A Note on Torsion Points on Ample Divisors on Abelian Varieties

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Abstract. — In the present paper, we consider torsion points on ample divisors on abelian varieties. We prove that, for each integer \( n \geq 2 \), an effective divisor of level \( n \) on an abelian variety does not contain the subgroup of \( n \)-torsion points. Moreover, we also discuss an application of this result to the study of the \( p \)-rank of cyclic coverings of curves in positive characteristic.

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Introduction

In the present paper, we consider torsion points on ample divisors on abelian varieties. The main result of the present paper is as follows [cf. Corollary 1.8, (i)].

**Theorem A.** — Let \( k \) be an algebraically closed field, \( A \) an abelian variety over \( k \), \( D \) an effective divisor on \( A \), and \( n \geq 2 \) an integer invertible in \( k \). Suppose that the effective divisor \( D \) is of level \( n \), i.e., that there exists an effective divisor \( D_1 \) on \( A \) such that \( D_1 \) gives rise to a principal polarization on \( A \), and, moreover, \( D \) is linearly equivalent to \( nD_1 \) [cf. Definition 1.3, (ii); also Remark 1.3.1]. Then the subgroup of \( n \)-torsion points of \( A \) is not contained in \( \text{Supp}(D) \).

Here, let us recall that R. Auffarth, G. P. Pirola, and R. S. Manni proved that if \( D \) is an effective divisor on an abelian variety of dimension \( g \geq 1 \) over the field of complex numbers that gives rise to a principal polarization on the abelian variety, then, for each integer \( n \geq 3 \), the set of \( n \)-torsion (respectively, 2-torsion) points on \( \text{Supp}(D) \) is of

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The cardinality \( \leq n^{2g} - (g + 1)n^g \) (respectively, \( \leq 2^{2g} - 2g - 2^g \)) \( \leq n^g \) \( g \) \( n^g \) \( g \) \( n^g < 2^{2g} \)) \( \leq 2^{2g} - 2g - 2^g \) \( n^g < 2^{2g} \)) \[ \text{cf. [1], Theorem 1.1}. \] Theorem A may be regarded as a partial generalization of this result \[ \text{cf. Remark 1.8.1}. \]

In §2 of the present paper, we apply Theorem A and Raynaud’s theory of theta divisors \[ \text{cf. [5]}. \] to obtain an application to the study of the \( p \)-rank of cyclic coverings of curves in positive characteristic. One consequence of our application is as follows \[ \text{cf. Theorem 2.7, (i)}. \]

**Theorem B.** — Let \( p \) be an odd prime number, \( k \) an algebraically closed field of characteristic \( p \), and \( X \) a projective smooth curve over \( k \) of genus \( \geq 2 \). Then there exist a positive integer \( n \) such that \( p - 1 \in n\mathbb{Z} \) and a finite étale cyclic covering of \( X \) of degree \( n \) whose Jacobian variety is of positive \( p \)-rank.

Here, let us recall that M. Raynaud proved that, in the situation of Theorem B, the étale fundamental group of \( X \) is not pro-prime-to-\( p \) \[ \text{cf. [5], Corollaire 4.3.2}. \] In §2 of the present paper, we also derive a refinement of this result from Theorem B \[ \text{cf. Remark 2.8.1}. \]

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**1. Torsion Points on Ample Divisors on Abelian Varieties**

In the present §1, we discuss torsion points on ample divisors on abelian varieties and prove the main result of the present paper \[ \text{cf. Corollary 1.8 below}. \] In the present §1, let \( g \) be a positive integer, \( k \) an algebraically closed field,

\[ A \]

an abelian variety over \( k \) of dimension \( g \), \( n \) a positive integer, and

\[ \mathcal{L} \]

an ample invertible sheaf on \( A \) of separable type \[ \text{cf. [2], p.289}. \]

**Definition 1.1.** — We shall write \( A[n] \subseteq A \) for the closed subgroup scheme of \( A \) obtained by forming the kernel of the endomorphism of \( A \) given by multiplication by \( n \).

