RIMS-1903

Integrable Connections II: Divided Power Stratifications

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July 2019

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Abstract. — In the present paper, we prove that the divided power stratification structures are strictly integrable and of standard type. In particular, as an application of the first fundamental correspondence, we obtain a natural bijection between the set of PD-stratifications and the set of PD-integrable PD-connections on objects of weakly integrable categories fibered in groupoids over categories of schemes. This bijection may be regarded as a generalization of the well-known equivalence concerning integrable connections and divided power stratifications.

Contents

Introduction

I.0. — Let S be a scheme and X a scheme which is *smooth* and *separated* over S. Write Sch_S for the category of schemes over S and morphisms of schemes over S [cf. [2], Definition 1.6, (i)] and

$$^{\mathrm{PD}}P^1 \subseteq X \times_S X$$

for the closed subscheme of $X \times_S X$ defined by the quasi-coherent ideal of $\mathcal{O}_{X\times_S X}$ obtained by forming the square of the quasi-coherent ideal that defines the diagonal closed subscheme $X \subseteq X \times_S X$ of $X \times_S X$ [cf. Definition 2.1, (ii); [1], Remark 4.2]. For each $i \in \{1, 2\}$, write, moreover,

$$^{\mathrm{PD}}\mathrm{pr}^1_i \colon {}^{\mathrm{PD}}P^1 \longrightarrow X$$

2010 Mathematics Subject Classification. — 14D15.
Key words and phrases. — divided power stratification structure, curvature, integrable connection, divided power stratification.

for the morphism over S obtained by forming the composite of the natural closed immersion ${}^{\mathrm{PD}}P^1 \hookrightarrow X \times_S X$ and the projection $X \times_S X \to X$ onto the i-th factor [cf. Lemma 2.2; [2], Definition 2.3, (ii)].

I.1. — In the remainder of the present Introduction, let

$$\mathcal{E}$$

be a locally free $\mathcal{O}_X$-module. Let us first recall that a *connection* on $\mathcal{E}$ [i.e., relative to X/S] is defined to be a homomorphism of $(X \to S)^{-1}\mathcal{O}_S$-modules

$$\nabla \colon \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

that satisfies the equality $\nabla(fe) = f\nabla(e) + e \otimes df$ — where f, e are local sections of $\mathcal{O}_X$, $\mathcal{E}$, respectively [cf. Definition 1.6]. On the other hand, it is well-known [cf., e.g., [1], Proposition 2.9] that giving a connection on $\mathcal{E}$ [i.e., a homomorphism of $(X \to S)^{-1}\mathcal{O}_S$-modules as above] is *equivalent* to giving an *isomorphism of* $\mathcal{O}_{{}^{\mathrm{PD}}P^1}$*-modules*

$$({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\mathcal{E} \xrightarrow{\sim} ({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\mathcal{E}$$

that restricts, on the diagonal closed subscheme $X \subseteq {}^{\mathrm{PD}}P^1$, to the *identity automorphism* of $\mathcal{E}$ [cf. Proposition 2.6, (ii)]. Moreover, such an isomorphism $({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\mathcal{E} \xrightarrow{\sim} ({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\mathcal{E}$ is naturally related to the notion of a *divided power stratification* [cf., e.g., [1], Definition 4.3] on $\mathcal{E}$.

Now let us observe that since the above "second" definition of the notion of a *connection* [i.e., an isomorphism $({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\mathcal{E} \xrightarrow{\sim} ({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\mathcal{E}$ as above] and the definition of the notion of a *divided power stratification* of [1], Definition 4.3, are "sufficiently abstract", one may apply these definitions [not only to locally free modules as above but also] to an object of a category fibered in groupoids over the category Sch_S. In the remainder of the present Introduction, let $\mathcal{F} \to \mathrm{Sch}_S$ be a category fibered in groupoids over Sch_S and ξ an object of $\mathcal{F}$ over X. Thus, one may define a PD-*connection* on ξ to be an isomorphism

$$({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\xi \xrightarrow{\sim} ({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\xi$$

in $\mathcal{F}$ over the identity automorphism of ${}^{\mathrm{PD}}P^1$ that restricts to the identity automorphism of $\xi = (({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\xi)|_X = (({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\xi)|_X$ [cf. Definition 2.5; [2], Definition 4.1, (iii)]. Moreover, one may also define a PD-*stratification* on ξ to be a collection of data similar to a divided power stratification defined in [1], Definition 4.3 [cf. Definition 2.5; [2], Definition 4.6]. Write

$${}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi), \qquad {}^{\mathrm{PD}}\mathrm{Strt}(\xi)$$

for the sets of PD-connections, PD-stratifications on ξ, respectively [cf. Definition 2.5; [2], Remark 4.1.1; [2], Definition 4.4; [2], Definition 4.6].

I.2. — Next, let us recall that it is well-known [cf., e.g., [1], Theorem 4.8] that, for a given connection ∇ on the locally free $\mathcal{O}_X$-module $\mathcal{E}$, the following two conditions are *equivalent*:

(†) The connection ∇ is *integrable* — i.e., the *curvature* of the connection ∇

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \xrightarrow{\nabla^1} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/S}$$

[cf., e.g., the discussion preceding [1], Theorem 2.15] is *zero*.

(‡) The connection ∇ extends to a *uniquely determined divided power stratification* on $\mathcal{E}$.

Here, let us observe that the usual definition of the *curvature* — hence also of the *integrability* — of a connection *relies*, at least *a priori*, on the fact that $\mathcal{E}$ is an $\mathcal{O}_X$-module. In particular, these definitions of *curvature* and *integrability* cannot be applied, at least in any immediate way, to a connection on an object of $\mathcal{F}$ discussed in §I.1.

Moreover, let us recall that the proof of the above relationship between integrable connections and divided power stratifications given in [1], §4, *relies*, at least *a priori*, on the fact that $\mathcal{E}$ is an $\mathcal{O}_X$-module [cf. the argument concerning the ring of *divided power differential operators* on $\mathcal{E}$ in the proof of [1], Theorem 4.8]. In particular, this proof cannot be applied, at least in any immediate way, in a situation in which we work with the category $\mathcal{F}$ fibered in groupoids over Sch_S as in §I.1.

Thus, one may pose the following two questions:

(A) What is a suitable definition of *"curvature"* [and *"integrability"*] of a PD-connection [cf. §I.1] on an object of $\mathcal{F}$?

(B) What is a suitable condition on $\mathcal{F}$ that leads us to a situation in which one may obtain a natural bijection between the set of *"integrable"* PD-*connections* [cf. (A)] and the set of PD-*stratifications* [cf. §I.1], i.e., on a fixed object of $\mathcal{F}$?

Note that the question (A) (respectively, (B)) is related to the discussion of [3], Definition 2.3, and Remark 4.3.2 of the present paper (respectively, [3], Question 2.7).

I.3. — With regard to the question (A) of §I.2, we define the PD-*curvature* of a PD-connection — that is an automorphism of a certain object of $\mathcal{F}$ — by applying [2], Definition 4.7, (i) [cf. Definition 2.5; [2], Definition 4.7, (i)]. Note that we observe in Remark 4.3.1, (i), that this definition essentially *generalizes* the usual definition of the *curvature* of a connection on a module. Moreover, we define a PD-*integrable* PD-*connection* by applying [2], Definition 4.7, (ii), i.e., to be a PD-connection whose PD-curvature is the identity automorphism [cf. Definition 2.5; [2], Definition 4.7, (ii)]. Write

$$^{\mathrm{PD}}\mathrm{IntCnn}(\xi) \subseteq {}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi)$$

for the set of PD-integrable PD-connections on ξ [cf. Definition 2.5; [2], Definition 4.7, (ii)].

With regard to the question (B) of §I.2, we defined, in [2], the notion of a *weakly integrable* category fibered in groupoids over Sch_S [cf. [2], Definition 1.8]. Moreover, we prove that each of

- the category fibered in groupoids of *locally free modules* [cf. Definition 1.1] and
- the category fibered in groupoids of *smooth schemes* [cf. Definition 1.7]

gives an example of a *weakly integrable* category fibered in groupoids over Sch_S [cf. Proposition 1.5; Proposition 1.11].

One important result of the present paper — that may be regarded as an application of the *first fundamental correspondence* of [2], Definition 5.7 — is as follows [cf. Corollary 4.3].

THEOREM A. — *Let S be a scheme, X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a **weakly integrable** *[cf. [2], Definition 1.8] category fibered in groupoids over Sch_S, and ξ an object of $\mathcal{F}$ over X. Then the natural map*

$${}^{\mathrm{PD}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathrm{PD}}\mathrm{IntCnn}(\xi)$$

[cf. Definition 2.5; [2], Definition 4.6; [2], Definition 4.7, (ii); [2], Lemma 4.8] is **bijective**.

Moreover, one may prove that, as in the case of *connections on modules*, if X is *of relative dimension* ≤ 1 over S, then every PD-connection on an object over X of a *weakly integrable* category fibered in groupoids over Sch_S is PD-*integrable* [cf. Corollary 4.2]. In particular, we also obtain the following result [cf. Corollary 4.4].

THEOREM B. — *Let S be a scheme, X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S, and ξ an object of $\mathcal{F}$ over X. Suppose that the following two conditions are satisfied:

(1) *The category $\mathcal{F}$ fibered in groupoids over Sch_S is* **weakly integrable** *[cf. [2], Definition 1.8].*

(2) *The scheme X is* **of relative dimension ≤ 1** *over S.*

Then the natural map

$${}^{\mathrm{PD}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi)$$

[cf. Definition 2.5; [2], Definition 4.4; [2], Definition 4.6] is **bijective**.

Thus, we obtain *generalizations* of the equivalence of the two conditions (†) and (‡) in §I.2, i.e., Theorem A and Theorem B.

ACKNOWLEDGMENTS

This research was supported by JSPS KAKENHI Grant Number 18K03239 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1. TWO EXAMPLES OF INTEGRABLE CATEGORY FIBERED IN GROUPOIDS

In the present §1, we give two examples of *integrable* [cf. [2], Definition 1.7] categories fibered in groupoids [cf. Proposition 1.5 and Proposition 1.11 below]. In the present §1, let S be a scheme. Thus, we have the category

$$\mathrm{Sch}_S$$

of schemes over S and morphisms of schemes over S [cf. [2], Definition 1.6, (i)].

DEFINITION 1.1. — We shall write

$$\mathrm{LcFr} \longrightarrow \mathrm{Sch}_S$$

for the category fibered in groupoids over Sch_S defined as follows:

- An object of the category LcFr is a pair $(X, \mathcal{E})$ consisting of a scheme X over S and a locally free $\mathcal{O}_X$-module $\mathcal{E}$.
- If $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ are objects of the category LcFr, then a morphism $(X, \mathcal{E}) \to (X', \mathcal{E}')$ in the category LcFr is defined to be a pair (f, ϕ) consisting of a morphism $f\colon X \to X'$ of schemes over S and an isomorphism $\phi\colon \mathcal{E} \xrightarrow{\sim} f^*\mathcal{E}'$ of $\mathcal{O}_X$-modules.
- The functor $\mathrm{LcFr} \to \mathrm{Sch}_S$ sends "$(X, \mathcal{E})$" to "X" and "(f, ϕ)" to "f".

