On Indigenous Bundles in Characteristic Three

Yuichiro Hoshi

February 2022

Abstract. — In the present paper, we prove three results concerning indigenous bundles on hyperbolic curves in characteristic three. The first result is a result concerning a relationship between the square Hasse invariants of indigenous bundles and the torsor structure of the Schwarz torsor in characteristic three. One immediate consequence of this first result is that the isomorphism class of an indigenous bundle in characteristic three is completely determined by the associated square Hasse invariant. The second result is a result concerning the ordinariness of nilpotent admissible indigenous bundles in characteristic three. This result asserts that, for a given nilpotent admissible indigenous bundle in characteristic three, it is ordinary if and only if the associated Hasse defect is parabolically ordinary. The third result is a result concerning a relationship between strongly spiked indigenous bundles and Tango curves in characteristic three. One immediate consequence of this third result is that if a projective hyperbolic curve in characteristic three admits a global differential whose square coincides with the square Hasse invariant of a strongly spiked indigenous bundles, then the curve is a Tango curve.

Contents

Introduction

Let us first recall that the notion of an *indigenous bundle* is one of the main notions in the theory of *hyperbolically ordinary curves* [cf., e.g., [9], [10]]. In the present paper, we prove three results concerning indigenous bundles on hyperbolic curves *in characteristic* 3. Throughout the present paper, let p be an *odd* prime number and (g, r) a pair of nonnegative integers such that $2g - 2 + r$ is *positive*; moreover, we shall use the notation "ω" to denote the *relative cotangent sheaf.*

The first main result of the present paper is a result concerning a relationship between the *square Hasse invariants* of indigenous bundles and the torsor structure of the *Schwarz torsor* in characteristic 3. Write $\mathcal{M}$ for the moduli stack of hyperbolic curves of type (g, r) in characteristic p, $(\mathcal{C}, \mathcal{D})$ for the universal hyperbolic curve of type (g, r) over $\mathcal{M}$,

2010 Mathematics Subject Classification. — 14G17.
Key words and phrases. — indigenous bundle, hyperbolic curve, square Hasse invariant, Schwarz torsor, hyperbolically ordinary, strongly spiked, Tango curve, p-adic Teichmüller theory.

$\pi\colon \mathcal{C} \to \mathcal{M}$ for the structure morphism of $\mathcal{C}$ over $\mathcal{M}$, and

$$\mathcal{S} \longrightarrow \mathcal{M}$$

for the *Schwarz torsor* over $\mathcal{M}$ [cf. [10, Introduction, §0.4]], i.e., the moduli stack of hyperbolic curves of type (g, r) in characteristic p equipped with *indigenous bundles* [cf. [9, Chapter I, Definition 2.2]]. Then it is well-known [cf. [9, Chapter I, Corollary 2.9], [10, Introduction, §0.4]] that the Schwarz torsor $\mathcal{S}$ admits a *natural structure of torsor under* $\mathcal{G} \stackrel{\text{def}}{=} \pi_*(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2}(\mathcal{D}))$ *over* $\mathcal{M}$. Write, moreover,

$$\mathcal{V} \longrightarrow \mathcal{M}$$

for the vector bundle over $\mathcal{M}$ associated to the locally free coherent $\mathcal{O}_{\mathcal{M}}$-module obtained by forming the $\mathcal{O}_{\mathcal{M}}$-dual of the [necessarily locally free coherent] $\mathcal{O}_{\mathcal{M}}$-module $\pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes(p-1)})$ and

$$\text{sq-Hss}\colon \mathcal{S} \longrightarrow \mathcal{V}$$

for the morphism of stacks over $\mathcal{M}$ obtained by considering the *square Hasse invariants* [cf. [9, Chapter II, Proposition 2.6, (1)]] of the indigenous bundles parametrized by the Schwarz torsor. Now let us observe that if $p = 3$, then the natural inclusion $\mathcal{O}_{\mathcal{C}} \hookrightarrow \mathcal{O}_{\mathcal{C}}(\mathcal{D})$ determines an injective homomorphism $\mathcal{G} = \pi_*(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2}(\mathcal{D})) \hookrightarrow \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes 2}) = \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes(p-1)})$. In particular, one obtains an action of $\mathcal{G}$ on the vector bundle $\mathcal{V}$ over $\mathcal{M}$ whenever $p = 3$.

The first main result of the present paper is as follows [cf. Theorem 1.7].

Theorem A. — *Suppose that $p = 3$. Then the morphism of stacks over $\mathcal{M}$*

$$\text{sq-Hss}\colon \mathcal{S} \longrightarrow \mathcal{V}$$

*is **compatible** with the respective actions of $\mathcal{G}$. In particular, this morphism is a **closed immersion**.*

One immediate consequence of Theorem A is that the isomorphism class of an indigenous bundle in characteristic 3 is *completely determined* by the associated *square Hasse invariant*. Moreover, one may derive [1, Theorem A] from Theorem A [cf. Remark 1.7.2].