**Lemma 1.2.** — The following four conditions are equivalent:

1. There exist an ample invertible sheaf \( \mathcal{L}_1 \) on \( A \) of degree one \[ \text{cf. [2], p.289, (III)}. \] and an isomorphism \( \mathcal{L} \cong \mathcal{L}_1^{\otimes n} \).

2. There exist an ample invertible sheaf \( \mathcal{L}_1 \) on \( A \) and an isomorphism \( \mathcal{L} \cong \mathcal{L}_1^{\otimes n} \). Moreover, the invertible sheaf \( \mathcal{L} \) is of degree \( n^g \).
(3) There exist an ample invertible sheaf $\mathcal{L}_1$ on $A$ and an isomorphism $\mathcal{L} \cong \mathcal{L}_1^\otimes n$. Moreover, the equality $H(\mathcal{L}) = A[n]$ [cf. [2], p.288, Definition] holds.


Proof. — The equivalence (1) $\iff$ (2) follows from [2], p.289, (II). Moreover, the equivalence (3) $\iff$ (4) follows from [3], p.214, Theorem 3. Next, since the group scheme $A[n]$ is of degree $n^2g$ over $k$ [cf. [3], p.60, Proposition, (1)], the implication (3) $\Rightarrow$ (2) follows from [2], p.289, (IV).

Finally, we verify the implication (2) $\Rightarrow$ (3). Suppose that condition (2) is satisfied. Then since $\mathcal{L}$ is isomorphic to $\mathcal{L}_1^\otimes n$ [cf. condition (2)], the homomorphisms $A(n \cdot \Lambda(\mathcal{L})) = H(\mathcal{L})$. On the other hand, since $\mathcal{L}$ is of degree $n^g$ [cf. condition (2)], it follows from [2], p.289, (IV), that the group scheme $H(\mathcal{L})$ is of degree $n^2g$ over $k$. Thus, since the group scheme $A[n]$ is of degree $n^2g$ over $k$ [cf. [3], p.60, Proposition, (1)], the equality $H(\mathcal{L}) = A[n]$, hence also condition (3), holds, as desired. This completes the proof of the implication (2) $\Rightarrow$ (3), hence also of Lemma 1.2. □

**Definition 1.3.**

(i) We shall say that the ample invertible sheaf $\mathcal{L}$ of separable type is of level $n$ if $\mathcal{L}$ satisfies the four conditions [i.e., with respect to the fixed “n”] in the statement of Lemma 1.2.

(ii) We shall say that an effective divisor $D$ on $A$ is of level $n$ if the invertible sheaf $\mathcal{O}_A(D)$ is [ample, of separable type, and] of level $n$.

**Remark 1.3.1.** — Let $\mathcal{M}$ be an invertible sheaf on $A$. Then it is immediate that $\mathcal{M}$ gives rise to a principal polarization on $A$ if and only if $\mathcal{M}$ is [ample, of separable type, and] of level one.

**Lemma 1.4.** — Let $\mathcal{M}$ be an invertible sheaf on $A$ algebraically equivalent to $\mathcal{L}$. Then the following hold:

(i) There exists a closed point $a \in A$ of $A$ such that $\mathcal{L}$ is isomorphic to $T_a^*\mathcal{M}$ [cf. [2], p.288, Definition].

(ii) The invertible sheaf $\mathcal{M}$ is ample and of separable type.

(iii) Suppose that $\mathcal{M}$ is of level $n$ [cf. (ii)]. Then $\mathcal{L}$ is of level $n$.