LEMMA 1.2. — *Let X, $\underline{X}$ be schemes over S; $\underline{X} \hookrightarrow X$ a* **square-nilpotent** [*cf.* [2], *Definition* 1.2, (ii)] *closed immersion over S; $\mathcal{E}_1$, $\mathcal{E}_2$ locally free $\mathcal{O}_X$-modules; $\phi\colon \mathcal{E}_1|_{\underline{X}} \xrightarrow{\sim} \mathcal{E}_2|_{\underline{X}}$ an isomorphism of $\mathcal{O}_{\underline{X}}$-modules. Write $\xi_1 \stackrel{\mathrm{def}}{=} (X, \mathcal{E}_1)$, $\xi_2 \stackrel{\mathrm{def}}{=} (X, \mathcal{E}_2)$ for the objects of the category* LcFr *determined by $\mathcal{E}_1$, $\mathcal{E}_2$, respectively; $\mathcal{I}_X \subseteq \mathcal{O}_X$ for the conormal sheaf of the square-nilpotent closed immersion $\underline{X} \hookrightarrow X$ [i.e., the quasi-coherent ideal of $\mathcal{O}_X$ that defines the closed subscheme of X determined by the closed immersion $\underline{X} \hookrightarrow X$]. Then the following hold:*

(i) *There exists an element*

$$o(\underline{X} \hookrightarrow X, \mathcal{E}_1, \mathcal{E}_2, \phi) \in H^1\big(\underline{X}, \mathcal{I}_X \otimes_{\mathcal{O}_{\underline{X}}} \mathcal{H}om_{\mathcal{O}_{\underline{X}}}(\mathcal{E}_1|_{\underline{X}}, \mathcal{E}_2|_{\underline{X}})\big)$$

that satisfies the following condition: Let Y be a scheme over S and $f\colon Y \to X$ a morphism over S. Write $\underline{Y} \stackrel{\mathrm{def}}{=} Y \times_X \underline{X} \hookrightarrow Y$ for the [necessarily square-nilpotent] closed immersion over S obtained by forming the base-change of the closed immersion $\underline{X} \hookrightarrow X$ by the morphism f, $\underline{f}\colon \underline{Y} \to \underline{X}$ for the morphism over S obtained by forming the base-change of the morphism f by the closed immersion $\underline{X} \hookrightarrow X$, and $\mathcal{I}_Y \subseteq \mathcal{O}_Y$ for the conormal sheaf of the square-nilpotent closed immersion $\underline{Y} \hookrightarrow Y$. Then it holds that the set

$$\mathrm{Lift}_{\underline{Y} \hookrightarrow Y}(f^*\xi_1, f^*\xi_2; \underline{f}^*\phi)$$

[*cf.* [2], *Definition* 1.6, (iii)] *is* **nonempty** *if and only if the pull-back*

$$\underline{f}^* o(\underline{X} \hookrightarrow X, \mathcal{E}_1, \mathcal{E}_2, \phi) \in H^1\Big(\underline{Y}, \mathcal{I}_Y \otimes_{\mathcal{O}_{\underline{Y}}} \mathcal{H}om_{\mathcal{O}_{\underline{Y}}}\big(\underline{f}^*(\mathcal{E}_1|_{\underline{X}}), \underline{f}^*(\mathcal{E}_2|_{\underline{X}})\big)\Big)$$

is **zero**.

(ii) *Suppose that the set*

$$\mathrm{Lift}_{\underline{X} \hookrightarrow X}(\xi_1, \xi_2; \phi)$$

is **nonempty**. *Then the set* $\mathrm{Lift}_{\underline{X} \hookrightarrow X}(\xi_1, \xi_2; \phi)$ *has a* **natural structure of torsor** *under the module*

$$\Gamma\big(\underline{X}, \mathcal{I}_X \otimes_{\mathcal{O}_{\underline{X}}} \mathcal{H}om_{\mathcal{O}_{\underline{X}}}(\mathcal{E}_1|_{\underline{X}}, \mathcal{E}_2|_{\underline{X}})\big).$$

PROOF. — These assertions follow from elementary deformation theory. □

LEMMA 1.3. — *The category* LcFr *fibered in groupoids over* Sch_S *satisfies condition* (1) *of* [2], *Definition* 1.7.

PROOF. — Let

$$\begin{array}{ccc} \underline{X} & \xrightarrow{\ \underline{f}\ } & \underline{Y} \\ {\scriptstyle j_X}\big\downarrow & & \big\downarrow{\scriptstyle j_Y} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

be a *cartesian* diagram in Sch_S such that the morphism j_Y is a *square-nilpotent* closed immersion, and, moreover, the morphism f is *conormally strict* with respect to the square-nilpotent closed immersion j_Y [cf. [2], Definition 1.3], which thus implies that there exist schemes $\overline{X}$, $\overline{Y}$ and closed immersions $i_X : \overline{X} \hookrightarrow X$, $i_Y : \overline{Y} \hookrightarrow Y$ that satisfy conditions (3), (4) of [2], Definition 1.3. Moreover, let $\mathcal{E}_1$, $\mathcal{E}_2$ be locally free $\mathcal{O}_Y$-modules; $\phi\colon \mathcal{E}_1|_{\underline{Y}} \xrightarrow{\sim} \mathcal{E}_2|_{\underline{Y}}$ an isomorphism of $\mathcal{O}_{\underline{Y}}$-modules. Write $\mathcal{I}_X$, $\mathcal{I}_Y$ for the conormal sheaves of the square-nilpotent closed immersions $j_X : \underline{X} \hookrightarrow X$, $j_Y : \underline{Y} \hookrightarrow Y$, respectively; $\xi_1 \overset{\mathrm{def}}{=} (X, \mathcal{E}_1)$, $\xi_2 \overset{\mathrm{def}}{=} (X, \mathcal{E}_2)$ for the objects of the category LcFr determined by $\mathcal{E}_1$, $\mathcal{E}_2$, respectively. Then, to verify Lemma 1.3, it suffices to verify the *bijectivity* of the map induced by f

$$\mathrm{Lift}_{j_Y}(\xi_1, \xi_2; \phi) \longrightarrow \mathrm{Lift}_{j_X}(f^*\xi_1, f^*\xi_2; \underline{f}^*\phi).$$

To this end, suppose that the codomain of this map is *nonempty*.

First, let us verify the following claim:

Claim 1.3.A: The domain of the map under consideration is *nonempty*.

To this end, let us recall from condition (3) of [2], Definition 1.3, that the closed immersions $\iota_X : \overline{X} \hookrightarrow X$, $\iota_Y : \overline{X} \hookrightarrow Y$ induce *isomorphisms* of modules

$$H^1\big(X, \mathcal{I}_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{E}_1, f^*\mathcal{E}_2)\big) \xrightarrow{\ \sim\ } H^1\Big(\overline{X}, i_X^*\mathcal{I}_X \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{H}om_{\mathcal{O}_{\overline{X}}}\big((f^*\mathcal{E}_1)|_{\overline{X}}, (f^*\mathcal{E}_2)|_{\overline{X}}\big)\Big),$$

$$H^1\big(Y, \mathcal{I}_Y \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{E}_1, \mathcal{E}_2)\big) \xrightarrow{\ \sim\ } H^1\big(\overline{Y}, i_Y^*\mathcal{I}_Y \otimes_{\mathcal{O}_{\overline{Y}}} \mathcal{H}om_{\mathcal{O}_{\overline{Y}}}(\mathcal{E}_1|_{\overline{Y}}, \mathcal{E}_2|_{\overline{Y}})\big),$$

respectively. Moreover, it follows from condition (4) of [2], Definition 1.3, and [2], Lemma 1.4, that the homomorphism induced by f

$$H^1\big(\overline{Y}, i_Y^*\mathcal{I}_Y \otimes_{\mathcal{O}_{\overline{Y}}} \mathcal{H}om_{\mathcal{O}_{\overline{Y}}}(\mathcal{E}_1|_{\overline{Y}}, \mathcal{E}_2|_{\overline{Y}})\big) \longrightarrow H^1\Big(\overline{X}, i_X^*\mathcal{I}_X \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{H}om_{\mathcal{O}_{\overline{X}}}\big((f^*\mathcal{E}_1)|_{\overline{X}}, (f^*\mathcal{E}_2)|_{\overline{X}}\big)\Big)$$

is an *isomorphism*. Thus, since [we have assumed that] the codomain of the map under consideration is *nonempty*, it follows from Lemma 1.2, (i), that the domain of the map under consideration is *nonempty*, as desired. This completes the proof of Claim 1.3.A.

Next, let us verify the desired *bijectivity*. To this end, let us recall from condition (3) of [2], Definition 1.3, that the closed immersions $\iota_X : \overline{X} \hookrightarrow X$, $\iota_Y : \overline{X} \hookrightarrow Y$ induce *isomorphisms* of modules

$$\Gamma\big(X, \mathcal{I}_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{E}_1, f^*\mathcal{E}_2)\big) \xrightarrow{\ \sim\ } \Gamma\Big(\overline{X}, i_X^*\mathcal{I}_X \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{H}om_{\mathcal{O}_{\overline{X}}}\big((f^*\mathcal{E}_1)|_{\overline{X}}, (f^*\mathcal{E}_2)|_{\overline{X}}\big)\Big),$$

$$\Gamma\big(Y, \mathcal{I}_Y \otimes_{\mathcal{O}_Y} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{E}_1, \mathcal{E}_2)\big) \xrightarrow{\sim} \Gamma\big(\overline{Y}, i_Y^* \mathcal{I}_Y \otimes_{\mathcal{O}_{\overline{Y}}} \mathcal{H}om_{\mathcal{O}_{\overline{Y}}}(\mathcal{E}_1|_{\overline{Y}}, \mathcal{E}_2|_{\overline{Y}})\big),$$

respectively. Moreover, it follows from condition (4) of [2], Definition 1.3, and [2], Lemma 1.4, that the homomorphism induced by f

$$\Gamma\big(\overline{Y}, i_Y^* \mathcal{I}_Y \otimes_{\mathcal{O}_{\overline{Y}}} \mathcal{H}om_{\mathcal{O}_{\overline{Y}}}(\mathcal{E}_1|_{\overline{Y}}, \mathcal{E}_2|_{\overline{Y}})\big) \longrightarrow \Gamma\Big(\overline{X}, i_X^* \mathcal{I}_X \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{H}om_{\mathcal{O}_{\overline{X}}}\big((f^*\mathcal{E}_1)|_{\overline{X}}, (f^*\mathcal{E}_2)|_{\overline{X}}\big)\Big)$$

is an *isomorphism*. Thus, it follows from Lemma 1.2, (ii), together with Claim 1.3.A, that the map under consideration is *bijective*, as desired. This completes the proof of Lemma 1.3. □

LEMMA 1.4. — *The category* LcFr *fibered in groupoids over* Sch_S *satisfies condition* (2) *of* [2], *Definition* 1.7.

