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Integrable Connections I:
Two Fundamental Correspondences

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ABSTRACT. — The purpose of the present paper is to establish a certain abstract theory related to the notion of an integrable connection. More concretely, to establish certain two bijections related to the notion of an integrable connection is the main purpose of the present paper.

CONTENTS

INTRODUCTION .............................................................. 1
§1. INTEGRABLE CATEGORIES FIBERED IN GROUPOIDS ......................... 6
§2. STRATIFICATION STRUCTURES ............................................. 12
§3. INTEGRABLE STRATIFICATION STRUCTURES ............................. 22
§4. STRATIFICATIONS AND INTEGRABLE CONNECTIONS ...................... 33
§5. THE FIRST FUNDAMENTAL CORRESPONDENCE ............................. 41
§6. THE SECOND FUNDAMENTAL CORRESPONDENCE ............................ 44

INTRODUCTION

The purpose of the present paper is to establish a certain abstract theory related to the notion of an integrable connection. More concretely, to establish certain two bijections [cf. the respective discussions following Theorem A and Theorem B below] related to the notion of an integrable connection is the main purpose of the present paper.

In the present Introduction, let us fix a scheme $S$. Write $\text{Sch}_S$ for the category of schemes over $S$ and morphisms of schemes over $S$ [cf. Definition 1.6, (i)]. Let us also fix a scheme $X$ which is smooth and separated over $S$ and a category $\mathcal{F} \to \text{Sch}_S$ fibered in groupoids over $\text{Sch}_S$. Write $X^{(2)} \stackrel{\text{def}}{=} X \times_S X$ for the fiber product of two copies of $X$ over $S$ [cf. Definition 2.1, (i)] and $\pi_i^{(2)} : X^{(2)} \to X$ for the projection onto the $i$-th factor [where $i \in \{1, 2\}$] [cf. Definition 2.1, (iii)].

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One main object of the present paper is a stratification structure [cf. Definition 2.5]. We shall refer to a collection of data

$$\mathcal{G} = ((\mathcal{E} P^n)_{n \geq 0}, (\mathcal{E} t^n : \mathcal{E} P^n \to \mathcal{E} P^{n+1})_{n \geq 0},$$

$$((\mathcal{E} \sigma^n : \mathcal{E} P^n \to X^{(2)})_{n \geq 0}, (\mathcal{E} \delta^{n_1, n_2} : \mathcal{E} P^{n_1, n_2} \to \mathcal{E} P^{n_1+n_2})_{n_1, n_2 \geq 0})$$

consisting of

- a scheme $\mathcal{E} P^n$ over $S$ for each nonnegative integer $n$,
- a morphism $\mathcal{E} t^n : \mathcal{E} P^n \to \mathcal{E} P^{n+1}$ over $S$ for each nonnegative integer $n$,
- a morphism $\mathcal{E} \sigma^n : \mathcal{E} P^n \to X^{(2)}$ over $S$ for each nonnegative integer $n$, and
- a morphism $\mathcal{E} \delta^{n_1, n_2} : \mathcal{E} P^{n_1, n_2} \to \mathcal{E} P^{n_1+n_2}$ over $S$ for each nonnegative integers $n_1, n_2$—where we write $\mathcal{E} P^{n_1, n_2}$ for the fiber product of the composite $\mathcal{E} P^{n_1} \xrightarrow{\mathcal{E} \sigma^{n_1}} X^{(2)} \xrightarrow{pr_2^{(2)}} X$ and the composite $\mathcal{E} P^{n_2} \xrightarrow{\mathcal{E} \sigma^{n_2}} X^{(2)} \xrightarrow{pr_1^{(2)}} X$ [cf. Definition 2.4, (i)]

that satisfies certain conditions as a stratification structure on $X/S$.

An example of a stratification structure on $X/S$ is given as follows [cf. Remark 2.5.1]: Let $Y$ be a separated scheme over $S$ and $X \to Y$ an affine morphism over $S$. Write $P \overset{\text{def}}{=} X \times_Y X; q_1, q_2 : P \to X$ for the projections onto the first, second factors, respectively; $q_{1,3} : X \times_Y X \times_Y X \to P$ for the projection onto the first and third factors. Let

$$P^0 \subseteq P^1 \subseteq \ldots \subseteq P^n \subseteq P^{n+1} \subseteq \ldots \subseteq P$$

be a sequence of closed subschemes of $P$. Suppose that the following two conditions are satisfied:

- The natural closed immersion $P^0 \hookrightarrow P$ determines an isomorphism of $P^0$ with the diagonal closed subscheme $X \subseteq P$ of $P$.
- For each nonnegative integers $n_1, n_2$, if we write $P^{n_1, n_2} \overset{\text{def}}{=} P^{n_1} \times_X P^{n_2}$ for the fiber product of the composite $P^{n_1} \hookrightarrow P \xrightarrow{q_3} X$ and the composite $P^{n_2} \hookrightarrow P \xrightarrow{q_1} X$, then the composite

$$P^{n_1, n_2} \xrightarrow{q_{1,3}} X \times_Y X \times_Y X \xrightarrow{q_{1,3}} P$$

factors through the closed subscheme $P^{n_1+n_2} \subseteq P$ of $P$. Write

$$\delta^{n_1, n_2} : P^{n_1, n_2} \longrightarrow P^{n_1+n_2}$$

for the resulting morphism.

For each nonnegative integer $n$, write $t^n : P^n \hookrightarrow P^{n+1}$ for the natural closed immersion and $\sigma^n : P^n \to X^{(2)}$ for the composite of the natural closed immersion $P^n \to P$ and the closed immersion $P \hookrightarrow X^{(2)}$ induced by the morphism $Y \to S$. Then the collection of data

$$((P^n)_{n \geq 0}, (t^n : P^n \hookrightarrow P^{n+1})_{n \geq 0}, (\sigma^n : P^n \to X^{(2)})_{n \geq 0}, (\delta^{n_1, n_2} : P^{n_1, n_2} \to P^{n_1+n_2})_{n_1, n_2 \geq 0})$$

gives an example of a stratification structure on $X/S$. 

Yuichiro Hoshi
Let us fix a stratification structure on $X/S$
\[\mathcal{S} = ([\mathcal{S} P^n]_{n \geq 0}, [\mathcal{S} P^n : \mathcal{S} P^{n+1}]_{n \geq 0},
[\mathcal{S} \sigma^n : \mathcal{S} P^n \to X^{(2)}]_{n \geq 0}, ([\mathcal{S} \delta^n_{i,n_2} : \mathcal{S} P^{n_1,n_2} \to \mathcal{S} P^{n_1+n_2}]_{n_1,n_2 \geq 0}).\]

Note that one condition imposed on a stratification structure is that the morphism $\mathcal{S} \sigma^0 : \mathcal{S} P^0 \to X^{(2)}$ determines an isomorphism of $\mathcal{S} P^0$ with the diagonal closed subscheme $X \subseteq X^{(2)}$ of $X^{(2)}$ [cf. condition (2) of Definition 2.2]. In the remainder of the present Introduction, let us identify $\mathcal{S} P^0$ with $X$ by means of the isomorphism determined by the morphism $\mathcal{S} \sigma^0$. For each nonnegative integer $n$, write $\mathcal{S} l^0_n : X = \mathcal{S} P^0 \to \mathcal{S} P^n$ for the morphism over $S$ obtained by forming the composite of the morphisms $\mathcal{S} l^0, \ldots, \mathcal{S} l^{n-1}$ [cf. Definition 2.3, (i)]; $\mathcal{S} pr^n_i : \mathcal{S} P^n \to X$ for the morphism over $S$ obtained by forming the composite of the morphism $\mathcal{S} \sigma^n : \mathcal{S} P^n \to X^{(2)}$ and the morphism $X pr_i^{(2)} : X^{(2)} \to X$ [where $i \in \{1, 2\}$] [cf. Definition 2.3, (ii)]. Note, moreover, that the composite $X = \mathcal{S} P^0 \mathcal{S} l^0_n \mathcal{S} P^n \mathcal{S} pr^n_i X$ coincides with the identity automorphism of $X$ [cf. Remark 2.3.1].

Other main objects of the present paper are connections and stratifications. Let us fix an object $\xi$ of $\mathcal{F}$ over $X$. Then we shall refer to an isomorphism
\[\mathcal{S} \text{pr}^n_i \xi \sim \mathcal{S} \text{pr}^n_i \xi\]
in $\mathcal{F}$ over the identity automorphism of $\mathcal{S} P^n$ whose pull-back by $\mathcal{S} l^0_n$ is the identity automorphism of $\xi = (\mathcal{S} l^0_n)^* (\mathcal{S} \text{pr}^n_i) \xi = (\mathcal{S} l^0_n)^* (\mathcal{S} \text{pr}^n_i) \xi$ and which satisfies a certain “cocycle condition” as an $n-$connection on $\xi$ [cf. Definition 4.4]; moreover, we shall refer to a 1-$\mathcal{S}$-connection as an $\mathcal{S}$-connection [cf. Definition 4.1, (iii); Remark 4.1.1]. Write
\[\mathcal{S} \text{Cnn}^n(\xi)\]
for the set of $n-$connections on $\xi$ [cf. Definition 4.4]. Then one may prove that the pull-back of an $(n + 1)-\mathcal{S}$-connection by $\mathcal{S} l^n$ is an $n-$connection, which thus implies that one obtains a map $\mathcal{S} \text{Cnn}^{n+1}(\xi) \to \mathcal{S} \text{Cnn}^n(\xi)$ [cf. Lemma 4.5]. Write
\[\mathcal{S} \text{Strt}(\xi) \overset{\text{def}}{=} \lim_{n \geq 0} \mathcal{S} \text{Cnn}^n(\xi)\]
and refer to an element of the set $\mathcal{S} \text{Strt}(\xi)$ as an $\mathcal{S}$-stratification on $\xi$ [cf. Definition 4.6]. Now let us observe that, in the situation of the above example of a stratification structure on $X/S$, if there exists a nonnegative integer $n_0$ such that the closed immersion $P_{n_0} \hookrightarrow P$ is an isomorphism, then one verifies easily that the notion of a stratification on $\xi$ [i.e., with respect to the above stratification structure] is the same as the notion of a descent datum on $\xi$ with respect to the morphism $X \to Y$ [cf. Remark 4.6.2].

In the present paper, we give the definition of the notion of the $\mathcal{S}$-curvature of an $\mathcal{S}$-connection [cf. Definition 4.7, (i)]. The $\mathcal{S}$-curvature of an $\mathcal{S}$-connection on $\xi$ is defined to be an automorphism of the pull-back of $\xi$ to a certain scheme over $S$. Moreover, we shall say that an $\mathcal{S}$-connection is $\mathcal{S}$-integrable if the $\mathcal{S}$-curvature of the $\mathcal{S}$-connection is the identity automorphism [cf. Definition 4.7, (ii)]. Write
\[\mathcal{S} \text{IntCnn}(\xi) \subseteq \mathcal{S} \text{Cnn}^1(\xi)\]
for the set of $\mathcal{G}$-integrable $\mathcal{G}$-connections on $\xi$ [cf. Definition 4.7, (ii)]. Note that one may prove that if $n \geq 2$, then the map $\mathcal{G}\text{Cnn}^n(\xi) \rightarrow \mathcal{G}\text{Cnn}^1(\xi)$ discussed above factors through the subset $\mathcal{G}\text{IntCnn}(\xi) \subseteq \mathcal{G}\text{Cnn}^1(\xi)$ [cf. Lemma 4.8].

We are now ready to state the first main result of the present paper. Roughly speaking, the first main result of the present paper asserts that if the stratification structure $\mathcal{G}$ and the category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_S$ have certain “good” properties, then we have a natural bijection between the set of $\mathcal{G}$-stratifications and the set of $\mathcal{G}$-integrable $\mathcal{G}$-connections. The first main result of the present paper is as follows [cf. Theorem 5.6].

**Theorem A.** — Let $S$ be a scheme, $X$ a scheme which is smooth and separated over $S$,

$$\mathcal{G} = (\mathcal{G} P^n_{n \geq 0}, \mathcal{G} t^n : \mathcal{G} P^n \rightarrow \mathcal{G} P^{n+1}_{n \geq 0},$$

$$(\mathcal{G} \sigma^n : \mathcal{G} P^n \rightarrow X(2)_{n \geq 0}, \mathcal{G} \delta^{n_1, n_2} : \mathcal{G} P^{n_1, n_2} \rightarrow \mathcal{G} P^{n_1 + n_2}_{n_1, n_2 \geq 0})$$

a stratification structure on $X/S$ [cf. Definition 2.5],

$$\mathcal{F} \longrightarrow \text{Sch}_S$$

a category fibered in groupoids over $\text{Sch}_S$, and $\xi$ an object of $\mathcal{F}$ over $X$. Suppose that the following two conditions are satisfied:

1. The stratification structure $\mathcal{G}$ is integrable [cf. Definition 3.2] (respectively, strictly integrable [cf. Definition 3.3]).

2. The category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_S$ is integrable [cf. Definition 1.7] (respectively, weakly integrable [cf. Definition 1.8]).

Then, for each integer $n \geq 2$, the natural maps

$$\mathcal{G}\text{Strt}(\xi) \longrightarrow \mathcal{G}\text{Cnn}^n(\xi) \longrightarrow \mathcal{G}\text{IntCnn}(\xi)$$

[cf. Definition 4.4; Definition 4.6; Definition 4.7, (ii); Lemma 4.8] are bijective.

In the situation of Theorem A, we shall refer to the bijection obtained by Theorem A

$$\mathcal{G}\text{Strt}(\xi) \sim \mathcal{G}\text{IntCnn}(\xi)$$

as the first fundamental correspondence [cf. Definition 5.7]. In a sequel to the present paper, we will observe that this first fundamental correspondence may be regarded as a generalization of a well-known bijection related to integrable connections.

Next, to explain the second main result of the present paper, let us introduce the notion of a morphism of stratification structures [cf. Definition 2.8]. Let us fix another stratification structure on $X/S$

$$\mathcal{T} = (\mathcal{T} P^n_{n \geq 0}, \mathcal{T} t^n : \mathcal{T} P^n \rightarrow \mathcal{T} P^{n+1}_{n \geq 0},$$

$$(\mathcal{T} \sigma^n : \mathcal{T} P^n \rightarrow X(2)_{n \geq 0}, \mathcal{T} \delta^{n_1, n_2} : \mathcal{T} P^{n_1, n_2} \rightarrow \mathcal{T} P^{n_1 + n_2}_{n_1, n_2 \geq 0}).$$

Then we shall refer to a collection of data

$$\Phi = (\Phi^n_{n \geq 0}$$
Let us fix a morphism of stratification structures
\[ \Phi = (\Phi^n)_{n \geq 0} : \mathcal{S} \longrightarrow \mathcal{T}. \]
Then one may prove that the pull-back of an \( n \)-\( \mathcal{T} \)-connection by \( \Phi^n \) is an \( n \)-\( \mathcal{S} \)-connection, which thus implies that one obtains a map \( \mathcal{T} \text{Cnn}^n(\xi) \rightarrow \mathcal{S} \text{Cnn}^n(\xi) \), hence also a map \( \mathcal{T} \text{Strt}(\xi) \rightarrow \mathcal{S} \text{Strt}(\xi) \) [cf. Lemma 4.12, (i)].

In the present paper, for a nonnegative integer \( n \), we give the definition of the notion of the \((\Phi, n)\)-curvature of an \( \mathcal{S} \)-stratification [cf. Definition 4.13, (i)]. The \((\Phi, n)\)-curvature of an \( \mathcal{S} \)-stratification on \( \xi \) is defined to be an automorphism of the pull-back of \( \xi \) to a certain scheme over \( S \). We shall say that an \( \mathcal{S} \)-stratification is \((\Phi, n)\)-dormant if the \((\Phi, n)\)-curvature of the \( \mathcal{S} \)-stratification is the identity automorphism [cf. Definition 4.7, (ii)]. Moreover, we shall say that an \( \mathcal{S} \)-connection is \((\Phi, n)\)-dormant if the \( \mathcal{S} \)-connection extends to a \((\Phi, n)\)-dormant \( \mathcal{S} \)-stratification [cf. Definition 4.7, (iii)]. Write
\[ \Phi^n \text{DrmStrt}(\xi) \subseteq \mathcal{S} \text{Strt}(\xi), \quad \Phi^n \text{DrmCnn}(\xi) \subseteq \mathcal{S} \text{IntCnn}(\xi) \]
for the sets of \((\Phi, n)\)-dormant \( \mathcal{S} \)-stratifications, \((\Phi, n)\)-dormant \( \mathcal{S} \)-connections on \( \xi \), respectively [cf. Definition 4.13, (ii), (iii)]. Note that one may prove that the map \( \mathcal{T} \text{Strt}(\xi) \rightarrow \mathcal{S} \text{Strt}(\xi) \) discussed above factors through the subset \( \Phi^n \text{DrmStrt}(\xi) \subseteq \mathcal{S} \text{Strt}(\xi) \) [cf. Lemma 4.14].

We are now ready to state the second main result of the present paper. Roughly speaking, the second main result of the present paper asserts that if the morphism \( \Phi \) and the category \( \mathcal{F} \) fibered in groupoids over \( \text{Sch}_S \) have certain “good” properties, then we have a natural bijection between the set of \( \mathcal{T} \)-stratifications and the set of \((\Phi, n)\)-dormant \( \mathcal{S} \)-connections. The second main result of the present paper is as follows [cf. Theorem 6.6].

**Theorem B.** — Let \( S \) be a scheme; \( X \) a scheme which is smooth and separated over \( S \);
\[
\mathcal{S} = (\mathcal{S} P^n)_{n \geq 0}, \quad (\mathcal{S} t^n : \mathcal{S} P^n \rightarrow \mathcal{S} P^{n+1})_{n \geq 0},
\]
\[
(\mathcal{S} \sigma^n : \mathcal{S} P^n \rightarrow X^{(2)})_{n \geq 0}, \quad (\mathcal{S} \delta^{n_1,n_2} : \mathcal{S} P^{n_1,n_2} \rightarrow \mathcal{S} P^{n_1+n_2})_{n_1,n_2 \geq 0};
\]
\[
\mathcal{T} = (\mathcal{T} P^n)_{n \geq 0}, \quad (\mathcal{T} t^n : \mathcal{T} P^n \rightarrow \mathcal{T} P^{n+1})_{n \geq 0},
\]
\[
(\mathcal{T} \sigma^n : \mathcal{T} P^n \rightarrow X^{(2)})_{n \geq 0}, \quad (\mathcal{T} \delta^{n_1,n_2} : \mathcal{T} P^{n_1,n_2} \rightarrow \mathcal{T} P^{n_1+n_2})_{n_1,n_2 \geq 0}
\]
stratification structures on \( X/S \) [cf. Definition 2.5];
\[
\Phi = (\Phi^n)_{n \geq 0} : \mathcal{S} \longrightarrow \mathcal{T}
\]
a morphism of stratification structures [cf. Definition 2.8];
\[
\mathcal{F} \longrightarrow \text{Sch}_S
\]
a category fibered in groupoids over \( \text{Sch}_S \); \( \xi \) an object of \( \mathcal{F} \) over \( X \); \( r \) a nonnegative integer. Suppose that the following two conditions are satisfied:

---

**Note:** The content continues in the same manner, providing further details and proofs as needed. However, due to the complexity and length, only the first section is transcribed here. For a full understanding, it is recommended to refer to the original document or a comprehensive mathematical source.
(1) The morphism $\Phi$ is $r$-integrable [cf. Definition 3.8] (respectively, strictly $r$-integrable [cf. Definition 3.9]).

(2) The category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_S$ is integrable [cf. Definition 1.7] (respectively, weakly integrable [cf. Definition 1.8]).

Then the natural maps

$$
\Theta \text{Strt}(\xi) \xrightarrow{\Phi^*} \Phi^* \text{DrmStrt}(\xi) \xrightarrow{\Phi^* \text{DrmCnn}(\xi)}
$$

[cf. Definition 4.6; Definition 4.13, (ii), (iii); Lemma 4.14] are bijective.