Proof. — First, we verify assertion (i). Let us first observe that the homomorphism $A(k) \to \text{Pic}^0(A)$ determined by $\Lambda(\mathcal{L})$ is surjective [cf. [2], p.289, (IV)]. Thus, there exists a closed point $a \in A$ of $A$ such that $\mathcal{M} \otimes \mathcal{O}_A \mathcal{L}^{-1}$ is isomorphic to $T_a^*\mathcal{L} \otimes \mathcal{O}_A \mathcal{L}^{-1}$. Thus, we conclude that $\mathcal{L}$ is isomorphic to $T_a^*\mathcal{M}$, as desired. This completes the proof of assertion (i). Assertions (ii), (iii) follow from assertion (i). This completes the proof of Lemma 1.4. □
LEMMA 1.5. — Suppose that \( \mathcal{L} \) is of level \( n \). Then the following two conditions are equivalent:

1. The inequality \( n > 1 \) holds.
2. The invertible sheaf \( \mathcal{L} \) is generated by the global sections.

Proof. — The implication (1) \( \Rightarrow \) (2) follows immediately from [3], pp.57-58, Application 1, (iii). Next, to verify the implication (2) \( \Rightarrow \) (1), assume that condition (2) is satisfied but that condition (1) is not satisfied [i.e., that \( n = 1 \)]. Then it follows from [2], p.289, (II), that \( \Gamma(A, \mathcal{L}) \) is of dimension one. Thus, since \( \mathcal{L} \) is generated by the global sections [cf. condition (2)], the invertible sheaf \( \mathcal{L} \) is trivial. In particular, since [we have assumed that] \( \mathcal{L} \) is ample, we conclude that \( g = 0 \). Thus, since [we have assumed that] \( g > 0 \), we obtain a contradiction, as desired. This completes the proof of the implication (2) \( \Rightarrow \) (1), hence also of Lemma 1.5.

One main technical observation of the present paper is as follows.

LEMMA 1.6. — Let \( D \) be an effective divisor on \( A \) obtained by forming the zero locus of a nonzero global section of the invertible sheaf \( \mathcal{L} \). Write \( H(D) \subseteq H(\mathcal{L}) \) for the subgroup of \( H(\mathcal{L}) \) consisting of \( a \in A \) such that \( T^*_aD = D \). Let \( H \subseteq H(\mathcal{L}) \) be a subgroup of \( H(\mathcal{L}) \) such that \( H + H(D) \) (def \( = \{ h + h_d \in H(\mathcal{L}) | h \in H, h_d \in H(D) \} \) ) = \( H(\mathcal{L}) \). Suppose that the inclusion \( H \subseteq \text{Supp}(D) \)

holds. Then the subset \( H \subseteq A \) of \( A \) is contained in the base locus of the [complete linear system associated to the] invertible sheaf \( \mathcal{L} \).

Proof. — Let \( s \in \Gamma(A, \mathcal{L}) \) be a nonzero global section of \( \mathcal{L} \) whose zero locus is given by \( D \).

Here, let us recall the exact sequence

\[
0 \longrightarrow k^* \longrightarrow \mathcal{G}(\mathcal{L}) \longrightarrow H(\mathcal{L}) \longrightarrow 0
\]

in [2], p.290, concerning the theta group \( \mathcal{G}(\mathcal{L}) \) associated to \( \mathcal{L} \). It follows from the definition of \( \mathcal{G}(\mathcal{L}) \) that there exists a natural action of \( \mathcal{G}(\mathcal{L}) \) on the linear space \( \Gamma(A, \mathcal{L}) \) over \( k \), which restricts to the natural action of the subgroup \( k^* \subseteq \mathcal{G}(\mathcal{L}) \) on \( \Gamma(A, \mathcal{L}) \) [cf. [2], p.295, Definition]. In particular,

(a) for each \( a \in H(\mathcal{L}) \), if \( \tilde{a} \in \mathcal{G}(\mathcal{L}) \) is a lifting of \( a \in H(\mathcal{L}) \), then the zero locus of the nonzero global section \( \tilde{a} \cdot s \in \Gamma(A, \mathcal{L}) \) is given by \( T^*_aD \).

