Pseudo-rigid \( p \)-Torsion
Finite Flat Commutative Group Schemes

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ABSTRACT. — Let $p$ be a prime number and $k$ a perfect field of characteristic $p$. In the present paper, we study deformations of finite flat commutative group schemes over $k$ to the ring $W$ of Witt vectors with coefficients in $k$. We prove that, for a given $p$-torsion finite flat commutative group scheme over $k$, it holds that the group scheme is pseudo-rigid — i.e., roughly speaking, has a unique, up to isomorphism over $W$, deformation to $W$ — if and only if the group scheme is either étale, multiplicative, or superspecial.

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INTRODUCTION

Let $p$ be a prime number and $k$ a perfect field of characteristic $p$. In the present paper, we study deformations of finite flat commutative group schemes over $k$ to the ring $W \overset{\text{def}}{=} W(k)$ of Witt vectors with coefficients in $k$. More specifically, we study pseudo-rigid $p$-torsion finite flat commutative group schemes over $k$.

A finite flat commutative group scheme over $W$ is one fundamental object in the study of arithmetic geometry from the point of view of Galois representations. Now let us observe that, for two finite flat commutative group schemes $H_1$, $H_2$ over $W$, if $H_1$ is isomorphic to $H_2$ over $W$, then it is immediate that $H_1 \times_W k$ is isomorphic to $H_2 \times_W k$ over $k$. However, in this situation, the existence of an isomorphism $H_1 \times_W k \overset{\sim}{\to} H_2 \times_W k$ over $k$ does in general not imply the existence of an isomorphism $H_1 \overset{\sim}{\to} H_2$ over $W$. Put another way, the isomorphism class of a finite flat commutative group scheme over $W$ is
in general not determined by the isomorphism class of the special fiber over \( k \). A central problem discussed in the present paper is as follows:

Give a sufficient and necessary condition for a finite flat commutative group scheme that ensures this converse implication.

Let \( G \) be a \( p \)-torsion finite flat commutative group scheme over \( k \). Then we shall say that \( G \) is pseudo-rigid [cf. Definition 1.7] if the following two conditions are satisfied:

- There exists a \( p \)-torsion finite flat commutative group scheme \( H \) over \( W \) such that \( H \times_W k \) is isomorphic to \( G \) over \( k \).
- If \( H_1, H_2 \) are \( p \)-torsion finite flat commutative group schemes over \( W \) such that \( H_i \times_W k \) is isomorphic to \( G \) over \( k \) for each \( i \in \{1, 2\} \), then \( H_1 \) is isomorphic to \( H_2 \) over \( W \).

Thus, roughly speaking, we say that \( G \) is pseudo-rigid if \( G \) has a deformation to \( W \), and, moreover, arbitrary two deformations of \( G \) to \( W \) are isomorphic over \( W \).

A typical example of a pseudo-rigid finite flat commutative group scheme is an \( \acute{e} \text{tale} \) commutative group scheme over \( k \). Moreover, one verifies immediately, by considering the Cartier dual, that a multiplicative [cf. Definition 4.9, (i)] finite flat commutative group scheme over \( k \) is pseudo-rigid. The main result of the present paper gives a sufficient and necessary condition for a \( p \)-torsion finite flat commutative group scheme over \( k \) to be pseudo-rigid. The main result of the present paper is as follows [cf. Theorem 4.11]:

\textbf{THEOREM.} — Let \( G \) be a \( p \)-torsion finite flat commutative group scheme over \( k \). Suppose that either \( G \) or the Cartier dual of \( G \) is connected whenever \( p = 2 \). Then it holds that \( G \) is pseudo-rigid if and only if \( G \) is either \( \acute{e} \text{tale} \) over \( k \), multiplicative, or superspecial [cf. Definition 4.9, (ii)].

The present paper is organized as follows: In \$1 \$, we introduce the notion of pseudo-rigidity, that is one central notion of the present paper. In \$2 \$, we give a review of a certain linear algebra theory for \( p \)-torsion finite flat commutative group schemes over \( k \). In \$3 \$, we give a review of a certain linear algebra theory for \( p \)-torsion finite flat commutative group schemes over \( W \). In \$4 \$, we prove the main result of the present paper.

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\section{\textit{p}-Torsion Finite Flat Commutative Group Schemes}

In the present \$1 \$, we introduce the notion of pseudo-rigidity [cf. Definition 1.7 below], that is one central notion of the present paper.

In the present \$1 \$, let \( p \) be a prime number and \( k \) a perfect field of characteristic \( p \). Write \( W \overset{\text{def}}{=} W(k) \) for the ring of Witt vectors with coefficients in \( k \). Let \( R \) be a \( W \)-algebra.
**Definition 1.1.** — We shall write
\[ \text{ffg}_R^{[p]} \]
for the category defined as follows:
- An object of the category \( \text{ffg}_R^{[p]} \) is defined to be a \( p \)-torsion finite flat commutative group scheme over \( R \).
- A morphism in the category \( \text{ffg}_R^{[p]} \) is defined to be a homomorphism between finite flat commutative group schemes over \( R \).

**Definition 1.2.** — We shall write
\[ \text{cffg}_R^{[p]} \subseteq \text{ffg}_R^{[p]} \]
for the full subcategory of \( \text{ffg}_R^{[p]} \) consisting of connected \( p \)-torsion finite flat commutative group schemes over \( R \).

