A Note on the Existence of Tango Curves

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ABSTRACT. In the present paper, we prove that, for an odd prime number \( p \) and a positive integer \( g \) such that \( g - 1 \) is divisible by \( p \), there exists a Tango curve of genus \( g \) in characteristic \( p \).

INTRODUCTION

Throughout the present paper, let \( p \) be an odd prime number and \( k \) an algebraically closed field of characteristic \( p \). Let us recall that a Tango curve over \( k \) is defined to be a projective smooth curve over \( k \) that admits a rational function \( f \) such that the divisor associated to the rational differential \( df \) is nonzero and of order divisible by \( p \) at each closed point of the curve [cf., e.g., [2, §2.1, [3, §3], [5, Definition 3.1.1, (ii)]]]. In the present paper, we prove the following result.

**Theorem 1.** Let \( g \) be a positive integer. Then the following two conditions are equivalent:

1. The integer \( g - 1 \) is divisible by \( p \).
2. There exists a Tango curve of genus \( g \) over \( k \).

Note that Theorem 1 determines “the complete list” discussed in [5, Remark 3.1.2], i.e., “the complete list of \( g \)’s such that there is a Tango curve of genus \( g \”).

One immediate application of Theorem 1 is as follows. The following corollary is a formal consequence of Theorem 1 and [4, Theorem B].

**Corollary 2.** Let \( g \geq 2 \) be an integer such that \( g - 1 \) is divisible by \( p \). Then the moduli stack of projective smooth curves of genus \( g \) over \( k \) equipped with Tango structures [cf. [4, Definition 5.1.1]] may be represented by a smooth Deligne-Mumford stack over \( k \) of pure dimension \( 2(g - 1)(p + 1)/p \), that is finite over the moduli stack of projective smooth curves of genus \( g \) over \( k \). In particular, the substack of the moduli stack of projective smooth curves of genus \( g \) over \( k \) that parametrizes Tango curves is a closed substack of pure codimension \( (g - 1)(p - 2)/p \).

A PROOF

Let us first observe that it follows from [1, Theorem A] that, to verify Theorem 1, it suffices to verify the following result, i.e., a “higher level version” of Theorem 1.

**Theorem 3.** Let \( g \) and \( N \) be positive integers. Then the following two conditions are equivalent:

1. The integer \( g - 1 \) is divisible by \( p^N \).

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(2) There exists a projective smooth curve of genus \( g \) over \( k \) that admits a Tango function of level \( N \) [cf. [1, Definition 1.3]].

In the remainder of the present paper, we give a proof of Theorem 3. To this end, let \( g \) and \( N \) be positive integers. Write \( q \overset{\text{def}}{=} p^N \). Let us first observe that since [we have assumed that] \( p \) is odd, it follows from [1, Corollary 1.10] that the implication (2) \( \Rightarrow \) (1) holds. In the remainder of the present paper, to verify the implication (1) \( \Rightarrow \) (2), let us prove that,

\[ (*) \text{for each nonnegative integer } n, \text{there exists a projective smooth curve } C \text{ of genus } \frac{q^n + 1}{k} \text{ over } k \text{ that admits a Tango function of level } N. \]

To this end, let \( n \) be a nonnegative integer.

Let us begin our construction of “C” with an ordinary elliptic curve \((E, o)\) over \( k \). [Note that it is well-known that an ordinary elliptic curve over \( k \) exists.] Thus, the elliptic curve \((E, o)\) admits a closed point \( e \) that is \( p^N \)-torsion but not \( p^{N-1} \)-torsion [which thus implies that \( e \neq o \)]. In particular, \( (*) \) follows immediately from our assumption that \( e \) is not \( p^{N-1} \)-torsion [i.e., which thus implies that the rational function \( f_E \) cannot be written as the “\( p \)-th power” of a rational function on \( E \)].

Write \( R(f_E) \) for the ramification divisor of the separable [cf. Lemma 4] morphism \( f_E: E \to \mathbb{P}^1_k \).

**Lemma 4.** The finite morphism \( f_E: E \to \mathbb{P}^1_k \) over \( k \) is separable [i.e., generically étale].

**Proof.** This assertion follows immediately from our assumption that \( e \) is not \( p^{N-1} \)-torsion [i.e., which thus implies that the rational function \( f_E \) cannot be written as the “\( p \)-th power” of a rational function on \( E \)]. \( \square \)

Write \( R(f_E) \) for the ramification divisor of the separable [cf. Lemma 4] morphism \( f_E: E \to \mathbb{P}^1_k \).

**Lemma 5.** The ramification divisor \( R(f_E) \) is given by \( q[o] + q[e] \).

**Proof.** Since the morphism \( f_E \) is of degree \( q \) [cf. \((†)\)], it follows from the Riemann-Hurwitz formula that the divisor \( R(f_E) \) is of degree \( 2q \). On the other hand, one verifies immediately from \((†)\) that \( q[o] + q[e] \leq R(f_E) \). In particular, Lemma 5 holds. \( \square \)

**Lemma 6.** The morphism \( f_E: E \to \mathbb{P}^1_k \) is étale over \( \mathbb{P}^1_k \setminus \{ f_E(o), f_E(e) \} \).

