb{G}'$ *of semi-graphs.*

Let $\mathbb{G} = (\mathrm{Vert}(\mathbb{G}), \mathrm{Cusp}(\mathbb{G}), \mathrm{Node}(\mathbb{G}), \{\zeta_e\}_e)$ be a semi-graph. We shall refer to an element of $\mathrm{Vert}(\mathbb{G})$ (respectively, $\mathrm{Cusp}(\mathbb{G})$; $\mathrm{Node}(\mathbb{G})$; $\mathrm{Cusp}(\mathbb{G}) \sqcup \mathrm{Node}(\mathbb{G})$) as a *vertex* (respectively, a *cusp*; a *node*; an *edge*) of $\mathbb{G}$. For a vertex v of $\mathbb{G}$ and an edge e of $\mathbb{G}$, we shall say that e *abuts* to v if $v \in \mathrm{Im}(\zeta_e)$. We shall say that $\mathbb{G}$ is a *graph* if $\mathrm{Cusp}(\mathbb{G}) = \emptyset$. We shall say that $\mathbb{G}$ is *finite* if each of the sets $\mathrm{Vert}(\mathbb{G})$, $\mathrm{Cusp}(\mathbb{G})$, and $\mathrm{Node}(\mathbb{G})$ is finite. In a case where $\mathbb{G}$ is finite, we shall say that $\mathbb{G}$ is *connected* if, for every v, $w \in \mathrm{Vert}(\mathbb{G})$, there exist vertices $v_0, \ldots, v_r$ of $\mathbb{G}$ and nodes $e_1, \ldots, e_r$ of $\mathbb{G}$ such that $v_0 = v$, $v_r = w$, and, for each $1 \le i \le r$, the node e_i abuts to both v_{i-1} and v_i. We shall say that $\mathbb{G}$ is *untangled* if, for every $e \in \mathrm{Node}(\mathbb{G})$, the image of ζ_e is of cardinality two.

We shall refer to a projective system consisting of finite semi-graphs as a *profinite semi-graph*. Let $\widetilde{\mathbb{G}} = (\mathbb{G}_\lambda = (\mathrm{Vert}(\mathbb{G}_\lambda), \mathrm{Cusp}(\mathbb{G}_\lambda), \mathrm{Node}(\mathbb{G}_\lambda), \{\zeta_{\lambda,e_\lambda}\}_{e_\lambda}))_\lambda$ be a profinite semi-graph. We shall refer to an element of the projective limit of the $\mathrm{Vert}(\mathbb{G}_\lambda)$'s (respectively, $\mathrm{Cusp}(\mathbb{G}_\lambda)$'s; $\mathrm{Node}(\mathbb{G}_\lambda)$'s; $\mathrm{Cusp}(\mathbb{G}_\lambda) \sqcup \mathrm{Node}(\mathbb{G}_\lambda)$'s) as a *vertex* (respectively, a *cusp*; a *node*; an *edge*) of $\widetilde{\mathbb{G}}$. For a vertex $v = (v_\lambda)_\lambda$ of $\widetilde{\mathbb{G}}$ and an edge $e = (e_\lambda)_\lambda$ of $\widetilde{\mathbb{G}}$, we shall say that e *abuts* to v if e_λ abuts to v_λ for every λ. We shall say that $\widetilde{\mathbb{G}}$ is a *profinite graph* if each of the $\mathbb{G}_\lambda$'s is a graph. We shall say that $\widetilde{\mathbb{G}}$ is *connected* if each of the $\mathbb{G}_\lambda$'s is connected.

1. EXTENSIONS DETERMINED BY OUTER REPRESENTATIONS OF PSC-TYPE

A basic reference for the theory of *semi-graphs of anabelioids of PSC-type* is [7]. We shall use the terms "*semi-graph of anabelioids of* [*pro*-Σ] *PSC-type*", "*PSC-fundamental group of a semi-graph of anabelioids of* [*pro*-Σ] *PSC-type*", "*finite étale covering of semi-graphs of anabelioids of* [*pro*-Σ] *PSC-type*", "*vertex*", "*edge*", "*cusp*", and "*node*" as they are defined in [7], Definition 1.1. Also, we shall refer to the "PSC-fundamental group of a semi-graph of anabelioids of [pro-Σ] PSC-type" simply as the "fundamental group" [of the semi-graph of anabelioids of [pro-Σ] PSC-type]. That is to say, we shall refer to the maximal pro-Σ quotient of the fundamental group of a semi-graph of anabelioids of pro-Σ PSC-type [as a semi-graph of anabelioids] as the "fundamental group of the semi-graph of anabelioids of pro-Σ PSC-type".

In the present §1, let Σ be a nonempty set of prime numbers and

$$\mathcal{G}$$

a semi-graph of anabelioids of pro-Σ PSC-type. Let us fix a universal pro-Σ covering $\widetilde{\mathcal{G}} \to \mathcal{G}$ of $\mathcal{G}$. Write $\widetilde{\mathbb{G}}$ for the underlying profinite semi-graph of $\widetilde{\mathcal{G}}$ [i.e., the projective system consisting of the underlying semi-graphs of the connected finite étale subcoverings of $\widetilde{\mathcal{G}} \to \mathcal{G}$] and

$$\Pi_{\mathcal{G}}$$

for the fundamental group of $\mathcal{G}$ determined by $\widetilde{\mathcal{G}} \to \mathcal{G}$.

DEFINITION 1.1.

(i) We shall write

$$\mathrm{Vert}(\widetilde{\mathcal{G}}), \quad \mathrm{Node}(\widetilde{\mathcal{G}})$$

for the sets of *vertices*, *nodes* of $\widetilde{\mathcal{G}}$, i.e., of the profinite semi-graph $\widetilde{\mathbb{G}}$, respectively. Moreover, we shall write

$$\mathrm{VN}(\widetilde{\mathcal{G}}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\widetilde{\mathcal{G}}) \sqcup \mathrm{Node}(\widetilde{\mathcal{G}}).$$

(ii) Let $\widetilde{z} \in \mathrm{VN}(\widetilde{\mathcal{G}})$ be an element of $\mathrm{VN}(\widetilde{\mathcal{G}})$. Then we shall write

$$\Pi_{\widetilde{z}} \subseteq \Pi_{\mathcal{G}}$$

for the VCN-subgroup of $\Pi_{\mathcal{G}}$ associated to $\widetilde{z} \in \mathrm{VN}(\widetilde{\mathcal{G}})$ [cf. [4], Definition 2.1, (i)], i.e., the stabilizer of $\widetilde{z} \in \mathrm{VN}(\widetilde{\mathcal{G}})$ with respect to the natural action of $\Pi_{\mathcal{G}}$ on $\mathrm{VN}(\widetilde{\mathcal{G}})$.

(iii) We shall write

$$\Pi_{\mathcal{G}}^{\mathrm{ab/node}}$$

for the quotient of the abelianization $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ of $\Pi_{\mathcal{G}}$ by the [necessarily normal closed] subgroup topologically generated by the images of $\Pi_{\widetilde{e}} \subseteq \Pi_{\mathcal{G}}$, where $\widetilde{e}$ ranges over the nodes of $\widetilde{\mathcal{G}}$.

REMARK 1.1.1. — Let us recall that it follows from the well-known structure of the maximal pro-Σ quotient of the admissible fundamental group of a pointed stable curve over an algebraically closed field of characteristic $\notin \Sigma$ [cf. also [7], Example 2.5] that the quotient $\Pi_{\mathcal{G}}^{\mathrm{ab/node}}$ is *torsion-free* [cf. also [7], Remark 1.1.4].

LEMMA 1.2. — *Let $J \subseteq \Pi_{\mathcal{G}}$ be a* **nontrivial procyclic** *closed subgroup of $\Pi_{\mathcal{G}}$. Then the following two conditions are equivalent:*

(1) *There exists a [uniquely determined — cf.* [3], *Lemma* 1.5] *node $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ of $\widetilde{\mathcal{G}}$ such that $J \subseteq \Pi_{\widetilde{e}}$.*

(2) *For every connected finite étale subcovering $\mathcal{H} \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$, the image of the composite*

$$J \cap \Pi_{\mathcal{H}} \hookrightarrow \Pi_{\mathcal{H}} \twoheadrightarrow \Pi_{\mathcal{H}}^{\mathrm{ab/node}}$$

is **trivial**.

PROOF. — This follows immediately from a similar argument to the argument applied in the proof of [3], Lemma 1.6. □

LEMMA 1.3. — *The following hold:*

(i) *There exists a connected finite étale subcovering $\mathcal{H} \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$ such that the underlying semi-graph of $\mathcal{H}$ is* **untangled**.

(ii) *If the underlying semi-graph of* $\mathcal{G}$ *is* **untangled**, *then the underlying semi-graph of a connected finite étale subcovering of* $\widetilde{\mathcal{G}} \to \mathcal{G}$ *is* **untangled**.

PROOF. — Assertion (i) follows from the fourth paragraph of the discussion entitled "Curves" in [8], §0. Assertion (ii) is immediate. This completes the proof of Lemma 1.3. □

In the remainder of the present §1, let I be a profinite group and $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ an *outer representation of pro-*Σ *PSC-type* [cf. [3], Definition 2.1, (i)], i.e., a homomorphism of profinite groups, which thus determines a homomorphism

$$I \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}).$$

DEFINITION 1.4.