**Definition 1.3.** — Let \( G \) be an object of the category \( \text{ffg}_R^{[p]} \). Then we shall write
\[ G^D \]
for the object of the category \( \text{ffg}_R^{[p]} \) obtained by forming the Cartier dual of \( G \).

**Remark 1.3.1.** — One verifies immediately that the assignment “\( G \rightsquigarrow G^D \)” defines a [contravariant] functor \( \text{ffg}_R^{[p]} \rightarrow \text{ffg}_R^{[p]} \), that gives an equivalence of categories.

**Definition 1.4.** — Let \( G \) be an object of the category \( \text{ffg}_W^{[p]} \). Then one verifies immediately that \( G \times_W R \) may be regarded as an object of the category \( \text{ffg}_R^{[p]} \). Moreover, one also verifies immediately that the assignment “\( G \rightsquigarrow G \times_W R \)” defines a functor \( \text{ffg}_W^{[p]} \rightarrow \text{ffg}_R^{[p]} \). We shall write
\[ \text{rdct} : \text{ffg}_W^{[p]} \longrightarrow \text{ffg}_k^{[p]} \]
for this functor in the case where we take the \( W \)-algebra “\( R \)” to be \( k = W/pW \).

In the remainder of the present §1, let \( G \) be an object of the category \( \text{ffg}_k^{[p]} \).

**Definition 1.5.** — Let \( H \) be an object of the category \( \text{ffg}_k^{[p]} \). Then we shall say that \( H \) is a deformation of \( G \) to \( W \) if \( G \) is isomorphic to \( \text{rdct}(H) \) in \( \text{ffg}_k^{[p]} \).

**Definition 1.6.** — We shall say that \( G \) is deformable if there exists a deformation of \( G \) to \( W \).
The following notion is one central notion of the present paper.

**Definition 1.7.** — We shall say that $G$ is *pseudo-rigid* if the following two conditions are satisfied:

1. The object $G$ is deformable.
2. If $H_1, H_2$ are deformations of $G$ to $W$, then $H_1$ is isomorphic to $H_2$ in $\text{ffg}_W^{[p]}$.

**Remark 1.7.1.**

(i) Let us observe that an arbitrary deformation [cf. condition (1) of Definition 1.7] of a *pseudo-rigid* object of the category $\text{ffg}_k^{[p]}$ may be regarded as an object $H$ of the category $\text{ffg}_W^{[p]}$ that satisfies the following condition [cf. condition (2) of Definition 1.7]:

For an object $H'$ of the category $\text{ffg}_W^{[p]}$, it holds that $H$ is isomorphic to $H'$ in $\text{ffg}_W^{[p]}$ if and only if $H \times_W (W/pW)$ is isomorphic to $H' \times_W (W/pW)$ in $\text{ffg}_W^{[p]/pW}$.

(ii) Suppose that $p \neq 2$. Then it follows from [6, §3] that the functor $\text{ffg}_W^{[p]} \to \text{ffg}_W^{[1/p]}$ of Definition 1.4 in the case where we take the $W$-algebra “$R$” to be $W[1/p]$ gives an equivalence of categories. In particular, one may conclude from [4, Corollary 8] that, for two objects $H_1, H_2$ of the category $\text{ffg}_W^{[p]}$, it holds that $H_1$ is isomorphic to $H_2$ in $\text{ffg}_W^{[p]}$ if and only if $H_1 \times_W (W/p^2W)$ is isomorphic to $H_2 \times_W (W/p^2W)$ in $\text{ffg}_W^{[p]/p^2W}$.

2. Review of a Linear Algebra Theory for Group Schemes over $k$

In the present §2, we maintain the notational conventions introduced at the beginning of the preceding §1. Write, moreover, $\text{Fr}$ for the $p$-th power Frobenius automorphism of $k$.

In the present §2, we give a review of a certain linear algebra theory for $p$-torsion finite flat commutative group schemes over $k$ [cf. Definition 2.1 and Proposition 2.5 below].

**Definition 2.1.** — We shall write

$$\mathcal{D}m^{[p]}$$

for the category defined as follows:

- An object of the category $\mathcal{D}m^{[p]}$ is defined to be a collection $\mathcal{D} = (M_\mathcal{D} = M, F_\mathcal{D} = F, V_\mathcal{D} = V)$ of data consisting of a finitely generated $k$-module $M$ and endomorphisms $F, V$ of the module $M$ such that the following three conditions are satisfied:
  - The endomorphism $F$ is Fr-semilinear.
  - The endomorphism $V$ is Fr$^{-1}$-semilinear.
  - The sequence of modules

$$M \xrightarrow{F} M \xrightarrow{V} M \xrightarrow{F} M$$

forms a complex.
Let \((M_1, F_1, V_1), (M_2, F_2, V_2)\) be objects of the category \(\mathcal{Dm}^{[p]}\). Then a morphism \(f : M_1 \to M_2\) of \(k\)-modules such that the equalities \(f \circ F_1 = F_2 \circ f, f \circ V_1 = V_2 \circ f\) hold.

In the remainder of the present \(\S 2\), let \(D = (M, F, V)\) be an object of the category \(\mathcal{Dm}^{[p]}\).

