Homotopy Sequences for Varieties over Curves

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Abstract. — In the present paper, we discuss the homotopy sequences for varieties over curves. We prove that, for instance, for a morphism from a normal variety to a certain smooth curve over a field of characteristic zero, it holds that the induced outer homomorphism between the étale fundamental groups satisfies some group-theoretic conditions if and only if the homotopy sequence for the morphism of varieties is exact. Moreover, we also give a refinement of a known result concerning the Grothendieck conjecture for hyperbolic polycurves of dimension two by means of our study of homotopy sequences.

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Introduction

The exactness of the homotopy sequences for certain *proper* morphisms was established in [16], Exposé X, Corollaire 1.4. Moreover, *K. Mitsui* [cf. [11], Theorem 1.1; [11], Theorem 4.22] and the author of the present paper [cf. [7], Theorem 2; [8], Proposition 1.10, (i)] have established the exactness of the homotopy sequences for certain *not necessarily proper* morphisms. [Note that, in [7], Theorem 2, the morphism "$f^{\log}$" of log schemes is assumed to be *proper*; on the other hand, by applying the *log purity theorem* of [7], Proposition B.7 — that gives a natural outer isomorphism between the log fundamental group of a suitable log scheme and the étale fundamental group of the interior of the log scheme — one may derive, from [7], Theorem 2, the exactness of the homotopy sequences for certain *not necessarily proper* morphisms of schemes — i.e., as opposed to log schemes — in suitable situations.] In the present paper, we discuss a *"group-theoretic" approach* to the study of homotopy sequences.

Here, let us observe that each of the above results concerning the exactness of homotopy sequences gives us a statement of the following form:

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> If a morphism of schemes satisfies suitable *scheme-theoretic conditions*, then the homotopy sequence for the morphism of schemes is *exact*.

On the other hand, one main purpose of the study of the present paper is to give a statement of the following form:

> For a morphism of schemes, if the induced outer homomorphism between the étale fundamental groups satisfies suitable *group-theoretic conditions*, then the homotopy sequence for the morphism of schemes is *exact*.

Note that a requirement for such a statement arises naturally in, for instance, an attempt to solve the *Grothendieck conjecture for hyperbolic polycurves* [cf., e.g., the various arguments of [8], [14]; also the arguments of §3 of the present paper].

Here, let us explain a property

$$\mathbb{P}_{\nexists\twoheadrightarrow\infty}$$

of a profinite group [cf. Definition 2.5, (i)] that plays an important role in the present paper. We shall say that a profinite group G has the property $\mathbb{P}_{\nexists\twoheadrightarrow\infty}$ if, for an arbitrary open subgroup $H \subseteq G$ of G, there exists a prime number l_H, depending on H, such that H has no quotient which is free pro-l_H and not topologically finitely generated. One verifies easily that if a profinite group G has one of the following three properties, then G has the property $\mathbb{P}_{\nexists\twoheadrightarrow\infty}$ [cf. Remark 2.5.1]:

- The profinite group G is *topologically finitely generated.*
- An open subgroup of the profinite group G is *abelian.*
- An open subgroup of the profinite group G is *pro*-Σ for some set Σ of prime numbers *not equal* to the set of all prime numbers.

The first main result — in the case where the base field is *of characteristic zero* — is as follows [cf. Theorem 2.8, also Remark 2.8.1]. In the present paper, if X is a connected locally noetherian scheme, then we shall write

$$\Pi_X$$

for the étale fundamental group of X [for some choice of basepoint] [cf. Definition 1.1, (i)]; moreover, if $f\colon X \to Y$ is a morphism of connected locally noetherian schemes, then we shall write

$$\Delta_f \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_X \to \Pi_Y)$$

for the kernel of the outer homomorphism $\Pi_X \to \Pi_Y$ induced by f [cf. Definition 1.1, (ii)].

THEOREM A. — *Let k be a field* **of characteristic zero***, S a* **smooth curve** *over k [cf. Definition 2.2, (ii)] whose base-change to an algebraic closure of k is* **isomorphic neither** *to the projective line* **nor** *to the affine line, X a* **normal variety** *over k [cf. Definition 2.2, (i)], $f\colon X \to S$ a morphism over k, and $\overline{\eta} \to S$ a geometric generic point of S. Then the following four conditions are* **equivalent***:*

(1) *The outer homomorphism $\Pi_X \twoheadrightarrow \Pi_S$ induced by f is* **surjective***. Moreover, the profinite group Δ_f is* **topologically finitely generated***.*

(2) *The outer homomorphism $\Pi_X \twoheadrightarrow \Pi_S$ induced by f is* **surjective***. Moreover, the profinite group Δ_f* **has** *the property $\mathbb{P}_{\nexists\twoheadrightarrow\infty}$ [cf. Definition 2.5, (i)].*

(3) *The morphism f is* **surjective** *and* **generically geometrically irreducible**. *Moreover, the sequence of irreducible [hence also connected] schemes*

$$X \times_S \overline{\eta} \xrightarrow{\mathrm{pr}_1} X \xrightarrow{f} S$$

induces an **exact** *sequence of profinite groups and outer homomorphisms*

$$\Pi_{X \times_S \overline{\eta}} \longrightarrow \Pi_X \longrightarrow \Pi_S \longrightarrow 1.$$

(4) *The morphism f is* **dominant** *and* **generically geometrically irreducible**. *Moreover, the sequence of irreducible [hence also connected] schemes*

$$X \times_S \overline{\eta} \xrightarrow{\mathrm{pr}_1} X \xrightarrow{f} S$$

induces an **exact** *sequence of profinite groups and outer homomorphisms*

$$\Pi_{X \times_S \overline{\eta}} \longrightarrow \Pi_X \longrightarrow \Pi_S \longrightarrow 1.$$

In §3 of the present paper, we apply [some considerations that lead us to] Theorem A to the study of the *Grothendieck conjecture for hyperbolic polycurves*. The second main result of the present paper is as follows [cf. Theorem 3.3].

THEOREM B. — *Let p be a prime number, k a* **sub-p-adic** *field [cf., e.g., [8], Definition 3.1], $\overline{k}$ an algebraic closure of k, X a* **hyperbolic polycurve of dimension 2** *over k [cf. [8], Definition 2.1, (ii)], and Z a* **normal variety** *over k [cf. Definition 2.2, (i)]. Write $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$. Let*

$$\alpha \colon \Pi_Z \longrightarrow \Pi_X$$

be an **open** *homomorphism of profinite groups over G_k [i.e., an open homomorphism $\alpha \colon \Pi_Z \to \Pi_X$ of profinite groups such that the induced outer homomorphism $\Pi_Z \to \Pi_X$ fits into a commutative diagram of profinite groups and outer homomorphisms*

$$\begin{tikzcd} \Pi_Z \arrow[rr] \arrow[dr] & & \Pi_X \arrow[dl] \\ & G_k & \end{tikzcd}$$

— where the left-hand, right-hand arrows to G_k are the outer homomorphisms induced by the structure morphisms of the normal varieties Z, X over k, respectively]. Suppose that the kernel of α **has** *the property $\mathbb{P}_{\nexists \twoheadrightarrow \infty}$ [cf. Definition 2.5, (i)]. Then there exists a* **uniquely determined dominant** *morphism*

$$Z \longrightarrow X$$

over k from which α **arises**. *In particular, the normal variety Z is* **of dimension $\geq$ 2**.

Note that Theorem B may be regarded as a *refinement* of [8], Theorem A, in the case where condition (2) is satisfied [cf. Remark 3.3.1]. Finally, in §3, we observe that if one drops one of

- the assumption that the hyperbolic polycurve X is *of dimension* 2 and

- the assumption that the kernel of α *has the property* $\mathbb{P}_{\nexists \twoheadrightarrow \infty}$

in the statement of Theorem B, then the conclusion of Theorem B no longer holds in general [cf. Remark 3.3.2; Remark 3.3.4, (i)].