Now let us fix a subset \( \tilde{H} \subseteq \mathcal{G}(\mathcal{L}) \) of \( \mathcal{G}(\mathcal{L}) \) such that the composite \( \tilde{H} \hookrightarrow \mathcal{G}(\mathcal{L}) \twoheadrightarrow H(\mathcal{L}) \) determines a bijection \( \tilde{H} \cong H \). Then since [we have assumed that] the inclusion \( H \subseteq \text{Supp}(D) \) holds, it follows from (a) that,

(b) for every \( \tilde{a} \in \tilde{H} \), the subset \( H \subseteq A \) [i.e., the subset “\( T^*_aH \)” of \( A \) — where we write \( a \) for the image of \( \tilde{a} \in \tilde{H} \) in \( H \)] is contained in the zero locus of the nonzero global section \( \tilde{a} \cdot s \in \Gamma(A, \mathcal{L}) \).
Next, let us observe that it follows immediately from (a), together with our assumption that \( H + H(D) = H(L) \), that

(c) the linear subspace of \( \Gamma(A, L) \) generated by the \( \mathcal{G}(L) \)-orbit of \( s \in \Gamma(A, L) \) coincides with the linear subspace of \( \Gamma(A, L) \) generated by the subset \( \{ \tilde{a} \cdot s \}_{\tilde{a} \in \mathcal{H}} \subseteq \Gamma(A, L) \).

On the other hand, it follows from [2], p.297, Theorem 2, that the action of \( \mathcal{G}(L) \) on \( \Gamma(A, L) \) is irreducible. Thus, we conclude from (c) that

(d) the subset \( \{ \tilde{a} \cdot s \}_{\tilde{a} \in \mathcal{H}} \subseteq \Gamma(A, L) \) generates the linear space \( \Gamma(A, L) \).

Thus, it follows from (b) and (d) that the subset \( H \subseteq A \) is contained in the base locus of the invertible sheaf \( L \), as desired. This completes the proof of Lemma 1.6. \( \square \)

**Theorem 1.7.** — Let \( k \) be an algebraically closed field, \( A \) an abelian variety over \( k \), and \( D \) an effective divisor on \( A \). Suppose that the invertible sheaf \( \mathcal{O}_A(D) \) is ample, of separable type [cf. [2], p.289], and generated by the global sections. Then the following hold:

(i) Recall the closed subgroup scheme \( H(\mathcal{O}_A(D)) \subseteq A \) of \( A \) defined in [2], p.288, Definition. Then \( H(\mathcal{O}_A(D)) \) is not contained in \( \text{Supp}(D) \).

(ii) Write \( \deg(D) \) for the degree of the ample invertible sheaf \( \mathcal{O}_A(D) \) [cf. [2], p.289, (III)]. Then \( A[\deg(D)] \) [cf. Definition 1.1] is not contained in \( \text{Supp}(D) \).

**Proof.** — Assertion (i) follows from Lemma 1.6. Assertion (ii) follows from assertion (i), together with the inclusion \( H(\mathcal{O}_A(D)) \subseteq A[\deg(D)] \) [cf. [2], p.289, (IV); [2], p.293, Theorem 1; also the first Definition in [2], p.294]. \( \square \)

The main result of the present paper is as follows.

**Corollary 1.8.** — Let \( k \) be an algebraically closed field, \( A \) an abelian variety over \( k \), \( D \) an effective divisor on \( A \), and \( n \) a positive integer invertible in \( k \). Suppose that the effective divisor \( D \) is of level \( n \) [cf. Definition 1.3, (ii)]. Then the following hold:

(i) Suppose that \( n \geq 2 \). Then \( A[n] \) is not contained in \( \text{Supp}(D) \).

(ii) Suppose that \( n = 1 \). Then, for each integer \( m \geq 2 \) invertible in \( k \), \( A[m] \) is not contained in \( \text{Supp}(D) \).