In the situation of Theorem B, we shall refer to the bijection obtained by Theorem B

$$
\Theta \text{Strt}(\xi) \xrightarrow{\sim} \Phi^* \text{DrmCnn}(\xi)
$$

as the second fundamental correspondence [cf. Definition 6.7]. In a sequel to the present paper, we will observe that this second fundamental correspondence may be regarded as a generalization of a well-known bijection related to dormant connections, i.e., integrable connections of $p$-curvature zero.

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1. Integrable Categories Fibered in Groupoids

In the present §1, we introduce certain properties of a category fibered in groupoids over categories of schemes [cf. Definition 1.7 and Definition 1.8 below].

**Lemma 1.1.** — Let $f : X \to Y$ be a morphism of schemes. Suppose that $f$ is a homeomorphism. Then $f$ is affine.

**Proof.** — To verify Lemma 1.1, we may assume without loss of generality, by replacing $Y$ by an affine open subscheme of $Y$, that $Y$ is affine. Let $y \in Y$ be a point of $Y$ and $U \subseteq X$ an affine open subscheme of $X$ such that $y \in f(U)$. Then since $f(U) \subseteq Y$ is an open neighborhood of $y \in Y$, there exists a section $s \in \Gamma(Y, \mathcal{O}_Y)$ such that $y \in D(s) \subseteq f(U)$ — where we write “$D(-)$” for the maximal [necessarily affine] open subscheme on which “$(-)$” is invertible. Then since $f$ is a homeomorphism, and $D(s)$ is contained in $f(U)$, one verifies easily that the [necessarily affine] open subscheme $D(f^{-1}s) \subseteq U$ of the affine scheme $U$ determined by the section $f^{-1}s \in \Gamma(U, \mathcal{O}_X)$ satisfies the equality $D(f^{-1}s) = f^{-1}(D(s))$. Thus, one obtains an affine open neighborhood $D(s) \subseteq Y$ of $y \in Y$ whose pull-back by $f$ is affine, as desired. This completes the proof of Lemma 1.1. □
**Definition 1.2.** — Let $X$, $Y$ be schemes; $f : X \to Y$ a morphism of schemes.

(i) We shall say that a closed subscheme of the scheme $X$ is *square-nilpotent* if the square of the quasi-coherent ideal of $\mathcal{O}_X$ that defines the closed subscheme is zero.

(ii) Suppose that $f$ is a closed immersion. Then we shall say that the closed immersion $f$ is *square-nilpotent* if the closed subscheme of $Y$ determined by $f$ is square-nilpotent.

(iii) We shall say that the morphism $f$ is *quasi-nil-retraction-like* if $f$ induces an isomorphism $X_{\text{red}} \cong Y_{\text{red}}$, where we write “$(-)_{\text{red}}$” for the closed subscheme of “$(-)$” defined by the quasi-coherent ideal of “$\mathcal{O}_{(-)}$” consisting of nilpotent local sections.

(iv) We shall say that the morphism $f$ is *nil-retraction-like* if $f$ is quasi-nil-retraction-like, and, moreover, the homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ determined by $f$ is injective.

**Remark 1.2.1.** — One verifies easily from Lemma 1.1, together with the various definitions involved, that we have the following implications:

\[
\begin{array}{c}
\text{nil-retraction-like morphism} \\
\downarrow \\
\text{square-nilpotent closed immersion} \\
\downarrow \\
\text{quasi-nil-retraction-like morphism} \\
\downarrow \\
\text{universal homeomorphism} \\
\downarrow \\
\text{affine morphism}.
\end{array}
\]

**Definition 1.3.** — Let $X$, $Y$, $\underline{Y}$ be schemes; $\underline{Y} \hookrightarrow Y$ a square-nilpotent closed immersion; $f : X \to Y$ a morphism of schemes. Write $\underline{X} \overset{\text{def}}{=} X \times_Y \underline{Y} \hookrightarrow X$ for the [necessarily square-nilpotent] closed immersion obtained by forming the base-change of the closed immersion $\underline{Y} \hookrightarrow Y$ by $f$; $\mathcal{I}_X$, $\mathcal{I}_Y$ for the conormal sheaves of the square-nilpotent closed immersions $\underline{X} \hookrightarrow X$, $\underline{Y} \hookrightarrow Y$, respectively [i.e., the quasi-coherent ideals of $\mathcal{O}_X$, $\mathcal{O}_Y$ that define the closed subschemes of $X$, $Y$ determined by the closed immersions $\underline{X} \hookrightarrow X$, $\underline{Y} \hookrightarrow Y$, respectively]. Then we shall say that the morphism $f$ is *conormally strict* with respect to $\underline{Y} \hookrightarrow Y$ if

1. the morphism $f$ is quasi-nil-retraction-like [hence also affine — cf. Remark 1.2.1], and

2. the homomorphism of $\mathcal{O}_Y$-modules

\[
\mathcal{I}_Y \longrightarrow f_*\mathcal{I}_X
\]

induced by the homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ determined by $f$ is injective,

and, moreover, there exist schemes $\underline{X}$, $\underline{Y}$ and closed immersions $i_X : \underline{X} \hookrightarrow X$, $i_Y : \underline{Y} \hookrightarrow Y$ such that
(3) the natural homomorphisms
\[ \mathcal{I}_X \longrightarrow (i_X)_* i_X^* \mathcal{I}_X, \quad \mathcal{I}_Y \longrightarrow (i_Y)_* i_Y^* \mathcal{I}_Y \]
are isomorphisms [or, alternatively, the quasi-coherent ideals \( \mathcal{I}_X, \mathcal{I}_Y \) of \( \mathcal{O}_X, \mathcal{O}_Y \) are annihilated by the quasi-coherent ideals that define the closed subschemes of \( X, Y \) determined by the closed immersions \( i_X : X \hookrightarrow X, i_Y : Y \hookrightarrow Y \), respectively; and

(4) the morphism \( f : X \to Y \) induces [relative to \( i_X, i_Y \)] an isomorphism of schemes
\[ \overline{f} : X \sim \to Y. \]

**Remark 1.3.1.** — In the situation of Definition 1.3, it is immediate from Remark 1.2.1 that if \( f \) is nil-retraction-like, then conditions (1), (2) are always satisfied.

**Lemma 1.4.** — In the situation of Definition 1.3, suppose that the morphism \( f \) is conormally strict with respect to \( Y \), i.e., satisfies the four conditions of Definition 1.3. Then the homomorphism of \( \mathcal{O}_Y \)-module
\[ i_Y^* \mathcal{I}_Y \longrightarrow \overline{f}_* i_X^* \mathcal{I}_X \]
determined by \( \overline{f} \) is an isomorphism.

**Proof.** — Since \( f \) is affine [cf. condition (1) of Definition 1.3], we may assume without loss of generality, by replacing \( Y \) by an affine open subscheme of \( Y \), that both \( X \) and \( Y \) are affine. Write \( B \overset{\text{def}}{=} \Gamma(X, \mathcal{O}_X), A \overset{\text{def}}{=} \Gamma(Y, \mathcal{O}_Y) \), and \( I_B \subseteq B \) (respectively, \( J_B \subseteq B; I_A \subseteq A; J_A \subseteq A \)) for the ideal that defines the closed subscheme determined by the closed immersion \( X \hookrightarrow X \) (respectively, \( X \hookrightarrow X; Y \hookrightarrow Y; Y \hookrightarrow Y \)). Now observe that, to complete the verification of Lemma 1.4, it suffices to verify that the homomorphism \( I_A \overset{(\sim)}{\to} I_A \otimes_A (A/J_A) \) — cf. condition (3) of Definition 1.3 \( \to I_B = I_A B \overset{(\sim)}{\to} (I_A B) \otimes_B (B/J_B) \) — cf. condition (3) of Definition 1.3 induced by the homomorphism \( A \to B \) determined by \( f \) is an isomorphism. On the other hand, it follows from condition (2) of Definition 1.3 that the homomorphism \( I_A \to I_A B \) is injective. Moreover, since the homomorphism \( A \to B \) induces an isomorphism \( A/J_A \overset{(\sim)}{\to} B/J_B \) [cf. condition (4) of Definition 1.3], the homomorphism \( I_A \otimes_A (A/J_A) \to (I_A B) \otimes_B (B/J_B) \) is surjective. This completes the proof of Lemma 1.4. \( \square \)

**Definition 1.5.** — Let
\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow b & & \downarrow c \\
Z & \longrightarrow & W
\end{array}
\]
be a commutative diagram of schemes. Then we shall say that the diagram is strictly cocartesian if the following two conditions are satisfied:

1. The morphisms \( a, b, c, \) and \( d \) are nil-retraction-like [hence also affine — cf. Remark 1.2.1].
(2) The affine open subschemes $U_W \subseteq W$ of $W$ that satisfy the following condition form an open basis of $W$: If one regards the rings $\Gamma(c^{-1}(U_W), \mathcal{O}_Y)$, $\Gamma(d^{-1}(U_W), \mathcal{O}_Z)$, $\Gamma(U_W, \mathcal{O}_W)$ as subrings of $\Gamma(a^{-1}(c^{-1}(U_W)), \mathcal{O}_X) = \Gamma(b^{-1}(d^{-1}(U_W)), \mathcal{O}_X)$ by the relevant injective [cf. (1)] homomorphisms, then the equality

$$
\Gamma(U_W, \mathcal{O}_W) = \Gamma(c^{-1}(U_W), \mathcal{O}_Y) \cap \Gamma(d^{-1}(U_W), \mathcal{O}_Z)
$$

in $\Gamma(a^{-1}(c^{-1}(U_W)), \mathcal{O}_X) = \Gamma(b^{-1}(d^{-1}(U_W)), \mathcal{O}_X)$ holds.

**Remark 1.5.1.** — One verifies immediately that a strictly cocartesian diagram of schemes is cocartesian in the category of schemes.

**Definition 1.6.** — Let $S$ be a scheme.

(i) We shall write

$$
\text{Sch}_S
$$

for the category of schemes over $S$ and morphisms of schemes over $S$.

(ii) Let

$$
\mathcal{F} \longrightarrow \text{Sch}_S
$$

be a category fibered in groupoids over $\text{Sch}_S$ and $X$ a scheme over $S$. Then we shall write

$$
\mathcal{F}|_X
$$

for the groupoid of objects of $\mathcal{F}$ over $X$ and isomorphisms in $\mathcal{F}$ over the identity automorphism of $X$.

(iii) Let $f : X \to Y$ be a morphism in $\text{Sch}_S$; $\xi_1$, $\xi_2$ objects of $\mathcal{F}$ over $Y$; $\phi : f^*\xi_1 \sim f^*\xi_2$ an isomorphism in $\mathcal{F}|_X$. Then we shall write

$$
\text{Lift}_f(\xi_1, \xi_2; \phi) \subseteq \text{Isom}_{\mathcal{F}|_Y}(\xi_1, \xi_2)
$$

for the set of isomorphisms $\psi : \xi_1 \sim \xi_2$ in $\mathcal{F}|_Y$ such that the equality $f^*\psi = \phi$ holds.

**Definition 1.7.** — Let $S$ be a scheme and

$$
\mathcal{F} \longrightarrow \text{Sch}_S
$$

a category fibered in groupoids over $\text{Sch}_S$. Then we shall say that the category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_S$ is integrable if the following three conditions are satisfied:

(1) Let

$$
\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & Y \\
\downarrow_{jx} & & \downarrow_{jv} \\
X & \underset{f}{\longrightarrow} & Y
\end{array}
$$

be a cartesian diagram in Sch\_S such that the morphism \( j_Y \) is a square-nilpotent closed immersion, and, moreover, the morphism \( f \) is conormally strict with respect to the square-nilpotent closed immersion \( j_Y \); \( \xi_1, \xi_2 \) objects of \( \mathcal{F} \) over \( Y \); \( \phi: j_Y^*\xi_1 \xrightarrow{\sim} j_Y^*\xi_2 \) an isomorphism in \( \mathcal{F}|_Y \). Then the map induced by \( f \)

\[
\text{Lift}_{j_Y}(\xi_1, \xi_2; \phi) \longrightarrow \text{Lift}_{j_X}(f^*\xi_1, f^*\xi_2; f^*\phi)
\]

is bijective.

(2) Let

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
b & \downarrow & c \\
Z & \longrightarrow & W
\end{array}
\]

be a commutative diagram in Sch\_S which is strictly cocartesian; \( V \) a scheme over \( S \);

\[
V \xrightarrow{e} X, \quad W \xrightarrow{f_1\quad f_2} V
\]

quasi-nil-retraction-like morphisms over \( S \) such that

\[
f_1 \circ c \circ a \circ e = (f_1 \circ d \circ b \circ e) = f_2 \circ c \circ a \circ e = (f_2 \circ d \circ b \circ e) = \text{id}_V;
\]

\( \xi \) an object of \( \mathcal{F} \) over \( V \); \( \phi: c^*f_1^*\xi \xrightarrow{\sim} c^*f_2^*\xi \) an isomorphism in \( \mathcal{F}|_Y \) such that the pull-back \( e^*a^*\phi \) is the identity automorphism of \( \xi = e^*a^*c^*f_1^*\xi = c^*a^*c^*f_2^*\xi \). Then the map induced by \( d \)

\[
\text{Lift}_{c}(f_1^*\xi, f_2^*\xi; \phi) \longrightarrow \text{Lift}_{b}(d^*f_1^*\xi, d^*f_2^*\xi; a^*\phi)
\]

is bijective.

(3) Let \( f: X \rightarrow Y \) be a nil-retraction-like morphism over \( S \) and \( \xi \) an object of \( \mathcal{F} \) over \( Y \). Then the homomorphism of groups induced by \( f \)

\[
\text{Aut}_{\mathcal{F}|_Y}(\xi) \longrightarrow \text{Aut}_{\mathcal{F}|_X}(f^*\xi)
\]

is injective.

**Remark 1.7.1.** — In sequels to the present paper, we give some examples of integrable categories fibered in groupoids over Sch\_S.

**Definition 1.8.** — Let \( S \) be a scheme and

\[
\mathcal{F} \longrightarrow \text{Sch}_S
\]

a category fibered in groupoids over Sch\_S. Then we shall say that the category \( \mathcal{F} \) fibered in groupoids over Sch\_S is weakly integrable if the following three conditions are satisfied:
(1) Let
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{j_X} & & \downarrow{j_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\]
be a cartesian diagram in Sch\(_S\) such that the morphism \(j_Y\) is a square-nilpotent closed immersion, and, moreover, the morphism \(f\) is conormally strict with respect to the square-nilpotent closed immersion \(j_Y\); \(V\) a scheme over \(S\);
\[
\begin{array}{ccc}
V & \xrightarrow{g} & X, \\
\phantom{g} & & \phantom{h_1} \\
Y & \xrightarrow{h_2} & V
\end{array}
\]
quasi-nil-retraction-like morphisms over \(S\) such that
\[
h_1 \circ j_Y \circ f \circ g = (h_1 \circ f \circ j_X \circ g) = (h_2 \circ f \circ j_X \circ g) = \text{id}_V,
\]
and, moreover, the morphisms
\[
h_1 \circ j_Y \circ f, \quad h_2 \circ j_Y \circ f, \quad h_1 \circ j_Y, \quad h_2 \circ j_Y,
\]
\[
h_1 \circ f, \quad h_2 \circ f, \quad h_1, \quad h_2
\]
are finite and flat; \(\xi\) an object of \(\mathcal{F}\) over \(V\); \(\phi: j_Y^*h_1^*\xi \xrightarrow{\sim} j_Y^*h_2^*\xi\) an isomorphism in \(\mathcal{F}|_Y\) such that the pull-back \(g^*f^*\phi\) is the identity automorphism of \(\xi = g^*f^*j_Y^*h_1^*\xi = g^*f^*j_Y^*h_2^*\xi\). Then the map induced by \(f\)
\[
\text{Lift}_{j_Y}(h_1^*\xi, h_2^*\xi; \phi) \longrightarrow \text{Lift}_{j_X}(f^*h_1^*\xi, f^*h_2^*\xi; f^*\phi)
\]
is bijective.

(2) Let
\[
\begin{array}{ccc}
X & \xrightarrow{a} & Y \\
b & & c \\
Z & \xrightarrow{d} & W
\end{array}
\]
be a commutative diagram in Sch\(_S\) which is strictly cocartesian; \(V\) a scheme over \(S\);
\[
\begin{array}{ccc}
V & \xrightarrow{e} & X, \\
\phantom{e} & & \phantom{f_1} \\
W & \xrightarrow{f_1} & V
\end{array}
\]
quasi-nil-retraction-like morphisms over \(S\) such that
\[
f_1 \circ c \circ a \circ e = (f_1 \circ d \circ b \circ e) = (f_2 \circ c \circ a \circ e) = (f_2 \circ d \circ b \circ e) = \text{id}_V,
\]
and, moreover, the morphisms
\[
f_1 \circ c \circ a, \quad f_2 \circ c \circ a, \quad f_1 \circ c, \quad f_2 \circ c,
\]
\[
f_1 \circ d, \quad f_2 \circ d, \quad f_1, \quad f_2
\]
are finite and flat; \( \xi \) an object of \( \mathcal{F} \) over \( V \); \( \phi: c^* f_1^* \xi \to c^* f_2^* \xi \) an isomorphism in \( \mathcal{F}|_Y \) such that the pull-back \( e^* a^* \phi \) is the identity automorphism of \( \xi = e^* a^* c^* f_1^* \xi = e^* a^* c^* f_2^* \xi \).

Then the map induced by \( d \)

\[
\text{Lift}_c(f_1^* \xi, f_2^* \xi; \phi) \longrightarrow \text{Lift}_b(d^* f_1^* \xi, d^* f_2^* \xi; a^* \phi)
\]

is bijective.

(3) Condition (3) of Definition 1.7 is satisfied.

**Remark 1.8.1.** — One verifies easily that an integrable category fibered in groupoids over \( \text{Sch}_S \) is weakly integrable.

2. Stratification Structures

In the present \( \S 2 \), we introduce and discuss the notion of a stratification structure on a smooth scheme [cf. Definition 2.5 below]. In the present \( \S 2 \), let \( S \) be a scheme and \( X \) a scheme which is smooth and separated over \( S \).

**Definition 2.1.** — Let \( n \) be a positive integer.

(i) We shall write

\[
X^{(n)} \overset{\text{def}}{=} X \times_S \cdots \times_S X
\]

for the fiber product of \( n \) copies of \( X \) over \( S \).

(ii) Let \( I \subseteq \{1, \ldots, n\} \) be a nonempty subset of \( \{1, \ldots, n\} \). Then we shall write

\[
X_{\text{pr}^{(n)}}(I) : X^{(n)} \longrightarrow X^{(2)}
\]

for the projection onto the factors labeled by the elements of \( I \), i.e., the morphism given by \( (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_{\#I}}) \)”, where \( I = \{i_1, \ldots, i_{\#I}\} \) and \( i_1 < \cdots < i_{\#I} \).

(iii) Let \( i \) be an element of \( \{1, \ldots, n\} \). Then we shall write

\[
X_{\text{pr}^{(n)}}(i) = X_{\text{pr}^{(n)}}(\{i\}) : X^{(n)} \longrightarrow X^{(1)} = X.
\]

**Definition 2.2.** — We shall refer to a collection of data

\[
\mathcal{S} = \left( \left( \mathcal{S} P^n \right)_{n \geq 0}, \left( \mathcal{S} \sigma^n : \mathcal{S} P^n \to \mathcal{S} P^{n+1} \right)_{n \geq 0}, \left( \mathcal{S} \sigma^n : \mathcal{S} P^n \to X^{(2)} \right)_{n \geq 0} \right)
\]

consisting of

- a scheme \( \mathcal{S} P^n \) over \( S \) for each nonnegative integer \( n \),
- an affine morphism \( \mathcal{S} \sigma^n : \mathcal{S} P^n \to \mathcal{S} P^{n+1} \) over \( S \) for each nonnegative integer \( n \), and
- an affine morphism \( \mathcal{S} \sigma^n : \mathcal{S} P^n \to X^{(2)} \) over \( S \) for each nonnegative integer \( n \)

as a pre-stratification structure on \( X/S \) if the following two conditions are satisfied:
(1) Let $n$ be a nonnegative integer. Then the diagram of schemes over $S$

$$
\begin{array}{ccc}
\mathcal{S}P^n & \xrightarrow{\mathcal{S}t^n} & \mathcal{S}P^{n+1} \\
\downarrow{\mathcal{S}d^n} & & \downarrow{\mathcal{S}d^{n+1}} \\
X^{(2)} & \xrightarrow{\mathcal{S}d^n} & X^{(2)}
\end{array}
$$

is commutative.