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1. Some Generalities on Homotopy Sequences

In the present §1, we discuss some generalities on homotopy sequences [cf. Lemma 1.3 and Lemma 1.5 below].

Definition 1.1.

(i) Let X be a connected locally noetherian scheme. Then we shall write

$$\Pi_X$$

for the étale fundamental group of X [for some choice of basepoint].

(ii) Let X, Y be connected locally noetherian schemes; $f\colon X \to Y$ a morphism of schemes. Then we shall write

$$\Delta_f = \Delta_{X/Y} \stackrel{\text{def}}{=} \operatorname{Ker}(\Pi_X \to \Pi_Y) \subseteq \Pi_X$$

for the kernel of the outer homomorphism $\Pi_X \to \Pi_Y$ induced by f.

Definition 1.2. — We shall say that a scheme is *of CLN-type* if the scheme is connected, locally noetherian, and normal.

Lemma 1.3. — *Let X, S be schemes* **of CLN-type***; $f\colon X \to S$ a morphism of schemes which is* **of finite type***,* **separated***, and* **dominant***; $\overline{\eta} \to S$ a geometric generic point of S. Suppose that the following two conditions are satisfied:*

(a) *The outer homomorphism $\Pi_X \to \Pi_S$ induced by f is* **surjective**.

(b) *The geometric generic fiber $X \times_S \overline{\eta}$ of f is* **reduced** [*cf. Remark* 1.3.1 *below*].

Consider the following four conditions:

(1) *Let $Y \to X$ be a connected finite étale covering of X. Then the normalization of S in Y is* **étale** *over S.*

(2) *Let $Y \to X$ be a connected finite étale covering of X. If the base-change $Y \times_S \overline{\eta} \to X \times_S \overline{\eta}$ of $Y \to X$ by $\overline{\eta} \to S$ has a* **splitting***, then the normalization of S in Y is* **étale** *over S.*

(3) *The morphism f is* **generically geometrically irreducible**. *Moreover, for a connected finite étale covering $Y \to X$ of X, if the base-change $Y \times_S \overline{\eta} \to X \times_S \overline{\eta}$ of $Y \to X$ by $\overline{\eta} \to S$ has a* **splitting**, *then there exists a connected finite étale covering of S whose pull-back by $X \to S$ is* **isomorphic** *to Y over X.*

(4) *The morphism f is* **generically geometrically irreducible**. *Moreover, the sequence of irreducible [hence also connected] schemes*

$$X \times_S \overline{\eta} \xrightarrow{\mathrm{pr}_1} X \xrightarrow{f} S$$

induces an **exact** *sequence of profinite groups and outer homomorphisms*

$$\Pi_{X \times_S \overline{\eta}} \longrightarrow \Pi_X \longrightarrow \Pi_S \longrightarrow 1.$$

Then the implications

$$(1) \Longrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4)$$

hold.

Proof. — The implication

$$(1) \Longrightarrow (2)$$

is immediate. Next, we verify the implication

$$(2) \Longrightarrow (3).$$

To this end, suppose that condition (2) is satisfied, and take a connected finite étale covering $Y \to X$ of X whose base-change $Y \times_S \overline{\eta} \to X \times_S \overline{\eta}$ by $\overline{\eta} \to S$ has a *splitting*. Now let us write

$$T \longrightarrow S$$

for the normalization of S in Y. Let us first verify the following claim:

> Claim 1.3.A: The normalization $T \to S$ of S in Y is a *connected finite étale covering* of S.

To this end, let us observe that it is immediate that T is *connected*. Next, let us observe that it follows from [2], Proposition 4.6.1, together with the condition (b), that the function field of X, hence also of Y, is *separable* over the function field of S. Thus, it follows from [9], §33, Lemma 1, that the morphism $T \to S$ is *finite*. In particular, since [it follows from condition (2) that] $T \to S$ is *étale*, we conclude that Claim 1.3.A holds, as desired. This completes the proof of Claim 1.3.A.

Next, we verify the following claim:

> Claim 1.3.B: The morphism f is *generically geometrically irreducible*.

To this end, let us observe that it follows from Claim 1.3.A [i.e., in the case where we take "Y" of Claim 1.3.A to be X] that the normalization of S in X is a *connected finite étale covering* of S. Thus, since the outer homomorphism $\Pi_X \twoheadrightarrow \Pi_S$ induced by f is *surjective* [cf. the condition (a)], we conclude that the normalization of S in X *coincides* with S, which thus implies that the function field of S is *algebraically closed* in the function field of X. In particular, it follows from [2], Proposition 4.5.9, and [3], Proposition 9.7.8, that f is *generically geometrically irreducible*, as desired. This completes the proof of Claim 1.3.B.

Next, we verify the following claim:

> Claim 1.3.C: The fiber product $X \times_S T$ is *of CLN-type*.

To this end, let us observe that one verifies easily from Claim 1.3.A that $X \times_S T$ is *locally noetherian* and *normal*. Next, it follows from the *surjectivity* of the outer homomorphism $\Pi_X \to \Pi_S$ induced by f [cf. the condition (a)], together with Claim 1.3.A, that $X \times_S T$ is *connected*. This completes the proof of Claim 1.3.C.

Next, we verify the following claim:

> Claim 1.3.D: The natural morphism $Y \to X \times_S T$ is a *connected finite étale covering* [hence *closed* and *open*; thus, this morphism $Y \to X \times_S T$ is *surjective* — cf. Claim 1.3.C].

To this end, let us first observe that since both Y and $X \times_S T$ are *finite* over X [cf. Claim 1.3.A], it follows that the morphism $Y \to X \times_S T$ is *finite*. In particular, since [one verifies easily that] both $Y \to X$ and $X \times_S T \to X$ are *surjective*, by considering the fibers of $Y \to X \times_S T \to X$ at the generic point of X [cf. also Claim 1.3.C], we conclude that the morphism $Y \to X \times_S T$ is *dominant*. On the other hand, since $Y \to X$ is *unramified*, it follows from [4], Proposition 17.3.3, (v), that the morphism $Y \to X \times_S T$ is *unramified*. Thus, since $X \times_S T$ is *normal* [cf. Claim 1.3.C], it follows from [16], Exposé I, Corollaire 9.11, that the morphism $Y \to X \times_S T$ is *étale*, as desired. This completes the proof of Claim 1.3.D.

Finally, we verify the following claim:

> Claim 1.3.E: The natural morphism $Y \to X \times_S T$ is an *isomorphism*.

To this end, write d for the degree of the connected finite covering $T \to S$ [cf. Claim 1.3.A]. Then since $X \times_S \overline{\eta}$ is *connected* [cf. Claim 1.3.B], one verifies easily that the number of the connected components of $(X \times_S T) \times_S \overline{\eta}$ is *equal* to d. Moreover, since the geometric fiber of $Y \to T$ at an arbitrary geometric generic point of T is *connected* [cf. [2], Proposition 4.5.9], the number of the connected components of $Y \times_S \overline{\eta}$ is *equal* to d. Thus, since $Y \to X \times_S T$ is *surjective* [cf. Claim 1.3.D], this morphism $Y \to X \times_S T$ determines a *bijection* between the set of connected components of $Y \times_S \overline{\eta}$ and the set of connected components of $(X \times_S T) \times_S \overline{\eta}$. On the other hand, let us recall that we have assumed that the base-change $Y \times_S \overline{\eta} \to X \times_S \overline{\eta}$ of $Y \to X$ by $\overline{\eta} \to S$ has a *splitting*. Thus, by considering the connected component of $Y \times_S \overline{\eta}$ obtained by forming the image of a splitting of $Y \times_S \overline{\eta} \to X \times_S \overline{\eta}$, we conclude that the connected finite étale covering $Y \to X \times_S T$ [cf. Claim 1.3.D] is *of degree one*, i.e., an *isomorphism*, as desired. This completes the proof of Claim 1.3.E, hence also of the implication (2) $\Rightarrow$ (3) [cf. also Claim 1.3.A].

