RIMS-1910

On the Geometric Subgroups of the Étale Fundamental Groups of Varieties over Real Closed Fields

By

Yuichiro HOSHI, Takahiro MUROTANI, and Shota TSUJIMURA

December 2019

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

ON THE GEOMETRIC SUBGROUPS OF THE ÉTALE FUNDAMENTAL GROUPS OF VARIETIES OVER REAL CLOSED FIELDS

YUICHIRO HOSHI, TAKAHIRO MUROTANI, AND SHOTA TSUJIMURA

DECEMBER 2019

ABSTRACT. In the present paper, we establish a "group-theoretic" algorithm for reconstructing, from the étale fundamental group of a suitable proper normal variety over a real closed field, the geometric subgroup of the étale fundamental group of the proper normal variety.

CONTENTS

INTRODUCTION

In the present Introduction, let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k, and X an algebraic variety over k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group [determined by the algebraic closure $\overline{k}$] of k and $\pi_1(X)$, $\pi_1(X \times_k \overline{k})$ for the respective étale fundamental groups [relative to appropriate choices of basepoints] of X, $X \times_k \overline{k}$. Thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \pi_1(X \times_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1$$

[cf. [5], Exposé IX, Théorème 6.1].

Anabelian geometry is, in a word, an area of arithmetic geometry in which one studies the geometry of geometric objects of interest from the point of view of purely group-theoretic properties of the étale fundamental groups. Put another way, roughly speaking, anabelian geometry discusses the issue of how much information concerning the geometry of geometric objects of interest [e.g., "X" as above] is included in the knowledge of the étale fundamental groups [e.g., "$\pi_1(X)$" as above].

Here, let us recall that one form of anabelian geometry is *"relative anabelian geometry"*, in which instead of starting from the profinite group $\pi_1(X)$, one starts from the profinite group $\pi_1(X)$

2010 *Mathematics Subject Classification.* 14G27.

Key words and phrases. real closed field, étale fundamental group, anabelian geometry, absolute anabelian geometry.

equipped with the surjective homomorphism $\pi_1(X) \twoheadrightarrow G_k$ that appears in the above displayed exact sequence. By contrast, *"absolute anabelian geometry"*, that is one form of anabelian geometry, refers to the study of the geometry of X as reflected solely in the profinite group $\pi_1(X)$ [cf. [2], Introduction].

In various studies of absolute anabelian geometry, as the "first step" of the argument of the "reconstruction" of the geometry of the variety X that starts from the profinite group $\pi_1(X)$, one often attempts to give a *"group-theoretic" characterization* of the *geometric subgroup* of $\pi_1(X)$, i.e., the closed subgroup $\pi_1(X \times_k \overline{k}) \subseteq \pi_1(X)$ of $\pi_1(X)$ that appears in the above displayed exact sequence. In [2], *S. Mochizuki* has established a *"group-theoretic" algorithm* for reconstructing — from the étale fundamental group of a *smooth variety* over a finite extension of either the field $\mathbb{Q}$ of rational numbers [cf. [2], Theorem 2.6, (vi)] or the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$ for some prime number p [cf. [2], Theorem 2.6, (v)] — the geometric subgroup of the étale fundamental group of the smooth variety. The main purpose of the present paper is the establishment of a similar *"group-theoretic" reconstruction algorithm* for a *suitable proper normal variety over a real closed field.*

In the remainder of the present Introduction, suppose that k is *real closed* [i.e., is a field such that the k-algebra $k[t]/(t^2+1)$ is an *algebraically closed field*], and that X is a *proper normal variety* over k [cf. Definition 2.1]. In the present paper, the condition $(\mathscr{I})$ defined in Definition 3.3 plays a central role. We shall say that the proper normal variety X satisfies the condition $(\mathscr{I})$ if $q_Y \neq 2q_X$ for each connected finite étale double covering $Y \to X$ of X, where we write "$q_{(-)}$" for the *irregularity* [cf. Definition 3.1] of the proper normal variety "$(-)$" [i.e., over the algebraic closure of k in the function field of "$(-)$" — cf. Remark 2.1.1]. In §4 of the present paper, we prove that, for instance, each of

- a fiber product of finitely many *proper smooth curves of positive genus* over a real closed field,
- a *torsor over an abelian variety of positive dimension* over a real closed field, and
- a proper normal variety over the field $\mathbb{R}$ of real numbers such that if we write X^{an} for the complex analytic space associated to the proper normal variety $X \times_{\mathbb{R}} \mathbb{C}$ over the field $\mathbb{C}$ of complex numbers, then the first homology group $H_1(X^{\mathrm{an}}, \mathbb{Z})$ with integer coefficients of the topological space X^{an} is *infinite* and has *no nontrivial* 2*-torsion* element

satisfies the condition $(\mathscr{I})$ [cf. Proposition 4.1, Remark 4.1.1].

Some portion of the main result of the present paper may be summarized as follows [cf. Theorem 3.7].

> *There exists a "group-theoretic" algorithm*
>
> $$\Pi \qquad \rightsquigarrow \qquad \Delta(\Pi) \subseteq \Pi$$
>
> *for constructing — from a profinite group* Π *isomorphic to the étale fundamental group of a proper normal variety over a real closed field which satisfies the condition* $(\mathscr{I})$ *— a closed subgroup* $\Delta(\Pi) \subseteq \Pi$ *of* Π *that satisfies the following condition: In the above situation, if the proper normal variety* X *over the real closed field* k *satisfies the condition* $(\mathscr{I})$*, and one then applies this "group-theoretic" algorithm to the profinite group* $\pi_1(X)$*, then the equality*
>
> $$\Delta(\pi_1(X)) = \pi_1(X \times_k \overline{k})$$
>
> *of closed subgroups of* $\pi_1(X)$ *holds.*

Moreover, one immediate application of the "group-theoretic" reconstruction algorithm established in the present paper is as follows [cf. Corollary 3.8, Corollary 4.2].

Theorem. *For each $\Box \in \{\circ, \bullet\}$, let $k_\Box$ be a* ***real closed*** *field, $\overline{k}_\Box$ an algebraic closure of $k_\Box$, and $X_\Box$ a* ***proper normal variety*** *over $k_\Box$; write $\pi_1(X_\Box \times_{k_\Box} \overline{k}_\Box) \subseteq \pi_1(X_\Box)$ for the respective étale fundamental groups [relative to appropriate choices of basepoints] of $X_\Box \times_{k_\Box} \overline{k}_\Box$, $X_\Box$. Suppose that, for each $\Box \in \{\circ, \bullet\}$, one of the following three conditions is satisfied:*

(1) *The proper normal variety $X_\Box$ satisfies the condition $(\mathscr{I})$, i.e., it holds that $q_{Y_\Box} \neq 2q_{X_\Box}$ for each connected finite étale* ***double*** *covering $Y_\Box \to X_\Box$ of $X_\Box$.*
(2) *The abelianization of the maximal pro-2 quotient of the profinite group $\pi_1(X_\Box \times_{k_\Box} \overline{k}_\Box)$ is* ***torsion-free****.*
(3) *The field $k_\Box$ is* ***isomorphic*** *to $\mathbb{R}$. Moreover, if we write $X_\Box^{\mathrm{an}}$ for the complex analytic space associated to the proper normal variety $X_\Box \times_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$, then the first homology group $H_1(X_\Box^{\mathrm{an}}, \mathbb{Z})$ with integer coefficients of the topological space $X_\Box^{\mathrm{an}}$ has* ***no nontrivial*** ***2-torsion*** *element.*

