A Note on an Anabelian Open Basis for a Smooth Variety

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Abstract. — Schmidt and Stix proved that every smooth variety over a field finitely generated over the field of rational numbers has an open basis for the Zariski topology consisting of "anabelian" varieties. This was predicted by Grothendieck in his letter to Faltings. In the present paper, we generalize this result to smooth varieties over generalized sub-p-adic fields. Moreover, we also discuss an absolute version of this result.

Contents

Introduction

Schmidt and *Stix* proved that every smooth variety over a field finitely generated over $\mathbb{Q}$ has an open basis for the Zariski topology consisting of "*anabelian*" varieties [cf. [9], Corollary 1.7]. This was predicted by *Grothendieck* in his letter to Faltings [cf. [1]]. In the present paper, we generalize this result to a smooth variety over a *generalized sub-p-adic field* — i.e., a field isomorphic to a subfield of a field finitely generated over the p-adic completion of a maximal unramified extension of $\mathbb{Q}_p$ — by means of some techniques of [2].

Let k be a perfect field and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$. We shall say that a smooth variety over k has a *relatively anabelian open basis* [cf. Definition 3.3] if there exists an open basis for the Zariski topology of the variety such that, for arbitrary members U and V of the open basis, the natural map

$$\mathrm{Isom}_k(U,V) \longrightarrow \mathrm{Isom}_{G_k}(\Pi_U,\Pi_V)/\mathrm{Inn}(\Delta_{V/k})$$

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is bijective — where we write "$\Pi_{(-)}$" for the étale fundamental group [relative to an appropriate choice of basepoint] of "$(-)$" [cf. Definition 2.1, (i)] and "$\Delta_{(-)/k}$" for the kernel of the outer surjection "$\Pi_{(-)} \twoheadrightarrow G_k$" induced by the structure morphism of "$(-)$" [cf. Definition 2.1, (ii)].

One main result of the present paper — that may be regarded as a *substantial refinement* of the above prediction by Grothendieck — is as follows [cf. Corollary 3.4, (i)].

THEOREM A. — *Every smooth variety over a* **generalized sub-p-adic field***, for some prime number p, has a* **relatively anabelian open basis***.*

In [9], Corollary 1.7, Schmidt and Stix proved Theorem A in the case where the base field is finitely generated over $\mathbb{Q}$. The proof of Theorem A gives an *alternative proof* of [9], Corollary 1.7.

Each of [9], Corollary 1.7, and Theorem A of the present paper is proved as a consequence of an *anabelian property of a certain hyperbolic polycurve*. Let us recall that we shall say that a smooth variety X over k is a *hyperbolic polycurve* [cf. Definition 1.9] if there exist a positive integer d and a factorization of the structure morphism of X

$$X = X_d \longrightarrow X_{d-1} \longrightarrow \dots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow \mathrm{Spec}(k) = X_0$$

such that, for each $i \in \{1, \dots, d\}$, the morphism $X_i \to X_{i-1}$ is a hyperbolic curve. In [9], Schmidt and Stix discussed an anabelian property of a *strongly hyperbolic Artin neighborhood* [cf. [9], Definition 6.1], i.e., a smooth variety X over k whose structure morphism has a factorization $X = X_d \to X_{d-1} \to \dots \to X_2 \to X_1 \to \mathrm{Spec}(k) = X_0$ such that, for each $i \in \{1, \dots, d\}$,

- the morphism $X_i \to X_{i-1}$ is a hyperbolic curve,
- the morphism $X_i \to X_{i-1}$ is not proper, and
- the smooth variety X_i may be embedded into the product of finitely many hyperbolic curves over k.

Schmidt and Stix proved that if k is finitely generated over $\mathbb{Q}$, and X and Y are strongly hyperbolic Artin neighborhoods over k, then the natural map $\mathrm{Isom}_k(X, Y) \to \mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$ is *bijective* [cf. [9], Corollary 1.6].

In [2], the author of the present paper discussed an anabelian property of a hyperbolic polycurve *of lower dimension*. The author of the present paper proved that if k is *sub-p-adic* — i.e., a field isomorphic to a subfield of a field finitely generated over $\mathbb{Q}_p$ — for some prime number p, and X and Y are hyperbolic polycurves over k, then the natural map $\mathrm{Isom}_k(X, Y) \to \mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$ is *bijective* whenever either X or Y is *of dimension* ≤ 4 [cf. [2], Theorem B].

In the present paper, in order to prove Theorem A, we discuss an anabelian property of a hyperbolic polycurve *of strictly decreasing type* [cf. Definition 1.10, (ii)], i.e., a hyperbolic polycurve X over k whose structure morphism has a factorization $X = X_d \to X_{d-1} \to \dots \to X_2 \to X_1 \to \mathrm{Spec}(k) = X_0$ such that,

- for each $i \in \{1, \dots, d\}$, the morphism $X_i \to X_{i-1}$ is a hyperbolic curve of type (g_i, r_i), and,

- for each $i \in \{2, \ldots, d\}$, the inequality $2g_{i-1} + \max\{0, r_{i-1} - 1\} > 2g_i + \max\{0, r_i - 1\}$ holds.

The main ingredient of the proof of Theorem A is the following anabelian result [cf. Theorem 2.4], which was essentially proved in [2], §4 [cf., e.g., [2], Theorem 4.3].