**Definition 2.2.** We shall say that \(D\) is *connected* if the endomorphism \(F\) of \(M\) is nilpotent.

**Definition 2.3.** We shall write
\[
M^D \overset{\text{def}}{=} \text{Hom}_k(M, k)
\]
for the [necessarily finitely generated] \(k\)-module obtained by forming the \(k\)-dual of \(M\),
\[
F^D : M^D \longrightarrow M^D
\]
for the [necessarily \(\text{Fr}\)-semilinear] endomorphism of \(M^D\) given by mapping \(\phi \in M^D\) to the element of \(M^D\) obtained by forming the composite
\[
M \xrightarrow{V} M \xrightarrow{\phi} k \xrightarrow{\text{Fr}} k,
\]
and
\[
V^D : M^D \longrightarrow M^D
\]
for the [necessarily \(\text{Fr}^{-1}\)-semilinear] endomorphism of \(M^D\) given by mapping \(\phi \in M^D\) to the element of \(M^D\) obtained by forming the composite
\[
M \xrightarrow{F} M \xrightarrow{\phi} k \xrightarrow{\text{Fr}^{-1}} k.
\]

**Definition 2.4.** One verifies immediately that the collection \((M^D, F^D, V^D)\) of data may be regarded as an object of the category \(\mathcal{Dm}^{[p]}\). We shall write
\[
D^D
\]
for this object.

**Remark 2.4.1.** One verifies immediately that the assignment \(\sim \) defines a [contravariant] functor \(\mathcal{Dm}^{[p]} \to \mathcal{Dm}^{[p]}\), that gives an *equivalence* of categories.

**Proposition 2.5.** There exists a contravariant functor
\[
\mathcal{M} : \mathcal{Rfg}^{[p]} \longrightarrow \mathcal{Dm}^{[p]}
\]
that gives an *equivalence* of categories and, for each object \(G\) of the category \(\mathcal{Rfg}^{[p]}\), satisfies the following three conditions:
(1) It holds that \( G \) is connected if and only if \( \mathcal{M}(G) \) is connected.

(2) The object \( \mathcal{M}(G^D) \) is naturally isomorphic [cf. Remark 1.3.1 and Remark 2.4.1] to the object \( \mathcal{M}(G)^D \) in \( \mathfrak{Dm}[p] \).

(3) It holds that \( G \) is \'{e}tale over \( k \) if and only if \( F_{\mathcal{M}(G)} \) is an isomorphism.

Proof. — This assertion follows from, for instance, [2, Chapitre III, §1]. □

3. Review of a Linear Algebra Theory for Group Schemes over \( W \)

In the present §3, we maintain the notational conventions introduced at the beginning of the preceding §2.

In the present §3, we give a review of a certain linear algebra theory for \( p \)-torsion finite flat commutative group schemes over \( W \) [cf. Definition 3.7 and Proposition 3.12 below].

Definition 3.1. — Let \( \mathbb{D} = (M, F, V) \) be an object of the category \( \mathfrak{Dm}[p] \). Then we shall write

\[
C_{\mathbb{D}} = C \overset{\text{def}}{=} \text{Im}(F) \subseteq M, \quad H_{\mathbb{D}} = H \overset{\text{def}}{=} \text{Im}(V) \subseteq M
\]

for the \( k \)-submodules of \( M \) obtained by forming the images of the \( \text{Fr} \)-, \( \text{Fr}^{-1} \)-semilinear endomorphisms \( F, V \), respectively.

Lemma 3.2. — Let \( \mathbb{D}_1, \mathbb{D}_2 \) be objects of the category \( \mathfrak{Dm}[p] \) and \( f : \mathbb{D}_1 \to \mathbb{D}_2 \) a morphism in the category \( \mathfrak{Dm}[p] \). Then the inclusions \( f(C_{\mathbb{D}_1}) \subseteq C_{\mathbb{D}_2}, f(H_{\mathbb{D}_1}) \subseteq H_{\mathbb{D}_2} \) hold.

Proof. — This assertion follows from the definition of morphisms in the category \( \mathfrak{Dm}[p] \). □

Lemma 3.3. — Let \( \mathbb{D} \) be an object of the category \( \mathfrak{Dm}[p] \). Then the following assertions hold:

(i) The equalities \( \dim_k(C_{\mathbb{D}}) = \dim_k(H_{\mathbb{D}'D}), \dim_k(H_{\mathbb{D}}) = \dim_k(C_{\mathbb{D}'D}) \) hold.

(ii) It holds that the equality \( C_{\mathbb{D}} = H_{\mathbb{D}} \) holds if and only if the equality \( C_{\mathbb{D}'D} = H_{\mathbb{D}'D} \) holds.

Proof. — These assertions follow immediately from the various definitions involved. □

Definition 3.4. — Let \( \mathbb{D} = (M, F, V) \) be an object of the category \( \mathfrak{Dm}[p] \). Then we shall say that \( \mathbb{D} \) is deformable if the complex \( M \xrightarrow{F} M \xrightarrow{V} M \xrightarrow{F} M \) [cf. Definition 2.1] forms an exact sequence.
**Lemma 3.5.** — Let \( \mathbb{D} \) be an object of the category \( \mathfrak{Dm}^{[p]} \). Then the following four conditions are equivalent:

1. The object \( \mathbb{D} \) is deformable.
2. The equality \( \dim_k(M_{\mathbb{D}}) = \dim_k(C_{\mathbb{D}}) + \dim_k(H_{\mathbb{D}}) \) holds.
3. The object \( \mathbb{D} \) is deformable.
4. The equality \( \dim_k(M_{\mathbb{D}} D) = \dim_k(C_{\mathbb{D}} D) + \dim_k(H_{\mathbb{D}} D) \) holds.