**Proof.** — Let us recall from condition (4) of Lemma 1.2 that the equality \( H(\mathcal{O}_A(D)) = A[n] \) holds. Thus, assertion (i) follows from Lemma 1.5 and Theorem 1.7, (i).

Next, we verify assertion (ii). Let \( m \geq 2 \) be an integer invertible in \( k \). Then since \( D \) is of level one, it is immediate that \( mD \) is of level \( m \). Thus, since \( \text{Supp}(mD) = \text{Supp}(D) \), it follows from assertion (i) that \( A[m] \) is not contained in \( \text{Supp}(D) \), as desired. This completes the proof of assertion (ii), hence also of Corollary 1.8. \( \square \)

**Remark 1.8.1.** — R. Auffarth, G. P. Pirola, and R. S. Manni proved that, in the situation of Corollary 1.8, if, moreover, \( k \) is the field of complex numbers, and \( n = 1 \) [i.e., the divisor \( D \) gives rise to a principal polarization on \( A \)] — cf. Remark 1.3.1, then,
for each integer \( m \geq 3 \), the set \( A[m] \cap \text{Supp}(D) \) (respectively, \( A[2] \cap \text{Supp}(D) \)) is of cardinality \( \leq n^{2g} - (g + 1)n^g \) (respectively, \( \leq 2^{2g} - 2^{g-1}g - 2^g < 2^{2g} \)) — where we write \( g \) for the dimension of \( A \) [cf. [1], Theorem 1.1]. Corollary 1.8 may be regarded as a partial generalization of this result.

2. Application: \( p \)-Rank of Cyclic Coverings of Curves

In the present \( \S 2 \), we apply the main result of the present paper and Raynaud’s theory of theta divisors [cf. [5]] to obtain an application to the study of the \( p \)-rank of cyclic coverings of curves in positive characteristic [cf. Theorem 2.7 below]. In the present \( \S 2 \), let \( p \) be a prime number, \( k \) an algebraically closed field of characteristic \( p \), \( g \geq 2 \) an integer,

\[
X
\]

a projective smooth curve over \( k \) of genus \( g \), \( n \geq 2 \) an integer invertible in \( k \), and

\[
\mathcal{L}
\]
an invertible sheaf on \( X \) of order \( n \).

**Definition 2.1.** — We shall write \( X^F \) for the projective smooth curve over \( k \) obtained by forming the base-change of \( X \) by the absolute Frobenius endomorphism of \( k \), \( \mathcal{L}^F \) for the invertible sheaf on \( X^F \) obtained by forming the base-change of \( \mathcal{L} \) by the absolute Frobenius endomorphism of \( k \), and \( \Phi: X \to X^F \) for the relative Frobenius morphism associated to \( X \) over \( k \).

**Remark 2.1.1.** — Let us recall that we have a natural isomorphism of invertible sheaves on \( X \)

\[
\mathcal{L}^{\otimes p} \sim \Phi^* \mathcal{L}^F
\]
given by, for each local section \( l \) of \( \mathcal{L} \), mapping \( l^{\otimes p} \) to \( \Phi^{-1} l^F \) — where we write \( l^F \) for the local section of \( \mathcal{L}^F \) obtained by forming the base-change of the local section \( l \) by the absolute Frobenius endomorphism of \( k \). Let us identify \( \mathcal{L}^{\otimes p} \) with \( \Phi^* \mathcal{L}^F \) by means of this isomorphism.

**Definition 2.2.**

(i) Let \( i \) be an element of \( \{1, \ldots, n\} \). Then we shall write

\[
\gamma_{\mathcal{L}, i}: H^1(X^F, (\mathcal{L}^F)^{\otimes i}) \longrightarrow H^1(X, \mathcal{L}^{\otimes m})
\]

for the \( k \)-linear homomorphism obtained by applying \( "H^1(X^F, (-)^{\otimes \Phi}, (\mathcal{L}^F)^{\otimes i})" \) to the homomorphism \( \mathcal{O}_{X^F} \to \Phi^* \mathcal{O}_X \) determined by \( \Phi \) [cf. also Remark 2.1.1].

