The Absolute Anabelian Geometry of Quasi-tripods

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Abstract. — In the present paper, we study the absolute anabelian geometry of hyperbolic orbicurves. The first main result of the present paper shows the absolute version of the Grothendieck conjecture for quasi-tripods — e.g., hyperbolic curves of genus less than two — over, for instance, finitely generated extensions of mixed-characteristic local fields. Moreover, we prove some absolute anabelian results for certain hyperbolic polycurves as applications of the first main result. Finally, we also show the absolute version of the Grothendieck conjecture for MLF-isotrivial hyperbolic orbicurves over finitely generated extensions of mixed-characteristic local fields.

Contents

Introduction

In the present Introduction, let p be a prime number and k a field of characteristic zero. We shall say that [cf. Definition 1.1 and Definition 1.2]

• the field k is *algebraic* if k is algebraic over the [unique] subfield of k isomorphic to the field of rational numbers,

• the field k is *p-adic local* if k is isomorphic to a finite extension of the field of fractions of the ring of Witt vectors with coefficients in a finite field of characteristic p,

• the field k is *mixed-characteristic local* if k is l-adic local for some prime number l,

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• the field k is *sub-p-adic* if k is isomorphic to a subfield of a field finitely generated over a p-adic local field,

• the field k is *generalized sub-p-adic* if k is isomorphic to a subfield of a field finitely generated over the field of fractions of the ring of Witt vectors with coefficients in an algebraic extension of a finite field of characteristic p, and

• the field k is *strictly sub-p-adic* if k is sub-p-adic and contains a p-adic local field.

The [*relative version* of the] Grothendieck conjecture for hyperbolic [orbi]curves over generalized sub-p-adic fields was proved by *Mochizuki* [cf. [12], Theorem A, and [13], Theorem 4.12; also Theorem 4.5 of the present paper]. In the present paper, we study the *absolute anabelian geometry* of hyperbolic orbicurves over generalized sub-p-adic fields. In particular, we discuss the *absolute version* of the Grothendieck conjecture for *quasi-tripods* over certain generalized sub-p-adic fields.

Let X be a hyperbolic orbicurve [cf. Definition 3.5, (i)] over k. Then we shall say that X is a *quasi-tripod* [cf. Definition 3.7] if there exist a positive integer n and a sequence $(X = X_1, X_2, \ldots, X_{n-1}, X_n)$ of hyperbolic orbicurves such that X_n is a tripod [i.e., a hyperbolic curve of type $(0, 3)$ — cf. Definition 3.4, (ii)], and, moreover, for each $i \in \{1, \ldots, n-1\}$, the hyperbolic orbicurve X_i is related to the hyperbolic orbicurve X_{i+1} in one of the following four ways:

(1) There exists a finite étale covering $X_{i+1} \to X_i$.

(2) There exists a finite étale covering $X_i \to X_{i+1}$.

(3) There exists an open immersion $X_i \hookrightarrow X_{i+1}$.

(4) There exists a morphism $X_i \to X_{i+1}$ such that the induced morphism between the associated coarse spaces is an isomorphism.

The notion of a quasi-tripod may be regarded as a generalization of the notion of a hyperbolic curve *of Belyi type* defined in [16], Definition 2.3, (ii). More precisely, if k is mixed-characteristic local, and X is a hyperbolic curve over k, then it holds that X is *of Belyi type* if and only if X is a *quasi-tripod* and, moreover, may be defined over an *algebraic* subfield of k. Here, one verifies easily from elementary algebraic geometry the following three assertions:

• An arbitrary nonempty open substack of a *quasi-tripod* is a *quasi-tripod*.

• For each point of a hyperbolic curve over k, there exists an open neighborhood of the point which is a *quasi-tripod*.

• Every hyperbolic curve *of genus* ≤ 1 over k is a *quasi-tripod*.

The first main result of the present paper, i.e., Corollary 5.6, shows the *absolute version* of the Grothendieck conjecture for *quasi-tripods* over certain generalized sub-p-adic fields. One consequence of Corollary 5.6 is as follows:

THEOREM A. — *For $\Box \in \{\circ, \bullet\}$, let $p_\Box$ be a prime number, $k_\Box$ a field of characteristic zero, and $X_\Box$ a hyperbolic orbicurve over $k_\Box$; write $\Pi_{X_\Box}$ for the étale fundamental group* [*relative to an appropriate choice of basepoint*] *of $X_\Box$. Let*

$$\alpha\colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Suppose that either $X_\circ$ *or* $X_\bullet$ *is a* **quasi-tripod**. *Suppose, moreover, that one of the following three conditions is satisfied:*

(1) *For each* $\Box \in \{\circ, \bullet\}$*, the field* $k_\Box$ *is* **algebraic**, **generalized sub-$p_\Box$-adic**, *and* **Hilbertian**.

(2) *For each* $\Box \in \{\circ, \bullet\}$*, the field* $k_\Box$ *is* **finitely generated** *and* **transcendental** *over a field that is* **algebraic** *and* **sub-$p_\Box$-adic**.

(3) *For each* $\Box \in \{\circ, \bullet\}$*, the field* $k_\Box$ *is* **strictly sub-$p_\Box$-adic**.

Then the isomorphism α **arises** *from a* **unique isomorphism** $X_\circ \xrightarrow{\sim} X_\bullet$ *of stacks.*

Note that Theorem A may be regarded as a *generalization* of the absolute anabelian result for hyperbolic curves *of Belyi type* over *mixed-characteristic local* fields proved by *Mochizuki*, i.e., [16], Corollary 2.3, in the case where condition (b) is satisfied.

Before proceeding, we pause to observe [cf. Remark 5.6.1] that if one replaces conditions (1), (2), and (3) in the statement of Theorem A by, for instance, the condition that

for each $\Box \in \{\circ, \bullet\}$, the field $k_\Box$ is *sub-$p_\Box$-adic*,

then the conclusion of Theorem A *no longer holds* in general. Indeed, if one writes $\mathbb{Q}$ for the field of rational numbers, $\mathbb{Q}_p$ for the p-adic completion of $\mathbb{Q}$, and F for the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_p$, then it is immediate that both $\mathbb{Q}_p$ and F are *sub-p-adic*. On the other hand, it follows from *Krasner's lemma* [cf., e.g., [20], Lemma 8.1.6] that the natural inclusion $F \hookrightarrow \mathbb{Q}_p$ induces an *outer isomorphism* between the respective absolute Galois groups. In particular, for an arbitrary hyperbolic curve X [e.g., a tripod, that is a *quasi-tripod*] over F, the first projection $X \times_F \mathbb{Q}_p \to X$ induces an *outer isomorphism* between the respective étale fundamental groups. However, one verifies easily that the scheme $X \times_F \mathbb{Q}_p$ is *not isomorphic* to the scheme X.

In [4], the author of the present paper studied the relative version of the Grothendieck conjecture for *hyperbolic polycurves* [cf. Definition 6.1 of the present paper] over sub-p-adic fields. In §6 of the present paper, we prove some absolute anabelian results for certain hyperbolic polycurves as applications of the first main result of the present paper. The first application, i.e., Corollary 6.4, is an absolute anabelian result for *configuration spaces* of hyperbolic curves [cf. Definition 6.2]. This result may be derived from the first main result of the present paper, together with the anabelian techniques applied in [18], §1. One consequence of Corollary 6.4 is as follows:

THEOREM B. — *For* $\Box \in \{\circ, \bullet\}$*, let* $d_\Box$ *be a positive integer,* $p_\Box$ *a prime number,* $k_\Box$ *a field of characteristic zero, and* $C_\Box$ *a hyperbolic curve over* $k_\Box$*; write* $X_\Box$ *for the* $d_\Box$*-th* **configuration space** *of* $C_\Box$ *and* $\Pi_{X_\Box}$ *for the étale fundamental group [relative to an appropriate choice of basepoint] of* $X_\Box$*. Let*

$$\alpha \colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Suppose that one of the following three conditions is satisfied:

(A) *Either* $C_\circ$ *or* $C_\bullet$ *is a* **quasi-tripod**.

(B) *Either* $C_\circ$ *or* $C_\bullet$ *is* **affine**. *Moreover, the inequality* $\max\{d_\circ, d_\bullet\} \geq 2$ *holds.*

(C) *The inequality* $\max\{d_\circ, d_\bullet\} \geq 3$ *holds.*

Suppose, moreover, that one of the three conditions (1)*,* (2)*, and* (3) *in the statement of Theorem* A *is satisfied. Then the isomorphism* α **arises** *from a* **unique isomorphism** $X_\circ \xrightarrow{\sim} X_\bullet$ *of schemes.*

The second application, i.e., Corollary 6.10, is a result concerning an *absolutely anabelian open basis*. Let us recall that *Schmidt* and *Stix* proved that every smooth variety over a field finitely generated over the field of rational numbers has an open basis for the Zariski topology consisting of "*anabelian*" varieties [cf. [22], Corollary 1.7]. This was predicted by *Grothendieck* in his letter to Faltings. Moreover, the author of the present paper generalized this result of *Schmidt* and *Stix* to smooth varieties over *generalized sub-p-adic fields* [cf. [8], Theorem A].

In [8], the author of the present paper also discussed an absolute version of the above prediction due to *Grothendieck*. We shall say that a smooth variety over k has an *absolutely anabelian open basis* [cf. Definition 6.9] if there exists an open basis for the Zariski topology of the variety such that, for arbitrary members U and V of the open basis, the natural map

$$\mathrm{Isom}(U, V) \longrightarrow \mathrm{Isom}(\Pi_U, \Pi_V)/\mathrm{Inn}(\Pi_V)$$

is bijective — where we write "$\Pi_{(-)}$" for the étale fundamental group [relative to an appropriate choice of basepoint] of "$(-)$". One consequence of Corollary 6.10 — that may be regarded as a *generalization* of some results of [8], hence also of [22], Corollary 1.7 — is as follows:

THEOREM C. — *Suppose that one of the following three conditions is satisfied:*

(1) *The field* k *is* **algebraic***,* **generalized sub-*p*-adic***, and* **Hilbertian***.*

(2) *The field* k *is* **finitely generated** *and* **transcendental** *over a field that is* **algebraic** *and* **sub-*p*-adic***.*

(3) *The field* k *is* **strictly sub-*p*-adic***.*

Then every smooth variety of positive dimension over the field k *has an* **absolutely anabelian open basis***.*

Finally, in §7 of the present paper, we also study the *absolute anabelian geometry* of *MLF-isotrivial* hyperbolic orbicurves. Let X be a hyperbolic orbicurve over k. Then we shall say that [cf. Definition 3.6]

• the hyperbolic orbicurve X is *strictly MLF-isotrivial* if there exist a mixed-characteristic local subfield $k_0 \subseteq k$ of k, a hyperbolic orbicurve X_0 over k_0, and an isomorphism $X \xrightarrow{\sim} X_0 \times_{k_0} k$ over k, and

• the hyperbolic orbicurve X is *MLF-isotrivial* if there exists a finite extension K of k such that the hyperbolic orbicurve $X \times_k K$ over K is strictly MLF-isotrivial.

The main result of §7 of the present paper, i.e., Theorem 7.1, shows the *absolute version* of the Grothendieck conjecture for *MLF-isotrivial* hyperbolic orbicurves over fields that are *strictly sub-p-adic* but *not p-adic local*. One consequence of Theorem 7.1 is as follows:

THEOREM D. — *For* $\square \in \{\circ, \bullet\}$*, let* $p_\square$ *be a prime number,* $k_\square$ *a field that is* **strictly sub-$p_\square$-adic** *but* **not $p_\square$-adic local***, and* $X_\square$ *a* **hyperbolic orbicurve** *over* $k_\square$*; write* $\Pi_{X_\square}$ *for the étale fundamental group [relative to an appropriate choice of basepoint] of* $X_\square$*. Let*

$$\alpha \colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Suppose that one of the following two conditions is satisfied:

(1) *Both* $X_\circ$ *and* $X_\bullet$ *are* **MLF-isotrivial**.

(2) *The hyperbolic orbicurve* $X_\circ$ *is* **strictly MLF-isotrivial***, and, moreover, the unique maximal* $p_\circ$*-adic local subfield of* $k_\circ$ *[cf. Proposition* 2.11*,* (iii)*] is* **absolutely abelian** *[cf.* [7]*, Definition* 4.2*,* (ii)*].*

Then the isomorphism α **arises** *from a* **unique isomorphism** $X_\circ \xrightarrow{\sim} X_\bullet$ *of stacks.*

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1. SOME GENERALITIES ON QUASI-p-ADIC LOCAL FIELDS

In the present §1, we discuss some generalities on *quasi-p-adic local fields* [cf. Definition 1.2, (ii)]. In the present §1, let p be a prime number and k a field of characteristic zero.

DEFINITION 1.1.

(i) We shall write $k_{\mathrm{rtn}} \subseteq k$ for the [unique] subfield of k isomorphic to the field of rational numbers. We shall write (p) for the [unique] nonarchimedean prime of k_{rtn} of residue characteristic p.

(ii) We shall write $k_{\mathrm{alg}} \subseteq k$ for the [unique] maximal subfield of k algebraic over k_{rtn}.

(iii) We shall say that k is *algebraic* if the equality $k = k_{\mathrm{alg}}$ holds.

(iv) We shall say that k is a *number field* if k is finite over k_{rtn}.

REMARK 1.1.1. — One verifies easily the following three assertions:

(i) If k is *algebraic*, then an arbitrary algebraic extension of k is *algebraic*.

(ii) If k is a *number field*, then an arbitrary finite extension of k is a *number field*.

(iii) A *number field* is *algebraic*.

DEFINITION 1.2.

(i) We shall say that k is *p-adic local* if k is isomorphic to a finite extension of the field of fractions of the ring of Witt vectors with coefficients in a finite field of characteristic p. We shall say that k is *mixed-characteristic local* if k is l-adic local for some prime number l.

(ii) We shall say that k is *quasi-p-adic local* if k is isomorphic to a finite extension of the field of fractions of the ring of Witt vectors with coefficients in an algebraic extension of a finite field of characteristic p.

(iii) We shall say that k is *sub-p-adic* if k is isomorphic to a subfield of a field finitely generated over a p-adic local field.

(iv) We shall say that k is *generalized sub-p-adic* if k is isomorphic to a subfield of a field finitely generated over a quasi-p-adic local field.

(v) We shall say that k is *strictly sub-p-adic* if k is sub-p-adic and contains a p-adic local field.

REMARK 1.2.1. — One verifies easily the following three assertions:

(i) If k is *p-adic local* (respectively, *quasi-p-adic local*), then an arbitrary finite extension of k is *p-adic local* (respectively, *quasi-p-adic local*).

(ii) If k is *sub-p-adic* (respectively, *generalized sub-p-adic*; *strictly sub-p-adic*), then an arbitrary finitely generated extension of k is *sub-p-adic* (respectively, *generalized sub-p-adic*; *strictly sub-p-adic*).

(iii) The following implications hold:

$$\begin{array}{ccccc} & & & & \text{number field} \\ & & & & \Downarrow \\ p\text{-adic local} & \Longrightarrow & \text{strictly sub-}p\text{-adic} & \Longrightarrow & \text{sub-}p\text{-adic} \\ \Downarrow & & & & \Downarrow \\ \text{quasi-}p\text{-adic local} & & \Longrightarrow & & \text{generalized sub-}p\text{-adic} \end{array}$$

LEMMA 1.3. — *It holds that the field k is* **p-adic local** (*respectively,* **quasi-p-adic local**) *if and only if there exists a prime $\mathfrak{p}$ of k_{alg} over (p) such that the degree* (*respectively, ramification index*) *of $\mathfrak{p}$ over (p) is* **finite**, *and, moreover, the natural inclusion $k_{\mathrm{alg}} \hookrightarrow k$ determines an* **isomorphism** *of the completion of k_{alg} at the prime $\mathfrak{p}$ with k.*

PROOF. — This assertion follows immediately from the definitions of the notion of a p-adic local field and the notion of a quasi-p-adic local field. □

LEMMA 1.4. — *Let $F \subseteq k$ be an* **algebraic** *subfield of k* [*so $k_{\mathrm{rtn}} \subseteq F$*] *and $\mathfrak{p}$ a prime of F over (p) such that the ramification index of $\mathfrak{p}$ over (p) is* **finite**. *Write $F^{\wedge}$ for the* *completion of F at the prime $\mathfrak{p}$ and $\mathcal{O}_{F^\wedge} \subseteq F^\wedge$ for the valuation ring of $F^\wedge$ with respect to the valuation determined by $\mathfrak{p}$. Then the following hold:*

(i) *The subring of $F^\wedge$ generated by the subset*

$$\bigcup_{l:\, prime} \Big(\bigcap_{n \geq 1} (F^\wedge)^{l^n}\Big) \subseteq F^\wedge$$

coincides *with the subring $\mathcal{O}_{F^\wedge} \subseteq F^\wedge$.*

(ii) *There exists a* **unique** *prime number l such that $l \cdot \mathcal{O}_{F^\wedge} \neq \mathcal{O}_{F^\wedge}$. Moreover, this unique prime number* **coincides** *with p.*

(iii) *The collection $\{p^n \cdot \mathcal{O}_{F^\wedge}\}_{n \geq 1}$ of ideals of $\mathcal{O}_{F^\wedge}$ determines a* **complete discrete valuation** *on $F^\wedge$. Moreover, this resulting complete discrete valuation on $F^\wedge$* **coincides** *with the valuation on $F^\wedge$ determined by $\mathfrak{p}$. In particular, the prime of F determined by this complete discrete valuation on $F^\wedge$* **coincides** *with the prime $\mathfrak{p}$.*

PROOF. — First, we verify assertion (i). Write $\boldsymbol{\mu} \subseteq \mathcal{O}_{F^\wedge}^\times$ for the subgroup consisting of roots of unity in $F^\wedge$, $\mathfrak{m} \subseteq \mathcal{O}_{F^\wedge}$ for the maximal ideal of the local ring $\mathcal{O}_{F^\wedge}$, $S \stackrel{\text{def}}{=} \bigcup_{l:\,\text{prime}} (\bigcap_{n \geq 1} (F^\wedge)^{l^n}) \subseteq F^\wedge$, and $R \subseteq F^\wedge$ for the subring of $F^\wedge$ generated by $S \subseteq F^\wedge$. Then since the ramification index of $\mathfrak{p}$ over (p) is *finite*, the quotient $(F^\wedge)^\times / \mathcal{O}_{F^\wedge}^\times$ is *cyclic* and *infinite*. Thus, the inclusion $S \subseteq \mathcal{O}_{F^\wedge}$, hence also the inclusion $R \subseteq \mathcal{O}_{F^\wedge}$, holds. Next, let us observe that one verifies immediately the inclusion $\boldsymbol{\mu} \subseteq S$. Moreover, one also verifies immediately the equality $(1 + \mathfrak{m})^l = 1 + \mathfrak{m}$ for an arbitrary prime number l not equal to p, hence also the inclusion $1 + \mathfrak{m} \subseteq S$. Thus, assertion (i) follows from the [easily verified] fact that the ring $\mathcal{O}_{F^\wedge}$ is *generated* by $\boldsymbol{\mu}$ and $1 + \mathfrak{m}$. This completes the proof of assertion (i). Assertions (ii), (iii) are immediate. This completes the proof of Lemma 1.4. □

DEFINITION 1.5. — We shall write $\mathcal{O}_k \subseteq k$ for the subring of k generated by the subset

$$\bigcup_{l:\,\text{prime}} \Big(\bigcap_{n \geq 1} k^{l^n}\Big) \subseteq k.$$

REMARK 1.5.1. — If the field k is *quasi-p-adic local*, then it follows from Lemma 1.3 and Lemma 1.4, (i), that the natural inclusion $\mathcal{O}_k \hookrightarrow k$ determines an *isomorphism* of the field of fractions of $\mathcal{O}_k$ with k.