Next, we verify the implication

$$(3) \Longrightarrow (2).$$

To this end, suppose that condition (3) is satisfied, take a connected finite étale covering $Y \to X$ of X whose base-change $Y \times_S \overline{\eta} \to X \times_S \overline{\eta}$ by $\overline{\eta} \to S$ has a *splitting*, and write

$T \to S$ for the normalization of S in Y. Then it follows from condition (3) that there exist a connected finite étale covering $S_Y \to S$ and an isomorphism $Y \xrightarrow{\sim} X \times_S S_Y$ over X. Now let us observe that it follows from [2], Proposition 4.6.1, together with the condition (b), that the function field of X, hence also of Y, is *separable* over the function field of S. Thus, since [one verifies easily that] S_Y is *normal*, and [we have assumed that] the morphism $X \to S$, hence also $Y \to S_Y$, is *generically geometrically irreducible*, it follows from [2], Proposition 4.5.9, that the morphism $S_Y \to S$ *factors* through a *birational* morphism $S_Y \to T$ over S. In particular, since T is *normal*, and the morphism $S_Y \to S$, hence also $S_Y \to T$, is *finite*, it follows from [1], Corollaire 4.4.9, that the *birational finite* morphism $S_Y \to T$ is an *isomorphism*, as desired. This completes the proof of the implication (3) $\Rightarrow$ (2).

Finally, we verify the equivalence

$$(3) \Longleftrightarrow (4).$$

To this end, suppose that the morphism f is *generically geometrically irreducible*. Now let us observe that since the composite $X \times_S \overline{\eta} \xrightarrow{\mathrm{pr}_1} X \xrightarrow{f} S$ *factors* through the geometric point $\overline{\eta} \to S$, the image of the composite $\Pi_{X \times_S \overline{\eta}} \to \Pi_X \to \Pi_S$ is *trivial*. Moreover, we have assumed that the outer homomorphism $\Pi_X \to \Pi_S$ induced by f is *surjective* [cf. the condition (a)]. Thus, we conclude that condition (4) is *equivalent* to the inclusion

$$\Delta_f \subseteq \mathrm{Im}(\Pi_{X \times_S \overline{\eta}} \to \Pi_X).$$

On the other hand, it is immediate that this inclusion is *equivalent* to condition (3). This completes the proof of the equivalence (3) $\Leftrightarrow$ (4), hence also of Lemma 1.3. $\square$

Remark 1.3.1. — In the situation of Lemma 1.3, suppose that the function field of S is *of characteristic zero*. Then since [we have assumed that] X is *normal*, it follows from [2], Proposition 4.6.1, that the condition (b) is always satisfied.

Definition 1.4. — Let $\mathbb{P}$ be a property of a profinite group [e.g., "topologically finitely generated", "abelian", or "pro-Σ for some set Σ of prime numbers"].

(i) We shall say that the property $\mathbb{P}$ is *stable under taking open subquotients* if the following condition is satisfied: If a profinite group G has the property $\mathbb{P}$, then every quotient of an arbitrary open subgroup of G [i.e., by a normal closed subgroup of the open subgroup] has the property $\mathbb{P}$.

(ii) Let X be a scheme of CLN-type. Then we shall say that X *satisfies the condition* $(*_{\mathbb{P}})$ if the following condition is satisfied: For a scheme Y of CLN-type and a finite and generically étale surjective morphism $f \colon Y \to X$, if the profinite group Δ_f has the property $\mathbb{P}$, then f is étale.

Lemma 1.5. — *Let X, S be schemes* **of CLN-type**; *$f \colon X \to S$ a morphism of schemes which is* **of finite type**, **separated**, *and* **dominant**; *$\overline{\eta} \to S$ a geometric generic point of S. Suppose that the following two conditions are satisfied:*

(a) *The outer homomorphism* $\Pi_X \to \Pi_S$ *induced by* f *is* **surjective**.

(b) *The geometric generic fiber* $X \times_S \overline{\eta}$ *of* f *is* **reduced** [*cf. Remark* 1.3.1].

Suppose, moreover, that there exists a property $\mathbb{P}$ *of a profinite group such that*

(1) *the property* $\mathbb{P}$ *is* **stable under taking open subquotients**,

(2) *the kernel* Δ_f **has** *the property* $\mathbb{P}$, *and, moreover,*

(3) *the scheme* S *of CLN-type* **satisfies** *the condition* $(*_{\mathbb{P}})$.

Then the following assertions hold:

(i) *Let* $Y \to X$ *be a connected finite étale covering of* X. *Then the normalization of* S *in* Y *is* **étale** *over* S.

(ii) *The morphism* f *is* **generically geometrically irreducible**. *Moreover, the sequence of irreducible* [*hence also connected*] *schemes*

$$X \times_S \overline{\eta} \xrightarrow{\mathrm{pr}_1} X \xrightarrow{f} S$$

induces an **exact** *sequence of profinite groups and outer homomorphisms*

$$\Pi_{X \times_S \overline{\eta}} \longrightarrow \Pi_X \longrightarrow \Pi_S \longrightarrow 1.$$

PROOF. — First, we verify assertion (i). Write $T \to S$ for the normalization of S in Y. First, let us verify the following claim:

> Claim 1.5.A: The morphism $T \to S$ is *finite*, *generically étale*, and *surjective*.

To this end, let us observe that it follows from [2], Proposition 4.6.1, together with the condition (b), that the function field of X, hence also of Y, is *separable* over the function field of S. Thus, the morphism $T \to S$ is *finite* [cf. [9], §33, Lemma 1] and *generically étale*, which thus implies that f is *closed* and *dominant*. In particular, the morphism $T \to S$ is *surjective*, as desired. This completes the proof of Claim 1.5.A.

Next, let us observe that we have a commutative diagram of profinite groups

$$\begin{array}{ccccccc}
1 & \longrightarrow & \Delta_{Y/S} & \longrightarrow & \Pi_Y & \longrightarrow & \Pi_S \\
 & & \downarrow & & \downarrow & & \| \\
1 & \longrightarrow & \Delta_{T/S} & \longrightarrow & \Pi_T & \longrightarrow & \Pi_S
\end{array}$$

— where the horizontal sequences are *exact*. It follows immediately from [similar arguments to the arguments applied in the proofs of] [8], Lemma 1.6, and [8], Lemma 1.9, that the middle vertical arrow $\Pi_Y \to \Pi_T$ of this diagram, hence also the left-hand vertical arrow $\Delta_{Y/S} \to \Delta_{T/S}$ of this diagram, is *surjective*. In particular, since [it follows from conditions (1), (2) that] the profinite group $\Delta_{Y/S}$ has the property $\mathbb{P}$, it follows from condition (1) that the profinite group $\Delta_{T/S}$ has the property $\mathbb{P}$. Thus, it follows from Claim 1.5.A and condition (3) that the morphism $T \to S$ is *étale*, as desired. This completes the proof of assertion (i).

Assertion (ii) follows from the implication (1) $\Rightarrow$ (4) of Lemma 1.3, together with assertion (i). This completes the proof of Lemma 1.5. $\square$

2. Homotopy Sequences for Varieties over Curves

In the present §2, we obtain an application of the results of §1 concerning the homotopy sequences for normal varieties over smooth curves [cf. Theorem 2.8 below].

Lemma 2.1. — *Let k be an algebraically closed field, X a* **hyperbolic curve** *over k [cf., e.g., [8], Definition 2.1, (i)], and l a prime number* **invertible** *in k. Then an arbitrary* **nontrivial normal** *closed subgroup of the maximal pro-l quotient of Π_X* **of infinite index** *is* **free pro-l** *and* **not topologically finitely generated**.