Let

$$\alpha \colon \pi_1(X_\circ) \xrightarrow{\sim} \pi_1(X_\bullet)$$

be an ***isomorphism*** *of profinite groups. Then the equality $\alpha(\pi_1(X_\circ \times_{k_\circ} \overline{k}_\circ)) = \pi_1(X_\bullet \times_{k_\bullet} \overline{k}_\bullet)$ holds.*

Finally, in §4 of the present paper, we verify that there exists a proper normal variety (respectively, nonproper smooth curve) over a real closed field such that the geometric subgroup of the étale fundamental group of the proper normal variety (respectively, nonproper smooth curve) is *not characteristic* as the subgroup of the étale fundamental group [i.e., is *not preserved* by some automorphism of the étale fundamental group]. In particular, one may conclude that

> it is *impossible* to establish any "group-theoretic" reconstruction algorithm as above, i.e., for reconstructing the geometric subgroup of the étale fundamental group of the proper normal variety (respectively, nonproper smooth curve)

[cf. Remark 4.2.1] (respectively, [cf. Remark 4.2.2]).

The present paper is organized as follows: In §1, we discuss the Galois representations that arise from abelian varieties over real closed fields. In §2, we introduce and discuss the geometrically pro-$\mathscr{C}$ étale fundamental groups of proper normal varieties over real closed fields. In §3, we establish a *"group-theoretic" algorithm* for reconstructing — from [a profinite group isomorphic to] the étale fundamental group of a suitable proper normal variety over a real closed field — the [normal closed subgroup that corresponds to the] geometric subgroup of the étale fundamental group of the proper normal variety. In §4, we give some examples of proper normal varieties that satisfy the condition $(\mathscr{I})$. Moreover, we also discuss necessity of some conditions that appear in the statement of the main result of the present paper.

Acknowledgments. The first author was supported by JSPS KAKENHI Grant Number 18K03239. The second author was supported by JSPS KAKENHI Grant Number 19J10214. The third author was supported by JSPS KAKENHI Grant Number 18J10260. This research was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1. THE TATE MODULES OF ABELIAN VARIETIES OVER REAL CLOSED FIELDS

In the present §1, we discuss the Galois representations that arise from *abelian varieties over real closed fields* [cf. Proposition 1.4 below]. Let k be a *real closed* field [i.e., a field such that the k-algebra $k[t]/(t^2+1)$ is an *algebraically closed field*] and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group [determined by the algebraic closure $\overline{k}$] of k.

Proposition 1.1. *Let p be a prime number. Then the p-adic cyclotomic character $G_k \to \mathbb{Z}_p^\times$ determines an **isomorphism** $G_k \xrightarrow{\sim} \{\pm 1\} \subseteq \mathbb{Z}_p^\times$. In particular, the group G_k is **isomorphic** to the group $\mathbb{Z}/2\mathbb{Z}$.*

Proof. This assertion follows from the assumption that the field k is real closed. □

Lemma 1.2. *Let R be a $\mathbb{Z}[1/2]$-algebra and M a finitely generated R-module equipped with an action of G_k over R. Then there exists a **unique decomposition** $M = M_+ \oplus M_-$ of M by R-submodules such that this decomposition is **compatible** with the action of G_k, and, moreover, the resulting action $G_k \to \mathrm{Aut}_R(M_+)$ (respectively, $G_k \to \mathrm{Aut}_R(M_-)$) is **trivial** (respectively, determines an **isomorphism** $G_k \xrightarrow{\sim} \{\pm 1\} \subseteq \mathrm{Aut}_R(M_-)$ whenever the R-submodule M_- is **nontrivial**).*

Proof. Since G_k is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [cf. Proposition 1.1], this assertion follows from elementary algebra. □

Lemma 1.3. *Let M be a **finitely generated free** $\mathbb{Z}_2$-module equipped with an action of G_k over $\mathbb{Z}_2$. Thus, by applying Lemma 1.2 to the $\mathbb{Q}_2$-module $V \stackrel{\text{def}}{=} M \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$ $(\supseteq M)$ equipped with the action of G_k [i.e., determined by the action of G_k on M], one obtains a decomposition $V = V_+ \oplus V_-$ as in Lemma 1.2. Then there exists an exact sequence of $\mathbb{Z}_2$-modules equipped with actions of G_k over $\mathbb{Z}_2$*

$$1 \longrightarrow \mathbb{Z}_2(1)^{\oplus \dim_{\mathbb{Q}_2}(V_-)} \longrightarrow M \longrightarrow \mathbb{Z}_2^{\oplus \dim_{\mathbb{Q}_2}(V_+)} \longrightarrow 1$$

— where "(1)" denotes a Tate twist.

Proof. This assertion is immediate from the condition imposed on the decomposition of Lemma 1.2. □

Proposition 1.4. *Let g be a positive integer, A an **abelian variety** over k of dimension g, and p a prime number. Write T_pA for the p-adic Tate module of A [on which G_k acts naturally]. Then the following hold:*

(i) *Suppose that $p \neq 2$. Then there exists a G_k**-equivariant isomorphism***

$$T_pA \xrightarrow{\sim} \mathbb{Z}_p^{\oplus g} \oplus \mathbb{Z}_p(1)^{\oplus g}$$

— where "(1)" denotes a Tate twist.

(ii) *Suppose that $p = 2$. Then there exists an exact sequence of $\mathbb{Z}_2$-modules equipped with actions of G_k over $\mathbb{Z}_2$*

$$1 \longrightarrow \mathbb{Z}_2(1)^{\oplus g} \longrightarrow T_2A \longrightarrow \mathbb{Z}_2^{\oplus g} \longrightarrow 1$$

— where "(1)" denotes a Tate twist.

(iii) *Write $T_pA \twoheadrightarrow Q$ for the **maximal** G_k**-stable torsion-free** quotient of T_pA on which G_k acts **trivially**. Then the equality $\mathrm{rank}_{\mathbb{Z}_p}(Q) = g$ holds.*

Proof. First, we verify assertions (i), (ii). Let us first recall that it is well-known [cf. the discussion following [5], Exposé XI, Théorème 2.1] that the $\mathbb{Z}_p$-module T_pA is *free of rank* $2g$. Thus, it follows from Lemma 1.2 and Lemma 1.3 that, to verify assertions (i), (ii), it suffices to verify that the $\mathbb{Q}_p$-module $(T_pA \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)_+$ [cf. Lemma 1.2] is *of dimension* g. On the other hand, again by Lemma 1.2 and Lemma 1.3, this assertion follows immediately from the existence of a G_k-*equivariant* isomorphism $T_pA \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA, \mathbb{Q}_p)(1)$ obtained by, for instance, considering a *polarization* on the abelian variety A over k. This completes the proofs of assertions (i), (ii). Assertion (iii) follows from assertions (i), (ii). This completes the proof of Proposition 1.4. □

2. The Étale Fundamental Groups of Varieties over Real Closed Fields

In the present §2, we introduce and discuss the *geometrically pro-$\mathscr{C}$ étale fundamental groups* [cf. Definition 2.3, (ii), below] of proper normal varieties [cf. Definition 2.1 below] over real closed fields.