DEFINITION 1.6. — Suppose that k is quasi-p-adic local.

(i) We shall write p_k for the [unique — cf. Lemma 1.3 and Lemma 1.4, (ii)] prime number such that $p_k \cdot \mathcal{O}_k \neq \mathcal{O}_k$. [So $p_k = p$ — cf. Lemma 1.3 and Lemma 1.4, (ii).]

(ii) The collection $\{p_k^n \cdot \mathcal{O}_k\}_{n \geq 1}$ of ideals of $\mathcal{O}_k$ determines a complete discrete valuation on k [cf. Lemma 1.3 and Lemma 1.4, (ii), (iii)]. We shall write $\mathfrak{p}_k$ for the prime of k_{alg} determined by this complete discrete valuation on k.

2. Anabelian Results for Finitely Generated Transcendental Extensions

In the present §2, we prove some *anabelian results* for finitely generated transcendental extensions. In particular, we discuss a consequence of anabelian results proved by *Mochizuki* and *Pop* [cf. Theorem 2.14 below]. In the present §2, let p be a prime number, k a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\overline{k}$; $\Omega \subseteq \overline{k}$ for the algebraic closure of k_{rtn} [hence also of k_{alg}] in $\overline{k}$;

$$G_k^{\mathrm{alg}} \stackrel{\text{def}}{=} \mathrm{Gal}(\Omega/k_{\mathrm{alg}}) \subseteq G_k^{\mathrm{rtn}} \stackrel{\text{def}}{=} \mathrm{Gal}(\Omega/k_{\mathrm{rtn}})$$

for the respective absolute Galois groups of k_{alg}, k_{rtn} determined by the algebraic closure Ω. Thus, the natural inclusion $\Omega \hookrightarrow \overline{k}$ determines a *surjective* homomorphism of profinite groups

$$G_k \longrightarrow\!\!\!\!\!\!\rightarrow G_k^{\mathrm{alg}},$$

by means of which we shall regard the profinite group G_k^{alg} as a quotient of the profinite group G_k.

Lemma 2.1. — *Suppose that k is* **quasi-p-adic local**. *Then the following hold:*

(i) *The natural surjective homomorphism $G_k \twoheadrightarrow G_k^{\mathrm{alg}}$ is an* **isomorphism**.

(ii) *Write $\widetilde{\mathfrak{p}}_k$ for the nonarchimedean prime of Ω determined by, relative to the natural inclusion $\Omega \hookrightarrow \overline{k}$, the unique extension to $\overline{k}$ of the complete discrete valuation on k of Definition* 1.6, (ii)*; $I_{\widetilde{\mathfrak{p}}_k} \subseteq D_{\widetilde{\mathfrak{p}}_k} \subseteq G_k^{\mathrm{rtn}}$ for the inertia, decomposition subgroups of G_k^{rtn} associated to $\widetilde{\mathfrak{p}}_k$, respectively. Then the following two conditions are satisfied:*

(1) *The inclusion $G_k^{\mathrm{alg}} \subseteq D_{\widetilde{\mathfrak{p}}_k}$ holds.*

(2) *The closed subgroup $G_k^{\mathrm{alg}} \subseteq G_k^{\mathrm{rtn}}$ contains an* **open** *subgroup of $I_{\widetilde{\mathfrak{p}}_k} \subseteq G_k^{\mathrm{rtn}}$.*

If, moreover, the field k is **p-adic local**, *then the closed subgroup $G_k^{\mathrm{alg}} \subseteq G_k^{\mathrm{rtn}}$ contains an* **open** *subgroup of $D_{\widetilde{\mathfrak{p}}_k} \subseteq G_k^{\mathrm{rtn}}$.*

Proof. — These assertions follow immediately — in light of Lemma 1.3 — from *Krasner's lemma* [cf., e.g., [20], Lemma 8.1.6]. □

Lemma 2.2. — *Suppose that k is* **quasi-p-adic local**. *Then the set of primes of k_{alg} over (p) of* **finite ramification index** *is* **of cardinality one**, *i.e., consists of the prime $\mathfrak{p}_k$ of Definition* 1.6, (ii).

Proof. — This assertion follows immediately — in light of Lemma 1.3 — from Lemma 2.1, (ii), and [15], Proposition 2.3, (iv). □

Lemma 2.3. — *Suppose that k is* **sub-p-adic** (*respectively,* **generalized sub-p-adic**). *Then there exists a prime $\widetilde{\mathfrak{p}}$ of Ω over (p) such that if one writes $I_{\widetilde{\mathfrak{p}}} \subseteq D_{\widetilde{\mathfrak{p}}} \subseteq G_k^{\mathrm{rtn}}$ for the inertia, decomposition subgroups of G_k^{rtn} associated to $\widetilde{\mathfrak{p}}$, respectively, then the closed subgroup $G_k^{\mathrm{alg}} \subseteq G_k^{\mathrm{rtn}}$ contains an* **open** *subgroup of $D_{\widetilde{\mathfrak{p}}} \subseteq G_k^{\mathrm{rtn}}$ (respectively, $I_{\widetilde{\mathfrak{p}}} \subseteq G_k^{\mathrm{rtn}}$).*

Proof. — Let K_0 be a *p-adic local* (respectively, *quasi-p-adic local*) field, K a field finitely generated over K_0, and $k \hookrightarrow K$ an injective homomorphism of fields. Then observe that, to verify Lemma 2.3, we may assume without loss of generality, by replacing k by K, that $k = K$. Next, observe that, to verify Lemma 2.3, we may assume without loss of generality, by replacing K_0 by the algebraic closure of K_0 in K, that $(K_0)_{\text{alg}} = K_{\text{alg}}$. Thus, Lemma 2.3 follows from Lemma 2.1, (ii). This completes the proof of Lemma 2.3. □

Definition 2.4. — Let G be a group and $H \subseteq G$ a subgroup of G.

(i) We shall write

$$\begin{aligned} & Z_G(H) \stackrel{\text{def}}{=} \{\, g \in G \,|\, ghg^{-1} = h \text{ for every } h \in H \,\} \\ \subseteq\ & N_G(H) \stackrel{\text{def}}{=} \{\, g \in G \,|\, gHg^{-1} = H \,\} \\ \subseteq\ & C_G(H) \stackrel{\text{def}}{=} \{\, g \in G \,|\, [H : H \cap (gHg^{-1})] < \infty,\ [gHg^{-1} : H \cap (gHg^{-1})] < \infty \,\} \end{aligned}$$

for the *centralizer*, *normalizer*, and *commensurator* of H in G, respectively. We shall write $Z(G) \stackrel{\text{def}}{=} Z_G(G)$ for the *center* of G.

(ii) We shall say that H is *commensurably terminal* in G if the equality $H = C_G(H)$, or, alternatively, the inclusion $C_G(H) \subseteq H$, holds.

(iii) Suppose that G is a profinite group. Then we shall say that G is *slim* if the equality $Z_G(H) = \{1\}$ holds for every open subgroup $H \subseteq G$ of G. Note that one verifies immediately that it holds that G is slim if and only if every open subgroup of G is center-free [i.e., has a trivial center].

Lemma 2.5. — *Let $\widetilde{\mathfrak{p}}$ be a nonarchimedean prime of Ω. Write $I_{\widetilde{\mathfrak{p}}} \subseteq D_{\widetilde{\mathfrak{p}}} \subseteq G_k^{\text{rtn}}$ for the inertia, decomposition subgroups of G_k^{rtn} associated to $\widetilde{\mathfrak{p}}$, respectively. Then the following hold:*

(i) *Let $H \subseteq D_{\widetilde{\mathfrak{p}}}$ be a* **nontrivial** *[not necessarily closed] subgroup of $D_{\widetilde{\mathfrak{p}}}$. Then the inclusion $C_{G_k^{\text{rtn}}}(H) \subseteq D_{\widetilde{\mathfrak{p}}}$ holds. In particular, the closed subgroup $D_{\widetilde{\mathfrak{p}}} \subseteq G_k^{\text{rtn}}$ is* **commensurably terminal** *in G_k^{rtn}.*

(ii) *Let $J \subseteq I_{\widetilde{\mathfrak{p}}}$ be an* **open** *subgroup of $I_{\widetilde{\mathfrak{p}}}$. Then the action of $N_{D_{\widetilde{\mathfrak{p}}}}(J)$ on $N_{D_{\widetilde{\mathfrak{p}}}}(J) \cap I_{\widetilde{\mathfrak{p}}}$ by conjugation is* **faithful**.

(iii) *Let $J \subseteq I_{\widetilde{\mathfrak{p}}}$ be an* **open** *subgroup of $I_{\widetilde{\mathfrak{p}}}$. Then the equality $Z_{G_k^{\text{rtn}}}(J) = \{1\}$ holds.*

Proof. — First, we verify assertion (i). Since $D_{\widetilde{\mathfrak{p}}}$ is *torsion-free* [cf. [15], Proposition 2.3, (iii)], the group H is *infinite*. Let $\gamma \in C_{G_k^{\text{rtn}}}(H)$ be an element of the commensurator of H in G_k^{rtn}. Write $\widetilde{\mathfrak{q}}$ for the nonarchimedean prime of Ω obtained by forming the conjugate of $\widetilde{\mathfrak{p}}$ by γ and $D_{\widetilde{\mathfrak{q}}} \subseteq G_k^{\text{rtn}}$ for the decomposition subgroup of G_k^{rtn} associated to $\widetilde{\mathfrak{q}}$. Then since H is *infinite* as already verified above, the intersection $H \cap (\gamma H \gamma^{-1})$, hence also the intersection $D_{\widetilde{\mathfrak{p}}} \cap D_{\widetilde{\mathfrak{q}}}$, is *nontrivial*. In particular, it follows from [15], Proposition 2.3, (iv), that $\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{q}}$, i.e., that $\gamma \in D_{\widetilde{\mathfrak{p}}}$, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since J is *open* in $I_{\widetilde{\mathfrak{p}}}$, one verifies easily that the closed subgroup $N_{D_{\widetilde{\mathfrak{p}}}}(J) \subseteq D_{\widetilde{\mathfrak{p}}}$ of $D_{\widetilde{\mathfrak{p}}}$ is *open*. Thus, it follows immediately from [20], Proposition 7.5.2, and [20], Proposition 7.5.4, (ii), that the natural outer action of $N_{D_{\widetilde{\mathfrak{p}}}}(J)/(N_{D_{\widetilde{\mathfrak{p}}}}(J) \cap$

$I_{\widetilde{\mathfrak{p}}}$) on $N_{D_{\widetilde{\mathfrak{p}}}}(J) \cap I_{\widetilde{\mathfrak{p}}}$ is *faithful*. On the other hand, let us recall from [13], Lemma 4.14, that $I_{\widetilde{\mathfrak{p}}}$ is *slim*, which thus implies that $N_{D_{\widetilde{\mathfrak{p}}}}(J) \cap I_{\widetilde{\mathfrak{p}}}$ is *center-free*. Thus, assertion (ii) holds. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). It follows from assertion (i) that $Z_{G_k^{\mathrm{rtn}}}(J) = Z_{D_{\widetilde{\mathfrak{p}}}}(J)$. Let $\gamma \in Z_{G_k^{\mathrm{rtn}}}(J) = Z_{D_{\widetilde{\mathfrak{p}}}}(J)$ be an element of the centralizer of J in G_k^{rtn}. Then since $I_{\widetilde{\mathfrak{p}}}$ is *slim* [cf. the proof of assertion (ii)], the action of $N_{D_{\widetilde{\mathfrak{p}}}}(J) \cap I_{\widetilde{\mathfrak{p}}}$ on J by conjugation is *faithful*. Thus, since [one verifies easily that] the resulting *injective* homomorphism $N_{D_{\widetilde{\mathfrak{p}}}}(J) \cap I_{\widetilde{\mathfrak{p}}} \hookrightarrow \mathrm{Aut}(J)$ is *compatible* with the respective natural actions of γ on $N_{D_{\widetilde{\mathfrak{p}}}}(J) \cap I_{\widetilde{\mathfrak{p}}}$ and $\mathrm{Aut}(J)$ [i.e., by conjugation], we conclude that γ *centralizes* $N_{D_{\widetilde{\mathfrak{p}}}}(J) \cap I_{\widetilde{\mathfrak{p}}}$. Thus, it follows from assertion (ii) that $\gamma = 1$, as desired. This completes the proof of assertion (iii), hence also of Lemma 2.5. $\square$

In the remainder of the present §2, for $\square \in \{\circ, \bullet\}$, let $p_\square$ be a prime number, $k_\square$ a field of characteristic zero, and $\overline{k}_\square$ an algebraic closure of $k_\square$; write $\Omega_\square \subseteq \overline{k}_\square$ for the algebraic closure of $(k_\square)_{\mathrm{rtn}}$ [hence also of $(k_\square)_{\mathrm{alg}}$] in $\overline{k}_\square$ and

$$G_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square) \twoheadrightarrow G_{k_\square}^{\mathrm{alg}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}\big(\Omega_\square/(k_\square)_{\mathrm{alg}}\big) \subseteq G_{k_\square}^{\mathrm{rtn}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}\big(\Omega_\square/(k_\square)_{\mathrm{rtn}}\big)$$

for the respective absolute Galois groups of $k_\square$, $(k_\square)_{\mathrm{alg}}$, $(k_\square)_{\mathrm{rtn}}$ determined by the algebraic closures $\overline{k}_\square$, $\Omega_\square$, $\Omega_\square$.

LEMMA 2.6. — *Suppose that, for each $\square \in \{\circ, \bullet\}$, the field $k_\square$ is* **quasi-$p_\square$-adic local**. *Let $\phi\colon (k_\bullet)_{\mathrm{alg}} \hookrightarrow (k_\circ)_{\mathrm{alg}}$ be an injective homomorphism of fields. Then the following hold:*

(i) *The equality $p_\circ = p_\bullet$ holds.*

(ii) *The homomorphism ϕ is* **compatible** *with the primes $\mathfrak{p}_{k_\bullet}$, $\mathfrak{p}_{k_\circ}$ of Definition* 1.6, (ii).

PROOF. — First, we verify assertion (i). It follows from Lemma 2.1, (i), and condition (2) of Lemma 2.1, (ii), together with [20], Proposition 7.5.1, that the profinite group $G_{k_\circ}^{\mathrm{alg}}$ contains a closed subgroup that is *free pro-$p_\circ$* and *not topologically finitely generated*. Thus, since [it is immediate that] the outer homomorphism $G_{k_\circ}^{\mathrm{alg}} \to G_{k_\bullet}^{\mathrm{alg}}$ of profinite groups induced by ϕ is *injective*, it follows from condition (1) of Lemma 2.1, (ii), that the decomposition subgroup $D_{\widetilde{\mathfrak{p}}_{k_\bullet}}$ associated to the prime $\widetilde{\mathfrak{p}}_{k_\bullet}$ introduced in the statement of Lemma 2.1 [i.e., in the case where we take the "$(k, \overline{k})$" of Lemma 2.1 to be $(k_\bullet, \overline{k}_\bullet)$] contains a closed subgroup that is *free pro-$p_\circ$* and *not topologically finitely generated*. In particular, it follows immediately from [20], Proposition 7.5.1, and [20], Proposition 7.5.2, that $p_\circ = p_\bullet$, as desired. This completes the proof of assertion (i). Assertion (ii) follows from assertion (i) and Lemma 2.2. This completes the proof of Lemma 2.6. $\square$

PROPOSITION 2.7. — *Suppose that, for each $\square \in \{\circ, \bullet\}$, the field $k_\square$ is* **quasi-$p_\square$-adic local**. *Then the natural map*

$$\mathrm{Isom}(k_\bullet, k_\circ) \longrightarrow \mathrm{Isom}\big((k_\bullet)_{\mathrm{alg}}, (k_\circ)_{\mathrm{alg}}\big)$$

is **bijective**.

PROOF. — The surjectivity of the map under consideration follows immediately from Lemma 2.6, (ii), together with Lemma 1.3 and Lemma 2.2. Next, to verify the injectivity of the map under consideration, observe that it is immediate that every isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of fields is *compatible* with the respective *topologies* determined by the complete discrete valuations on $k_\bullet$, $k_\circ$ discussed in Definition 1.6, (ii). On the other hand, the subfields $(k_\bullet)_{\mathrm{alg}}$, $(k_\circ)_{\mathrm{alg}}$ are *dense* in $k_\bullet$, $k_\circ$ with respect to these *topologies*, respectively. Thus, the desired injectivity holds. This completes the proof of Proposition 2.7. □

PROPOSITION 2.8. — *Suppose that $k_\circ$ satisfies one of the following two conditions:*

(1) *The field $k_\circ$ is* **algebraic** *and* **generalized sub-$\boldsymbol{p}_\circ$-adic**.

(2) *The field $k_\circ$ is* **quasi-$\boldsymbol{p}_\circ$-adic local**.

Then the natural map

$$\mathrm{Isom}(k_\bullet, k_\circ) \longrightarrow \mathrm{Isom}(G_{k_\circ}, G_{k_\bullet})/\mathrm{Inn}(G_{k_\bullet})$$

is **injective**.

PROOF. — Let us first observe that, to verify Proposition 2.8, we may assume without loss of generality, by considering the difference of two elements of the domain of the map under consideration whose images coincide, that $k_\circ = k_\bullet$. Next, let us observe that it follows from Lemma 2.1, (i), and Proposition 2.7 that, to verify Proposition 2.8, we may assume without loss of generality, by replacing $k_\circ$ by $(k_\circ)_{\mathrm{alg}}$, that $k_\circ$ satisfies condition (1). Then the desired injectivity is a formal consequence of Lemma 2.3 and Lemma 2.5, (iii). This completes the proof of Proposition 2.8. □

LEMMA 2.9. — *Suppose that, for each $\square \in \{\circ, \bullet\}$, the field $k_\square$ is* **quasi-$\boldsymbol{p}_\square$-adic local**. *Let $\alpha \colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ be an outer isomorphism of profinite groups. Then the following two conditions are equivalent:*

(1) *The outer isomorphism $\alpha \colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$* **arises** *from an isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of fields.*

(2) *The composite*

$$G_{k_\circ}^{\mathrm{alg}} \underset{\sim}{\longleftarrow} G_{k_\circ} \underset{\sim}{\xrightarrow{\alpha}} G_{k_\bullet} \underset{\sim}{\longrightarrow} G_{k_\bullet}^{\mathrm{alg}}$$

— where the first and third arrows are the isomorphisms of Lemma 2.1, (i) *—* **arises** *from an isomorphism $(k_\bullet)_{\mathrm{alg}} \xrightarrow{\sim} (k_\circ)_{\mathrm{alg}}$ of fields.*

PROOF. — This assertion follows immediately from Lemma 2.1, (i), and Proposition 2.7. □

LEMMA 2.10. — *Suppose that, for each $\square \in \{\circ, \bullet\}$, the field $k_\square$ is* **quasi-$\boldsymbol{p}_\square$-adic local**. *Suppose, moreover, that one of the following two conditions is satisfied:*

(1) *For each $\square \in \{\circ, \bullet\}$, the field $k_\square$ is* **$\boldsymbol{p}_\square$-adic local**.