PROOF. — Let

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow[d]{} & W \end{array}$$

be a commutative diagram in Sch_S which is *strictly cocartesian* [cf. [2], Definition 1.5]; V a scheme over S;

$$V \xrightarrow{e} X, \qquad W \underset{f_2}{\overset{f_1}{\rightrightarrows}} V$$

quasi-nil-retraction-like morphisms [cf. [2], Definition 1.2, (iii)] over S such that

$$f_1 \circ c \circ a \circ e = (f_1 \circ d \circ b \circ e =)\, f_2 \circ c \circ a \circ e = (f_2 \circ d \circ b \circ e =)\, \mathrm{id}_V;$$

$\mathcal{E}$ a locally free $\mathcal{O}_V$-module; $\phi\colon c^*\mathcal{E}_1 \xrightarrow{\sim} c^*\mathcal{E}_2$ an isomorphism of $\mathcal{O}_Y$-modules — where we write $\mathcal{E}_i \stackrel{\mathrm{def}}{=} f_i^*\mathcal{E}$ $[i \in \{1, 2\}]$ — such that the pull-back $e^*a^*\phi$ is the identity automorphism of $\mathcal{E} = e^*a^*c^*\mathcal{E}_1 = e^*a^*c^*\mathcal{E}_2$. Write $\xi_1 \stackrel{\mathrm{def}}{=} (W, \mathcal{E}_1)$, $\xi_2 \stackrel{\mathrm{def}}{=} (W, \mathcal{E}_2)$ for the objects of the category LcFr determined by $\mathcal{E}_1$, $\mathcal{E}_2$, respectively. Then, to verify Lemma 1.4, it suffices to verify the *bijectivity* of the map induced by d

$$\mathrm{Lift}_c(\xi_1, \xi_2; \phi) \longrightarrow \mathrm{Lift}_b(d^*\xi_1, d^*\xi_2; a^*\phi).$$

Now let us observe that since each of the morphisms a, b, c, d, e, f_1, f_2 is an *affine* morphism [cf. [2], Remark 1.2.1], to verify the desired *bijectivity*, we may assume without loss of generality, by replacing V by a suitable open subscheme of V, that

- the scheme V — hence also the schemes X, Y, Z, and W — is *affine*,
- both $M_1 \stackrel{\mathrm{def}}{=} \Gamma(W, \mathcal{E}_1)$ and $M_2 \stackrel{\mathrm{def}}{=} \Gamma(W, \mathcal{E}_2)$ are *flat* $\Gamma(W, \mathcal{O}_W)$-modules, and, moreover,
- if one regards the rings $A_Y \stackrel{\mathrm{def}}{=} \Gamma(Y, \mathcal{O}_Y)$, $A_Z \stackrel{\mathrm{def}}{=} \Gamma(Z, \mathcal{O}_Z)$, $A_W \stackrel{\mathrm{def}}{=} \Gamma(W, \mathcal{O}_W)$ as subrings of $A_X \stackrel{\mathrm{def}}{=} \Gamma(X, \mathcal{O}_X)$ by the relevant injective homomorphisms [cf. condition (1) of [2], Definition 1.5], then the equality

$$A_W = A_Y \cap A_Z$$

in A_X holds [cf. condition (2) of [2], Definition 1.5].

Then since the homomorphism $A_W \to A_Z$ is *injective*, the *injectivity* of the map under consideration is immediate. Moreover, to verify the *surjectivity* of the map under consideration, it suffices to verify the following claim:

> Claim 1.4.A: Suppose that we are given an isomorphism $\phi_Y\colon M_1 \otimes_{A_W} A_Y \xrightarrow{\sim} M_2 \otimes_{A_W} A_Y$ of A_Y-modules and an isomorphism $\phi_Z\colon M_1 \otimes_{A_W} A_Z \xrightarrow{\sim} M_2 \otimes_{A_W} A_Z$ of A_Z-modules such that $\phi_Y \otimes_{A_Y} A_X = \phi_Z \otimes_{A_Z} A_X$. Then there exists an isomorphism $\phi_W\colon M_1 \xrightarrow{\sim} M_2$ of A_W-modules such that $\phi_W \otimes_{A_W} A_Y = \phi_Y$ and $\phi_W \otimes_{A_W} A_Z = \phi_Z$.

To this end, let us observe that it follows immediately from the equality $\phi_Y \otimes_{A_Y} A_X = \phi_Z \otimes_{A_Z} A_X$ that the diagram of natural injective homomorphisms of modules

$$\begin{array}{ccccc} & & M_1 \otimes_{A_W} A_Y & \xrightarrow[\sim]{\phi_Y} & M_2 \otimes_{A_W} A_Y & & \\ & \nearrow & & & & \searrow & \\ M_1 & & & & & & M_2 \otimes_{A_W} A_X \\ & \searrow & & & & \nearrow & \\ & & M_1 \otimes_{A_W} A_Z & \xrightarrow[\sim]{\phi_Z} & M_2 \otimes_{A_W} A_Z & & \end{array}$$

is *commutative*, which thus implies that the image of these two composite [i.e., from M_1 to $M_2 \otimes_{A_W} A_X$] is *contained* in the intersection $(M_2 \otimes_{A_W} A_Y) \cap (M_2 \otimes_{A_W} A_Z)$ [i.e., in $M_2 \otimes_{A_W} A_X$]. Thus, since M_2 is a *flat* A_W-module, we conclude immediately from the equality $A_W = A_Y \cap A_Z$ that the above two composites determine a [single] homomorphism $\phi_W\colon M_1 \to M_2$ of A_W-modules such that $\phi_W \otimes_{A_W} A_Y = \phi_Y$ and $\phi_W \otimes_{A_W} A_Z = \phi_Z$. Moreover, by applying a similar argument to this argument to ϕ_Y^{-1} and ϕ_Z^{-1}, we conclude that ϕ_W is an *isomorphism*. This completes the proof of Claim 1.4.A, hence also of Lemma 1.4. □

PROPOSITION 1.5. — *The category* LcFr *fibered in groupoids over* Sch_S *is* **integrable** [*cf.* [2], *Definition* 1.7], *hence also* [*cf.* [2], *Remark* 1.8.1] **weakly integrable** [*cf.* [2], *Definition* 1.8].

PROOF. — Let us observe that one verifies easily that the category LcFr fibered in groupoids over Sch_S satisfies condition (3) of [2], Definition 1.7. Thus, Proposition 1.5 follows from Lemma 1.3 and Lemma 1.4. This completes the proof of Proposition 1.5. □

DEFINITION 1.6. — Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. Then we shall refer to a homomorphism of $(X \to S)^{-1}\mathcal{O}_S$-modules

$$\nabla\colon \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

as a *classical connection* on $\mathcal{E}$ if the equality $\nabla(fe) = f\nabla(e) + e \otimes df$ — where f, e are local sections of $\mathcal{O}_X$, $\mathcal{E}$, respectively — holds.

DEFINITION 1.7. — We shall write

$$\mathrm{SmSch} \longrightarrow \mathrm{Sch}_S$$

for the category fibered in groupoids over Sch_S defined as follows:

- An object of the category SmSch is a smooth morphism $Z \to X$ of schemes over S.
- If $\xi\colon Z \to X$ and $\xi'\colon Z' \to X'$ are objects of the category SmSch, then a morphism $(\xi\colon Z \to X) \to (\xi'\colon Z' \to X')$ in the category SmSch is defined to be a pair (f_Z, f_X) consisting of morphisms $f_Z\colon Z \to Z'$ and $f_X\colon X \to X'$ of schemes over S such that the diagram in Sch_S

$$\begin{array}{ccc} Z & \xrightarrow{f_Z} & Z' \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi'} \\ X & \xrightarrow[f_X]{} & X' \end{array}$$

is commutative and induces an isomorphism $Z \xrightarrow{\sim} X \times_{X'} Z'$.

- The functor $\mathrm{SmSch} \to \mathrm{Sch}_S$ sends "$Z \to X$" to "X" and "(f_Z, f_X)" to "f_X".

LEMMA 1.8. — *Let X, $\underline{X}$ be schemes over S; $\underline{X} \hookrightarrow X$ a* **square-nilpotent** *closed immersion over S; $\xi_1\colon Z_1 \to X$, $\xi_2\colon Z_2 \to X$ smooth morphisms over S; $\phi\colon Z_1 \times_X \underline{X} \xrightarrow{\sim} Z_2 \times_X \underline{X}$ an isomorphism over $\underline{X}$. Write $\underline{Z} \stackrel{\mathrm{def}}{=} Z_1 \times_X \underline{X}$, $\underline{\xi}\colon \underline{Z} \to \underline{X}$ for the [necessarily smooth] morphism over S obtained by forming the base-change of the morphism ξ_1 by the closed immersion $\underline{X} \hookrightarrow X$, and $\mathcal{I}_X \subseteq \mathcal{O}_X$ for the conormal sheaf of the square-nilpotent closed immersion $\underline{X} \hookrightarrow X$. Then the following hold:*

(i) *There exists an element*

$$o(\underline{X} \hookrightarrow X, \xi_1, \xi_2, \phi) \in H^1(\underline{Z}, \underline{\xi}^*\mathcal{I}_X \otimes_{\mathcal{O}_{\underline{Z}}} \mathcal{T}_{\underline{Z}/\underline{X}})$$

that satisfies the following condition: Let Y be a scheme over S and $f\colon Y \to X$ a morphism over S. Write $\underline{Y} \stackrel{\mathrm{def}}{=} Y \times_X \underline{X} \hookrightarrow Y$ for the [necessarily square-nilpotent] closed immersion over S obtained by forming the base-change of the closed immersion $\underline{X} \hookrightarrow X$ by the morphism f, $\underline{f}\colon \underline{Y} \to \underline{X}$ for the morphism over S obtained by forming the base-change of the morphism f by the closed immersion $\underline{X} \hookrightarrow X$, $\underline{f}_Z\colon \underline{Z} \times_{\underline{X}} \underline{Y} \to \underline{Z}$ for the morphism over S obtained by forming the base-change of the morphism $\underline{f}$ by the morphism $\underline{\xi}$, $\underline{\xi}_Y\colon \underline{Z} \times_{\underline{X}} \underline{Y} \to \underline{Y}$ for the morphism over S obtained by forming the base-change of the morphism $\underline{\xi}$ by the morphism $\underline{f}$, and $\mathcal{I}_Y \subseteq \mathcal{O}_Y$ for the conormal sheaf of the square-nilpotent closed immersion $\underline{Y} \hookrightarrow Y$. Then it holds that the set

$$\mathrm{Lift}_{\underline{Y} \hookrightarrow Y}(f^*\xi_1, f^*\xi_2; \underline{f}^*\phi)$$

is **nonempty** *if and only if the pull-back*

$$\underline{f}_Z^* o(\underline{X} \hookrightarrow X, \xi_1, \xi_2, \phi) \in H^1(\underline{Z} \times_{\underline{X}} \underline{Y}, \underline{\xi}_Y^*\mathcal{I}_Y \otimes_{\mathcal{O}_{\underline{Z} \times_{\underline{X}} \underline{Y}}} \mathcal{T}_{\underline{Z} \times_{\underline{X}} \underline{Y}/\underline{Y}})$$

is **zero**.

(ii) *Suppose that the set*

$$\mathrm{Lift}_{\underline{X}\hookrightarrow X}(\xi_1, \xi_2; \phi)$$

is **nonempty**. *Then the set* $\mathrm{Lift}_{\underline{X}\hookrightarrow X}(\xi_1, \xi_2; \phi)$ *has a* **natural structure of torsor** *under the module*

$$\Gamma(\underline{Z}, \underline{\xi}^* \mathcal{I}_X \otimes_{\mathcal{O}_{\underline{Z}}} \mathcal{T}_{\underline{Z}/\underline{X}}).$$

PROOF. — These assertions follow from elementary deformation theory. □

LEMMA 1.9. — *The category* SmSch *fibered in groupoids over* Sch_S *satisfies condition* (1) *of* [2], *Definition* 1.7.

PROOF. — This assertion follows immediately from a similar argument to the argument applied in the proof of Lemma 1.3, together with Lemma 1.8. □

LEMMA 1.10. — *The category* SmSch *fibered in groupoids over* Sch_S *satisfies condition* (2) *of* [2], *Definition* 1.7.