(i) Since $\Pi_{\mathcal{G}}$ is topologically finitely generated and center-free [cf. [7], Remark 1.1.3], the outer representation ρ determines an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} I \longrightarrow I \longrightarrow 1$$

[cf. the discussion entitled "Profinite Groups" in §0]. We shall write

$$\Pi_{\rho} \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}} \overset{\mathrm{out}}{\rtimes} I$$

for the middle profinite group of this exact sequence.

(ii) Let $\widetilde{z} \in \mathrm{VN}(\widetilde{\mathcal{G}})$ be an element of $\mathrm{VN}(\widetilde{\mathcal{G}})$. Then we shall write

$$I_{\widetilde{z}} \overset{\mathrm{def}}{=} Z_{\Pi_{\rho}}(\Pi_{\widetilde{z}}) \subseteq D_{\widetilde{z}} \overset{\mathrm{def}}{=} N_{\Pi_{\rho}}(\Pi_{\widetilde{z}}) \subseteq \Pi_{\rho}$$

for the *inertia*, *decomposition subgroups of* Π_{ρ} *associated to* $\widetilde{z}$, respectively [cf. [3], Definition 2.2, (i), (iii)].

(iii) Let $H \subseteq \Pi_{\rho}$ be an open subgroup of Π_{ρ}. Then the open subgroup $H \cap \Pi_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$ corresponds to a connected finite étale subcovering $\mathcal{H} \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$. Moreover, one verifies easily that if we write $I_H \subseteq I$ for the image of $H \subseteq \Pi_{\rho}$ in I, then the resulting exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{H}} \longrightarrow H \longrightarrow I_H \longrightarrow 1$$

determines an outer representation $I_H \to \mathrm{Aut}(\mathcal{H})$ of pro-Σ PSC-type. We shall refer to this resulting outer representation of pro-Σ PSC-type as the *outer representation of pro-*Σ *PSC-type determined by the open subgroup* $H \subseteq \Pi_{\rho}$ *of* Π_{ρ}.

REMARK 1.4.1. — Note that the exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_{\rho} \longrightarrow I \longrightarrow 1$$

determines an action of I on the abelianization $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ of $\Pi_{\mathcal{G}}$.

REMARK 1.4.2. — One verifies immediately from [7], Proposition 1.2, (i), that, for each $\widetilde{z} \in \mathrm{VN}(\widetilde{\mathcal{G}})$, the decomposition subgroup $D_{\widetilde{z}} \subseteq \Pi_\rho$ associated to $\widetilde{z}$ coincides with the stabilizer of $\widetilde{z} \in \mathrm{VN}(\widetilde{\mathcal{G}})$ with respect to the natural action of Π_ρ on $\mathrm{VN}(\widetilde{\mathcal{G}})$.

LEMMA 1.5. — *The following hold:*

(i) *For every $\widetilde{z} \in \mathrm{VN}(\widetilde{\mathcal{G}})$, the equality $D_{\widetilde{z}} \cap \Pi_{\mathcal{G}} = \Pi_{\widetilde{z}}$ holds.*

(ii) *For every $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the equality $I_{\widetilde{v}} \cap \Pi_{\mathcal{G}} = \{1\}$ holds.*

(iii) *For every $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the composite $I_{\widetilde{v}} \hookrightarrow \Pi_\rho \twoheadrightarrow I$ is* **injective**. *In particular, if I is* **abelian**, *then $I_{\widetilde{v}}$ is* **abelian**.

(iv) *For every $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the natural inclusions $\Pi_{\widetilde{v}}$, $I_{\widetilde{v}} \hookrightarrow D_{\widetilde{v}}$ determine an* **injection** $\Pi_{\widetilde{v}} \times I_{\widetilde{v}} \hookrightarrow D_{\widetilde{v}}$.

PROOF. — Assertion (i) follows formally from the *commensurable terminality* of $\Pi_{\widetilde{z}}$ in $\Pi_{\mathcal{G}}$ [cf. [7], Proposition 1.2, (ii)]. Assertion (ii) follows from [3], Lemma 2.3, (i). Assertions (iii), (iv) follow from assertion (ii). This completes the proof of Lemma 1.5. □

LEMMA 1.6. — *For every $\widetilde{z}_1$, $\widetilde{z}_2 \in \mathrm{VN}(\widetilde{\mathcal{G}})$, the following two conditions are equivalent:*

(1) *The equality $\widetilde{z}_1 = \widetilde{z}_2$ holds.*

(2) *The equality $D_{\widetilde{z}_1} = D_{\widetilde{z}_2}$ holds.*

PROOF. — The implication (1) ⇒ (2) is immediate. The implication (2) ⇒ (1) follows from [7], Proposition 1.2, (i) [cf. also [7], Remark 1.1.3], together with Lemma 1.5, (i). This completes the proof of Lemma 1.6. □

Next, let us recall some fundamental conditions imposed on outer representations of PSC-type [cf. [3], Definition 2.4; [6], Definition 1.3]:

DEFINITION 1.7.

(i) We shall say that ρ is *of IPSC-type* [cf. [3], Definition 2.4, (i)] [where the "IPSC" stands for "inertial pointed stable curve"] if ρ is isomorphic [cf. [3], Definition 2.1, (ii)] to the outer representation of PSC-type determined by [cf. [3], Remark 2.1.1] a pro-Σ IPSC-extension [i.e., roughly speaking, an extension that arises from a stable log curve over a log point — cf. [8], Definition 1.2, (ii)]. We shall say that ρ is *of PIPSC-type* [cf. [6], Definition 1.3] [where the "PIPSC" stands for "potentially inertial pointed stable curve"] if the following two conditions are satisfied:

(1) The profinite group I is isomorphic, as an abstract profinite group, to $\widehat{\mathbb{Z}}^\Sigma$.

(2) The restriction of ρ to some open subgroup of I is of IPSC-type.

(ii) We shall say that ρ is *of VA-type* [cf. [3], Definition 2.4, (ii)] [where the "VA" stands for "verticially admissible"] if condition (1) in (i) and the following condition are satisfied:

(3) For every $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the [necessarily injective — cf. Lemma 1.5, (iii)] composite $I_{\widetilde{v}} \hookrightarrow \Pi_\rho \twoheadrightarrow I$ is an open homomorphism.

We shall say that ρ is *of SVA-type* [cf. [3], Definition 2.4, (ii)] [where the "SVA" stands for "strictly verticially admissible"] if condition (1) in (i) and the following condition are satisfied:

(3′) For every $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the [necessarily injective — cf. Lemma 1.5, (iii)] composite $I_{\widetilde{v}} \hookrightarrow \Pi_\rho \twoheadrightarrow I$ is surjective.

(iii) We shall say that ρ is *of NN-type* [cf. [3], Definition 2.4, (iii)] [where the "NN" stands for "nodally nondegenerate"] if ρ is of VA-type, and, moreover, the following condition is satisfied:

(4) For every $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, if we write $\widetilde{v}_1, \widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ for the two distinct vertices of $\widetilde{\mathcal{G}}$ to which $\widetilde{e}$ abuts, then the natural inclusions $I_{\widetilde{v}_1}, I_{\widetilde{v}_2} \hookrightarrow I_{\widetilde{e}}$ determine an open injection $I_{\widetilde{v}_1} \times I_{\widetilde{v}_2} \hookrightarrow I_{\widetilde{e}}$.

We shall say that ρ is *of SNN-type* [cf. [3], Definition 2.4, (iii)] [where the "SNN" stands for "strictly nodally nondegenerate"] if ρ is of SVA-type and of NN-type.

LEMMA 1.8. — *The following hold:*

(i) *The following implications hold:*

$$\begin{array}{ccccc} \rho \textit{ is } \textbf{of IPSC-type} & \Longrightarrow & \rho \textit{ is } \textbf{of SNN-type} & \Longrightarrow & \rho \textit{ is } \textbf{of SVA-type} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \rho \textit{ is } \textbf{of PIPSC-type} & \Longrightarrow & \rho \textit{ is } \textbf{of NN-type} & \Longrightarrow & \rho \textit{ is } \textbf{of VA-type}. \end{array}$$

(ii) *If ρ is* **of SVA-type**, *then the three vertical implications in* (i) *are* **equivalences**.

(iii) *Suppose that I is* **isomorphic**, *as an abstract profinite group, to $\widehat{\mathbb{Z}}^\Sigma$. Let $H \subseteq \Pi_\rho$ be an open subgroup of Π_ρ. Write ρ_H for the outer representation of pro-Σ PSC-type determined by H [cf. Definition* 1.4, (iii)*]. Then it holds that ρ is* **of PIPSC-type** (*respectively,* **of VA-type**; **of NN-type**) *if and only if ρ_H is* **of PIPSC-type** (*respectively,* **of VA-type**; **of NN-type**).