(2) *The field* $k_\bullet$ *is isomorphic to a finite extension of the field of fractions of the ring of Witt vectors with coefficients in an* **algebraic closure** *of a finite field of characteristic* $p_\bullet$.

Then an arbitrary injective homomorphism $k_\bullet \hookrightarrow k_\circ$ *of fields is* **finite**.

PROOF. — Let $\phi\colon k_\bullet \hookrightarrow k_\circ$ be an injective homomorphism of fields. Write $\phi_{\mathrm{alg}}\colon (k_\bullet)_{\mathrm{alg}} \hookrightarrow (k_\circ)_{\mathrm{alg}}$ for the injective homomorphism of fields determined by ϕ. Then it follows immediately from Lemma 2.1, (i), (ii), that ϕ_{alg} is *finite*. Thus, it follows from Lemma 2.6, (ii), together with Lemma 1.3 and Lemma 2.2, that ϕ is *finite*, as desired. This completes the proof of Lemma 2.10. □

PROPOSITION 2.11. — *The following hold:*

(i) *The following three conditions are equivalent:*

(1) *The field* k *is* **strictly sub-*p*-adic**.

(2) *The field* k *contains a* ***p*-adic local** *field and is* **sub-*l*-adic** *for some prime number* l.

(3) *The field* k *is* **finitely generated** *over a* ***p*-adic local** *field.*

(ii) *Suppose that* k *is* **strictly sub-*p*-adic** *and contains an* ***l*-adic local** *field for some prime number* l. *Then the equality* $l = p$ *holds.*

(iii) *Suppose that* k *is* **strictly sub-*p*-adic**. *Then there exists a* **unique maximal mixed-characteristic local** *subfield of* k. *Moreover, this mixed-characteristic local subfield is* ***p*-adic local** *and* **algebraically closed** *in* k, *and the field* k *is* **finitely generated** *over this mixed-characteristic local subfield.*

(iv) *Suppose that* k *contains a field isomorphic to the field of fractions of the ring of Witt vectors with coefficients in an* **algebraic closure** *of a finite field of characteristic* p. *Then it holds that* k *is* **generalized sub-*l*-adic** *for some prime number* l *if and only if* k *is* **finitely generated** *over a* **quasi-*p*-adic local** *field.*

PROOF. — First, we verify assertion (i). The implications (3) ⇒ (1) ⇒ (2) are immediate. To verify the implication (2) ⇒ (3), suppose that condition (2) is satisfied, i.e., that there exist a *p-adic local* subfield $k_0 \subseteq k$ of k, an *l-adic local* field K_0, a field K finitely generated over K_0, and an injective homomorphism $k \hookrightarrow K$ of fields. Now let us observe that, to verify condition (3), we may assume without loss of generality, by replacing K_0 by the algebraic closure of K_0 in K, that K_0 is *algebraically closed* in K. Then since [one verifies easily that] the quotient $K^\times/K_0^\times$ is a *free* module, the equality $\mathcal{O}_{K_0} = \mathcal{O}_K$ of the subrings of Definition 1.5 holds. Thus, the injective homomorphism $k \hookrightarrow K$ restricts to an injective homomorphism $\mathcal{O}_{k_0} \hookrightarrow (\mathcal{O}_k \hookrightarrow \mathcal{O}_K =)\ \mathcal{O}_{K_0}$, hence also [cf. Remark 1.5.1] an injective homomorphism $k_0 \hookrightarrow K_0$. In particular, it follows from Lemma 2.10 that the resulting injective homomorphism $k_0 \hookrightarrow K_0$ is *finite*, which thus implies that K, hence also k, is *finitely generated* over k_0, as desired. This completes the proof of the implication (2) ⇒ (3), hence also of assertion (i).

Next, we verify assertion (ii). Let $k_0 \subseteq k$ be a *p-adic local* subfield of k over which k is *finitely generated* [cf. assertion (i)] and $k_l \subseteq k$ an *l-adic local* subfield of k. Now let us observe that, to verify assertion (ii), we may assume without loss of generality, by replacing k_0 by the algebraic closure of k_0 in k, that k_0 is *algebraically closed* in k, which thus implies [cf. the proof of assertion (i)] that $\mathcal{O}_{k_0} = \mathcal{O}_k$. Thus, the natural inclusion $k_l \hookrightarrow k$ restricts to an injective homomorphism $\mathcal{O}_{k_l} \hookrightarrow (\mathcal{O}_k =) \mathcal{O}_{k_0}$, hence also [cf. Remark 1.5.1] an injective homomorphism $k_l \hookrightarrow k_0$. In particular, it follows immediately from Lemma 2.6, (i), that $l = p$, as desired. This completes the proof of assertion (ii).

Assertion (iii) follows from assertions (i), (ii) [cf. also the proof of assertion (ii)]. Assertion (iv) follows immediately from a similar argument to the argument applied in the proof of [the equivalence (2) $\Leftrightarrow$ (3) in] assertion (i). This completes the proof of Proposition 2.11. $\square$

Definition 2.12. — Suppose that k is strictly sub-p-adic.

(i) We shall write $k_{\mathrm{MLF}} \subseteq k$ for the [unique] maximal [necessarily p-adic local] mixed-characteristic local subfield of k [cf. Proposition 2.11, (iii)]. Thus, it follows from Proposition 2.11, (iii), that the subfield k_{MLF} is algebraically closed in k, and the field k is finitely generated over k_{MLF}.

(ii) We shall write d_k for the extension degree of k_{MLF} over the [unique] minimal mixed-characteristic local subfield of k_{MLF} [i.e., over the subfield of k_{MLF} obtained by forming the closure of $k_{\mathrm{rtn}} \subseteq k_{\mathrm{MLF}}$ with respect to the topology on k_{MLF} determined by the complete discrete valuation discussed in Definition 1.6, (ii)].

(iii) We shall write e_k for the absolute ramification index of the mixed-characteristic local field k_{MLF}, i.e., for $\sharp(k^{\times}_{\mathrm{MLF}}/(\mathcal{O}^{\times}_{k_{\mathrm{MLF}}} \cdot p^{\mathbb{Z}}_{k_{\mathrm{MLF}}}))$ [cf. Definition 1.5 and Definition 1.6, (i)].

Lemma 2.13. — *Let k_0 be a subfield of k. Suppose that the field k is* **finitely generated** *and* **transcendental** *over k_0, and that the subfield k_0 is* **algebraically closed** *in k. Write $\overline{k}_0 \subseteq \overline{k}$ for the algebraic closure of k_0 in $\overline{k}$. Thus, the natural inclusion $\overline{k}_0 \hookrightarrow \overline{k}$ determines a* **surjective** *homomorphism $G_k \twoheadrightarrow G_{k_0} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_0/k_0)$ of profinite groups. Write, moreover, $\Delta \subseteq G_k$ for the kernel of this surjective homomorphism $G_k \twoheadrightarrow G_{k_0}$. Then the following hold:*

(i) *The profinite group Δ is* **slim**.

(ii) *The natural outer action of G_{k_0} on Δ is* **faithful**.

(iii) *Let α, β be automorphisms of the profinite group G_k that preserve the closed subgroup $\Delta \subseteq G_k$. Then the equality $\alpha|_\Delta = \beta|_\Delta$ implies the equality $\alpha = \beta$.*

Proof. — First, we verify assertion (i). Let us first observe that since [it is immediate that] every open subgroup of Δ is the absolute Galois group of a finite extension of $k \cdot \overline{k}_0$ in $\overline{k}$, to verify assertion (i), it suffices to verify that Δ is *center-free*. Next, let us observe that since [one verifies easily that] an extension of a *center-free* group by a *center-free* group is *center-free*, to verify assertion (i), we may assume without loss of generality, by applying induction on the transcendental degree over k_0 and replacing k by a subfield of k *of transcendental degree one* over k_0, that k is *of transcendental degree one* over k_0. Let C be a projective smooth model of k over k_0; x, $y \in C$ *distinct* closed points of C; $I(x)$, $I(y) \subseteq \Delta$ inertia subgroups of Δ associated to x, $y \in C$, respectively. Then it follows immediately from the well-known structure of the étale fundamental group of an algebraic curve over an algebraically closed field of characteristic zero that $I(x) \cap I(y) = \{1\}$, and, moreover, both $I(x)$ and $I(y)$ are *commensurably terminal* in Δ. Thus, we conclude that $Z(\Delta) \subseteq C_\Delta(I(x)) \cap C_\Delta(I(y)) = I(x) \cap I(y) = \{1\}$, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $t \in k \setminus k_0$ be an element of k *transcendental* over k_0. Then it follows immediately from [9], Lemma 2.2, together with assertion (i), that, to verify assertion (ii), we may assume without loss of generality, by replacing k by the subfield $k_0(t) \subseteq k$ of k, that $k = k_0(t)$. For each $a \in \overline{k}_0$, write $\mathcal{I}(a)$ for the Δ-conjugacy class of inertia subgroups of Δ associated to the discrete $\overline{k}_0$-valuation on $k \cdot \overline{k}_0 = \overline{k}_0(t)$ determined by the *prime* element $t - a \in \overline{k}_0[t]$ of $\overline{k}_0[t]$. Then it is immediate that the assignment "$\overline{k}_0 \ni a \mapsto \mathcal{I}(a)$" is *compatible* with the natural action of G_{k_0} on $\overline{k}_0$ and the action of G_{k_0} on the set $\{\mathcal{I}(a)\}_{a \in \overline{k}_0}$ determined by the natural outer action of G_{k_0} on Δ. Thus, since the natural action of G_{k_0} on $\overline{k}_0$ is *faithful*, we conclude that the action of G_{k_0} on the set $\{\mathcal{I}(a)\}_{a \in \overline{k}_0}$, hence also the natural outer action of G_{k_0} on Δ, is *faithful*, as desired. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). It follows immediately from assertions (i), (ii) that the action of G_k on Δ by conjugation is *faithful*. Thus, since [one verifies easily that] the resulting *injective* homomorphism $G_k \hookrightarrow \mathrm{Aut}(\Delta)$ is *compatible* with the respective natural actions of $\alpha \circ \beta^{-1}$ on G_k and $\mathrm{Aut}(\Delta)$, we conclude that assertion (iii) holds. This completes the proof of assertion (iii), hence also of Lemma 2.13. $\square$

Anabelian results proved by *Mochizuki* and *Pop*, together with some results of the present §2 of the present paper, lead us to the following theorem:

THEOREM 2.14. — *For $\square \in \{\circ, \bullet\}$, let $p_\square$ be a prime number, $k_\square$ a field of characteristic zero, and $\overline{k}_\square$ an algebraic closure of $k_\square$. Moreover, for $\square \in \{\circ, \bullet\}$, write $\Omega_\square \subseteq \overline{k}_\square$ for the algebraic closure of $(k_\square)_{\mathrm{alg}}$ in $\overline{k}_\square$;*

$$G_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square) \twoheadrightarrow G^{\mathrm{alg}}_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}\big(\Omega_\square/(k_\square)_{\mathrm{alg}}\big)$$

for the respective absolute Galois groups of $k_\square$, $(k_\square)_{\mathrm{alg}}$ determined by the algebraic closures $\overline{k}_\square$, $\Omega_\square$; $\Delta_\square \subseteq G_{k_\square}$ for the kernel of the surjective homomorphism $G_{k_\square} \twoheadrightarrow G^{\mathrm{alg}}_{k_\square}$. Suppose that, for each $\square \in \{\circ, \bullet\}$, there exists a subfield $(k_\square)_0 \subseteq k_\square$ of $k_\square$ that satisfies the following two conditions:

(1) *The field $k_\square$ is* **finitely generated** *and* **transcendental** *over $(k_\square)_0$.*

(2) *The subfield $(k_\square)_0$ is* **algebraically closed** *in $k_\square$.*

Suppose, moreover, that one of the following five conditions is satisfied:

(a) *For each $\square \in \{\circ, \bullet\}$, the field $(k_\square)_0$ is* **algebraic** *and* **sub-$p_\square$-adic**.

(b) *For each $\square \in \{\circ, \bullet\}$, the field $k_\square$ is* **of transcendental degree one** *over $(k_\square)_0$, and, moreover, the field $(k_\square)_0$ is* **algebraic** *and* **generalized sub-$p_\square$-adic**.

(c) *For each $\square \in \{\circ, \bullet\}$, the field $k_\square$ is* **of transcendental degree ≥ 3** *over $(k_\square)_0$, and, moreover, the field $(k_\square)_0$ is* **algebraic** *and* **generalized sub-$p_\square$-adic**.

(d) *For each $\square \in \{\circ, \bullet\}$, the field $(k_\square)_0$ is* **$p_\square$-adic local**.

(e) *For each* $\Box \in \{\circ, \bullet\}$*, the field* $k_\Box$ *is* **of transcendental degree one** *over* $(k_\Box)_0$*, and, moreover, the field* $(k_\Box)_0$ *is* **quasi-$p_\Box$-adic local**.

Write

$$\mathrm{Isom}(k_\bullet,\, k_\circ)$$

for the set of isomorphisms $k_\bullet \xrightarrow{\sim} k_\circ$ *of fields [which necessarily restrict to isomorphisms*

$$(k_\bullet)_{\mathrm{alg}} = \big((k_\bullet)_0\big)_{\mathrm{alg}} \xrightarrow{\ \sim\ } (k_\circ)_{\mathrm{alg}} = \big((k_\circ)_0\big)_{\mathrm{alg}}$$

— cf. (2) *— of subfields] and*

$$\mathrm{OutIsom}_{\text{alg-fld}}(G_{k_\circ},\, G_{k_\bullet})$$

for the set of outer isomorphisms $\alpha\colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ *of profinite groups such that*

- *the outer isomorphism* α **restricts** *to a* $G_{k_\bullet}$*-conjugacy class of isomorphisms* $\Delta_\circ \xrightarrow{\sim} \Delta_\bullet$ *of closed subgroups, and, moreover,*
- *the resulting outer isomorphism* $G_{k_\circ}^{\mathrm{alg}} \xrightarrow{\sim} G_{k_\bullet}^{\mathrm{alg}}$ **arises** *from an isomorphism* $(k_\bullet)_{\mathrm{alg}} \xrightarrow{\sim} (k_\circ)_{\mathrm{alg}}$ *of fields.*

Then the natural map

$$\mathrm{Isom}(k_\bullet,\, k_\circ) \longrightarrow \mathrm{OutIsom}_{\text{alg-fld}}(G_{k_\circ},\, G_{k_\bullet})$$

is **bijective**.

PROOF. — Let us first observe that if condition (a) (respectively, (b); (c); (d); (e)) is satisfied, then the injectivity of the map under consideration follows immediately — in light of Lemma 2.1, (i), and Proposition 2.8 — from the injectivity portion of [12], Theorem B (respectively, [13], Theorem 4.12; [21], Theorem 1.2, (4); [12], Theorem B; [13], Theorem 4.12). The remainder of the proof of Theorem 2.14 is devoted to verifying the surjectivity of the map under consideration.

If condition (a) is satisfied, then the desired surjectivity follows immediately from the surjectivity portion of [12], Theorem B.

Next, suppose that condition (b) is satisfied. To verify the desired surjectivity, let α be an element of $\mathrm{OutIsom}_{\text{alg-fld}}(G_{k_\circ}, G_{k_\bullet})$. For $\Box \in \{\circ, \bullet\}$, let $C_\Box$ be a projective smooth model of $k_\Box$ over $(k_\Box)_0$. Then it follows immediately from similar arguments to the arguments applied in the proofs of [19], Corollary 1.11, (a), (b), that the outer isomorphism α determines a *bijection* between the set of cuspidal inertia subgroups of $G_{k_\circ}$ associated to closed points of $C_\circ$ and the set of cuspidal inertia subgroups of $G_{k_\bullet}$ associated to closed points of $C_\bullet$. Thus, the desired surjectivity follows immediately from the surjectivity portion of [13], Theorem 4.12. This completes the proof of the desired surjectivity in the case where condition (b) is satisfied.

Next, suppose that condition (c) is satisfied. To verify the desired surjectivity, let $\alpha\colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ be an isomorphism of profinite groups that *restricts* to an isomorphism $\Delta_\circ \xrightarrow{\sim} \Delta_\bullet$ of closed subgroups. Then it follows immediately from [21], Theorem 1.2, (4), that the outer isomorphism $\Delta_\circ \xrightarrow{\sim} \Delta_\bullet$ determined by the resulting isomorphism arises from an isomorphism $k_\bullet \cdot \Omega_\bullet \xrightarrow{\sim} k_\circ \cdot \Omega_\circ$ of fields. On the other hand, since [it is immediate that] the outer isomorphism $\Delta_\circ \xrightarrow{\sim} \Delta_\bullet$ is *compatible* with the respective natural outer actions of $G_{k_\circ}/\Delta_\circ = G_{k_\circ}^{\mathrm{alg}}$, $G_{k_\bullet}/\Delta_\bullet = G_{k_\bullet}^{\mathrm{alg}}$ on $\Delta_\circ$, $\Delta_\bullet$ [relative to some isomorphism $G_{k_\circ}^{\mathrm{alg}} \xrightarrow{\sim} G_{k_\bullet}^{\mathrm{alg}}$], it follows from the injectivity of the map under consideration already verified above that the isomorphism $k_\bullet \cdot \Omega_\bullet \xrightarrow{\sim} k_\circ \cdot \Omega_\circ$ is *compatible* with the respective natural actions of $G_{k_\circ}^{\mathrm{alg}}$, $G_{k_\bullet}^{\mathrm{alg}}$ on $k_\bullet \cdot \Omega_\bullet$, $k_\circ \cdot \Omega_\circ$ [relative to some isomorphism $G_{k_\circ}^{\mathrm{alg}} \xrightarrow{\sim} G_{k_\bullet}^{\mathrm{alg}}$]. In particular, by considering the $G_{k_\circ}^{\mathrm{alg}}$-, $G_{k_\bullet}^{\mathrm{alg}}$-invariants, respectively, we conclude that the isomorphism $k_\bullet \cdot \Omega_\bullet \xrightarrow{\sim} k_\circ \cdot \Omega_\circ$ of fields *restricts* to an isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of subfields. Thus, it follows immediately from Lemma 2.13, (iii), that the outer isomorphism $G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ determined by α is *contained* in the image of the map under consideration, as desired. This completes the proof of the desired surjectivity in the case where condition (c) is satisfied.