Definition 2.1. We shall say that a scheme V over a field F of characteristic zero is a *proper normal variety* over F if the scheme V is normal, and, moreover, the structure morphism $V \to \mathrm{Spec}(F)$ is proper and geometrically connected.

Remark 2.1.1. Let V be a proper normal variety over a field F of characteristic zero and $W \to V$ a connected finite étale covering of V. Then one verifies easily that W is a *proper normal variety* over a(n) [necessarily finite] extension of F obtained by forming the algebraic closure of F in the function field of W.

Definition 2.2. Let G be a profinite group. Then we shall write $(G \twoheadrightarrow)\ G^{\mathrm{ab}}$ for the *abelianization* of the profinite group G [i.e., the maximal abelian quotient of G whose kernel is closed in G] and $(G \twoheadrightarrow)\ G^{\mathrm{ab/tor}}$ for the maximal abelian torsion-free quotient of G whose kernel is closed in G:

$$G \twoheadrightarrow G^{\mathrm{ab}} \twoheadrightarrow G^{\mathrm{ab/tor}}.$$

In the remainder of the present §2, let k be a *real closed* field, $\overline{k}$ an algebraic closure of k, and X a *proper normal variety* over k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group [determined by the algebraic closure $\overline{k}$] of k. Write, moreover, $\pi_1(X)$, $\pi_1(X \times_k \overline{k})$ for the étale fundamental groups [relative to appropriate choices of basepoints] of X, $X \times_k \overline{k}$, respectively. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \pi_1(X \times_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1$$

[cf. [5], Exposé IX, Théorème 6.1]. Moreover, in the remainder of the present §2, let $\mathscr{C}$ be a full formation of finite groups, i.e., a family of finite groups that is closed under taking quotients, subgroups, and extensions.

Definition 2.3.

(i) We shall write Δ_X for the *pro-$\mathscr{C}$ geometric étale fundamental group* of X, i.e., the maximal pro-$\mathscr{C}$ quotient of $\pi_1(X \times_k \overline{k})$.

(ii) We shall write Π_X for the *geometrically pro-$\mathscr{C}$ étale fundamental group* of X, i.e., the quotient of $\pi_1(X)$ by the normal closed subgroup of $\pi_1(X)$ obtained by forming the kernel of the natural surjective homomorphism $(\pi_1(X) \supseteq)\ \pi_1(X \times_k \overline{k}) \twoheadrightarrow \Delta_X$. Thus, the exact sequence preceding the present Definition 2.3 determines an exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1,$$

which thus determines, by conjugation, an action of G_k on Δ_X^{ab}, hence also an action of G_k on $\Delta_X^{\mathrm{ab/tor}}$. We shall always regard Δ_X^{ab}, $\Delta_X^{\mathrm{ab/tor}}$ as G_k-modules by means of these actions, respectively.

(iii) We shall write

$$1 \longrightarrow {}_2\Delta_X \longrightarrow {}_2\Pi_X \longrightarrow G_k \longrightarrow 1$$

for the exact sequence of (ii) in the case where we take the full formation "$\mathscr{C}$" to be the full formation consisting of 2-groups.

Remark 2.3.1. Let us recall from [5], Exposé X, Théorème 2.9, that the profinite group Δ_X, hence also [cf. Proposition 1.1] the profinite group Π_X, is *topologically finitely generated*. In particular, the $\widehat{\mathbb{Z}}$-modules Δ_X^{ab}, Π_X^{ab}, ${}_2\Delta_X^{\mathrm{ab}}$, ${}_2\Pi_X^{\mathrm{ab}}$ are *finitely generated*. Moreover, the $\mathbb{Z}_2$-modules ${}_2\Delta_X^{\mathrm{ab/tor}}$, ${}_2\Pi_X^{\mathrm{ab/tor}}$ [cf. also Lemma 2.4, (i), below] are *finitely generated* and *free*.

Lemma 2.4. *The following hold:*

(i) *Suppose that the full formation $\mathscr{C}$ **contains** [a group isomorphic to] $\mathbb{Z}/2\mathbb{Z}$. Write T for the **maximal pro-**2 quotient of the profinite group Π_X. Then the natural surjective homomorphism $\pi_1(X) \twoheadrightarrow T$ determines an **isomorphism** ${}_2\Pi_X \xrightarrow{\sim} T$.*

(ii) *The restriction of the natural surjective homomorphism $\Pi_X \twoheadrightarrow \Pi_X^{\mathrm{ab}}$ to the closed subgroup $\Delta_X \subseteq \Pi_X$ determines an **isomorphism** of*
- *the **maximal** G_k**-stable** quotient of Δ_X^{ab} on which G_k acts **trivially***

with
- *the image of $\Delta_X \subseteq \Pi_X$ in Π_X^{ab}.*

(iii) *Write ${}_2\Delta_X^{\mathrm{ab}} \twoheadrightarrow Q$ for the **maximal** G_k**-stable torsion-free** quotient of ${}_2\Delta_X^{\mathrm{ab}}$ on which G_k acts **trivially**. Then the equality $\mathrm{rank}_{\mathbb{Z}_2}(Q) = \mathrm{rank}_{\mathbb{Z}_2}({}_2\Pi_X^{\mathrm{ab/tor}})$ holds.*

Proof. Assertions (i), (ii) follow immediately from the fact that G_k is an abelian 2-group [cf. Proposition 1.1]. Assertion (iii) follows from assertion (ii). This completes the proof of Lemma 2.4. □

Lemma 2.5. *The following hold:*

(i) *The following two conditions are equivalent:*
  (1) *Either that the full formation $\mathscr{C}$ does **not contain** [any group isomorphic to] $\mathbb{Z}/2\mathbb{Z}$, or that the pro-2 group ${}_2\Delta_X$ is **trivial**.*
  (2) *The maximal pro-2 quotient of Δ_X is **trivial**.*

(ii) *Suppose that conditions (1), (2) of (i) are satisfied. Then the following condition is satisfied:*
  (3) *The set of open subgroups of Π_X **of index** 2 consists of a **single** element.*

(iii) *Suppose that condition (3) of (ii) is satisfied [which is the case if, for instance, conditions (1), (2) of (i) are satisfied — cf. (ii)]. Then the open subgroup $\Delta_X \subseteq \Pi_X$ of Π_X is the **unique** [cf. condition (3)] open subgroup of Π_X **of index** 2.*

Proof. These assertions follow immediately from the fact that G_k is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [cf. Proposition 1.1]. □

Definition 2.6. Let Π be a profinite group.

(i) We shall say that the profinite group Π *satisfies* $(\mathscr{U})$ if the set of open subgroups of Π of index 2 consists of a single element.

(ii) Suppose that the profinite group Π satisfies the condition $(\mathscr{U})$. Then we shall write $\Delta(\Pi) \subseteq \Pi$ for the unique open subgroup of Π of index 2.