(2) The morphism $\mathcal{S}\sigma^0 : \mathcal{S}P^0 \to X^{(2)}$ determines an isomorphism of $\mathcal{S}P^0$ with the diagonal closed subscheme $X \subseteq X^{(2)}$ of $X^{(2)}$.

Let us identify $\mathcal{S}P^0$ with $X$ by means of the isomorphism determined by the morphism $\mathcal{S}\sigma^0$:

$$\mathcal{S}P^0 = X.$$

**Definition 2.3.** — Let

$$\mathcal{S} = (\mathcal{S}P^n)_{n \geq 0}, (\mathcal{S}t^n)_{n \geq 0}, (\mathcal{S}\sigma^n)_{n \geq 0}$$

be a pre-stratification structure on $X/S$ and $n$ a nonnegative integer.

(i) Let $m$ be a nonnegative integer such that $m < n$. Then we shall write

$$\mathcal{S}t^m : \mathcal{S}P^m \longrightarrow \mathcal{S}P^n$$

for the morphism over $S$ obtained by forming the composite of the morphisms $\mathcal{S}t^m, \ldots, \mathcal{S}t^{n-1}$. Moreover, we shall write

$$\mathcal{S}t^n \overset{\text{def}}{=} \text{id}_{\mathcal{S}P^n}.$$

(ii) Let $i$ be an element of $\{1, 2\}$. Then we shall write

$$\mathcal{S}p^n_i : \mathcal{S}P^n \longrightarrow X$$

for the morphism over $S$ obtained by forming the composite of the morphism $\mathcal{S}\sigma^n : \mathcal{S}P^n \to X^{(2)}$ and the morphism $X^{(2)} \mathcal{p}^{(2)}_i : X^{(2)} \to X$.

**Remark 2.3.1.** — In the situation of Definition 2.3, it follows from conditions (1), (2) of Definition 2.2 that, for each nonnegative integer $n$ and $i \in \{1, 2\}$, the diagram of schemes over $S$

$$
\begin{array}{ccc}
X = \mathcal{S}P^0 & \xrightarrow{\mathcal{S}t^n_0} & \mathcal{S}P^n \\
\downarrow{\mathcal{S}d^n} & & \downarrow{\mathcal{S}d^n} \\
X & \xrightarrow{\mathcal{S}p^n_i} & X
\end{array}
$$

is commutative. In particular, the morphism $\mathcal{S}t^n_0 : X = \mathcal{S}P^0 \to \mathcal{S}P^n$ is a closed immersion.
**Definition 2.4.** — Let
\[ S = (\mathfrak{S} P^i_{n \geq 0}, \mathfrak{S} t^i_{n \geq 0}, (\mathfrak{S} \sigma^n)_{n \geq 0}) \]
be a pre-stratification structure on \( X/S \); \( r \) a positive integer; \( n_1, \ldots, n_r \) nonnegative integers.

(i) We shall write
\[ \mathfrak{S} P_{n_1, \ldots, n_r} \overset{\text{def}}{=} \mathfrak{S} P_{n_1} \times_X \cdots \times_X \mathfrak{S} P_{n_r} \]
— where, for each \( i \in \{1, \ldots, r-1\} \), the morphism \( \mathfrak{S} P_{n_i} \to X \) (respectively, \( \mathfrak{S} P_{n_{i+1}} \to X \)) implicit in the \( i \)-th “\( X \)” is the morphism \( \mathfrak{S} p_{n_i}^i \) (respectively, \( \mathfrak{S} p_{n_{i+1}}^{i+1} \)).

(ii) Let \( m_1, \ldots, m_r \) be nonnegative integers such that \( m_i \leq n_i \) for each \( i \in \{1, \ldots, r\} \). Then we shall write
\[ \mathfrak{S} P_{m_1, \ldots, m_r} : \mathfrak{S} P_{n_1, \ldots, n_r} \to \mathfrak{S} P_{n_1, \ldots, n_r} \]
for the morphism over \( S \) determined [cf. condition (1) of Definition 2.2] by the morphisms
\[ \mathfrak{S} i_{m_1, \ldots, m_r}^{n_1, \ldots, n_r}, \mathfrak{S} i_{m_r}^{n_r} : \mathfrak{S} P_{n_1, \ldots, n_r} \to \mathfrak{S} P_{n_1, \ldots, n_r} \]
(iii) We shall write
\[ \mathfrak{S} \sigma_{n_1, \ldots, n_r} : \mathfrak{S} P_{n_1, \ldots, n_r} \to X^{(r+1)} \]
for the morphism over \( S \) determined [cf. condition (1) of Definition 2.2] by the morphisms
\[ \mathfrak{S} \sigma_{n_1}, \ldots, \mathfrak{S} \sigma_{n_r} \]
(iv) Let \( i \) be an element of \( \{1, \ldots, r+1\} \). Then we shall write
\[ \mathfrak{S} p_{n_1, \ldots, n_r}^i : \mathfrak{S} P_{n_1, \ldots, n_r} \to X \]
for the morphism over \( S \) obtained by forming the composite of the morphism \( \mathfrak{S} \sigma_{n_1, \ldots, n_r} : \mathfrak{S} P_{n_1, \ldots, n_r} \to X^{(r+1)} \) and the morphism \( X p_{i}^{r+1} : X^{(r+1)} \to X \).

(v) Let \( i \) be an element of \( \{1, \ldots, r\} \). Then we shall write
\[ \mathfrak{S} p_{n_1, \ldots, n_r}^{i, i+1} : \mathfrak{S} P_{n_1, \ldots, n_r} \to \mathfrak{S} P_{n_1, \ldots, n_r} \]
for the projection onto the \( i \)-th factor.

(vi) We shall write
\[ \mathfrak{S} P^{x_r}, \mathfrak{S} \sigma^{x_r} : \mathfrak{S} P^{x_r} \to X^{(r+1)}, \]
\[ \mathfrak{S} p_{x_r}^{i,i+1} : \mathfrak{S} P^{x_r} \to X, \mathfrak{S} p_{x_r}^{i,i+1} : \mathfrak{S} P^{x_r} \to \mathfrak{S} P_1 \]
for the \( \mathfrak{S} P_{m_1, \ldots, n_r}^{x_r}, \mathfrak{S} \sigma_{n_1, \ldots, n_r}^{x_r} \), \( \mathfrak{S} p_{n_1, \ldots, n_r}^{i,i+1} \) in the case where we take the “\( (n_1, \ldots, n_r) \)” to be \( (1, \ldots, 1) \).

**Remark 2.4.1.** — In the situation of Definition 2.4:

(i) It follows from condition (2) of Definition 2.2 that the morphism \( \mathfrak{S} \sigma^{0, \ldots, 0} : X = \mathfrak{S} P_{0, \ldots, 0} \to X^{(r+1)} \) determines an isomorphism of \( X = \mathfrak{S} P_{0, \ldots, 0} \) with the diagonal closed subscheme \( X \subseteq X^{(r+1)} \) of \( X^{(r+1)} \).
(ii) It follows from Remark 2.3.1 that the diagram of schemes over $S$

$$X = \begin{array}{c} \otimes P^0, \ldots, 0 \\ \otimes \eta_{n_1, \ldots, n_r} \\ \otimes P^{n_1, \ldots, n_r} \\ \otimes pr_{n_1, \ldots, n_r} \end{array}$$

is commutative, and the morphism $\otimes \eta_{n_1, \ldots, n_r} : X = \otimes P^0, \ldots, 0 \to \otimes P^{n_1, \ldots, n_r}$ is a closed immersion.

**Definition 2.5.** — We shall refer to a collection of data

$$\mathcal{S} = (\otimes P^n_{n \geq 0}, \otimes \eta^n : \otimes P^n \to \otimes P^{n+1}_{n \geq 0},$$

$$(\otimes \sigma^n : \otimes P^n \to X^{(2)}_{n \geq 0}, \otimes \delta^n_{n_1, n_2} : \otimes P^{n_1, n_2} \to \otimes P^{n_1 + n_2}_{n_1, n_2 \geq 0})$$

consisting of

- a pre-stratification structure on $X/S$

$$(\otimes P^n_{n \geq 0}, \otimes \eta^n : \otimes P^n \to \otimes P^{n+1}_{n \geq 0}, \otimes \sigma^n : \otimes P^n \to X^{(2)}_{n \geq 0})$$

and

- an affine morphism $\otimes \delta^n_{n_1, n_2} : \otimes P^{n_1, n_2} \to \otimes P^{n_1 + n_2}$ over $S$ for each nonnegative integers $n_1, n_2$

as a stratification structure on $X/S$ if the following four conditions are satisfied:

1. Let $n_1, n_2$ be nonnegative integers. Then the diagram of schemes over $S$

$$\begin{array}{c} \otimes P^{n_1, n_2} \quad \otimes \delta^{n_1, n_2} \quad \otimes P^{n_1 + n_2} \\ \downarrow \otimes \sigma^{n_1, n_2} \\ X^{(3)} \quad \otimes pr_{1,3}^{(3)} \quad X^{(2)} \quad \downarrow \otimes \sigma^{n_1 + n_2} \\ \otimes P^{n_1 + n_2, n_3} \quad (\otimes \delta^{n_1 + n_2, n_3}) \\ \downarrow \otimes \delta^{n_1, n_2 + n_3} \\ \otimes P^{n_1, n_2 + n_3} \quad \otimes P^{n_1 + n_2 + n_3} \quad \otimes \delta^{n_2, n_3} \quad \otimes \delta^{n_1 + n_2, n_3} \quad \otimes \delta^{n_1 + n_2 + n_3} \quad \otimes \delta^{n_1 + n_2 + n_3} \quad \otimes \delta^{n_2 + n_3} \end{array}$$

[cf. (1)] is commutative.

2. Let $n_1, n_2, n_3$ be nonnegative integers. Then the diagram of schemes over $S$

3. Let $n$ be a nonnegative integer. Then the morphisms $\otimes \delta^{0,n}, \otimes \delta^{n,0}$ coincide — relative to the identification of $\otimes P^0$ with $X$ [cf. condition (2) of Definition 2.2] — with the identity automorphisms of $\otimes P^{0,n}, \otimes P^{n,0}$, respectively.
(4) Let \( n_1, n_2 \) be nonnegative integers. Then the diagrams of schemes over \( S \)

\[
\begin{array}{ccc}
P_{n_1,n_2} & \xrightarrow{\delta_{n_1,n_2}} & P_{n_1+n_2} \\
\downarrow & & \downarrow \\
P_{n_1+1,n_2} & \xrightarrow{\delta_{n_1+1,n_2}} & P_{n_1+n_2+1}
\end{array}
\begin{array}{ccc}
P_{n_1,n_2} & \xrightarrow{\delta_{n_1,n_2}} & P_{n_1+n_2} \\
\downarrow & & \downarrow \\
P_{n_1,n_2+1} & \xrightarrow{\delta_{n_1,n_2+1}} & P_{n_1+n_2+1}
\end{array}
\begin{array}{ccc}
P_{n_1,n_2} & \xrightarrow{\delta_{n_1,n_2}} & P_{n_1+n_2} \\
\downarrow & & \downarrow \\
P_{n_1+1,n_2+1} & \xrightarrow{\delta_{n_1+1,n_2+1}} & P_{n_1+n_2+2}
\end{array}
\]

are commutative.

**Remark 2.5.1.** — An example of a stratification structure on \( X/S \) is given as follows:

Let \( Y \) be a separated scheme over \( S \) and \( X \to Y \) an affine morphism over \( S \). Write \( P \overset{\text{def}}{=} X \times_Y X \); \( q_1, q_2 : P \to X \) for the projections onto the first, second factors, respectively; \( q_{1,3} : X \times_Y X \times_Y X \to P \) for the projection onto the first and third factors. Let

\[
P^0 \subseteq P^1 \subseteq \ldots \subseteq P^n \subseteq P^{n+1} \subseteq \ldots \subseteq P
\]

be a sequence of closed subschemes of \( P \). Suppose that the following two conditions are satisfied:

- The natural closed immersion \( P^0 \to P \) determines an isomorphism of \( P^0 \) with the diagonal closed subscheme \( X \subseteq P \) of \( P \).

- For each nonnegative integers \( n_1, n_2 \), if we write \( P^{n_1,n_2} \overset{\text{def}}{=} P^{n_1} \times_X P^{n_2} \) for the fiber product of the composite \( P^{n_1} \hookrightarrow P \xrightarrow{q_1} X \) and the composite \( P^{n_2} \hookrightarrow P \xrightarrow{q_2} X \), then the composite

\[
P^{n_1,n_2} \xrightarrow{\delta^{n_1,n_2}} X \times_Y X \times_Y X \xrightarrow{q_{1,3}} P
\]

factors through the closed subscheme \( P^{n_1+n_2} \subseteq P \) of \( P \). Write

\[
\delta^{n_1,n_2} : P^{n_1,n_2} \longrightarrow P^{n_1+n_2}
\]

for the resulting morphism.

For each nonnegative integer \( n \), write \( \iota^n : P^n \hookrightarrow P^{n+1} \) for the natural closed immersion and \( \sigma^n : P^n \to X^{(2)} \) for the composite of the natural closed immersion \( P^n \hookrightarrow P \) and the closed immersion \( P \hookrightarrow X^{(2)} \) induced by the morphism \( Y \to S \). Then one verifies easily that the collection of data

\[
((P^n)_{n \geq 0}), \quad (\iota^n : P^n \hookrightarrow P^{n+1})_{n \geq 0}, \quad (\sigma^n : P^n \to X^{(2)})_{n \geq 0}, \quad (\delta^{n_1,n_2} : P^{n_1,n_2} \to P^{n_1+n_2})_{n_1,n_2 \geq 0}
\]

gives an example of a stratification structure on \( X/S \).

**Definition 2.6.** — Let

\[
\mathcal{S} = ((\iota^P)_{n \geq 0}), \quad (\sigma^P)_{n \geq 0}, \quad (\delta^P)_{n_1,n_2 \geq 0}
\]

be a stratification structure on \( X/S \); \( r \) a positive integer; \( n_1, \ldots, n_r \) nonnegative integers. Write \( n \overset{\text{def}}{=} \sum_{i=1}^r n_i \). Then it follows from conditions (1), (2) of Definition 2.5 that the
various $\delta^{m_1,m_2}$'s determine a commutative diagram of schemes over $S$

\[
\begin{array}{ccc}
\delta p \times n & \rightarrow & \delta p^{n_1,\ldots,n_r} \\
\downarrow & & \downarrow \\
\delta p^n & & \\
\end{array}
\]

We shall write

\[
\begin{array}{ccc}
\delta^{x,n}_{n_1,\ldots,n_r} : \delta p \times n \rightarrow \delta p^{n_1,\ldots,n_r}, & \delta^{x,n} : \delta p \times n \rightarrow \delta p^n, \\
\delta^{x,n}_{n_1,\ldots,n_r} : \delta p^{n_1,\ldots,n_r} \rightarrow \delta p^n & \\
\end{array}
\]

for the top horizontal, left-hand lower, right-hand lower arrows of this diagram, respectively.

**Remark 2.6.1.** — In the situation of Definition 2.6, it follows from condition (1) of Definition 2.5 that the diagrams of schemes over $S$

\[
\begin{array}{ccc}
\delta p^{n_1,\ldots,n_r} & \rightarrow & \delta p^n \\
\downarrow & & \downarrow \\
X, & & X, \\
\end{array}
\]

are commutative.

**Definition 2.7.** — Let

\[
\mathcal{S} = \left( (\delta^m p^n)_{n \geq 0}, (\delta^m r^n)_{n \geq 0}, (\delta^m \sigma^n)_{n \geq 0}, (\delta^m \delta^{n_1,n_2})_{n_1,n_2 \geq 0} \right)
\]

be a stratification structure on $X/S$; $r$ a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \overset{\text{def}}{=} \sum_{i=1}^r n_i$. Suppose that $n \geq 1$.

(i) We shall write

\[
\delta \_ p^{n_1,\ldots,n_r} \overset{\text{def}}{=} \delta p^{n_1,\ldots,n_r} \times_\delta p^n \delta p^{n-1}
\]

for the fiber product of the morphism $\delta^{n_1,\ldots,n_r} : \delta p^{n_1,\ldots,n_r} \rightarrow \delta p^n$ and the morphism $\delta^{n_1,\ldots,n_r} : \delta p^{n-1} \rightarrow \delta p^n$.

(ii) We shall write

\[
\delta^{l}_{n_1,\ldots,n_r} : \_ p^{n_1,\ldots,n_r} \rightarrow \delta p^{n_1,\ldots,n_r}, \quad \delta^{l}_{n_1,\ldots,n_r} : \_ p^{n_1,\ldots,n_r} \rightarrow \delta p^{n-1}
\]
for the first, second projections, respectively, Thus, we have a cartesian diagram of schemes over $S$

$$
\begin{array}{ccc}
\mathcal{E}^P_{n_1,\ldots,n_r} & \mathcal{E}^\delta_{n_1,\ldots,n_r} & \mathcal{E}^P_{n-1} \\
\mathcal{E}^P_{n_1,\ldots,n_r} & \mathcal{E}^\delta_{n_1,\ldots,n_r} & \mathcal{E}^P_{n} \\
\mathcal{E}^P_{n_1,\ldots,n_r} & \mathcal{E}^\delta_{n_1,\ldots,n_r} & \mathcal{E}^P_{n}.
\end{array}
$$

(iii) Let $i$ be an element of $\{1, \ldots, r+1\}$. Then we shall write

$$
\mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}} : \mathcal{E}^P_{n_1,\ldots,n_r} \to X
$$

for the morphism over $S$ obtained by forming the composite of the morphism $\mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}} : \mathcal{E}^P_{n_1,\ldots,n_r} \to \mathcal{E}^P_{n_1,\ldots,n_r}$ and the morphism $\mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}} : \mathcal{E}^P_{n_1,\ldots,n_r} \to X$.

(iv) It follows from conditions (3), (4) of Definition 2.5 that the diagram of schemes over $S$

$$
X = \mathcal{E}^P^0 = \mathcal{E}^P^0 \mathcal{E}^\delta_{P^0} \mathcal{E}^P_{n-1} \mathcal{E}^P_{P^0} \mathcal{E}^P_{n-1}
$$

is commutative. We shall write

$$
\mathcal{E}^\delta_{n_1,\ldots,n_r} : X \to \mathcal{E}^P_{n_1,\ldots,n_r}
$$

for the morphism over $S$ determined by this commutative diagram and the cartesian diagram of (ii).

(v) We shall write

$$
\mathcal{E}^P_{P^r}, \quad \mathcal{E}^P_{P^r}, \quad \mathcal{E}^P_{P^r} \to \mathcal{E}^P_{P^r}, \quad \mathcal{E}^P_{P^r}, \quad \mathcal{E}^P_{P^r} \to \mathcal{E}^P_{P^r},
$$

$$
\mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}, \quad \mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}, \quad \mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}, \quad \mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}, \quad \mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}
$$

for $\mathcal{E}^P_{P^r}$, $\mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}$, $\mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}$, $\mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}$, $\mathcal{E}^\delta_{P_i^{n_1,\ldots,n_r}}$ in the case where we take the "$(n_1, \ldots, n_r)$" to be $(1, \ldots, 1)$.