PROOF. — Assertion (i) follows from [3], Remark 2.4.2, and [6], Remark 1.6.2. Next, we verify assertion (ii). Now it is immediate that the middle and right-hand vertical implications in assertion (i) are *equivalences* under the assumption that ρ is *of SVA-type*. On the other hand, it follows immediately from [4], Corollary 5.9, (i), (iii), that the left-hand vertical implication in assertion (i) is an *equivalence* under the assumption that ρ is *of SVA-type*. This completes the proof of assertion (ii). Finally, assertion (iii) follows immediately from a similar argument to the argument applied in the proof of [6], Lemma 1.5. This completes the proof of Lemma 1.8. $\square$

LEMMA 1.9. — *Suppose that ρ is* **of VA-type**. *Let $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ be a vertex of $\widetilde{\mathcal{G}}$. Then the closed subgroup $I_{\widetilde{v}} \subseteq D_{\widetilde{v}}$ of $D_{\widetilde{v}}$ coincides with the* **local center** *of $D_{\widetilde{v}}$.*

PROOF. — Let us first observe that it follows from Lemma 1.5, (i), and condition (3) of Definition 1.7, (ii), that the subgroup $\Pi_{\widetilde{v}} \times I_{\widetilde{v}} \subseteq D_{\widetilde{v}}$ of $D_{\widetilde{v}}$ [cf. Lemma 1.5, (iv)] is *open*. Thus, since $I_{\widetilde{v}}$ is *abelian* [cf. Lemma 1.5, (iii)], we conclude that $I_{\widetilde{v}}$ is *contained* in the local center of $D_{\widetilde{v}}$.

Next, let $\gamma \in D_{\widetilde{v}}$ be an element of the *local center* of $D_{\widetilde{v}}$. Thus, the element γ *centralizes* some open subgroup of $D_{\widetilde{v}}$, hence also some open subgroup of $\Pi_{\widetilde{v}}$. Now let us recall that $\Pi_{\widetilde{v}}$ is *slim* [cf. [7], Remark 1.1.3]. Thus, since $\Pi_{\widetilde{v}}$ is *normal* in $D_{\widetilde{v}}$, the element γ *centralizes* $\Pi_{\widetilde{v}}$, i.e., $\gamma \in I_{\widetilde{v}}$, as desired. This completes the proof of Lemma 1.9. □

LEMMA 1.10. — *Suppose that ρ is* **of VA-type**. *Let $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ be a vertex of $\widetilde{\mathcal{G}}$ and $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ a node of $\widetilde{\mathcal{G}}$. Then the following two conditions are equivalent:*

(1) *The node $\widetilde{e}$* **abuts** *to the vertex $\widetilde{v}$.*

(2) *The intersection $D_{\widetilde{e}} \cap D_{\widetilde{v}}$ is* **not procyclic**.

PROOF. — First, we verify the implication (1) ⇒ (2). Suppose that condition (1) is satisfied. Then it is immediate that $\Pi_{\widetilde{e}} \subseteq \Pi_{\widetilde{v}}$, which thus implies that $I_{\widetilde{v}} \subseteq I_{\widetilde{e}}$. In particular, it follows from Lemma 1.5, (ii), that $\Pi_{\widetilde{e}} \times I_{\widetilde{v}} \subseteq D_{\widetilde{e}} \cap D_{\widetilde{v}}$. Now recall that both $\Pi_{\widetilde{e}}$ and $I_{\widetilde{v}}$ are *isomorphic*, as abstract profinite groups, to $\widehat{\mathbb{Z}}^{\Sigma}$ [cf. [7], Remark 1.1.3; condition (3) of Definition 1.7, (ii)]. Thus, we conclude that condition (2) is satisfied. This completes the proof of the implication (1) ⇒ (2).

Next, we verify the implication (2) ⇒ (1). Suppose that condition (1) is *not satisfied*. Then it follows from [3], Lemma 1.7, together with Lemma 1.5, (i), that $D_{\widetilde{e}} \cap D_{\widetilde{v}} \cap \Pi_{\mathcal{G}} = \{1\}$, which thus implies that the composite $D_{\widetilde{e}} \cap D_{\widetilde{v}} \hookrightarrow \Pi_\rho \twoheadrightarrow I$ is *injective*. Thus, since I is *procyclic*, condition (2) is *not satisfied*. This completes the proof of the implication (2) ⇒ (1), hence also of Lemma 1.10. □

LEMMA 1.11. — *Suppose that ρ is* **of NN-type**. *Let $\widetilde{e}_1, \widetilde{e}_2 \in \mathrm{Node}(\widetilde{\mathcal{G}})$ be nodes of $\widetilde{\mathcal{G}}$. Then the following two conditions are equivalent:*

(1) *It holds that $\widetilde{e}_1 \neq \widetilde{e}_2$, but there exists a [uniquely determined] vertex of $\widetilde{\mathcal{G}}$ to which both $\widetilde{e}_1$ and $\widetilde{e}_2$* **abut**.

(2) *It holds that $D_{\widetilde{e}_1} \neq D_{\widetilde{e}_2}$, but $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2} \neq \{1\}$.*

PROOF. — This assertion follows from [3], Proposition 3.8, (i). □

LEMMA 1.12. — *In the situation of Lemma 1.11, suppose that the two conditions in the statement of Lemma 1.11 are satisfied. Write $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ for the unique vertex of condition (1) of Lemma 1.11. Then the following hold:*

(i) *The intersection $I_{\widetilde{v}} \cap D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}$ is* **open** *in both $I_{\widetilde{v}}$ and $D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}$. In particular, the equality $C_{\Pi_\rho}(I_{\widetilde{v}}) = C_{\Pi_\rho}(D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2})$ holds.*

(ii) *The inclusion $C_{\Pi_\rho}(I_{\widetilde{v}}) \subseteq D_{\widetilde{v}}$ holds.*

(iii) *The inclusion $D_{\widetilde{v}} \subseteq N_{\Pi_\rho}(I_{\widetilde{v}})$ holds.*

(iv) *The equality $C_{\Pi_\rho}(D_{\widetilde{e}_1} \cap D_{\widetilde{e}_2}) = D_{\widetilde{v}}$ holds.*

PROOF. — Assertion (i) follows from [3], Proposition 3.8, (ii). Assertion (ii) follows immediately from the final equivalence of [3], Remark 3.5.1. Assertion (iii) follows formally — in light of Lemma 1.8, (i) — from Lemma 1.9. Assertion (iv) follows from assertions (i), (ii), (iii). This completes the proof of Lemma 1.12. □

LEMMA 1.13. — *The following hold:*

(i) *It holds that the semi-graph obtained by forming the quotient, by the natural action of I, of the underlying semi-graph of $\mathcal{G}$ is* **untangled** *if and only if the following condition is satisfied: Let $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ be a node of $\widetilde{\mathcal{G}}$. Write $\widetilde{v}_1$, $\widetilde{v}_2 \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ for the two distinct vertices of $\widetilde{\mathcal{G}}$ to which $\widetilde{e}$* **abuts**. *Then the Π_ρ-conjugacy class of the pair $(D_{\widetilde{v}_1}, D_{\widetilde{v}_2})$ does* **not coincide** *with the Π_ρ-conjugacy class of the pair $(D_{\widetilde{v}_2}, D_{\widetilde{v}_1})$.*

(ii) *Suppose that ρ is* **of SVA-type**, *and that the underlying semi-graph of $\mathcal{G}$ is* **untangled**. *Then the semi-graph obtained by forming the quotient, by the natural action of I, of the underlying semi-graph of $\mathcal{G}$ is* **untangled**.

PROOF. — Assertion (i) follows immediately from the definition of the condition "untangled". Next, to verify assertion (ii), let us observe that since ρ is *of SVA-type*, it is immediate that the natural action of I on the underlying semi-graph of $\mathcal{G}$ is *trivial*. Thus, assertion (ii) follows. This completes the proof of Lemma 1.13. □

LEMMA 1.14. — *The following hold:*

(i) *Suppose that $\mathcal{G}$ is* **not noncuspidal** [*i.e., has a cusp — cf.* [7], *Definition* 1.1, (i)], *and that I is* **isomorphic**, *as an abstract profinite group, to $\widehat{\mathbb{Z}}^{\Sigma}$. Then, for each open subgroup $H \subseteq \Pi_\rho$ of Π_ρ and each prime number l, it holds that $H^3(H, \mathbb{F}_l) = \{0\}$.*

(ii) *Suppose that $\mathcal{G}$ is* **noncuspidal** [*i.e., has no cusp — cf.* [7], *Definition* 1.1, (i)], *and that ρ is* **of SVA-type**. *Let $l \in \Sigma$ be an element of Σ. Then it holds that $H^3(\Pi_\rho, \mathbb{F}_l) \neq \{0\}$.*

PROOF. — Let $H \subseteq \Pi_\rho$ be an open subgroup of Π_ρ. Thus, by applying the notation of Definition 1.4, (iii), we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{H}} \longrightarrow H \longrightarrow I_H \longrightarrow 1,$$

which thus gives rise to a spectral sequence

$$E_2^{i,j} = H^i(I_H, H^j(\Pi_{\mathcal{H}}, \mathbb{F}_l)) \Longrightarrow H^{i+j}(H, \mathbb{F}_l) = E^{i+j}.$$

Suppose that we are in the situation of assertion (i). Then both $\Pi_{\mathcal{H}}$ and I_H are *free pro-Σ* [cf. [7], Remark 1.1.3]. Thus, it holds that $E_2^{i,j} = \{0\}$ whenever either $i \geq 2$ or $j \geq 2$, which thus implies that $E^3 = \{0\}$, as desired. This completes the proof of assertion (i).