3. The Geometric Subgroups for Varieties over Real Closed Fields

In the present §3, we establish a *"group-theoretic" algorithm* for reconstructing — from [a profinite group isomorphic to] the étale fundamental group of a suitable proper normal variety over a real closed field — the [normal closed subgroup that corresponds to the] *geometric subgroup* of the étale fundamental group of the proper normal variety [cf. Theorem 3.7 below]. Let k be a *real closed* field, $\overline{k}$ an algebraic closure of k, X a *proper normal variety* over k [cf. Definition 2.1], and $\mathscr{C}$ a full formation of finite groups. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group [determined by the algebraic closure $\overline{k}$] of k. Now let us recall the exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1$$

of Definition 2.3, (ii).

Definition 3.1. Let F be a field of characteristic zero and V a proper normal variety over F. Then we shall write $q_V \stackrel{\mathrm{def}}{=} \dim_F H^1(V, \mathscr{O}_V)$ for the *irregularity* of V.

Remark 3.1.1. In the situation of Definition 3.1, let $\overline{F}$ be an algebraic closure of F. Then it is well-known [cf., e.g., [2], Proposition A.6, (iii), and its proof; also our assumption that F is *of characteristic zero*] that the irregularity q_V of V coincides with the *dimension of the Albanese variety* of the proper normal variety $V \times_F \overline{F}$ over $\overline{F}$ [cf., e.g., [2], Definition A.1, (ii); [2], Proposition A.6, (i)].

One main technical observation of the present paper is as follows.

Lemma 3.2. *The following hold:*

(i) *There exist an abelian variety A over k of dimension q_X and a* ***G_k-equivariant isomorphism*** $\Delta_X^{\mathrm{ab/tor}} \xrightarrow{\sim} \Delta_A^{\mathrm{ab}}$ [*cf. Definition* 2.3, (ii)].

(ii) *The free $\mathbb{Z}_2$-module ${}_2\Delta_X^{\mathrm{ab/tor}}$* [*cf. Remark* 2.3.1] *is* ***of rank*** $2q_X$.

(iii) *The free $\mathbb{Z}_2$-module ${}_2\Pi_X^{\mathrm{ab/tor}}$* [*cf. Remark* 2.3.1] *is* ***of rank*** q_X.

Proof. Assertion (i) follows immediately — in light of [2], Remark A.11.1 — from [2], Proposition A.6, (iv) [cf. also Remark 3.1.1 of the present paper]. Assertion (ii) follows from assertion (i), together with the well-known [cf. the discussion following [5], Exposé XI, Théorème 2.1] fact that if one writes $\widehat{\mathbb{Z}}^{\mathscr{C}}$ for the pro-$\mathscr{C}$ completion of $\widehat{\mathbb{Z}}$, then the $\widehat{\mathbb{Z}}^{\mathscr{C}}$-module Δ_B^{ab} is *free of rank* $2g$ whenever B is an abelian variety over k of dimension g.

Finally, we verify assertion (iii). Write ${}_2\Delta_X^{\mathrm{ab}} \twoheadrightarrow Q$ for the *maximal G_k-stable torsion-free* quotient of ${}_2\Delta_X^{\mathrm{ab}}$ on which G_k acts *trivially*. Then it follows from Lemma 2.4, (iii), that, to verify assertion (iii), it suffices to verify that $\mathrm{rank}_{\mathbb{Z}_2}(Q) = q_X$. On the other hand, this follows from assertion (i) and Proposition 1.4, (iii). This completes the proof of assertion (iii), hence also of Lemma 3.2. $\square$

Definition 3.3. We shall say that the proper normal variety X over k *satisfies* $(\mathscr{I})$ if $q_Y \neq 2q_X$ for each connected finite étale double covering $Y \to X$ of X [cf. also Remark 2.1.1].

Remark 3.3.1. It follows immediately from Lemma 3.2, (ii), together with the fact that G_k is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [cf. Proposition 1.1], that the following two conditions are equivalent:

(1) The proper normal variety X over k satisfies the condition $(\mathscr{I})$.

(2) It holds that $\mathrm{rank}_{\mathbb{Z}_2}(H^{\mathrm{ab/tor}}) \neq 2 \cdot \mathrm{rank}_{\mathbb{Z}_2}({}_2\Delta_X^{\mathrm{ab/tor}})$ for each open subgroup $H \subseteq {}_2\Delta_X$ of ${}_2\Delta_X$ of index either 1 or 2.

Definition 3.4. We shall say that a profinite group Π *satisfies* $(\mathscr{I})$ if there exist a proper normal variety V over a real closed field and a full formation $\mathscr{F}$ of finite groups such that the proper normal variety V satisfies the condition $(\mathscr{I})$, and, moreover, the profinite group Π is isomorphic to the geometrically pro-$\mathscr{F}$ étale fundamental group of V.

Lemma 3.5. *Suppose that the proper normal variety X over k satisfies the condition $(\mathscr{I})$. Let $J \subseteq {}_2\Pi_X$ be a normal open subgroup of ${}_2\Pi_X$. Then the following two conditions are equivalent:*

(1) *The equality $J = {}_2\Delta_X$ holds.*

(2) *The open subgroup J is of index 2 in ${}_2\Pi_X$, and, moreover, the equality $\mathrm{rank}_{\mathbb{Z}_2}(J^{\mathrm{ab/tor}}) = 2 \cdot \mathrm{rank}_{\mathbb{Z}_2}({}_2\Pi_X^{\mathrm{ab/tor}})$ holds.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 3.2, (ii), (iii), together with the fact that G_k is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [cf. Proposition 1.1]. Next, to verify the implication (2) $\Rightarrow$ (1), suppose that condition (2) is satisfied, but condition (1) is *not* satisfied. Write $Y \to X$ for the connected finite étale covering of X [necessarily of degree 2 — cf. condition (2)] that corresponds to the normal open subgroup $J \subseteq {}_2\Pi_X$ of ${}_2\Pi_X$. Then since [we have assumed that] $J \neq {}_2\Delta_X$, or, alternatively, the composite $J \hookrightarrow {}_2\Pi_X \twoheadrightarrow G_k$ is *surjective* [cf. Proposition 1.1], it follows that Y is a proper normal variety *over k*, which thus implies [cf. Lemma 3.2, (iii)] that the free $\mathbb{Z}_2$-module $J^{\mathrm{ab/tor}}$ is of rank q_Y. In particular, again by Lemma 3.2, (iii), it follows from condition (2) that the equality $q_Y = 2q_X$ holds. Thus, since [we have assumed that] the proper normal variety X over k satisfies the condition $(\mathscr{I})$, we obtain a *contradiction*. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Lemma 3.5. ☐

Definition 3.6. Let Π be a profinite group that satisfies the condition $(\mathscr{I})$.

(i) We shall write $Q(\Pi)$ for the maximal pro-2 quotient of Π.

(ii) It follows immediately — in light of Lemma 2.4, (i), and Lemma 2.5, (i), (ii), (iii) — from Lemma 3.5 that the set of normal open subgroups $J \subseteq Q(\Pi)$ of $Q(\Pi)$ that satisfy the following condition consists of a single element: The normal open subgroup J is of index 2 in $Q(\Pi)$, and, moreover, the equality $\mathrm{rank}_{\mathbb{Z}_2}(J^{\mathrm{ab/tor}}) = 2 \cdot \mathrm{rank}_{\mathbb{Z}_2}(Q(\Pi)^{\mathrm{ab/tor}})$ holds. We shall write $\Delta(Q(\Pi)) \subseteq Q(\Pi)$ for the unique element of this set.