Theorem B. — *Suppose that k is* **generalized sub-p-adic**, *for some prime number p. Let X and Y be* **hyperbolic polycurves of strictly decreasing type** *over k. Then the natural map*

$$\mathrm{Isom}_k(X, Y) \longrightarrow \mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$$

is **bijective**.

In the present paper, we also discuss an *absolute version* of an anabelian open basis for a smooth variety. We shall say that a smooth variety over k has an *absolutely anabelian open basis* [cf. Definition 3.3] if there exists an open basis for the Zariski topology of the variety such that, for arbitrary members U and V of the open basis, the natural map

$$\mathrm{Isom}(U, V) \longrightarrow \mathrm{Isom}(\Pi_U, \Pi_V)/\mathrm{Inn}(\Pi_V)$$

is bijective. In [9], Schmidt and Stix essentially proved that every smooth variety over a field finitely generated over $\mathbb{Q}$ has an absolutely anabelian open basis [cf. Corollary 3.4, (ii); also Remark 3.4.1, (i)]. In the present paper, we prove the following result concerning an absolutely anabelian open basis for a smooth variety by means of some results obtained in the study of *absolute anabelian geometry*, i.e., in [5] and [6] [cf. Corollary 3.4, (iii)].

Theorem C. — *Every smooth variety* **of positive dimension** *over a* **finite** *extension of $\mathbb{Q}_p$, for some prime number p, has an* **absolutely anabelian open basis**.

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1. Hyperbolic Polycurves of Strictly Decreasing Type

In the present §1, we introduce and discuss the notion of *hyperbolic polycurves* [cf. Definition 1.9 below] *of strictly decreasing type* [cf. Definition 1.10, (ii), below]. In particular, we prove that every smooth variety of positive dimension over an infinite perfect field has an open basis for the Zariski topology such that each member of the open basis has a *tripodal unit* [cf. Definition 1.3 below] and a structure of *hyperbolic polycurve of strictly decreasing type* [cf. Lemma 1.12 below].

In the present §1, let k be a perfect field.

DEFINITION 1.1.

(i) We shall say that k is a *mixed-characteristic local field* if k is isomorphic to a finite extension of $\mathbb{Q}_p$, for some prime number p.

(ii) Let p be a prime number. Then we shall say that k is *generalized sub-p-adic* if k is isomorphic to a subfield of a field finitely generated over the p-adic completion of a maximal unramified extension of $\mathbb{Q}_p$ [cf. [3], Definition 4.11].

DEFINITION 1.2. — We shall say that a scheme X over k is a *normal* (respectively, *smooth*) *variety* over k if X is of finite type and separated over k, and, moreover, every geometric fiber of the structure morphism $X \to \mathrm{Spec}(k)$ [hence also the scheme X] is normal (respectively, regular) and connected.

REMARK 1.2.1. — Let X be a normal (respectively, smooth) variety over k.

(i) One verifies immediately that an arbitrary nonempty open subscheme of X is a normal (respectively, smooth) variety over k.

(ii) Let $Y \to X$ be a connected finite étale covering of X. Then one verifies immediately that Y is a normal (respectively, smooth) variety over the [necessarily finite] extension of k obtained by forming the algebraic closure of k in the function field of Y.

DEFINITION 1.3. — Let X be a normal variety over k. Then we shall say that a regular function f on X is a *tripodal unit* if f is nonconstant [i.e., $\notin k$], and, moreover, both f and $1 - f$ are invertible.

LEMMA 1.4. — *Let X be a normal variety over k. Then the following assertions hold:*

(i) *Let $x \in X$ be a point of X. Then there exists an open neighborhood $U \subseteq X$ of $x \in X$ such that U has a* **tripodal unit**.

(ii) *Let Y be a normal variety over k and $Y \to X$ a* **dominant** *morphism over k. Suppose that X has a* **tripodal unit**. *Then Y has a* **tripodal unit**.

PROOF. — These assertions follow immediately from the various definitions involved. □

DEFINITION 1.5. — Let S be a scheme. Then we shall say that a scheme X over S is a *smooth curve* [of type (g, r)] over S if there exist

- a pair of nonnegative integers (g, r),
- a scheme X^{cpt} over S that is smooth, proper, and of relative dimension one over S, and
- a [possibly empty] closed subscheme $D \subseteq X^{\mathrm{cpt}}$ of X^{cpt} that is finite and étale over S

such that

• each geometric fiber of X^{cpt} over S is connected [hence also a smooth proper curve] and of genus g,

• the finite étale covering of S obtained by forming the composite $D \hookrightarrow X^{\text{cpt}} \to S$ is of degree r, and

• the scheme X is isomorphic to $X^{\text{cpt}} \setminus D$ over S.

REMARK 1.5.1. — It is immediate that a smooth curve over k is a smooth variety over k.

DEFINITION 1.6. — Let n be an integer and S a scheme. Then we shall say that a smooth curve X over S is *of rank* n if X is of type (g, r), and, moreover, the equality $n = 2g + \max\{0, r-1\}$ holds.

DEFINITION 1.7. — Let S be a scheme. Then we shall say that a smooth curve X over S is a *hyperbolic curve* over S if the following condition is satisfied: The smooth curve X over S is of type (g, r), and, moreover, the inequality $2g - 2 + r > 0$ holds [which is the case if, for instance, the smooth curve X over S is of rank ≥ 3].