If condition (d) (respectively, (e)) is satisfied, then the desired surjectivity follows immediately — in light of Lemma 2.1, (i), and Lemma 2.9, together with the definition of the set "$\mathrm{OutIsom}_{\text{alg-fld}}(G_{k_\circ}, G_{k_\bullet})$" — from a similar argument to the argument applied in the proof of the desired surjectivity in the case where condition (a) (respectively, (b)) is satisfied. This completes the proof of Theorem 2.14. $\square$

3. Quasi-tripods

In the present §3, we introduce the notion of a *quasi-tripod* [cf. Definition 3.7 below], that is one central object of the present paper. In the present §3, let k be a field of characteristic zero.

Definition 3.1. — We shall say that a scheme X over k is a *smooth variety* over k if X is smooth, of finite type, separated, and geometrically connected over k.

Definition 3.2. — Let S be a scheme. Then we shall say that a scheme X over S is a *hyperbolic curve* [of type (g, r)] over S if there exist

- a pair (g, r) of nonnegative integers,
- a scheme X^+ over S that is smooth, proper, geometrically connected, and of relative dimension one over S, and
- a [possibly empty] closed subscheme $D \subseteq X^+$ of X^+ that is finite and étale over S

such that

- the inequality $2g - 2 + r > 0$ holds,
- each geometric fiber of X^+ over S is [a necessarily smooth proper curve] of genus g,
- the finite étale covering of S obtained by forming the composite $D \hookrightarrow X^+ \to S$ is of degree r, and
- the scheme X is isomorphic to $X^+ \setminus D$ over S.

Remark 3.2.1. — It is immediate that a *hyperbolic curve* over k is a *smooth variety* over k.

REMARK 3.2.2. — Suppose that we are in the situation of Definition 3.2.

(i) One verifies immediately from elementary algebraic geometry that if S is the *spectrum of a field*, then both X^+ and D as in Definition 3.2 are *uniquely determined up to canonical isomorphism.*

(ii) It follows from (i) that if S is the *spectrum of a field*, then (g, r) as in Definition 3.2 is *uniquely determined.* Thus, by considering a fiber of $X \to S$, one may conclude that if S is *connected*, then (g, r) as in Definition 3.2 is *uniquely determined.*

DEFINITION 3.3. — Let X be a hyperbolic curve over k. Then we shall refer to X^+ as in Definition 3.2 [which is uniquely determined up to canonical isomorphism — cf. Remark 3.2.2, (i)] as the *smooth compactification* of X.

DEFINITION 3.4. — Let X be a hyperbolic curve over S.

(i) We shall say that X is *split* if there exists D as in Definition 3.2 such that D is either empty or isomorphic, over S, to the disjoint union of finitely many copies of S.

(ii) We shall say that X is a *tripod* if X is of type $(0, 3)$.

(iii) Suppose that X is of type (g, r). Then we shall refer to the positive integer $2g + \max\{0, r - 1\}$ as the *rank* of X.

REMARK 3.4.1. — It follows from Remark 3.2.2, (ii), that the *rank* of a hyperbolic curve over a connected scheme is *uniquely determined.*

DEFINITION 3.5.

(i) We shall say that an algebraic stack X over k is a *hyperbolic orbicurve* over k if there exist

- a finite étale Galois covering $Y \to X$ whose domain Y is a hyperbolic curve over the [necessarily finite] extension of k obtained by forming the algebraic closure of k in the function field of Y and
- a dense open substack $U \subseteq X$ of X that is a scheme and is geometrically connected over k.

(ii) Let X be a hyperbolic orbicurve over k and $Y \to X$ as in (i). Then we shall refer to the stack-theoretic quotient of the smooth compactification of Y by the natural action of $\mathrm{Gal}(Y/X)$ as the *smooth compactification of X relative to $Y \to X$.*

REMARK 3.5.1. — Let X be a *smooth variety* (respectively, *hyperbolic curve*; *hyperbolic orbicurve*) over k. Then one verifies easily the following two assertions:

(i) An arbitrary nonempty open substack of X is a *smooth variety* (respectively, *hyperbolic curve*; *hyperbolic orbicurve*) over k.

(ii) Let $Y \to X$ be a connected finite étale covering of X. Then Y is a *smooth variety* (respectively, *hyperbolic curve*; *hyperbolic orbicurve*) over the [necessarily finite] extension of k obtained by forming the algebraic closure of k in the function field of Y.

DEFINITION 3.6. — Let X be a hyperbolic orbicurve over k.

(i) We shall say that X is *strictly NF-isotrivial* (respectively, *strictly MLF-isotrivial*) if there exist a subfield $k_0 \subseteq k$ of k that is a number field (respectively, mixed-characteristic local field), a hyperbolic orbicurve X_0 over k_0, and an isomorphism $X \xrightarrow{\sim} X_0 \times_{k_0} k$ over k.

(ii) We shall say that X is *NF-isotrivial* (respectively, *MLF-isotrivial*) if there exists a finite extension K of k such that the hyperbolic orbicurve $X \times_k K$ over K is strictly NF-isotrivial (respectively, strictly MLF-isotrivial).

DEFINITION 3.7. — We shall say that a hyperbolic orbicurve X over k is a *quasi-tripod* if there exist a positive integer n and a sequence $(X = X_1, X_2, \ldots, X_{n-1}, X_n)$ of hyperbolic orbicurves such that X_n is a tripod, and, moreover, for each $i \in \{1, \ldots, n-1\}$, the hyperbolic orbicurve X_i is related to the hyperbolic orbicurve X_{i+1} in one of the following four ways:

(1) There exists a finite étale covering $X_{i+1} \to X_i$.

(2) There exists a finite étale covering $X_i \to X_{i+1}$.

(3) There exists an open immersion $X_i \hookrightarrow X_{i+1}$.

(4) There exists a morphism $X_i \to X_{i+1}$ such that the induced morphism between the associated coarse spaces is an isomorphism.

REMARK 3.7.1. — Let X be a hyperbolic curve over a *mixed-characteristic local* field. Then it is immediate that the following two conditions are equivalent:

(1) The hyperbolic curve X is a *strictly NF-isotrivial quasi-tripod*.

(2) The hyperbolic curve X is *of Belyi type* [cf. [16], Definition 2.3, (ii)].

REMARK 3.7.2. — One verifies easily the following three assertions:

(i) An arbitrary nonempty open substack of a *quasi-tripod* is a *quasi-tripod*.

(ii) For each point of a hyperbolic curve over k, there exists an open neighborhood of the point which is a *quasi-tripod*.

(iii) Every hyperbolic curve *of genus* ≤ 1 over k is a *quasi-tripod* [cf. also the proof of [8], Proposition 2.8].

4. Review of Some Anabelian Results for Hyperbolic Orbicurves

In the present §4, we discuss some immediate consequences of the *Grothendieck conjecture for hyperbolic curves over generalized sub-p-adic fields* proved in [13] by *Mochizuki* [cf. Lemma 4.4, Theorem 4.5, Lemma 4.6 below]. In the present §4, let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k, and X a *hyperbolic orbicurve* over k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\overline{k}$, Π_X for the étale fundamental group [relative to an appropriate choice of basepoint] of X, and $\Delta_X \subseteq \Pi_X$ for the kernel of the natural surjective outer homomorphism $\Pi_X \twoheadrightarrow G_k$.

Lemma 4.1. — *The following hold:*

(i) *It holds that X is a* **hyperbolic curve** *over k if and only if the profinite group Δ_X is* **torsion-free***.*

(ii) *Let $H \subseteq \Delta_X$ be an open subgroup of Δ_X. Write $Y_H \to X \times_k \overline{k}$ for the connected finite étale covering of $X \times_k \overline{k}$ corresponding to the open subgroup $H \subseteq \Delta_X$. Then it holds that the profinite group H is* **torsion-free** *and* **not topologically generated by the cuspidal inertia subgroups** *of H [i.e., associated to the cusps of Y_H] if and only if the hyperbolic orbicurve Y_H is a* **hyperbolic curve of genus ≥ 1***.*

(iii) *There exists a* **characteristic** *open subgroup $H \subseteq \Delta_X$ of Δ_X [which thus implies that H is* **normal** *in Π_X] such that H is* **torsion-free** *and* **not topologically generated by the cupidal inertia subgroups** *of H. Moreover, an arbitrary open subgroup of Δ_X contained in the open subgroup H is* **torsion-free** *and* **not topologically generated by the cupidal inertia subgroups***.*

(iv) *Let H be as in* (iii)*. Write Q for the quotient of H by the normal closed subgroup normally topologically generated by the cuspidal inertia subgroups of H and $\Lambda \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\widehat{\mathbb{Z}}}(H^2(Q,\widehat{\mathbb{Z}}),\widehat{\mathbb{Z}})$. Then the profinite module Λ is a* **free $\widehat{\mathbb{Z}}$-module of rank one***.*

(v) *Suppose that we are in the situation of* (iv)*. Then the action of Π_X on Λ by conjugation [cf. the fact that H is* **normal** *in Π_X — cf.* (iii)*]* **factors** *through the natural surjective outer homomorphism $\Pi_X \twoheadrightarrow G_k$, and, moreover, the resulting character $G_k \to \widehat{\mathbb{Z}}^\times$ [cf.* (iv)*] coincides with the* **cyclotomic character** *of G_k.*

Proof. — Assertion (i) follows from [17], Lemma 4.1, (iv). Assertion (ii) follows immediately from assertion (i), together with the well-known structure of the étale fundamental group of an algebraic curve over an algebraically closed field of characteristic zero. Next, since the profinite group Δ_X is *topologically finitely generated* [cf. [17], Proposition 2.2], assertion (iii) follows immediately — in light of assertion (ii) — from elementary algebraic geometry, together with the definition of the notion of a hyperbolic orbicurve. Assertions (iv), (v) follow immediately — in light of assertion (ii) — from the elementary theory of étale cohomology of algebraic curves [cf. also the proof of the existence of the natural isomorphism between étale cohomology and group cohomology discussed in [2], Lemma 4.2, (iii)]. This completes the proof of Lemma 4.1. □

PROPOSITION 4.2. — *Write* $\Omega \subseteq \overline{k}$ *for the algebraic closure of* k_{alg} *in* $\overline{k}$, $G_k^{\mathrm{alg}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\Omega/k_{\mathrm{alg}})$ *for the absolute Galois group of* k_{alg} *determined by the algebraic closure* Ω, $\Delta \subseteq G_k$ *for the kernel of the surjective homomorphism* $G_k \twoheadrightarrow G_k^{\mathrm{alg}}$ *induced by the natural inclusion* $\Omega \hookrightarrow \overline{k}$ [*cf. the discussion at the beginning of* §2], *and*

$$\rho\colon G_k \longrightarrow \mathrm{Out}(\Delta_X)$$

for the natural outer action of G_k *on* Δ_X. *Then the inclusion* $\mathrm{Ker}(\rho) \subseteq \Delta$ *holds.*

PROOF. — Write $G_k^{\mathrm{rtn}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\Omega/k_{\mathrm{rtn}})$ for the absolute Galois group of k_{rtn} determined by the algebraic closure Ω.

Let us first observe that it follows immediately from Lemma 4.1, (iii), (iv), (v), that the cyclotomic character $G_k \to \widehat{\mathbb{Z}}^\times$ of G_k *factors* through the outer action ρ. Thus, to verify Proposition 4.2, we may assume without loss of generality, by replacing k by the finite extension of k obtained by adjoining to k the fourth roots of unity in $\overline{k}$, that the closed subgroup $G_k^{\mathrm{alg}} \subseteq G_k^{\mathrm{rtn}}$ of G_k^{rtn} does *not contain* an arbitrary decomposition subgroup of G_k^{rtn} associated to the *unique archimedean* prime of k_{rtn}.

Next, let us recall from [11], Theorem C, (ii), that if X is a *hyperbolic curve* over k, then Proposition 4.2 holds. Thus, since the profinite group Δ_X is *topologically finitely generated* [cf. [17], Proposition 2.2] and *slim* [cf. [17], Proposition 2.3, (i)], it follows from [3], Lemma 23, (i), (iii), together with the definition of the notion of a hyperbolic orbicurve, that the closed subgroup $\mathrm{Ker}(\rho) \cap \Delta \subseteq \mathrm{Ker}(\rho)$ of $\mathrm{Ker}(\rho)$ is *open*. In particular, the image of the composite $\mathrm{Ker}(\rho) \hookrightarrow G_k \twoheadrightarrow G_k/\Delta \xrightarrow{\sim} G_k^{\mathrm{alg}}$ is *finite*. Thus, it follows from [20], Theorem 12.1.7, that the image of the composite $\mathrm{Ker}(\rho) \hookrightarrow G_k \twoheadrightarrow G_k/\Delta \xrightarrow{\sim} G_k^{\mathrm{alg}}$ is *trivial*, as desired. This completes the proof of Proposition 4.2. □

LEMMA 4.3. — *In the situation of Proposition* 4.2, *the following hold:*

(i) *If* X *is either* **strictly NF-isotrivial** *or* **strictly MLF-isotrivial**, *then the equality* $\mathrm{Ker}(\rho) = \Delta$ *holds.*

(ii) *If* X *is either* **NF-isotrivial** *or* **MLF-isotrivial**, *then the closed subgroup* $\mathrm{Ker}(\rho) \subseteq \Delta$ *of* Δ [*cf. Proposition* 4.2] *is* **open**.

(iii) *Let* X_0 *be a hyperbolic orbicurve over* k_{alg} *such that there exists an isomorphism* $X \xrightarrow{\sim} X_0 \times_{k_{\mathrm{alg}}} k$ *over* k [*which thus implies that* X *is* **strictly NF-isotrivial**]. *Write* Π_{X_0} *for the étale fundamental group* [*relative to an appropriate choice of basepoint*] *of* X_0. *Then the outer homomorphism* $\Pi_X \to \Pi_{X_0}$ *induced by the composite of the isomorphism* $X \xrightarrow{\sim} X_0 \times_{k_{\mathrm{alg}}} k$ *and the first projection* $X_0 \times_{k_{\mathrm{alg}}} k \to X_0$ *is a* **surjective** *outer homomorphism whose* **kernel** *coincides with the centralizer* $Z_{\Pi_X}(\Delta_X) \subseteq \Pi_X$ *of* Δ_X *in* Π_X.

PROOF. — Assertions (i), (ii) follow from Lemma 2.1, (i), and Proposition 4.2, together with the various definitions involved. Finally, since the profinite group Δ_X is *topologically finitely generated* [cf. [17], Proposition 2.2] and *slim* [cf. [17], Proposition 2.3, (i)], assertion (iii) follows immediately from assertion (i) and [3], Lemma 4, (i). This completes the proof of Lemma 4.3. □

LEMMA 4.4. — *Suppose that k is* **generalized sub-$\boldsymbol{p}$-adic** *for some prime number p. Let X^+ be a smooth compactification of X. Then the following hold:*

(i) *Let x, $y \in X^+$ be closed points of X^+. Then the following three conditions are equivalent:*

(1) *The equality $x = y$ holds.*

(2) *The Π_X-conjugacy class of* **decomposition subgroups** *associated to x* **coincides** *with the Π_X-conjugacy class of* **decomposition subgroups** *associated to y.*

(3) *There exist decomposition subgroups $D(x)$, $D(y) \subseteq \Pi_X$ of Π_X associated to x, y, respectively, such that the image of the composite $D(x) \cap D(y) \hookrightarrow \Pi_X \twoheadrightarrow G_k$ is* **open** *in G_k.*

(ii) *An arbitrary* **decomposition subgroup** *of Π_X associated to a closed point of X^+ is* **commensurably terminal** *in Π_X.*

(iii) *Let $x \in X^+$ be a closed point of X^+. Then it holds that the image of x in the coarse space associated to X^+ is* **$\boldsymbol{k}$-rational** *if and only if a decomposition subgroup of Π_X associated to x maps* **surjectively** *onto G_k.*

PROOF. — First, we verify assertion (i). The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate. Next, let us observe that it is immediate that, to verify the implication (3) $\Rightarrow$ (1), we may assume without loss of generality, by replacing X by a suitable connected finite étale covering of X, that X is a *hyperbolic curve.* Then the implication (3) $\Rightarrow$ (1) follows immediately from [13], Theorem 4.12, together with a similar argument to the argument applied in the proof of [12], Theorem C. This completes the proof of assertion (i). Assertion (ii) is a formal consequence of assertion (i) [cf. also the proof of Lemma 2.5, (i)]. Assertion (iii) follows immediately from the various definitions involved. This completes the proof of Lemma 4.4. $\square$

THEOREM 4.5. — *For $\square \in \{\circ, \bullet\}$, let $p_\square$ be a prime number, $k_\square$ a* **generalized sub-$\boldsymbol{p_\square}$-adic** *field, $\overline{k}_\square$ an algebraic closure of $k_\square$, and $X_\square$ a* **hyperbolic orbicurve** *over $k_\square$; write $\Pi_{X_\square}$ for the étale fundamental group [relative to an appropriate choice of basepoint] of $X_\square$, $G_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square)$ for the absolute Galois group of $k_\square$ determined by the algebraic closure $\overline{k}_\square$, and $\Delta_{X_\square} \subseteq \Pi_{X_\square}$ for the kernel of the natural surjective outer homomorphism $\Pi_{X_\square} \twoheadrightarrow G_{k_\square}$. Let*

$$\alpha\colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups that **restricts** *to an isomorphism $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ of closed subgroups. Write*

$$\alpha_G\colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$$

for the isomorphism induced by α. Suppose that the isomorphism α_G **arises** *from an isomorphism $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields that restricts to an isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of subfields. Then the isomorphism α* **arises** *from a* **unique isomorphism** *$X_\circ \xrightarrow{\sim} X_\bullet$ of stacks that lies over the isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$.*

PROOF. — First, we verify the uniqueness of such an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ of stacks. Let us observe that, to verify the desired uniqueness, we may assume without loss of generality, by replacing $X_\circ$ by a suitable connected finite étale covering of $X_\circ$ and replacing $X_\bullet$ by the corresponding connected finite étale covering of $X_\bullet$ by α, that both $X_\circ$ and $X_\bullet$ are *hyperbolic curves*. Then the desired uniqueness follows from Lemma 4.4, (i), and [5], Lemma 3.5. Next, we verify the existence of such an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ of stacks. Let us observe that it follows from the uniqueness already verified above that, to verify the desired existence, we may assume without loss of generality — by applying *Galois descent*, replacing $X_\circ$ by a suitable connected finite étale covering of $X_\circ$, and replacing $X_\bullet$ by the corresponding connected finite étale covering of $X_\bullet$ by α — that both $X_\circ$ and $X_\bullet$ are *hyperbolic curves*. Then the desired existence follows immediately from [13], Theorem 4.12. This completes the proof of Theorem 4.5. □

LEMMA 4.6. — *For $\square \in \{\circ, \bullet\}$, let $p_\square$ be a prime number, $k_\square$ a* **generalized sub-$p_\square$-adic** *field, $\overline{k}_\square$ an algebraic closure of $k_\square$, and $X_\square$ a* **hyperbolic orbicurve** *over $k_\square$; write $\Pi_{X_\square}$ for the étale fundamental group [relative to an appropriate choice of basepoint] of $X_\square$, $G_{k_\square} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square)$ for the absolute Galois group of $k_\square$ determined by the algebraic closure $\overline{k}_\square$, and $\Delta_{X_\square} \subseteq \Pi_{X_\square}$ for the kernel of the natural surjective outer homomorphism $\Pi_{X_\square} \twoheadrightarrow G_{k_\square}$. Let*

$$\alpha\colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups that **restricts** *to an isomorphism $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ of closed subgroups. Write*

$$\alpha_G\colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$$

for the isomorphism induced by α. Then the following hold:

(i) *The isomorphism α determines a* **bijection** *between the set of* **cuspidal inertia subgroups** *of $\Delta_{X_\circ}$ and the set of* **cuspidal inertia subgroups** *of $\Delta_{X_\bullet}$. Moreover, the isomorphism α also determines a* **bijection** *between the set of* **cuspidal decomposition** *subgroups of $\Pi_{X_\circ}$ and the set of* **cuspidal decomposition** *subgroups of $\Pi_{X_\bullet}$.*

(ii) *The isomorphism α_G is* **compatible** *with the respective* **cyclotomic characters** *of $G_{k_\circ}$, $G_{k_\bullet}$.*

(iii) *It holds that $X_\circ$ is a* **quasi-tripod** *if and only if $X_\bullet$ is a* **quasi-tripod**.