Proof. — Let N be a *nontrivial normal* closed subgroup of the maximal pro-l quotient of Π_X *of infinite index*. Let us first recall from [12], Proposition 2.3, (i), that N is *not topologically finitely generated*. Next, let us observe that since N is *of infinite index*, and the maximal pro-l quotient of Π_X is *topologically finitely generated* [cf. [12], Proposition 2.2], there exists a sequence of *open* subgroups of the maximal pro-l quotient of Π_X

$$\cdots \subseteq H_{n+1} \subseteq H_n \subseteq \cdots \subseteq H_2 \subseteq H_1$$

such that $H_i \neq H_{i+1}$ for an *arbitrary* positive integer i, and, moreover, the equality $\bigcap_i H_i = N$ holds, which thus implies [cf. [13], Proposition 6.5.5] that the natural homomorphism

$$\varinjlim_i H^2(H_i, \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^2(N, \mathbb{Z}/l\mathbb{Z})$$

— where the module $\mathbb{Z}/l\mathbb{Z}$ is regarded as being equipped with the *trivial* actions of N and the H_i's — is an *isomorphism*. In particular, to verify that N is *free pro-l*, it follows from [13], Theorem 7.7.4, that it suffices to verify that the image of the natural homomorphism

$$H^2(H_n, \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^2(H_{n+1}, \mathbb{Z}/l\mathbb{Z})$$

is *zero* for an arbitrary positive integer n. To this end, write $X_{n+1} \to X_n \to X$ for the connected finite étale coverings of X that correspond to the open subgroups $H_{n+1} \subseteq H_n$ of the maximal pro-l quotient of Π_X, respectively. Now let us recall that the homomorphism under consideration fits into a commutative diagram

$$\begin{tikzcd} H^2(H_n, \mathbb{Z}/l\mathbb{Z}) \arrow[r] \arrow[d, "\wr"'] & H^2(H_{n+1}, \mathbb{Z}/l\mathbb{Z}) \arrow[d, "\wr"] \\ H^2_{\acute{e}t}(X_n, \mathbb{Z}/l\mathbb{Z}) \arrow[r] & H^2_{\acute{e}t}(X_{n+1}, \mathbb{Z}/l\mathbb{Z}) \end{tikzcd}$$

— where the vertical arrows are *isomorphisms* [cf., e.g., [6], Lemma 4.2, (iii)], and the lower horizontal arrow is the homomorphism induced by the connected finite étale covering $X_{n+1} \to X_n$. Thus, since the module $H^2(H_n, \mathbb{Z}/l\mathbb{Z})$ is *annihilated* by l, and [it is well-know — cf., e.g., the first paragraph of the proof of [10], Chapter V, Theorem 2.1, (a), if X is not affine; [10], Chapter VI, Theorem 7.2, if X is affine — that] the lower horizontal arrow of this diagram *factors* through the endomorphism of $H^2_{\acute{e}t}(X_n, \mathbb{Z}/l\mathbb{Z})$ given by multiplication by the degree of the covering $X_{n+1} \to X_n$ — i.e., the index $[H_n : H_{n+1}]$ ($\neq 1$) — we conclude that the image of the upper horizontal arrow of the above diagram is *zero*, as desired. This completes the proof of Lemma 2.1. □

DEFINITION 2.2. — Let k be a field and X a scheme over k.

(i) We shall say that X is a *normal variety* over k if X is of finite type, separated, geometrically connected, and geometrically normal over k.

(ii) We shall say that X is a *smooth curve* over k if there exist a normal variety X^{cpt} over k which is proper over k and of dimention one, a [possible empty] closed subscheme $D \subseteq X^{\mathrm{cpt}}$ of X^{cpt} which is étale over k, and an isomorphism $X \xrightarrow{\sim} X^{\mathrm{cpt}} \setminus D$ over k.

REMARK 2.2.1. — One verifies easily that a *smooth curve* over a field is *smooth* and a *normal variety* over the field.

LEMMA 2.3. — *Let k be an algebraically closed field, X a* **smooth curve** *over k, and l a prime number* **invertible** *in k. Suppose that X is* **isomorphic neither** *to the projective line over k* **nor** *to the affine line over k. Then the maximal pro-l quotient of Π_X, hence also Π_X itself, is* **infinite**.

PROOF. — This assertion follows from, for instance, the isomorphism (1-3) and the exact sequence (1-5) of [15], Remark 1.3. □

LEMMA 2.4. — *Let k be an algebraically closed field, X and Y* **smooth curves** *over k, l a prime number* **invertible** *in k, and $f\colon X \to Y$ a* **dominant** *and* **generically étale** *morphism over k. Suppose that the following two conditions are satisfied:*

(1) *The smooth curve Y is* **isomorphic neither** *to the projective line over k* **nor** *to the affine line over k.*

(2) *The outer homomorphism from the maximal pro-l quotient of Π_X to the maximal pro-l quotient of Π_Y induced by f is an outer* **isomorphism**.

Then f is a **finite étale covering**.

PROOF. — Let us write X^{cpt} (respectively, Y^{cpt}) for the smooth compactification of X (respectively, Y), g_X (respectively, g_Y) for the genus of X^{cpt} (respectively, Y^{cpt}), and r_X (respectively, r_Y) for the cardinality of the complement in X^{cpt} (respectively, Y^{cpt}) of X (respectively, Y). Then since f is *dominant*, it follows from elementary theory of algebraic curves that

$$g_X \geq g_Y \text{ and } r_X \geq r_Y.$$

Thus, it follows immediately from conditions (1), (2), together with [15], Proposition 1.1 [cf. also the isomorphism (1-3) and the exact sequence (1-5) of [15], Remark 1.3], that

$$g_X = g_Y \text{ and } r_X = r_Y.$$

In particular, since f is *generically étale*, it follows immediately from condition (1), together with elementary theory of algebraic curves, that f is a *finite étale covering*, as desired. This completes the proof of Lemma 2.4. □

Definition 2.5.

(i) Let Σ be a [possibly empty] set of prime numbers. Then we shall write

$$\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$$

for the property of a profinite group defined as follows: A profinite group G has the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$ if, for an arbitrary open subgroup $H \subseteq G$ of G, there exists a prime number l_H, dependeing on H, such that $l_H \notin \Sigma$, and, moreover, there is no quotient of H which is free pro-l_H and not topologically finitely generated. Moreover, we shall write

$$\mathbb{P}_{\nexists \twoheadrightarrow \infty} \stackrel{\text{def}}{=} \mathbb{P}^{\exists \notin \emptyset}_{\nexists \twoheadrightarrow \infty}.$$

(ii) Let $\mathbb{P}$ be a property of a profinite group. Then we shall say that a profinite group G has the property *almost* $\mathbb{P}$ if an open subgroup of G has the property $\mathbb{P}$.

Remark 2.5.1. — One verifies easily that, in the situation of Definition 2.5, (i), if Σ is *not equal* to the set of all prime numbers, and a profinite group G has one of the following three properties, then G has the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$:

(1) The profinite group G is *topologically finitely generated*.

(2) The profinite group G is *almost abelian*.

(3) The profinite group G is *almost pro-*Σ' for some set Σ' of prime numbers such that the union $\Sigma \cup \Sigma'$ does *not coincide* with the set of all prime numbers.