LEMMA 1.8. — *Let n_0 be an integer, S a normal variety over k, X a smooth curve over S, and $x \in X$ a closed point of X. Then there exist an open subscheme $U_S \subseteq S$ of S and a closed subscheme $E \subseteq U_X \stackrel{\text{def}}{=} X \times_S U_S$ of U_X such that*

• *the point $x \in X$ is contained in the open subscheme $U_X \setminus E \subseteq X$ of X, and, moreover,*

• *the composite $E \hookrightarrow U_X \to U_S$ is a* **finite étale covering of degree $> n_0$** *— which thus implies that the composite $U_X \setminus E \hookrightarrow U_X \to U_S$ is a* **smooth curve of rank $\geq n_0$**.

PROOF. — Let X^{cpt} and D be as in Definition 1.5. Let us first observe that, by applying induction on n_0, we may assume without loss of generality that $n_0 = 0$. Write $s \in S$ for the closed point obtained by forming the image of $x \in X$ in S; $X_s^{\text{cpt}} \subseteq X^{\text{cpt}}$ for the closed subscheme of X^{cpt} obtained by forming the fiber of $X^{\text{cpt}} \to S$ at $s \in S$; $\eta \in S$ for the generic point of S; X_η^{cpt} for the fiber of $X^{\text{cpt}} \to S$ at $\eta \in S$. Then since X^{cpt} is *smooth*, *proper*, and *of relative dimension one* over S, there exist an open neighborhood $V \subseteq X^{\text{cpt}}$ of $x \in X \subseteq X^{\text{cpt}}$ and a morphism $f\colon V \to \mathbb{P}^1_S$ over S such that f is *étale* at $x \in X \subseteq X^{\text{cpt}}$ and restricts to a finite flat morphism $f_\eta\colon X_\eta^{\text{cpt}} \to \mathbb{P}^1_\eta$ over η.

For each closed point $a \in \mathbb{P}^1_k$ of $\mathbb{P}^1_k$, write $E_a \subseteq X^{\text{cpt}}$ for the scheme-theoretic closure in X^{cpt} of the closed subscheme of X_η^{cpt} obtained by pulling back the *reduced* closed subscheme of $\mathbb{P}^1_k$ whose support consists of $a \in \mathbb{P}^1_k$ by the composite of $f_\eta\colon X_\eta^{\text{cpt}} \to \mathbb{P}^1_\eta$ and the natural projection $\mathbb{P}^1_\eta \to \mathbb{P}^1_k$. Now let us observe that since X^{cpt} is *proper* over S,

(a) the composite $E_a \hookrightarrow X^{\text{cpt}} \to S$ is *finite*.

Next, let us observe that since f is *étale* at $x \in X \subseteq X^{\text{cpt}}$, one verifies immediately that there exists a closed point $a_0 \in \mathbb{P}^1_k$ of $\mathbb{P}^1_k$ such that

(b) both $\{x\} \cap E_{a_0}$ and $X_s^{\mathrm{cpt}} \cap D \cap E_{a_0}$ are *empty*, and, moreover,

(c) the intersection $X_s^{\mathrm{cpt}} \cap E_{a_0} \subseteq X^{\mathrm{cpt}}$ is contained in the *étale locus* of f.

Thus, since the intersection $X_s^{\mathrm{cpt}} \cap E_{a_0} \subseteq X^{\mathrm{cpt}}$ is contained in $V \subseteq X^{\mathrm{cpt}}$ [cf. (c)], we may assume without loss of generality, by replacing S by a suitable open neighborhood of $s \in S$, that

(d) the closed subscheme $E_{a_0} \subseteq X^{\mathrm{cpt}}$ *coincides* with the closed subscheme of X^{cpt} obtained by pulling back the *reduced* closed subscheme of $\mathbb{P}^1_k$ whose support consists of $a_0 \in \mathbb{P}^1_k$ by the composite of $f \colon V \to \mathbb{P}^1_S$ and the natural projection $\mathbb{P}^1_S \to \mathbb{P}^1_k$.

In particular, since k is *perfect*, it follows from (c) and (d) that we may assume without loss of generality, by replacing S by a suitable open neighborhood of $s \in S$, that

(e) the composite $E_{a_0} \hookrightarrow X^{\mathrm{cpt}} \to S$ is *étale*.

Write $U_S \subseteq S$ for the open subscheme of S obtained by forming the complement in S of the image of the intersection $D \cap E_{a_0} \subseteq X^{\mathrm{cpt}}$ in S. Then it follows from (a), (b), (e) that

- the subscheme $E \stackrel{\mathrm{def}}{=} E_{a_0} \times_S U_S \subseteq U_X \stackrel{\mathrm{def}}{=} X \times_S U_S$ of U_X is *closed* and *nonempty*,
- the point $x \in X$ is contained in $U_X \subseteq X$ but is not contained in $E \subseteq U_X$, and
- the composite $E \hookrightarrow U_X \to U_S$ is *finite* and *étale*,

as desired. This completes the proof of Lemma 1.8. $\square$

DEFINITION 1.9. — Let S be a scheme. Then we shall say that a scheme X over S is a *hyperbolic polycurve* over S if there exist a positive integer d and a [not necessarily unique] factorization of the structure morphism $X \to S$ of X

$$X = X_d \longrightarrow X_{d-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow S = X_0$$

such that, for each $i \in \{1, \ldots, d\}$, the morphism $X_i \to X_{i-1}$ is a hyperbolic curve. We shall refer to a factorization of $X \to S$ as above as a *sequence of parametrizing morphisms* for X over S.

