A NOTE ON THE EXISTENCE OF TANGO CURVES

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ABSTRACT. In the present paper, we prove that, for an odd prime number p and a positive integer g such that $g-1$ is divisible by p, there exists a Tango curve of genus g in characteristic p.

INTRODUCTION

Throughout the present paper, let p be an *odd* prime number and k an algebraically closed field of characteristic p. Let us recall that a *Tango curve* over k is defined to be a projective smooth curve over k that admits a rational function f such that the divisor associated to the rational differential df is nonzero and of order divisible by p at each closed point of the curve [cf., e.g., [2, §2.1], [3, §3], [5, Definition 3.1.1, (ii)]]. In the present paper, we prove the following result.

Theorem 1. *Let g be a positive integer. Then the following two conditions are equivalent:*

(1) *The integer $g-1$ is divisible by p.*
(2) *There exists a Tango curve of genus g over k.*

Note that Theorem 1 determines "the complete list" discussed in [5, Remark 3.1.2], i.e., "the complete list of g's such that there is a Tango curve of genus g".

One immediate application of Theorem 1 is as follows. The following corollary is a formal consequence of Theorem 1 and [4, Theorem B].

Corollary 2. *Let $g \geq 2$ be an integer such that $g-1$ is divisible by p. Then the moduli stack of projective smooth curves of genus g over k equipped with Tango structures* [*cf.* [4, Definition 5.1.1]] *may be represented by a smooth Deligne-Mumford stack over k of pure dimension $2(g-1)(p+1)/p$, that is finite over the moduli stack of projective smooth curves of genus g over k. In particular, the substack of the moduli stack of projective smooth curves of genus g over k that parametrizes Tango curves is a closed substack of pure codimension $(g-1)(p-2)/p$.*

A PROOF

Let us first observe that it follows from [1, Theorem A] that, to verify Theorem 1, it suffices to verify the following result, i.e., a "higher level version" of Theorem 1.

Theorem 3. *Let g and N be positive integers. Then the following two conditions are equivalent:*

(1) *The integer $g-1$ is divisible by p^N.*

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(2) *There exists a projective smooth curve of genus g over k that admits a Tango function of level N* [*cf.* [1, Definition 1.3]].

In the remainder of the present paper, we give a proof of Theorem 3. To this end, let g and N be positive integers. Write $q \stackrel{\mathrm{def}}{=} p^N$. Let us first observe that since [we have assumed that] p is *odd*, it follows from [1, Corollary 1.10] that the implication (2) $\Rightarrow$ (1) holds. In the remainder of the present paper, to verify the implication (1) $\Rightarrow$ (2), let us prove that,

($*$) for each nonnegative integer n, there exists a projective smooth curve C *of genus* $qn+1$ over k that admits a *Tango function of level* N.

To this end, let n be a nonnegative integer.

Let us begin our construction of "C" with an *ordinary* elliptic curve (E,o) over k. [Note that it is well-known that an *ordinary* elliptic curve over k exists.] Thus, the elliptic curve (E,o) admits a closed point e that is p^N*-torsion* but *not* p^{N-1}*-torsion* [which thus implies that $e \neq o$]. In particular,

(†) there exists a rational function $f_E \colon E \to \mathbb{P}^1_k$ such that the associated divisor is given by $q[o]-q[e]$ — where we write "$[-]$" for the principal divisor determined by the closed point "$(-)$".

Lemma 4. *The finite morphism* $f_E \colon E \to \mathbb{P}^1_k$ *over k is separable* [*i.e., generically étale*].

Proof. This assertion follows immediately from our assumption that e is *not* p^{N-1}*-torsion* [i.e., which thus implies that the rational function f_E cannot be written as the "p-th power" of a rational function on E]. □

Write $R(f_E)$ for the ramification divisor of the *separable* [cf. Lemma 4] morphism $f_E \colon E \to \mathbb{P}^1_k$.

Lemma 5. *The ramification divisor* $R(f_E)$ *is given by* $q[o]+q[e]$.

Proof. Since the morphism f_E is *of degree* q [cf. (†)], it follows from the *Riemann-Hurwitz formula* that the divisor $R(f_E)$ is *of degree* $2q$. On the other hand, one verifies immediately from (†) that $q[o]+q[e] \leq R(f_E)$. In particular, Lemma 5 holds. □

Lemma 6. *The morphism* $f_E \colon E \to \mathbb{P}^1_k$ *is étale over* $\mathbb{P}^1_k \setminus \{f_E(o), f_E(e)\}$.