Lemma 2.6. — *Let Σ be a set of prime numbers and G a profinite group. Then the following assertions hold:*

(i) *Let $H \subseteq G$ be an open subgroup of G. Then the profinite group G* **has** *the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$ if and only if the profinite group H* **has** *the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$.*

(ii) *The property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$ is* **stable under taking open subquotients***.*

Proof. — First, we verify assertion (i). It is immediate that if G has the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$, then H has the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$. Suppose that H has the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$. To complete the verification of assertion (i), assume that G does *not have* the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$, i.e., that there exists an open subgroup $U \subseteq G$ of G such that, for an arbitrary prime number $l \notin \Sigma$, the profinite group U has a *free pro-l* and *not topologically finitely generated* quotient $U \twoheadrightarrow Q$. Then since $U \cap H$ is an *open* subgroup of U, the image of the composite $U \cap H \hookrightarrow U \twoheadrightarrow Q$ is an *open* subgroup of Q. In particular, this image is *free pro-l* [cf. [13], Corollary 7.7.5] and *not topologically finitely generated*. Thus, since $U \cap H$ is an *open* subgroup of H, we obtain a *contradiction*. This completes the proof of assertion (i).

Next, we verify assertion (ii). It follows from assertion (i) that, to verify assertion (ii), it suffices to verify that if a profinite group has the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$, then an arbitrary quotient of the profinite group has the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$. On the other hand, this is immediate. This completes the proof of assertion (ii), hence also of Lemma 2.6. $\square$

LEMMA 2.7. — *Let k be an algebraically closed field and X a* **smooth curve** *over k. Suppose that X is* **isomorphic neither** *to the projective line over k* **nor** *to the affine line over k. Write Σ for the set of prime numbers* **not invertible** *in k. [So $\sharp\Sigma \leq 1$.] Then the following assertions hold:*

(i) *Let Y be a* **hyperbolic curve** *over k and $f\colon Y \to X$ a* **dominant** *and* **generically étale** *morphism over k. Suppose that there exists a prime number $l \notin \Sigma$ such that there is* **no** *quotient of Δ_f which is* **free pro-l** *and* **not topologically finitely generated**. *Then f is a* **finite étale covering**.

(ii) *The smooth curve X* **satisfies** *the condition $(*_{\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}})$ [cf. Definition 1.4, (ii)].*

PROOF. — First, we verify assertion (i). Let us first observe that since [we have assumed that] X is *isomorphic neither* to the projective line over k *nor* to the affine line over k, it follows from elementary theory of algebraic curves that an arbitrary connected finite étale covering of X is *isomorphic neither* to the projective line over k *nor* to the affine line over k. Thus, we may assume without loss of generality, by replacing X by the connected finite étale covering of X that corresponds to the image of some homomorphism $\Pi_Y \to \Pi_X$ induced by f, that

the outer homomorphism $\Pi_Y \to \Pi_X$ is *surjective.*

Let $l \notin \Sigma$ be as in the statement of assertion (i). Then we have an *exact* sequence of profinite groups

$$\Delta_f^{(l)} \longrightarrow \Pi_Y^{(l)} \longrightarrow \Pi_X^{(l)} \longrightarrow 1$$

— where we write "$(-)^{(l)}$" for the maximal pro-l quotient of "$(-)$". Now let us recall from Lemma 2.3 that the profinite group $\Pi_X^{(l)}$ is *infinite*, which thus implies that the image of the first arrow of the above exact sequence is *of infinite index* in $\Pi_Y^{(l)}$. Thus, it follows immediately from Lemma 2.1, together with our assumption on the quotients of Δ_f, that the image of the first arrow of the above exact sequence is *trivial.* In particular, the outer homomorphism $\Pi_Y^{(l)} \to \Pi_X^{(l)}$ induced by f is an outer *isomorphism.* Thus, it follows from Lemma 2.4 that f is a *finite étale covering*, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that it follows immediately from assertion (i) that, to verify assertion (ii), it suffices to verify the following claim:

> Claim 2.7.A: Let Y be a scheme *of CLN-type* and $f\colon Y \to X$ a *finite* and *generically étale surjective* morphism. [So one verifies easily that Y has a natural structure of a smooth curve over k with respect to which f is a morphism *over k*.] If Y, hence also X, is *not a hyperbolic curve* over k, then f is *étale.*

To this end, let us first observe that since [we have assumed that] X is *isomorphic neither* to the projective line over k *nor* to the affine line over k, it follows immediately from elementary theory of algebraic curves that both Y and X are either

- *proper* over k and *of genus one*, or

• *isomorphic* to the complement in the projective line over k of *distinct two* closed points.

Thus, since [we have assumed that] f is *generically étale* and *finite*, again by elementary theory of algebraic curves, we conclude immediately that f is *étale*, as desired. This completes the proof of assertion (ii), hence also of Lemma 2.7. $\square$

Theorem 2.8. — *Let k be a field, S a* **smooth curve** *over k [cf. Definition 2.2, (ii)] whose base-change to an algebraic closure of k is* **isomorphic neither** *to the projective line* **nor** *to the affine line, X a* **normal variety** *over k [cf. Definition 2.2, (i)], $f\colon X \to S$ a morphism over k, and $\overline{\eta} \to S$ a geometric generic point of S. Suppose that the geometric generic fiber $X \times_S \overline{\eta}$ of f is [either empty or]* **reduced** *[cf. Remark 2.8.1 below]. Consider the following four conditions:*

(1) *The outer homomorphism $\Pi_X \to \Pi_S$ induced by f is* **surjective**. *Moreover, the profinite group Δ_f is* **topologically finitely generated**.

(2) *The outer homomorphism $\Pi_X \to \Pi_S$ induced by f is* **surjective**. *Moreover, if we write Σ for the set of prime numbers* **not invertible** *in k [so $\sharp\Sigma \leq 1$], then the profinite group Δ_f* **has** *the property $\mathbb{P}^{\exists \notin \Sigma}_{\nexists \twoheadrightarrow \infty}$ defined in Definition 2.5, (i) [which is the case if, for instance, either*

- *the profinite group Δ_f is* **topologically finitely generated**,
- *the profinite group Δ_f is* **almost abelian**, *or*
- *the profinite group Δ_f is* **almost pro-Σ'** *for some [possibly empty] set Σ' of prime numbers such that there exists a prime number* **not contained in** *Σ' and* **invertible in** *k*

— cf. Remark 2.5.1].

(3) *The morphism f is* **surjective** *and* **generically geometrically irreducible**. *Moreover, the sequence of irreducible [hence also connected] schemes*

$$X \times_S \overline{\eta} \xrightarrow{\mathrm{pr}_1} X \xrightarrow{f} S$$

induces an **exact** *sequence of profinite groups and outer homomorphisms*

$$\Pi_{X \times_S \overline{\eta}} \longrightarrow \Pi_X \longrightarrow \Pi_S \longrightarrow 1.$$

(4) *The morphism f is* **dominant** *and* **generically geometrically irreducible**. *Moreover, the sequence of irreducible [hence also connected] schemes*

$$X \times_S \overline{\eta} \xrightarrow{\mathrm{pr}_1} X \xrightarrow{f} S$$

induces an **exact** *sequence of profinite groups and outer homomorphisms*

$$\Pi_{X \times_S \overline{\eta}} \longrightarrow \Pi_X \longrightarrow \Pi_S \longrightarrow 1.$$

Then the following assertions hold:

(i) *The implications*

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$$

hold.

(ii) *Suppose that either k is* **of characteristic zero**, *or the geometric generic fiber $X \times_S \overline{\eta}$ of f is* **proper** *over $\overline{\eta}$. Then the equivalences*

$$(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4)$$

hold.