(iii) We shall write $\Delta(\Pi) \subseteq \Pi$ for the normal open subgroup of Π obtained by forming the pull-back of $\Delta(Q(\Pi)) \subseteq Q(\Pi)$ by the natural surjective homomorphism $\Pi \twoheadrightarrow Q(\Pi)$.

Remark 3.6.1. Let Π be a profinite group. Suppose that Π satisfies either the condition $(\mathscr{U})$ [cf. Definition 2.6, (i)] or the condition $(\mathscr{I})$. Then, by applying Definition 2.6, (ii), or Definition 3.6, (iii), to Π, we obtain an open subgroup $\Delta(\Pi) \subseteq \Pi$ of Π [i.e., of index 2].

The main result of the present paper is as follows.

Theorem 3.7. *Let X be a **proper normal variety** over a **real closed** field [cf. Definition 2.1] and $\mathscr{C}$ a full formation of finite groups. Write Π_X for the **geometrically pro-$\mathscr{C}$ étale fundamental group** of X [cf. Definition 2.3, (ii)] and $\Delta_X \subseteq \Pi_X$ for the **pro-$\mathscr{C}$ geometric étale fundamental group** of X [cf. Definition 2.3, (i)]. Suppose that one of the following two conditions is satisfied:*

(1) *The set of open subgroups of* Π_X ***of index*** 2 *consists of a* ***single*** *element* [*which thus implies that the profinite group* Π_X *satisfies the condition* $(\mathscr{U})$ — *cf. Definition* 2.6, (i)].

(2) *It holds that* $q_Y \neq 2q_X$ [*cf. Definition* 3.1] *for each connected finite étale* ***double*** *covering* $Y \to X$ *of* X [*which thus implies that the profinite group* Π_X *satisfies the condition* $(\mathscr{I})$ — *cf. Definition* 3.4].

Then the equality $\Delta_X = \Delta(\Pi_X)$ [*cf. Remark* 3.6.1] *holds.*

Proof. This assertion follows from Lemma 2.5, (iii), and Lemma 3.5. □

Remark 3.7.1. The main result of the present paper, i.e., Theorem 3.7, may be summarized as follows:

For suitable choices of a proper normal variety X over a real closed field and a full formation $\mathscr{C}$ of finite groups, there exists a *"group-theoretic" algorithm*

$$\Pi_X \qquad \rightsquigarrow \qquad \Delta_X \subseteq \Pi_X$$

for reconstructing — from [a profinite group isomorphic to] the geometrically pro-$\mathscr{C}$ étale fundamental group Π_X of X — the [normal closed subgroup that corresponds to the] geometric subgroup $\Delta_X \subseteq \Pi_X$ of Π_X.

An immediate application of the main result of the present paper is as follows.

Corollary 3.8. *For each* $\Box \in \{\circ, \bullet\}$, *let* $k_\Box$ *be a* ***real closed*** *field,* $X_\Box$ *a* ***proper normal variety*** *over* $k_\Box$, *and* $\mathscr{C}_\Box$ *a full formation of finite groups; write* $\Pi_\Box$ *for the* ***geometrically pro-$\mathscr{C}_\Box$ étale fundamental group*** *of* $X_\Box$ *and* $\Delta_\Box \subseteq \Pi_\Box$ *for the* ***pro-$\mathscr{C}_\Box$ geometric étale fundamental group*** *of* $X_\Box$. *Suppose that, for each* $\Box \in \{\circ, \bullet\}$, *one of the following two conditions is satisfied:*

(1) *The set of open subgroups of* $\Pi_\Box$ ***of index*** 2 *consists of a* ***single*** *element.*

(2) *It holds that* $q_{Y_\Box} \neq 2q_{X_\Box}$ *for each connected finite étale* ***double*** *covering* $Y_\Box \to X_\Box$ *of* $X_\Box$.

Let

$$\alpha \colon \Pi_\circ \xrightarrow{\sim} \Pi_\bullet$$

be an ***isomorphism*** *of profinite groups. Then the equality* $\alpha(\Delta_\circ) = \Delta_\bullet$ *holds.*

Proof. This assertion follows from Theorem 3.7. □

4. Some Examples

In the present §4, we give some examples of proper normal varieties that satisfy the condition $(\mathscr{I})$ [cf. Proposition 4.1 and Remark 4.1.1 below]. Moreover, we also discuss *necessity* of some conditions that appear in the statement of Theorem 3.7 [cf. Remark 4.2.1 and Remark 4.2.2 below]. Write $\mathbb{R}$ for the field of real numbers [that is, as is well-known, a *real closed* field], $\mathbb{C}$ for the field of complex numbers [that is, as is well-known, an algebraic closure of $\mathbb{R}$], and $G_\mathbb{R} \stackrel{\text{def}}{=} \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ for the absolute Galois group [determined by the algebraic closure $\mathbb{C}$] of $\mathbb{R}$.

Proposition 4.1. *Let* k *be a real closed field and* X *a proper normal variety over* k [*cf. Definition* 2.1]. *Suppose that one of the following three conditions is satisfied:*

(1) *The pro-*2 *group* ${}_2\Delta_X$ [*cf. Definition* 2.3, (iii)] *is* ***abelian*** *and* ***infinite***.

(2) *The* $\mathbb{Z}_2$*-module* ${}_2\Delta_X^{\mathrm{ab}}$ *is* ***nontrivial*** *and* ***torsion-free***.

(3) *The field k is* ***isomorphic*** *to $\mathbb{R}$. Moreover, if we write X^{an} for the complex analytic space associated to the proper normal variety $X \times_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$, then the first homology group $H_1(X^{\mathrm{an}}, \mathbb{Z})$ with integer coefficients of the topological space X^{an} is* ***infinite*** *and has* ***no nontrivial*** ***2-torsion*** *element.*

Then the proper normal variety X satisfies the condition $(\mathscr{I})$ [cf. Definition 3.3].

Proof. Write $d \stackrel{\text{def}}{=} \mathrm{rank}_{\mathbb{Z}_2}({}_2\Delta_X^{\mathrm{ab/tor}})$. Let $H \subseteq {}_2\Delta_X$ be an open subgroup of ${}_2\Delta_X$ of index either 1 or 2. Thus, it follows from Remark 3.3.1 that, to verify the desired assertion [i.e., that the proper normal variety X satisfies the condition $(\mathscr{I})$], it suffices to verify that $\mathrm{rank}_{\mathbb{Z}_2}(H^{\mathrm{ab/tor}}) \neq 2d$.

First, we verify Proposition 4.1 in the case where condition (1) is satisfied. Suppose that condition (1) is satisfied. Then since ${}_2\Delta_X$ is abelian [cf. condition (1)], the equality $\mathrm{rank}_{\mathbb{Z}_2}(H^{\mathrm{ab/tor}}) = d$ holds [cf. also Remark 2.3.1]. Thus, since $d > 0$ [cf. condition (1)], we obtain that $\mathrm{rank}_{\mathbb{Z}_2}(H^{\mathrm{ab/tor}}) = d < 2d$, as desired. This completes the proof of Proposition 4.1 in the case where condition (1) is satisfied.