(ii) We shall say that the invertible sheaf \( \mathcal{L} \) is new-ordinary if, for every element \( i \in \{1, \ldots, n - 1\} \) with \( n\mathbb{Z} + i\mathbb{Z} = \mathbb{Z} \), the homomorphism \( \gamma_{\mathcal{L}, i} \) of (i) is an isomorphism.
REM 2.2.1.  

(i) One verifies immediately from the theory of finite étale cyclic coverings and generalized Hasse-Witt invariants [cf., e.g., [6], §2.1, or [7], pp.73-74] that

the existence of a new-ordinary invertible sheaf on $X$ of order $n$

is equivalent to

the existence of a new-ordinary finite étale cyclic covering of $X$ of degree $n$,

i.e., a finite étale cyclic covering of $X$ of degree $n$ that has a new ordinary part

in the sense of [6], Définition 2.1.1,

which thus implies

the existence of a finite étale cyclic covering of $X$ of degree $n$ whose Jacobian variety is of $p$-rank $\geq (g-1) \cdot \varepsilon/\{\mathbb{Z}/n\mathbb{Z}\}^\times (> 0)$.

(ii) Suppose that $p - 1 \in n\mathbb{Z}$. Then each trivialization $\iota$ of $L^\otimes n$ determines an isomorphism of invertible sheaves on $X$

$$\iota^{(p-1)/n}: L^{\otimes p} \sim \sim L.$$ 

Thus, the homomorphism $\gamma_{L,\iota}$ may be “identified” with the homomorphism

$$H^1(X^F, (L^F)^{\otimes i}) \longrightarrow H^1(X, L^{\otimes i}).$$

In particular, one verifies immediately from the theory of finite étale cyclic coverings and generalized Hasse-Witt invariants [cf., e.g., [6], §2.1, or [7], pp.73-74] that

the existence of an invertible sheaf $M$ on $X$ of order $n$ such that the homomorphism $\gamma_{M,\iota}$ is an isomorphism for some $i \in \{1, \ldots, n-1\}$

implies

the existence of a finite étale cyclic covering of $X$ of degree $n$ whose Jacobian variety is of $p$-rank $\geq \dim_k H^1(X, L^{\otimes i}) = g - 1 (> 0)$.

In the remainder of the present §2, write $J^F$ for the Jacobian variety of $X^F$ and $B^F$ for the $O_X$-module obtained by forming the cokernel of the homomorphism $O_{X^F} \to \Phi_*O_X$ determined by $\Phi$. Moreover, let us fix a universal invertible sheaf $P^F$ on $X^F \times_k J^F$ of degree zero.

DEF 2.3. — We shall write

$$\Theta_{B^F} \subseteq J^F$$

for the closed subscheme of $J^F$ defined by the zeroth Fitting ideal of the coherent $O_{J^F}$-module

$$\mathbb{R}^1(X^F \times_k J^F \xrightarrow{pr_2} J^F)_* (P^F \otimes_{O_{X^F \times_k J^F}} (X^F \times_k J^F \xrightarrow{pr_1} X^F)^* B^F)$$

[cf. also [7], Remark 1.1].
Proposition 2.4. — The following hold:

(i) The closed subscheme $\Theta_{BF} \subseteq J^F$ of $J^F$ forms a [necessarily effective] divisor on $J^F$ of level $p - 1$ [cf. Definition 1.3, (ii)].

(ii) Let $x \in J^F$ be a closed point of $J^F$ and $\mathcal{M}^F$ an invertible sheaf on $X^F$ of degree zero whose isomorphism class corresponds to $x \in J^F$. Then the following three conditions are equivalent:

1. The closed point $x \in J^F$ is not contained in $\Theta_{BF}$.
2. The equality $\Gamma(X^F, \mathcal{M}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{H}^F) = \{0\}$ holds.
3. The equality $H^1(X^F, \mathcal{M}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{H}^F) = \{0\}$ holds.