PROOF. — Let us first observe that one verifies immediately from [16], Exposé IX, Théorème 6.1, that, to verify Theorem 2.8, we may assume without loss of generality, by replacing k by the algebraic closure of k determined by the composite $\overline{\eta} \to S \to \mathrm{Spec}(k)$, that

$$k \text{ is } \textit{algebraically closed.}$$

Now we verify assertion (i). The implications

$$(1) \Longrightarrow (2), \qquad (3) \Longrightarrow (4)$$

are immediate. Next, to verify the implication

$$(2) \Longrightarrow (4),$$

suppose that condition (2) is satisfied. Let us first recall from Lemma 2.3 that the profinite group Π_S is *nontrivial*. In particular, since [we have assumed that] the outer homomorphism $\Pi_X \to \Pi_S$ is *surjective*, we conclude that the morphism f is *nonconstant*, i.e., *dominant*. Thus, the implication (2) $\Rightarrow$ (4) follows from Lemma 1.5, (ii), and Lemma 2.7, (ii) [cf. also Lemma 2.6, (ii)]. This completes the proof of the implication (2) $\Rightarrow$ (4).

Next, to verify the implication

$$(2) \Longrightarrow (3),$$

suppose that condition (2), hence also condition (4) [cf. the preceding paragraph], is satisfied, and write $U \subseteq S$ for the image of the morphism $f\colon X \to S$. [Thus, U is an *open* subscheme of S — cf. condition (4).] Then we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_f & \longrightarrow & \Pi_X & \longrightarrow & \Pi_S & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_{U/S} & \longrightarrow & \Pi_U & \longrightarrow & \Pi_S & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*. Next, let us observe that since the resulting [necessarily *surjective*] morphism $X \to U$ is *generically geometrically irreducible* [cf. condition (4)], it follows from [a similar argument to the argument applied in the proof of] [8], Lemma 1.6, that the middle vertical arrow $\Pi_X \to \Pi_U$ of this diagram, hence also the left-hand vertical arrow $\Delta_f \to \Delta_{U/S}$ of this diagram, is *surjective*. In particular, since the profinite group Δ_f has the property $\mathbb{P}^{\exists\not\in\Sigma}_{\nexists\twoheadrightarrow\infty}$ [cf. condition (2)], it follows from Lemma 2.6, (ii), that the profinite group $\Delta_{U/S}$ has the property $\mathbb{P}^{\exists\not\in\Sigma}_{\nexists\twoheadrightarrow\infty}$, which thus implies that there exists a prime number $l \notin \Sigma$ such that there is *no* quotient of $\Delta_{U/S}$ which is *free pro-l* and *not topologically finitely generated*.

Next, let us observe that the lower horizontal sequence of the above diagram determines an *exact* sequence of profinite groups

$$\Delta_{U/S}^{(l)} \longrightarrow \Pi_U^{(l)} \longrightarrow \Pi_S^{(l)} \longrightarrow 1$$

— where we write "$(-)^{(l)}$" for the maximal pro-l quotient of "$(-)$". Now assume that $U \neq S$. Then since [we have assumed that] S is *isomorphic neither* to the projective line over k *nor* to the affine line over k, one verifies easily that U is a *hyperbolic curve* over k. Moreover, it follows from Lemma 2.3 that the profinite group $\Pi_S^{(l)}$ is *infinite*, which thus implies that the image of the first arrow of the above exact sequence is *of infinite index* in $\Pi_U^{(l)}$. Thus, it follows immediately from Lemma 2.1, together with our assumption on the quotients of $\Delta_{U/S}$, that the image of the first arrow of the above exact sequence is *trivial*. In particular, the outer homomorphism $\Pi_U^{(l)} \to \Pi_S^{(l)}$ induced by the open immersion $U \hookrightarrow S$ is an outer *isomorphism*. Thus, it follows from Lemma 2.4 that $U = S$, in *contradiction* to our assumption that $U \neq S$. This completes the proof of the implication (2) $\Rightarrow$ (3), hence also of assertion (i).

Next, we verify assertion (ii). Suppose that condition (4) is satisfied. Thus, the outer homomorphism $\Pi_X \to \Pi_S$ induced by f is *surjective*. Next, let us observe that it follows from condition (4) that the outer homomorphism $\Pi_{X \times_S \overline{\eta}} \to \Pi_X$ induced by $X \times_S \overline{\eta} \to X$ determines a Π_X-conjugacy class of *surjective* homomorphisms $\Pi_{X \times_S \overline{\eta}} \twoheadrightarrow \Delta_f$. Thus, since $\Pi_{X \times_S \overline{\eta}}$ is *topologically finitely generated* [cf. [8], Lemma 1.7, and [16], Exposé X, Théorème 2.9], we conclude that condition (1) is satisfied, as desired. This completes the proof of assertion (ii), hence also of Theorem 2.8. □

REMARK 2.8.1. — In the situation of Theorem 2.8, suppose that k is *of characteristic zero*. Then it follows from Remark 1.3.1 that the assumption that the geometric generic fiber $X \times_S \overline{\eta}$ of f is [either empty or] *reduced* in the statement of Theorem 2.8 is always satisfied.

REMARK 2.8.2. — Theorem 2.8 may be regarded as a *refinement* of the implication (2) $\Rightarrow$ (3) of [8], Lemma 2.11.

REMARK 2.8.3. — Let k be an algebraically closed field of characteristic zero. Write S for *either the projective line over k or the affine line over k*. Let $f \colon X \stackrel{\text{def}}{=} \operatorname{Spec}(k) \to S$ be a k-rational point of S [i.e., a splitting of the structure morphism of S]. Then one verifies easily that the morphism f *satisfies* condition (1) of Theorem 2.8 but does *not satisfy* condition (4) of Theorem 2.8.

3. AN ANABELIAN RESULT ON HYPERBOLIC POLYCURVES OF DIMENSION TWO

In the present §3, we obtain an application of the results of §1 concerning the Grothendieck conjecture for hyperbolic polycurves of dimension two [cf. Theorem 3.3 below].

DEFINITION 3.1. — Let $\mathbb{P}$ be a property of a profinite group. Then we shall say that the property $\mathbb{P}$ is *stable under taking extensions of topologically finitely generated profinite groups* if the following condition is satisfied: If a profinite group G has the property $\mathbb{P}$, then an arbitrary extension of a topologically finitely generated profinite group by G [i.e, a profinite group E that fits into an exact sequence of profinite groups

$$1 \longrightarrow G \longrightarrow E \longrightarrow H \longrightarrow 1$$

— where H is topologically finitely generated] has the property $\mathbb{P}$.

REMARK 3.1.1. — Let $\mathbb{P}$ be a property of a profinite group which is *stable under taking open subquotients* and *stable under taking extensions of topologically finitely generated profinite groups*. Suppose that there exists a profinite group that has the property $\mathbb{P}$. Then one verifies easily that an *arbitrary topologically finitely generated* profinite group has the property $\mathbb{P}$.

LEMMA 3.2. — *Let Σ be a set of prime numbers. Then the property $\mathbb{P}^{\exists\not\in\Sigma}_{\not\exists\twoheadrightarrow\infty}$ on a profinite group defined in Definition* 2.5, (i), *is* **stable under taking extensions of topologically finitely generated profinite groups**.

PROOF. — Let G be a profinite group that has the property $\mathbb{P}^{\exists\not\in\Sigma}_{\not\exists\twoheadrightarrow\infty}$ and

$$1 \longrightarrow G \longrightarrow E \longrightarrow H \longrightarrow 1$$

an exact sequence of profinite groups such that H is *topologically finitely generated*. Now, to verify Lemma 3.2, assume that the profinite group E does *not have* the property $\mathbb{P}^{\exists\not\in\Sigma}_{\not\exists\twoheadrightarrow\infty}$, i.e., that there exists an open subgroup $U \subseteq E$ of E such that, for an arbitrary prime number $l \not\in \Sigma$, the profinite group U has a *free pro-l* and *not topologically finitely generated* quotient $U \twoheadrightarrow Q_l$.