**Proof.** — This assertion follows from the various definitions involved. \( \square \)

**Definition 3.6.** — Let \( \mathbb{D} \) be an object of the category \( \mathfrak{Dm}^{[p]} \). Suppose that \( \mathbb{D} \) is deformable. Then we shall refer to a \( k \)-linear splitting of the natural surjective homomorphism \( M_{\mathbb{D}} \rightarrow M_{\mathbb{D}} = C_{\mathbb{D}} \) as a deformation structure on \( \mathbb{D} \).

**Remark 3.6.1.** — Let \( \mathbb{D} \) be an object of the category \( \mathfrak{Dm}^{[p]} \).

(i) Suppose that \( \mathbb{D} \) is deformable. Let \( L \subseteq M_{\mathbb{D}} \) be a \( k \)-submodule of \( M_{\mathbb{D}} \). Then one verifies immediately that the pair \( (L, \mathbb{D}) \) is a finite Honda system in the sense of [3, §9.4] if and only if \( L \) determines a deformation structure on \( \mathbb{D} \) in the sense of Definition 3.6.

(ii) One also verifies immediately from the discussion of (i) that \( \mathbb{D} \) is deformable if and only if there exists a \( k \)-submodule \( L \subseteq M_{\mathbb{D}} \) of \( M_{\mathbb{D}} \) such that the pair \( (L, \mathbb{D}) \) is a finite Honda system in the sense of [3, §9.4].

**Definition 3.7.** — We shall write \( \mathfrak{fHs}^{[p]} \) for the category defined as follows:

- An object of the category \( \mathfrak{fHs}^{[p]} \) is defined to be a collection \( \mathbb{H} = (\mathbb{D}, s) \) of data consisting of a deformable object \( \mathbb{D} \) of the category \( \mathfrak{Dm}^{[p]} \) and a deformation structure \( s \) on \( \mathbb{D} \).

- Let \( (\mathbb{D}_1, s_1) \), \( (\mathbb{D}_2, s_2) \) be objects of the category \( \mathfrak{fHs}^{[p]} \). Then a morphism \( (\mathbb{D}_1, s_1) \rightarrow (\mathbb{D}_2, s_2) \) in the category \( \mathfrak{fHs}^{[p]} \) is defined to be a morphism \( f : \mathbb{D}_1 \rightarrow \mathbb{D}_2 \) in the category \( \mathfrak{Dm}^{[p]} \) such that if one writes \( f_{/C} : M_{\mathbb{D}_1}/C_{\mathbb{D}_1} \rightarrow M_{\mathbb{D}_2}/C_{\mathbb{D}_2} \) for the homomorphism of \( k \)-modules determined by \( f \) [cf. Lemma 3.2], then the equality \( f \circ s_1 = s_2 \circ f_{/C} \) holds.

**Definition 3.8.** — We shall write
\[
\mathcal{R} : \mathfrak{fHs}^{[p]} \longrightarrow \mathfrak{Dm}^{[p]}
\]
for the functor defined by the assignment \( "(\mathbb{D}, s) \sim \mathbb{D}" \).

**Definition 3.9.** — Let \( \mathbb{H} \) be an object of the category \( \mathfrak{fHs}^{[p]} \). Then we shall say that \( \mathbb{H} \) is connected if the object \( \mathcal{R}(\mathbb{H}) \) of the category \( \mathfrak{Dm}^{[p]} \) is connected.
**Definition 3.10.** — We shall write
\[ \text{cf}\mathfrak{H}^p \subseteq \mathfrak{H}^p \]
for the full subcategory of \( \mathfrak{H}^p \) consisting of connected objects of \( \mathfrak{H}^p \).

**Definition 3.11.** — Let \( \mathbb{H} = (\mathbb{D}, s) \) be an object of the category \( \mathfrak{H}^p \). Then one verifies immediately that the \( k \)-submodule of \( M_D \) obtained by forming the kernel of the surjective homomorphism \( M_D \to \text{Hom}_k(M_D/C_D, k) \) of \( k \)-modules induced by the injective homomorphism \( s: M_D/C_D \hookrightarrow M_D \) determines a deformation structure \( s_D \) on \( D_D \). We shall write
\[ D_D \overset{\text{def}}{=} (D^D, s^D) \]
for the resulting object of \( \mathfrak{H}^p \).

**Remark 3.11.1.** — One verifies immediately that the assignment “\( \mathbb{H} \leadsto D_D \)” defines a [contravariant] functor \( \mathfrak{H}^p \to \mathfrak{H}^p \), that gives an equivalence of categories.