REMARK 1.9.1. — It is immediate that a hyperbolic polycurve over k is a smooth variety over k.

DEFINITION 1.10. — Let S be a scheme and X a hyperbolic polycurve over S.

(i) We shall say that a sequence $X = X_d \to X_{d-1} \to \ldots \to X_2 \to X_1 \to S = X_0$ of parametrizing morphisms for X over S is *of strictly decreasing type* if the following condition is satisfied: The inequalities $n_1 > n_2 > \cdots > n_{d-1} > n_d$ hold whenever the hyperbolic curve $X_i \to X_{i-1}$ is of rank n_i for each $i \in \{1, \ldots, d\}$.

(ii) We shall say that the hyperbolic polycurve X over S is *of strictly decreasing type* if there exists a sequence of parametrizing morphisms for X over S of strictly decreasing type.

Lemma 1.11. — *Let n_0 be an integer, X a smooth variety over k, and $x \in X$ a point of X. Suppose that k is* **infinite**, *and that X is* **of positive dimension**. *Then there exists an open neighborhood $U \subseteq X$ of $x \in X$ that satisfies the following three conditions:*

(1) *The smooth variety U has a* **tripodal unit**.

(2) *The smooth variety U has a structure of* **hyperbolic polycurve** *over k.*

(3) *There exists a sequence $U = U_d \to U_{d-1} \to \ldots \to U_2 \to U_1 \to \mathrm{Spec}(k) = U_0$ of parametrizing morphisms for U over k [cf. (2)] such that this sequence is* **of strictly decreasing type**, *and, moreover, the hyperbolic curve U over U_{d-1} is* **of rank $\geq n_0$**.

Proof. — We prove Lemma 1.11 by induction on the dimension of X. If X is *of dimension one*, then Lemma 1.11 follows from Lemma 1.4, (i), (ii), and Lemma 1.8. In the remainder of the proof of Lemma 1.11, suppose that X is *of dimension* ≥ 2, and that the induction hypothesis is in force.

Next, let us observe that we may assume without loss of generality, by replacing $x \in X$ by a closed point of the closure of $\{x\} \subseteq X$ in X, that $x \in X$ is a *closed* point of X. Moreover, it follows from Lemma 1.4, (i), that we may assume without loss of generality, by replacing X by a suitable open neighborhood of $x \in X$, that

(a) the smooth variety X [hence also an arbitrary nonempty open subscheme of X — cf. Lemma 1.4, (ii)] has a *tripodal unit*.

Next, since [we have assumed that] k is *perfect*, it follows from a similar argument to the argument applied in the proof of [11], Exposé XI, Proposition 3.3 [i.e., as in the proof of [9], Lemma 6.3], that we may assume without loss of generality, by replacing X by a suitable open neighborhood of $x \in X$, that there exists a smooth variety S over k such that X has a structure of *smooth curve* over S, by means of which let us regard X as a scheme over S. Thus, it follows from Lemma 1.8 that we may assume without loss of generality, by replacing X by a suitable open neighborhood of $x \in X$, that

(b) the smooth curve X over S is *of rank* $\geq \max\{3, n_0\}$, hence also a *hyperbolic curve* over S.

Write n^X ($\geq n_0$) for the rank of the hyperbolic curve X over S [cf. (b)]. Then since S is *of dimension* $\dim(X) - 1$, it follows from the induction hypothesis that we may assume without loss of generality, by replacing S by a suitable open neighborhood of the image of $x \in X$ in S, that

(c) the smooth variety S has a structure of *hyperbolic polycurve* over k, and

(d) there exists a sequence $S = S_{d-1} \to S_{d-2} \to \ldots \to S_2 \to S_1 \to \mathrm{Spec}(k) = S_0$ of parametrizing morphisms for S over k [cf. (c)] such that this sequence is *of strictly decreasing type*, and, moreover, the hyperbolic curve S over S_{d-2} is *of rank* $> n^X$.

Now let us observe that it follows from (a) that X satisfies condition (1). Moreover, it follows from (b), (c), (d) that X satisfies conditions (2), (3). This completes the proof of Lemma 1.11. □

Lemma 1.12. — *Let X be a smooth variety over k. Suppose that k is* **infinite**, *and that X is* **of positive dimension**. *Then there exists an open basis for the Zariski topology* *of X such that each member of the open basis has a* **tripodal unit** *and a structure of* **hyperbolic polycurve of strictly decreasing type** *over k.*

PROOF. — This assertion follows from Lemma 1.11. $\square$

2. SOME ANABELIAN RESULTS FOR HYPERBOLIC POLYCURVES

In the present §2, we prove some anabelian results for *hyperbolic polycurves of strictly decreasing type* [cf. Theorem 2.4, Theorem 2.6 below]. Moreover, we also prove an anabelian result for *hyperbolic curves of pseudo-Belyi type* [cf. Definition 2.7, Theorem 2.9 below].

In the present §2, let k be a field of characteristic zero and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

DEFINITION 2.1. — Let X be a connected locally noetherian scheme.

(i) We shall write

$$\Pi_X$$

for the *étale fundamental group* [relative to an appropriate choice of basepoint] of X.