(iii) The underlying closed subset of the closed subscheme $\Theta_{BF} \subseteq J^F$ of $J^F$ is stabilized by the automorphism of $J^F$ given by multiplication by $-1$.

Proof. — First, we verify assertion (i). It follows from [5], Théorème 4.1.1, that the closed subscheme $\Theta_{BF} \subseteq J^F$ of $J^F$ forms a [necessarily effective] divisor on $J^F$. Moreover, since [it is well-known that] the “classical theta divisor” on $J^F$ gives rise to a principal polarization on $J^F$, it follows from [5], Proposition 1.8.1, (2) [cf. also [5], §4], together with Lemma 1.4, (iii), of the present paper [cf. also Remark 1.3.1 of the present paper], that the divisor determined by $\Theta_{BF} \subseteq J^F$ is of level $p - 1$, as desired. This completes the proof of assertion (i).

Assertion (ii) follows immediately from the definition of the closed subscheme $\Theta_{BF} \subseteq J^F$ [cf. also [5], §4; [7], Lemma 1.2]. Finally, we verify assertion (iii). Let us recall from the discussion preceding [5], Théorème 4.1.1, that there exists an isomorphism $\mathcal{B}^F \cong \mathcal{H}om_{\mathcal{O}_{X^F}}(\mathcal{B}^F, \Omega^1_{X^F/k})$ of $\mathcal{O}_{X^F}$-modules. Thus, assertion (iii) follows immediately from assertion (ii), together with Serre duality. This completes the proof of assertion (iii), hence also of Proposition 2.4.

Lemma 2.5. — The following hold:

(i) Suppose that $p \neq 2$. Then $J^F[p - 1]$ [cf. Definition 1.1] is not contained in $\Theta_{BF}$.

(ii) Suppose that $p = 2$. Then, for each odd integer $m \geq 3$, $J^F[m]$ is not contained in $\Theta_{BF}$.

Proof. — These assertions follow from Corollary 1.8 and Proposition 2.4, (i).

Lemma 2.6. — The following hold:

(i) Let $i$ be an element of $\{1, \ldots, n\}$. Then it holds that the homomorphism $\gamma_{\mathcal{L}, i}$ is an isomorphism if and only if the closed point of $J^F$ that corresponds to $(\mathcal{L}^F)^{\otimes i}$ is not contained in $\Theta_{BF} \subseteq J^F$.

(ii) It holds that the Jacobian variety of $X$ is ordinary if and only if the identity element of $J^F$ is not contained in $\Theta_{BF} \subseteq J^F$.

(iii) It holds that the invertible sheaf $\mathcal{L}$ is new-ordinary if and only if, for every element $i \in \{1, \ldots, n - 1\}$ with $n\mathbb{Z} + i\mathbb{Z} = \mathbb{Z}$, the closed point of $J^F$ that corresponds to $(\mathcal{L}^F)^{\otimes i}$ is not contained in $\Theta_{BF} \subseteq J^F$. 

(iv) Suppose that \( n \in \{2, 3, 4, 6\} \). Then it holds that the invertible sheaf \( \mathcal{L} \) is new-ordinary if and only if there exists an element \( i \in \{1, \ldots, n-1\} \) such that \( n\mathbb{Z} + i\mathbb{Z} = \mathbb{Z} \), and, moreover, the closed point of \( J^F \) that corresponds to \( (\mathcal{L}^F)^{\otimes i} \) is not contained in \( \Theta_{BF} \subseteq J^F \).

**Proof.** — Assertion (i) follows immediately from Proposition 2.4, (ii), together with the definition of the \( \mathcal{O}_{X^F} \)-module \( B^F \). Assertions (ii), (iii) follow from assertion (i) [cf. also [6], §2.1]. Finally, we verify assertion (iv). The necessity follows from assertion (iii). The sufficiency follows from Proposition 2.4, (iii), and assertion (iii). This completes the proof of assertion (iv), hence also of Lemma 2.6. □

One interesting application of the main result of the present paper is as follows.