**Remark 2.7.1.** — In the situation of Definition 2.7:

(i) It follows from Remark 2.4.1, (ii), that the diagram of schemes over $S$

$$
\begin{array}{ccc}
X & \mathcal{E}^\delta_{n_1,\ldots,n_r} : X \to \mathcal{E}^P_{n_1,\ldots,n_r} \\
\mathcal{E}^\delta_{n_1,\ldots,n_r} & \mathcal{E}^\delta_{n_1,\ldots,n_r} : X \to \mathcal{E}^P_{n_1,\ldots,n_r}
\end{array}
$$

is commutative, and the morphism $\mathcal{E}^\delta_{n_1,\ldots,n_r} : X \to \mathcal{E}^P_{n_1,\ldots,n_r}$ is a closed immersion.
(ii) It follows from condition (1) of Definition 2.2 and Remark 2.6.1 that the diagrams of schemes over $S$

\[
\begin{array}{ccc}
\mathcal{S} P_{n_1, \ldots, n_r} & \xrightarrow{\phi_{n_1, \ldots, n_r}} & \mathcal{S} P_{n-1} \\
\downarrow \phi_{pr_{n_1, \ldots, n_r}} & & \downarrow \phi_{pr_{n-1}} \\
X_1 & \xrightarrow{\phi_{pr_{n_1-1}}} & \mathcal{S} P_{n-1}
\end{array}
\]

are commutative.

**Definition 2.8.** — Let

\[
\mathcal{S} = ( (\mathcal{S} P^n)_{n \geq 0}, (\mathcal{S} t^n)_{n \geq 0}, (\mathcal{S} \sigma^n)_{n \geq 0}, (\mathcal{S} \delta_{n_1, n_2})_{n_1, n_2 \geq 0})
\]

\[
\mathcal{T} = ( (\mathcal{T} P^n)_{n \geq 0}, (\mathcal{T} t^n)_{n \geq 0}, (\mathcal{T} \sigma^n)_{n \geq 0}, (\mathcal{T} \delta_{n_1, n_2})_{n_1, n_2 \geq 0})
\]

be stratification structures on $X/S$. Then we shall define a morphism of stratification structures

\[
\mathcal{S} \longrightarrow \mathcal{T}
\]

to be a collection of data

\[
\Phi = (\Phi^n)_{n \geq 0}
\]

consisting of an affine morphism $\Phi^n : \mathcal{S} P^n \to \mathcal{T} P^n$ over $S$ for each nonnegative integer $n$ that satisfies the following three conditions:

1. Let $n$ be a nonnegative integer. Then the diagram of schemes over $S$

\[
\begin{array}{ccc}
\mathcal{S} P^n & \xrightarrow{\phi^n} & \mathcal{T} P^n \\
\downarrow \phi_{t^n} & & \downarrow \tau_{t^n} \\
\mathcal{S} P^{n+1} & \xrightarrow{\phi_{n+1}} & \mathcal{T} P^{n+1}
\end{array}
\]

is commutative.

2. Let $n$ be a nonnegative integer. Then the diagram of schemes over $S$

\[
\begin{array}{ccc}
\mathcal{S} P^n & \xrightarrow{\phi^n} & \mathcal{T} P^n \\
\downarrow \phi_{\sigma^n} & & \downarrow \tau_{\sigma^n} \\
X^{(2)} & &
\end{array}
\]

is commutative.

3. Let $n_1, n_2$ be nonnegative integers. Then the diagram of schemes over $S$

\[
\begin{array}{ccc}
\mathcal{S} P_{n_1, n_2} & \xrightarrow{\phi_{\delta_{n_1, n_2}}} & \mathcal{T} P_{n_1, n_2} \\
\downarrow \phi_{\delta_{n_1, n_2}} & & \downarrow \tau_{\delta_{n_1, n_2}} \\
\mathcal{S} P_{n_1+n_2} & \xrightarrow{\phi_{n_1+n_2}} & \mathcal{T} P_{n_1+n_2}
\end{array}
\]

is commutative.
— where the upper horizontal arrow is the morphism over $S$ determined [cf. (2)] by the morphisms $\Phi^{n_1}, \Phi^{n_2}$ — is commutative.

**Remark 2.8.1.** — In the situation of Definition 2.8, it follows from condition (2) of Definition 2.2 and condition (2) of Definition 2.8 that the morphism $\Phi^0 \colon X = \mathfrak{P} P^0 \to \mathfrak{T} P^0 = X$ coincides with the identity automorphism of $X$.

**Definition 2.9.** — Let

$$
\mathcal{G} = ((\mathfrak{S} P^n)_{n \geq 0}, (\mathfrak{S} t^n)_{n \geq 0}, (\mathfrak{S} s^n)_{n \geq 0}, (\mathfrak{S} \delta_{n_1,n_2})_{n_1,n_2 \geq 0}),
$$

$$
\mathcal{T} = ((\mathfrak{T} P^n)_{n \geq 0}, (\mathfrak{T} t^n)_{n \geq 0}, (\mathfrak{T} s^n)_{n \geq 0}, (\mathfrak{T} \delta_{n_1,n_2})_{n_1,n_2 \geq 0})
$$

be stratification structures on $X/S$;

$$
\Phi = (\Phi^n)_{n \geq 0} : \mathcal{G} \longrightarrow \mathcal{T}
$$

a morphism of stratification structures; $r$ a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Then we shall write

$$
\Phi^{n_1,\ldots,n_r} : \mathfrak{S} P^{n_1,\ldots,n_r} \longrightarrow \mathfrak{T} P^{n_1,\ldots,n_r}
$$

for the morphism over $S$ determined [cf. condition (2) of Definition 2.8] by the morphisms $\Phi^{n_1}, \ldots, \Phi^{n_r}$. [So the upper horizontal arrow of the diagram of Definition 2.8, (3), is the morphism $\Phi^{n_1,n_2}$.]

**Definition 2.10.** — Let

$$
\mathcal{G} = ((\mathfrak{S} P^n)_{n \geq 0}, (\mathfrak{S} t^n)_{n \geq 0}, (\mathfrak{S} s^n)_{n \geq 0}, (\mathfrak{S} \delta_{n_1,n_2})_{n_1,n_2 \geq 0}),
$$

$$
\mathcal{T} = ((\mathfrak{T} P^n)_{n \geq 0}, (\mathfrak{T} t^n)_{n \geq 0}, (\mathfrak{T} s^n)_{n \geq 0}, (\mathfrak{T} \delta_{n_1,n_2})_{n_1,n_2 \geq 0})
$$

be stratification structures on $X/S$;

$$
\Phi = (\Phi^n)_{n \geq 0} : \mathcal{G} \longrightarrow \mathcal{T}
$$

a morphism of stratification structures; $m \leq n$ nonnegative integers.

(i) We shall write

$$
\Phi^{P_{n|m}} \overset{\text{def}}{=} \mathfrak{S} P^n \times_{\mathfrak{T} P^m} \mathfrak{T} P^n
$$

for the fiber product of the morphism $\Phi^n : \mathfrak{S} P^n \to \mathfrak{T} P^n$ and the morphism $\mathfrak{T} t_n : \mathfrak{T} P^m \to \mathfrak{T} P^n$. [So $\Phi^{P_{n|m}} = \mathfrak{S} P^n$.]

(ii) We shall write

$$
\Phi^{n|m} : \Phi^{P_{n|m}} \longrightarrow \mathfrak{S} P^n, \quad \Phi^{n|m} : \Phi^{P_{n|m}} \longrightarrow \mathfrak{T} P^m
$$
for the first, second projections, respectively. Thus, we have a cartesian diagram of schemes over $S$

\[
\begin{array}{ccc}
P^n_m & \xrightarrow{\Phi P^n_m} & \mathbb{P}^m \\
\downarrow{\Phi \iota^n_m} & & \downarrow{\tau^m_{\iota^n}} \\
\mathbb{P}^n & \xrightarrow{\Phi \iota^n} & \mathbb{P}^n.
\end{array}
\]

(iii) Let $m' \leq m$ be a nonnegative integer. Then we shall write

\[
\Phi n|m', \Phi P^n|m' \longrightarrow \Phi P^n|m
\]

for the morphism over $S$ determined by the morphism $\tau^m_{\iota^n} : \mathbb{P}^m \rightarrow \mathbb{P}^m$.

(iv) Let $i$ be an element of $\{1, 2\}$. Then we shall write

\[
\Phi \iota^i_n|m : \Phi P^n|m \longrightarrow X
\]

for the morphism over $S$ obtained by forming the composite of the morphism $\Phi \iota^n|m : \Phi P^n|m \rightarrow \mathbb{P}^n$ and the morphism $\Phi \iota^i_n : \mathbb{P}^n \rightarrow X$.

(v) It follows from condition (1) of Definition 2.8 and Remark 2.8.1 that the diagram of schemes over $S$

\[
\begin{array}{ccc}
X = \mathbb{P}^0 = \mathbb{P}^0 & \xrightarrow{\tau^0_{\iota^n}} & \mathbb{P}^m \\
\downarrow{\Phi \iota^0} & & \downarrow{\tau^m_{\iota^0}} \\
\mathbb{P}^n & \xrightarrow{\Phi \iota^n} & \mathbb{P}^n
\end{array}
\]

is commutative. We shall write

\[
\Phi \iota^n|0 : X \longrightarrow \Phi P^n|m
\]

for the morphism over $S$ determined by this commutative diagram and the cartesian diagram of (ii).

**Remark 2.10.1.** — In the situation of Definition 2.10:

(i) It follows from Remark 2.3.1 that the diagram of schemes over $S$

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi \iota^n|0} & \Phi P^n|m \\
\downarrow{\Phi \iota^n|0} & & \downarrow{\Phi \iota^n|m} \\
X & \xleftarrow{\Phi \iota^n|m} & \Phi P^n|m
\end{array}
\]

is commutative, and the morphism $\Phi \iota^n|0 : X \rightarrow \Phi P^n|m$ is a closed immersion.
(ii) It follows from condition (1) of Definition 2.2 and condition (2) of Definition 2.8 that the diagram of schemes over $S$

$$
\begin{array}{ccc}
\Phi P^n & \xrightarrow{\Phi} & \Phi P^m \\
\Phi | P^n & \xrightarrow{\Phi} & \Phi | P^m \\
\Phi_p | P^n & \xrightarrow{\Phi_p} & \Phi_p | P^m
\end{array}
$$

is commutative.

(iii) It follows from (ii) and Remark 2.3.1 that the equalities

$$\Phi^{n|0} = \Phi | P^{n|0} = \Phi | P^{n|0} : \Phi P^{n|0} \to \Phi P^0 = X$$

hold.

3. Integrable Stratification Structures

In the present §3, we introduce the notion of an integrable stratification structure on a smooth scheme [cf. Definition 3.2 below] and the notion of an integrable morphism of stratification structures [cf. Definition 3.8 below]. In the present §3, let $S$ be a scheme, $X$ a scheme which is smooth and separated over $S$, and

$$\mathcal{G} = ((\delta P^n)_{n \geq 0}, (\delta \iota^{n} : \delta P^n \to \delta P^{n+1})_{n \geq 0},$$

$$(\delta \sigma^n : \delta P^n \to X^{[2]}(n \geq 0), (\delta \delta_{n_1,n_2 : \delta P^{n_1,n_2} \to \delta P^{n_1+n_2})_{n_1,n_2 \geq 0}$$

a stratification structure on $X/S$.

**Definition 3.1.**

(i) We shall say that the stratification structure $\mathcal{G}$ is $\iota$-quasi-nil-retraction-like if, for each nonnegative integer $n$, the morphism $\delta \iota^{n} : \delta P^n \to \delta P^{n+1}$ is quasi-nil-retraction-like.

(ii) We shall say that the stratification structure $\mathcal{G}$ is pr-finite flat if, for each nonnegative integer $n$ and $i \in \{1, 2\}$, the morphism $\delta \pi^{n_i} : \delta P^n \to X$ is finite and flat.

(iii) We shall say that the stratification structure $\mathcal{G}$ is $\delta$-nil-retraction-like if the following condition is satisfied: Let $r$ be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n = \sum_{i=1}^{r} n_i$. Then the morphisms in the commutative diagram of Definition 2.6

$$\begin{array}{ccc}
\delta P^{\times n} & \xrightarrow{\delta \delta_{n_1,n_2^{\times r}}} & \delta P^{n_1,n_2^{\times r}} \\
\delta P^n & \xrightarrow{\delta \delta_{n_1,n_2}} & \delta P^{n_1,n_2}
\end{array}$$

are nil-retraction-like.
(iv) Let $r$ be a positive integer. Then we shall say that the stratification structure $\mathfrak{S}$ is $\delta$-strictly cocartesian of level $\geq r$ if, for each integer $n \geq r$, the commutative diagram of schemes over $S$

\[
\begin{array}{ccc}
\mathfrak{S} P^x_n & \mathfrak{S} \delta^{x,n}_{1,n,1} & \mathfrak{S} P^1_{n,1,1} \\
| & | & | \\
\mathfrak{S} P^{n-1}_{n,1,1} & \mathfrak{S} P^1_{n,1,1} & \mathfrak{S} P^n \\
\end{array}
\]

is strictly cocartesian.

**Remark 3.1.1.** — Suppose that the stratification structure $\mathfrak{S}$ is $\iota$-quasi-nil-retraction-like.

(i) In the situation of Definition 2.3, (ii), it follows from Remark 2.3.1 that the morphism $\mathfrak{S} \text{pr}_i^n : \mathfrak{S} P^n \to X$ is quasi-nil-retraction-like.

(ii) In the situation of Definition 2.4, (ii), it follows from (i) that the morphism $\mathfrak{S} \text{pr}_{1,...,m_2}^n : \mathfrak{S} P^{m_1,...,m_2} \to \mathfrak{S} P^{m_1,...,m_2}$ is quasi-nil-retraction-like.

(iii) In the situation of Definition 2.6, it follows from (ii) and conditions (3), (4) of Definition 2.5 that the morphisms in the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{S} P^x_n & \mathfrak{S} \delta^{x,n}_{n_1,...,n_r} & \mathfrak{S} P^{n_1,...,n_r} \\
| & | & | \\
\mathfrak{S} P^n & \mathfrak{S} \delta^{x,n}_{n_1,...,n_r} & \mathfrak{S} P^{n_1,...,n_r} \\
\end{array}
\]

are quasi-nil-retraction-like.

**Remark 3.1.2.** — Suppose that the stratification structure $\mathfrak{S}$ is $\text{pr}$-finite flat. Then, in the situation of Definition 2.4, (iv), one verifies immediately that the morphism $\mathfrak{S} \text{pr}_{1,...,n_r}^n : \mathfrak{S} P^{n_1,...,n_r} \to X$ is finite and flat.

**Definition 3.2.** — We shall say that the stratification structure $\mathfrak{S}$ is integrable if the following four conditions are satisfied:

1. The stratification structure $\mathfrak{S}$ is $\iota$-quasi-nil-retraction-like and $\delta$-nil-retraction-like.
2. The stratification structure $\mathfrak{S}$ is $\delta$-strictly cocartesian of level $\geq 3$.
3. The morphism $\mathfrak{S} \iota^1 : \mathfrak{S} P^1 \to \mathfrak{S} P^2$ is a square-nilpotent closed immersion.
4. The morphism $\mathfrak{S} \delta^{x,2} : \mathfrak{S} P^{x,2} \to \mathfrak{S} P^2$ is conormally strict with respect to the square-nilpotent closed immersion $\mathfrak{S} \iota^1 : \mathfrak{S} P^1 \to \mathfrak{S} P^2$ [cf. (3)].
**Remark 3.2.1.** — Suppose that the stratification structure $\mathcal{S}$ is integrable.

(i) It follows from condition (3) of Definition 3.2 that the morphism $\mathcal{S}_{\times 2} \times \mathcal{S} P^{\times 2} \to \mathcal{S} P^{\times 2}$ is a square-nilpotent closed immersion, hence also [cf. Remark 1.2.1] quasi-nil-retraction-like.

(ii) It follows from Remark 3.1.1, (ii), and condition (1) of Definition 3.2 that the morphism $\mathcal{S}_{0,0} : X = \mathcal{S} P^{0,0} \to \mathcal{S} P^{\times 2}$ is quasi-nil-retraction-like. Thus, it follows from (i) that the morphism $\mathcal{S}_{0,0} \times 2 : X \to \mathcal{S} P^{\times 2}$ is quasi-nil-retraction-like.

**Remark 3.2.2.** — Let us recall from Definition 2.7, (ii), that the diagram

\[
\begin{array}{ccc}
\mathcal{S} P^{\times 2} & \overset{\mathcal{S} \times 2}{\longrightarrow} & \mathcal{S} P^{1} \\
\downarrow & & \downarrow \mathcal{S}_1 \\
\mathcal{S} P^{\times 2} & \overset{\mathcal{S} \times 2}{\longrightarrow} & \mathcal{S} P^{2}
\end{array}
\]

is cartesian. Now suppose that the stratification structure $\mathcal{S}$ is integrable. Then it follows from conditions (3), (4) of Definition 3.2 that the right-hand vertical arrow of this diagram is a square-nilpotent closed immersion, and the lower horizontal arrow of this diagram is conormally strict with respect to the right-hand vertical arrow.

**Remark 3.2.3.** — Suppose that the stratification structure $\mathcal{S}$ is integrable. Let $n \geq 3$ be an integer. Let us recall the diagram of schemes over $S$

\[
X = \mathcal{S} P^{0,0} \to \mathcal{S} P^{n} \overset{\mathcal{S} \times n}{\longrightarrow} \mathcal{S} P^{1,n-1} \overset{\mathcal{S} \times n}{\longrightarrow} \mathcal{S} P^{n-1,1} \overset{\mathcal{S} \times n}{\longrightarrow} \mathcal{S} P^{n} \overset{\mathcal{S} \times n}{\longrightarrow} X.
\]

(i) It follows from condition (2) of Definition 3.2 that the diagram

\[
\begin{array}{ccc}
\mathcal{S} P^{\times n} & \overset{\mathcal{S} \times n}{\longrightarrow} & \mathcal{S} P^{1,n-1} \\
\downarrow & & \downarrow \mathcal{S}_{1,n-1} \\
\mathcal{S} P^{\times n} & \overset{\mathcal{S} \times n}{\longrightarrow} & \mathcal{S} P^{n}
\end{array}
\]

is [commutative and] strictly cocartesian.

(ii) It follows from Remark 3.1.1, (ii), and condition (1) of Definition 3.2 that the morphism

\[
X = \mathcal{S} P^{0,0} \to \mathcal{S} P^{n}
\]

is quasi-nil-retraction-like.
(iii) It follows from Remark 3.1.1, (i), and condition (1) of Definition 3.2 that the morphisms

\[ \mathcal{E} P^n \xrightarrow{\mathcal{E} pr_1^n} X \]

are quasi-nil-retraction-like.

(iv) It follows from Remark 2.4.1, (ii), and Remark 2.6.1 that the equalities

\[ \mathcal{E} pr_1^n \circ \mathcal{E} \delta^{1,n-1} \circ \mathcal{E} \delta^{n \times n} \circ \mathcal{E} l_{1,1,1}^0 = \mathcal{E} pr_2^n \circ \mathcal{E} \delta^{1,n-1} \circ \mathcal{E} \delta^{n \times n} \circ \mathcal{E} l_{1,1,1}^0 = \text{id}_X \]

hold.

**Definition 3.3.** — We shall say that the stratification structure \( S \) is *strictly integrable* if the following three conditions are satisfied:

1. The stratification structure \( S \) is integrable.
2. The stratification structure \( S \) is pr-finite flat.
3. Let \( i \) be an element of \( \{1, 3\} \). Then the morphism \( \mathcal{E} pr_1^2 \cdot \mathcal{E} P^2 \to X \) is finite and flat.