(ii) Let Y be a connected locally noetherian scheme and $f\colon X \to Y$ a morphism of schemes. Then we shall write

$$\Delta_f = \Delta_{X/Y} \subseteq \Pi_X$$

for the kernel of the outer homomorphism $\Pi_X \to \Pi_Y$ induced by f.

LEMMA 2.2. — *Let n be an integer, S a normal variety over k, and X a hyperbolic curve over S. Then the following two conditions are equivalent:*

(1) *The hyperbolic curve X over S is* **of rank $\boldsymbol{n}$**.

(2) *The abelianization of the profinite group $\Delta_{X/S}$ is a free $\widehat{\mathbb{Z}}$-module* **of rank $\boldsymbol{n}$**.

PROOF. — This assertion follows from [2], Proposition 2.4, (v). $\square$

LEMMA 2.3. — *Let X (respectively, Y) be a hyperbolic polycurve over k, $X = X_{d_X} \to X_{d_X-1} \to \ldots \to X_2 \to X_1 \to \mathrm{Spec}(k) = X_0$ (respectively, $Y = Y_{d_Y} \to Y_{d_Y-1} \to \ldots \to Y_2 \to Y_1 \to \mathrm{Spec}(k) = Y_0$) a sequence of parametrizing morphisms for X (respectively, Y) over k* **of strictly decreasing type**, *and*

$$\alpha\colon \Pi_X \xrightarrow{\sim} \Pi_Y$$

an isomorphism of profinite groups. Suppose that $k = \overline{k}$. Then the following assertions hold:

(i) *Suppose that the inclusion $\alpha(\Delta_{X/X_{d_X-1}}) \subseteq \Delta_{Y/Y_{d_Y-1}}$ holds. Then the equality $\alpha(\Delta_{X/X_{d_X-1}}) = \Delta_{Y/Y_{d_Y-1}}$ holds.*

(ii) *Suppose that either X or Y is* **of dimension one**. *Then both X and Y are* **of dimension one**.

(iii) *The isomorphism α restricts to an* **isomorphism** $\Delta_{X/X_{d_X-1}} \xrightarrow{\sim} \Delta_{Y/Y_{d_Y-1}}$.

(iv) *The equality $d_X = d_Y$ holds.*

Proof. — First, we verify assertion (i). Since the inclusion $\alpha(\Delta_{X/X_{d_X-1}}) \subseteq \Delta_{Y/Y_{d_Y-1}}$ holds, it follows from [2], Proposition 2.4, (iii), (iv), that the [necessarily *normal*] closed subgroup $\alpha(\Delta_{X/X_{d_X-1}}) \subseteq \Delta_{Y/Y_{d_Y-1}}$ of $\Delta_{Y/Y_{d_Y-1}}$ is *open*, which thus implies that the closed subgroup $\Delta_{Y/Y_{d_Y-1}}/\alpha(\Delta_{X/X_{d_X-1}}) \subseteq \Pi_Y/\alpha(\Delta_{X/X_{d_X-1}})$ of $\Pi_Y/\alpha(\Delta_{X/X_{d_X-1}})$ is *finite*. Thus, since $\Pi_Y/\alpha(\Delta_{X/X_{d_X-1}})$ is isomorphic to $\Pi_{X_{d_X-1}}$ [cf. [2], Proposition 2.4, (i)], which is *torsion-free* [cf. [2], Proposition 2.4, (iii)], we conclude that $\alpha(\Delta_{X/X_{d_X-1}}) = \Delta_{Y/Y_{d_Y-1}}$, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that we may assume without loss of generality, by replacing (X, Y) by (Y, X) if necessary, that Y is *of dimension one*. Then since $\alpha(\Delta_{X/X_{d_X-1}}) \subseteq \Pi_Y = \Delta_{Y/Y_{d_Y-1}}$, it follows from assertion (i) that the restriction of α to the closed subgroup $\Delta_{X/X_{d_X-1}} \subseteq \Pi_X$ of Π_X is *surjective*. Thus, since α is an *isomorphism*, it follows that $\Delta_{X/X_{d_X-1}} = \Pi_X$. In particular, it follows immediately from [2], Proposition 2.4, (i), (iii), that X is *of dimension one*, as desired. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Let us first observe that if either X or Y is *of dimension one*, which thus implies [cf. assertion (ii)] that both X and Y are *of dimension one*, then assertion (iii) is immediate. Thus, we may assume without loss of generality that both X and Y are *of dimension* ≥ 2.