PROOF. — Let

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ Z & \xrightarrow[d]{} & W \end{array}$$

be a commutative diagram in Sch_S which is *strictly cocartesian*; V a scheme over S;

$$V \xrightarrow{e} X, \qquad W \underset{f_2}{\overset{f_1}{\rightrightarrows}} V$$

quasi-nil-retraction-like morphisms over S such that

$$f_1 \circ c \circ a \circ e = (f_1 \circ d \circ b \circ e =)\, f_2 \circ c \circ a \circ e = (f_2 \circ d \circ b \circ e =)\, \mathrm{id}_V;$$

T a scheme over S; $\xi \colon T \to V$ a smooth morphism over S; $\phi \colon T_1 \times_W Y \xrightarrow{\sim} T_2 \times_W Y$ an isomorphism over Y — where we write T_i for the fiber product of ξ and f_i [$i \in \{1, 2\}$] — such that the pull-back $e^*a^*\phi$ is the identity automorphism of $T = e^*a^*c^*T_1 = e^*a^*c^*T_2$. Write $\xi_1 \colon T_1 \to W$, $\xi_2 \colon T_2 \to W$ for the objects of the category SmSch determined by T_1, T_2, respectively. Then, to very Lemma 1.10, it suffices to verify the *bijectivity* of the map induced by d

$$\mathrm{Lift}_c(\xi_1, \xi_2; \phi) \longrightarrow \mathrm{Lift}_b(d^*\xi_1, d^*\xi_2; a^*\phi).$$

Now let us observe that since each of the morphisms a, b, c, d, e, f_1, f_2 is an *affine* morphism [cf. [2], Remark 1.2.1], to verify the desired *bijectivity*, we may assume without loss of generality, by replacing V by a suitable open subscheme of V, that

- the scheme V — hence also the schemes X, Y, Z, and W — is *affine*, and, moreover,

• if one regards the rings $A_Y \stackrel{\text{def}}{=} \Gamma(Y, \mathcal{O}_Y)$, $A_Z \stackrel{\text{def}}{=} \Gamma(Z, \mathcal{O}_Z)$, $A_W \stackrel{\text{def}}{=} \Gamma(W, \mathcal{O}_W)$ as subrings of $A_X \stackrel{\text{def}}{=} \Gamma(X, \mathcal{O}_X)$ by the relevant injective homomorphisms [cf. condition (1) of [2], Definition 1.5], then the equality

$$A_W = A_Y \cap A_Z$$

in A_X holds [cf. condition (2) of [2], Definition 1.5].

Moreover, let us observe that since [we have assumed that] the pull-back $e^*a^*\phi$ is the *identity automorphism* of T, one verifies easily that every element of each of the two sets $\mathrm{Lift}_c(\xi_1, \xi_2; \phi)$, $\mathrm{Lift}_b(d^*\xi_1, d^*\xi_2; a^*\phi)$ induces the *"identity automorphism"* of the underlying topological space, i.e., relative to the "identifications" of the underlying topological spaces of T, T_1, T_2, $T_1 \times_W Z$, and $T_2 \times_W Z$ determined by the relevant morphisms of schemes [cf. [2], Remark 1.2.1]. In particular, to verify the desired *bijectivity*, we may assume without loss of generality, by replacing T by a suitable open subscheme of T, that T is *affine* [which thus implies that $\Gamma(T, \mathcal{O}_T)$ is *flat* over $\Gamma(V, \mathcal{O}_V)$]. Then the desired *bijectivity* follows immediately from a similar argument to the argument applied in the proof of Lemma 1.4. This completes the proof of Lemma 1.10. □

PROPOSITION 1.11. — *The category* SmSch *fibered in groupoids over* Sch_S *is* **integrable**, *hence also* [*cf.* [2], *Remark* 1.8.1] **weakly integrable**.

PROOF. — Let us observe that one verifies easily that the category SmSch fibered in groupoids over Sch_S satisfies condition (3) of [2], Definition 1.7. Thus, Proposition 1.11 follows from Lemma 1.9 and Lemma 1.10. This completes the proof of Proposition 1.11. □

2. DIVIDED POWER STRATIFICATION STRUCTURES

In the present §2, we introduce and discuss the notion of a *divided power stratification structure* on a smooth scheme [cf. Definition 2.5 below]. In the present §2, let S be a scheme and X a scheme which is *smooth* and *separated* over S.

DEFINITION 2.1.

(i) We shall write

$$\left({}^{\mathrm{PD}}P \longrightarrow X^{(2)},\ {}^{\mathrm{PD}}\mathcal{I}^{[1]} \subseteq \mathcal{O}_{{}^{\mathrm{PD}}P},\ \left((-)^{[n]} \colon {}^{\mathrm{PD}}\mathcal{I}^{[1]} \longrightarrow \mathcal{O}_{{}^{\mathrm{PD}}P}\right)_{n \geq 0}\right)$$

for the divided power envelope of the diagonal closed immersion $X \hookrightarrow X^{(2)}$ with respect to a divided power structure on a quasi-coherent ideal of $\mathcal{O}_S$ [cf. Remark 2.1.1 below]. Thus, we have a sequence of quasi-coherent ideals of $\mathcal{O}_{{}^{\mathrm{PD}}P}$

$$\ldots \subseteq {}^{\mathrm{PD}}\mathcal{I}^{[n+1]} \subseteq {}^{\mathrm{PD}}\mathcal{I}^{[n]} \subseteq \ldots \subseteq {}^{\mathrm{PD}}\mathcal{I}^{[2]} \subseteq {}^{\mathrm{PD}}\mathcal{I}^{[1]} \subseteq \mathcal{O}_{{}^{\mathrm{PD}}P}.$$

(ii) Let n be a nonnegative integer. Then we shall write

$${}^{\mathrm{PD}}P^n \subseteq {}^{\mathrm{PD}}P$$

for the closed subscheme of ${}^{\mathrm{PD}}P$ defined by the quasi-coherent ideal ${}^{\mathrm{PD}}\mathcal{I}^{[n+1]} \subseteq \mathcal{O}_{{}^{\mathrm{PD}}P}$ of $\mathcal{O}_{{}^{\mathrm{PD}}P}$. Thus, we have a sequence of closed subschemes of ${}^{\mathrm{PD}}P$

$$ {}^{\mathrm{PD}}P \supseteq \ldots \supseteq {}^{\mathrm{PD}}P^{n+1} \supseteq {}^{\mathrm{PD}}P^{n} \supseteq \ldots \supseteq {}^{\mathrm{PD}}P^{2} \supseteq {}^{\mathrm{PD}}P^{1} \supseteq {}^{\mathrm{PD}}P^{0}. $$

(iii) Let n be a nonnegative integer. Then we shall write

$$ {}^{\mathrm{PD}}\iota^{n} \colon {}^{\mathrm{PD}}P^{n} \hookrightarrow {}^{\mathrm{PD}}P^{n+1} $$

for the natural closed immersion over S.

(iv) Let n be a nonnegative integer. Then we shall write

$$ {}^{\mathrm{PD}}\sigma^{n} \colon {}^{\mathrm{PD}}P^{n} \longrightarrow X^{(2)} $$

for the morphism over S obtained by forming the composite of the natural closed immersion ${}^{\mathrm{PD}}P^{n} \hookrightarrow {}^{\mathrm{PD}}P$ and the natural morphism ${}^{\mathrm{PD}}P \to X^{(2)}$.

REMARK 2.1.1. — It follows from [1], Corollary 3.22, that an arbitrary divided power structure on a quasi-coherent ideal of $\mathcal{O}_S$ *extends* to $\mathcal{O}_X$. Thus, it follows from [1], Remarks 3.20, (6), that the divided power envelope of the diagonal closed immersion $X \hookrightarrow X^{(2)}$ does *not depend* on the choice of a divided power structure on a quasi-coherent ideal of $\mathcal{O}_S$. In particular, one may take the "divided power structure" of Definition 2.1, (i), to be the "trivial divided power structure" [i.e., on the zero ideal of $\mathcal{O}_S$].

LEMMA 2.2. — *The collection of data*

$$ \big(({}^{\mathrm{PD}}P^{n})_{n\geq 0},\ ({}^{\mathrm{PD}}\iota^{n} \colon {}^{\mathrm{PD}}P^{n} \to {}^{\mathrm{PD}}P^{n+1})_{n\geq 0},\ ({}^{\mathrm{PD}}\sigma^{n} \colon {}^{\mathrm{PD}}P^{n} \to X^{(2)})_{n\geq 0}\big) $$

forms a **pre-stratification structure** [*cf.* [2]*, Definition* 2.2] *on* X/S*.*

PROOF. — It is immediate from the definitions of "${}^{\mathrm{PD}}\iota^{n}$" and "${}^{\mathrm{PD}}\sigma^{n}$" that the collection of data under consideration satisfies condition (1) of [2], Definition 2.2. Moreover, it follows from [1], Remarks 3.20, (4), and [1], Corollary 3.22, that the collection of data under consideration satisfies condition (2) of [2], Definition 2.2. This completes the proof of Lemma 2.2. □

DEFINITION 2.3. — Let n_1, n_2 be nonnegative integers. Thus, we have the scheme ${}^{\mathrm{PD}}P^{n_1,n_2}$ over S [cf. [2], Definition 2.4, (i), in the case where we take the "$\mathfrak{S}$" to be the pre-stratification structure of Lemma 2.2]. Now let us recall from the discussion preceding [1], Definition 4.3, that we have a morphism ${}^{\mathrm{PD}}P^{n_1,n_2} \to {}^{\mathrm{PD}}P^{n_1+n_2}$ over S. We shall write

$$ {}^{\mathrm{PD}}\delta^{n_1,n_2} \colon {}^{\mathrm{PD}}P^{n_1,n_2} \longrightarrow {}^{\mathrm{PD}}P^{n_1+n_2} $$

for this morphism.

PROPOSITION 2.4. — *The collection of data*

$$\mathrm{PD} = \big(({}^{\mathrm{PD}}P^n)_{n\geq 0},\ ({}^{\mathrm{PD}}\iota^n\colon {}^{\mathrm{PD}}P^n \to {}^{\mathrm{PD}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathrm{PD}}\sigma^n\colon {}^{\mathrm{PD}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathrm{PD}}\delta^{n_1,n_2}\colon {}^{\mathrm{PD}}P^{n_1,n_2} \to {}^{\mathrm{PD}}P^{n_1+n_2})_{n_1,\,n_2\geq 0}\big)$$

forms a **stratification structure** [*cf.* [2], *Definition* 2.5] *on* X/S.

PROOF. — This assertion follows immediately from Lemma 2.2, together with the definition of "${}^{\mathrm{PD}}\delta^{n_1,n_2}$". □

DEFINITION 2.5. — We shall refer to the stratification structure of Proposition 2.4

$$\mathrm{PD} = \big(({}^{\mathrm{PD}}P^n)_{n\geq 0},\ ({}^{\mathrm{PD}}\iota^n\colon {}^{\mathrm{PD}}P^n \to {}^{\mathrm{PD}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathrm{PD}}\sigma^n\colon {}^{\mathrm{PD}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathrm{PD}}\delta^{n_1,n_2}\colon {}^{\mathrm{PD}}P^{n_1,n_2} \to {}^{\mathrm{PD}}P^{n_1+n_2})_{n_1,\,n_2\geq 0}\big)$$

as the *divided power stratification structure* on X/S.