(iv) *Suppose that either $X_\circ$ or $X_\bullet$ is a* **quasi-tripod** *[which thus implies that both $X_\circ$ and $X_\bullet$ are* **quasi-tripods** *— cf.* (iii)*]. Then, after replacing $k_\circ$ by a suitable finite extension of $k_\circ$ in $\overline{k}_\circ$ and replacing $k_\bullet$ by the corresponding finite extension of $k_\bullet$ in $\overline{k}_\bullet$ by α_G, there exist* **split tripods** *$T_\circ$, $T_\bullet$ over $k_\circ$, $k_\bullet$, respectively, and a commutative diagram of profinite groups*

$$\begin{array}{ccc} \Pi_{T_\circ} & \xrightarrow{\sim} & \Pi_{T_\bullet} \\ \downarrow & & \downarrow \\ G_{k_\circ} & \xrightarrow[\alpha_G]{\sim} & G_{k_\bullet} \end{array}$$

— *where the vertical arrows are the natural surjective outer homomorphisms, the horizontal arrows are* **isomorphisms**, *and, for* $\square \in \{\circ, \bullet\}$, *we write* $\Pi_{T_\square}$ *for the étale fundamental group [relative to an appropriate choice of basepoint] of* $T_\square$.

In the remainder of the statement of Lemma 4.6, *suppose that either* $X_\circ$ *or* $X_\bullet$ *is a* **hyperbolic curve** [*which thus implies that both* $X_\circ$ *and* $X_\bullet$ *are* **hyperbolic curves** — *cf. Lemma* 4.1, (i)].

(v) *Suppose that, for* $\square \in \{\circ, \bullet\}$, *the hyperbolic curve* $X_\square$ *over* $k_\square$ *is of type* $(g_\square, r_\square)$. *Then the equality* $(g_\circ, r_\circ) = (g_\bullet, r_\bullet)$ *holds.*

(vi) *It holds that* $X_\circ$ *is* **affine** (*respectively,* **split**; *a* **tripod**) *if and only if* $X_\bullet$ *is* **affine** (*respectively,* **split**; *a* **tripod**).

PROOF. — First, we verify assertion (i). Let us first observe that, to verify assertion (i), it follows immediately from Lemma 4.4, (ii), that we may assume without loss of generality, by replacing $X_\circ$ by a suitable connected finite étale covering of $X_\circ$ and replacing $X_\bullet$ by the corresponding connected finite étale covering of $X_\bullet$ by α, that both $X_\circ$ and $X_\bullet$ are *hyperbolic curves*. Next, let us recall from [17], Lemma 4.5, that, to verify assertion (i), it suffices to verify that, for $\square \in \{\circ, \bullet\}$, the *$p_\circ$-adic cyclotomic character* $G_{k_\square} \to \mathbb{Z}_{p_\circ}^\times$ of $G_{k_\square}$ has *open image*. To this end, let us recall that it is well-known that, for $\square \in \{\circ, \bullet\}$ and a prime number l, the restriction, to a suitable open subgroup of $\Pi_{X_\square}$, of the action, by conjugation, of $\Pi_{X_\square}$ on the *"determinant"* of $\Delta_{X_\square}^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l$ is isomorphic to a *positive power of the l-adic cyclotomic character*. In particular, since [one verifies easily that] the $p_\circ$-adic cyclotomic character of $G_{k_\circ}$ has *open image*, it follows that $p_\circ$-adic cyclotomic character of $G_{k_\bullet}$ has *open image*, as desired. This completes the proof of assertion (i).

Assertion (ii) follows immediately from assertion (i) and Lemma 4.1, (iii), (iv), (v). Next, we verify assertion (v). It follows from assertion (i) and Lemma 4.4, (i), that $r_\circ = r_\bullet$. Thus, since [it is well-known that], for $\square \in \{\circ, \bullet\}$, the abelianization of the profinite group $\Delta_{X_\square}$ is a *free $\widehat{\mathbb{Z}}$-module of rank* $2g_\square + \max\{0, r_\square - 1\}$, one may conclude the equality $(g_\circ, r_\circ) = (g_\bullet, r_\bullet)$ of assertion (v), as desired. This completes the proof of assertion (v). Assertion (vi) follows from assertions (i), (v) and Lemma 4.4, (iii).

Assertions (iii), (iv) follow — in light of assertion (vi) — from [17], Theorem 4.7, (i), (ii), together with [17], Example 4.8, (i). This completes the proof of Lemma 4.6. $\square$

5. THE ABSOLUTE ANABELIAN GEOMETRY OF QUASI-TRIPODS

In the present §5, we discuss the étale fundamental groups of *tripods over generalized sub-p-adic fields* [cf. Theorem 5.4 below]. Moreover, we also give a proof of the first main result of the present paper [cf. Corollary 5.6 below].

In the present §5, let p be a prime number, k a *generalized sub-p-adic* field, and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\overline{k}$, $\overline{k}/G_k$ for the set of G_k-orbits of $\overline{k}$, $P \stackrel{\mathrm{def}}{=} \mathbb{P}_k^1$ for the projective line over k, and P^{cl} for the set of closed points of P. By abuse of notation, we shall regard k as a subset of $\overline{k}/G_k$ in an evident way. Then the standard coordinate on the projective line P determines a bijection between P^{cl} and $(\overline{k}/G_k) \cup \{\infty\}$. We shall identify P^{cl} with $(\overline{k}/G_k) \cup \{\infty\}$ by means of this bijection:

$$P^{\mathrm{cl}} = (\overline{k}/G_k) \cup \{\infty\}.$$

Write, moreover, $T \subseteq P$ for the *split tripod* over k obtained by forming the complement in P of the closed subset determined by $\{0, 1, \infty\} \subseteq (\overline{k}/G_k) \cup \{\infty\} = P^{\mathrm{cl}}$ and $T^{\mathrm{cl}} \subseteq P^{\mathrm{cl}}$ for the set of closed points of T. Thus, the identification of P^{cl} with $(\overline{k}/G_k) \cup \{\infty\}$ determines an identification of T^{cl} with $(\overline{k}/G_k) \setminus \{0, 1\}$:

$$T^{\mathrm{cl}} = (\overline{k}/G_k) \setminus \{0, 1\}.$$

Finally, if $U \subseteq P$ is a nonempty open subscheme of P, then write Π_U for the étale fundamental group [relative to an appropriate choice of basepoint] of U and $\Delta_U \subseteq \Pi_U$ for the kernel of the natural surjective outer homomorphism $\Pi_U \twoheadrightarrow G_k$.

DEFINITION 5.1. — Let $S \subseteq \overline{k}/G_k$ be a finite subset of $\overline{k}/G_k$.

(i) We shall write $T_S \subseteq T$ for the open subscheme of T obtained by forming the complement in T of the closed subset determined by $S \setminus (S \cap \{0, 1\}) \subseteq T^{\mathrm{cl}}$.

(ii) Let a be an element of $(\overline{k}/G_k) \cup \{\infty\}$. Then we shall write $\mathcal{D}_{T_S}(a)$ for the Π_{T_S}-conjugacy class of decomposition subgroups of Π_{T_S} associated to the closed point of P $(\supseteq T_S)$ determined by $a \in (\overline{k}/G_k) \cup \{\infty\}$.

LEMMA 5.2. — *Suppose that k is* **algebraic**. *Let a, $b \in k$ be two [not necessarily distinct] elements of k. Then, after replacing k by a suitable finite extension of k in $\overline{k}$, there exist a finite subset $S \subseteq \overline{k}/G_K$ of $\overline{k}/G_K$ and a finite étale covering $T_S \to T$ over k such that a, $b \in S$.*

PROOF. — This assertion follows from [14], Corollary 1.2. □

LEMMA 5.3. — *Let $a \in k^\times$ be an element of $k^\times$. Then the following hold:*

(i) *There exists an automorphism of Π_T over G_k that maps $\mathcal{D}_T(0)$ (respectively, $\mathcal{D}_T(1)$; $\mathcal{D}_T(\infty)$) to $\mathcal{D}_T(\infty)$ (respectively, $\mathcal{D}_T(1)$; $\mathcal{D}_T(0)$). Moreover, an arbitrary such automorphism of Π_T maps $\mathcal{D}_T(a)$ to $\mathcal{D}_T(1/a)$.*

(ii) *There exists an automorphism of Π_T over G_k that maps $\mathcal{D}_T(0)$ (respectively, $\mathcal{D}_T(1)$; $\mathcal{D}_T(\infty)$) to $\mathcal{D}_T(1)$ (respectively, $\mathcal{D}_T(0)$; $\mathcal{D}_T(\infty)$). Moreover, an arbitrary such automorphism of Π_T maps $\mathcal{D}_T(a)$ to $\mathcal{D}_T(1-a)$.*

(iii) *There exists an automorphism of $\Pi_{T_{\{-1\}}}$ over G_k that maps $\mathcal{D}_{T_{\{-1\}}}(0)$ (respectively, $\mathcal{D}_{T_{\{-1\}}}(1)$; $\mathcal{D}_{T_{\{-1\}}}(\infty)$) to $\mathcal{D}_{T_{\{-1\}}}(0)$ (respectively, $\mathcal{D}_{T_{\{-1\}}}(-1)$; $\mathcal{D}_{T_{\{-1\}}}(\infty)$). Moreover, an arbitrary such automorphism of $\Pi_{T_{\{-1\}}}$ maps $\mathcal{D}_{T_{\{-1\}}}(a)$ to $\mathcal{D}_{T_{\{-1\}}}(-a)$.*

(iv) *Let $b \in k$ be an element of k. Then there exists an automorphism of $\Pi_{T_{\{b\}}}$ over G_k that maps $\mathcal{D}_{T_{\{b\}}}(0)$ (respectively, $\mathcal{D}_{T_{\{b\}}}(1)$; $\mathcal{D}_{T_{\{b\}}}(\infty)$) to $\mathcal{D}_{T_{\{b\}}}(\infty)$ (respectively, $\mathcal{D}_{T_{\{b\}}}(b)$; $\mathcal{D}_{T_{\{b\}}}(0)$). Moreover, an arbitrary such automorphism of $\Pi_{T_{\{b\}}}$ maps $\mathcal{D}_{T_{\{b\}}}(a)$ to $\mathcal{D}_{T_{\{b\}}}(b/a)$.*

PROOF. — First, we verify assertion (i). By considering the outer automorphism of Π_T induced by the automorphism of T over k given by "$t \mapsto 1/t$", one may conclude the existence of an automorphism of the desired type. Moreover, the final portion of assertion (i) follows from Lemma 4.4, (i), and Theorem 4.5, together with elementary algebraic geometry. This completes the proof of assertion (i).

Assertion (ii) (respectively, (iii); (iv)) follows from a similar argument to the argument applied in the proof of assertion (i), together with the existence of the automorphism of T (respectively, $T_{\{-1\}}$; $T_{\{b\}}$) over k given by "$t \mapsto 1-t$" (respectively, "$t \mapsto -t$"; "$t \mapsto b/t$"). This completes the proof of Lemma 5.3. $\square$

THEOREM 5.4. — *For $\square \in \{\circ, \bullet\}$, let $p_\square$ be a prime number, $k_\square$ a* **generalized sub-$p_\square$-adic** *field, $\overline{k}_\square$ an algebraic closure of $k_\square$, and $X_\square$ a hyperbolic orbicurve over $k_\square$. Moreover, for $\square \in \{\circ, \bullet\}$, write $\Omega_\square \subseteq \overline{k}_\square$ for the algebraic closure of $(k_\square)_{\mathrm{alg}}$ in $\overline{k}_\square$;*

$$G_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square) \twoheadrightarrow G^{\mathrm{alg}}_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}\big(\Omega_\square/(k_\square)_{\mathrm{alg}}\big)$$

for the respective absolute Galois groups of $k_\square$, $(k_\square)_{\mathrm{alg}}$ determined by the algebraic closures $\overline{k}_\square$, $\Omega_\square$; $\Pi_{X_\square}$ for the étale fundamental group [relative to an appropriate choice of basepoint] of $X_\square$; $\Delta_{X_\square} \subseteq \Pi_{X_\square}$ for the kernel of the natural surjective outer homomorphism $\Pi_{X_\square} \twoheadrightarrow G_{k_\square}$. Let

$$\alpha\colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups that **restricts** *to an isomorphism $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ of closed subgroups. Write*

$$\alpha_G\colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$$

for the isomorphism induced by α. Suppose that either $X_\circ$ or $X_\bullet$ is a **split tripod** *[which thus implies that both $X_\circ$ and $X_\bullet$ are* **split tripods** *— cf. Lemma 4.6, (vi)]. Then the isomorphism α_G* **restricts** *to an isomorphism of the kernel of the natural surjective homomorphism $G_{k_\circ} \twoheadrightarrow G^{\mathrm{alg}}_{k_\circ}$ with the kernel of the natural surjective homomorphism $G_{k_\bullet} \twoheadrightarrow G^{\mathrm{alg}}_{k_\bullet}$. Moreover, the resulting isomorphism $G^{\mathrm{alg}}_{k_\circ} \xrightarrow{\sim} G^{\mathrm{alg}}_{k_\bullet}$* **arises** *from an isomorphism $\Omega_\bullet \xrightarrow{\sim} \Omega_\circ$ of fields that restricts to an isomorphism $(k_\bullet)_{\mathrm{alg}} \xrightarrow{\sim} (k_\circ)_{\mathrm{alg}}$ of subfields.*

PROOF. — For $\square \in \{\circ, \bullet\}$, write $T_\square$ for the "T" of the discussion at the beginning of the present §5, i.e., in the case where we take the "k" of the discussion at the beginning of the present §5 to be $k_\square$. Then it follows from elementary algebraic geometry that, for $\square \in \{\circ, \bullet\}$, there exists an isomorphism $\iota_\square\colon X_\square \xrightarrow{\sim} T_\square$ over $k_\square$ such that the bijection [cf. Lemma 4.6, (i)] between $\{\mathcal{D}_{T_\circ}(0), \mathcal{D}_{T_\circ}(1), \mathcal{D}_{T_\circ}(\infty)\}$ and $\{\mathcal{D}_{T_\bullet}(0), \mathcal{D}_{T_\bullet}(1), \mathcal{D}_{T_\bullet}(\infty)\}$ determined by the composite $\Pi_{T_\circ} \xleftarrow{\sim} \Pi_{X_\circ} \overset{\alpha}{\xrightarrow{\sim}} \Pi_{X_\bullet} \xrightarrow{\sim} \Pi_{T_\bullet}$ — where the first, third arrows are isomorphisms over $G_{k_\circ}$, $G_{k_\bullet}$ that arise from $\iota_\circ$, $\iota_\bullet$, respectively — maps $\mathcal{D}_{T_\circ}(*)$ to $\mathcal{D}_{T_\bullet}(*)$ for each $* \in \{0, 1, \infty\}$. In the remainder of the proof of Theorem 5.4, for $\square \in \{\circ, \bullet\}$, we shall identify $X_\square$ with $T_\square$ by means of such an isomorphism $\iota_\square$.

Next, for $\square \in \{\circ, \bullet\}$, write $\rho_\square \colon G_{k_\square} \to \mathrm{Out}(\Delta_{T_\square})$ for the natural outer action of $G_{k_\square}$ on $\Delta_{T_\square}$. Thus, the isomorphism α determines a commutative diagram

$$\begin{array}{ccc} G_{k_\circ} & \xrightarrow[\sim]{\alpha_G} & G_{k_\bullet} \\ {\scriptstyle \rho_\circ}\downarrow & & \downarrow{\scriptstyle \rho_\bullet} \\ \mathrm{Out}(\Delta_{T_\circ}) & \xrightarrow[\sim]{} & \mathrm{Out}(\Delta_{T_\bullet}). \end{array}$$

Now observe that since, for $\square \in \{\circ, \bullet\}$, the split tripod $T_\square$ is *strictly NF-isotrivial*, it follows from Lemma 4.3, (i), that the outer action $\rho_\square$ factors through the natural surjective homomorphism $G_{k_\square} \twoheadrightarrow G^{\mathrm{alg}}_{k_\square}$, and the resulting outer action $G^{\mathrm{alg}}_{k_\square} \to \mathrm{Out}(\Delta_{T_\square})$ is *injective*. Thus, we conclude from the above diagram that the isomorphism α_G *restricts* to an isomorphism of the kernel of the natural surjective homomorphism $G_{k_\circ} \twoheadrightarrow G^{\mathrm{alg}}_{k_\circ}$ with the kernel of the natural surjective homomorphism $G_{k_\bullet} \twoheadrightarrow G^{\mathrm{alg}}_{k_\bullet}$. Moreover, to verify Theorem 5.4, we may assume without loss of generality — by replacing $\Pi_{X_\circ}$, $\Pi_{X_\bullet}$ by the quotients $\Pi_{X_\circ}/Z_{\Pi_{X_\circ}}(\Delta_{X_\circ})$, $\Pi_{X_\bullet}/Z_{\Pi_{X_\bullet}}(\Delta_{X_\bullet})$, respectively [cf. Lemma 4.3, (iii)] — that both $k_\circ$ and $k_\bullet$ are *algebraic*.