Write $n^X_{d_X}$ (respectively, $n^X_{d_X-1}$; $n^Y_{d_Y}$; $n^Y_{d_Y-1}$) for the rank of the hyperbolic curve $X \to X_{d_X-1}$ (respectively, $X_{d_X-1} \to X_{d_X-2}$; $Y \to Y_{d_Y-1}$; $Y_{d_Y-1} \to Y_{d_Y-2}$). Thus, since $n^X_{d_X-1} > n^X_{d_X}$ and $n^Y_{d_Y-1} > n^Y_{d_Y}$, we may assume without loss of generality, by replacing (X, Y) by (Y, X) if necessary, that $n^X_{d_X} < n^Y_{d_Y-1}$. Then since the given sequence $Y = Y_{d_Y} \to Y_{d_Y-1} \to \ldots \to Y_2 \to Y_1 \to \mathrm{Spec}(k) = Y_0$ of parametrizing morphisms for Y over k is *of strictly decreasing type*, by applying a similar argument to the argument in the proof of Claim 4.2.B.1 in the proof of [2], Lemma 4.2, (ii) [cf. also Lemma 2.2 of the present paper], we conclude that $\alpha(\Delta_{X/X_{d_X-1}}) \subseteq \Delta_{Y/Y_{d_Y-1}}$. Thus, it follows from assertion (i) that $\alpha(\Delta_{X/X_{d_X-1}}) = \Delta_{Y/Y_{d_Y-1}}$, as desired. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). Let us first observe that we may assume without loss of generality, by replacing (X, Y) by (Y, X) if necessary, that $d_X \leq d_Y$. Next, it follows immediately from assertion (iii) and [2], Proposition 2.4, (i), that we may assume without loss of generality — by replacing Π_X, Π_Y by $\Pi_{X_1} = \Pi_X/\Delta_{X/X_1}$, $\Pi_{Y_{d_Y-d_X+1}} = \Pi_Y/\Delta_{Y/Y_{d_Y-d_X+1}} = \Pi_Y/\alpha(\Delta_{X/X_1})$, respectively — that X is *of dimension one*. Then assertion (iv) follows from assertion (ii). This completes the proof of assertion (iv), hence also of Lemma 2.3. □

The first main anabelian result of the present paper is as follows.

THEOREM 2.4. — *Let X and Y be* **hyperbolic polycurves of strictly decreasing type** *over k and*

$$\alpha \colon \Pi_X \xrightarrow{\sim} \Pi_Y$$

an isomorphism of profinite groups over G_k. Suppose that k is **generalized sub-p-adic**, *for some prime number p. Then there exists a* **unique isomorphism** $X \xrightarrow{\sim} Y$ *over k from which α* **arises**.

PROOF. — This assertion follows immediately — in light of Lemma 2.3, (iii), (iv), and [3], Theorem 4.12 — from [2], Proposition 3.2, (i), and a similar argument to the argument applied in the proof of [2], Lemma 4,2, (iii). $\square$

REMARK 2.4.1. — Let Π be a profinite group over G_k. Suppose that k is *generalized sub-p-adic*, for some prime number p. Then one immediate consequence of Theorem 2.4 is that the set of k-isomorphism classes of *hyperbolic polycurves of strictly decreasing type* over k whose étale fundamental groups are isomorphic to Π over G_k is *of cardinality* ≤ 1. On the other hand, in [8], *Sawada* proved that the set of k-isomorphism classes of *hyperbolic polycurves* over k whose étale fundamental groups are isomorphic to Π over G_k is *finite* [cf. the main result of [8]].

Next, let us recall the following important consequence of some results of [5] and [6].

LEMMA 2.5. — *Let X (respectively, Y) be a normal variety over a* **mixed-characteristic local field** k_X *(respectively, k_Y) and $\overline{k}_X$ (respectively, $\overline{k}_Y$) an algebraic closure of k_X (respectively, k_Y). Write $G_{k_X} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_X/k_X)$ and $G_{k_Y} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_Y/k_Y)$. Let*

$$\alpha \colon \Pi_X \longrightarrow \Pi_Y$$

be an **open** *homomorphism of profinite groups. Suppose that α restricts to an* **open** *homomorphism $\Delta_{X/k_X} \to \Delta_{Y/k_Y}$, which thus implies that α induces a [necessarily* **open**] *homomorphism of profinite groups*

$$\alpha_G \colon G_{k_X} \longrightarrow G_{k_Y}.$$

Suppose, moreover, that there exists a connected finite étale covering $Y' \to Y$ of Y such that Y' has a **tripodal unit**. *Then there exists a* **unique isomorphism of fields** $\overline{k}_Y \xrightarrow{\sim} \overline{k}_X$ *which restricts to a finite [necessarily injective] homomorphism $k_Y \hookrightarrow k_X$ and from which the open homomorphism $\alpha_G \colon G_{k_X} \to G_{k_Y}$* **arises**.

PROOF. — Let us first observe that it follows from our assumption that we may assume without loss of generality, by replacing Π_Y by a suitable open subgroup of Π_Y, that Y has a *tripodal unit*. Next, let us observe that a tripodal unit of Y determines a *dominant* morphism from Y to a *tripod* T over k_Y, i.e., a hyperbolic curve over k_Y *of type* $(0, 3)$. Thus, we may assume without loss of generality, by replacing α by the composite of α and a [necessarily *open*] homomorphism $\Pi_Y \to \Pi_T$ that arises from a dominant morphism $Y \to T$ over k, that Y is a *tripod* over k_Y. Then Lemma 2.5 follows from a similar argument to the argument applied in the proof of [5], Theorem 3.5, (iii), together with the assertion $(*^{\text{A-qLT}})$ of [5], Remark 3.8.1, whose proof was given in [6], Appendix. This completes the proof of Lemma 2.5. $\square$

The second main anabelian result of the present paper is as follows.

Theorem 2.6. — *Let X (respectively, Y) be a* **hyperbolic polycurve of strictly decreasing type** *over a field k_X (respectively, k_Y) and*

$$\alpha\colon \Pi_X \xrightarrow{\sim} \Pi_Y$$

an isomorphism of profinite groups. Suppose that one of the following two conditions is satisfied:

(1) *Both k_X and k_Y are* **finitely generated** *over $\mathbb{Q}$.*

(2) *Both k_X and k_Y are* **mixed-characteristic local fields***, and, moreover, either X or Y has a connected finite étale covering that has a* **tripodal unit**.