**Proof.** This assertion is an immediate consequence of Lemma 5. \( \square \)

Next, let us observe that it follows from the well-known structure of the maximal pro-prime-to-\( p \) quotient of the abelianization of the étale fundamental group of the smooth curve \( E \setminus \{ o, e \} \) that

\[ (‡) \text{there exist a projective smooth curve } C \text{ over } k \text{ and a finite morphism } f_C: C \to E \text{ of degree } q n + 1 \text{ over } k \text{ such that the morphism } f_C \text{ is étale over } E \setminus \{ o, e \}, \text{ and, moreover, for each } x \in \{ o, e \}, \text{ the fiber } f_C^{-1}(x) \text{ consists of a single closed point } x_C \text{ of } C. \]

**Lemma 7.** The curve \( C \) is of genus \( qn + 1 \).

**Proof.** This assertion follows from \((‡)\) and the Riemann-Hurwitz formula. \( \square \)

Write \( f \overset{\text{def}}{=} f_E \circ f_C: C \to \mathbb{P}^1_k \) for the composite of the morphisms \( f_E \) and \( f_C \).

**Lemma 8.** Let \( x \in E \) be either \( o \in E \) or \( e \in E \). Let \( t_{f_E(x)} \) be a uniformizer of the local ring \( \mathcal{O}_{\mathbb{P}^1_k, f_E(x)} \).

Then there exists a uniformizer \( t_{x_C} \) of the local ring \( \mathcal{O}_{C, x_C} \) and units \( u_1, u_2 \) of the local ring \( \mathcal{O}_{C, x_C} \) such that the homomorphism \( \mathcal{O}_{\mathbb{P}^1_k, f_E(x)} \to \mathcal{O}_{C, x_C} \) induced by the morphism \( f \) maps \( t_{f_E(x)} \in \mathcal{O}_{\mathbb{P}^1_k, f_E(x)} \) to

\[ u_2^{q(qn+1)} t_{x_C}^{(q+1)(qn+1)} + u_1 t_{x_C}^{(q+1)(qn+1)} \in \mathcal{O}_{C, x_C}. \]
Proof. Let us first observe that one verifies immediately from (†) and Lemma 5 that there exist a uniformizer $t_x$ of the local ring $\mathcal{O}_{E,x}$ and a unit $v_1$ of the local ring $\mathcal{O}_{E,x}$ such that the homomorphism $\mathcal{O}_{\mathbb{P}^1, f_E(x)} \to \mathcal{O}_{E,x}$ induced by the morphism $f_E$ maps $t_{f_E(x)} \in \mathcal{O}_{\mathbb{P}^1, f_E(x)}$ to

$$t_x^q + v_1 t_x^{q+1} \in \mathcal{O}_{E,x}.$$  

Moreover, let us also observe that one verifies immediately from (‡) that there exist a uniformizer $t_{x_C}$ of the local ring $\mathcal{O}_{C,x_C}$ and a unit $v_2$ of the local ring $\mathcal{O}_{C,x_C}$ such that the homomorphism $\mathcal{O}_{E,x} \to \mathcal{O}_{C,x_C}$ induced by the morphism $f_C$ maps $t_x \in \mathcal{O}_{E,x}$ to

$$v_2 t_{x_C}^{q+1} \in \mathcal{O}_{C,x_C}.$$  

In particular, Lemma 8 holds. □

Lemma 9. The rational function $f : C \to \mathbb{P}^1_k$ is a Tango function of level $N$.

Proof. Let us observe that it follows from Lemma 6 and (‡) that the morphism $f : C \to \mathbb{P}^1_k$ is étale over $\mathbb{P}^1_k \setminus \{f_E(o), f_E(e)\}$. Thus, Lemma 9 follows immediately from Lemma 8 and [1, Proposition 1.7]. □

The assertion (∗) follows from Lemma 7 and Lemma 9. This completes the proof of the implication (1) ⇒ (2), hence also of Theorem 3.

Remark 10. As discussed in the proof of Lemma 9, the morphism $f : C \to \mathbb{P}^1_k$ is étale over $\mathbb{P}^1_k \setminus \{f_E(o), f_E(e)\}$. Thus, it follows immediately from (†) and Lemma 8 that the divisor associated to the rational differential $df$ is given by $q(qn + n + 1)[o_C] − q(qn − n + 1)[e_C]$. Moreover, it follows from (†) and (‡) that the divisor associated to the rational function $f$ is given by $q(qn + 1)[o_C] − q(qn + 1)[e_C]$. Thus, we conclude that the divisor associated to the logarithmic differential $df/f$ of $f$ is given by $qn[o_C] + qn[e_C]$. In particular, the logarithmic differential $df/f$ is regular everywhere.

REFERENCES


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