PROPOSITION 2.6. — *Let* $\mathcal{E}$ *be a locally free* $\mathcal{O}_X$*-module. Write* $\xi = (X, \mathcal{E})$ *for the object of the category* LcFr *fibered in groupoids over* Sch_S *of Definition* 1.1 *determined by* $\mathcal{E}$. *Then the following hold:*

(i) *The notion of a* **divided power stratification** *on the locally free* $\mathcal{O}_X$*-module* $\mathcal{E}$ *in the sense of* [1], *Definition* 4.3, *is the* **same** *as the notion of a* **PD-stratification** *on* ξ *in the sense of Definition* 2.5 *and* [2], *Definition* 4.6*:*

$$\{\textit{divided power stratifications on } \mathcal{E} \textit{ in the sense of } [1],\ \textit{Definition } 4.3\} = {}^{\mathrm{PD}}\mathrm{Strt}(\xi)$$

[*cf. Definition* 2.5*;* [2], *Definition* 4.6].

(ii) *There exists a natural* **bijection** *between*

- *the set of* **classical connections** [*cf. Definition* 1.6] *on the locally free* $\mathcal{O}_X$*-module* $\mathcal{E}$ *and*
- *the set of* **PD-connections** *on* ξ [*cf. Definition* 2.5*;* [2], *Definition* 4.1, (iii)]

$${}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi)$$

[*cf. Definition* 2.5*;* [2], *Remark* 4.1.1*;* [2], *Definition* 4.4]

that is **compatible**, *in the evident sense, with the identification of* (i)

$$\{\textit{divided power stratifications on } \mathcal{E} \textit{ in the sense of } [1],\ \textit{Definition } 4.3\} = {}^{\mathrm{PD}}\mathrm{Strt}(\xi).$$

PROOF. — These assertions follow immediately — in light of [1], Proposition 2.9, and [1], Remark 4.2 — from the various definitions involved. □

LEMMA 2.7. — *Let* n *be a nonnegative integer. Then the closed immersion* ${}^{\mathrm{PD}}\iota^n\colon {}^{\mathrm{PD}}P^n \hookrightarrow {}^{\mathrm{PD}}P^{n+1}$ *is* **square-nilpotent**. *In particular, the stratification structure* PD *is* **ι-quasi-nil-retraction-like** [*cf.* [2], *Definition* 3.1, (i)].

PROOF. — This assertion follows from [1], Proposition 3.25 [cf. also [2], Remark 1.2.1]. □

LEMMA 2.8. — *Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \overset{\text{def}}{=} \sum_{i=1}^{r} n_i$. Then the morphisms in the diagram of schemes over S*

$$\begin{array}{ccc} {}^{\mathrm{PD}}P^{\times n} & \xrightarrow{{}^{\mathrm{PD}}\delta^{\times n}_{n_1,\ldots,n_r}} & {}^{\mathrm{PD}}P^{n_1,\ldots,n_r} \\ & {\scriptstyle {}^{\mathrm{PD}}\delta^{\times n}} \searrow \quad \swarrow {\scriptstyle {}^{\mathrm{PD}}\delta^{n_1,\ldots,n_r}} & \\ & {}^{\mathrm{PD}}P^{n} & \end{array}$$

of [2], *Definition* 2.6 [*i.e., in the case where we take the "$\mathfrak{S}$" to be the divided power stratification structure* PD], *are* **quasi-nil-retraction-like**.

PROOF. — This assertion follows from Lemma 2.7 and [2], Remark 3.1.1, (iii). □

3. DIVIDED POWER POLYNOMIAL ALGEBRAS

In the present §3, we discuss *divided power polynomial algebras*. The results obtained in the present §3 will be applied in §4 to prove that the divided power stratification structures have some good properties.

In the present §3, let R be a ring and A an R-algebra. Suppose that there exist a nonnegative integer d and d elements $a_1, \ldots, a_d \in A$ of A such that the a_i's determine an *étale* morphism $R[s_1, \ldots, s_d] \to A$ over R — where $s_1, \ldots, s_d$ are indeterminates. Write

$$A^{(2)} \overset{\text{def}}{=} A \otimes_R A;$$

$$I \subseteq A^{(2)}$$

for the kernel of the multiplication $A^{(2)} \twoheadrightarrow A$;

$${}^{X}_{\sharp}\mathrm{pr}^{(2)}_1,\ {}^{X}_{\sharp}\mathrm{pr}^{(2)}_2 \colon A \longrightarrow A^{(2)}$$

for the homomorphisms given by mapping $a \in A$ to $a \otimes 1,\ 1 \otimes a \in A^{(2)}$, respectively. Thus, for each nonnegative integer n, if one regards the ring $A^{(2)}/I^{n+1}$ as an A-algebra by ${}^{X}_{\sharp}\mathrm{pr}^{(2)}_1$ (respectively, ${}^{X}_{\sharp}\mathrm{pr}^{(2)}_2$), then the images of

$${}^{X}_{\sharp}\mathrm{pr}^{(2)}_2(a_1) - {}^{X}_{\sharp}\mathrm{pr}^{(2)}_1(a_1), \ldots, {}^{X}_{\sharp}\mathrm{pr}^{(2)}_2(a_d) - {}^{X}_{\sharp}\mathrm{pr}^{(2)}_1(a_d)$$

in $A^{(2)}/I^{n+1}$ determine an *isomorphism* $A[t_1, \ldots, t_d]/(t_1, \ldots, t_d)^{n+1} \xrightarrow{\sim} A^{(2)}/I^{n+1}$ of A-algebras — where $t_1, \ldots, t_d$ are indeterminates.

DEFINITION 3.1.

(i) We shall write

$$\left({}^{\mathrm{PD}}A \overset{\text{def}}{=} A\langle x_1, \ldots, x_d\rangle,\ {}^{\mathrm{PD}}I^{[1]} \subseteq {}^{\mathrm{PD}}A,\ \left((-)^{[n]} \colon {}^{\mathrm{PD}}I^{[1]} \longrightarrow {}^{\mathrm{PD}}A\right)_{n \geq 0}\right)$$

for the divided power polynomial A-algebra on the indeterminates $\{x_1, \ldots, x_d\}$ [cf., e.g., [1], Theorem 3.9]. Thus, we have a sequence of ideals of ${}^{\mathrm{PD}}A$

$$\ldots \subseteq {}^{\mathrm{PD}}I^{[n+1]} \subseteq {}^{\mathrm{PD}}I^{[n]} \subseteq \ldots \subseteq {}^{\mathrm{PD}}I^{[2]} \subseteq {}^{\mathrm{PD}}I^{[1]} \subseteq {}^{\mathrm{PD}}A$$

[cf. [1], Definition 3.24].

(ii) Let n be a nonnegative integer. Then we shall write

$${}^{\mathrm{PD}}A^n \stackrel{\mathrm{def}}{=} {}^{\mathrm{PD}}A/{}^{\mathrm{PD}}I^{[n+1]}$$

for the quotient of ${}^{\mathrm{PD}}A$ by the ideal ${}^{\mathrm{PD}}I^{[n+1]} \subseteq {}^{\mathrm{PD}}A$ of ${}^{\mathrm{PD}}A$.

(iii) Let n be a nonnegative integer. Then we shall write

$${}^{\mathrm{PD}}_{\sharp}\iota^n \colon {}^{\mathrm{PD}}A^{n+1} \longrightarrow\!\!\!\!\to {}^{\mathrm{PD}}A^n$$

for the natural surjective homomorphism.

(iv) Let n be a nonnegative integer. Then if one regards the ring $A^{(2)}/I^{n+1}$ as an A-algebra by ${}^{X}_{\sharp}\mathrm{pr}_1^{(2)}$, then it follows from the discussion preceding Definition 3.1 that we have a homomorphism of A-algebras

$$A^{(2)}/I^{n+1} \longrightarrow {}^{\mathrm{PD}}A^n$$

given by, for each $i \in \{1, \ldots, d\}$, mapping ${}^{X}_{\sharp}\mathrm{pr}_2^{(2)}(a_i) - {}^{X}_{\sharp}\mathrm{pr}_1^{(2)}(a_i)$ to x_i. We shall write

$${}^{\mathrm{PD}}_{\sharp}\sigma^n \colon A^{(2)} \longrightarrow {}^{\mathrm{PD}}A^n$$

for the homomorphism obtained by forming the composite of the natural surjective homomorphism $A^{(2)} \twoheadrightarrow A^{(2)}/I^{n+1}$ and this homomorphism $A^{(2)}/I^{n+1} \to {}^{\mathrm{PD}}A^n$.

LEMMA 3.2. — *Write*

$$X \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A) \longrightarrow S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R).$$

In particular, we are in the situation of [2], *§*2*, hence also the situation of §*2 *of the present paper, which thus implies that we are given the morphisms*

$${}^{X}\mathrm{pr}_1^{(2)}, {}^{X}\mathrm{pr}_2^{(2)} \colon X^{(2)} \longrightarrow X$$

of [2]*, Definition* 2.1*,* (iii)*, and the pre-stratification structure on X/S of Lemma* 2.2 *of the present paper*

$$\big(({}^{\mathrm{PD}}P^n)_{n\geq 0},\ ({}^{\mathrm{PD}}\iota^n \colon {}^{\mathrm{PD}}P^n \to {}^{\mathrm{PD}}P^{n+1})_{n\geq 0},\ ({}^{\mathrm{PD}}\sigma^n \colon {}^{\mathrm{PD}}P^n \to X^{(2)})_{n\geq 0}\big).$$

Then there exist **isomorphisms** *of schemes*

$$X^{(2)} \xrightarrow{\sim} \mathrm{Spec}(A^{(2)}), \qquad {}^{\mathrm{PD}}P^n \xrightarrow{\sim} \mathrm{Spec}({}^{\mathrm{PD}}A^n)$$

— where n is a nonnegative integer — such that the morphisms of schemes

$${}^{X}\mathrm{pr}_1^{(2)}, \qquad {}^{X}\mathrm{pr}_2^{(2)}, \qquad {}^{\mathrm{PD}}\iota^n, \qquad {}^{\mathrm{PD}}\sigma^n$$

coincide — *relative to the isomorphisms of schemes under consideration — with the morphisms of schemes induced by the homomorphisms*

$$ {}^{X}_{\sharp}\mathrm{pr}_1^{(2)}, \qquad {}^{X}_{\sharp}\mathrm{pr}_2^{(2)}, \qquad {}^{\mathrm{PD}}_{\sharp}\iota^n, \qquad {}^{\mathrm{PD}}_{\sharp}\sigma^n $$

defined in the discussion preceding Definition 3.1 *and Definition* 3.1*, respectively.*

PROOF. — This assertion follows immediately from a similar argument to the argument applied in the proof of [1], Proposition 3.32, together with the various definitions involved. □

DEFINITION 3.3. — Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \stackrel{\mathrm{def}}{=} \sum_{i=1}^{r} n_i$.