**Proposition 3.12.** — There exist contravariant functors
\[ M: \mathfrak{fg}^p_k \to \mathfrak{Dm}^p, \quad \mathcal{LM}: \mathfrak{fg}^p_W \to \mathfrak{Hs}^p \]
that satisfy the following five conditions:

1. The functor \( M \) gives an equivalence of categories and satisfies the three conditions in the statement of Proposition 2.5.
2. The functor \( \mathcal{LM} \) restricts to an equivalence of categories
\[ \mathfrak{fg}^p_W \sim \text{cf}\mathfrak{H}^p \]
Moreover, the functor \( \mathcal{LM}: \mathfrak{fg}^p_W \to \mathfrak{Hs}^p \) gives an equivalence of categories whenever \( p \neq 2 \).

3. Let \( G \) be an object of the category \( \mathfrak{fg}^p_W \). Then it holds that \( G \) is connected if and only if \( \mathcal{LM}(G) \) is connected.

4. Let \( G \) be an object of the category \( \mathfrak{fg}^p_W \). Then \( \mathcal{LM}(G^D) \) is naturally isomorphic [cf. Remark 1.3.1 and Remark 3.11.1] to \( \mathcal{LM}(G) \) in \( \mathfrak{Hs}^p \).

5. Let \( G \) be an object of the category \( \mathfrak{fg}^p_W \). Then \( \mathcal{M}(\text{rdct}(G)) \) is naturally isomorphic to \( R(\mathcal{LM}(G)) \) in \( \mathfrak{Hs}^p \).

**Proof.** — This assertion follows from, for instance, [3, §9] and [1, §1] [cf. also Remark 3.6.1, (i), (ii), of the present paper]. \( \square \)
Lemma 3.13. — Let $G$ be an object of the category $\text{ffg}_k^{[p]}$. Then the following assertions hold:

(i) If $G$ is deformable, then $\mathcal{M}(G)$ is deformable.

(ii) Suppose that either $G$ or $G^D$ is connected whenever $p = 2$. Then it holds that $G$ is deformable if and only if $\mathcal{M}(G)$ is deformable.

Proof. — These assertions follow immediately from Proposition 3.12.

4. Pseudo-rigid $p$-Torsion Finite Flat Commutative Group Schemes

In the present §4, we prove the main result of the present paper [cf. Theorem 4.11 below].

In the present §4, we maintain the notational conventions introduced at the beginning of §2. Moreover, let $\mathbb{D} = (M, F, V)$ be a deformable object of the category $\mathbb{Dm}^{[p]}$ such that $M \neq \{0\}$. Thus, we have a commutative diagram of $k$-modules

$$
\begin{array}{ccc}
0 & \longrightarrow & C \cap H \longrightarrow H \longrightarrow H/(C \cap H) \longrightarrow 0 \\
0 & \longrightarrow & C \longrightarrow M \longrightarrow M/C \longrightarrow 0
\end{array}
$$

— where the horizontal sequences are exact, and the vertical arrows are injective. Now let us observe that it is immediate that the natural homomorphism $\text{Aut}_{\mathbb{Dm}^{[p]}}(\mathbb{D}) \to \text{Aut}_k(M)$ of groups is injective. Let us regard $\text{Aut}_{\mathbb{Dm}^{[p]}}(\mathbb{D})$ as a subgroup of $\text{Aut}_k(M)$ by means of this injective homomorphism:

$$
\text{Aut}_{\mathbb{Dm}^{[p]}}(\mathbb{D}) \subseteq \text{Aut}_k(M).
$$

Lemma 4.1. — The automorphism of $M$ contained in the subgroup $\text{Aut}_{\mathbb{Dm}^{[p]}}(\mathbb{D}) \subseteq \text{Aut}_k(M)$ preserves the $k$-submodules $C$, $H \subseteq M$ of $M$. In particular, we have a natural action of the group $\text{Aut}_{\mathbb{Dm}^{[p]}}(\mathbb{D})$ on the set of deformation structures on $\mathbb{D}$.

Proof. — This assertion follows from Lemma 3.2.

Lemma 4.2. — Let $G$ be an object of $\text{ffg}_k^{[p]}$ such that $\mathcal{M}(G)$ [cf. Proposition 3.12] is isomorphic to $\mathbb{D}$ in $\mathbb{Dm}^{[p]}$. Suppose that either $G$ or $G^D$ is connected whenever $p = 2$. Then it holds that $G$ is pseudo-rigid if and only if the action of the group $\text{Aut}_{\mathbb{Dm}^{[p]}}(\mathbb{D})$ on the set of deformation structures on $\mathbb{D}$ [cf. Lemma 4.1] is transitive.

Proof. — This assertion follows immediately from Proposition 3.12.

Lemma 4.3. — The following assertions hold:

(i) The assignment “$s \sim s^D$” defined in Definition 3.11 determines a bijection between the set of deformation structures on $\mathbb{D}$ and the set of deformation structures on $\mathbb{D}^D$. 
(ii) The bijection of (i) is compatible with the respective actions of \(\text{Aut}_{\mathcal{Dm}^{[p]}}(D)\), \(\text{Aut}_{\mathcal{Dm}^{[p]}}(D^p)\) [i.e., relative to the natural isomorphism \(\text{Aut}_{\mathcal{Dm}^{[p]}}(D) \sim \text{Aut}_{\mathcal{Dm}^{[p]}}(D^p)\) of groups — cf. Remark 2.4.1].