Then there exists a **unique isomorphism** *$X \xrightarrow{\sim} Y$ from which α* **arises**.

Proof. — Suppose that condition (1) (respectively, (2)) is satisfied. Let us first observe that it follows from a similar argument to the argument applied in the proof of [2], Corollary 3.20, (i) (respectively, from [5], Corollary 2.8, (ii)), that α restricts to an isomorphism $\Delta_{X/k_X} \xrightarrow{\sim} \Delta_{Y/k_Y}$. Moreover, since [we have assumed that] condition (1) (respectively, (2)) is satisfied, it follows immediately from [2], Proposition 3.19, (ii) [i.e., the main result of [7] — cf. also [10] for a survey on [7]] (respectively, Lemma 2.5), that we may assume without loss of generality that $k_X = k_Y$, and that the isomorphism α lies over the *identity automorphism* of the absolute Galois group of $k_X = k_Y$. Thus, it follows from Theorem 2.4 that there exists a *unique isomorphism* $X \xrightarrow{\sim} Y$ from which α *arises*, as desired. This completes the proof of Theorem 2.6. $\square$

In the remainder of the present §2, let us consider a *refinement* of Theorem 2.6 in the case where condition (2) is satisfied, and, moreover, Y is *of dimension one*.

Definition 2.7. — We shall say that a hyperbolic curve X over k is *of pseudo-Belyi type* if there exists a connected finite étale covering $Y \to X$ of X such that Y has a tripodal unit.

Remark 2.7.1. — Let X be a hyperbolic curve over a *mixed-characteristic local field.* Then it is immediate that the following two conditions are equivalent:

(1) The hyperbolic curve X is of pseudo-Belyi type and defined over a finite extension of $\mathbb{Q}$.

(2) The hyperbolic curve X is of quasi-Belyi type [cf. [4], Definition 2.3, (iii)].

Remark 2.7.2. — Let X be a hyperbolic curve over k. Then it follows from Lemma 1.4 that the following assertions hold:

(i) Let $x \in X$ be a point of X. Then there exists an open neighborhood $U \subseteq X$ of $x \in X$ such that U is a hyperbolic curve over k of pseudo-Belyi type.

(ii) Let Y be a hyperbolic curve over k and $Y \to X$ a dominant morphism over k. Suppose that X is of pseudo-Belyi type. Then Y is of pseudo-Belyi type.

PROPOSITION 2.8. — *Every hyperbolic curve of genus* ≤ 1 *over* k *is* **of pseudo-Belyi type**.

PROOF. — It is immediate that every hyperbolic curve of genus 0 over k is of pseudo-Belyi type. Let X be a hyperbolic curve of genus 1 over k and (X^{cpt}, D) a pair as in Definition 1.5 [i.e., for the smooth curve X over k]. Then we may assume without loss of generality, by replacing k by a suitable finite extension of k, that every point of D is k-rational. Thus, it follows from Remark 2.7.2, (ii), that we may assume without loss of generality, by replacing D by a single k-rational point of D, that D consists of a single k-rational point, and X^{cpt} has a structure of *elliptic curve whose origin is given by the closed subscheme* $D \subseteq X^{\mathrm{cpt}}$. Write $2_X \colon X^{\mathrm{cpt}} \to X^{\mathrm{cpt}}$ for the endomorphism of the elliptic curve (X^{cpt}, D) given by multiplication by 2 and $E \subseteq X^{\mathrm{cpt}}$ for the pull-back of $D \subseteq X^{\mathrm{cpt}}$ by $2_X \colon X^{\mathrm{cpt}} \to X^{\mathrm{cpt}}$. Then we may assume without loss of generality, by replacing k by a suitable finite extension of k, that every point of E is k-rational. Moreover, one verifies easily that the endomorphism 2_X of X^{cpt} determines a *connected finite étale covering* $X^{\mathrm{cpt}} \setminus E \to X^{\mathrm{cpt}} \setminus D \cong X$. Then, by considering the quotient of $X^{\mathrm{cpt}} \setminus E$ by the automorphism of $X^{\mathrm{cpt}} \setminus E$ determined by the automorphism of the elliptic curve (X^{cpt}, D) given by multiplication by -1, one may conclude that the hyperbolic curve $X^{\mathrm{cpt}} \setminus E$ has a tripodal unit. In particular, the hyperbolic curve X is of pseudo-Belyi type, as desired. This completes the proof of Proposition 2.8. □

THEOREM 2.9. — *Let X be a* **normal variety** *over a* **mixed-characteristic local field** k_X*, Y a* **hyperbolic curve** *over a* **mixed-characteristic local field** k_Y*, and*

$$\alpha \colon \Pi_X \longrightarrow \Pi_Y$$

an **open** *homomorphism of profinite groups. Suppose that the following two conditions are satisfied:*

(1) *The open homomorphism α restricts to an* **open** *homomorphism* $\Delta_{X/k_X} \to \Delta_{Y/k_Y}$ [*which is the case if, for instance, the open homomorphism α is an* **isomorphism** — *cf.* [5], *Corollary* 2.8, (ii)].