(i) Let $m_1, \ldots, m_r$ be nonnegative integers such that $m_i \leq n_i$ for each $i \in \{1, \ldots, r\}$; j an element of $\{1, \ldots, r+1\}$; j' an element of $\{1, \ldots, r\}$. Then it follows from Lemma 3.2 that we obtain rings

$$ {}^{\mathrm{PD}}A^{n_1,\ldots,n_r}, \qquad {}^{\mathrm{PD}}A^{\times r} $$

that "correspond" — relative to the isomorphisms of schemes of Lemma 3.2 — to the schemes

$$ {}^{\mathrm{PD}}P^{n_1,\ldots,n_r}, \qquad {}^{\mathrm{PD}}P^{\times r} $$

defined in [2], Definition 2.4, (i), (vi) [i.e., in the case where we take the "$\mathfrak{S}$" to be the pre-stratification structure of Lemma 2.2], respectively, and homomorphisms

$$ {}^{\mathrm{PD}}_{\sharp}\iota^{m_1,\ldots,m_r}_{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n_1,\ldots,n_r} \longrightarrow {}^{\mathrm{PD}}A^{m_1,\ldots,m_r}, $$

$$ {}^{\mathrm{PD}}_{\sharp}\mathrm{pr}^{n_1,\ldots,n_r}_{j} \colon A \longrightarrow {}^{\mathrm{PD}}A^{n_1,\ldots,n_r}, \qquad {}^{\mathrm{PD}}_{\sharp}\mathrm{pr}^{n_1,\ldots,n_r}_{\{j',j'+1\}} \colon {}^{\mathrm{PD}}A^{n_{j'}} \longrightarrow {}^{\mathrm{PD}}A^{n_1,\ldots,n_r}, $$

$$ {}^{\mathrm{PD}}_{\sharp}\mathrm{pr}^{\times r}_{j} \colon A \longrightarrow {}^{\mathrm{PD}}A^{\times r}, \qquad {}^{\mathrm{PD}}_{\sharp}\mathrm{pr}^{\times r}_{\{j',j'+1\}} \colon {}^{\mathrm{PD}}A^{1} \longrightarrow {}^{\mathrm{PD}}A^{\times r} $$

that "correspond" — relative to the isomorphisms of schemes of Lemma 3.2 — to the morphisms of schemes

$$ {}^{\mathrm{PD}}\iota^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}, \qquad {}^{\mathrm{PD}}\mathrm{pr}^{n_1,\ldots,n_r}_{j}, \qquad {}^{\mathrm{PD}}\mathrm{pr}^{n_1,\ldots,n_r}_{\{j',j'+1\}}, \qquad {}^{\mathrm{PD}}\mathrm{pr}^{\times r}_{j}, \qquad {}^{\mathrm{PD}}\mathrm{pr}^{\times r}_{\{j',j'+1\}} $$

defined in [2], Definition 2.4, (ii), (iv), (v), (vi) [i.e., in the case where we take the "$\mathfrak{S}$" to be the pre-stratification structure of Lemma 2.2], respectively.

(ii) It follows from Lemma 3.2 that we obtain homomorphisms

$$ {}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n_1,\ldots,n_r} \longrightarrow {}^{\mathrm{PD}}A^{\times n}, \qquad {}^{\mathrm{PD}}_{\sharp}\delta^{\times n} \colon {}^{\mathrm{PD}}A^{n} \longrightarrow {}^{\mathrm{PD}}A^{\times n}, $$

$$ {}^{\mathrm{PD}}_{\sharp}\delta^{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n} \longrightarrow {}^{\mathrm{PD}}A^{n_1,\ldots,n_r} $$

that "correspond" — relative to the isomorphisms of schemes of Lemma 3.2 — to the morphisms of schemes

$$ {}^{\mathrm{PD}}\delta^{\times n}_{n_1,\ldots,n_r}, \qquad {}^{\mathrm{PD}}\delta^{\times n}, \qquad {}^{\mathrm{PD}}\delta^{n_1,\ldots,n_r} $$

defined in [2], Definition 2.6 [i.e., in the case where we take the "$\mathfrak{S}$" to be the stratification structure of Definition 2.5], respectively.

(iii) Let i be an element of $\{1, \ldots, r+1\}$. Suppose that $n \geq 1$. Then it follows from Lemma 3.2 that we obtain rings

$$^{\mathrm{PD}}\underline{A}^{n_1,\ldots,n_r}, \qquad ^{\mathrm{PD}}\underline{A}^{\times r}$$

that "correspond" — relative to the isomorphisms of schemes of Lemma 3.2 — to the schemes

$$^{\mathrm{PD}}\underline{P}^{n_1,\ldots,n_r}, \qquad ^{\mathrm{PD}}\underline{P}^{\times r}$$

defined in [2], Definition 2.7, (i), (v) [i.e., in the case where we take the "$\mathfrak{S}$" to be the stratification structure of Definition 2.5], respectively, and homomorphisms

$$^{\mathrm{PD}}_{\sharp}\underline{\iota}^{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n_1,\ldots,n_r} \longrightarrow {}^{\mathrm{PD}}\underline{A}^{n_1,\ldots,n_r}, \qquad ^{\mathrm{PD}}_{\sharp}\underline{\delta}^{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n-1} \longrightarrow {}^{\mathrm{PD}}\underline{A}^{n_1,\ldots,n_r},$$

$$^{\mathrm{PD}}_{\sharp}\underline{\mathrm{pr}}_i^{n_1,\ldots,n_r} \colon A \longrightarrow {}^{\mathrm{PD}}\underline{A}^{n_1,\ldots,n_r}, \qquad ^{\mathrm{PD}}_{\sharp}\underline{\iota}^0_{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}\underline{A}^{n_1,\ldots,n_r} \longrightarrow A,$$

$$^{\mathrm{PD}}_{\sharp}\underline{\iota}^{\times r} \colon {}^{\mathrm{PD}}A^{\times r} \longrightarrow {}^{\mathrm{PD}}\underline{A}^{\times r}, \qquad ^{\mathrm{PD}}_{\sharp}\underline{\delta}^{\times r} \colon {}^{\mathrm{PD}}A^{r-1} \longrightarrow {}^{\mathrm{PD}}\underline{A}^{\times r},$$

$$^{\mathrm{PD}}_{\sharp}\underline{\mathrm{pr}}_i^{\times r} \colon A \longrightarrow {}^{\mathrm{PD}}\underline{A}^{\times r}, \qquad ^{\mathrm{PD}}_{\sharp}\underline{\iota}^0_{\times r} \colon {}^{\mathrm{PD}}\underline{A}^{\times r} \longrightarrow A$$

that "correspond" — relative to the isomorphisms of schemes of Lemma 3.2 — to the morphisms of schemes

$$^{\mathrm{PD}}\underline{\iota}^{n_1,\ldots,n_r}, \qquad ^{\mathrm{PD}}\underline{\delta}^{n_1,\ldots,n_r}, \qquad ^{\mathrm{PD}}\underline{\mathrm{pr}}_i^{n_1,\ldots,n_r}, \qquad ^{\mathrm{PD}}\underline{\iota}^0_{n_1,\ldots,n_r},$$

$$^{\mathrm{PD}}\underline{\iota}^{\times r}, \qquad ^{\mathrm{PD}}\underline{\delta}^{\times r}, \qquad ^{\mathrm{PD}}\underline{\mathrm{pr}}_i^{\times r}, \qquad ^{\mathrm{PD}}\underline{\iota}^0_{\times r}$$

defined in [2], Definition 2.7, (ii), (iii), (iv), (v) [i.e., in the case where we take the "$\mathfrak{S}$" to be the stratification structure of Definition 2.5], respectively.

DEFINITION 3.4. — Let n be a positive integer.

(i) We shall write

$$[n, d]$$

for the set of maps $\{1, \ldots, n\} \to \{0, \ldots, d\}$.

(ii) We shall define an equivalence relation $\sim$ on the set $[n, d]$ as follows: For $\mathfrak{m}$, $\mathfrak{n} \in [n, d]$, we write $\mathfrak{m} \sim \mathfrak{n}$ if $\sharp\mathfrak{m}^{-1}(\{i\}) = \sharp\mathfrak{n}^{-1}(\{i\})$ for every $i \in \{0, \ldots, d\}$ [cf. Remark 3.6.1 below].

(iii) Let $\overline{\mathfrak{m}}$ be an element of $[n, d]/\sim$. Then we shall write

$$x^{[\overline{\mathfrak{m}}]} \stackrel{\mathrm{def}}{=} \prod_{i=0}^{d} x_i^{[\sharp\mathfrak{m}^{-1}(\{i\})]} \in {}^{\mathrm{PD}}A^n$$

— where we write

$$x_0 \stackrel{\mathrm{def}}{=} 1,$$

and $\mathfrak{m} \in \overline{\mathfrak{m}} \subseteq [n,d]$. Note that it is immediate that the element "$x^{[\overline{\mathfrak{m}}]}$" does not depend on the choice of $\mathfrak{m} \in \overline{\mathfrak{m}}$, i.e., depends only on $\overline{\mathfrak{m}}$.

(iv) Let $\mathfrak{m}$ be an element of $[n,d]$. Then we shall write

$$x_{\otimes \mathfrak{m}} \stackrel{\text{def}}{=} x_{\mathfrak{m}(1)} \otimes \cdots \otimes x_{\mathfrak{m}(n)} \in {}^{\text{PD}}A^{\times n}$$

— where we write

$$x_0 \stackrel{\text{def}}{=} 1.$$

(v) Let $\overline{\mathfrak{m}}$ be an element of $[n,d]/\sim$. Then we shall write

$$x_{\otimes \overline{\mathfrak{m}}} \stackrel{\text{def}}{=} \sum_{\mathfrak{m} \in \overline{\mathfrak{m}}} x_{\otimes \mathfrak{m}} \in {}^{\text{PD}}A^{\times n}.$$

LEMMA 3.5. — *Let n be a positive integer. Then the following hold:*

(i) *If one regards ${}^{\text{PD}}A^n$ as an A-module by the homomorphism ${}^{\text{PD}}_{\sharp}\text{pr}^n_1$ (respectively, ${}^{\text{PD}}_{\sharp}\text{pr}^n_2$), then the A-module ${}^{\text{PD}}A^n$ is* **free**, *and the subset $\{x^{[\overline{\mathfrak{m}}]}\}_{\overline{\mathfrak{m}} \in [n,d]/\sim} \subseteq {}^{\text{PD}}A^n$ of ${}^{\text{PD}}A^n$ forms a* **basis** *of the free A-module ${}^{\text{PD}}A^n$.*

(ii) *If one regards ${}^{\text{PD}}A^{\times n}$ as an A-module by the homomorphism ${}^{\text{PD}}_{\sharp}\text{pr}^{\times n}_1$ (respectively, ${}^{\text{PD}}_{\sharp}\text{pr}^{\times n}_{n+1}$), then the A-module ${}^{\text{PD}}A^{\times n}$ is* **free**, *and the subset $\{x_{\otimes \mathfrak{m}}\}_{\mathfrak{m} \in [n,d]} \subseteq {}^{\text{PD}}A^{\times n}$ of ${}^{\text{PD}}A^{\times n}$ forms a* **basis** *of the free A-module ${}^{\text{PD}}A^{\times n}$.*

PROOF. — Assertion (i) follows from (4) of [1], Theorem 3.9. Assertion (ii) follows from assertion (i). ☐

DEFINITION 3.6. — Let n be a positive integer.

(i) We shall write

$$\mathfrak{S}_n$$

for the group of self-bijections of the set $\{1,\ldots,n\}$ and

$$\mathfrak{S}_{1,n-1} \qquad \text{(respectively, } \mathfrak{S}_{n-1,1}\text{)}$$

for the subgroup of $\mathfrak{S}_n$ obtained by forming the stabilizer of $1 \in \{1,\ldots,n\}$ (respectively, $n \in \{1,\ldots,n\}$).