Next, we verify Proposition 4.1 in the case where condition (2) is satisfied. Suppose that condition (2) is satisfied. Let F_2^d be a *free pro-*2 *group of rank d*. Then it follows from condition (2) that $d = \mathrm{rank}_{\mathbb{Z}/2\mathbb{Z}}({}_2\Delta_X^{\mathrm{ab}} \otimes_{\mathbb{Z}_2} (\mathbb{Z}/2\mathbb{Z})) > 0$. Thus, it follows from [3], Theorem 7.8.1, that there exists a *surjective* homomorphism $F_2^d \twoheadrightarrow {}_2\Delta_X$. Now since $d > 0$, if H is of index 1 in ${}_2\Delta_X$ [i.e., $H = {}_2\Delta_X$], then it follows that $\mathrm{rank}_{\mathbb{Z}_2}(H^{\mathrm{ab/tor}}) = d \neq 2d$. Thus, in the remainder of the proof of Proposition 4.1 in the case where condition (2) is satisfied, we may assume without loss of generality that H is of index 2 in ${}_2\Delta_X$. Write $\widetilde{H} \subseteq F_2^d$ for the open subgroup of F_2^d obtained by forming the pull-back of $H \subseteq {}_2\Delta_X$ by the *surjective* homomorphism $F_2^d \twoheadrightarrow {}_2\Delta_X$. Then since $\widetilde{H}$ is of index 2 in F_2^d, it follows from [3], Theorem 3.6.2, that the pro-2 group $\widetilde{H}$ is isomorphic to a *free pro-*2 *group of rank* $2d-1$. Thus, we conclude that $2d > 2d-1 = \mathrm{rank}_{\mathbb{Z}_2}(\widetilde{H}^{\mathrm{ab/tor}}) \geq \mathrm{rank}_{\mathbb{Z}_2}(H^{\mathrm{ab/tor}})$, as desired. This completes the proof of Proposition 4.1 in the case where condition (2) is satisfied.

Finally, we verify Proposition 4.1 in the case where condition (3) is satisfied. Suppose that condition (3) is satisfied. Write $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$ for the topological fundamental group [relative to an appropriate choice of basepoint] of the topological space X^{an}. Then it follows from [5], Exposé XII, Corollaire 5.2, that $\pi_1(X \times_{\mathbb{R}} \mathbb{C})$ is isomorphic to the profinite completion of $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$. In particular, it follows from the *Hurewicz theorem* that the $\mathbb{Z}_2$-module ${}_2\Delta_X^{\mathrm{ab}}$ is isomorphic to the $\mathbb{Z}_2$-module $H_1(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Thus, since [we have assumed that] the [necessarily finitely generated] module $H_1(X^{\mathrm{an}}, \mathbb{Z})$ is infinite and has no nontrivial 2-torsion element, we conclude that the $\mathbb{Z}_2$-module ${}_2\Delta_X^{\mathrm{ab}}$ is nontrivial and torsion-free. In particular, the proper normal variety X satisfies condition (2), hence also the condition $(\mathscr{I})$, as desired. This completes the proof of Proposition 4.1 in the case where condition (3) is satisfied, hence also Proposition 4.1. □

Remark 4.1.1.

(i) It follows from [5], Exposé X, Théorème 2.6, together with [5], Exposé X, Corollaire 1.7, that every fiber product of finitely many *proper smooth curves of positive genus* over a real closed field satisfies condition (2) in the statement of Proposition 4.1, hence also [cf. Proposition 4.1] the condition $(\mathscr{I})$. Thus, again by [5], Exposé X, Théorème 2.6, together with [5], Exposé X, Corollaire 1.7 [cf. also Lemma 2.5, (ii), of the present paper], every fiber product of finitely many *proper smooth curves* over a real closed field satisfies either condition (1) or condition (2) in the statement of Theorem 3.7 [i.e., for an arbitrary choice of "$\mathscr{C}$"].

(ii) It follows from the discussion following [5], Exposé XI, Théorème 2.1, that every *torsor over an abelian variety of positive dimension* over a real closed field satisfies condition (2) in the statement of Proposition 4.1, hence also [cf. Proposition 4.1] the condition $(\mathscr{I})$. Thus, it follows from Lemma 2.5, (ii), that every *torsor over an abelian variety* over a real closed field satisfies either condition (1) or condition (2) in the statement of Theorem 3.7 [i.e., for an arbitrary choice of "$\mathscr{C}$"].

Corollary 4.2. *For each $\square \in \{\circ, \bullet\}$, let $k_\square$ be a* ***real closed*** *field, $X_\square$ a* ***proper normal variety*** *over $k_\square$, and $\mathscr{C}_\square$ a full formation of finite groups; write $\Pi_\square$ for the* ***geometrically pro-$\mathscr{C}_\square$ étale fundamental group*** *of $X_\square$ [cf. Definition 2.3, (ii)] and $\Delta_\square \subseteq \Pi_\square$ for the* ***pro-$\mathscr{C}_\square$ geometric étale fundamental group*** *of $X_\square$ [cf. Definition 2.3, (i)]. Suppose that, for each $\square \in \{\circ, \bullet\}$, one of the following seven conditions is satisfied:*

(1) *The full formation $\mathscr{C}_\square$ does* ***not contain*** *[any group isomorphic to] $\mathbb{Z}/2\mathbb{Z}$.*
(2) *The maximal pro-2 quotient of $\Delta_{X_\square}$ is* ***trivial****.*
(3) *The maximal pro-2 quotient of $\Delta_{X_\square}$ is* ***abelian*** *and* ***infinite****.*
(4) *The abelianization [cf. Definition 2.2] of the maximal pro-2 quotient of $\Delta_{X_\square}$ is* ***torsion-free****.*
(5) *The field $k_\square$ is* ***isomorphic*** *to $\mathbb{R}$. Moreover, if we write $X_\square^{\mathrm{an}}$ for the complex analytic space associated to the proper normal variety $X_\square \times_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$, then the first homology group $H_1(X_\square^{\mathrm{an}}, \mathbb{Z})$ with integer coefficients of the topological space $X_\square^{\mathrm{an}}$ has* ***no nontrivial*** *2-**torsion** element.*
(6) *The proper normal variety $X_\square$ is isomorphic to the fiber product of finitely many* ***proper smooth curves*** *over $k_\square$.*
(7) *The proper normal variety $X_\square$ is isomorphic to a* ***torsor over an abelian variety*** *over $k_\square$.*

Let

$$\alpha \colon \Pi_\circ \xrightarrow{\sim} \Pi_\bullet$$

be an ***isomorphism*** *of profinite groups. Then the equality $\alpha(\Delta_\circ) = \Delta_\bullet$ holds.*

Proof. Let us recall from [3], Theorem 7.8.1, that, for a given pro-2 group G, it holds that the pro-2 group G is *trivial* if and only if the $\mathbb{Z}_2$-module G^{ab} is *trivial*. Thus, Corollary 4.2 follows — in light of Lemma 2.5, (ii); Proposition 4.1; Remark 4.1.1 — from Corollary 3.8. ☐

Remark 4.2.1. In Theorem 3.7, we have established a *"group-theoretic" reconstruction algorithm*

$$\pi_1(X) \quad \rightsquigarrow \quad \pi_1(X \times_k \overline{k}) \subseteq \pi_1(X)$$

for a proper normal variety X over a real closed field k that satisfies either condition (1) or condition (2) in the statement of Theorem 3.7 [cf. Remark 3.7.1]. Here, let us observe that

> there exists a proper normal variety over a real closed field such that it is *impossible* to establish a similar "group-theoretic" reconstruction algorithm for the proper normal variety.