Next, suppose that $H = \Pi_\rho$, and that we are in the situation of assertion (ii). Then since I_H is *free pro-Σ*, and $\Pi_{\mathcal{H}}$ is *isomorphic* to the maximal pro-Σ quotient of the étale fundamental group of a *proper* hyperbolic curve over an algebraically closed field of characteristic $\notin \Sigma$ [cf. [7], Remark 1.1.3], it holds that $E_2^{i,j} = \{0\}$ whenever either $i \geq 2$ or $j \geq 3$, which thus implies that

$$E_2^{1,2} = H^1(I_H, H^2(\Pi_{\mathcal{H}}, \mathbb{F}_l)) = H^1(I_H, \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Lambda_{\mathcal{H}}, \mathbb{F}_l)) \cong H^3(H, \mathbb{F}_l) = E^3$$

[cf. [4], Definition 3.8, (i)]. Now let us recall that since [we have assumed that] ρ is *of SVA-type*, it follows immediately from [4], Corollary 3.9, (ii), that the action of I_H on $\Lambda_{\mathcal{H}}$ is *trivial*. Thus, since $\Lambda_{\mathcal{H}}$ is isomorphic, as an abstract module, to $\widehat{\mathbb{Z}}^{\Sigma}$, we conclude that $H^3(H, \mathbb{F}_l) \cong \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\widehat{\mathbb{Z}}^{\Sigma}, \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\widehat{\mathbb{Z}}^{\Sigma}, \mathbb{F}_l)) \cong \mathbb{F}_l \neq \{0\}$, as desired. This completes the proof of assertion (ii), hence also of Lemma 1.14. $\square$

2. Maximal Abelian Torsion-free Quotients

In the present §2, we maintain the notational conventions introduced at the beginning of the preceding §1. Moreover, let I be a profinite group and $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ an outer representation of pro-Σ PSC-type. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_{\rho} \longrightarrow I \longrightarrow 1.$$

In the present §2, we discuss the quotient $\Pi_{\rho}^{\text{ab-free}}$ of Π_{ρ}.

Definition 2.1.

(i) We shall write

$$\Pi_{\mathcal{G}}^{\text{ab-free}/\rho}$$

for the [uniquely determined] *maximal torsion-free quotient* of $\Pi_{\mathcal{G}}^{\text{ab}}$ whose kernel is closed in $\Pi_{\mathcal{G}}^{\text{ab}}$ and on which I acts trivially [cf. Remark 1.4.1].

(ii) We shall write

$$\Pi_{\rho}^{\underline{\text{ab/node}}}$$

for the quotient of Π_{ρ} by the kernel of the natural surjection $(\Pi_{\rho} \supseteq)\ \Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\text{ab/node}}$. Thus, this quotient and the exact sequence at the beginning of the present §2 determine an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{G}}^{\text{ab/node}} \longrightarrow \Pi_{\rho}^{\underline{\text{ab/node}}} \longrightarrow I \longrightarrow 1.$$

Lemma 2.2. — *Suppose that ρ is* **of VA-type**. *Then the exact sequence at the beginning of the present* §2 *determines an* **exact sequence** *of profinite groups*

$$1 \longrightarrow \Pi_{\mathcal{G}}^{\text{ab-free}/\rho} \longrightarrow \Pi_{\rho}^{\text{ab-free}} \longrightarrow I \longrightarrow 1.$$

Proof. — Let us first recall that I is *free pro-Σ*. In particular, the surjection $\Pi_{\rho} \twoheadrightarrow I$ has a *splitting*. Thus, Lemma 2.2 follows immediately from the fact that I is *abelian* and *torsion-free*. This completes the proof of Lemma 2.2. $\square$

Lemma 2.3. — *Suppose that ρ is* **of SVA-type**. *Then the following hold:*

(i) *For each $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, write $Q_{\widetilde{v}} \subseteq \Pi_{\rho}^{\underline{\text{ab/node}}}$ for the image of $I_{\widetilde{v}} \subseteq \Pi_{\rho}$ in the quotient $\Pi_{\rho}^{\underline{\text{ab/node}}}$. Then the closed subgroup $Q_{\widetilde{v}} \subseteq \Pi_{\rho}^{\underline{\text{ab/node}}}$ does* **not depend** *on the choice of $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$.*

(ii) *Write* $Q_{\mathrm{Vert}} \subseteq \Pi_\rho^{\mathrm{ab/node}}$ *for the closed subgroup topologically generated by the images of* $I_{\widetilde{v}} \subseteq \Pi_\rho$ *— where* $\widetilde{v}$ *ranges over the vertices of* $\widetilde{\mathcal{G}}$ *— in the quotient* $\Pi_\rho^{\mathrm{ab/node}}$*. Then the closed subgroup* $Q_{\mathrm{Vert}} \subseteq \Pi_\rho^{\mathrm{ab/node}}$ *is* **normal** *and coincides with the* **image of a splitting** *of the surjection* $\Pi_\rho^{\mathrm{ab/node}} \twoheadrightarrow I$*.*

(iii) *The profinite group* $\Pi_\rho^{\mathrm{ab/node}}$ *is* **abelian** *and* **torsion-free***.*

PROOF. — First, we verify assertion (i). Let us first observe that since the profinite semi-graph $\widetilde{\mathbb{G}}$ is *connected*, to verify assertion (i), it suffices to verify that

> for $\widetilde{v}$, $\widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, if there exists a node $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ that abuts to both $\widetilde{v}$ and $\widetilde{w}$, then $Q_{\widetilde{v}} = Q_{\widetilde{w}}$.

To this end, let us recall that, in the above situation, since [we have assumed that] ρ is *of SVA-type*, it follows from [3], Remark 2.7.1, that

$$D_{\widetilde{e}} \ = \ I_{\widetilde{v}} \times \Pi_{\widetilde{e}} \ = \ I_{\widetilde{w}} \times \Pi_{\widetilde{e}}.$$

In particular, the respective images of $D_{\widetilde{e}}$, $I_{\widetilde{v}}$, and $I_{\widetilde{w}}$ in $\Pi_\rho^{\mathrm{ab/node}}$ *coincide*, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that it is immediate that a Π_ρ-conjugate of $I_{\widetilde{v}}$ is $I_{\widetilde{w}}$ for some $\widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Thus, the closed subgroup $Q_{\mathrm{Vert}} \subseteq \Pi_\rho^{\mathrm{ab/node}}$ is *normal*. Next, since [we have assumed that] ρ is *of SVA-type*, it follows that, for each $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$, the closed subgroup $I_{\widetilde{v}} \subseteq \Pi_\rho$ coincides with the *image of a splitting* of the surjection $\Pi_\rho \twoheadrightarrow I$. Thus, it follows from assertion (i) that the closed subgroup $Q_{\mathrm{Vert}} \subseteq \Pi_\rho^{\mathrm{ab/node}}$ coincides with the *image of a splitting* of the surjection $\Pi_\rho^{\mathrm{ab/node}} \twoheadrightarrow I$. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Let us first recall that $\Pi_{\mathcal{G}}^{\mathrm{ab/node}}$ is *abelian* and *torsion-free* [cf. Remark 1.1.1]. Thus, since I is *abelian* and *torsion-free*, it follows from assertion (ii), together with the exact sequence of Definition 2.1, (ii), that $\Pi_\rho^{\mathrm{ab/node}}$ is *abelian* and *torsion-free*, as desired. This completes the proof of assertion (iii), hence also of Lemma 2.3. □

One main technical observation of the present paper is as follows:

LEMMA 2.4. — *The following hold:*

(i) *Suppose that* ρ *is* **of SVA-type***. Then the natural surjection* $\Pi_\rho \twoheadrightarrow \Pi_\rho^{\mathrm{ab/node}}$ **factors** *through the natural surjection* $\Pi_\rho \twoheadrightarrow \Pi_\rho^{\text{ab-free}}$*:*

$$\Pi_\rho \ \twoheadrightarrow \ \Pi_\rho^{\text{ab-free}} \ \twoheadrightarrow \ \Pi_\rho^{\mathrm{ab/node}}.$$

(ii) *Suppose that* ρ *is* **of IPSC-type***, and that* $\mathcal{G}$ *is* **noncuspidal***. Then the quotient* $\Pi_\rho \twoheadrightarrow \Pi_\rho^{\text{ab-free}}$ **coincides** *with the quotient* $\Pi_\rho \twoheadrightarrow \Pi_\rho^{\mathrm{ab/node}}$*:*

$$\Pi_\rho^{\text{ab-free}} \ = \ \Pi_\rho^{\mathrm{ab/node}}.$$

(iii) *Suppose that* ρ *is* **of PIPSC-type**, *and that* $\mathcal{G}$ *is* **noncuspidal**. *Then the natural surjection* $\Pi_\rho \twoheadrightarrow \Pi_\rho^{\text{ab-free}}$ **factors** *through the natural surjection* $\Pi_\rho \twoheadrightarrow \Pi_\rho^{\underline{\text{ab/node}}}$:

$$\Pi_\rho \twoheadrightarrow \Pi_\rho^{\underline{\text{ab/node}}} \twoheadrightarrow \Pi_\rho^{\text{ab-free}}.$$

PROOF. — Assertion (i) is an immediate consequence of Lemma 2.3, (iii). Next, we verify assertion (ii). Since an outer representation *of IPSC-type* is *of SVA-type* [cf. Lemma 1.8, (i)], it follows from Lemma 2.2 and assertion (i) that, to verify assertion (ii), it suffices to verify that

> the natural surjection $\Pi_\mathcal{G} \twoheadrightarrow \Pi_\mathcal{G}^{\text{ab-free}/\rho}$ *factors* through the natural surjection $\Pi_\mathcal{G} \twoheadrightarrow \Pi_\mathcal{G}^{\text{ab/node}}$.