Proof. This assertion is an immediate consequence of Lemma 5. □

Next, let us observe that it follows from the well-known structure of the maximal pro-prime-to-p quotient of the abelianization of the étale fundamental group of the smooth curve $E \setminus \{o,e\}$ that

(‡) there exist a projective smooth curve C over k and a finite morphism $f_C \colon C \to E$ *of degree* $qn+1$ over k such that the morphism f_C is *étale* over $E \setminus \{o,e\}$, and, moreover, for each $x \in \{o,e\}$, the fiber $f_C^{-1}(x)$ consists of a *single* closed point x_C of C.

Lemma 7. *The curve C is of genus* $qn+1$.

Proof. This assertion follows from (‡) and the *Riemann-Hurwitz formula*. □

Write $f \stackrel{\mathrm{def}}{=} f_E \circ f_C \colon C \to \mathbb{P}^1_k$ for the composite of the morphisms f_E and f_C.

Lemma 8. *Let* $x \in E$ *be either* $o \in E$ *or* $e \in E$. *Let* $t_{f_E(x)}$ *be a uniformizer of the local ring* $\mathcal{O}_{\mathbb{P}^1_k, f_E(x)}$. *Then there exist a uniformizer* t_{x_C} *of the local ring* $\mathcal{O}_{C,x_C}$ *and units* u_1, u_2 *of the local ring* $\mathcal{O}_{C,x_C}$ *such that the homomorphism* $\mathcal{O}_{\mathbb{P}^1_k, f_E(x)} \to \mathcal{O}_{C,x_C}$ *induced by the morphism f maps* $t_{f_E(x)} \in \mathcal{O}_{\mathbb{P}^1_k, f_E(x)}$ *to*

$$u_2^q t_{x_C}^{q(qn+1)} + u_1 t_{x_C}^{(q+1)(qn+1)} \in \mathcal{O}_{C,x_C}.$$

Proof. Let us first observe that one verifies immediately from ($\dagger$) and Lemma 5 that there exist a uniformizer t_x of the local ring $\mathscr{O}_{E,x}$ and a unit v_1 of the local ring $\mathscr{O}_{E,x}$ such that the homomorphism $\mathscr{O}_{\mathbb{P}^1_k, f_E(x)} \to \mathscr{O}_{E,x}$ induced by the morphism f_E maps $t_{f_E(x)} \in \mathscr{O}_{\mathbb{P}^1_k, f_E(x)}$ to

$$t_x^q + v_1 t_x^{q+1} \in \mathscr{O}_{E,x}.$$

Moreover, let us also observe that one verifies immediately from ($\ddagger$) that there exist a uniformizer t_{x_C} of the local ring $\mathscr{O}_{C,x_C}$ and a unit v_2 of the local ring $\mathscr{O}_{C,x_C}$ such that the homomorphism $\mathscr{O}_{E,x} \to \mathscr{O}_{C,x_C}$ induced by the morphism f_C maps $t_x \in \mathscr{O}_{E,x}$ to

$$v_2 t_{x_C}^{qn+1} \in \mathscr{O}_{C,x_C}.$$

In particular, Lemma 8 holds. □

Lemma 9. *The rational function $f\colon C \to \mathbb{P}^1_k$ is a Tango function of level N.*

Proof. Let us observe that it follows from Lemma 6 and ($\ddagger$) that the morphism $f\colon C \to \mathbb{P}^1_k$ is *étale* over $\mathbb{P}^1_k \setminus \{f_E(o), f_E(e)\}$. Thus, Lemma 9 follows immediately, in light of ($\dagger$) and ($\ddagger$), from Lemma 8 and [1, Proposition 1.7]. □

The assertion ($*$) follows from Lemma 7 and Lemma 9. This completes the proof of the implication (1) $\Rightarrow$ (2), hence also of Theorem 3.

Remark 10. As discussed in the proof of Lemma 9, the morphism $f\colon C \to \mathbb{P}^1_k$ is *étale* over $\mathbb{P}^1_k \setminus \{f_E(o), f_E(e)\}$. Thus, it follows immediately from ($\dagger$) and Lemma 8 that the divisor associated to the rational differential df is given by $q(qn+n+1)[o_C] - q(qn-n+1)[e_C]$. Moreover, it follows from ($\dagger$) and ($\ddagger$) that the divisor associated to the rational function f is given by $q(qn+1)[o_C] - q(qn+1)[e_C]$. Thus, we conclude that the divisor associated to the logarithmic differential df/f of f is given by $qn[o_C] + qn[e_C]$. In particular, the logarithmic differential df/f is *regular everywhere*.

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