An example of such a proper normal variety is given as follows: Let X be a(n) [necessarily projective smooth] *Enriques surface* over $\mathbb{R}$ that has an $\mathbb{R}$-*rational point*. [Note that one verifies easily that such an Enriques surface exists.] Now let us recall from [5], Exposé IX, Théorème 6.1, that the sequence of profinite groups

$$1 \longrightarrow \pi_1(X \times_{\mathbb{R}} \mathbb{C}) \longrightarrow \pi_1(X) \longrightarrow G_{\mathbb{R}} \longrightarrow 1$$

is exact. Moreover, it is well-known [cf., e.g., [1], Chapter VIII, Lemma 15.1, (ii); [5], Exposé XII, Corollaire 5.2] that the group $\pi_1(X \times_{\mathbb{R}} \mathbb{C})$ is isomorphic to the group $\mathbb{Z}/2\mathbb{Z}$. In particular, since the group $G_{\mathbb{R}}$ is isomorphic to the group $\mathbb{Z}/2\mathbb{Z}$ [cf. Proposition 1.1], and an $\mathbb{R}$-rational point of X gives rise to a splitting of the above exact sequence, we conclude that the group $\pi_1(X)$ is isomorphic to the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus, one verifies immediately that there exists an automorphism of $\pi_1(X)$ that does not preserve the open subgroup $\pi_1(X \times_{\mathbb{R}} \mathbb{C}) \subseteq \pi_1(X)$ of $\pi_1(X)$. In particular, it is *impossible* to establish a "group-theoretic" reconstruction algorithm as in Theorem 3.7 for the Enriques surface X over $\mathbb{R}$.

Remark 4.2.2. As discussed in Remark 4.1.1, (i), a *proper smooth curve* X over a real closed field k is subject to the *"group-theoretic" reconstruction algorithm*

$$\pi_1(X) \quad \rightsquigarrow \quad \pi_1(X \times_k \overline{k}) \subseteq \pi_1(X)$$

of Theorem 3.7 [cf. Remark 3.7.1]. On the other hand,

> there exists a *nonproper smooth curve* over a real closed field such that it is *impossible* to establish a similar "group-theoretic" reconstruction algorithm for the *nonproper smooth curve*.

An example of such a nonproper smooth curve is given as follows: Write X for the spectrum of the $\mathbb{R}$-algebra

$$\mathbb{R}[x,y,z]/(x^2+y^2+1, xz-1).$$

Then one verifies easily that X is a *smooth curve* over $\mathbb{R}$ that satisfies condition (2) in the statement of Lemma 4.3, (i), below, which thus implies [cf. Lemma 4.3, (i), below] that the étale fundamental group $\pi_1(X)$ of X is a *free* profinite group. In particular, since [one also verifies easily that] the scheme $X \times_{\mathbb{R}} \mathbb{C}$ is isomorphic to the complement in the projective line over $\mathbb{C}$ of distinct 4 closed points, it follows from Lemma 4.3, (ii), below that $\pi_1(X)$ is a *free profinite group of rank* 2. Thus, it follows from Lemma 4.4, (i), (ii), below that there exists an automorphism of $\pi_1(X)$ that does not preserve the open subgroup $\pi_1(X \times_{\mathbb{R}} \mathbb{C}) \subseteq \pi_1(X)$ of $\pi_1(X)$. In particular, it is *impossible* to establish a "group-theoretic" reconstruction algorithm as in Theorem 3.7 for the smooth curve X over $\mathbb{R}$.

Lemma 4.3. *Let X be a smooth curve over $\mathbb{R}$. Then the following hold:*

- (i) *The following two conditions are equivalent:*
  - (1) *The étale fundamental group $\pi_1(X)$ of X is a **free** profinite group.*
  - (2) *The smooth curve X is **not proper** over $\mathbb{R}$ and, moreover, has **no** $\mathbb{R}$**-rational point**.*
- (ii) *Let d be a positive integer. Suppose that the smooth curve X has **no** $\mathbb{R}$**-rational point**, and that $X \times_{\mathbb{R}} \mathbb{C}$ is **isomorphic** to the complement in the projective line over $\mathbb{C}$ of distinct $2d$ closed points. Then the étale fundamental group $\pi_1(X)$ of X is a **free** profinite group **of rank** d.*

Proof. Let us recall from [5], Exposé IX, Théorème 6.1, that the sequence of profinite groups

$$1 \longrightarrow \pi_1(X \times_{\mathbb{R}} \mathbb{C}) \longrightarrow \pi_1(X) \longrightarrow G_{\mathbb{R}} \longrightarrow 1$$

is exact.

First, we verify the implication (1) $\Rightarrow$ (2) in assertion (i). Suppose that condition (1) is satisfied. Then since $\pi_1(X \times_{\mathbb{R}} \mathbb{C})$ is of index 2 in the *free* profinite group $\pi_1(X)$ [cf. Proposition 1.1], the profinite group $\pi_1(X \times_{\mathbb{R}} \mathbb{C})$ is *free*. Thus, it follows immediately from [5], Exposé X, Théorème

2.6, that X is *not proper* over $\mathbb{R}$. Moreover, if X has an $\mathbb{R}$-*rational point*, then the natural surjective homomorphism $\pi_1(X) \twoheadrightarrow G_{\mathbb{R}}$ has a splitting; in particular, since the group $G_{\mathbb{R}}$ is nontrivial and finite [cf. Proposition 1.1], the *free* profinite group $\pi_1(X)$ has a *nontrivial torsion* element, which thus implies that we obtain a *contradiction*. This completes the proof of the implication (1) $\Rightarrow$ (2) in assertion (i).

Next, we verify the implication (2) $\Rightarrow$ (1) in assertion (i). Suppose that condition (2) is satisfied. Write X^{an} for the complex analytic space associated to the smooth curve $X \times_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$ and $Q \stackrel{\mathrm{def}}{=} X^{\mathrm{an}}/G_{\mathbb{R}}$ for the quotient space of X^{an} by the natural action of $G_{\mathbb{R}}$. Write, moreover, $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$, $\pi_1^{\mathrm{top}}(Q)$ for the topological fundamental groups [relative to appropriate choices of basepoints] of the topological spaces X^{an}, Q, respectively. Then since the natural action of the group $G_{\mathbb{R}}$ of order 2 [cf. Proposition 1.1] on X^{an} has no fixed point [cf. condition (2)], the natural surjective map $X^{\mathrm{an}} \twoheadrightarrow Q$ is a *double covering map*, which thus determines an *exact* sequence of groups

$$1 \longrightarrow \pi_1^{\mathrm{top}}(X^{\mathrm{an}}) \longrightarrow \pi_1^{\mathrm{top}}(Q) \longrightarrow G_{\mathbb{R}} \longrightarrow 1.$$

Moreover, since X^{an}, hence also Q, is a *noncompact* [cf. condition (2); [5], Exposé XII, Proposition 3.2] *topological surface*, it is well-known [cf., e.g., [4], §4.2.2] that $\pi_1^{\mathrm{top}}(Q)$ is a *free* group.