**Proof.** — These assertions follow immediately from the various definitions involved [cf. also Remark 3.11.1]. □

**Definition 4.4.** — We shall write
\[ U(D) \subseteq \text{Aut}_k(M) \]
for the subgroup of \(\text{Aut}_k(M)\) consisting of [necessarily unipotent] automorphisms of the \(k\)-module \(M\) that preserve the \(k\)-submodules \(C\) and \(H\) of \(M\) and, moreover, induce the identity automorphisms of the four subquotients \(C, H, M/C,\) and \(M/H\) of \(M\).

**Lemma 4.5.** — The following assertions hold:

(i) The inclusion \(U(D) \subseteq \text{Aut}_{\mathcal{Dm}^{[p]}}(D)\) holds.

(ii) Suppose that the equality \(C = H\) holds. Then the action of \(U(D)\) on the set of deformation structures on \(M\) [cf. (i) and Lemma 4.1] is transitive.

**Proof.** — Assertion (i) follows immediately from the definition of the category \(\mathcal{Dm}^{[p]}\) and the definition of the notion of deformability. Next, we verify assertion (ii). Let us first observe that it follows immediately from the various definitions involved that the set of deformation structures on \(D\) [i.e., the set of \(k\)-linear splittings of the natural surjective homomorphism \(M \to M/C\)] has a natural structure of \(\text{Hom}_k(M/C, C)\)-torsor. Moreover, one also verifies immediately that the equality \(C = H\), together with the definition of the subgroup \(U(D)\), implies that there exists an isomorphism \(U(D) \sim \text{Hom}_k(M/C, C)\) of groups, that is compatible with the respective actions on the set of deformation structures on \(D\). Thus, we conclude that the action of \(U(D)\) on the set of deformation structures on \(M\) is transitive, as desired. This completes the proof of assertion (ii), hence also of Lemma 4.5. □

**Definition 4.6.** — Let \(s\) be a deformation structure on \(D\). Then we shall say that \(s\) is \(H\)-full if the image of the \(k\)-submodule \(H/(C \cap H) \subseteq M/C\) by \(s: M/C \to M\) is contained in the \(k\)-submodule \(H \subseteq M\) [cf. the diagram in the discussion at the beginning of the present §4].

**Lemma 4.7.** — The following assertions hold:

(i) A deformation structure on \(D\) obtained as an element of the \(\text{Aut}_{\mathcal{Dm}^{[p]}}(D)\)-orbit of an \(H\)-full deformation structure on \(D\) is \(H\)-full.

(ii) There exists an \(H\)-full deformation structure on \(D\).

(iii) The following two conditions are equivalent:

(1) Either the equality \(C = \{0\}\) or the inclusion \(H \subseteq C\) holds.
An arbitrary deformation structure on \( D \) is \( H \)-full.

(iv) Suppose that condition (2) in (iii) is **not** satisfied. Then the action of the group \( \text{Aut}_{\mathcal{D}_m}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is **not** transitive.

**Proof.** — Assertion (i) follows from Lemma 4.1. Assertion (ii) follows from the elementary theory of linear algebra. Next, we verify assertion (iii). If the equality \( C = \{0\} \) holds, then the equality \( H = M \) holds [cf. Lemma 3.5], which thus implies that condition (2) is satisfied. Moreover, if the inclusion \( H \subseteq C \) holds, then the \( k \)-module \( H/(C \cap H) \) is zero, which thus implies that condition (2) is satisfied. This completes the proof of the implication (1) \( \Rightarrow \) (2).

Next, to verify the implication (2) \( \Rightarrow \) (1), suppose that condition (1) is **not** satisfied. Let \( s \) be an \( H \)-full deformation structure on \( D \) [cf. assertion (ii)] and \( e_1, \ldots, e_d \) elements of \( M/C \) that form a basis of the finitely generated \( k \)-module \( M/C \) such that \( e_1, \ldots, e_h \) form a basis of the \( k \)-submodule \( H/(C \cap H) \) for some \( h \in \{0, \ldots, d\} \). Now let us observe that since \( \{0\} \subseteq C \), the inequality \( h \neq 0 \) holds. Moreover, let us also observe that since \( \{0\} \subseteq C \), the equality \( C \neq \{0\} \) [which thus implies that \( H \neq M \) — cf. Lemma 3.5] there exists an element \( m \in M \setminus H \). Then, by considering the assignment \( (e_1, e_2, \ldots, e_d) \mapsto (s(e_1) + m, s(e_2), \ldots, s(e_d)) \), one may obtain a deformation structure on \( D \) that is **not** \( H \)-full, as desired. This completes the proof of the implication (2) \( \Rightarrow \) (1), hence also of assertion (iii).

Finally, we verify assertion (iv). It follows from assertion (ii) that there exists a deformation structure on \( D \) that is \( H \)-full. Moreover, it follows from our assumption that there exists a deformation structure on \( D \) that is **not** \( H \)-full. Thus, it follows from assertion (i) that the action of the group \( \text{Aut}_{\mathcal{D}_m}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is **not** transitive, as desired. This completes the proof of assertion (iv), hence also of Lemma 4.7.

One main technical observation of the present paper is as follows:

**Lemma 4.8.** — It holds that the action of the group \( \text{Aut}_{\mathcal{D}_m}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is transitive if and only if one of the following three equalities holds:

\[
C = \{0\}, \quad H = \{0\}, \quad C = H.
\]

**Proof.** — Let us begin the proof of Lemma 4.8 with the following claim:

Claim 4.8.A: Suppose that \( C = \{0\} \). Then the action of the group \( \text{Aut}_{\mathcal{D}_m}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is transitive.