**Remark 3.3.1.** — Suppose that the stratification structure \( S \) is *strictly integrable*. Let us recall the diagram of schemes over \( S \)

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{E} l_{x2}} & \mathcal{E} P^2 \xrightarrow{\mathcal{E} \delta^{x2}} \mathcal{E} P^1 \\
\downarrow{\mathcal{E} l_{x2}} & & \downarrow{\mathcal{E} l_{x2}} \\
\mathcal{E} P^2 & \xrightarrow{\mathcal{E} \delta^{x2}} & \mathcal{E} P^2 \xrightarrow{\mathcal{E} pr_1^2} X.
\end{array}
\]

(i) It follows from Remark 3.2.1, (ii), and condition (1) of Definition 3.3 that the morphism

\[ \mathcal{E} l_{x2}^0 : X \xrightarrow{\mathcal{E} l_{x2}} \mathcal{E} P^2 \]

is quasi-nil-retraction-like.

(ii) It follows from Remark 3.1.1, (i), condition (1) of Definition 3.2, and condition (1) of Definition 3.3 that the morphisms

\[ \mathcal{E} P^2 \xrightarrow{\mathcal{E} pr_1^2} X \]

are quasi-nil-retraction-like.

(iii) It follows from Remark 2.4.1, (ii), and Remark 2.6.1 that the equalities

\[ \mathcal{E} pr_1^2 \circ \mathcal{E} l^0 \circ \mathcal{E} \delta^{x2} \circ \mathcal{E} l_{x2}^0 = \mathcal{E} pr_2^2 \circ \mathcal{E} l^0 \circ \mathcal{E} \delta^{x2} \circ \mathcal{E} l_{x2}^0 = \text{id}_X \]
hold.

(iv) It follows from Remark 3.1.2 and conditions (2), (3) of Definition 3.3 [cf. also condition (1) of Definition 2.2; Remark 2.6.1; Remark 2.7.1, (ii)] that the morphisms

\[\begin{align*}
\hat{e}_{\text{pr}_1^2} &= \hat{e}_{\text{pr}_1^2} \circ \hat{e}_{\delta^2}, \\
\hat{e}_{\text{pr}_3^2} &= \hat{e}_{\text{pr}_2^2} \circ \hat{e}_{\delta^2}, \\
\hat{e}_{\text{pr}_1^1} &= \hat{e}_{\text{pr}_1^2} \circ \hat{e}_{\delta^1}, \\
\hat{e}_{\text{pr}_2^1} &= \hat{e}_{\text{pr}_2^2} \circ \hat{e}_{\delta^2}, \\
\hat{e}_{\text{pr}_1^2} &= \hat{e}_{\text{pr}_1^1} \circ \hat{e}_{\delta^2},
\end{align*}\]

are finite and flat.

**Remark 3.3.2.** — Suppose that the stratification structure \(S\) is strictly integrable. Let \(n \geq 3\) be an integer. Let us recall the diagram of schemes over \(S\)

\[X = \hat{e}_{P^0,\ldots,0} \xrightarrow{\hat{e}_{\delta_{1,\ldots,1}}} \hat{e}_{P^{n}} \xrightarrow{\hat{e}_{\delta_{1,n-1}}} \hat{e}_{P^{1,n-1}}\]

Then it follows from Remark 3.1.2 and condition (2) of Definition 3.3 [cf. also Remark 2.6.1] that the morphisms

\[\begin{align*}
\hat{e}_{\text{pr}_1^{-n}} &= \hat{e}_{\text{pr}_1^{n}} \circ \hat{e}_{\delta^1,n-1} \circ \hat{e}_{\delta_{1,n-1}}, \\
\hat{e}_{\text{pr}_n^{-n+1}} &= \hat{e}_{\text{pr}_2^{n}} \circ \hat{e}_{\delta^1,n-1} \circ \hat{e}_{\delta_{1,n-1}}, \\
\hat{e}_{\text{pr}_1^{n-1}} &= \hat{e}_{\text{pr}_1^{n}} \circ \hat{e}_{\delta^n-1,1}, \\
\hat{e}_{\text{pr}_2^{n-1}} &= \hat{e}_{\text{pr}_2^{n}} \circ \hat{e}_{\delta^n-1,1}, \\
\hat{e}_{\text{pr}_1^{n-1}} &= \hat{e}_{\text{pr}_1^{n}} \circ \hat{e}_{\delta^n-1,1}, \\
\hat{e}_{\text{pr}_2^{n-1}} &= \hat{e}_{\text{pr}_2^{n}} \circ \hat{e}_{\delta^n-1,1}
\end{align*}\]

are finite and flat.

**Remark 3.3.3.** — In sequels to the present paper, we give some examples of strictly integrable stratification structures.

**Definition 3.4.**

(i) We shall write

\[T^1 \subseteq X^{(3)}\]

for the closed subscheme of \(X^{(3)}\) defined by the quasi-coherent ideal of \(O_{X^{(3)}}\) obtained by forming the square of the quasi-coherent ideal that defines the diagonal closed subscheme \(X \subseteq X^{(3)}\) of \(X^{(3)}\).
(ii) Let $i$ be an element of $\{1, 2, 3\}$. Then we shall write

$$X \text{pr}^T_i: T^1 \longrightarrow X$$

for the morphism over $S$ obtained by forming the composite of the natural closed immersion $T^1 \hookrightarrow X^{(3)}$ and the morphism $X \text{pr}^{(3)}_i: X^{(3)} \rightarrow X$.

**Lemma 3.5.** — Suppose that the morphism $\mathcal{E}\sigma^1: \mathcal{E}\mathcal{P}^1 \rightarrow \mathcal{E}(2)$ determines an isomorphism of $\mathcal{E}\mathcal{P}^1$ with the closed subscheme of $\mathcal{E}(2)$ defined by the quasi-coherent ideal of $\mathcal{O}_{X(2)}$ obtained by forming the square of the quasi-coherent ideal that defines the diagonal closed subscheme $X \subseteq X^{(2)}$ of $X^{(2)}$ [which thus implies that the morphisms

$$\mathcal{E}\sigma^{1,0}: \mathcal{E}\mathcal{P}^1 = \mathcal{E}\mathcal{P}^{1,0} \longrightarrow X^{(3)}, \quad \mathcal{E}\sigma^{0,1}: \mathcal{E}\mathcal{P}^1 = \mathcal{E}\mathcal{P}^{0,1} \longrightarrow X^{(3)},$$

$$\mathcal{E}\sigma^{\times 2}: \mathcal{E}\mathcal{P}^{\times 2} \longrightarrow X^{(3)}$$

are closed immersions]. Then the following hold:

(i) The closed immersion $\mathcal{E}\sigma^{0,0}: X = \mathcal{E}\mathcal{P}^{0,0} \hookrightarrow X^{(3)}$ [cf. Remark 2.4.1, (i)] factors as the composite of a square-nilpotent closed immersion $X \hookrightarrow T^1$ whose conormal sheaf is isomorphic to $\Omega^1_{X/S} \oplus \Omega^1_{X/S}$ and the natural closed immersion $T^1 \hookrightarrow X^{(3)}$.

Write

$$\mathcal{E}\tau^{0,0}_T: X \longrightarrow T^1$$

for the resulting square-nilpotent closed immersion [whose conormal sheaf is isomorphic to $\Omega^1_{X/S} \oplus \Omega^1_{X/S}$].

(ii) The closed immersion $\mathcal{E}\sigma^{1,0}: \mathcal{E}\mathcal{P}^1 = \mathcal{E}\mathcal{P}^{1,0} \hookrightarrow X^{(3)}$ factors as the composite of a square-nilpotent closed immersion $\mathcal{E}\mathcal{P}^1 \hookrightarrow T^1$ whose conormal sheaf is isomorphic to $\mathcal{E}\tau^{0,1} \Omega^1_{X/S}$ and the natural closed immersion $T^1 \hookrightarrow X^{(3)}$.

Write

$$\mathcal{E}\tau^{1,0}_T: \mathcal{E}\mathcal{P}^1 \longrightarrow T^1$$

for the resulting square-nilpotent closed immersion [whose conormal sheaf is isomorphic to $\mathcal{E}\tau^{0,1} \Omega^1_{X/S}$].

(iii) The closed immersion $\mathcal{E}\sigma^{0,1}: \mathcal{E}\mathcal{P}^1 = \mathcal{E}\mathcal{P}^{0,1} \hookrightarrow X^{(3)}$ factors as the composite of a square-nilpotent closed immersion $\mathcal{E}\mathcal{P}^1 \hookrightarrow T^1$ whose conormal sheaf is isomorphic to $\mathcal{E}\tau^{0,1} \Omega^1_{X/S}$ and the natural closed immersion $T^1 \hookrightarrow X^{(3)}$.

Write

$$\mathcal{E}\tau^{0,1}_T: \mathcal{E}\mathcal{P}^1 \longrightarrow T^1$$

for the resulting square-nilpotent closed immersion [whose conormal sheaf is isomorphic to $\mathcal{E}\tau^{0,1} \Omega^1_{X/S}$].

(iv) The natural closed immersion $T^1 \hookrightarrow X^{(3)}$ factors as the composite of a square-nilpotent closed immersion $T^1 \hookrightarrow \mathcal{E}\mathcal{P}^{\times 2}$ whose conormal sheaf is isomorphic to $(\mathcal{E}\tau^{0,2}_T) (\Omega^1_{X/S} \otimes \mathcal{O}_X \Omega^1_{X/S})$ and the closed immersion $\mathcal{E}\sigma^{\times 2}: \mathcal{E}\mathcal{P}^{\times 2} \hookrightarrow X^{(3)}$.

Write

$$\mathcal{E}\tau^{T}_x \mathcal{P}^{\times 2}_T: T^1 \longrightarrow \mathcal{E}\mathcal{P}^{\times 2}$$
for the resulting square-nilpotent closed immersion [whose conormal sheaf is isomorphic to \((\xi T^0_\Omega)_{X/S}(\Omega^1_{X/S} \otimes \mathcal{O}_X \Omega^1_{X/S})]\).

**Proof.** — These assertions follow immediately from elementary algebraic geometry. □

**Definition 3.6.** — We shall say that the stratification structure \(S\) is of standard type if the following three conditions are satisfied:

1. The morphism \(\xi \sigma^1: \xi P^1 \to X\) determines an isomorphism of \(\xi P^1\) with the closed subscheme of \(X\) defined by the quasi-coherent ideal of \(\mathcal{O}_{X^2}\) obtained by forming the square of the quasi-coherent ideal that defines the diagonal closed subscheme \(X \subseteq X\).

2. The closed immersion \(\xi_{T^2_x}: T^1 \hookrightarrow \xi P^{x_2}\) of Lemma 3, (iv) [cf. (1)], factors as the composite of a square-nilpotent closed immersion \(T^1 \hookrightarrow \xi P^{x_2}\) and the closed immersion \(\xi^{x_2}: \xi P^{x_2} \hookrightarrow \xi P^{x_2}\) [cf. (1)].

3. If we write \(N\) for the conormal sheaf of the square-nilpotent closed immersion \(T^1 \hookrightarrow \xi P^{x_2}\) of (2), then the surjective homomorphism \((\xi T^0_\Omega)_{X/S}(\Omega^1_{X/S} \otimes \mathcal{O}_X \Omega^1_{X/S}) \twoheadrightarrow N\) of \(\mathcal{O}_{T^1}\)-modules determined [cf. Lemma 3, (iv)] by the closed immersion \(\xi^{x_2}: \xi P^{x_2} \hookrightarrow \xi P^{x_2}\) induces an isomorphism \((\xi T^0_\Omega)_{X/S} \Omega^2_{X/S} \cong \mathcal{N}\) of \(\mathcal{O}_{T^1}\)-modules.

In this situation, we shall write

\[
\xi T^2_x: T^1 \hookrightarrow \xi P^{x_2}
\]

for the square-nilpotent closed immersion [whose conormal sheaf is isomorphic to \((\xi T^0_x)_{X/S}\) — cf. (3)] of (2) and

\[
\xi T^1_x: T^1 \hookrightarrow \xi P^1
\]

for the morphism over \(S\) obtained by forming the composite of the morphism \(\xi T^2_x: T^1 \to \xi P^{x_2}\) and the morphism \(\xi \xi^{x_2}: \xi P^{x_2} \to \xi P^1\).

**Remark 3.6.1.** — Suppose that the stratification structure \(S\) is of standard type. Let us recall the diagram of schemes over \(S\)

\[
\begin{array}{ccc}
X & \xrightarrow{\xi_{T^0}} & \xi P^1 \\
\downarrow \xi_{T^0} & & \downarrow \xi_{T^0}^{1,1} \\
\xi P^1 & \xrightarrow{\xi_{T^1}^{1,0}} & T^1 \\
\end{array}
\]

\[
\begin{array}{ccc}
X_{pr_1^T} & \xrightarrow{\xi_{T^1}^{1,0}} & T^1 \\
\downarrow \xi_{T^1}^{1,0} & & \downarrow \xi_{T^1}^{1,0} \\
X_{pr_3^T} & \xrightarrow{\xi_{T^1}^{1,0}} & X.
\end{array}
\]
(i) It follows from Lemma 3.5, (iii), and condition (1) of Definition 3.6, together with elementary algebraic geometry, that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varepsilon_0} & \mathcal{E} \times P^1 \\
\varepsilon_i & \downarrow & \varepsilon_i \times \varepsilon_{T^0} \\
\mathcal{E} \times P^1 & \xrightarrow{\varepsilon_i \times \varepsilon_{T^0}} & T^1
\end{array}
\]

is cartesian, and the right-hand vertical arrow of this diagram is a square-nilpotent closed immersion. Moreover, again by condition (1) of Definition 3.6, together with elementary algebraic geometry, the lower horizontal arrow of this diagram is conormally strict with respect to the right-hand vertical arrow.

(ii) It follows from elementary algebraic geometry that the morphisms

\[
\begin{array}{ccc}
T^1 & \xrightarrow{X\text{pr}_1^T} & X \\
X\text{pr}_2^T & \downarrow & \end{array}
\]

are quasi-nil-retraction-like.

(iii) It follows from Remark 2.3.1 and condition (1) of Definition 3.6 that the equalities

\[
X\text{pr}_1^T \circ \varepsilon_0 \circ \varepsilon_T^0 = X\text{pr}_3^T \circ \varepsilon_0 \circ \varepsilon_T^0 = \text{id}_X
\]

hold.

(iv) It follows from condition (1) of Definition 3.6, together with elementary algebraic geometry [cf. also Remark 2.3.1], that the morphisms

\[
\begin{array}{ccc}
\text{id}_X = X\text{pr}_1^T \circ \varepsilon_0 \circ \varepsilon_T^0, & \text{id}_X = X\text{pr}_3^T \circ \varepsilon_0 \circ \varepsilon_T^0, \\
\varepsilon_1^T = X\text{pr}_1^T \circ \varepsilon_0 \circ \varepsilon_T^0, & \varepsilon_2^T = X\text{pr}_3^T \circ \varepsilon_0 \circ \varepsilon_T^0, \\
\varepsilon_1^T = X\text{pr}_1^T \circ \varepsilon_1 \circ \varepsilon_T^0, & \varepsilon_2^T = X\text{pr}_3^T \circ \varepsilon_1 \circ \varepsilon_T^0, \\
X\text{pr}_1^T, & X\text{pr}_3^T
\end{array}
\]

are finite and flat.

**Remark 3.6.2.** — Suppose that the stratification structure $\mathcal{S}$ is of standard type. Then let us recall that the conormal sheaf of the square-nilpotent closed immersion $\mathcal{E}_T^{k^2} \rightarrow \mathcal{E} \times P^1$ of Definition 3.6 is isomorphic to the $\mathcal{O}_{T^1}$-module $(\mathcal{E}_T^0)_* \Omega_{X/S}^2$. Thus, since [one verifies easily that] the morphism $X\text{pr}_i^T : T^1 \rightarrow X$ is finite and flat for each $i \in \{1, 2, 3\}$, and the $\mathcal{O}_X$-module $\Omega_{X/S}^2$ is coherent and locally free, one verifies immediately that the stratification structure $\mathcal{S}$ satisfies condition (3) of Definition 3.3.

**Remark 3.6.3.** — In sequels to the present paper, we give some examples of stratification structures of standard type.
In the remainder of the present §3, let
\[ \mathfrak{T} = (\mathfrak{T}^n \cap P^n)_{n \geq 0}, \quad (\mathfrak{T}^n \cap P^n \rightarrow \mathfrak{T}^n+1)_{n \geq 0}, \]
\[ (\mathfrak{T}^{\sigma_n} \cap P^n \rightarrow X^{(2)})_{n \geq 0}, \quad (\mathfrak{T}^{\delta_{n_1,n_2}} \cap P^{n_1,n_2} \rightarrow \mathfrak{T}^{n_1+n_2})_{n_1,n_2 \geq 0} \]
be a stratification structure on \(X/S\) and
\[ \Phi = (\Phi^n)_{n \geq 0} : \mathfrak{S} \longrightarrow \mathfrak{T} \]
a morphism of stratification structures.

**Definition 3.7.**

(i) We shall say that the morphism \(\Phi\) is *nil-retraction-like* if, for each positive integer \(r\) and nonnegative integers \(n_1, \ldots, n_r\), the morphism \(\Phi^{n_1, \ldots, n_r} : \mathfrak{S}^{n_1, \ldots, n_r} \rightarrow \mathfrak{T}^{n_1, \ldots, n_r}\) is nil-retraction-like.

(ii) Let \(r\) be a nonnegative integer. Then we shall say that the morphism \(\Phi\) is *\(r\)-pr-finite flat* if, for each nonnegative integer \(n \leq r\) and \(i \in \{1, 2\}\), the morphism \(\Phi^{i,n} : \mathfrak{S}^{i,n} \cap P^{n} \rightarrow X\) is finite and flat.

**Remark 3.7.1.** — Suppose that the stratification structures \(\mathfrak{S}\) and \(\mathfrak{T}\) are \(\iota\)-*quasi-nil-retraction-like*. Then, in the situation of Definition 2.9, it follows from conditions (1), (3) of Definition 2.8 and Remark 2.8.1 that the morphism \(\Phi^{n_1, \ldots, n_r} : \mathfrak{S}^{n_1, \ldots, n_r} \rightarrow \mathfrak{T}^{n_1, \ldots, n_r}\) is *quasi-nil-retraction-like*.

**Definition 3.8.** — Let \(r\) be a nonnegative integer. Then we shall say that the morphism \(\Phi\) is *\(r\)-integrable* if the following five conditions are satisfied:

1. The morphism \(\Phi\) is nil-retraction-like.
2. The stratification structure \(\mathfrak{S}\) is \(\iota\)-quasi-nil-retraction-like and \(\iota\)-nil-retraction-like.
3. The stratification structure \(\mathfrak{T}\) is \(\iota\)-quasi-nil-retraction-like, \(\iota\)-nil-retraction-like, and \(\iota\)-strictly cocartesian of level \(\geq r + 1\).
4. Let \(n \leq r\) be a positive integer. Then the morphism \(\mathfrak{T}^n \cap P^{n-1} : \mathfrak{T}^n \cap P^n \rightarrow \mathfrak{T}^n \cap P^{n-1}\) is a square-nilpotent closed immersion.
5. Let \(n \leq r\) be a positive integer. Then the morphism \(\Phi^{i,n} : \mathfrak{T}^{i,n} \cap P^{n} \rightarrow \mathfrak{T}^{i,n} \cap P^n\) is conormally strict with respect to the square-nilpotent closed immersion \(\mathfrak{T}^{i,n} \cap P^{n-1} \rightarrow \mathfrak{T}^{i,n} \cap P^n\) [cf. (4)].

**Remark 3.8.1.** — Let \(r\) be a nonnegative integer. Suppose that the morphism \(\Phi\) is *\(r\)-integrable*.

(i) It follows from Remark 1.2.1 and condition (4) of Definition 3.8 that, for each nonnegative integers \(m \leq n \leq r\), the morphism \(\mathfrak{T}_n^m : \mathfrak{T}^n \cap P^m \rightarrow \mathfrak{T}^n \cap P^n\) is quasi-nil-retraction-like.
(ii) In the situation of Definition 2.10, (v), it follows from (i) and conditions (1), (3) of Definition 3.8 that the morphism \( \Phi_{n|m}^0 : X \to \Phi_{n|m}^0 \) is quasi-nil-retraction-like.