Next, let us observe that it follows from [5], Exposé XII, Corollaire 5.2, that $\pi_1(X \times_{\mathbb{R}} \mathbb{C})$ is isomorphic to the profinite completion of $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$. Moreover, it follows immediately from the various definitions involved [cf., especially, the definition of Q] that an *isomorphism* of $\pi_1(X \times_{\mathbb{R}} \mathbb{C})$ with the profinite completion of $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$ extends to an *isomorphism* of $\pi_1(X)$ with the profinite completion of $\pi_1^{\mathrm{top}}(Q)$. Thus, since $\pi_1^{\mathrm{top}}(Q)$ is a *free* group, the étale fundamental group $\pi_1(X)$ of X is a *free* profinite group, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1) in assertion (i), hence also of assertion (i).

Finally, we verify assertion (ii). Suppose that the smooth curve X has no $\mathbb{R}$-rational point, and that $X \times_{\mathbb{R}} \mathbb{C}$ is isomorphic to the complement in the projective line over $\mathbb{C}$ of distinct $2d$ closed points. Let us observe that it follows from assertion (i) that $\pi_1(X)$ is a free profinite group. Moreover, it follows immediately from [5], Exposé XII, Corollaire 5.2, that $\pi_1(X \times_{\mathbb{R}} \mathbb{C})$ is a *free profinite group of rank* $2d-1$. Thus, since $\pi_1(X \times_{\mathbb{R}} \mathbb{C})$ is of index 2 in $\pi_1(X)$ [cf. Proposition 1.1], it follows from [3], Theorem 3.6.2, that $\pi_1(X)$ is a *free profinite group of rank* d, as desired. This completes the proof of assertion (ii), hence also of Lemma 4.3. ☐

Lemma 4.4. *Let* d *be a positive integer and* G *a* ***free*** *profinite group* ***of rank*** d. *Then the following hold:*

(i) *Suppose that* $d \geq 2$. *Then the set of open subgroups of* G ***of index*** 2 *is* ***of cardinality*** ≥ 2.
(ii) *Let* $H_1, H_2 \subseteq G$ *be open subgroups of* G ***of index*** 2. *Then there exists an automorphism* α *of the profinite group* G *such that* $\alpha(H_1) = H_2$.

Proof. Let $\{g_1, \ldots, g_d\} \subseteq G$ be a free generator of G. Write $\pi\colon G \twoheadrightarrow V \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} (\mathbb{Z}/2\mathbb{Z})$ for the natural surjective homomorphism. Then it is immediate that V has a natural structure of $\mathbb{Z}/2\mathbb{Z}$-module of dimension d; moreover, the subset $\{\pi(g_1), \ldots, \pi(g_d)\} \subseteq V$ forms a basis of the $\mathbb{Z}/2\mathbb{Z}$-module V. For a subset $S \subseteq \{1, \ldots, d\}$, write $\chi_S\colon V \to \mathbb{Z}/2\mathbb{Z}$ for the $\mathbb{Z}/2\mathbb{Z}$-linear homomorphism given by, for each $i \in \{1, \ldots, d\}$, mapping $\pi(g_i) \in V$ to $1 \in \mathbb{Z}/2\mathbb{Z}$ (respectively, $0 \in \mathbb{Z}/2\mathbb{Z}$) if $i \in S$ (respectively, $i \notin S$). Then one verifies easily that, for an arbitrary open subgroup of G of index 2, there exists a *unique* nonempty subset $S \subseteq \{1, \ldots, d\}$ such that the open subgroup coincides with $\pi^{-1}(\mathrm{Ker}(\chi_S))$. Thus, assertion (i) holds.

Next, to verify assertion (ii), let us observe that it follows from the proof of assertion (i) that, for each $i \in \{1,2\}$, there exists a nonempty subset $S_i \subseteq \{1,\ldots,d\}$ such that $H_i = \pi^{-1}(\mathrm{Ker}(\chi_{S_i}))$. Moreover, let us also observe that one verifies easily that, for a nonempty subset $S = \{i_1,\ldots,i_{\sharp S}\} \subseteq \{1,\ldots,d\}$, the automorphism of G given by, for each $i \in \{1,\ldots,d\}$, mapping $g_i \in G$ to $g_i \in G$ (respectively, to $g_i \cdot g_{i_1} \in G$) if $i \notin S \setminus \{i_1\}$ (respectively, $i \in S \setminus \{i_1\}$) maps $\pi^{-1}(\mathrm{Ker}(\chi_S)) \subseteq G$ bijectively onto $\pi^{-1}(\mathrm{Ker}(\chi_{\{i_1\}})) \subseteq G$. Thus, to verify assertion (ii), we may assume without loss of generality that, for each $i \in \{1,2\}$, the subset $S_i \subseteq \{1,\ldots,d\}$ is of cardinality 1, i.e., that $S_1 = \{a\}$ and $S_2 = \{b\}$ for some a, $b \in \{1,\ldots,d\}$. Thus, one verifies immediately that the automorphism of G given by, for each $i \in \{1,\ldots,d\}$, mapping $g_i \in G$ to $g_i \in G$ (respectively, to $g_a \in G$; to $g_b \in G$) if $i \notin \{a,b\}$ (respectively, $i = b$; $i = a$) maps $H_1 \subseteq G$ bijectively onto $H_2 \subseteq G$, as desired. This completes the proof of assertion (ii), hence also of Lemma 4.4. □

References

[1] W. Barth, C. Peters, and A. Van de Ven, Compact complex surfaces. Ergebnisse der Mathematik und ihrer Grenzgebiete **(3)**, **4**. *Springer-Verlag*, *Berlin*, 1984.

[2] S. Mochizuki, Topics in absolute anabelian geometry I: generalities. *J. Math. Sci. Univ. Tokyo* **19** (2012), no. **2**, 139-242.

[3] L. Ribes and P. Zalesskii, Profinite groups. Ergebnisse der Mathematik und ihrer Grenzgebiete. **3**. Folge. A Series of Modern Surveys in Mathematics, **40**. *Springer-Verlag*, *Berlin*, 2000.

[4] J. Stillwell, Classical topology and combinatorial group theory. Second edition. Graduate Texts in Mathematics, **72**. *Springer-Verlag*, *New York*, 1993.

[5] Revêtements étales et groupe fondamental (SGA 1). Séminaire de géométrie algébrique du Bois Marie 1960-61. Directed by A. Grothendieck. With two papers by M. Raynaud. Updated and annotated reprint of the 1971 original. Documents Mathématiques (Paris), **3**. *Société Mathématique de France*, *Paris*, 2003.

(Yuichiro Hoshi) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

Email address: yuichiro@kurims.kyoto-u.ac.jp

(Takahiro Murotani) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

Email address: murotani@kurims.kyoto-u.ac.jp

(Shota Tsujimura) RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

Email address: stsuji@kurims.kyoto-u.ac.jp