(ii) The assignment

$$\mathfrak{S}_n \ni \sigma \longmapsto (x_{i_1} \otimes \cdots \otimes x_{i_n} \mapsto x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(n)}})$$

— where we write

$$x_0 \stackrel{\text{def}}{=} 1,$$

and i_j is an element of $\{0,\ldots,d\}$ for each $j \in \{1,\ldots,n\}$ — determines an action of the group $\mathfrak{S}_n$ on the subset $\{x_{\otimes \mathfrak{m}}\}_{\mathfrak{m} \in [n,d]} \subseteq {}^{\text{PD}}A^{\times n}$ of ${}^{\text{PD}}A^{\times n}$ discussed in Lemma 3.5, (ii). Thus, it follows from Lemma 3.5, (ii), that if one regards ${}^{\text{PD}}A^{\times n}$ as an A-module by the homomorphism ${}^{\mathrm{PD}}_{\sharp}\mathrm{pr}_1^{\times n}$, then this action determines an action of the group $\mathfrak{S}_n$ on the A-module ${}^{\mathrm{PD}}A^{\times n}$. For a subgroup $G \subseteq \mathfrak{S}_n$ of $\mathfrak{S}_n$, we shall write

$$({}^{\mathrm{PD}}A^{\times n})^G \subseteq {}^{\mathrm{PD}}A^{\times n}$$

for the A-submodule of G-invariants.

REMARK 3.6.1. — Let n be a positive integer. Then the action of the group $\mathfrak{S}_n$ on the set $\{1, \ldots, n\}$ induces an action of the group $\mathfrak{S}_n$ on the set $[n, d]$. Moreover, one verifies easily that the equivalence relation $\sim$ on the set $[n, d]$ of Definition 3.4, (ii), *coincides* with the equivalence relation on the set $[n, d]$ determined by this action of the group $\mathfrak{S}_n$ on the set $[n, d]$.

LEMMA 3.7. — *Let n be a positive integer. Then the following hold:*

(i) *Let $\overline{\mathfrak{m}}$ be an element of $[n, d]/\sim$. Then the homomorphism ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n} \colon {}^{\mathrm{PD}}A^n \to {}^{\mathrm{PD}}A^{\times n}$* **maps** *$x^{[\overline{\mathfrak{m}}]} \in {}^{\mathrm{PD}}A^n$ to $x_{\otimes\overline{\mathfrak{m}}} \in {}^{\mathrm{PD}}A^{\times n}$:*

$$ {}^{\mathrm{PD}}_{\sharp}\delta^{\times n}(x^{[\overline{\mathfrak{m}}]}) = x_{\otimes\overline{\mathfrak{m}}}. $$

(ii) *The homomorphism ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n} \colon {}^{\mathrm{PD}}A^n \to {}^{\mathrm{PD}}A^{\times n}$ is* **injective**.

(iii) *The image of the injective* [*cf.* (ii)] *homomorphism ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n} \colon {}^{\mathrm{PD}}A^n \hookrightarrow {}^{\mathrm{PD}}A^{\times n}$* **coincides** *with the submodule $({}^{\mathrm{PD}}A^{\times n})^{\mathfrak{S}_n} \subseteq {}^{\mathrm{PD}}A^{\times n}$ of ${}^{\mathrm{PD}}A^{\times n}$.*

PROOF. — Assertion (i) follows from a straightforward calculation [cf. also the discussion preceding [1], Definition 4.3, concerning the homomorphism "δ"]. Assertions (ii), (iii) follow immediately from assertion (i), together with Lemma 3.5, (i), (ii) [cf. also Remark 3.6.1]. ☐

LEMMA 3.8. — *Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \stackrel{\mathrm{def}}{=} \sum_{i=1}^{r} n_i$. Let us recall the commutative diagram of rings*

$$\begin{array}{ccc} & {}^{\mathrm{PD}}A^n & \\ {}^{\mathrm{PD}}_{\sharp}\delta^{n_1,\ldots,n_r} \swarrow & & \searrow {}^{\mathrm{PD}}_{\sharp}\delta^{\times n} \\ {}^{\mathrm{PD}}A^{n_1,\ldots,n_r} & \xrightarrow[{}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{n_1,\ldots,n_r}]{} & {}^{\mathrm{PD}}A^{\times n} \end{array}$$

[*cf.* [2], *Definition* 2.6*, i.e., in the case where we take the "$\mathfrak{S}$" to be the stratification structure of Definition* 2.5]. *Then the homomorphisms in this diagram are* **injective**.

PROOF. — Let us first observe that since the homomorphism ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}$ is *injective* [cf. Lemma 3.7, (ii)], to verify Lemma 3.8, it suffices to verify that the homomorphism ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{n_1,\ldots,n_r}$ is *injective*. Thus, by applying Lemma 3.5, (i), (ii), and induction on r [cf. the definition of the homomorphism ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{n_1,\ldots,n_r}$], to verify Lemma 3.8, we may assume without loss of generality that $r = 1$. On the other hand, it follows from Lemma 3.7, (ii), that if $r = 1$, then the homomorphism ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{n_1,\ldots,n_r}$ is *injective*, as desired. This completes the proof of Lemma 3.8. □

LEMMA 3.9. — *Suppose that $n \geq 3$. Let us recall the commutative diagram of* **injective** *[cf. Lemma 3.8] homomorphisms of rings*

$$\begin{array}{ccc} {}^{\mathrm{PD}}A^{n} & \xhookrightarrow{{}^{\mathrm{PD}}_{\sharp}\delta^{1,n-1}} & {}^{\mathrm{PD}}A^{1,n-1} \\ {\scriptstyle {}^{\mathrm{PD}}_{\sharp}\delta^{n-1,1}}\big\downarrow & \searrow {\scriptstyle {}^{\mathrm{PD}}_{\sharp}\delta^{\times n}} & \big\downarrow{\scriptstyle {}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{1,n-1}} \\ {}^{\mathrm{PD}}A^{n-1,1} & \xhookrightarrow[{}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{n-1,1}]{} & {}^{\mathrm{PD}}A^{\times n}. \end{array}$$

Then the intersection of the image of ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{1,n-1}$ and the image of ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}_{n-1,1}$ **coincides** *with the image of ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}$.*

PROOF. — Let z be an element of ${}^{\mathrm{PD}}A^{\times n}$ contained in the intersection under consideration. Then it follows immediately from Lemma 3.7, (iii), that z is *contained* in both $({}^{\mathrm{PD}}A^{\times n})^{\mathfrak{S}_{1,n-1}}$ and $({}^{\mathrm{PD}}A^{\times n})^{\mathfrak{S}_{n-1,1}}$. Thus, since [one verifies easily from our assumption that $n \geq 3$ that] the group $\mathfrak{S}_n$ is *generated* by the subgroups $\mathfrak{S}_{1,n-1}$ and $\mathfrak{S}_{n-1,1}$, again by Lemma 3.7, (iii), we conclude that z is *contained* in the image of ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}$, as desired. This completes the proof of Lemma 3.9. □

LEMMA 3.10. — *Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \stackrel{\mathrm{def}}{=} \sum_{i=1}^{r} n_i$. Suppose that $n \geq 1$. Then the kernel of the surjective homomorphism ${}^{\mathrm{PD}}_{\sharp}\underline{\iota}^{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n_1,\ldots,n_r} \twoheadrightarrow {}^{\mathrm{PD}}\underline{A}^{n_1,\ldots,n_r}$ — i.e., the ideal of ${}^{\mathrm{PD}}A^{n_1,\ldots,n_r}$ generated by the image of ${}^{\mathrm{PD}}I^{[n]}/{}^{\mathrm{PD}}I^{[n+1]} \subseteq {}^{\mathrm{PD}}A^{n}$ by ${}^{\mathrm{PD}}_{\sharp}\delta^{n_1,\ldots,n_r}$ — is* **annihilated** *by the kernel of the surjective homomorphism ${}^{\mathrm{PD}}_{\sharp}\iota^{0,\ldots,0}_{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n_1,\ldots,n_r} \twoheadrightarrow {}^{\mathrm{PD}}A^{0,\ldots,0} = A$.*

PROOF. — Let us first observe that it follows from [1], Lemma 3.5, [1], Lemma 3.7, and [1], Proposition 3.25, together with the various definitions involved, that we have a *natural divided power structure* on the ideal of ${}^{\mathrm{PD}}A^{n}$ (respectively, ${}^{\mathrm{PD}}A^{n_1,\ldots,n_r}$) obtained by forming the kernel of the surjective homomorphism ${}^{\mathrm{PD}}_{\sharp}\iota^{0}_{n} \colon {}^{\mathrm{PD}}A^{n} \twoheadrightarrow {}^{\mathrm{PD}}A^{0} = A$ (respectively, ${}^{\mathrm{PD}}_{\sharp}\iota^{0,\ldots,0}_{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n_1,\ldots,n_r} \twoheadrightarrow {}^{\mathrm{PD}}A^{0,\ldots,0} = A$), by means of which let us regard the ring ${}^{\mathrm{PD}}A^{n}$ (respectively, ${}^{\mathrm{PD}}A^{n_1,\ldots,n_r}$) as a divided power ring. Moreover, one verifies immediately from the discussion preceding [1], Definition 4.3, that the homomorphism ${}^{\mathrm{PD}}_{\sharp}\delta^{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}A^{n} \to {}^{\mathrm{PD}}A^{n_1,\ldots,n_r}$ is *compatible* with the respective divided power structures. Thus, Lemma 3.10 follows immediately from the [easily verified] fact that the ideal "$I^{[n]}$" defined in [1], Definition 3.24, for the divided power ring ${}^{\mathrm{PD}}A^{n_1,\ldots,n_r}$ in the case where we take the "n" of [1], Definition 3.24, to be $n + 1$ is *zero*. This completes the proof of Lemma 3.10. □

Lemma 3.11. — *Let n be a nonnegative integer and p a prime number. Suppose that the ring R is a ring* **over a field of characteristic p**. *Write $J \subseteq A^{(2)}/I^{n+1}$ for the ideal of $A^{(2)}$ generated by* [*the images of*]

$$ {}^{X}_{\sharp}\mathrm{pr}_2^{(2)}(a_1)^p - {}^{X}_{\sharp}\mathrm{pr}_1^{(2)}(a_1)^p, \ldots, {}^{X}_{\sharp}\mathrm{pr}_2^{(2)}(a_d)^p - {}^{X}_{\sharp}\mathrm{pr}_1^{(2)}(a_d)^p. $$

Then the homomorphism $A^{(2)}/I^{n+1} \to {}^{\mathrm{PD}}A^n$ of Definition 3.1, (iv), **factors** *through the natural surjective homomorphism $A^{(2)}/I^{n+1} \twoheadrightarrow A^{(2)}/(I^{n+1} + J)$.*

Proof. — It follows from the definition of the homomorphism $A^{(2)}/I^{n+1} \to {}^{\mathrm{PD}}A^n$ of Definition 3.1, (iv), that, to verify Lemma 3.11, it suffices to verify that $x_i^p = 0$ for every $i \in \{1, \ldots, d\}$. On the other hand, this follows immediately from the equality "$n!\gamma_n(x) = x^n$" in [1], Definition 3.1. This completes the proof of Lemma 3.11. □

4. An Application of the First Fundamental Correspondence

In the present §4, we prove [cf. Theorem 4.1 below] that the divided power stratification structure PD is *strictly integrable* [cf. [2], Definition 3.3] and *of standard type* [cf. [2], Definition 3.6]. Moreover, we discuss an application of the first fundamental correspondence of [2], Definition 5.7 [cf. Corollary 4.3 below].

The main result of the present paper is as follows.