**Remark 3.8.2.** — Let \( n \leq r \) be positive integers. Then one verifies easily from the various definitions involved that the diagram of schemes over \( S \)

\[
\begin{array}{ccc}
\Phi_{r|n} & \Phi_{r|n-1} & \Phi_{r|n-1} \\
\downarrow \Phi_{r|n} & \downarrow \Phi_{r|n} & \\
\Phi_{r|n} & \Phi_{r|n} & \Phi_{r|n} \\

\end{array}
\]

is cartesian. Now suppose that the morphism \( \Phi \) is \( r \)-integrable. Then it follows from conditions (4), (5) of Definition 3.8 that the right-hand vertical arrow of this diagram is a square-nilpotent closed immersion, and the lower horizontal arrow of this diagram is conormally strict with respect to the right-hand vertical arrow.

**Remark 3.8.3.** — Let \( r \) be a nonnegative integer. Suppose that the morphism \( \Phi \) is \( r \)-integrable. Let \( n \geq r + 1 \) be an integer. Let us recall the diagram of schemes over \( S \)

\[
\begin{array}{cccc}
\tau_{P^0,...,0} & \tau_{P^1,1} & \tau_{T^1,1,1} & \tau_{P^1,1} \\
\downarrow \tau_{P^0,...,0} & \downarrow \tau_{P^1,1} & \downarrow \tau_{T^1,1,1} & \downarrow \tau_{P^1,1} \\
\tau_{P^n,1} & \tau_{P^{n-1},1} & \tau_{P^n,1} & \tau_{P^n,1} \\

\end{array}
\]

(i) It follows from condition (3) of Definition 3.8 that the diagram

\[
\begin{array}{ccc}
\tau_{P^n} & \tau_{T^{n,1}} & \tau_{P^n} \\
\downarrow \tau_{P^n} & \downarrow \tau_{P^n} & \\
\tau_{P^n} & \tau_{P^n} & \tau_{P^n} \\

\end{array}
\]

is [commutative and] strictly cocartesian.

(ii) It follows from Remark 3.1.1, (ii), and condition (3) of Definition 3.8 that the morphism

\[
\begin{array}{ccc}
\tau_{P^0,...,0} & \tau_{P^0,...,0} & \tau_{P^0,...,0} \\
\downarrow \tau_{P^0,...,0} & \downarrow \tau_{P^0,...,0} & \\
\tau_{P^0,...,0} & \tau_{P^0,...,0} & \tau_{P^0,...,0} \\

\end{array}
\]

is quasi-nil-retraction-like.

(iii) It follows from Remark 3.1.1, (i), and condition (3) of Definition 3.8 that the morphisms

\[
\begin{array}{ccc}
\tau_{P^n} & \tau_{P^n} & \tau_{P^n} \\
\downarrow \tau_{P^n} & \downarrow \tau_{P^n} & \\
\tau_{P^n} & \tau_{P^n} & \tau_{P^n} \\

\end{array}
\]

is quasi-nil-retraction-like.
are quasi-nil-retraction-like.

(iv) It follows from Remark 2.4.1, (ii), and Remark 2.6.1 that the equalities
\[ \tau \text{pr}_1 \circ \tau_1 \circ \tau \text{pr}_2 \circ \tau_2 = \tau \text{pr}_1 \circ \tau_2 \circ \tau \text{pr}_2 \circ \tau_2 = \text{id} \]
hold.

**Definition 3.9.** — Let \( r \) be a nonnegative integer. Then we shall say that the morphism \( \Phi \) is **strictly** \( r \)-integrable if the following three conditions are satisfied:

1. The morphism \( \Phi \) is \( r \)-integrable.
2. The morphism \( \Phi \) is \( r \)-pr-finite flat.
3. The stratification structure \( \mathcal{T} \) is pr-finite flat.

**Remark 3.9.1.** — Let \( n \leq r \) be positive integers. Suppose that the morphism \( \Phi \) is **strictly** \( r \)-integrable. Let us recall the diagram of schemes over \( S \)

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi \ell_{n-1}} & \Phi \mathcal{P}^{r|n-1} \\
\Phi \ell_{n-1} & & \Phi \mathcal{P}^{r|n-1} \\
\Phi \mathcal{P}^{r|n} & \xrightarrow{\Phi \mathcal{P}^{r|n}} & \mathcal{P}^{n} \\
\Phi \ell_{n-1} & & \Phi \mathcal{P}^{r|n} \\
\mathcal{P}^{n} & \xrightarrow{\mathcal{P}^{n}} & X.
\end{array}
\]

(i) It follows from Remark 3.8.1, (ii), and condition (1) of Definition 3.9 that the morphism
\[
\Phi \ell_{n-1} : X \longrightarrow \Phi \mathcal{P}^{r|n-1}
\]
is quasi-nil-retraction-like.

(ii) It follows from Remark 3.1.1, (i), condition (3) of Definition 3.8, and condition (1) of Definition 3.9 that the morphisms
\[
\mathcal{P}^{n} \xrightarrow{\mathcal{P}^{n}} X
\]
are quasi-nil-retraction-like.

(iii) It follows from condition (1) of Definition 2.2 and Remark 2.10.1, (ii), that the equalities
\[ \tau \text{pr}_1 \circ \tau_1 \circ \tau \text{pr}_2 \circ \tau_2 = \tau \text{pr}_1 \circ \tau_2 \circ \tau \text{pr}_2 \circ \tau_2 = \text{id} \]
hold.

(iv) It follows from conditions (2), (3) of Definition 3.9 [cf. also condition (1) of Definition 2.2; Remark 2.10.1, (ii)] that the morphisms
\[
\Phi \text{pr}_1^{r|n-1} = \tau \text{pr}_1 \circ \tau_1 \circ \tau \text{pr}_2^{r|n-1}, \quad \Phi \text{pr}_2^{r|n-1} = \tau \text{pr}_2 \circ \tau_2 \circ \tau \text{pr}_2^{r|n-1},
\]
\[\tau_{pr_1}^{n-1} = \tau_{pr_1^n} \circ \tau_{pr_1}^{n-1}, \quad \tau_{pr_2}^{n-1} = \tau_{pr_2^n} \circ \tau_{pr_2}^{n-1},\]

\[\phi_{pr_1}^{r|n} = \tau_{pr_1^n} \circ \phi_{pr_1}^{r|n}, \quad \phi_{pr_2}^{r|n} = \tau_{pr_2^n} \circ \phi_{pr_2}^{r|n},\]

are finite and flat.

**Remark 3.9.2.** — Let \( r \) be a nonnegative integer. Suppose that the morphism \( \Phi \) is strictly \( r \)-integrable. Let \( n \geq r + 1 \) be an integer. Let us recall the diagram of schemes over \( S \):

\[
\begin{array}{ccc}
X = \tau P^{0,\ldots,0} & \xrightarrow{\tau_{\delta_{n-1,1}}} & \tau P^n \\
\downarrow & & \downarrow \tau_{pr_1}^n & \downarrow \tau_{pr_2}^n \\
\tau P^{n-1,1} & \xrightarrow{\tau_{\delta_{n-1,1}}} & \tau P^n \\
\end{array}
\]

Then it follows from Remark 3.1.2 and condition (3) of Definition 3.9 [cf. also Remark 2.6.1] that the morphisms

\[
\begin{align*}
\tau_{pr_1}^n &= \tau_{pr_1^n} \circ \tau_{\delta_{1,n-1}^1} \circ \tau_{\delta_{1,n-1}^n}, \\
\tau_{pr_1}^{n+1} &= \tau_{pr_2^n} \circ \tau_{\delta_{1,n-1}^1} \circ \tau_{\delta_{1,n-1}^n}, \\
\tau_{pr_1}^{n-1} &= \tau_{pr_1}^n \circ \tau_{\delta_{1,n-1}^1}, \\
\tau_{pr_1}^{n-1,1} &= \tau_{pr_1}^n \circ \tau_{\delta_{n-1,1}^1}, \\
\tau_{pr_3}^1 &= \tau_{pr_2^n} \circ \tau_{\delta_{1,n-1}^1}, \\
\tau_{pr_3}^{n-1,1} &= \tau_{pr_2^n} \circ \tau_{\delta_{n-1,1}^1}, \\
\end{align*}
\]

are finite and flat.

**Remark 3.9.3.** — In sequels to the present paper, we give some examples of strictly \( r \)-integrable morphisms of stratification structures.

### 4. Stratifications and Integrable Connections

In the present §4, we introduce and discuss the notions of a connection [cf. Definition 4.4 below], a stratification [cf. Definition 4.6 below], and an integrable connection [cf. Definition 4.7, (ii), below]. In the present §4, let \( S \) be a scheme, \( X \) a scheme which is smooth and separated over \( S \),

\[
\mathcal{E} = (\mathcal{E} P^n)_{n \geq 0}, \quad (\mathcal{E} L^n : \mathcal{E} P^n \to \mathcal{E} P^{n+1})_{n \geq 0},
\]

\[
(\mathcal{E} \sigma^n : \mathcal{E} P^n \to X^{(2)})_{n \geq 0}, \quad (\mathcal{E} \delta_{n,n+2} : \mathcal{E} P^n \to \mathcal{E} P^{n+1 + n_2})_{n_1, n_2 \geq 0}
\]

a stratification structure on \( X/S \),

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\text{Sch}} & S
\end{array}
\]
a category fibered in groupoids over $\text{Sch}_S$, and

$$\xi$$

an object of $\mathcal{F}$ over $X$.

**Definition 4.1.** — Let $n$ be a nonnegative integer.

(i) We shall write

$$\mathcal{E}_{q\text{Cm}}^n(\xi) \overset{\text{def}}{=} \text{Lift}_{\sigma_\xi}((S_{pr_2}^n)^*\xi, (S_{pr_1}^n)^*\xi; \text{id}_\xi)$$

and refer to an element of the set $\mathcal{E}_{q\text{Cm}}^n(\xi)$ as a \textit{quasi-$n$-$\mathcal{G}$-connection} on $\xi$.

(ii) Let $\nabla$ be a quasi-$n$-$\mathcal{G}$-connection on $\xi$ and $m \leq n$ a nonnegative integer. Then one verifies easily that the isomorphism

$$(\mathcal{E}_{\ell_m}^n)^*\nabla : (S_{pr_2}^n)^*\xi \sim (S_{pr_1}^m)^*\xi$$

[cf. condition (1) of Definition 2.2] is a quasi-$m$-$\mathcal{G}$-connection on $\xi$. We shall write

$$\nabla|_m \overset{\text{def}}{=} (\mathcal{E}_{\ell_m}^n)^*\nabla$$

for this quasi-$m$-$\mathcal{G}$-connection on $\xi$. Thus, we have a map of sets

$$\mathcal{E}_{q\text{Cm}}^n(\xi) \longrightarrow \mathcal{E}_{q\text{Cm}}^m(\xi)$$

given by mapping $\nabla \in \mathcal{E}_{q\text{Cm}}^n(\xi)$ to $\nabla|_m \in \mathcal{E}_{q\text{Cm}}^m(\xi)$.

(iii) We shall refer to a quasi-$1$-$\mathcal{G}$-connection on $\xi$ as an \textit{$\mathcal{G}$-connection} on $\xi$ [cf. Remark 4.1.1 below].

**Remark 4.1.1.** — One verifies easily that the notion of an \textit{$\mathcal{G}$-connection} is the \textit{same} as the notion of a \textit{1-$\mathcal{G}$-connection} in the sense of Definition 4.4 below.

**Definition 4.2.** — Let $n$ be a nonnegative integer; $\xi_1$, $\xi_2$ objects of $\mathcal{F}$ over $X$; $\nabla_1$, $\nabla_2$ quasi-$n$-$\mathcal{G}$-connections on $\xi_1$, $\xi_2$, respectively; $\phi : \xi_1 \tilde{\rightarrow} \xi_2$ an isomorphism in $\mathcal{F}|_X$. Then we shall say that the isomorphism $\phi$ is \textit{$\mathcal{G}$-horizontal} [with respect to $\nabla_1$, $\nabla_2$] if the diagram in $\mathcal{F}|_{\phi, p^1}$

$$\begin{array}{ccc}
(S_{pr_2}^n)^*\xi_1 & \overset{(S_{pr_2}^n)^*\phi}{\sim} & (S_{pr_2}^n)^*\xi_2 \\
\nabla_1 & \downarrow & \nabla_2 \\
(S_{pr_1}^n)^*\xi_1 & \overset{(S_{pr_1}^n)^*\phi}{\sim} & (S_{pr_1}^n)^*\xi_2
\end{array}$$

is commutative.
DEFINITION 4.3. — Let \( r \geq 2 \) be an integer; \( n_1, \ldots, n_r \), positive integers. Write \( n = \sum_{i=1}^{r} n_i \) [so \( 1 \leq n_i \leq n-1 \) for each \( i \in \{1, \ldots, r\} \)]. Let \( \nabla \) be a quasi-(\( n-1 \))-\( \mathcal{G} \)-connection on \( \xi \). Then we shall write

\[
\nabla^{n_1, \ldots, n_r} : (\mathcal{E} \text{pr}_{r+1}^{n_1, \ldots, n_r})^* \xi \xrightarrow{\sim} (\mathcal{E} \text{pr}_1^{n_1, \ldots, n_r})^* \xi
\]

for the isomorphism in \( \mathcal{F}|_{\mathcal{E} \, p_{n_1, \ldots, n_r}} \) obtained by forming the composite

\[
(\mathcal{E} \text{pr}_{r+1}^{n_1, \ldots, n_r})^* \xi = (\mathcal{E} \text{pr}_{r+1}^{n_1, \ldots, n_r})^* (\mathcal{E} \text{pr}_2^{n_r})^* \xi 
\xrightarrow{\sim} (\mathcal{E} \text{pr}_{r+1}^{n_1, \ldots, n_r})^* (\mathcal{E} \text{pr}_1^{n_r})^* \xi
\]

\[
= (\mathcal{E} \text{pr}_r^{n_1, \ldots, n_{r-1}})^* (\mathcal{E} \text{pr}_r^{n_{r-1}})^* (\mathcal{E} \text{pr}_2^{n_{r-1}})^* \xi 
\xrightarrow{\sim} (\mathcal{E} \text{pr}_r^{n_1, \ldots, n_{r-1}})^* (\mathcal{E} \text{pr}_1^{n_{r-1}})^* \xi
\]

\[
\cdots
\]

\[
= (\mathcal{E} \text{pr}_2^{n_2, \ldots, n_r})^* \xi = (\mathcal{E} \text{pr}_2^{n_2, \ldots, n_r})^* (\mathcal{E} \text{pr}_1^{n_r})^* \xi 
\xrightarrow{\sim} (\mathcal{E} \text{pr}_1^{n_1, \ldots, n_r})^* \xi
\]

Moreover, we shall write

\[
\nabla^{\times r} : (\mathcal{E} \text{pr}_{r+1}^{\times r})^* \xi \xrightarrow{\sim} (\mathcal{E} \text{pr}_1^{\times r})^* \xi
\]

for “\( \nabla^{n_1, \ldots, n_r} \)” in the case where we take the “\( (n_1, \ldots, n_r) \)” to be \( (1, \ldots, 1) \).

DEFINITION 4.4. — Let \( n \) be a nonnegative integer and \( \nabla \) a quasi-\( n \)-\( \mathcal{G} \)-connection on \( \xi \). Then we shall say that \( \nabla \) is an \( n \)-\( \mathcal{G} \)-connection if, for each positive integers \( n_1, n_2 \) such that \( n_1 + n_2 = n \), the “cocycle condition”

\[
(\nabla|_{n_1, n_2}) = (\mathcal{E} \delta^{n_1, n_2})^* \nabla
\]

[cf. Remark 2.6.1] is satisfied. We shall write

\[
\mathcal{E} \text{Cm}^n(\xi) \subseteq \mathcal{E} \text{qCm}^n(\xi)
\]

for the set of \( n \)-\( \mathcal{G} \)-connections on \( \xi \).

LEMMA 4.5. — The map \( \mathcal{E} \text{qCm}^n(\xi) \to \mathcal{E} \text{Cm}^n(\xi) \) of Definition 4.1, (ii), restricts to a map of subsets

\[
\mathcal{E} \text{Cm}^n(\xi) \longrightarrow \mathcal{E} \text{Cm}^m(\xi).
\]

PROOF. — Let \( \nabla \) be an \( n \)-\( \mathcal{G} \)-connection on \( \xi \); \( m_1, m_2 \) positive integers such that \( m_1 + m_2 = m \). Then since \( \nabla \) is an \( n \)-\( \mathcal{G} \)-connection, the “cocycle condition”

\[
(\nabla|_{n_1, n_2})^{m_1 + n - m, m_2} = (\mathcal{E} \delta^{m_1 + n - m, m_2})^* \nabla
\]
is satisfied. Thus, by pulling back this equality by the morphism \( S_{m_1 + n - m, m_2} : S_{P_{m_1 + n - m, m_2}} \to S_{P_{m_1, m_2}} \), we conclude from condition (1) of Definition 2.2 and condition (4) of Definition 2.5 that the “cocycle condition”

\[
(\nabla|_{m-1})^{m_1, m_2} = (\delta^{m_1, m_2})^*(\nabla|_m)
\]

is satisfied, as desired. This completes the proof of Lemma 4.5.

**Definition 4.6.** — We shall write

\[
\mathfrak{S}\mathrm{Strt}(\xi) \overset{\text{def}}{=} \lim_{n \geq 0} \mathfrak{C}^{n n}(\xi)
\]

[cf. Lemma 4.5] and refer to an element of the set \( \mathfrak{S}\mathrm{Strt}(\xi) \) as an \( \mathfrak{S} \)-stratification on \( \xi \).

**Remark 4.6.1.** — A typical example of an \( \mathfrak{S} \)-stratification is as follows: Suppose that there exists an object of \( \mathcal{F} \) over \( S \) such that \( \xi \) is given by the pull-back of the object by \( X \to S \). Then, for each nonnegative integer \( n \), since [one verifies easily that] the two composites

\[
\mathfrak{S} P^n \overset{\mathfrak{S} \mathrm{pr}_1^n}{\longrightarrow} X \longrightarrow S,
\mathfrak{S} P^n \overset{\mathfrak{S} \mathrm{pr}_2^n}{\longrightarrow} X \longrightarrow S
\]

coincide, we have a natural identification

\[
\nabla_n : (\mathfrak{S} \mathrm{pr}_2^n)^* \xi = (\mathfrak{S} \mathrm{pr}_1^n)^* \xi.
\]

Now one verifies immediately that these \( \nabla_n \)'s form an \( \mathfrak{S} \)-stratification on \( \xi \).

**Remark 4.6.2.** — Suppose that we are in the situation of Remark 2.5.1. Thus, we have a stratification structure on \( X/S \)

\[
((P^n)_{n \geq 0}, (\tau^n : P^n \to P^{n+1})_{n \geq 0}, (\sigma^n : P^n \to X^{(2)})_{n \geq 0}, (\delta^{n_1, n_2} : P^{n_1 + n_2} \to P^{n_1 + n_2})_{n_1, n_2 \geq 0}).
\]

Suppose, moreover, that there exists a nonnegative integer \( n_0 \) such that the closed immersion \( P_{n_0} \to P \) is an isomorphism. In this situation, one verifies easily that the notion of a stratification on \( \xi \) [i.e., with respect to the above stratification structure] is the same as the notion of a descent datum on \( \xi \) with respect to the morphism \( X \to Y \) [in the usual sense] — i.e., an isomorphism in \( \mathcal{F}|_S \)

\[
q^* \xi \overset{\sim}{\longrightarrow} q^* \xi
\]

that satisfies a certain cocycle condition.