On the other hand, this follows immediately from [7], Proposition 2.6 [i.e., essentially the "weight-monodromy conjecture for proper hyperbolic curves"]. This completes the proof of assertion (ii). Finally, assertion (iii) follows formally from assertion (ii), together with Lemma 2.2. This completes the proof of Lemma 2.4. □

LEMMA 2.5. — *Suppose that* ρ *is* **of PIPSC-type**, *and that* $\mathcal{G}$ *is* **noncuspidal**. *Let* $J \subseteq \Pi_\rho$ *be a* **nontrivial procyclic** *closed subgroup of* Π_ρ. *Then the following two conditions are equivalent:*

(1) *There exists a* [*uniquely determined — cf.* [3], *Lemma* 1.5] *node* $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ *of* $\widetilde{\mathcal{G}}$ *such that* $J \subseteq \Pi_{\widetilde{e}}$.

(2) *For every open subgroup* $H \subseteq \Pi_\rho$ *of* Π_ρ, *the image of the composite*

$$J \cap H \hookrightarrow H \twoheadrightarrow H^{\text{ab-free}}$$

is **trivial**.

PROOF. — This assertion follows immediately — in light of Lemma 1.8, (iii) — from Lemma 1.2 and Lemma 2.4, (ii), (iii). □

3. PROFINITE GROUPS OF PIPSC-TYPE

In the present §3, we maintain the notational conventions introduced at the beginning of the preceding §2. Thus, we are given an outer representation $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ of pro-Σ PSC-type and an exact sequence of profinite groups

$$1 \longrightarrow \Pi_\mathcal{G} \longrightarrow \Pi_\rho \longrightarrow I \longrightarrow 1.$$

In the present §3, we establish a "group-theoretic" algorithm for constructing, from a profinite group *of PIPSC-type* [cf. Definition 3.1 below] that is *noncuspidal* [cf. Definition 3.2 below], a certain *profinite graph* [cf. Theorem 3.13 below].

DEFINITION 3.1. — Let Π be a profinite group. Then we shall say that Π is *of* [*pro*-Σ] *PIPSC-type* if there exists an outer representation χ of pro-Σ PSC-type such that χ is of PIPSC-type [cf. Definition 1.7, (i)], and, moreover, the profinite group Π is isomorphic to the profinite group Π_χ determined by χ [cf. Definition 1.4, (i)].

Remark 3.1.1. — It follows from Lemma 1.8, (iii), that an open subgroup of a profinite group *of* [*pro*-Σ] *PIPSC-type* is *of* [*pro*-Σ] *PIPSC-type*.

Remark 3.1.2. — Let R be a strictly henselian discrete valuation ring. Write K for the field of fractions of R. Let K^{sep} be a separable closure of K and X a hyperbolic curve over K.

(i) Suppose that R is *of residue characteristic* $\notin \Sigma$, and that the action, by conjugation, of the étale fundamental group of X on the maximal pro-Σ quotient of the étale fundamental group of $X \times_K K^{\mathrm{sep}}$ *factors* through the maximal pro-Σ quotient of the étale fundamental group of X. Then one verifies easily that the maximal pro-Σ quotient of the étale fundamental group of X gives an example of a profinite group *of pro-*Σ *PIPSC-type*.

(ii) In particular, if R is *of residue characteristic zero*, then the étale fundamental group of X gives an example of a profinite group *of PIPSC-type*.

Definition 3.2. — Let Π be a profinite group of PIPSC-type. Then we shall say that Π is *noncuspidal* if there exist an open subgroup $H \subseteq \Pi$ of Π and a prime number l such that $H^3(H, \mathbb{F}_l) \neq \{0\}$.

Proposition 3.3. — *Suppose that* ρ *is* **of PIPSC-type** [*which thus implies that the profinite group* Π_ρ *is* **of PIPSC-type**]. *Then the following two conditions are equivalent:*

(1) *The profinite group* Π_ρ *of PIPSC-type is* **noncuspidal**.

(2) *The semi-graph of anabelioids* $\mathcal{G}$ *of pro-*Σ *PSC-type is* **noncuspidal**.

Proof. — This assertion follows from Lemma 1.14. $\square$

Remark 3.3.1. — It follows from Proposition 3.3, together with Remark 3.1.1, that an open subgroup of a profinite group *of PIPSC-type* that is *noncuspidal* is a profinite group *of PIPSC-type* that is *noncupidal*.

In the remainder of the present §3, suppose that

• the outer representation ρ is *of PIPSC-type* [which thus implies that the profinite group Π_ρ is *of PIPSC-type*], and that

• the semi-graph of anabelioids $\mathcal{G}$ of pro-Σ PSC-type is *noncuspidal*.

Moreover, let

$$\Pi$$

be a profinite group *of PIPSC-type* that is *noncuspidal*.

Definition 3.4.

(i) Let $J \subseteq \Pi$ be a closed subgroup of Π. Then we shall say that J is *nodal* if the following three conditions are satisfied:

(1) The closed subgroup J is nontrivial and procyclic.

(2) For every open subgroup $H \subseteq \Pi$ of Π, the image of the composite

$$J \cap H \hookrightarrow H \twoheadrightarrow H^{\text{ab-free}}$$

is trivial.

(3) If a closed subgroup $K \subseteq \Pi$ of Π satisfies conditions (1), (2) and contains J, then $J = K$.

(ii) We shall refer to a closed subgroup of Π obtained by forming the normalizer (respectively, centralizer) of a nodal closed subgroup of Π as a *nodal normalizer* (respectively, *nodal centralizer*) *subgroup* of Π.

(iii) We shall say that Π is *nonnodal* if there is no nodal closed subgroup of Π.

PROPOSITION 3.5. — *Let $J \subseteq \Pi_\rho$ be a closed subgroup of Π_ρ. Then the following hold:*

(i) *The following two conditions are equivalent:*

(i-1) *The closed subgroup J is* **nodal** *[i.e., in the sense of Definition* 3.4, (i)*]*.

(i-2) *There exists a node $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ of $\widetilde{\mathcal{G}}$ such that $J = \Pi_{\widetilde{e}}$.*

(ii) *The following two conditions are equivalent:*

(ii-1) *The closed subgroup J is a* **nodal normalizer subgroup**.

(ii-2) *There exists a node $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ of $\widetilde{\mathcal{G}}$ such that $J = D_{\widetilde{e}}$.*

(iii) *The following two conditions are equivalent:*

(iii-1) *The closed subgroup J is a* **nodal centralizer subgroup**.

(iii-2) *There exists a node $\widetilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ of $\widetilde{\mathcal{G}}$ such that $J = I_{\widetilde{e}}$.*

(iv) *The following two conditions are equivalent:*

(iv-1) *The profinite group Π_ρ of PIPSC-type is* **nonnodal**.

(iv-2) *The semi-graph of anabelioids $\mathcal{G}$ of pro-Σ PSC-type is* **nonnodal** *[i.e., has no node — cf.* [7], *Definition* 1.1, (i)*]*.

PROOF. — These assertions follow immediately — in light of Proposition 3.3 — from Lemma 2.5. □

DEFINITION 3.6.

(i) Suppose that Π is not nonnodal. Let $J \subseteq \Pi$ be a closed subgroup of Π. Then we shall say that J is a *verticial normalizer subgroup* of Π if there exist nodal normalizer subgroups D_1, $D_2 \subseteq \Pi$ of Π such that the following two conditions are satisfied:

(1) It holds that $D_1 \neq D_2$, but $D_1 \cap D_2 \neq \{1\}$.

(2) The closed subgroup J coincides with $C_\Pi(D_1 \cap D_2)$.

(ii) Suppose that Π is nonnodal. Let $J \subseteq \Pi$ be a closed subgroup of Π. Then we shall say that J is a *verticial normalizer subgroup* of Π if $J = \Pi$.

(iii) We shall refer to a closed subgroup of Π obtained by forming the local center of a verticial normalizer subgroup of Π as a *verticial centralizer subgroup* of Π.

PROPOSITION 3.7. — *Let $J \subseteq \Pi_\rho$ be a closed subgroup of Π_ρ. Then the following hold:*

(i) *The following two conditions are equivalent:*

(i-1) *The closed subgroup J is a* **verticial normalizer subgroup**.

(i-2) *There exists a vertex $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ of $\widetilde{\mathcal{G}}$ such that $J = D_{\widetilde{v}}$.*

(ii) *The following two conditions are equivalent:*

(ii-1) *The closed subgroup J is a* **verticial centralizer subgroup**.

(ii-2) *There exists a vertex $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ of $\widetilde{\mathcal{G}}$ such that $J = I_{\widetilde{v}}$.*

PROOF. — Assertion (i) follows immediately — in light of Lemma 1.8, (i), and Proposition 3.5, (ii) — from Lemma 1.12, (iv). Assertion (ii) follows immediately — in light of Lemma 1.8, (i), and assertion (i) — from Lemma 1.9. This completes the proof of Proposition 3.7. □

DEFINITION 3.8.

(i) We shall write

$$\widetilde{\mathrm{Vert}}(\Pi)$$

for the set of verticial normalizer subgroups of Π. Thus, we have an action of Π on $\widetilde{\mathrm{Vert}}(\Pi)$ by conjugation.

(ii) We shall write

$$\widetilde{\mathrm{Node}}(\Pi)$$

for the set of nodal normalizer subgroups of Π. Thus, we have an action of Π on $\widetilde{\mathrm{Node}}(\Pi)$ by conjugation.