Now let us observe that since G is *normal* in E, the image of the composite $U \cap G \hookrightarrow U \twoheadrightarrow Q_l$ is *normal* in Q_l. In particular, if this image of $U \cap G$ in Q_l is *nontrivial*, then this image in Q_l is *free pro-l* [cf. [13], Corollary 7.7.5] and *not topologically finitely generated* [cf. [13], Theorem 8.6.5]. Thus, since $U \cap G$ is an *open* subgroup of G, and [we have assumed that] G has the property $\mathbb{P}^{\exists\not\in\Sigma}_{\not\exists\twoheadrightarrow\infty}$, we conclude that there exists a prime number $l \not\in \Sigma$ such that the image of the composite $U \cap G \hookrightarrow U \twoheadrightarrow Q_l$ is *trivial*, which thus implies that we have a *surjective* homomorphism $U/(U \cap G) \twoheadrightarrow Q_l$.

On the other hand, since $U/(U \cap G)$ may be regarded as an *open* subgroup of the *topologically finitely generated* profinite group H, the quotient $U/(U \cap G)$ is *topologically finitely generated*. Thus, since Q_l is *not topologically finitely generated*, we obtain a *contradiction* by the above *surjective* homomorphism $U/(U \cap G) \twoheadrightarrow Q_l$. This completes the proof of Lemma 3.2. □

THEOREM 3.3. — *Let p be a prime number, k a* **sub-p-adic** *field [cf., e.g., [8], Definition 3.1], $\overline{k}$ an algebraic closure of k, X a* **hyperbolic polycurve of dimension 2** *over k [cf. [8], Definition 2.1, (ii)], and Z a* **normal variety** *over k [cf. Definition 2.2, (i)]. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$. Let*

$$\alpha \colon \Pi_Z \longrightarrow \Pi_X$$

be an **open** *homomorphism of profinite groups over G_k. Suppose that the kernel $\mathrm{Ker}(\alpha)$ of α* **has** *the property $\mathbb{P}_{\not\exists \twoheadrightarrow \infty}$ defined in Definition 2.5, (i) [which is the case if, for instance, either*

- *the profinite group $\mathrm{Ker}(\alpha)$ is* **topologically finitely generated**,
- *the profinite group $\mathrm{Ker}(\alpha)$ is* **almost abelian**, *or*
- *the profinite group $\mathrm{Ker}(\alpha)$ is* **almost pro-Σ** *for some [possibly empty] set Σ of prime numbers* **not equal** *to the set of all prime numbers*

— cf. Remark 2.5.1]. Then there exists a **uniquely determined dominant** *morphism*

$$Z \longrightarrow X$$

over k from which α **arises**. *In particular, the normal variety Z is* **of dimension ≥ 2**.

PROOF. — Let us first observe that one verifies easily [cf. also [8], Proposition 2.3] that, to verify Theorem 3.3, we may assume without loss of generality, by replacing X by the connected finite étale covering of X that corresponds to the image of α, that

the homomorphism α is *surjective*.

Fix a parametrizing morphism $X \to S$ for X [cf. [8], Definition 2.1, (ii)] and a [necessarily *surjective* — cf. [8], Proposition 2.4, (i)] homomorphism $\Pi_X \twoheadrightarrow \Pi_S$ *over* G_k induced by this parametrizing morphism $X \to S$. Write $\beta \colon \Pi_Z \stackrel{\alpha}{\twoheadrightarrow} \Pi_X \twoheadrightarrow \Pi_S$ for the composite of α and the fixed surjective homomorphism $\Pi_X \twoheadrightarrow \Pi_S$. Then it follows from [8], Theorem A, in the case where condition (1) is satisfied that there exists a *uniquely determined dominant* morphism

$$f \colon Z \longrightarrow S$$

over k from which β *arises*.

Now we verify the following claim:

Claim 3.3.A: The kernel Δ_f [i.e., the kernel of β] *has* the property $\mathbb{P}_{\not\exists \twoheadrightarrow \infty}$.

To this end, let us first observe that we have a commutative diagram of profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \Delta_f \arrow[r] \arrow[d] & \Pi_Z \arrow[r, "\beta"] \arrow[d, "\alpha"] & \Pi_S \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta_{X/S} \arrow[r] & \Pi_X \arrow[r] & \Pi_S \arrow[r] & 1
\end{tikzcd}$$

— where the horizontal sequences are *exact*. Thus, the kernel Δ_f is *isomorphic* to an extension of $\Delta_{X/S}$ by the kernel of α. Now let us recall from [8], Proposition 2.4, (iii), that the profinite group $\Delta_{X/S}$ is *topologically finitely generated*. Thus, since the property $\mathbb{P}_{\not\exists \twoheadrightarrow \infty}$ is *stable under taking extensions of topologically finitely generated profinite groups* [cf. Lemma 3.2], and [we have assumed that] the kernel of α has the property $\mathbb{P}_{\not\exists \twoheadrightarrow \infty}$, the profinite group Δ_f has the property $\mathbb{P}_{\nexists\twoheadrightarrow\infty}$, as desired. This completes the proof of Claim 3.3.A.

Next, we verify the following claim:

> Claim 3.3.B: Both the morphism $X \to S$ and $f\colon Z \to S$ are *generically geometrically irreducible.*

To this end, let us recall from the definition of a hyperbolic curve that the morphism $X \to S$ is *generically geometrically irreducible.* On the other hand, the assertion that the morphism $f\colon Z \to S$ is *generically geometrically irreducible* follows from Lemma 1.5, (ii) [cf. also Lemma 2.6, (ii), and Lemma 2.7, (ii)], together with Claim 3.3.A. This completes the proof of Claim 3.3.B.

Write $\eta \to S$ for the [morphism of schemes determined by the] generic point of S. Let $\overline{\eta} \to (\eta \to)\ S$ be a geometric generic point of S. Next, we verify the following claim:

> Claim 3.3.C: There exists a *surjective* homomorphism
>
> $$\alpha_\eta\colon \Pi_{Z\times_S\eta} \longrightarrow \Pi_{X\times_S\eta}$$
>
> [cf. Claim 3.3.B] over Π_η such that the diagram of profinite groups
>
> $$\begin{array}{ccc} \Pi_{Z\times_S\eta} & \xrightarrow{\alpha_\eta} & \Pi_{X\times_S\eta} \\ \downarrow & & \downarrow \\ \Pi_Z & \xrightarrow[\alpha]{} & \Pi_X \end{array}$$
>
> — where the vertical arrows are some [necessarily *surjective* — cf. [8], Lemma 1.2] homomorphisms over Π_S induced by $Z\times_S\eta \overset{\mathrm{pr}_1}{\to} Z$, $X\times_S\eta \overset{\mathrm{pr}_1}{\to} X$, respectively — *commutes.*

To this end, let us first observe that it follows immediately from the various definitions involved that we have a sequence of profinite groups over Π_η

$$\Pi_{Z\times_S\eta} \longrightarrow \Pi_Z \times_{\Pi_S} \Pi_\eta \xrightarrow{\alpha\times_{\mathrm{id}_{\Pi_S}}\mathrm{id}_{\Pi_\eta}} \Pi_X \times_{\Pi_S} \Pi_\eta \longleftarrow \Pi_{X\times_S\eta}.$$

On the other hand, it follows from [8], Proposition 2.4, (ii), that the third arrow of this sequence is an *isomorphism*; moreover, since α is *surjective*, one verifies easily that the second arrow of this sequence is *surjective.* Thus, to complete the verification of Claim 3.3.C, it suffices to verify that the first arrow of this sequence is *surjective*, or, alternatively [cf. [8], Lemma 1.5], the Π_Z-conjugacy class of homomorphisms $\Pi_{Z\times_S\overline{\eta}} \to \Delta_{Z/S}$ induced by $Z \times_S \overline{\eta} \overset{\mathrm{pr}_1}{\to} Z$ is *surjective.* On the other hand, this follows from Lemma 1.5, (ii) [cf. also Lemma 2.6, (ii), and Lemma 2.7, (ii)], together with Claim 3.3.A. This completes the proof of Claim 3.3.C.