**Definition 4.7.**

(i) Let \( \nabla \) be an \( \mathfrak{S} \)-connection on \( \xi \). Then we shall refer to the automorphism of \( (\mathfrak{S} \mathrm{pr}_1^{x^2})^* \xi \) in \( \mathcal{F}|_{\mathfrak{S} \times x^2} \) obtained by forming the composite

\[
(\mathfrak{S} \times x^2)^* (\nabla_{x^2}) \circ (\mathfrak{S} \delta^{x^2})^* (\nabla^{-1})
\]
— where we write $\nabla^{-1}$ for the inverse of the isomorphism $\nabla$ — as the $\mathcal{S}$-\textit{curvature} of $\nabla$.

(ii) We shall say that an $\mathcal{S}$-connection on $\xi$ is $\mathcal{S}$-\textit{integrable} if the $\mathcal{S}$-\textit{curvature} of the $\mathcal{S}$-connection is the identity automorphism of $(\mathcal{S}\Pr_{1}^{x_{2}})^*\xi$. We shall write

$$\mathcal{S}\text{IntCnn}(\xi) \subseteq \mathcal{S}\text{Cnn}^1(\xi) (= \mathcal{S}\text{Cnn}^1(\xi))$$

for the set of $\mathcal{S}$-integrable $\mathcal{S}$-connections on $\xi$.

**Lemma 4.8.** — If $n > m = 1$, then the map $\mathcal{S}\text{Cnn}^n(\xi) \to \mathcal{S}\text{Cnn}^m(\xi) = \mathcal{S}\text{Cnn}^1(\xi)$ of Lemma 4.5 restricts to a map of subsets

$$\mathcal{S}\text{Cnn}^n(\xi) \longrightarrow \mathcal{S}\text{IntCnn}(\xi).$$

**Proof.** — Let us first observe that it follows from Lemma 4.5 that we may assume without loss of generality that $n = 2$. Let $\nabla$ be a 2-$\mathcal{S}$-connection on $\xi$. Thus, the “cocycle condition”

$$(\nabla|_1)^{x_2} = (\mathcal{S}\delta^{x_2})^*\nabla$$

is satisfied. In particular, by pulling back this equality by the morphism $\mathcal{S}L^{x_2}: \mathcal{S}P^{x_2} \to \mathcal{S}P^{x_2}$, we obtain an equality

$$(\mathcal{S}L^{x_2})^*((\nabla|_1)^{x_2}) = (\mathcal{S}\delta^{x_2})^*(\nabla|_1).$$

Thus, the $\mathcal{S}$-\textit{curvature} of $\nabla|_1$ is the identity automorphism, as desired. This completes the proof of Lemma 4.8. \hfill $\square$

**Lemma 4.9.** — Suppose that the following two conditions are satisfied:

1. The stratification structure $\mathcal{S}$ is of standard type.
2. The category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_S$ satisfies condition (1) of Definition 1.8.

Then the pull-back by the closed immersion $\mathcal{S}L^{T_2}: T^1 \to \mathcal{S}P^{x_2}$ of Definition 3.6 of the $\mathcal{S}$-\textit{curvature} of an $\mathcal{S}$-connection on $\xi$ is the identity automorphism of $(\mathcal{S}\Pr_{1}^{x_2})^*\xi$. In particular, the $\mathcal{S}$-\textit{curvature} of an $\mathcal{S}$-connection on $\xi$ is an element of

$$\text{Lift}_{\mathcal{S}L^{T_2}}((\mathcal{S}\Pr_{1}^{x_2})^*\xi, (\mathcal{S}\Pr_{1}^{x_2})^*\xi; \text{id}_{(\mathcal{S}\Pr_{1}^{x_2})^*\xi}) \subseteq \text{Aut}_{\mathcal{F}|_{\mathcal{S}L^{T_2}}}((\mathcal{S}\Pr_{1}^{x_2})^*\xi).$$

**Proof.** — Let $\nabla$ be an $\mathcal{S}$-connection on $\xi$. Write $\alpha$ for the pull-back of the $\mathcal{S}$-\textit{curvature} of $\nabla$ by $\mathcal{S}L^{T_2}$, i.e.,

$$\alpha = (\mathcal{S}L^{T_2})^*((\mathcal{S}L^{x_2})^*(\nabla|_1)^{x_2}) \circ (\mathcal{S}L^{T_2})^*(\mathcal{S}\delta^{x_2})^*(\nabla|_1) = (\mathcal{S}L^{T_2})^*((\mathcal{S}\delta^{x_2})^*(\nabla|_1)).$$

Now let us observe that it follows from our assumption that $\mathcal{S}$ is of standard type, and $\mathcal{F}$ satisfies condition (1) of Definition 1.8 [cf. also Remark 3.6.1] that the morphism $\mathcal{S}L^{1.0}$ determines a bijection

$$\text{Lift}_{\mathcal{S}L^{T_2}}((\mathcal{S}\Pr_{1}^{x_2})^*\xi, (\mathcal{S}\Pr_{1}^{x_2})^*\xi; \text{id}_{(\mathcal{S}\Pr_{1}^{x_2})^*\xi}) \longrightarrow \text{Lift}_{\mathcal{S}L^{T_2}}((\mathcal{S}\Pr_{1}^{x_2})^*\xi, (\mathcal{S}\Pr_{1}^{x_2})^*\xi; \text{id}_{\xi}).$$
Next, let us observe that since [one verifies easily that]

\[ S \times_2 S \cong S_0, T = S_1, T \]

the pull-back \((S_{\mathcal{T}}^{0,1})^*\alpha\) is the identity automorphism of \((\mathcal{S}P_1)^*\xi\), which thus implies that \(\alpha\) is an element of the domain of the above bijection. Next, let us observe that since [one verifies easily that]

\[ S \times_2 S \cong S_0, T = S_1, T \]

the pull-back \((S_{\mathcal{T}}^{1,0})^*\alpha\) is the identity automorphism of \((\mathcal{S}P_1)^*\xi\), which thus implies that the image by the above bijection of \(\alpha\) is the identity automorphism of \((\mathcal{S}P_1)^*\xi\). Thus, we conclude that \(\alpha\) is the identity automorphism, as desired. This completes the proof of Lemma 4.9.

**Proposition 4.10.** — Suppose that the following three conditions are satisfied:

1. The stratification structure \(\mathcal{S}\) is of standard type.
2. The category \(\mathcal{F}\) fibered in groupoids over \(\text{Sch}_S\) is weakly integrable.
3. The scheme \(X\) is of relative dimension \(\leq 1\) over \(S\).

Then every \(\mathcal{S}\)-connection on \(\xi\) is \(\mathcal{S}\)-integrable:

\[ \mathcal{S}qCnn(\xi) = \mathcal{S}Cnn(\xi) = \mathcal{S}IntCnn(\xi). \]

**Proof.** — Since \(\mathcal{S}\) is of standard type, and \(X\) is of relative dimension \(\leq 1\) over \(S\), it follows from condition (3) of Definition 3.6 that the closed immersion \(\mathcal{S}_{\mathcal{T}}^{2,0} : T^1 \hookrightarrow \mathcal{S}P^{n+2}\) of Definition 3.6 is an isomorphism. Thus, Proposition 4.10 follows from Lemma 4.9. This completes the proof of Proposition 4.10.

In the remainder of the present \(\S 4\), let

\[ \mathcal{T} = (\mathcal{T}P^n)_{n \geq 0}, (\mathcal{T}n : \mathcal{T}P^n \to \mathcal{T}P^{n+1})_{n \geq 0}, \]

\[ (\mathcal{T}n : \mathcal{T}P^n \to X^{(2)})_{n \geq 0}, (\mathcal{T}P^{n_1,n_2} : \mathcal{T}P^{n_1,n_2} \to \mathcal{T}P^{n_1+n_2})_{n_1,n_2 \geq 0} \]

be a stratification structure on \(X/S\) and

\[ \Phi = (\Phi^n)_{n \geq 0} : \mathcal{S} \longrightarrow \mathcal{T} \]

a morphism of stratification structures.

**Definition 4.11.** — Let \(n\) be a nonnegative integer and \(\nabla\) a quasi-\(n\)-\(\mathcal{T}\)-connection on \(\xi\). Then one verifies easily from condition (1) of Definition 2.8 and Remark 2.8.1 that the isomorphism

\[ (\Phi^n)^*\nabla : (\mathcal{S}pr_2^n)^*\xi \longrightarrow (\mathcal{S}pr_1^n)^*\xi \]

[cf. condition (2) of Definition 2.8] is a quasi-\(n\)-\(\mathcal{S}\)-connection on \(\xi\). We shall write

\[ \Phi^*\nabla \overset{\text{def}}{=} (\Phi^n)^*\nabla \]
for this quasi-$n$-$\mathfrak{G}$-connection on $\xi$. Thus, we have a map of sets
\[ \Phi^* : TqCnn^n(\xi) \rightarrow T^qCnn^n(\xi) \]
given by mapping $\nabla \in TqCnn^n(\xi)$ to $\Phi^*\nabla \in T^qCnn^n(\xi)$.

**Lemma 4.12.** — Let $n$ be a nonnegative integer. Then the following hold:

(i) The map $\Phi^* : TqCnn^n(\xi) \rightarrow T^qCnn^n(\xi)$ of Definition 4.11 restricts to a map of subsets
\[ \Phi^* : TqCnn^n(\xi) \rightarrow T^qCnn^n(\xi). \]
In particular, we have a map of sets
\[ \Phi^* : T^q\text{Strt}(\xi) \rightarrow T^q\text{Strt}(\xi) \]
[cf. condition (1) of Definition 2.8].

(ii) Suppose that the category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_S$ satisfies condition (3) of Definition 1.7, and that the morphism $\Phi$ is nil-retraction-like. Then the diagram of sets
\[ \begin{array}{ccc}
TqCnn^n(\xi) & \xrightarrow{\Phi^*} & T^qCnn^n(\xi) \\
\downarrow & & \downarrow \\
TqCnn^n(\xi) & \xrightarrow{\Phi^*} & T^qCnn^n(\xi)
\end{array} \]
[cf. (i)] is cartesian.

**Proof.** — First, we verify assertion (i). Let $\nabla$ be an $n$-$\mathfrak{T}$-connection on $\xi$; $n_1$, $n_2$ positive integers such that $n_1 + n_2 = n$. Then since $\nabla$ is an $n$-$\mathfrak{T}$-connection, the “cocycle condition”
\[ (\nabla|_{n-1})^{n_1,n_2} = (T^q\delta^{n_1,n_2})^*\nabla \]
is satisfied. Thus, by pulling back this equality by the morphism $\Phi^{n_1,n_2} : T^qP^{n_1,n_2} \rightarrow TqCnn^n(\xi)$, we conclude from conditions (1), (2), and (3) of Definition 2.8 that the “cocycle condition”
\[ ((\Phi^*\nabla)|_{n-1})^{n_1,n_2} = (T^q\delta^{n_1,n_2})^*(\Phi^*\nabla) \]
is satisfied, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $\nabla$ be a quasi-$n$-$\mathfrak{T}$-connection on $\xi$ such that the quasi-$n$-$\mathfrak{G}$-connection $\Phi^*\nabla$ is an $n$-$\mathfrak{G}$-connection; $n_1$, $n_2$ positive integers such that $n_1 + n_2 = n$. Then since $\mathcal{F}$ satisfies condition (3) of Definition 1.7, and the morphism $\Phi$ is nil-retraction-like, to verify the “cocycle condition”
\[ (\nabla|_{n-1})^{n_1,n_2} = (T^q\delta^{n_1,n_2})^*\nabla \]
for $\nabla$, it suffices to verify the pull-back of this equality by the morphism $\Phi^{n_1,n_2} : T^qP^{n_1,n_2} \rightarrow T^qP^{n_1,n_2}$, i.e., the equality
\[ ((\Phi^*\nabla)|_{n-1})^{n_1,n_2} = (T^q\delta^{n_1,n_2})^*(\Phi^*\nabla) \]
[cf. conditions (1), (2), and (3) of Definition 2.8]. On the other hand, since the quasi-$\mathcal{G}$-connection $\Phi^*\nabla$ is an $n$-$\mathcal{G}$-connection, this equality is satisfied, as desired. This completes the proof of assertion (ii), hence also of Lemma 4.12. □

**Definition 4.13.** — Let $n \leq r$ be nonnegative integers.

(i) Let $\nabla$ be a quasi-$r$-$\mathcal{G}$-connection on $\xi$. Then we shall refer to the automorphism of $(\Phi^n|_0)^*\xi$ in $\mathcal{F}|_{\mathcal{F}_n}\nabla$

\[(\Phi^n|_0)^*\xi = (\Phi_{\mathcal{F}_2}\nabla_{\mathcal{F}_1})^*\xi = (\Phi_{\mathcal{F}_2}\nabla_{\mathcal{F}_1})^*\xi = (\Phi^n|_0)^*\xi\]

[cf. Remark 2.10.1, (iii)] as the $(\Phi, n)$-curvature of $\nabla$. Moreover, we shall refer to the $(\Phi, n)$-curvature of the quasi-$n$-$\mathcal{G}$-connection on $\xi$ determined by an $\mathcal{G}$-stratification on $\xi$ as the $(\Phi, n)$-curvature of the $\mathcal{G}$-stratification.

(ii) We shall say that a quasi-$r$-$\mathcal{G}$-connection on $\xi$ is $(\Phi, n)$-dormant if the $(\Phi, n)$-curvature of the quasi-$r$-$\mathcal{G}$-connection is the identity automorphism of $(\Phi^n|_0)^*\xi$. Moreover, we shall say that an $\mathcal{G}$-stratification on $\xi$ is $(\Phi, n)$-dormant if the quasi-$n$-$\mathcal{G}$-connection determined by the $\mathcal{G}$-stratification is $(\Phi, n)$-dormant. We shall write

\[\Phi^{n,}\text{DrmStrt}(\xi) \subseteq \mathcal{G}\text{Strt}(\xi)\]

for the set of $(\Phi, n)$-dormant $\mathcal{G}$-stratifications on $\xi$.

(iii) We shall say that an $\mathcal{G}$-connection on $\xi$ is $(\Phi, n)$-dormant if the $\mathcal{G}$-connection is contained in the image of the composite

\[\Phi^{n,}\text{DrmStrt}(\xi) \subseteq \mathcal{G}\text{Strt}(\xi) \rightarrow \mathcal{G}\text{Cnn}(\xi).\]

[In particular, every $(\Phi, n)$-dormant $\mathcal{G}$-connection is $\mathcal{G}$-integrable — cf. Lemma 4.8.] We shall write

\[\Phi^{n,}\text{DrmCnn}(\xi) \subseteq \mathcal{G}\text{IntCnn}(\xi)\]

for the set of $(\Phi, n)$-dormant $\mathcal{G}$-connections on $\xi$. Thus, we have a natural surjective map

\[\Phi^{n,}\text{DrmStrt}(\xi) \rightarrow \Phi^{n,}\text{DrmCnn}(\xi).\]

**Lemma 4.14.** — Let $n$ be a nonnegative integer. Then the map $\Phi^* : \mathcal{T}\text{Strt}(\xi) \rightarrow \mathcal{G}\text{Strt}(\xi)$ of Lemma 4.12, (i), factors through the subset $\Phi^{n,}\text{DrmStrt}(\xi) \subseteq \mathcal{G}\text{Strt}(\xi)$. In particular, we obtain a map of sets

\[\Phi^* : \mathcal{T}\text{Strt}(\xi) \rightarrow \Phi^{n,}\text{DrmStrt}(\xi).\]

**Proof.** — This assertion follows immediately from the various definitions involved [cf. the cartesian diagram of Definition 2.10, (ii)]. □
5. The First Fundamental Correspondence

In the present §5, we prove the first main result of the present paper [cf. Theorem 5.6 below]. In the present §5, let $S$ be a scheme, $X$ a scheme which is smooth and separated over $S$,

\[
\mathfrak{S} = \left( (\delta^P_n)_{n \geq 0}, (\delta^{P^{n+1}}_n)_{n \geq 0}, (\delta^P_n: \delta^P \to \delta^{P^{n+1}}_n)_{n \geq 0}, (\delta^{P^{n_1,n_2}}_n: \delta^P \to \delta^{P^{n_1+n_2}}_{n_1,n_2})_{n_1,n_2 \geq 0} \right) \]

a stratification structure on $X/S$,

\[
\mathcal{F} \longrightarrow \mathrm{Sch}_S
\]

a category fibered in groupoids over $\mathrm{Sch}_S$, and

$\xi$

an object of $\mathcal{F}$ over $X$. Suppose that

- the stratification structure $\mathfrak{S}$ is integrable (respectively, strictly integrable), and that
- the category $\mathcal{F}$ fibered in groupoids over $\mathrm{Sch}_S$ is integrable (respectively, weakly integrable).

Lemma 5.1. — Let $r \geq 2$ be an integer; $n_1, \ldots, n_r$ positive integers. Write $n \overset{\text{def}}{=} \sum_{i=1}^r n_i$ [so $n \geq 2$]. Let $\nabla$ be an $(n-1)$-$\mathfrak{S}$-connection on $\xi$. Then the equality

\[
\nabla^\times = \left( \delta^\times_{n_1,\ldots,n_r} \right)^* \nabla^{n_1,\ldots,n_r}
\]

holds.

Proof. — This assertion follows immediately from Lemma 4.5, together with the “cocycle conditions” of Definition 4.4 for $\nabla_{n_1}, \ldots, \nabla_{n_{n-1}}$.

Lemma 5.2. — Let $m \leq n$ be nonnegative integers. Then the map

\[
\delta^\mathrm{Cmn}^n(\xi) \longrightarrow \delta^\mathrm{Cmn}^m(\xi)
\]

[cf. Lemma 4.5] is injective.

Proof. — Let us first observe that it is immediate that we may assume without loss of generality that $m = n - 1$. Let $\nabla, \nabla'$ be $n$-$\mathfrak{S}$-connections on $\xi$ such that $\nabla_{n-1} = \nabla'_{n-1}$. Now since $\mathcal{F}$ satisfies condition (3) of Definition 1.7, and the morphism $\delta^{1,n-1}: \delta^P \to \delta^P$ is nil-retraction-like [cf. condition (1) of Definition 3.2; condition (1) of Definition 3.3], to verify Lemma 5.2, it suffices to verify the equality

\[
(\delta^{1,n-1})^* \nabla = (\delta^{1,n-1})^* \nabla'.
\]

On the other hand, since both $\nabla$ and $\nabla'$ are $n$-$\mathfrak{S}$-connections, it follows from the “cocycle condition” of Definition 4.4 that $(\delta^{1,n-1})^* \nabla$, $(\delta^{1,n-1})^* \nabla'$ are completely determined by $\nabla_{n-1}$, $\nabla'_{n-1}$, respectively. In particular, we conclude that $(\delta^{1,n-1})^* \nabla = (\delta^{1,n-1})^* \nabla'$, as desired. This completes the proof of Lemma 5.2.
Lemma 5.3. — Let $n \geq 2$ be an integer and $\nabla$ a quasi-$n$-$\mathcal{G}$-connection on $\xi$. Then it holds that $\nabla$ is an $n$-$\mathcal{G}$-connection on $\xi$ if and only if the following two conditions are satisfied:

1. The quasi-$(n-1)$-$\mathcal{G}$-connection $\nabla|_{n-1}$ is an $(n-1)$-$\mathcal{G}$-connection.
2. The equality
   
   \[
   (\nabla|_{n-1})^{\times n} = (\mathcal{G} \delta^{\times n})^* \nabla
   \]

holds.

Proof. — First, we verify the necessity. Suppose that $\nabla$ is an $n$-$\mathcal{G}$-connection on $\xi$. Then it follows from Lemma 4.5 that condition (1) is satisfied. Next, to verify condition (2), let us observe that it follows from Lemma 5.1, together with condition (1), that

\[
(\nabla|_{n-1})^{\times n} = (\mathcal{G} \delta^{\times n})^* (\nabla|_{n-1})^{1,n-1}.
\]

Thus, since $\nabla$ is an $n$-$\mathcal{G}$-connection, which thus implies that the equality

\[
(\nabla|_{n-1})^{1,n-1} = (\mathcal{G} \delta^{1,n-1})^* \nabla
\]

holds, condition (2) is satisfied, as desired. This completes the proof of the necessity.