(iii) We shall write

$$\widetilde{\mathrm{VN}}(\Pi) \stackrel{\mathrm{def}}{=} \widetilde{\mathrm{Vert}}(\Pi) \sqcup \widetilde{\mathrm{Node}}(\Pi).$$

Thus, the actions of Π on $\widetilde{\mathrm{Vert}}(\Pi)$ and $\widetilde{\mathrm{Node}}(\Pi)$ determine an action of Π on $\widetilde{\mathrm{VN}}(\Pi)$.

PROPOSITION 3.9. — *The following hold:*

(i) *The assignment "$\mathrm{Vert}(\widetilde{\mathcal{G}}) \ni \widetilde{v} \mapsto D_{\widetilde{v}}$" determines a* **$\boldsymbol{\Pi_\rho}$-equivariant bijection**

$$\mathrm{Vert}(\widetilde{\mathcal{G}}) \xrightarrow{\sim} \widetilde{\mathrm{Vert}}(\Pi_\rho).$$

(ii) *The assignment "$\mathrm{Node}(\widetilde{\mathcal{G}}) \ni \widetilde{e} \mapsto D_{\widetilde{e}}$" determines a* **$\boldsymbol{\Pi_\rho}$-equivariant bijection**

$$\mathrm{Node}(\widetilde{\mathcal{G}}) \xrightarrow{\sim} \widetilde{\mathrm{Node}}(\Pi_\rho).$$

(iii) *The assignment "$\mathrm{VN}(\widetilde{\mathcal{G}}) \ni \widetilde{z} \mapsto D_{\widetilde{z}}$" determines a* **$\boldsymbol{\Pi_\rho}$-equivariant bijection**

$$\mathrm{VN}(\widetilde{\mathcal{G}}) \xrightarrow{\sim} \widetilde{\mathrm{VN}}(\Pi_\rho).$$

PROOF. — Assertion (i) follows from Lemma 1.6 and Proposition 3.7, (i). Assertion (ii) follows from Lemma 1.6 and Proposition 3.5, (ii). Assertion (iii) follows from Lemma 1.6 and assertions (i), (ii). This completes the proof of Proposition 3.9. □

DEFINITION 3.10. — We shall say that Π is *untangled* if the following condition is satisfied: Let $N \subseteq \Pi$ be a nodal normalizer subgroup of Π and V_1, $V_2 \subseteq \Pi$ verticial normalizer subgroups of Π. Suppose that $V_1 \neq V_2$, and that neither $N \cap V_1$ nor $N \cap V_2$ is procyclic. Then the Π-conjugacy class of the pair (V_1, V_2) does not coincide with the Π-conjugacy class of the pair (V_2, V_1).

PROPOSITION 3.11. — *The following two conditions are equivalent:*

(1) *The profinite group Π_ρ of PIPSC-type is* **untangled**.

(2) *The semi-graph obtained by forming the quotient, by the natural action of I, of the underlying semi-graph of $\mathcal{G}$ is* **untangled**.

PROOF. — This assertion follows immediately — in light of Lemma 1.8, (i) — from Lemma 1.13, (i), together with Lemma 1.10 and Proposition 3.9, (i), (ii). □

DEFINITION 3.12.

(i) Let $H \subseteq \Pi$ be an open subgroup of Π. Suppose that H is untangled [cf. Remark 3.1.1; Remark 3.3.1]. Then let us define a graph

$$\mathbb{G}(H)$$

as follows:

(1) The set of vertices of $\mathbb{G}(H)$ is defined to be the set of H-conjugacy classes of verticial normalizer subgroups of H [cf. Remark 3.1.1; Remark 3.3.1].

(2) Let $N \in \widetilde{\mathrm{Node}}(H)$ be a nodal normalizer subgroup of H [cf. Remark 3.1.1; Remark 3.3.1]. Then it follows from Lemma 1.10 and Proposition 3.9, (i), (ii), that there are precisely two distinct elements V_1, $V_2 \in \widetilde{\mathrm{Vert}}(H)$ of $\widetilde{\mathrm{Vert}}(H)$ such that neither $N \cap V_1$ nor $N \cap V_2$ is procyclic. Write $e(N)$ for the set consisting of the H-conjugacy class of the pair (V_1, V_2) and the H-conjugacy class of the pair (V_2, V_1). Note that since [we have assumed that] H is untangled, it follows immediately from Lemma 1.10 and Proposition 3.9, (i), (ii), that,

(a) for each $N \in \widetilde{\mathrm{Node}}(H)$, the set $e(N)$ is of cardinality two, and,

(b) for each N_1, $N_2 \in \widetilde{\mathrm{Node}}(H)$, the following three conditions are equivalent:
• N_1 is an H-conjugate of N_2. • $e(N_1) = e(N_2)$. • $e(N_1) \cap e(N_2) \neq \emptyset$.

(3) The set of edges of $\mathbb{G}(H)$ is defined to be the set consisting of the $e(N)$'s of (2), where N ranges over the nodal normalizer subgroups of H [cf. (a), (b) of (2)]. [So it follows from (b) of (2) that the set of edges of $\mathbb{G}(H)$ is naturally identified with the set of H-conjugacy classes of nodal normalizer subgroups of H.]

(4) Let $N \in \widetilde{\mathrm{Node}}(H)$ be a nodal normalizer subgroup of H. Then the map $\zeta_{e(N)}\colon e(N) \to \mathrm{Vert}(\mathbb{G}(H))$ is defined, in the notation of (2), to be the map

$$e(N) = \{[V_1,\ V_2],\ [V_2,\ V_1]\} \longrightarrow \mathrm{Vert}(\mathbb{G}(H));\quad [V_i,\ V_j] \mapsto [V_i]$$

— where we write "$[-]$" for the H-conjugacy class of "$(-)$".

(ii) Let $H_1 \subseteq H_2 \subseteq \Pi$ be untangled open subgroups of Π. Let us define a map

$$\mathrm{Vert}(\mathbb{G}(H_1)) \longrightarrow \mathrm{Vert}(\mathbb{G}(H_2))$$
$$(\text{respectively,}\quad \mathrm{Node}(\mathbb{G}(H_1)) \longrightarrow \mathrm{Node}(\mathbb{G}(H_2)))$$

as follows: Let v (respectively, e) be a vertex (respectively, an edge) of the graph $\mathbb{G}(H_1)$. Let us take an element $V_1 \in \widetilde{\mathrm{Vert}}(H_1)$ (respectively, $E_1 \in \widetilde{\mathrm{Node}}(H_1)$) whose H_1-conjugacy class is given by v (respectively, class corresponds to e). Then it follows from [7], Proposition 1.2, (i), and Proposition 3.9, (i) (respectively, Proposition 3.9, (ii)), that there exists a unique element $V_2 \in \widetilde{\mathrm{Vert}}(H_2)$ (respectively, $E_2 \in \widetilde{\mathrm{Node}}(H_2)$) such that V_1 (respectively, E_1) is an open subgroup of V_2 (respectively, E_2). Then the image of v (respectively, e) by the map is defined to be the vertex (respectively, edge) given by (respectively, corresponding to) the H_2-conjugacy class of V_2 (respectively, E_2). [Note that one verifies easily that the H_2-conjugacy class of V_2 (respectively, E_2) does not depend on the choice of "V_1" (respectively, "E_1"), i.e., depends only on v (respectively, e).]

(iii) In the situation of (ii), it follows immediately from the various definitions involved that the maps defined in (ii) determine a morphism of graphs

$$\mathbb{G}(H_1) \longrightarrow \mathbb{G}(H_2).$$

We shall write

$$\widetilde{\mathbb{G}}(\Pi) \stackrel{\text{def}}{=} (\mathbb{G}(H))_{H\subseteq\Pi}$$

for the profinite graph consisting of the various $\mathbb{G}(H)$'s — where H ranges over the untangled open subgroups of Π. Thus, the actions of Π on $\widetilde{\mathrm{VN}}(\Pi)$ determines an action of Π on $\widetilde{\mathbb{G}}(\Pi)$.

The main result of the present paper is as follows:

THEOREM 3.13. — *Let Σ be a nonempty set of prime numbers,*

$$\mathcal{G}$$

a semi-graph of anabelioids of pro-Σ PSC-type, and $\widetilde{\mathcal{G}} \to \mathcal{G}$ a universal pro-Σ covering of $\mathcal{G}$. Write $\widetilde{\mathbb{G}}$ for the underlying profinite semi-graph of $\widetilde{\mathcal{G}}$ and $\Pi_{\mathcal{G}}$ for the fundamental group of $\mathcal{G}$ determined by $\widetilde{\mathcal{G}} \to \mathcal{G}$. Let I be a profinite group and $\rho\colon I \to \mathrm{Aut}(\mathcal{G})$ an outer representation of pro-Σ PSC-type. Suppose that ρ is **of PIPSC-type** [*which thus implies that the profinite group Π_ρ defined in Definition* 1.4, (i), *is* **of PIPSC-type**], *and that $\mathcal{G}$ is* **noncuspidal**. *Then the bijection $\mathrm{VN}(\widetilde{\mathcal{G}}) \xrightarrow{\sim} \widetilde{\mathrm{VN}}(\Pi_\rho)$ of Proposition* 3.9, (iii), *determines a* **Π_ρ-equivariant isomorphism** *of profinite graphs*

$$\widetilde{\mathbb{G}} \xrightarrow{\sim} \widetilde{\mathbb{G}}(\Pi_\rho).$$

PROOF. — This assertion follows immediately from the definition of the assignment "$\widetilde{\mathbb{G}}(-)$" [cf. also Lemma 1.3, (i), (ii); Lemma 1.8, (i); Lemma 1.13, (ii)]. $\square$

REMARK 3.13.1.