Next, let us verify the following assertion:

> Claim 5.4.A: There exists a *unique* map $\phi \colon k_\circ \to \overline{k}_\bullet/G_{k_\bullet}$ of sets that satisfies the following three conditions:
>
> (1) For each $a, b \in k_\circ$ — after replacing $k_\circ$ by a suitable finite extension [that may depend on the choice of $\{a, b\}$] of $k_\circ$ in $\overline{k}_\circ$ and replacing $k_\bullet$ by the corresponding finite extension of $k_\bullet$ in $\overline{k}_\bullet$ by α_G — there exists a commutative diagram of profinite groups
>
> $$\begin{array}{ccc} \Pi_{(T_\circ)_{\{a,b\}}} & \xrightarrow[\sim]{\alpha_{\{a,b\}}} & \Pi_{(T_\bullet)_{\{\phi(a),\phi(b)\}}} \\ \downarrow & & \downarrow \\ \Pi_{(T_\circ)_{\{b\}}} & \xrightarrow[\sim]{\alpha_{\{b\}}} & \Pi_{(T_\bullet)_{\{\phi(b)\}}} \\ \downarrow & & \downarrow \\ \Pi_{T_\circ} & \xrightarrow[\sim]{\alpha} & \Pi_{T_\bullet} \end{array}$$
>
> — where the horizontal arrows are *isomorphisms*, and the left-hand upper (respectively, right-hand upper; left-hand lower; right-hand lower) vertical arrow is a surjective homomorphism that arises from the natural open immersion $(T_\circ)_{\{a,b\}} \hookrightarrow (T_\circ)_{\{b\}}$ (respectively, $(T_\bullet)_{\{\phi(a),\phi(b)\}} \hookrightarrow (T_\bullet)_{\{\phi(b)\}}$; $(T_\circ)_{\{b\}} \hookrightarrow T_\circ$; $(T_\bullet)_{\{\phi(b)\}} \hookrightarrow T_\bullet$).
>
> (2) For each $a \in k_\circ$, the isomorphism α maps $\mathcal{D}_{T_\circ}(a)$ to $\mathcal{D}_{T_\bullet}(\phi(a))$.
>
> (3) The equality $(\phi(0), \phi(1)) = (0, 1)$ holds.
>
> Moreover, in the situation of (1), the isomorphism $\alpha_{\{b\}}$ (respectively, $\alpha_{\{b\}}$; $\alpha_{\{a,b\}}$) maps $\mathcal{D}_{(T_\circ)_{\{b\}}}(b)$ (respectively, $\mathcal{D}_{(T_\circ)_{\{b\}}}(a)$; $\mathcal{D}_{(T_\circ)_{\{a,b\}}}(a)$) to $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(b))$ (respectively, $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(a))$; $\mathcal{D}_{(T_\bullet)_{\{\phi(a),\phi(b)\}}}(\phi(a))$).

To this end, let us first observe that the uniqueness of such a map ϕ follows from condition (2) and Lemma 4.4, (i). Next, to verify the existence of such a map ϕ, let us observe that, for each $a, b \in k_\circ$, it follows from Lemma 5.2 and [17], Theorem 4.7, (i), (ii), together with [17], Example 4.8, (i), that — after replacing $k_\circ$ by a suitable finite extension [that may depend on the choice of $\{a, b\}$] of $k_\circ$ in $\overline{k}_\circ$ and replacing $k_\bullet$ by the corresponding finite extension of $k_\bullet$ in $\overline{k}_\bullet$ by α_G — there exists a commutative diagram as in condition (1) for some elements "$\phi(a)$" and "$\phi(b)$" of $\overline{k}_\bullet/G_{k_\bullet}$. Then since [we have assumed that] the isomorphism α maps $\mathcal{D}_{T_\circ}(*)$ to $\mathcal{D}_{T_\bullet}(*)$ for each $* \in \{0, 1, \infty\}$, by applying Lemma 4.6, (i), to the isomorphism $\alpha_{\{b\}}$, we conclude that the isomorphism $\alpha_{\{b\}}$ maps $\mathcal{D}_{(T_\circ)_{\{b\}}}(b)$ to $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(b))$. Thus, by applying Lemma 4.6, (i), to the isomorphism $\alpha_{\{a,b\}}$, we conclude that the isomorphism $\alpha_{\{a,b\}}$ maps $\mathcal{D}_{(T_\circ)_{\{a,b\}}}(a)$ to $\mathcal{D}_{(T_\bullet)_{\{\phi(a),\phi(b)\}}}(\phi(a))$, which thus implies that the isomorphism $\alpha_{\{b\}}$ maps $\mathcal{D}_{(T_\circ)_{\{b\}}}(a)$ to $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(a))$. In particular, it follows immediately from Lemma 4.4, (ii), that the "original α" [i.e., without replacing $k_\circ$, $k_\bullet$ by finite extensions of $k_\circ$, $k_\bullet$, respectively] maps $\mathcal{D}_{T_\circ}(a)$ to $\mathcal{D}_{T_\bullet}(\phi(a))$, as desired. This completes the proof of Claim 5.4.A.

Next, let us verify the following assertion:

> Claim 5.4.B: The map $\phi\colon k_\circ \to \overline{k}_\bullet/G_{k_\bullet}$ of sets in Claim 5.4.A determines a *bijection* $k_\circ \xrightarrow{\sim} k_\bullet$. Moreover, for each $a \in k_\circ^\times$ and $b \in k_\circ$, the equalities in $k_\bullet$
>
> $$\phi(1/a) = 1/\phi(a), \qquad \phi(1-a) = 1-\phi(a),$$
> $$\phi(-a) = -\phi(a), \qquad \phi(b/a) = \phi(b)/\phi(a)$$
>
> hold.

To this end, let us first observe that it follows from condition (2) of Claim 5.4.A and Lemma 4.4, (iii), that the image of ϕ is *contained* in the subset $k_\bullet \subseteq \overline{k}_\bullet/G_{k_\bullet}$. Next, let us observe that the injectivity of the map ϕ follows from condition (2) of Claim 5.4.A and Lemma 4.4, (i). Moreover, one may conclude the surjectivity [i.e., bijectivity] of the resulting injective map $k_\circ \hookrightarrow k_\bullet$ by applying Claim 5.4.A to the inverse of α.

Next, to verify the [fourth] equality $\phi(b/a) = \phi(b)/\phi(a)$, let us recall that, in the diagram of condition (1) of Claim 5.4.A, the isomorphism $\alpha_{\{b\}}$ maps

$$\mathcal{D}_{(T_\circ)_{\{b\}}}(0) \quad \big(\text{respectively, } \mathcal{D}_{(T_\circ)_{\{b\}}}(1); \quad \mathcal{D}_{(T_\circ)_{\{b\}}}(\infty); \quad \mathcal{D}_{(T_\circ)_{\{b\}}}(a)\big)$$

to

$$\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(0) \quad \Big(\text{respectively, } \mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(1); \quad \mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\infty); \quad \mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}\big(\phi(a)\big)\Big).$$

Let β be an automorphism of $\Pi_{(T_\circ)_{\{b\}}}$ over $G_{k_\circ}$ that maps

$$\mathcal{D}_{(T_\circ)_{\{b\}}}(0) \quad \big(\text{respectively, } \mathcal{D}_{(T_\circ)_{\{b\}}}(1); \quad \mathcal{D}_{(T_\circ)_{\{b\}}}(\infty); \quad \mathcal{D}_{(T_\circ)_{\{b\}}}(a)\big)$$

to

$$\mathcal{D}_{(T_\circ)_{\{b\}}}(\infty) \quad \big(\text{respectively, } \mathcal{D}_{(T_\circ)_{\{b\}}}(b); \quad \mathcal{D}_{(T_\circ)_{\{b\}}}(0); \quad \mathcal{D}_{(T_\circ)_{\{b\}}}(b/a)\big)$$

[cf. Lemma 5.3, (iv)]. Then since the isomorphism $\alpha_{\{b\}}$ maps $\mathcal{D}_{(T_\circ)_{\{b\}}}(b)$ to $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(b))$ [cf. the final portion of Claim 5.4.A], it is immediate that the automorphism $\alpha_{\{b\}} \circ \beta \circ \alpha_{\{b\}}^{-1}$

of $\Pi_{(T_\bullet)_{\{\phi(b)\}}}$ is an automorphism *over* $G_{k_\bullet}$ and, moreover, maps

$$\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(0) \quad \big(\text{respectively, } \mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(1); \quad \mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\infty)\big)$$

to

$$\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\infty) \quad \Big(\text{respectively, } \mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}\big(\phi(b)\big); \quad \mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(0)\Big).$$

Thus, again by Lemma 5.3, (iv), we conclude that the automorphism $\alpha_{\{b\}} \circ \beta \circ \alpha_{\{b\}}^{-1}$ maps $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(a))$ to $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(b)/\phi(a))$. In particular, since the isomorphism $\beta \circ \alpha_{\{b\}}^{-1}$ maps $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(a))$ to $\mathcal{D}_{(T_\circ)_{\{b\}}}(b/a)$ [cf. the final portion of Claim 5.4.A], the isomorphism $\alpha_{\{b\}}$ maps $\mathcal{D}_{(T_\circ)_{\{b\}}}(b/a)$ to $\mathcal{D}_{(T_\bullet)_{\{\phi(b)\}}}(\phi(b)/\phi(a))$, which thus implies that $\phi(b/a) = \phi(b)/\phi(a)$, as desired. This completes the proof of the equality $\phi(b/a) = \phi(b)/\phi(a)$.

The [first, second] equalities $\phi(1/a) = 1/\phi(a)$, $\phi(1-a) = 1-\phi(a)$ follow from a similar argument to the argument applied in the proof of the equality $\phi(b/a) = \phi(b)/\phi(a)$, together with Lemma 5.3, (i), (ii).

Next, observe that it follows from the [first] equality $\phi(1/a) = 1/\phi(a)$ already verified above, together with the equality $\phi(1) = 1$ [cf. condition (3) of Claim 5.4.A], that $\phi(-1) = -1$. Thus, the [third] equality $\phi(-a) = -\phi(a)$ follows from a similar argument to the argument applied in the proof of the equality $\phi(b/a) = \phi(b)/\phi(a)$, together with Lemma 5.3, (iii). This completes the proof of Claim 5.4.B.

Next, we verify the following assertion:

> Claim 5.4.C: The bijection $\phi\colon k_\circ \xrightarrow{\sim} k_\bullet$ of sets [cf. Claim 5.4.B] is an *isomorphism of fields*.

To this end, let us first recall from condition (3) of Claim 5.4.A that $\phi(0) = 0$, $\phi(1) = 1$. Moreover, it follows from Claim 5.4.B that, for each $a \in k_\circ^\times$ and $b \in k_\circ$,

$$\phi(a \cdot b) = \phi\big(b/(1/a)\big) = \phi(b)/\phi(1/a) = \phi(b)/\big(1/\phi(a)\big) = \phi(a) \cdot \phi(b),$$

$$\phi(a+b) = \phi\Big(a \cdot \big(1-(-b/a)\big)\Big) = \phi(a) \cdot \phi\big(1-(-b/a)\big) = \phi(a) \cdot \big(1-\phi(-b/a)\big)$$
$$= \phi(a) \cdot \big(1+\phi(b/a)\big) = \phi(a) \cdot \big(1+\phi(b)/\phi(a)\big) = \phi(a)+\phi(b),$$

as desired. This completes the proof of Claim 5.4.C.

By applying Claim 5.4.C to the various restrictions of α to the open subgroups of $\Pi_{T_\circ}$ that arise from the open subgroups of $G_{k_\circ}$, we obtain an *isomorphism* $\overline{k}_\circ \xrightarrow{\sim} \overline{k}_\bullet$ *of fields* [that restricts to an isomorphism $k_\circ \xrightarrow{\sim} k_\bullet$ of subfields]. Moreover, one verifies easily that, for $\Box \in \{\circ, \bullet\}$, the identification of the set of closed points of $T_\Box \times_{k_\Box} \overline{k}_\Box$ with $\overline{k}_\Box \setminus \{0,1\}$ by the standard coordinate on $T_\Box$ is $\Pi_{T_\Box}$-*equivariant*. Thus, we conclude that the isomorphism $\alpha_G\colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ *arises* from the inverse of the isomorphism $\overline{k}_\circ \xrightarrow{\sim} \overline{k}_\bullet$ of fields, as desired. This completes the proof Theorem 5.4. $\Box$

LEMMA 5.5. — *Let F be a field and $\overline{F}$ a separable closure of F. Write $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$ for the absolute Galois group of F determined by the separable closure $\overline{F}$. Consider the following three conditions:*

(1) *There exists a subfield* $F_0 \subseteq F$ *of* F *such that* F *is* **finitely generated** *and* **transcendental** *over* F_0.

(2) *The field* F *is* **Hilbertian**.

(3) *The profinite group* G_F *is* **nontrivial**, *and an arbitrary open subgroup of* G_F *has* **no nontrivial topologically finitely generated normal closed subgroup**. [*In particular, the profinite group* G_F *is* **not topologically finitely generated**.]

Then the implications

$$(1) \Longrightarrow (2) \Longrightarrow (3).$$

hold.

Proof. — The implication (1) ⇒ (2) follows from [1], Theorem 13.4.2. The implication (2) ⇒ (3) follows from [1], Corollary 12.2.3, and [1], Proposition 16.11.6. □

We are now ready to state and prove the first main result of the present paper.

Corollary 5.6. — *For* $\square \in \{\circ, \bullet\}$, *let* $p_\square$ *be a prime number,* $k_\square$ *a field of characteristic zero,* $\overline{k}_\square$ *an algebraic closure of* $k_\square$, *and* $X_\square$ *a hyperbolic orbicurve over* $k_\square$; *write* $\Pi_{X_\square}$ *for the étale fundamental group* [*relative to an appropriate choice of basepoint*] *of* $X_\square$, $G_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square)$ *for the absolute Galois group of* $k_\square$ *determined by the algebraic closure* $\overline{k}_\square$, *and* $\Delta_{X_\square} \subseteq \Pi_{X_\square}$ *for the kernel of the natural surjective outer homomorphism* $\Pi_{X_\square} \twoheadrightarrow G_{k_\square}$. *Let*

$$\alpha\colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Suppose that either $X_\circ$ *or* $X_\bullet$ *is a* **quasi-tripod**. *Suppose, moreover, that one of the following six conditions is satisfied:*

(1) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is* **algebraic** *and* **generalized sub-$p_\square$-adic**.

(2) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is* **finitely generated** *and* **transcendental** *over a field that is* **algebraic** *and* **sub-$p_\square$-adic**.

(3) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is* **finitely generated** *and* **of transcendental degree $\notin \{0, 2\}$** *over a field that is* **algebraic** *and* **generalized sub-$p_\square$-adic**.

(4) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is* **strictly sub-$p_\square$-adic**.

(5) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is* **quasi-$p_\square$-adic local**.

(6) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is* **finitely generated** *and* **of transcendental degree one** *over a* **quasi-$p_\square$-adic local** *field.*

If either (1) *or* (5) *is satisfied, then suppose, moreover, that one of the following three conditions is satisfied:*

(a) *The isomorphism* α **restricts** *to an isomorphism* $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ *of closed subgroups.*

(b) *For each* $\square \in \{\circ, \bullet\}$, *the profinite group* $G_{k_\square}$ *has* **no nontrivial topologically finitely generated normal closed subgroup**.

(c) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is* **Hilbertian**.

Then the following hold:

(i) *The isomorphism* α **restricts** *to an isomorphism* $\Delta_{X_\circ} \overset{\sim}{\to} \Delta_{X_\bullet}$ *of closed subgroups.*

(ii) *The isomorphism of profinite groups*

$$\alpha_G \colon G_{k_\circ} \overset{\sim}{\longrightarrow} G_{k_\bullet}$$

induced by α [*cf.* (i)] **arises** *from a* **unique isomorphism** $\overline{k}_\bullet \overset{\sim}{\to} \overline{k}_\circ$ *of fields that restricts to an isomorphism* $k_\bullet \overset{\sim}{\to} k_\circ$ *of subfields.*

(iii) *The isomorphism* α **arises** *from a* **unique isomorphism** $X_\circ \overset{\sim}{\to} X_\bullet$ *of stacks that lies over the isomorphism* $k_\bullet \overset{\sim}{\to} k_\circ$ *of* (ii).

PROOF. — First, we verify assertion (i). Now let us verify the following assertion:

Claim 5.6.A: The implications

$$(c) \Longrightarrow (b) \Longrightarrow (a)$$

hold.

To this end, let us first recall that the implication (c) $\Rightarrow$ (b) follows from Lemma 5.5. Next, to verify the implication (b) $\Rightarrow$ (a), suppose that condition (b) is satisfied. Then since $\Delta_{X_\circ}$ is *topologically finitely generated* [cf. [17], Proposition 2.2] and *normal* in $\Pi_{X_\circ}$, it follows from condition (b) that the image of the composite $\Delta_{X_\circ} \hookrightarrow \Pi_{X_\circ} \overset{\alpha}{\overset{\sim}{\to}} \Pi_{X_\bullet} \twoheadrightarrow G_{k_\bullet}$ is *trivial*, which thus implies that $\alpha(\Delta_{X_\circ}) \subseteq \Delta_{X_\bullet}$. Moreover, by applying a similar argument to this argument to the inverse of α, we conclude that $\alpha(\Delta_{X_\circ}) = \Delta_{X_\bullet}$, as desired. This completes the proof of the implication (b) $\Rightarrow$ (a), hence also of Claim 5.6.A.

If either (1) or (5) is satisfied, then assertion (i) follows from our assumption, together with Claim 5.6.A. Next, if either (2), (3), or (6) is satisfied, then it follows from Lemma 5.5 that condition (c) is satisfied, which thus implies [cf. Claim 5.6.A] that assertion (i) holds. Thus, to verify assertion (i), we may assume without loss of generality that condition (4) is satisfied. In particular, for each $\Box \in \{\circ, \bullet\}$, the field $k_\Box$ is *finitely generated* over the $p_\Box$-adic local subfield $(k_\Box)_{\mathrm{MLF}} \subseteq k_\Box$ of Definition 2.12, (i).

Now suppose that $k_\circ = (k_\circ)_{\mathrm{MLF}}$, i.e., that $k_\circ$ is $p_\circ$*-adic local*. Then it follows from [17], Theorem 2.6, (ii), that the profinite group $\Pi_{X_\circ}$, hence also the profinite group $\Pi_{X_\bullet}$, is *topologically finitely generated*. In particular, it follows immediately from Lemma 5.5 that $k_\bullet = (k_\bullet)_{\mathrm{MLF}}$, i.e., that $k_\bullet$ is $p_\bullet$*-adic local*. Thus, we conclude from [17], Corollary 2.8, (ii), that assertion (i) holds.