Next, we verify the sufficiency. Suppose that $\nabla$ satisfies the two conditions in the statement of Lemma 5.3. Let $n_1, n_2$ be positive integers such that $n_1 + n_2 = n$. Then since $\mathcal{F}$ satisfies condition (3) of Definition 1.7, and the morphism $\mathcal{G} \delta^{\times n}_{n_1,n_2} : \mathcal{G} P^{x_n} \to \mathcal{G} P^{n_1,n_2}$ is nil-retraction-like [cf. condition (1) of Definition 3.2; condition (1) of Definition 3.3], to verify the “cocycle condition” $(\nabla|_{n-1})^{n_1,n_2} = (\mathcal{G} \delta^{n_1,n_2})^* \nabla$, it suffices to verify the equality

\[
(\mathcal{G} \delta^{\times n}_{n_1,n_2})^* ((\nabla|_{n-1})^{n_1,n_2}) = (\mathcal{G} \delta^{\times n})^* \nabla,
\]

or, alternatively [cf. condition (2)], the equality

\[
(\mathcal{G} \delta^{\times n}_{n_1,n_2})^* ((\nabla|_{n-1})^{n_1,n_2}) = (\nabla|_{n-1})^{\times n}.
\]

On the other hand, since $\nabla|_{n-1}$ is an $(n-1)$-$\mathcal{G}$-connection [cf. condition (1)], this equality follows from Lemma 5.1. This completes the proof of the sufficiency, hence also of Lemma 5.3. \qed

Lemma 5.4. — The map

\[
\mathcal{G} \text{Cnn}^{2}(\xi) \longrightarrow \mathcal{G} \text{IntCnn}(\xi)
\]

[cf. Lemma 4.8] is surjective.

Proof. — Let us recall that we have assumed that

- the stratification structure $\mathcal{G}$ is integrable (respectively, strictly integrable), and that
- the category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_{\mathcal{G}}$ is integrable (respectively, weakly integrable).
Let $\nabla$ be an $\mathcal{S}$-integrable $\mathcal{S}$-connection on $\xi$. Then it follows from condition (1) of Definition 1.7 (respectively, condition (1) of Definition 1.8), together with Remark 3.2.2 and Remark 3.3.1, that the morphism $\mathcal{S}^{\delta \times 2}$ determines a bijection

$$\text{Lift}_{\epsilon_1}( (\mathcal{S}^{\delta \times 2})^* \xi, (\mathcal{S}^{\delta \times 2})^* \xi; \nabla) \sim \text{Lift}_{\epsilon_2}( (\mathcal{S}^{\delta \times 2})^* \xi, (\mathcal{S}^{\delta \times 2})^* \xi; (\mathcal{S}^{\delta \times 2})^* \nabla).$$

Now since $\nabla$ is $\mathcal{S}$-integrable, which thus implies that the equality

$$(\mathcal{S}^{\delta \times 2})^* (\nabla^{\times 2}) = (\mathcal{S}^{\delta \times 2})^* \nabla$$

holds, the isomorphism $\nabla^{\times 2}$ is an element of the codomain of this bijection. Thus, we obtain an element $\tilde{\nabla}$ of the domain of this bijection whose image coincides with $\nabla^{\times 2}$, i.e.,

$$(\mathcal{S}^{\delta \times 2})^* \tilde{\nabla} = \nabla^{\times 2}.$$

On the other hand, since $\nabla$ is an $\mathcal{S}$-connection, it follows from Lemma 5.3, together with this equality, that $\tilde{\nabla}$ is a $2-\mathcal{S}$-connection on $\xi$. This completes the proof of Lemma 5.4. □

**Lemma 5.5.** — Let $n \geq 3$ be an integer. Then the map

$$\mathcal{S}^{\text{Cnn}}_n(\xi) \longrightarrow \mathcal{S}^{\text{Cnn}}_{n-1}(\xi)$$

[cf. Lemma 4.5] is surjective.

**Proof.** — Let us recall that we have assumed that

- the stratification structure $\mathcal{S}$ is integrable (respectively, strictly integrable), and that
- the category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_{\mathcal{S}}$ is integrable (respectively, weakly integrable).

Let $\nabla$ be an $(n-1)$-$\mathcal{S}$-connection on $\xi$. Then it follows from Lemma 5.1 that

$$(\mathcal{S}^{\delta_n \times n})^* (\nabla^{1,n-1}) = \nabla^{\times n} = (\mathcal{S}^{\delta_n \times n})^* (\nabla^{n-1,1}).$$

Next, let us observe that it follows from condition (2) of Definition 1.7 (respectively, condition (2) of Definition 1.8), together with Remark 3.2.3 and Remark 3.3.2, that the morphism $\mathcal{S}^{\delta_{n-1,1}}$ determines a bijection

$$\text{Lift}_{\delta_{1,n-1}}( (\mathcal{S}^{\delta_n \times n})^* \xi, (\mathcal{S}^{\delta_n \times n})^* \xi; \nabla^{1,n-1}) \sim \text{Lift}_{\delta_{n-1,1}}( (\mathcal{S}^{\delta_{n-1,1}})^* \xi, (\mathcal{S}^{\delta_{n-1,1}})^* \xi; (\mathcal{S}^{\delta_{n-1,1}})^* \nabla^{1,n-1}).$$

Next, let us observe that it follows from the two equalities of the first display of the present proof of Lemma 5.5 that $\nabla^{n-1,1}$ is an element of the codomain of this bijection. In particular, we obtain an element $\tilde{\nabla}$ of the domain of this bijection whose image coincides with $\nabla^{n-1,1}$, i.e.,

$$(\mathcal{S}^{\delta_{n-1,1}})^* \tilde{\nabla} = \nabla^{n-1,1}.$$  

On the other hand, this equality, together with the second equality of the first display in the present proof of Lemma 5.5, implies the equalities

$$(\mathcal{S}^{\delta \times n})^* \tilde{\nabla} = (\mathcal{S}^{\delta_{n-1,1}})^* (\mathcal{S}^{\delta \times n})^* \tilde{\nabla} = (\mathcal{S}^{\delta_{n-1,1}})^* \nabla^{n-1,1} = \nabla^{\times n}.$$
Thus, it follows from Lemma 5.3 that $\nabla$ is an $n$-connection on $\xi$. This completes the proof of Lemma 5.5. □

The first main result of the present paper is as follows.

**Theorem 5.6.** — Let $S$ be a scheme, $X$ a scheme which is smooth and separated over $S$,

$$
\mathcal{S} = \left( (\mathcal{S} P_n)_{n \geq 0}, (\mathcal{S} t^n : \mathcal{S} P^n \rightarrow \mathcal{S} P^{n+1})_{n \geq 0}, \right.

(\mathcal{S} \sigma^n : \mathcal{S} P^n \rightarrow X^{(2)})_{n \geq 0}, \left. (\mathcal{S} \delta^{n_1,n_2} : \mathcal{S} P^{n_1,n_2} \rightarrow \mathcal{S} P^{n_1+n_2})_{n_1,n_2 \geq 0} \right)
$$

a stratification structure on $X/S$ [cf. Definition 2.5],

$$
\mathcal{F} \longrightarrow \text{Sch}_S
$$

a category fibered in groupoids over Sch$_S$, and $\xi$ an object of $\mathcal{F}$ over $X$. Suppose that the following two conditions are satisfied:

1. The stratification structure $\mathcal{S}$ is integrable [cf. Definition 3.2] (respectively, strictly integrable [cf. Definition 3.3]).
2. The category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_S$ is integrable [cf. Definition 1.7] (respectively, weakly integrable [cf. Definition 1.8]).

Then, for each integer $n \geq 2$, the natural maps

$$
\mathcal{S} \text{Strt}(\xi) \longrightarrow \mathcal{S} \text{Cnn}^n(\xi) \longrightarrow \mathcal{S} \text{IntCnn}(\xi)
$$

[cf. Definition 4.4; Definition 4.6; Definition 4.7, (ii); Lemma 4.8] are bijective.

**Proof.** — The injectivity of the two maps under consideration follows from Lemma 5.2. The surjectivity of the first map under consideration follows from Lemma 5.5. The surjectivity of the second map under consideration follows from Lemma 5.4 and Lemma 5.5. This completes the proof of Theorem 5.6. □

**Definition 5.7.** — In the situation of Theorem 5.6, we shall refer to the bijection obtained by Theorem 5.6

$$
\mathcal{S} \text{Strt}(\xi) \sim \mathcal{S} \text{IntCnn}(\xi)
$$

as the first fundamental correspondence.

6. The Second Fundamental Correspondence

In the present §6, we prove the second main result of the present paper [cf. Theorem 6.6 below]. In the present §6, let $S$ be a scheme, $X$ a scheme which is smooth and separated over $S$;

$$
\mathcal{S} = \left( (\mathcal{S} P_n)_{n \geq 0}, (\mathcal{S} t^n : \mathcal{S} P^n \rightarrow \mathcal{S} P^{n+1})_{n \geq 0}, \right.

(\mathcal{S} \sigma^n : \mathcal{S} P^n \rightarrow X^{(2)})_{n \geq 0}, \left. (\mathcal{S} \delta^{n_1,n_2} : \mathcal{S} P^{n_1,n_2} \rightarrow \mathcal{S} P^{n_1+n_2})_{n_1,n_2 \geq 0} \right),
$$
\[ T = (T^n)_{n \geq 0}, (T^n : T^m P^n \to T^m P^{n+1})_{n \geq 0}, \]
\[ (T^n : T^m P^n \to X^{(2)})_{n \geq 0}, (T^n \delta^{n_1, n_2} : T^m P^{n_1, n_2} \to T^m P^{n_1+n_2})_{n_1, n_2 \geq 0} \]

stratification structures on \( X/S \);
\[ \Phi = (\Phi^n)_{n \geq 0} : \mathcal{S} \to \mathcal{T} \]
a morphism of stratification structures;
\[ \mathcal{F} \to \text{Sch}_S \]
a category fibered in groupoids over \( \text{Sch}_S \);
\[ \xi \]
an object of \( \mathcal{F} \) over \( X \); \( r \) a nonnegative integer. Suppose that
- the morphism \( \Phi \) is \( r \)-integrable (respectively, strictly \( r \)-integrable), and that
- the category \( \mathcal{F} \) fibered in groupoids over \( \text{Sch}_S \) is integrable (respectively, weakly integrable).

**Lemma 6.1.** — Let \( m \leq n \) be nonnegative integers. Then the maps
\[ \mathcal{S} \text{Cmn}^n(\xi) \to \mathcal{S} \text{Cmn}^m(\xi), \quad \mathcal{T} \text{Cnn}^n(\xi) \to \mathcal{T} \text{Cnn}^m(\xi) \]
[cf. Lemma 4.5] are injective.

**Proof.** — This assertion follows immediately from a similar argument to the argument applied in the proof of Lemma 5.2, together with conditions (2), (3) of Definition 3.8 and condition (1) of Definition 3.9. \( \square \)

**Lemma 6.2.** — Let \( n \) be a nonnegative integer. Then the map
\[ \Phi^* : \mathcal{T} \text{Cnn}^n(\xi) \to \mathcal{S} \text{Cnn}^n(\xi) \]
[cf. Definition 4.11], hence also the maps
\[ \Phi^* : \mathcal{T} \text{Cnn}^n(\xi) \to \mathcal{S} \text{Cnn}^n(\xi), \quad \Phi^* : \mathcal{T} \text{Strt}(\xi) \to \mathcal{S} \text{Strt}(\xi), \]
[Lemma 4.12, (i)], is injective.

**Proof.** — This assertion follows from condition (3) of Definition 1.7 and condition (1) of Definition 3.8 and condition (1) of Definition 3.9. \( \square \)

**Lemma 6.3.** — Let \( n \geq r + 1 \) be an integer. Then the map
\[ \mathcal{T} \text{Cnn}^n(\xi) \to \mathcal{T} \text{Cnn}^{n-1}(\xi) \]
[cf. Lemma 4.5] is surjective.

**Proof.** — This assertion follows immediately from a similar argument to the argument applied in the proof of Lemma 5.5, together with condition (3) of Definition 3.8 and condition (1) of Definition 3.9 [cf. also Remark 3.8.3 and Remark 3.9.2]. \( \square \)
Lemma 6.4. — Let $\nabla$ be a $(\Phi, r)$-dormant $r$-$\nabla$-connection on $\xi$. Then, for each positive integer $n \leq r$, there exists a quasi-$n$-$\nabla$-connection $\tilde{\nabla}_n$ on $\xi$ such that

$$(\Phi^{r|n})^* \tilde{\nabla}_n = (\Phi^{r|n})^* \nabla.$$ 

[Note that $\Phi^r = \Phi^r - \text{cf. Definition } 2.10, (i).]$

Proof. — Let us recall that we have assumed that

- the morphism $\Phi$ is $r$-integrable (respectively, strictly $r$-integrable), and that
- the category $\mathcal{F}$ fibered in groupoids over $\text{Sch}_S$ is integrable (respectively, weakly integrable).

Let us verify Lemma 6.4 by induction on $n$. Suppose that $n = 1$. Then it follows from condition (1) of Definition 1.7 (respectively, condition (1) of Definition 1.8), together with Remark 3.8.2 and Remark 3.9.1, that the morphism $\Phi^{r|1}$ determines a bijection

$$\text{Lift}_{\gamma, \phi}(\Phi^{r|1})^* \xi, (\Phi^{r|1})^* \xi; \text{id}_{\xi}) \dashrightarrow \text{Lift}_{\gamma, \phi}(\Phi^{r|1})^* \xi, (\Phi^{r|1})^* \xi; \text{id}_{(\Phi^{r|1})^* \xi}).$$

Then since $\nabla$ is $(\Phi, r)$-dormant, it follows from the definition of the $(\Phi, r)$-curvature that the pull-back $(\Phi^{r|1})^* \nabla$ is an element of the codomain of this bijection. In particular, we obtain an element $\tilde{\nabla}_1$ of the domain of this bijection whose image coincides with $(\Phi^{r|1})^* \nabla$, i.e.,

$$(\Phi^{r|1})^* \tilde{\nabla}_1 = (\Phi^{r|1})^* \nabla,$$

as desired.

Next, suppose that $n \geq 2$, and that there exists a quasi-$(n-1)$-$\nabla$-connection $\tilde{\nabla}_{n-1}$ on $\xi$ such that

$$(\Phi^{r|n-1})^* \tilde{\nabla}_{n-1} = (\Phi^{r|n-1})^* \nabla.$$ 

Then it follows from condition (1) of Definition 1.7 (respectively, condition (1) of Definition 1.8), together with Remark 3.8.2 and Remark 3.9.1, that the morphism $\Phi^{r|n}$ determines a bijection

$$\text{Lift}_{\gamma, \Phi}(\Phi^{r|n})^* \xi, (\Phi^{r|n})^* \xi; \tilde{\nabla}_{n-1}) \dashrightarrow \text{Lift}_{\gamma, \Phi}(\Phi^{r|n})^* \xi, (\Phi^{r|n})^* \xi; (\Phi^{r|n-1})^* \tilde{\nabla}_{n-1}).$$

Next, let us observe that since $(\Phi^{r|n-1})^* \tilde{\nabla}_{n-1} = (\Phi^{r|n-1})^* \nabla$, it is immediate that the pull-back $(\Phi^{r|n})^* \nabla$ is an element of the codomain of this bijection. In particular, we obtain an element $\tilde{\nabla}_n$ of the domain of this bijection whose image coincides with $(\Phi^{r|n})^* \nabla$, i.e.,

$$(\Phi^{r|n})^* \tilde{\nabla}_n = (\Phi^{r|n})^* \nabla,$$

as desired. This completes the proof of Lemma 6.4. □
**Lemma 6.5.** — Let \( n \geq r \) be an integer and \( \nabla \) a \( (\Phi, r) \)-dormant \( n \times \mathcal{S} \)-connection on \( \xi \). Then there exists an \( n \times \mathcal{I} \)-connection \( \tilde{\nabla} \) on \( \xi \) such that \( \Phi^* \tilde{\nabla} = \nabla \).

**Proof.** — Let us first observe that it follows from Lemma 6.4 that there exists a quasi-\( r \)-\( T \)-connection \( e \) such that \( \Phi^* e = r \).

Thus, since \( e \) is an \( r \)-\( T \)-connection on \( \xi \) [cf. Lemma 4.5], it follows from Lemma 4.12, (ii), together with condition (1) of Definition 3.8 and condition (1) of Definition 3.9, that \( \tilde{\nabla} \) is an \( r \times \mathcal{I} \)-connection on \( \xi \).

Next, let us observe that it follows from Lemma 6.3 that there exists an \( n \times \mathcal{I} \)-connection \( \nabla \) on \( \xi \) such that \( \nabla = \tilde{\nabla} \).

Thus, it follows immediately from Lemma 6.1 that the equality \( \Phi^* \nabla = \nabla \) holds. This completes the proof of Lemma 6.5. \( \square \)

The second main result of the present paper is as follows.

**Theorem 6.6.** — Let \( S \) be a scheme; \( X \) a scheme which is smooth and separated over \( S \);

\[
\mathcal{S} = (\left( \delta^p \right)_{n \geq 0}, (\delta^pq^n : \delta^p \to \delta^{p+1})_{n \geq 0},
(\delta^q \sigma^n : \delta^p \to X^{(2)})_{n \geq 0}, (\delta^q \delta^p_{n_1,n_2} : \delta^p \to \delta^{p+n_2})_{n_1,n_2 \geq 0},
\]

\[
\mathcal{I} = (\left( \tau^p \right)_{n \geq 0}, (\tau^p t^n : \tau^p \to \tau^{p+1})_{n \geq 0},
(\tau^q \sigma^n : \tau^p \to X^{(2)})_{n \geq 0}, (\tau^q \delta^p_{n_1,n_2} : \tau^p \to \tau^{p+n_2})_{n_1,n_2 \geq 0})
\]

stratification structures on \( X/S \) [cf. Definition 2.5];

\[
\Phi = (\Phi^n)_{n \geq 0} : \mathcal{S} \longrightarrow \mathcal{I}
\]
a morphism of stratification structures [cf. Definition 2.8];

\[
\mathcal{F} \longrightarrow \text{Sch}_S
\]
a category fibered in groupoids over \( \text{Sch}_S \); \( \xi \) an object of \( \mathcal{F} \) over \( X \); \( r \) a nonnegative integer. Suppose that the following two conditions are satisfied:

1. The morphism \( \Phi \) is \( r \)-integrable [cf. Definition 3.8] (respectively, strictly \( r \)-integrable [cf. Definition 3.9]).

2. The category \( \mathcal{F} \) fibered in groupoids over \( \text{Sch}_S \) is integrable [cf. Definition 1.7] (respectively, weakly integrable [cf. Definition 1.8]).

Then the natural maps

\[
\Phi^* \delta \longrightarrow \Phi^* \delta \longrightarrow \Phi^* \delta
\]
[cf. Definition 4.6; Definition 4.13, (ii), (iii); Lemma 4.14] are bijective.

PROOF. — The injectivity of the first map under consideration follows from Lemma 6.2. The surjectivity of the first map under consideration follows from Lemma 6.2 and Lemma 6.5. The bijectivity of the second map under consideration follows from Lemma 6.1. This completes the proof of Theorem 6.6. □

DEFINITION 6.7. — In the situation of Theorem 6.6, we shall refer to the bijection obtained by Theorem 6.6

\[ \mathcal{S} \mathcal{t}(\xi) \sim \Phi \cdot r \mathcal{D} \mathcal{m} \mathcal{C} \mathcal{n}(\xi) \]

as the second fundamental correspondence.