(i) The main result of the present paper, i.e., Theorem 3.13, may be summarized as follows:

> There exists a *"group-theoretic"* algorithm
>
> $$\widetilde{\mathbb{G}}\colon\ \Pi \ \mapsto\ (\Pi \curvearrowright \widetilde{\mathbb{G}}(\Pi))$$
>
> for constructing, from a profinite group Π *of PIPSC-type* that is *noncuspidal*, a profinite graph $\widetilde{\mathbb{G}}(\Pi)$ equipped with an action of Π such that if one applies this algorithm to the profinite group Π_ρ arising from the outer representation ρ of PIPSC-type as in Theorem 3.13, then there exists a natural isomorphism of $\widetilde{\mathbb{G}}$ with $\widetilde{\mathbb{G}}(\Pi_\rho)$.

(ii) Here, let us recall that if we are in a situation in which the profinite group Π_ρ is equipped with the closed subgroup $\Pi_{\mathcal{G}} \subseteq \Pi_\rho$, then a similar reconstruction result to the reconstruction result summarized in (i) was already essentially obtained by *S. Mochizuki* and the author of the present paper in [5], Theorem 1.9, (ii), without the noncuspidal assumption. That is to say, roughly speaking, we already have a *"group-theoretic"* algorithm

$$(\Pi_{\mathcal{G}} \subseteq \Pi_\rho) \ \mapsto\ (\Pi_{\mathcal{G}} \subseteq \Pi_\rho \curvearrowright \widetilde{\mathbb{G}})$$

for constructing, from the profinite group Π_ρ equipped with the closed subgroup $\Pi_{\mathcal{G}} \subseteq \Pi_\rho$, the profinite graph $\widetilde{\mathbb{G}}$ equipped with the natural action of Π_ρ. From this point of view, the main result of the present paper, i.e., Theorem 3.13, may be regarded as an *"absolute version"* of the reconstruction result of [5], Theorem 1.9, (ii), in the noncuspidal case.

(iii) In the context of (ii), it is of interest to observe that, in general, one *cannot "reconstruct"* group-theoretically, from Π_ρ, the closed subgroup $\Pi_{\mathcal{G}} \subseteq \Pi_\rho$. Indeed, suppose that $\mathrm{Node}(\widetilde{\mathcal{G}}) = \emptyset$ [which thus implies that $\mathrm{Vert}(\widetilde{\mathcal{G}})$ is *of cardinality one*], and that ρ is *of IPSC-type* [which thus implies that ρ is *trivial*]. Write $\widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ for the unique vertex of $\widetilde{\mathcal{G}}$. Then it is immediate that the equality $\Pi_\rho = \Pi_{\mathcal{G}} \times I_{\widetilde{v}}$ holds. Thus, since the abelianization of $\Pi_{\mathcal{G}}$ is a *nontrivial free* $\widehat{\mathbb{Z}}^{\Sigma}$*-module* [cf. [7], Remark 1.1.4], which thus implies that there exists a *nontrivial* homomorphism $\Pi_{\mathcal{G}} \to I_{\widetilde{v}}$, one verifies easily that there exists an automorphism of $\Pi_\rho = \Pi_{\mathcal{G}} \times I_{\widetilde{v}}$ which does *not preserve* the closed subgroup $\Pi_{\mathcal{G}} \subseteq \Pi_\rho$. In particular, in this situation, one *cannot "reconstruct"* group-theoretically, from Π_ρ, the closed subgroup $\Pi_{\mathcal{G}} \subseteq \Pi_\rho$.

REMARK 3.13.2. — In general, one *cannot "reconstruct"* group-theoretically, from Π_ρ, the profinite *semi-graph* $\widetilde{\mathbb{G}}$ equipped with the natural action of Π_ρ if one does *not work with the assumption that* $\mathcal{G}$ *is noncuspidal.* Indeed, suppose that we are in the situation of Remark 3.13.1, (iii), but *remove* the assumption that $\mathcal{G}$ is *noncuspidal.* [Then, as in the situation of Remark 3.13.1, (iii), the equality $\Pi_\rho = \Pi_{\mathcal{G}} \times I_{\widetilde{v}}$ holds.] Suppose, moreover, that Σ is *of cardinality one*, and that $\mathcal{G}$ has *at least two cusps.* Let $\widetilde{e}$ be a cusp of $\widetilde{\mathcal{G}}$. Write $\Pi_{\widetilde{e}} \subseteq \Pi_{\mathcal{G}}$ for the VCN-subgroup of $\Pi_{\mathcal{G}}$ associated to $\widetilde{e}$ [cf. [4], Definition 2.1, (i)]. Then it follows from the well-known structure of the maximal pro-Σ quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic $\notin \Sigma$ [cf. [7], Remark 1.1.3] that

• the closed subgroup $\Pi_{\widetilde{e}} \subseteq \Pi_{\mathcal{G}}$ is *isomorphic*, as an abstract profinite group, to $\widehat{\mathbb{Z}}^{\Sigma}$, and

• the composite $\Pi_{\widetilde{e}} \hookrightarrow \Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab}}$ is a *split injection*;

moreover,

• $\Pi_{\mathcal{G}}$ is *free pro*-Σ, and

• there exists a *split injection* $\widehat{\mathbb{Z}}^{\Sigma} \hookrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab}}$ such that, for every VCN-subgroup of $\Pi_{\mathcal{G}}$ associated to a cusp of $\widetilde{\mathcal{G}}$, the intersection of the image of the VCN-subgroup in $\Pi_{\mathcal{G}}^{\mathrm{ab}}$ and the image of the split injection $\widehat{\mathbb{Z}}^{\Sigma} \hookrightarrow \Pi_{\mathcal{G}}^{\mathrm{ab}}$ is *trivial*.

Thus, one verifies easily that there exists an automorphism of $\Pi_\rho = \Pi_{\mathcal{G}} \times I_{\widetilde{v}}$ that maps $\Pi_{\widetilde{e}} \subseteq \Pi_{\mathcal{G}}$ to a closed subgroup of $\Pi_{\mathcal{G}}$ *not a VCN-subgroup of* $\Pi_{\mathcal{G}}$ *associated to a cusp of* $\widetilde{\mathcal{G}}$. In particular, in this situation, one *cannot "reconstruct"* group-theoretically, from Π_ρ, the VCN-subgroups of $\Pi_{\mathcal{G}}$ associated to cusps of $\widetilde{\mathcal{G}}$ [i.e., the collection of *stabilizers of the cusps* of $\widetilde{\mathbb{G}}$ with respect to the natural action of Π_ρ on $\widetilde{\mathbb{G}}$], hence also the profinite semi-graph $\widetilde{\mathbb{G}}$ equipped with the natural action of Π_ρ.

REMARK 3.13.3.

(i) In [1], the author of the present paper has established a *"group-theoretic"* algorithm for constructing, from the geometrically pro-p étale fundamental group of a proper hyperbolic curve over a p-adic local field, the set of vertices *of positive p-rank* of the dual graph of the special fiber of the geometric stable model of the curve [cf. [1], Theorem 3.7, (viii)]. Thus, the main result of the present paper, i.e., Theorem 3.13, may be regarded as a *"PIPSC-type analogue"* of the reconstruction result of [1], Theorem 3.7, (viii).

(ii) In [9], *Y. Yang* has essentially established a *"group-theoretic"* algorithm for constructing, from the admissible fundamental group of a pointed stable curve over an algebraically closed field *of positive characteristic*, the dual semi-graph of the pointed stable curve, i.e., the underlying semi-graph of the semi-graph of anabelioids of PSC-type determined by the pointed stable curve [cf. [9], Theorem 1.2]. Thus, the main result of the present paper, i.e., Theorem 3.13, may be regarded as a *"PIPSC-type analogue"* of the reconstruction result of [9], Theorem 1.2.

4. PIPSC-PAIRS

In the present §4, we maintain the notational conventions introduced at the beginning of §2. In the present §4, we study analogues of the discussions of *mono-anabelian transport for MLF-pairs* in [2] from the point of view of the present paper.

DEFINITION 4.1.

(i) We shall refer to a collection of data

$$\Pi \curvearrowright \mathbb{H}$$

consisting of a profinite semi-graph $\mathbb{H}$, a profinite group Π, and a continuous action of Π on $\mathbb{H}$ as a *profinite-*(*group-semi-graph*)*-pair*.

(ii) Let $\Pi_\circ \curvearrowright \mathbb{H}_\circ$, $\Pi_\bullet \curvearrowright \mathbb{H}_\bullet$ be profinite-(group-semi-graph)-pairs. Then we shall refer to a pair $\alpha = (\alpha_\Pi, \alpha_\mathbb{H})$ consisting of isomorphisms $\alpha_\Pi \colon \Pi_\circ \xrightarrow{\sim} \Pi_\bullet$, $\alpha_\mathbb{H} \colon \mathbb{H}_\circ \xrightarrow{\sim} \mathbb{H}_\bullet$ compatible with the respective actions of $\Pi_\circ$, $\Pi_\bullet$ on $\mathbb{H}_\circ$, $\mathbb{H}_\bullet$ as an *isomorphism* from $\Pi_\circ \curvearrowright \mathbb{H}_\circ$ to $\Pi_\bullet \curvearrowright \mathbb{H}_\bullet$.

DEFINITION 4.2.