Indeed, this assertion follows from the [easily verified] observation that the equality \( C = \{0\} \) implies that the set of deformation structures on \( \mathbb{D} \) is of cardinality one.

Next, I claim the following assertion:

Claim 4.8.B: Suppose that \( H = \{0\} \). Then the action of the group \( \text{Aut}_{\mathcal{D}_m}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is transitive.
Indeed, this assertion follows from the [easily verified] observation that it follows from Lemma 3.5 that the equality \( H = \{ 0 \} \) implies the equality \( C = M \), which thus implies that the set of deformation structures on \( \mathbb{D} \) is of cardinality one.

Next, I claim the following assertion:

Claim 4.8.C: Suppose that \( C = H \). Then the action of the group \( \text{Aut}_{\mathbb{D}[p]}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is transitive.

Indeed, this assertion follows from Lemma 4.5, (i), (ii).

It follows from Claim 4.8.A, Claim 4.8.B, and Claim 4.8.C that, to verify Lemma 4.8, we may assume without loss of generality that

\[
C \neq \{ 0 \}, \quad H \neq \{ 0 \}, \quad C \neq H,
\]

which thus implies [cf. Lemma 3.3, (i), (ii)] that

\[
C_{\mathbb{D}^D} \neq \{ 0 \}, \quad H_{\mathbb{D}^D} \neq \{ 0 \}, \quad C_{\mathbb{D}^D} \neq H_{\mathbb{D}^D}.
\]

Next, I claim the following assertion:

Claim 4.8.D: Suppose that the inclusion \( H \subseteq C \) does not hold. Then the action of the group \( \text{Aut}_{\mathbb{D}[p]}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is not transitive.

Indeed, since [we have assumed that] \( C \neq \{ 0 \} \), this assertion follows from Lemma 4.7, (iii), (iv).

Finally, I claim the following assertion:

Claim 4.8.E: The action of the group \( \text{Aut}_{\mathbb{D}[p]}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is not transitive.

To this end, let us first observe that it follows from Claim 4.8.D that we may assume without loss of generality that the inclusion \( H \subseteq C \) holds. Thus, since [we have assumed that] \( C \neq H \), the inequality \( \dim_k(H) < \dim_k(C) \) holds. Thus, it follows from Lemma 3.3, (i), that \( \dim_k(C_{\mathbb{D}^D}) = \dim_k(H) < \dim_k(C) = \dim_k(H_{\mathbb{D}^D}) \). In particular, the inclusion \( H_{\mathbb{D}^D} \subseteq C_{\mathbb{D}^D} \) does not hold. Thus, it follows from Claim 4.8.D [cf. also Lemma 3.5] that the action of the group \( \text{Aut}_{\mathbb{D}[p]}(\mathbb{D}^D) \) on the set of deformation structures on \( \mathbb{D}^D \) is not transitive. In particular, we conclude from Lemma 4.3, (ii), that the action of the group \( \text{Aut}_{\mathbb{D}[p]}(\mathbb{D}) \) on the set of deformation structures on \( \mathbb{D} \) is not transitive, as desired. This completes the proof of Claim 4.8.E, hence also of Lemma 4.8. \( \square \)

**Definition 4.9.** — Let \( G \) be an object of the category \( \text{ffg}_{k}^{[p]} \).

(i) We shall say that \( G \) is multiplicative if either \( G \) is zero or the following condition is satisfied: Let \( \overline{k} \) be an algebraic closure of \( k \). Then the finite flat commutative group scheme \( G \times_k \overline{k} \) over \( \overline{k} \) is isomorphic to the fiber product of finitely many copies of the finite flat commutative group scheme \( \mu_p \) over \( \overline{k} \).

(ii) We shall say that \( G \) is superspecial if either \( G \) is zero or the following condition is satisfied: Let \( \overline{k} \) be an algebraic closure of \( k \). Then there exist a positive integer \( r \).
and, for each $i \in \{1, \ldots, r\}$, a supersingular elliptic curve $E_i$ over $\overline{k}$ such that if, for each $i \in \{1, \ldots, r\}$, one writes $E_i[p]$ for the finite flat commutative group scheme over $k$ obtained by forming the kernel of the endomorphism of $E_i$ given by multiplication by $p$, then the finite flat commutative group scheme $G \times_k \overline{k}$ over $\overline{k}$ is isomorphic to the fiber product $E_i[p] \times_{\overline{k}} \cdots \times_{\overline{k}} E_i[p]$ over $\overline{k}$.

**Lemma 4.10.** — Let $G$ be an object of $\ffrak{fg}_k^{[p]}$ such that $\mathcal{M}(G)$ [cf. Proposition 3.12] is isomorphic to $\mathbb{D}$ in $\mathcal{Dm}^{[p]}$. Then the following two conditions are equivalent:

1. The finite flat commutative group scheme $G$ is superspecial.

2. The equality $C = H$ holds.

**Proof.** — Let us first observe that one verifies immediately from, for instance, [2, Chapitre III, §2] that, to verify Lemma 4.10, we may assume without loss of generality, by replacing $G$ by the base-change of $G$ to an algebraic closure of $k$, that $k$ is algebraically closed.