**Theorem 2.7.** — Let \( p \) be a prime number, \( k \) an algebraically closed field of characteristic \( p \), and \( X \) a projective smooth curve over \( k \) of genus \( \geq 2 \). Then the following hold:

(i) Suppose that \( p \neq 2 \). Then there exist a positive integer \( n \) such that \( p-1 \in n\mathbb{Z} \) and a finite étale cyclic covering of \( X \) of degree \( n \) whose Jacobian variety is of positive \( p \)-rank.

(ii) Suppose that the Jacobian variety of \( X \) is not ordinary. Let \( n \) be an integer such that \( (p, n) \in \{(2, 3), (3, 2)\} \). Then there exists a new-ordinary finite étale cyclic covering of \( X \) of degree \( n \), i.e., a finite étale cyclic covering of \( X \) of degree \( n \) that has a new ordinary part in the sense of [6], Définition 2.1.1.

**Proof.** — Assertion (i) follows immediately — in light of Remark 2.2.1, (ii) — from Lemma 2.5, (i), and Lemma 2.6, (i), (ii). Assertion (ii) follows immediately — in light of Remark 2.2.1, (i) — from Lemma 2.5, (i), (ii), and Lemma 2.6, (ii), (iv). □

**Remark 2.7.1.** — Some results closely related to the content of Theorem 2.7 are as follows: In the situation of Theorem 2.7, suppose that \( X \) is of genus \( g \) (\( \geq 2 \)). Then:

(i) M. Raynaud proved that if, moreover, \( l \) is a prime number such that \( l + 1 \geq (p-1)3^{g-1}! \), then there exists a new-ordinary finite étale cyclic covering of \( X \) of degree \( l \) [cf. [5], Théorème 4.3.1; also [7], Remark 3.11].

(ii) S. Nakajima proved that if, moreover, \((g, p) = (2, 2)\), and the Jacobian variety of \( X \) is not ordinary [i.e., the curve \( X \) is either of type I or of type II in the sense of [4], §6], then every finite étale cyclic covering of \( X \) of degree three is new-ordinary [i.e., the curve \( X \) is 3-ordinary in the sense of the discussion at the beginning of [4], §4] [cf. [4], §6].

**Corollary 2.8.** — Let \( p \) be a prime number, \( k \) an algebraically closed field of characteristic \( p \), and \( X \) a projective smooth curve over \( k \) of genus \( \geq 2 \). Write \( \pi_1(X) \) for the étale fundamental group [for some choice of basepoint] of \( X \),

\[
n_p \overset{\text{def}}{=} \begin{cases} p - 1 & \text{if } p \neq 2 \\ 3 & \text{if } p = 2, \end{cases}
\]
$N \subseteq \pi_1(X)$ for the normal open subgroup of $\pi_1(X)$ obtained by forming the kernel of the natural surjective homomorphism

$$\pi_1(X) \twoheadrightarrow \pi_1(X)^{ab} \otimes_{\mathbb{Z}} (\mathbb{Z}/n_p\mathbb{Z}),$$

and $Y \to X$ for the finite étale abelian covering that corresponds to the normal open subgroup $N \subseteq \pi_1(X)$. Then the Jacobian variety of $Y$ is of positive $p$-rank. In particular, the maximal pro-$p$ abelian quotient of $N$ is nontrivial.

**Proof.** — This assertion is a formal consequence of Theorem 2.7, (i), (ii). \hfill \Box

**Remark 2.8.1.** — *M. Raynaud* proved that, in the situation of Corollary 2.8, the profinite group $\pi_1(X)$ is not pro-prime-to-$p$ [cf. [5], Corollaire 4.3.2]. Corollary 2.8 may be regarded as a refinement of this result.

**References**


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