(i) If ρ is of PIPSC-type, then we shall refer to the profinite-(group-semi-graph)-pair

$$\Pi_\rho \curvearrowright \widetilde{\mathbb{G}}$$

as the *model* [*pro-*Σ] *PIPSC-pair* associated to ρ. Moreover, we shall say that the model PIPSC-pair associated to ρ is *noncuspidal* if $\mathcal{G}$ is noncuspidal.

(ii) We shall refer to a profinite-(group-semi-graph)-pair isomorphic [i.e., in the sense of Definition 4.1, (ii)] to the model [pro-Σ] PIPSC-pair (respectively, noncuspidal model [pro-Σ] PIPSC-pair) associated to an outer representation of pro-Σ PSC-type and of PIPSC-type as a [*pro-*Σ] *PIPSC-pair* (respectively, *noncuspidal* [*pro-*Σ] *PIPSC-pair*).

(iii) Let Π be a profinite group of [pro-Σ] PIPSC-type that is noncuspidal. Then, by Definition 3.12, (iii), one can construct, from Π, a profinite-(group-semi-graph)-pair

$$\Pi \curvearrowright \widetilde{\mathbb{G}}(\Pi).$$

Moreover, it follows from Theorem 3.13 that this profinite-(group-semi-graph)-pair is a [necessarily noncuspidal] [pro-Σ] PIPSC-pair. We shall refer to this PIPSC-pair as the *PIPSC-pair associated to* Π.

LEMMA 4.3. — *Let* $\Pi \curvearrowright \mathbb{H}$ *be a* **noncupidal PIPSC-pair**. [*So one verifies easily that the profinite group* Π *is* **of PIPSC-type** *and* **noncuspidal**.] *Then the following hold:*

(i) *The* **stabilizer** *of a vertex* (*respectively, an edge*) *of* $\mathbb{H}$ *with respect to the action of* Π *on* $\mathbb{H}$ *is a* **verticial** (*respectively,* **nodal**) **normalizer subgroup** *of* Π [*cf. Definition* 3.4, (ii)*; Definition* 3.6, (i)*,* (ii)].

(ii) *The resulting* [*cf.* (i)] *assignments*

$$\mathrm{Vert}(\mathbb{H}) \longrightarrow \widetilde{\mathrm{Vert}}(\Pi), \quad \mathrm{Node}(\mathbb{H}) \longrightarrow \widetilde{\mathrm{Node}}(\Pi)$$

determine a **Π-equivariant isomorphism**

$$\mathbb{H} \xrightarrow{\sim} \widetilde{\mathbb{G}}(\Pi).$$

PROOF. — Assertion (i) follows from Proposition 3.5, (ii), and Proposition 3.7, (i). Assertion (ii) follows immediately from Theorem 3.13. $\square$

DEFINITION 4.4. — Let $\Pi \curvearrowright \mathbb{H}$ be a noncuspidal PIPSC-pair. Then we shall write

$$\kappa(\Pi \curvearrowright \mathbb{H}) \colon (\Pi \curvearrowright \mathbb{H}) \xrightarrow{\sim} (\Pi \curvearrowright \widetilde{\mathbb{G}}(\Pi))$$

for the isomorphism of Lemma 4.3, (ii).

Remark 4.4.1. — The isomorphism "$\kappa(\Pi \curvearrowright \mathbb{H})$" of Definition 4.4 may be regarded as an analogue of the *Kummer poly-isomorphism* of [2], Definition 7.4, from the point of view of the present paper.

One important consequence of the main result of the present paper is as follows:

Theorem 4.5. — *Let* $\Pi_\circ \curvearrowright \mathbb{H}_\circ$, $\Pi_\bullet \curvearrowright \mathbb{H}_\bullet$ *be* **noncuspidal PIPSC-pairs**. *Then the natural map*

$$\mathrm{Isom}(\Pi_\circ \curvearrowright \mathbb{H}_\circ, \Pi_\bullet \curvearrowright \mathbb{H}_\bullet) \longrightarrow \mathrm{Isom}(\Pi_\circ, \Pi_\bullet)$$

is **bijective**.

Proof. — First, we verify the *surjectivity* of the map. Let $\alpha_\Pi \colon \Pi_\circ \xrightarrow{\sim} \Pi_\bullet$ be an isomorphism [i.e., an element of the codomain of the map under consideration]. Then one verifies easily from the *functoriality* of the assignment "$\widetilde{\mathbb{G}}(-)$" that α_Π induces an isomorphism of profinite-(group-semi-graph)-pairs

$$(\alpha_\Pi,\ \widetilde{\mathbb{G}}(\alpha_\Pi))\colon\ (\Pi_\circ \curvearrowright \widetilde{\mathbb{G}}(\Pi_\circ)) \ \xrightarrow{\sim}\ (\Pi_\bullet \curvearrowright \widetilde{\mathbb{G}}(\Pi_\bullet)).$$

Thus, by considering the isomorphisms $\kappa(\Pi_\circ \curvearrowright \mathbb{H}_\circ)$, $\kappa(\Pi_\bullet \curvearrowright \mathbb{H}_\bullet)$ of Definition 4.4, we obtain an element of the domain of the map under consideration whose image, via the map under consideration, *coincides* with the original isomorphism α_Π, as desired. This completes the proof of the *surjectivity*.

Next, we verify the *injectivity* of the map. Let us first observe that, to verify the *injectivity*, by considering the difference of two elements of the domain of the map under consideration whose images, via the map under consideration, coincide, it suffices to verify the following assertion:

> Let $(\alpha_\Pi, \alpha_\mathbb{H})$ be an automorphism of $\Pi_\circ \curvearrowright \mathbb{H}_\circ$. Suppose that α_Π is the *identity automorphism*. Then $\alpha_\mathbb{H}$ is the *identity automorphism*.

To this end, let us observe that it follows from the *functoriality* of the isomorphism "$\kappa(\Pi \curvearrowright \mathbb{H})$" of Definition 4.4 that the isomorphism $\kappa(\Pi_\circ \curvearrowright \mathbb{H}_\circ)$ is *compatible* with the automorphism $(\alpha_\Pi, \alpha_\mathbb{H})$, i.e., the diagram of isomorphisms of profinite-(group-semi-graph)-pairs

$$\begin{array}{ccc}
(\Pi_\circ \curvearrowright \mathbb{H}_\circ) & \xrightarrow{\kappa(\Pi_\circ \curvearrowright \mathbb{H}_\circ)} & (\Pi_\circ \curvearrowright \widetilde{\mathbb{G}}(\Pi_\circ)) \\
{\scriptstyle (\alpha_\Pi,\ \alpha_\mathbb{H})}\downarrow & & \downarrow{\scriptstyle (\alpha_\Pi,\ \widetilde{\mathbb{G}}(\alpha_\Pi))} \\
(\Pi_\circ \curvearrowright \mathbb{H}_\circ) & \xrightarrow[\kappa(\Pi_\circ \curvearrowright \mathbb{H}_\circ)]{} & (\Pi_\circ \curvearrowright \widetilde{\mathbb{G}}(\Pi_\circ))
\end{array}$$

commutes. Next, let us observe that since [we have assumed that] α_Π is the *identity automorphism*, the right-hand vertical arrow of this diagram is the *identity automorphism*. Thus, we conclude that $\alpha_\mathbb{H}$ is the *identity automorphism*, as desired. This completes the proof of the *injectivity*, hence also of Theorem 4.5. ☐

Remark 4.5.1. — The bijectivity of the map of Theorem 4.5 may be regarded as an analogue of the bijectivity of the map of [2], Theorem 7.6, (iv), from the point of view of the present paper.

REMARK 4.5.2. — A similar remark to [2], Remark 7.6.1 [i.e., concerning the technique of *mono-anabelian transport*], applies to the situation discussed in Theorem 4.5, as well. We leave the routine details of translating the remark into the language of the situation of Theorem 4.5 to the interested reader.

REFERENCES

[1] Y. Hoshi, On the pro-p absolute anabelian geometry of proper hyperbolic curves, *J. Math. Sci. Univ. Tokyo* **25** (2018), no. **1**, 1–34.
[2] Y. Hoshi, *Introduction to mono-anabelian geometry*, RIMS Preprint **1868** (January 2017).
[3] Y. Hoshi and S. Mochizuki, On the combinatorial anabelian geometry of nodally nondegenerate outer representations, *Hiroshima Math. J.* **41** (2011), no. **3**, 275–342.
[4] Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: inertia groups and profinite Dehn twists, *Galois-Teichmüller theory and arithmetic geometry*, 659–811, Adv. Stud. Pure Math., **63**, Math. Soc. Japan, Tokyo, 2012.
[5] Y. Hoshi and S. Mochizuki, *Topics surrounding the combinatorial anabelian geometry of hyperbolic curves II: tripods and combinatorial cuspidalization*, RIMS Preprint **1762** (November 2012).
[6] Y. Hoshi and S. Mochizuki, *Topics surrounding the combinatorial anabelian geometry of hyperbolic curves III: tripods and tempered fundamental groups*, RIMS Preprint **1763** (November 2012).
[7] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* (**2**) **59** (2007), no. **3**, 455–479.
[8] S. Mochizuki, Topics in absolute anabelian geometry II: decomposition groups and endomorphisms, *J. Math. Sci. Univ. Tokyo* **20** (2013), no. **2**, 171–269.
[9] Y. Yang, On the admissible fundamental groups of curves over algebraically closed fields of characteristic $p > 0$, *Publ. Res. Inst. Math. Sci.* **54** (2018), no. **3**, 649–678.

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