Thus, since [one verifies easily from Claim 3.3.B that] $Z \times_S \eta$ is a *normal variety* over η [i.e., over the function field of S], and the function field of S is *sub-p-adic*, it follows immediately from [8], Theorem A, in the case where condition (1) is satisfied and [8], Lemma 2.10, that there exists a *uniquely determined dominant* morphism

$$Z \longrightarrow X$$

over S from which α *arises*, as desired. This completes the proof of Theorem 3.3. $\square$

Remark 3.3.1. — Let us recall that [8], Theorem A, in the case where condition (2) is satisfied is nothing but Theorem 3.3 under the assumption that the kernel of α is *topologically finitely generated*. Thus, Theorem 3.3 may be regarded as a *refinement* of [8], Theorem A, in the case where condition (2) is satisfied.

Remark 3.3.2. — Suppose that we are in the situation of Theorem 3.3. Then one may conclude that

> if one drops the assumption that the hyperbolic polycurve X is *of dimension* 2, then the conclusion of Theorem 3.3 no longer holds in general

as follows: Let C_1, C_2, C_3 be proper hyperbolic curves over k. Then $Y \stackrel{\text{def}}{=} C_1 \times_k C_2 \times_k C_3$ is a *hyperbolic polycurve of dimension* 3 over k. Now it follows from [5], Exposé XII, Corollaire 3.5, that there exist a *normal variety* H *of dimension* 2 over k and a closed immersion $i\colon H \hookrightarrow Y$ over k such that i induces an outer *isomorphism*

$$\Pi_H \xrightarrow{\sim} \Pi_Y .$$

Now let us observe that

(a) the triple consisting of the hyperbolic polycurve Y over k, the normal variety H over k, and some fixed isomorphism $\Pi_H \xrightarrow{\sim} \Pi_Y$ over G_k contained in the outer isomorphism induced by i satisfies the condition imposed on the triple "(X, Z, α)" in the statement of Theorem 3.3 *except for the condition that "X" is of dimension* 2. On the other hand, there is *no dominant* morphism from H to Y [cf. $\dim(H) = 2 < 3 = \dim(Y)$], which thus implies that the fixed isomorphism $\Pi_H \xrightarrow{\sim} \Pi_Y$ over G_k *never arises* from a *dominant* morphism over k.

Moreover, let C_4 be a hyperbolic curve over k. Then one verifies easily [cf. [8], Proposition 2.4, (i)] that the composite $H \times_k C_4 \to Y$ of the first projection $H \times_k C_4 \to H$ and the closed immersion $i\colon H \hookrightarrow Y$ induces an outer *surjective* homomorphism

$$\Pi_{H \times_k C_4} \twoheadrightarrow \Pi_Y$$

whose kernel is *isomorphic* to $\Delta_{C_4/k}$, hence also *topologically finitely generated* [cf. [8], Proposition 2.4, (iii)]. Now let us also observe that

(b) the triple consisting of the hyperbolic polycurve Y over k, the normal variety $H \times_k C_4$ over k, and some fixed surjective homomorphism $\Pi_{H \times_k C_4} \twoheadrightarrow \Pi_Y$ over G_k contained in the outer surjective homomorphism induced by the above composite satisfies the condition imposed on the triple "(X, Z, α)" in the statement of Theorem 3.3 *except for the condition that "X" is of dimension* 2. On the other hand, it follows immediately from [8], Proposition 3.2, (ii), that the fixed surjective homomorphism $\Pi_{H \times_k C_4} \twoheadrightarrow \Pi_Y$ over G_k *never arises* from a *dominant* morphism over k.

Remark 3.3.3. — Let us observe that it follows from [8], Lemma 2.6, (i), and [8], Proposition 2.7, that the normal variety "H" over k of Remark 3.3.2 is *of LFG-type* [cf.

[8], Definition 2.5]. In particular, it follows from [8], Proposition 2.7, that the normal variety "$H \times_k C_4$" over k of Remark 3.3.2 is *of LFG-type*. Thus, the observation (a) of Remark 3.3.2 implies that

> if one drops assumption (3-iv) in the statement of [8], Theorem A, then the conclusion of [8], Theorem A [i.e., in the case where condition (3) is satisfied], no longer holds in general.

Moreover, the observation (b) of Remark 3.3.2 implies that

> if one drops assumption (3-ii) in the statement of [8], Theorem A, then the conclusion of [8], Theorem A [i.e., in the case where condition (3) is satisfied], no longer holds in general.

Finally, one may also conclude from a similar argument to the argument applied in the consideration related to (b) of Remark 3.3.2 that

> if one drops assumption (4-ii) in the statement of [8], Theorem A, then the conclusion of [8], Theorem A [i.e., in the case where condition (4) is satisfied], no longer holds in general.

REMARK 3.3.4.

(i) Suppose that we are in the situation of Theorem 3.3. Then one may conclude that

> if one drops the assumption that the kernel of α *has the property* $\mathbb{P}_{\nexists\twoheadrightarrow\infty}$, then the conclusion of Theorem 3.3 no longer holds in general

as follows: Let C_1 and C_2 be proper hyperbolic curves over k. Then $W \stackrel{\mathrm{def}}{=} C_1 \times_k C_2$ is a *hyperbolic polycurve of dimension* 2 over k. Now it follows from [5], Exposé XII, Corollaire 3.5, that there exist a *normal variety* J over k and a closed immersion $j\colon J \hookrightarrow W$ over k such that j is *not an isomorphism* [or, alternatively, *not dominant*] — which thus implies that J is *of dimension* ≤ 1 — but induces an outer *surjective* homomorphism

$$\Pi_J \longrightarrow\!\!\!\!\!\rightarrow \Pi_W.$$

Now let us observe that the triple consisting of the hyperbolic polycurve W over k, the normal variety J over k, and some fixed surjective homomorphism $\Pi_J \twoheadrightarrow \Pi_W$ over G_k contained in the outer surjective homomorphism induced by j satisfies the condition imposed on the triple "(X, Z, α)" in the statement of Theorem 3.3 *except for the condition that the kernel of "α" has the property* $\mathbb{P}_{\nexists\twoheadrightarrow\infty}$. On the other hand, there is *no dominant* morphism from J to W [cf. $\dim(J) \leq 1 < 2 = \dim(W)$], which thus implies that the fixed surjective homomorphism $\Pi_J \twoheadrightarrow \Pi_W$ over G_k *never arises* from a *dominant* morphism over k.

(ii) One may find

> an example of a triple consisting of a hyperbolic polycurve X *of dimension* 2 over k, a normal variety Z over k, and a *dominant* morphism $f\colon Z \to X$ over k such that the kernel Δ_f does *not have the property* $\mathbb{P}_{\nexists\twoheadrightarrow\infty}$

as follows: Let C_1, C_2, C_3 be hyperbolic curves over k; $f_C\colon C_2 \to C_3$ a *dominant* morphism over k that is *not a finite étale covering*. Then both $W_{1,2} \stackrel{\mathrm{def}}{=} C_1 \times_k C_2$ and $W_{1,3} \stackrel{\mathrm{def}}{=} C_1 \times_k C_3$ are *hyperbolic polycurves of dimension* 2 over k, and the morphism $f_W\colon W_{1,2} \to W_{1,3}$ over C_1, hence also over k, induced by f_C is *dominant*. Moreover, one verifies easily [cf. [8], Proposition 2.4, (i)] that $\Delta_{f_W} \cong \Delta_{f_C}$. On the other hand, it follows immediately from Lemma 2.7, (i), that the profinite group Δ_{f_C}, hence also Δ_{f_W}, does *not have the property* $\mathbb{P}_{\nexists \twoheadrightarrow \infty}$.

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