Now we verify the implication (1) $\Rightarrow$ (2). Let us first observe that it is immediate that, to verify the implication (1) $\Rightarrow$ (2), we may assume without loss of generality that $G$ is isomorphic to “$E_i[p]$” as in Definition 4.9, (ii). Then it is well-known [cf., e.g., [5, §5.6]] that condition (2) is satisfied, as desired. This completes the proof of the implication (1) $\Rightarrow$ (2).

Finally, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. Let us first observe that since the sequence $M \xrightarrow{F} M \xrightarrow{V} M \xrightarrow{F} M$ is exact, the Fr-, $F^{-1}$-semilinear endomorphisms $F$, $V$ determine Fr-, $F^{-1}$-semilinear isomorphisms $M/H \xrightarrow{\sim} C$, $M/C \xrightarrow{\sim} H$, respectively. Write $\overline{F}$, $\overline{V}$ for these Fr-, $F^{-1}$-semilinear isomorphisms $M/C = M/H \xrightarrow{\sim} C = H$ [cf. condition (2)], respectively. Then since $k$ is algebraically closed, and the composite

$$M/C \xrightarrow{\overline{F}} C \xrightarrow{\overline{V}^{-1}} M/C$$

is an $F^2$-semilinear isomorphism, it follows from [7, Exposé XXII, Proposition 1.1] that there exist elements $e_1, \ldots, e_d$ of $M/C$ such that these elements form a basis of the finitely generated $k$-module $M/C$, and, moreover, the equality $(\overline{V}^{-1} \circ \overline{F})(e_i) = e_i$, i.e., the equality $\overline{F}(e_i) = \overline{V}(e_i)$, holds for each $i \in \{1, \ldots, d\}$. For each $i \in \{1, \ldots, d\}$, let us fix a lifting $\tilde{e}_i \in M$ of $e_i \in M/C$ and write $M_i \subseteq M$ for the $k$-submodule of $M$ generated by $\overline{F}(e_i) = \overline{V}(e_i) \in C \subseteq M$ and the fixed lifting $\tilde{e}_i \in M$. Then one verifies immediately from condition (2), together with the various definitions involved, that

- the collection $\mathbb{D}_i \overset{\text{def}}{=} (M_i, F|_{M_i}, V|_{M_i})$ of data forms a deformable object of the category $\mathcal{Dm}^{[p]}$ such that $C_{\mathbb{D}_i} = H_{\mathbb{D}_i}$ for each $i \in \{1, \ldots, d\}$, and, moreover,

- the natural inclusions $M_i \hookrightarrow M$ — where $i$ ranges over the elements of $\{1, \ldots, d\}$ — determine an isomorphism $\mathbb{D}_1 \oplus \cdots \oplus \mathbb{D}_d \xrightarrow{\sim} \mathbb{D}$ in $\mathcal{Dm}^{[p]}$.

In particular, we conclude immediately from Proposition 3.12 that, to verify the implication (2) $\Rightarrow$ (1), we may assume without loss of generality, by replacing $\mathbb{D}$ by $\mathbb{D}_i$ for each $i \in \{1, \ldots, d\}$, that $\dim_k(M) = 2$. On the other hand, if $\dim_k(M) = 2$, then the
implication (2) \( \Rightarrow \) (1) is well-known [cf., e.g., [5, §5.6]]. This completes the proof of the implication (2) \( \Rightarrow \) (1), hence also of Lemma 4.10.

The main result of the present paper is as follows:

**Theorem 4.11.** — Let \( p \) be a prime number, \( k \) a perfect field of characteristic \( p \), and \( G \) a \( p \)-torsion finite flat commutative group scheme over \( k \). Suppose that either \( G \) or the Cartier dual of \( G \) is connected whenever \( p = 2 \). Then it holds that \( G \) is pseudo-rigid [cf. Definition 1.7] if and only if \( G \) is either étale over \( k \), multiplicative [cf. Definition 4.9, (i)], or superspecial [cf. Definition 4.9, (ii)].

**Proof.** — Let us first observe that it follows — in light of Lemma 3.13, (ii) — from Lemma 4.2 and Lemma 4.8 that \( G \) is pseudo-rigid if and only if \( \mathcal{M}(G) \) [cf. Proposition 3.12] is deformable, and, moreover, one of the following three equalities holds:

\[
C_{\mathcal{M}(G)} = \{0\}, \quad H_{\mathcal{M}(G)} = \{0\}, \quad C_{\mathcal{M}(G)} = H_{\mathcal{M}(G)}.
\]

On the other hand,

- by conditions (2), (3) of Proposition 2.5, Lemma 3.3, (i), and Lemma 3.5, the equality \( C_{\mathcal{M}(G)} = \{0\} \) is equivalent to the condition that \( G \) is multiplicative,

- by condition (3) of Proposition 2.5 and Lemma 3.5, the equality \( H_{\mathcal{M}(G)} = \{0\} \) is equivalent to the condition that \( G \) is étale over \( k \), and

- by Lemma 4.10, under the assumption that \( \mathcal{M}(G) \) is deformable, the equality \( C_{\mathcal{M}(G)} = H_{\mathcal{M}(G)} \) is equivalent to the condition that \( G \) is superspecial.

This completes the proof of Theorem 4.11. \( \square \)

**References**


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