Theorem 4.1. — *Let S be a scheme and X a scheme which is* **smooth** *and* **separated** *over S. Then the divided power stratification structure* [*cf. Definition* 2.5]

$$ \mathrm{PD} = \big(({}^{\mathrm{PD}}P^n)_{n\geq 0},\ ({}^{\mathrm{PD}}\iota^n\colon {}^{\mathrm{PD}}P^n \to {}^{\mathrm{PD}}P^{n+1})_{n\geq 0}, $$

$$ ({}^{\mathrm{PD}}\sigma^n\colon {}^{\mathrm{PD}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathrm{PD}}\delta^{n_1,n_2}\colon {}^{\mathrm{PD}}P^{n_1,n_2} \to {}^{\mathrm{PD}}P^{n_1+n_2})_{n_1,\,n_2\geq 0} \big) $$

is **strictly integrable** [*cf.* [2], *Definition* 3.3] *and* **of standard type** [*cf.* [2], *Definition* 3.6].

Proof. — Let us first observe that it follows from [1], Remark 4.2, that PD satisfies condition (1) of [2], Definition 3.6. Moreover, it follows immediately from Lemma 3.5, (i), (ii), and Lemma 3.7, (i), that PD satisfies conditions (2), (3) of [2], Definition 3.6. In particular, the stratification structure PD is *of standard type*.

Next, let us recall from Lemma 2.7 that PD is *ι-quasi-nil-retraction-like*. Moreover, it follows from Lemma 2.8 and Lemma 3.8 that PD is *δ-nil-retraction-like* [cf. [2], Definition 3.1, (iii)]. In particular, the stratification structure PD satisfies condition (1) of [2], Definition 3.2.

Moreover, since PD is *δ-nil-retraction-like* [cf. the second paragraph of the present proof of Theorem 4.1], it follows from Lemma 3.9 that PD is *δ-strictly cocartesian of level* ≥ 3 [cf. [2], Definition 3.1, (iv)]. In particular, the stratification structure PD satisfies condition (2) of [2], Definition 3.2.

Next, it follows from Lemma 2.7 that the stratification structure PD satisfies condition (3) of [2], Definition 3.2.

Next, let us verify that PD satisfies condition (4) of [2], Definition 3.2. Let us first observe that since [it follows from the second paragraph of the present proof of Theorem 4.1

that] the morphism ${}^{\mathrm{PD}}\delta^{\times 2}\colon {}^{\mathrm{PD}}P^{\times 2} \to {}^{\mathrm{PD}}P^2$ is *nil-retraction-like* [cf. [2], Definition 1.2, (iv)], it follows from [2], Remark 1.3.1, that the morphism ${}^{\mathrm{PD}}\delta^{\times 2}\colon {}^{\mathrm{PD}}P^{\times 2} \to {}^{\mathrm{PD}}P^2$ satisfies conditions (1), (2) of [2], Definition 1.3. Now let us take the "$\overline{X}$" (respectively, "$\overline{Y}$") of [2], Definition 1.3, to be X (respectively, X) and the closed immersion "i_X" (respectively, "i_Y") of [2], Definition 1.3, to be the closed immersion ${}^{\mathrm{PD}}\iota^{0,0}_{1,1}\colon X = {}^{\mathrm{PD}}P^{0,0} \hookrightarrow {}^{\mathrm{PD}}P^{\times 2}$ (respectively, ${}^{\mathrm{PD}}\iota^{0}_{2}\colon X = {}^{\mathrm{PD}}P^{0} \hookrightarrow {}^{\mathrm{PD}}P^{2}$). Then it follows from Lemma 3.10 that condition (3) of [2], Definition 1.3, is satisfied. Moreover, condition (4) of [2], Definition 1.3, is immediate. This completes the proof of the assertion that PD satisfies condition (4) of [2], Definition 3.2. In particular, the stratification structure PD satisfies condition (1) of [2], Definition 3.3.

Next, it follows from Lemma 3.5, (i), that PD is pr-*finite flat* [cf. [2], Definition 3.1, (ii)]. In particular, the stratification structure PD satisfies condition (2) of [2], Definition 3.3.

Thus, since PD satisfies condition (3) of [2], Definition 3.3 [cf. the first paragraph of the present proof of Theorem 4.1 and [2], Remark 3.6.2], we conclude that the stratification structure PD is *strictly integrable*. This completes the proof of Theorem 4.1. □

COROLLARY 4.2. — *Let S be a scheme, X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S*, and ξ an object of $\mathcal{F}$ over X. Suppose that the following two conditions are satisfied:*

(1) *The category $\mathcal{F}$ fibered in groupoids over* Sch_S *is* **weakly integrable** [*cf.* [2], *Definition* 1.8].

(2) *The scheme X is* **of relative dimension ≤ 1** *over S.*

Then every PD-*connection* [*cf. Definition* 2.5*;* [2], *Definition* 4.1, (iii)] *on ξ is* **PD-integrable** [*cf. Definition* 2.5*;* [2], *Definition* 4.7, (ii)]*:*

$${}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi) = {}^{\mathrm{PD}}\mathrm{IntCnn}(\xi)$$

[*cf. Definition* 2.5*;* [2], *Definition* 4.7, (ii)].

PROOF. — This assertion follows from Theorem 4.1 and [2], Proposition 4.10. □

One main application of the first fundamental correspondence of [2], Definition 5.7, is as follows.

COROLLARY 4.3. — *Let S be a scheme, X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a **weakly integrable** [*cf.* [2], *Definition* 1.8] *category fibered in groupoids over* Sch_S*, and ξ an object of $\mathcal{F}$ over X. Then the natural map*

$${}^{\mathrm{PD}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathrm{PD}}\mathrm{IntCnn}(\xi)$$

[*cf. Definition* 2.5*;* [2]*, Definition* 4.6*;* [2]*, Definition* 4.7*,* (ii)*;* [2]*, Lemma* 4.8] *is* **bijective**.

PROOF. — This assertion follows from Theorem 4.1 and [2], Theorem 5.6. □

REMARK 4.3.1. — Suppose that we are in the situation of Corollary 4.3.

(i) Suppose that one takes the "$\mathcal{F} \to \mathrm{Sch}_S$" of Corollary 4.3 to be the category $\mathrm{LcFr} \to \mathrm{Sch}_S$ fibered in groupoids of Definition 1.1 [cf. also Proposition 1.5], which thus implies that the object ξ corresponds to a *locally free* $\mathcal{O}_X$*-module* $\mathcal{E}$.

Let ∇ be a PD-connection on ξ. Then it follows from Proposition 2.6, (ii), that the PD-connection ∇ corresponds to a *classical connection* on $\mathcal{E}$, i.e., a certain homomorphism of $(X \to S)^{-1}\mathcal{O}_S$-modules [cf. Definition 1.6]

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}.$$

Now let us recall that the closed immersion ${}^{\mathrm{PD}}\underline{\iota}^T_{\times 2} \colon T^1 \hookrightarrow {}^{\mathrm{PD}}\underline{P}^{\times 2}$ of [2], Definition 3.6, is a *square-nilpotent* closed immersion whose conormal sheaf is *isomorphic* to the $\mathcal{O}_{T^1}$-module $({}^{\mathrm{PD}}\iota^0_T)_*\Omega^2_{X/S}$ [cf. Theorem 4.1; condition (3) of [2], Definition 3.6]. Thus, it follows from Lemma 1.2, (ii), that the subgroup

$$\mathrm{Lift}_{{}^{\mathrm{PD}}\underline{\iota}^T_{\times 2}}\big(({}^{\mathrm{PD}}\underline{\mathrm{pr}}_1^{\times 2})^*\xi, ({}^{\mathrm{PD}}\underline{\mathrm{pr}}_1^{\times 2})^*\xi; \mathrm{id}_{({}^X\mathrm{pr}_1^T)^*\xi}\big) \subseteq \mathrm{Aut}_{\mathcal{F}|_{{}^{\mathrm{PD}}\underline{P}^{\times 2}}}\big(({}^{\mathrm{PD}}\underline{\mathrm{pr}}_1^{\times 2})^*\xi\big)$$

may be naturally identified with the module

$$\Gamma\big(X, \Omega^2_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\big).$$

In particular, since the stratification structure PD is *of standard type* [cf. Theorem 4.1], and the category LcFr fibered in groupoids is *weakly integrable* [cf. Proposition 1.5], by applying [2], Lemma 4.9, we conclude that the PD-*curvature* [cf. Definition 2.5; [2], Definition 4.7, (i)] of the PD-connection ∇ may be naturally identified with a global section of

$$\Omega^2_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}).$$

Moreover, in this case, one verifies easily from a straightforward calculation that this global section of $\Omega^2_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\xi)$ *coincides*, up to sign, with the *curvature* of the corresponding classical connection on $\mathcal{E}$

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

in the usual sense [cf., e.g., the discussion preceding [1], Theorem 2.15].

In particular, we conclude from Proposition 2.6, (i), (ii), that the *bijection* of Corollary 4.3 may be regarded as a *generalization* of the equivalence [cf. [1], Theorem 4.8] between

- an integrable connection on $\mathcal{E}$ and
- a divided power stratification on $\mathcal{E}$

[i.e., in the case where the modules under consideration are locally free].

(ii) Suppose that one takes the "$\mathcal{F} \to \mathrm{Sch}_S$" of Corollary 4.3 to be the category $\mathrm{SmSch} \to \mathrm{Sch}_S$ fibered in groupoids of Definition 1.7 [cf. also Proposition 1.11], which thus implies that the object ξ is a smooth morphism $Z \to X$ of schemes over S. Let ∇ be a PD-connection on ξ. Then it follows immediately from a similar argument to the argument of (i), together with Lemma 1.8, (ii), that the PD-*curvature* of the PD-connection ∇ may be naturally identified with a global section of

$$\Omega^2_{X/S} \otimes_{\mathcal{O}_X} \xi_* \mathcal{T}_{Z/X}.$$

REMARK 4.3.2. — In [3], Definition 2.3, *B. Osserman* asserted that the *integrability* of a [PD-]connection [cf. Proposition 2.6, (ii)] concerns a certain "cocycle condition" on "$X_3^{(2)}$" — i.e., T^1 in the notational conventions of [2] [cf. [2], Definition 3.4, (i)]. However, this is *false*. Indeed, as we have already observed in [2], Lemma 4.9, the "cocycle condition" on "$X_3^{(2)}$" — i.e., T^1 in the notational conventions of [2] — for every PD-connection is *always satisfied* whenever the category $\mathcal{F}$ fibered in groupoids over Sch_S is *weakly integrable* [as in the case of LcFr — cf. Proposition 1.5]. As discussed in Remark 4.3.1, (i), and [2], Definition 4.7, (i), the *integrability* of a PD-connection concerns a certain "cocycle condition" on ${}^{\mathrm{PD}}\underline{P}^{\times 2}$ [i.e., as opposed to a certain "cocycle condition" on T^1].

COROLLARY 4.4. — *Let S be a scheme, X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S*, and ξ an object of $\mathcal{F}$ over X. Suppose that the following two conditions are satisfied:*

(1) *The category $\mathcal{F}$ fibered in groupoids over* Sch_S *is* **weakly integrable** [*cf.* [2], *Definition* 1.8].

(2) *The scheme X is* **of relative dimension ≤ 1** *over S.*

Then the natural map

$${}^{\mathrm{PD}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi)$$

[*cf. Definition* 2.5*;* [2]*, Definition* 4.4*;* [2]*, Definition* 4.6] *is* **bijective**.

PROOF. — This assertion follows from Corollary 4.2 and Corollary 4.3. □

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