Next, suppose that $k_\circ \neq (k_\circ)_{\mathrm{MLF}}$, i.e., that $k_\circ$ is *transcendental* over $(k_\circ)_{\mathrm{MLF}}$. Then it follows from Lemma 5.5 that the profinite group $\Pi_{X_\circ}$, hence also the profinite group $\Pi_{X_\bullet}$, is *not topologically finitely generated*. In particular, it follows from [17], Theorem 2.6, (ii), that $k_\bullet \neq (k_\bullet)_{\mathrm{MLF}}$, i.e., that $k_\bullet$ is *transcendental* over $(k_\bullet)_{\mathrm{MLF}}$. Thus, we conclude from Lemma 5.5 that condition (c) is satisfied, which thus implies [cf. Claim 5.6.A] that assertion (i) holds. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that it follows immediately from Proposition 2.8 and the injectivity portion of Theorem 2.14 that, by *Galois descent*, to verify assertion (ii), it suffices to verify that there exists a finite extension $K_\circ \subseteq \overline{k}_\circ$ of $k_\circ$ in $\overline{k}_\circ$ such that if one writes $K_\bullet \subseteq \overline{k}_\bullet$ for the corresponding finite extension of $k_\bullet$ in $\overline{k}_\bullet$ by α_G, then the restriction $\mathrm{Gal}(\overline{k}_\circ/K_\circ) \xrightarrow{\sim} \mathrm{Gal}(\overline{k}_\bullet/K_\bullet)$ of α_G *arises* from an isomorphism $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields that restricts to an isomorphism $K_\bullet \xrightarrow{\sim} K_\circ$ of subfields. Thus, it follows from Lemma 4.6, (iv), that, to verify assertion (ii), we may assume without loss of generality that both $X_\circ$ and $X_\bullet$ are *split tripods*. In particular, it follows from Theorem 2.14 and Theorem 5.4 that the isomorphism α_G *arises* from a *unique isomorphism* $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields that restricts to an isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of subfields, as desired. This completes the proof of assertion (ii). Assertion (iii) follows from assertion (ii) and Theorem 4.5. This completes the proof of Corollary 5.6. □

REMARK 5.6.1. — If one replaces conditions (1), (2), (3), (4), (5), and (6) in the statement of Corollary 5.6 by, for instance, the condition that

for each $\Box \in \{\circ, \bullet\}$, the field $k_\Box$ is *sub-$p_\Box$-adic*,

then the conclusion of Corollary 5.6 *no longer holds* in general. A counter example may be obtained as follows: Let p be a prime number. Write $\mathbb{Q}$ for the field of rational numbers, $\mathbb{Q}_p$ for the p-adic completion of $\mathbb{Q}$, and F for the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_p$. Then it is immediate that both $\mathbb{Q}_p$ and F are *sub-p-adic*. On the other hand, it follows from *Krasner's lemma* [cf., e.g., [20], Lemma 8.1.6] that the natural inclusion $F \hookrightarrow \mathbb{Q}_p$ induces an *outer isomorphism* between the respective absolute Galois groups. In particular, for an arbitrary hyperbolic curve X [e.g., a tripod, that is a *quasi-tripod*] over F, the first projection $X \times_F \mathbb{Q}_p \to X$ induces an *outer isomorphism* between the respective étale fundamental groups [that satisfies a similar condition to condition (a) in the statement of Corollary 5.6]. However, one verifies easily that the scheme $X \times_F \mathbb{Q}_p$ is *not isomorphic* to the scheme X.

REMARK 5.6.2. — Corollary 5.6, (iii), under the condition that,

> for each $\Box \in \{\circ, \bullet\}$, the field $k_\Box$ is *algebraic* and *sub-$p_\Box$-adic*, and, moreover, one of the three conditions (a), (b), and (c) in the statement of Corollary 5.6 is satisfied

may also be derived from [5], Theorem A, as follows: Let us first observe that it follows from a similar argument to the argument applied in the proof of Theorem 4.5 that we may assume without loss of generality, by replacing $X_\circ$ by a suitable connected finite étale covering of $X_\circ$ and replacing $X_\bullet$ by the corresponding connected finite étale covering of $X_\bullet$ by α, that both $X_\circ$ and $X_\bullet$ are *hyperbolic curves*. Next, observe that since an arbitrary sub-l-adic field for some prime number l is *Kummer-faithful* [cf. [19], Remark 1.5.4, (i)], both $k_\circ$ and $k_\bullet$ are *Kummer-faithful*. Moreover, since both $k_\circ$ and $k_\bullet$ are *algebraic*, and $X_\circ$ and $X_\bullet$ are *quasi-tripods* [cf. Lemma 4.6, (iii)], it follows immediately from a similar argument to the argument applied in the proof of [16], Proposition 2.4, (iii), that the isomorphism α is *point-theoretic*. Thus, it follows immediately from [5], Theorem A, that the isomorphism α *arises* from a *unique isomorphism* $X_\circ \xrightarrow{\sim} X_\bullet$ of stacks, as desired.

REMARK 5.6.3. — Corollary 5.6 may be regarded as a *generalization* of [16], Corollary 2.3, in the case where condition (b) is satisfied [cf. also Remark 3.7.1].

6. Applications to Certain Hyperbolic Polycurves

In the present §6, we discuss some *applications* of the first main result of the present paper, i.e., Corollary 5.6, to the absolute anabelian geometry of certain hyperbolic polycurves [cf. Corollary 6.4, Corollary 6.6, Corollary 6.10 below]. In the present §6, let k be a field of characteristic zero.

Definition 6.1. — Let S be a scheme. Then we shall say that a scheme X over S is a *hyperbolic polycurve* over S if there exist a positive integer d and a [not necessarily unique] factorization of the structure morphism $X \to S$

$$X = X_d \longrightarrow X_{d-1} \longrightarrow \dots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow S = X_0$$

such that, for each $i \in \{1, \dots, d\}$, the morphism $X_i \to X_{i-1}$ is a hyperbolic curve. We shall refer to a factorization of $X \to S$ as above as a *sequence of parametrizing morphisms* for X over S.

Remark 6.1.1. — It is immediate that a *hyperbolic polycurve* over k is a *smooth variety* over k.

Definition 6.2. — Let d be a positive integer and X a hyperbolic curve over k. Write $X^{\times d}$ for the fiber product over k of d copies of X and $\delta_X \subseteq X^{\times 2}$ for the diagonal divisor. Write, moreover, for a subset $S \subseteq \{1, \dots, d\}$ of cardinality two, $p_S \colon X^{\times d} \to X^{\times 2}$ for the projection onto the factors labeled by the elements of S. Then we shall refer to the open subscheme of $X^{\times d}$

$$X^{\times d} \setminus \bigcup_{S \subseteq \{1,\dots,d\},\ \sharp S = 2} p_S^{-1}(\delta_X) \subseteq X^{\times d}$$

as the *d-th configuration space* of X.

Remark 6.2.1. — One verifies easily that, in the situation of Definition 6.2, the d-th *configuration space* of X is a *hyperbolic polycurve* over k.

Here, let us recall the following result concerning the Grothendieck conjecture for configuration spaces of hyperbolic curves over generalized sub-p-adic fields:

Theorem 6.3. — *For $\Box \in \{\circ, \bullet\}$, let $d_\Box$ be a positive integer, $p_\Box$ a prime number, $k_\Box$ a* **generalized sub-$p_\Box$-adic** *field, $\overline{k}_\Box$ an algebraic closure of $k_\Box$, and $C_\Box$ a hyperbolic curve over $k_\Box$; write $X_\Box$ for the $d_\Box$-th* **configuration space** *of $C_\Box$, $\Pi_{X_\Box}$ for the étale fundamental group [relative to an appropriate choice of basepoint] of $X_\Box$, $G_{k_\Box} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\Box/k_\Box)$ for the absolute Galois group of $k_\Box$ determined by the algebraic closure $\overline{k}_\Box$, and $\Delta_{X_\Box} \subseteq \Pi_{X_\Box}$ for the kernel of the natural surjective outer homomorphism $\Pi_{X_\Box} \twoheadrightarrow G_{k_\Box}$. Let*

$$\alpha \colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups that **restricts** *to an isomorphism* $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ *of closed subgroups. Write*

$$\alpha_G \colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$$

for the isomorphism induced by α*. Suppose that the isomorphism* α_G **arises** *from an isomorphism* $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ *of fields that restricts to an isomorphism* $k_\bullet \xrightarrow{\sim} k_\circ$ *of subfields. Then the isomorphism* α **arises** *from a* **unique isomorphism** $X_\circ \xrightarrow{\sim} X_\bullet$ *of schemes that lies over the isomorphism* $k_\bullet \xrightarrow{\sim} k_\circ$*.*

PROOF. — This assertion follows immediately — in light of Theorem 4.5 — from [10], Theorem A, (ii), together with a similar argument to the argument applied in the proof of [18], Theorem 1.8, (i). □

COROLLARY 6.4. — *For* $\square \in \{\circ, \bullet\}$*, let* $d_\square$ *be a positive integer,* $p_\square$ *a prime number,* $k_\square$ *a field of characteristic zero,* $\overline{k}_\square$ *an algebraic closure of* $k_\square$*, and* $C_\square$ *a hyperbolic curve over* $k_\square$*; write* $X_\square$ *for the* $d_\square$*-th* **configuration space** *of* $C_\square$*,* $\Pi_{X_\square}$ *for the étale fundamental group [relative to an appropriate choice of basepoint] of* $X_\square$*,* $G_{k_\square} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square)$ *for the absolute Galois group of* $k_\square$ *determined by the algebraic closure* $\overline{k}_\square$*, and* $\Delta_{X_\square} \subseteq \Pi_{X_\square}$ *for the kernel of the natural surjective outer homomorphism* $\Pi_{X_\square} \twoheadrightarrow G_{k_\square}$*. Let*

$$\alpha \colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Suppose that one of the following three conditions is satisfied:

(A) *Either* $C_\circ$ *or* $C_\bullet$ *is a* **quasi-tripod***.*

(B) *Either* $C_\circ$ *or* $C_\bullet$ *is* **affine***. Moreover, the inequality* $\max\{d_\circ, d_\bullet\} \geq 2$ *holds.*

(C) *The inequality* $\max\{d_\circ, d_\bullet\} \geq 3$ *holds.*

Suppose, moreover, that one of the six conditions (1), (2), (3), (4), (5), *and* (6) *in the statement of Corollary* 5.6 *is satisfied. If either* (1) *or* (5) *in the statement of Corollary* 5.6 *is satisfied, then suppose, moreover, that one of the three conditions* (a), (b), *and* (c) *in the statement of Corollary* 5.6 *is satisfied. Then the following hold:*

(i) *The isomorphism* α **restricts** *to an isomorphism* $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ *of closed subgroups.*

(ii) *The isomorphism of profinite groups*

$$\alpha_G \colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$$

induced by α [*cf.* (i)] **arises** *from a* **unique isomorphism** $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ *of fields that restricts to an isomorphism* $k_\bullet \xrightarrow{\sim} k_\circ$ *of subfields.*

(iii) *The isomorphism* α **arises** *from a* **unique isomorphism** $X_\circ \xrightarrow{\sim} X_\bullet$ *of schemes that lies over the isomorphism* $k_\bullet \xrightarrow{\sim} k_\circ$ *of* (ii)*.*

PROOF. — Assertion (i) follows from a similar argument to the argument applied in the proof of Corollary 5.6, (i).

Next, we verify assertions (ii), (iii) in the case where condition (A) is satisfied. Suppose that condition (A) is satisfied. Then it follows from assertion (i) and [10], Theorem A, (ii), that, to verify assertion (ii), we may assume without loss of generality, by replacing $X_\Box$ by $C_\Box$ for each $\Box \in \{\circ, \bullet\}$, that $d_\Box = 1$. In particular, assertion (ii) follows from Corollary 5.6, (ii). Moreover, assertion (iii) follows from assertion (ii) and Theorem 6.3. This completes the proofs of assertions (ii), (iii) in the case where condition (A) is satisfied.

Finally, we verify assertions (ii), (iii) in the case where either (B) or (C) is satisfied. Suppose that either (B) or (C) is satisfied. Let us first observe that if condition (B) is satisfied, then it follows from assertion (i), Lemma 4.6, (vi), and [10], Theorem A, (i), that both $C_\circ$ and $C_\bullet$ are *affine*. Thus, assertions (ii), (iii) follow immediately — in light of assertion (i) and [10], Theorem A, (ii) — from Corollary 5.6, (ii), (iii), and a similar argument to the argument applied in the proof of [18], Corollary 1.11, (iii). This completes the proofs of assertions (ii), (iii) in the case where either (B) or (C) is satisfied, hence also of Corollary 6.4. □

DEFINITION 6.5. — Let S be a scheme and X a hyperbolic polycurve over S. Then we shall say that a sequence $X = X_d \to X_{d-1} \to \ldots \to X_2 \to X_1 \to S = X_0$ of parametrizing morphisms for X over S is *of strictly decreasing type* if the following condition is satisfied: If, for each $i \in \{1, \ldots, d\}$, the hyperbolic curve $X_i \to X_{i-1}$ is of rank n_i, then $n_1 > n_2 > \cdots > n_{d-1} > n_d$.

COROLLARY 6.6. — *For $\Box \in \{\circ, \bullet\}$, let $p_\Box$ be a prime number, $k_\Box$ a field of characteristic zero, $\overline{k}_\Box$ an algebraic closure of $k_\Box$, and $X_\Box$ a* **hyperbolic polycurve** *over $k_\Box$; write $\Pi_{X_\Box}$ for the étale fundamental group [relative to an appropriate choice of basepoint] of $X_\Box$, $G_{k_\Box} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}_\Box/k_\Box)$ for the absolute Galois group of $k_\Box$ determined by the algebraic closure $\overline{k}_\Box$, and $\Delta_{X_\Box} \subseteq \Pi_{X_\Box}$ for the kernel of the natural surjective outer homomorphism $\Pi_{X_\Box} \twoheadrightarrow G_{k_\Box}$. Let*

$$\alpha \colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Suppose that, for each $\Box \in \{\circ, \bullet\}$, there exists a sequence $X_\Box = (X_\Box)_{d_\Box} \to (X_\Box)_{d_\Box - 1} \to \ldots \to (X_\Box)_2 \to (X_\Box)_1 \to \mathrm{Spec}(k_\Box) = (X_\Box)_0$ of parametrizing morphisms for $X_\Box$ over $k_\Box$ **of strictly decreasing type** *such that either the hyperbolic curve $(X_\circ)_1$ over $k_\circ$ or the hyperbolic curve $(X_\bullet)_1$ over $k_\bullet$ is a* **quasi-tripod**. *Suppose, moreover, that one of the six conditions* (1), (2), (3), (4), (5), *and* (6) *in the statement of Corollary* 5.6 *is satisfied. If either* (1) *or* (5) *in the statement of Corollary* 5.6 *is satisfied, then suppose, moreover, that one of the three conditions* (a), (b), *and* (c) *in the statement of Corollary* 5.6 *is satisfied. Then the following hold:*

(i) *The isomorphism α* **restricts** *to an isomorphism $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ of closed subgroups.*

(ii) *The isomorphism of profinite groups*

$$\alpha_G \colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$$

induced by α [cf. (i)] **arises** *from a* **unique isomorphism** *$\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields that restricts to an isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of subfields.*

(iii) *The isomorphism α* **arises** *from a* **unique isomorphism** *$X_\circ \xrightarrow{\sim} X_\bullet$ of schemes that lies over the isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of* (ii).

PROOF. — Assertion (i) follows from a similar argument to the argument applied in the proof of Corollary 5.6, (i). Next, we verify assertion (ii). It follows from assertion (i) and [8], Lemma 2.3, (iii), (iv), that we may assume without loss of generality, by replacing $X_\square$ by $(X_\square)_1$ for each $\square \in \{\circ, \bullet\}$, that $d_\square = 1$. Thus, assertion (ii) follows from Corollary 5.6, (ii). This completes the proof of assertion (ii). Finally, assertion (iii) follows immediately from assertion (ii) and [8], Theorem 2.4. This completes the proof of Corollary 6.6. $\square$

LEMMA 6.7. — *Let n_0 be an integer, X a smooth variety over k, and $x \in X$ a point of X. Suppose that X is of positive dimension. Then there exists an open neighborhood $U \subseteq X$ of $x \in X$ that satisfies the following two conditions:*

(1) *The smooth variety U has a structure of* **hyperbolic polycurve** *over k.*

(2) *There exists a sequence $U = U_d \to U_{d-1} \to \ldots \to U_2 \to U_1 \to \mathrm{Spec}(k) = U_0$ of parametrizing morphisms for U over k [cf. (1)]* **of strictly decreasing type** *such that the hyperbolic curve U over U_{d-1} is* **of rank $\geq n_0$**, *and, moreover, the hyperbolic curve U_1 over k is a* **quasi-tripod**.

PROOF. — We prove Lemma 6.7 by induction on the dimension of X. If X is *of dimension one*, then Lemma 6.7 follows immediately from Remark 3.7.2, (i), (ii), and [8], Lemma 1.8. In the remainder of the proof of Lemma 6.7, suppose that X is *of dimension* ≥ 2, and that the induction hypothesis is in force.

Next, let us observe that we may assume without loss of generality, by replacing $x \in X$ by a closed point of the closure of $\{x\} \subseteq X$ in X, that $x \in X$ is a *closed* point of X. Moreover, it follows from [8], Lemma 1.11, that we may assume without loss of generality, by replacing X by a suitable open neighborhood of $x \in X$, that there exists a smooth variety S over k such that X has a structure of *hyperbolic curve* over S, by means of which we shall regard X as a scheme over S. Thus, it follows from [8], Lemma 1.8, that we may assume without loss of generality, by replacing X by a suitable open neighborhood of $x \in X$, that

(a) the hyperbolic curve X over S is *of rank* $\geq n_0$.

Write n^X ($\geq n_0$) for the rank of the hyperbolic curve X over S [cf. (a)]. Then since S is *of dimension* $\dim(X) - 1$, it follows from the induction hypothesis that we may assume without loss of generality, by replacing S by a suitable open neighborhood of the image of $x \in X$ in S, that

(b) the smooth variety S has a structure of *hyperbolic polycurve* over k, and

(c) there exists a sequence $S = S_{d-1} \to S_{d-2} \to \ldots \to S_2 \to S_1 \to \mathrm{Spec}(k) = S_0$ of parametrizing morphisms for S over k [cf. (b)] *of strictly decreasing type* such that the hyperbolic curve S over S_{d-2} is *of rank* $> n^X$, and, moreover, the hyperbolic curve S_1 over k is a *quasi-tripod*.

Now let us observe that it follows from (a), (b), (c) that X satisfies conditions (1), (2) in the statement of Lemma 6.7. This completes the proof of Lemma 6.7. $\square$

LEMMA 6.8. — *Let X be a smooth variety over k. Suppose that X is of positive dimension. Then there exists an open basis for the Zariski topology of X such that each member U of the open basis satisfies the following two conditions:*

(1) *The smooth variety U has a structure of* **hyperbolic polycurve** *over k.*

(2) *There exists a sequence $U = U_d \to U_{d-1} \to \ldots \to U_2 \to U_1 \to \mathrm{Spec}(k) = U_0$ of parametrizing morphisms for U over k [cf. (1)] such that this sequence is* **of strictly decreasing type***, and, moreover, the hyperbolic curve U_1 over k is a* **quasi-tripod***.*

PROOF. — This assertion follows from Lemma 6.7. □

DEFINITION 6.9. — We shall say that a smooth variety over k has an *absolutely anabelian open basis* if there exists an open basis for the Zariski topology of the variety such that, for arbitrary members U and V of the open basis, the natural map

$$\mathrm{Isom}(U, V) \longrightarrow \mathrm{Isom}(\Pi_U, \Pi_V)/\mathrm{Inn}(\Pi_V)$$

— where we write "$\Pi_{(-)}$" for the étale fundamental group [relative to an appropriate choice of basepoint] of "$(-)$" — is bijective.