(2) *The hyperbolic curve Y is* **of pseudo-Belyi type** [*which is the case if, for instance, the hyperbolic curve Y is* **of genus ≤ 1** — *cf. Proposition* 2.8].

Then there exists a **unique dominant** *morphism $X \to Y$ from which α* **arises**.

PROOF. — Let us first observe that it follows from condition (2) — together with [5], Remark 3.8.1, and [6], Appendix [cf. also the proof of Lemma 2.5 of the present paper] — that the extension Π_Y [i.e., of the absolute Galois group of k_Y] is *of A-qLT-type* [cf. [5], Definition 3.1, (v)]. Thus, Theorem 2.9 follows from [2], Proposition 3.2, (i), and [2], Corollary 3.20, (iii), i.e., in the case where conditions (1) and (iii-c) are satisfied [i.e., a partial generalization — to the case where the "domain" is the étale fundamental group of a *normal* variety — of [5], Corollary 3.8, in the case where the condition (g) is satisfied]. This completes the proof of Theorem 2.9. □

3. Existence of an Anabelian Open Basis

In the present §3, we prove that every smooth variety over a *generalized sub-p-adic field*, for some prime number p, has an open basis for the Zariski topology consisting of "*anabelian*" varieties [cf. Corollary 3.4, (i), below]. Moreover, we also discuss an absolute version of this result [cf. Corollary 3.4, (ii), (iii), below].

In the present §3, let k be a perfect field and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Definition 3.1.

(i) We shall say that a class $\mathcal{C}$ of smooth varieties over k is *relatively anabelian* over k if, for smooth varieties X, Y that belong to $\mathcal{C}$, the natural map

$$\mathrm{Isom}_k(X,Y) \longrightarrow \mathrm{Isom}_{G_k}(\Pi_X,\Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$$

is bijective.

(ii) We shall say that a class $\mathcal{C}$ of smooth varieties over fields is *absolutely anabelian* if, for smooth varieties X, Y that belong to $\mathcal{C}$ [note that the base field of X is not necessarily isomorphic to the base field of Y], the natural map

$$\mathrm{Isom}(X,Y) \longrightarrow \mathrm{Isom}(\Pi_X,\Pi_Y)/\mathrm{Inn}(\Pi_Y)$$

is bijective.

Corollary 3.2. — *The following assertions hold:*

(i) *Let k be a* **generalized sub-p-adic field**, *for some prime number p. Then the class consisting of* **hyperbolic polycurves of strictly decreasing type** *over k is* **relatively anabelian** *over k.*

(ii) *The class consisting of* **hyperbolic polycurves of strictly decreasing type** *over fields* **finitely generated** *over $\mathbb{Q}$ is* **absolutely anabelian**.

(iii) *The class consisting of* **hyperbolic polycurves of strictly decreasing type** *over* **mixed-characteristic local fields** *that have* **tripodal units** *is* **absolutely anabelian**.

Proof. — Assertion (i) follows from Theorem 2.4. Assertions (ii), (iii) follow from Theorem 2.6. This completes the proof of Corollary 3.2. □

Definition 3.3. — We shall say that a smooth variety X over k has a *relatively anabelian open basis* (respectively, an *absolutely anabelian open basis*) if there exist an open basis for the Zariski topology of X and a class $\mathcal{C}$ of smooth varieties over k (respectively, over fields) such that $\mathcal{C}$ is relatively anabelian over k (respectively, absolutely anabelian), and, moreover, each member of the open basis belongs to $\mathcal{C}$.

COROLLARY 3.4. — *The following assertions hold:*

(i) *Every smooth variety over a* **generalized sub-*p*-adic field**, *for some prime number p, has a* **relatively anabelian open basis**.

(ii) *Every smooth variety over a field* **finitely generated** *over* $\mathbb{Q}$ *has an* **absolutely anabelian open basis**.

(iii) *Every smooth variety* **of positive dimension** *over a* **mixed-characteristic local field** *has an* **absolutely anabelian open basis**.

PROOF. — Assertion (i) follows from Lemma 1.12 and Corollary 3.2, (i). Assertion (ii) in the case where the smooth variety is *of dimension zero* follows from [2], Proposition 3.19, (ii) [i.e., the main result of [7] — cf. also [10] for a survey on [7]]. Assertion (ii) in the case where the smooth variety is *of positive dimension* follows from Lemma 1.12 and Corollary 3.2, (ii). Assertion (iii) follows from Lemma 1.12 and Corollary 3.2, (iii). This completes the proof of Corollary 3.4. □

REMARK 3.4.1.

(i) In [9], Corollary 1.7, *Schmidt* and *Stix* proved the assertion that, in the terminology of the present paper,

> $(*)$ if k is a field *finitely generated* over $\mathbb{Q}$, then every smooth variety over k has a *relatively anabelian open basis*,

that may be regarded as an assertion weaker than Corollary 3.4, (ii). On the other hand, let us observe that one verifies immediately that Corollary 3.4, (ii), may also be easily derived from [9], Corollary 1.7, and [2], Proposition 3.19, (ii) [i.e., the main result of [7] — cf. also [10] for a survey on [7]].

(ii) The assertion $(*)$ of (i) was predicted by *Grothendieck* in his letter to Faltings [cf. [1]]. Here, let us observe that Corollary 3.4, (i), may be regarded as a *substantial refinement* of this prediction $(*)$ of (i).

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