COROLLARY 6.10. — *Let k be a field. Suppose that there exists a prime number p such that the field k satisfies one of the following five conditions:*

(1) *The field k is* **algebraic**, **generalized sub-p-adic**, *and* **Hilbertian**.

(2) *The field k is* **finitely generated** *and* **transcendental** *over a field that is* **algebraic** *and* **sub-p-adic**.

(3) *The field k is* **finitely generated** *and* **of transcendental degree $\not\in \{0, 2\}$** *over a field that is* **algebraic** *and* **generalized sub-p-adic**.

(4) *The field k is* **strictly sub-p-adic**.

(5) *The field k is* **finitely generated** *and* **of transcendental degree one** *over a* **quasi-p-adic local** *field.*

Then every smooth variety of positive dimension over the field k has an **absolutely anabelian open basis**.

PROOF. — This assertion follows from Corollary 6.6 and Lemma 6.8. □

REMARK 6.10.1. — Observe that an arbitrary finitely generated extension of the field of rational numbers satisfies either (1) or (2) in the statement of Corollary 6.10; moreover, an arbitrary mixed-characteristic local field satisfies condition (4) in the statement of Corollary 6.10. Thus, Corollary 6.10 may be regarded as a *generalization* of [8], Corollary 3.4, (ii), (iii) [i.e., in the case where the variety under consideration is of positive dimension], hence also of [22], Corollary 1.7 [cf. also [8], Remark 3.4.1].

7. The Absolute Anabelian Geometry of MLF-isotrivial Orbicurves

In the present §7, we discuss the absolute anabelian geometry of *MLF-isotrivial* hyperbolic orbicurves.

Theorem 7.1. — *For $\Box \in \{\circ, \bullet\}$, let $p_\Box$ be a prime number, $k_\Box$ a field that is* **strictly sub-$p_\Box$-adic** *but* **not $p_\Box$-adic local**, *$\overline{k}_\Box$ an algebraic closure of $k_\Box$, and $X_\Box$ a* **hyperbolic orbicurve** *over $k_\Box$; write $\Pi_{X_\Box}$ for the étale fundamental group [relative to an appropriate choice of basepoint] of $X_\Box$, $G_{k_\Box} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}_\Box/k_\Box)$ for the absolute Galois group of $k_\Box$ determined by the algebraic closure $\overline{k}_\Box$, and $\Delta_{X_\Box} \subseteq \Pi_{X_\Box}$ for the kernel of the natural surjective outer homomorphism $\Pi_{X_\Box} \twoheadrightarrow G_{k_\Box}$. Let*

$$\alpha \colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Suppose that one of the following four conditions is satisfied:

(1) *Both $X_\circ$ and $X_\bullet$ are* **MLF-isotrivial** *[cf. Definition* 3.6, (ii)*].*

(2) *The hyperbolic orbicurve $X_\circ$ is* **strictly MLF-isotrivial** *[cf. Definition* 3.6, (i)*], and, moreover, the inequality $d_{k_\circ} \leq d_{k_\bullet}$ [cf. Definition* 2.12, (ii)*] holds.*

(3) *The hyperbolic orbicurve $X_\circ$ is* **strictly MLF-isotrivial**, *and, moreover, the inequality $e_{k_\circ} \leq e_{k_\bullet}$ [cf. Definition* 2.12, (iii)*] holds.*

(4) *The hyperbolic orbicurve $X_\circ$ is* **strictly MLF-isotrivial**, *and, moreover, the mixed-characteristic local field $(k_\circ)_{\mathrm{MLF}}$ [cf. Definition* 2.12, (i)*] is* **absolutely abelian** *[cf.* [7], *Definition* 4.2, (ii)*].*

Then the following hold:

(i) *The isomorphism α* **restricts** *to an isomorphism $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ of closed subgroups.*

(ii) *The isomorphism of profinite groups*

$$\alpha_G \colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$$

induced by α [cf. (i)*]* **arises** *from a* **unique isomorphism** *$\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields that restricts to an isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of subfields.*

(iii) *The isomorphism α* **arises** *from a* **unique isomorphism** *$X_\circ \xrightarrow{\sim} X_\bullet$ of stacks that lies over the isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of* (ii)*.*

Proof. — Let us first observe that, for each $\Box \in \{\circ, \bullet\}$, since the field $k_\Box$ is *strictly sub-$p_\Box$-adic* but *not $p_\Box$-adic local*, the field $k_\Box$ is *finitely generated* and *transcendental* over the $p_\Box$-adic local subfield $(k_\Box)_0 \stackrel{\text{def}}{=} (k_\Box)_{\mathrm{MLF}} \subseteq k_\Box$ [cf. Definition 2.12, (i)]. Moreover, for each $\Box \in \{\circ, \bullet\}$, since the subfield $(k_\Box)_0 \subseteq k_\Box$ is *algebraically closed* in $k_\Box$ [cf. Definition 2.12, (i)], the equality $(k_\Box)_{\mathrm{alg}} = ((k_\Box)_0)_{\mathrm{alg}} \subseteq (k_\Box)_0$ holds. For $\Box \in \{\circ, \bullet\}$, write $(\overline{k}_\Box)_0 \subseteq \overline{k}_\Box$ for the algebraic closure of $(k_\Box)_0$ in $\overline{k}_\Box$ and $\Omega_\Box \subseteq \overline{k}_\Box$ for the algebraic closure of $(k_\Box)_{\mathrm{alg}}$ in $\overline{k}_\Box$. Thus, the natural inclusions $\Omega_\Box \hookrightarrow (\overline{k}_\Box)_0 \hookrightarrow \overline{k}_\Box$ determine homomorphisms of profinite groups

$$G_{k_\Box} \twoheadrightarrow G_{(k_\Box)_0} \stackrel{\text{def}}{=} \mathrm{Gal}\big((\overline{k}_\Box)_0/(k_\Box)_0\big) \xrightarrow{\sim} G^{\mathrm{alg}}_{k_\Box} \stackrel{\text{def}}{=} \mathrm{Gal}\big(\Omega_\Box/(k_\Box)_{\mathrm{alg}}\big)$$

— where the first arrow is *surjective*, and the second arrow is an *isomorphism* [cf. Lemma 2.1, (i)]. For $\Box \in \{\circ, \bullet\}$, write $\Delta_\Box \subseteq G_{k_\Box}$ for the kernel of the first surjective homomorphism $G_{k_\Box} \twoheadrightarrow G_{(k_\Box)_0}$.

Now we verify assertion (i). It follows from the discussion of the preceding paragraph, together with Lemma 5.5, that condition (c) in the statement of Corollary 5.6 is satisfied. Thus, it follows from a similar argument to the argument applied in the proof of Claim 5.6.A [i.e., in the proof of Corollary 5.6] that assertion (i) holds. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that it follows immediately from the injectivity portion of Theorem 2.14 that, by *Galois descent*, to verify assertion (ii), it suffices to verify that there exists a finite extension $K_\circ \subseteq \overline{k}_\circ$ of $k_\circ$ in $\overline{k}_\circ$ such that if one writes $K_\bullet \subseteq \overline{k}_\bullet$ for the corresponding finite extension of $k_\bullet$ in $\overline{k}_\bullet$ by α_G, then the restriction $\mathrm{Gal}(\overline{k}_\circ/K_\circ) \xrightarrow{\sim} \mathrm{Gal}(\overline{k}_\bullet/K_\bullet)$ of α_G *arises* from an isomorphism $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields that restricts to an isomorphism $K_\bullet \xrightarrow{\sim} K_\circ$ of subfields. Thus, to verify assertion (ii), we may assume without loss of generality that $X_\circ$ is *strictly MLF-isotrivial*.

For $\Box \in \{\circ, \bullet\}$, write $\rho_\Box \colon G_{k_\Box} \to \mathrm{Out}(\Delta_{X_\Box})$ for the natural outer action of $G_{k_\Box}$ on $\Delta_{X_\Box}$. Thus, the isomorphism α determines a commutative diagram

$$\begin{array}{ccc} G_{k_\circ} & \xrightarrow[\sim]{\alpha_G} & G_{k_\bullet} \\ {\scriptstyle \rho_\circ}\downarrow & & \downarrow{\scriptstyle \rho_\bullet} \\ \mathrm{Out}(\Delta_{X_\circ}) & \xrightarrow[\sim]{} & \mathrm{Out}(\Delta_{X_\bullet}) \end{array}$$

— where the horizontal arrows are *isomorphisms*. To verify assertion (ii), let us verify the following assertion:

> Claim 7.1.A: The isomorphism α_G *restricts* to an isomorphism $\Delta_\circ \xrightarrow{\sim} \Delta_\bullet$ of closed subgroups.

To this end, let us observe that since $X_\circ$ is *strictly MLF-isotrivial*, it follows from the diagram preceding Claim 7.1.A, together with Proposition 4.2 and Lemma 4.3, (i), that the isomorphism α_G determines a sequence of profinite groups

$$G_{(k_\circ)_0} \xleftarrow{\sim} G_{k_\circ}/\mathrm{Ker}(\rho_\circ) \xrightarrow{\sim} G_{k_\bullet}/\mathrm{Ker}(\rho_\bullet) \twoheadrightarrow G_{(k_\bullet)_0}$$

— where the first and second arrows are *isomorphisms*, and the third arrow is a *surjective* homomorphism whose kernel is $\Delta_\bullet/\mathrm{Ker}(\rho_\bullet)$.

Now let us discuss Claim 7.1.A in the case where condition (1) is satisfied. Since the hyperbolic orbicurve $X_\bullet$ is *MLF-isotrivial*, it follows from Lemma 4.3, (ii), that the closed subgroup $\mathrm{Ker}(\rho_\bullet) \subseteq \Delta_\bullet$ of $\Delta_\bullet$ is *open*, which thus implies that $\Delta_\bullet/\mathrm{Ker}(\rho_\bullet)$ is *finite*. Thus, since $G_{(k_\circ)_0}$, hence also $G_{k_\bullet}/\mathrm{Ker}(\rho_\bullet)$, is *torsion-free* [cf. [15], Proposition 2.3, (iii)], we conclude that $\Delta_\bullet = \mathrm{Ker}(\rho_\bullet)$, which thus implies that Claim 7.1.A holds. This completes the proof of Claim 7.1.A in the case where condition (1) is satisfied.

Next, suppose that condition (2) (respectively, (3); (4)) is satisfied. Then it follows from [7], Theorem B, (i), together with [6], Proposition 3.6 (respectively, [7], Theorem B, (ii), together with [6], Proposition 3.6; Lemma 4.6, (ii), and [7], Theorem 4.13, (ii), together with [6], Proposition 4.2, (iv), and [7], Proposition 4.9, (iii)), that the composite $G_{(k_\circ)_0} \twoheadrightarrow G_{(k_\bullet)_0}$ of the three homomorphisms in the display of the discussion at the beginning of the proof of Claim 7.1.A is an *isomorphism*. In particular, Claim 7.1.A holds. This completes the proof of Claim 7.1.A.

It follows from Claim 7.1.A that the isomorphism α_G induces an isomorphism $G_{(k_\circ)_0} \xrightarrow{\sim} G_{(k_\bullet)_0}$. Thus, since, for each $\square \in \{\circ, \bullet\}$, the field $k_\square$ is *finitely generated* and *transcendental* over the $p_\square$*-adic local* field $(k_\square)_0$, it follows from [17], Corollary 3.9 [cf. also [19], Appendix], that the isomorphism $\alpha_G \colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ *arises* from an isomorphism $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields that restricts to an isomorphism $k_\bullet \xrightarrow{\sim} k_\circ$ of subfields, as desired. This completes the proof of assertion (ii). Assertion (iii) follows from assertion (ii) and Theorem 4.5. This completes the proof of Theorem 7.1. □

COROLLARY 7.2. — *For* $\square \in \{\circ, \bullet\}$*, let* $p_\square$ *be a prime number,* $k_\square$ *a field that is* **strictly sub-$p_\square$-adic** *but* **not $p_\square$-adic local**, $\overline{k}_\square$ *an algebraic closure of* $k_\square$*, and* $X_\square$ *a* **hyperbolic polycurve** *over* $k_\square$*; write* $\Pi_{X_\square}$ *for the étale fundamental group [relative to an appropriate choice of basepoint] of* $X_\square$*,* $G_{k_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_\square/k_\square)$ *for the absolute Galois group of* $k_\square$ *determined by the algebraic closure* $\overline{k}_\square$*, and* $\Delta_{X_\square} \subseteq \Pi_{X_\square}$ *for the kernel of the natural surjective outer homomorphism* $\Pi_{X_\square} \twoheadrightarrow G_{k_\square}$*. Let*

$$\alpha \colon \Pi_{X_\circ} \xrightarrow{\sim} \Pi_{X_\bullet}$$

be an isomorphism of profinite groups. Suppose that one of the following eight conditions is satisfied:

(1) *For each* $\square \in \{\circ, \bullet\}$*, there exist a positive integer* $d_\square$*, an* **MLF-isotrivial** *hyperbolic curve* $C_\square$ *over* $k_\square$*, and an isomorphism of* $X_\square$ *with the* $d_\square$*-th* **configuration space** *of* $C_\square$ *over* $k_\square$*.*

(2) *For each* $\square \in \{\circ, \bullet\}$*, there exist a positive integer* $d_\square$*, a hyperbolic curve* $C_\square$ *over* $k_\square$*, and an isomorphism of* $X_\square$ *with the* $d_\square$*-th* **configuration space** *of* $C_\square$ *over* $k_\square$*. Moreover, the hyperbolic curve* $C_\circ$ *is* **strictly MLF-isotrivial***, and the inequality* $d_{k_\circ} \leq d_{k_\bullet}$ *holds.*

(3) *For each* $\square \in \{\circ, \bullet\}$*, there exist a positive integer* $d_\square$*, a hyperbolic curve* $C_\square$ *over* $k_\square$*, and an isomorphism of* $X_\square$ *with the* $d_\square$*-th* **configuration space** *of* $C_\square$ *over* $k_\square$*. Moreover, the hyperbolic curve* $C_\circ$ *is* **strictly MLF-isotrivial***, and the inequality* $e_{k_\circ} \leq e_{k_\bullet}$ *holds.*

(4) *For each* $\square \in \{\circ, \bullet\}$*, there exist a positive integer* $d_\square$*, a hyperbolic curve* $C_\square$ *over* $k_\square$*, and an isomorphism of* $X_\square$ *with the* $d_\square$*-th* **configuration space** *of* $C_\square$ *over* $k_\square$*. Moreover, the hyperbolic curve* $C_\circ$ *is* **strictly MLF-isotrivial***, and the mixed-characteristic local field* $(k_\circ)_{\mathrm{MLF}}$ *is* **absolutely abelian**.

(5) *For each* $\square \in \{\circ, \bullet\}$*, there exists a sequence* $X_\square = (X_\square)_{d_\square} \to (X_\square)_{d_\square - 1} \to \ldots \to (X_\square)_2 \to (X_\square)_1 \to \mathrm{Spec}(k_\square) = (X_\square)_0$ *of parametrizing morphisms for* $X_\square$ *over* $k_\square$ *such that this sequence is* **of strictly decreasing type***, and, moreover, the hyperbolic curve* $(X_\square)_1$ *is* **MLF-isotrivial**.

(6) *For each* $\square \in \{\circ, \bullet\}$*, there exists a sequence* $X_\square = (X_\square)_{d_\square} \to (X_\square)_{d_\square - 1} \to \ldots \to (X_\square)_2 \to (X_\square)_1 \to \mathrm{Spec}(k_\square) = (X_\square)_0$ *of parametrizing morphisms for* $X_\square$ *over* $k_\square$ **of strictly decreasing type**. *Moreover, the hyperbolic curve* $(X_\circ)_1$ *is* **strictly MLF-isotrivial***, and the inequality* $d_{k_\circ} \leq d_{k_\bullet}$ *holds.*

(7) *For each* $\square \in \{\circ, \bullet\}$, *there exists a sequence* $X_\square = (X_\square)_{d_\square} \to (X_\square)_{d_\square - 1} \to \ldots \to (X_\square)_2 \to (X_\square)_1 \to \mathrm{Spec}(k_\square) = (X_\square)_0$ *of parametrizing morphisms for* $X_\square$ *over* $k_\square$ **of strictly decreasing type**. *Moreover, the hyperbolic curve* $(X_\circ)_1$ *is* **strictly MLF-isotrivial**, *and the inequality* $e_{k_\circ} \leq e_{k_\bullet}$ *holds.*

(8) *For each* $\square \in \{\circ, \bullet\}$, *there exists a sequence* $X_\square = (X_\square)_{d_\square} \to (X_\square)_{d_\square - 1} \to \ldots \to (X_\square)_2 \to (X_\square)_1 \to \mathrm{Spec}(k_\square) = (X_\square)_0$ *of parametrizing morphisms for* $X_\square$ *over* $k_\square$ **of strictly decreasing type**. *Moreover, the hyperbolic curve* $(X_\circ)_1$ *is* **strictly MLF-isotrivial**, *and the mixed-characteristic local field* $(k_\circ)_{\mathrm{MLF}}$ *is* **absolutely abelian**.

Then the following hold:

(i) *The isomorphism* α **restricts** *to an isomorphism* $\Delta_{X_\circ} \xrightarrow{\sim} \Delta_{X_\bullet}$ *of closed subgroups.*

(ii) *The isomorphism of profinite groups*

$$\alpha_G \colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$$

induced by α [*cf.* (i)] **arises** *from a* **unique isomorphism** $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ *of fields that restricts to an isomorphism* $k_\bullet \xrightarrow{\sim} k_\circ$ *of subfields.*

(iii) *The isomorphism* α **arises** *from a* **unique isomorphism** $X_\circ \xrightarrow{\sim} X_\bullet$ *of schemes that lies over the isomorphism* $k_\bullet \xrightarrow{\sim} k_\circ$ *of* (ii).

PROOF. — Assertion (i) follows from a similar argument to the argument applied in the proof of Theorem 7.1, (i). Next, we verify assertion (ii). If either (1), (2), (3), or (4) (respectively, (5), (6), (7), or (8)) is satisfied, then it follows from assertion (i) and [10], Theorem A, (ii) (respectively, and [8], Lemma 2.3, (iii), (iv)), that, to verify assertion (ii), we may assume without loss of generality, by replacing $X_\square$ by $C_\square$ (respectively, $(X_\square)_1$) for each $\square \in \{\circ, \bullet\}$, that $d_\square = 1$. Thus, assertion (ii) follows from Theorem 7.1, (ii). This completes the proof of assertion (ii). Assertion (iii) in the case where either (1), (2), (3), or (4) (respectively, (5), (6), (7), or (8)) is satisfied follows from assertion (ii), together with Theorem 6.3 (respectively, with [8], Theorem 2.4). This completes the proof of Corollary 7.2. □

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