The second main result of the present paper is a result concerning the *ordinariness* of nilpotent admissible indigenous bundles in characteristic 3. Let S be a noetherian scheme over $\mathbb{F}_p$, (X, D) a hyperbolic curve of type (g, r) over S, and (P, ∇_P) an indigenous bundle on (X, D). Suppose that the indigenous bundle (P, ∇_P) is *nilpotent* [cf. [9, Chapter II, Definition 2.4]] and *admissible* [cf. [9, Chapter II, Definition 2.4]]. Then let us recall from [9, Chapter II, Proposition 2.6, (3)] that there exist a unique, up to isomorphism [cf. [9, Chapter II, Proposition 2.6, (4)]], invertible sheaf $\mathcal{H}$ on X and a global section χ of $\mathcal{H}$ [i.e., the *Hasse invariant* of (P, ∇_P)] such that the global section $\chi^{\otimes 2}$ of the square $\mathcal{H}^{\otimes 2}$ coincides with the square Hasse invariant of (P, ∇_P). We shall refer to the invertible sheaf on X

$$\mathcal{H}om_{\mathcal{O}_X}\big(\omega_{X/S}(D)^{\otimes(p-1)/2}, \mathcal{H}\big)$$

as the *Hasse defect* of (P, ∇_P). One may verify that the square of the Hasse defect of (P, ∇_P) is *trivial* [cf. Proposition 2.2].

The second main result of the present paper is as follows [cf. Theorem 2.4].

THEOREM B. — *Suppose that $p = 3$. Then the following two conditions are equivalent:*

(1) *The nilpotent admissible indigenous bundle (P, ∇_P) is* ***ordinary*** [*cf.* [9, Chapter II, Definition 3.1]].

(2) *The* ***Hasse defect*** *of (P, ∇_P) is* ***parabolically ordinary*** [*cf.* [1, Definition A.7]].

Here, let us recall that if the scheme S is the spectrum of an algebraically closed field, then condition (2) in the statement of Theorem B is the condition that either

• the Hasse defect of (P, ∇_P) is *trivial*, and the *Jacobian* variety of the curve X is an *ordinary* abelian variety, or

• the Hasse defect of (P, ∇_P) is *nontrivial*, and the associated *Prym* variety is an *ordinary* abelian variety.

Note that Theorem B *generalizes* [1, Proposition 4.4]. Moreover, one may derive, from Theorem B, the assertion that the [necessarily nilpotent admissible] indigenous bundle obtained by forming the projectivization of the *relative first de Rham cohomology module*, equipped with the *Gauss-Manin connection*, of the universal stable curve of type $(1, 1)$ over the *modular curve* over $\overline{\mathbb{F}}_3$ associated to the congruence subgroup $\Gamma(32) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is *not ordinary* [cf. Remark 2.4.2]. Furthermore, one may also derive, from Theorem B, the assertion that, for an arbitrary nilpotent admissible indigenous bundle on a hyperbolic curve over an algebraically closed field of characteristic 3, there exists a finite flat tamely ramified covering of the hyperbolic curve such that the pull-back by the cover of the indigenous bundle is *not ordinary* [cf. Corollary 2.6]. Note that this result yields a *negative* answer to the basic question in p-adic Teichmüller theory given as [10, Introduction, §2.1, (2)].

The third main result of the present paper is a result concerning a relationship between *strongly spiked* indigenous bundles and *Tango curves* in characteristic 3. Let k be an algebraically closed field of characteristic p and (X, D) a hyperbolic curve of type (g, r) over k. Then we shall say that a nilpotent active [cf. [10, Chapter II, Definition 1.1]] indigenous bundle on (X, D) is *strongly spiked* if the indigenous bundle is mildly spiked of strength $(p-1)(2g-2+r)$ [cf. [10, Chapter II, Definition 3.1]]. Moreover, we shall say that a global section s of the invertible sheaf $\omega_{X/k}(D)$ is *strongly spiked* if the global section $s^{\otimes(p-1)}$ of the invertible sheaf $\omega_{X/k}(D)^{\otimes(p-1)}$ coincides with the square Hasse invariant of a strongly spiked indigenous bundle on (X, D).

The third main result of the present paper is as follows [cf. Theorem 3.9].

THEOREM C. — *Suppose that $(p, r) = (3, 0)$ [which thus implies that $D = \emptyset$]. Then a global section of the invertible sheaf $\omega_{X/k}(D) = \omega_{X/k}$ is* ***strongly spiked*** *if and only if the global section may be written as the product of a primitive fourth root of unity and the* ***logarithmic differential of a Tango function of level* 1** [*cf.* [7, Definition 1.3]] *on X.*

One immediate consequence [cf. Corollary 3.10] of Theorem C is that if a projective hyperbolic curve in characteristic 3 admits a *strongly spiked* global differential, then the curve is a *Tango curve* [cf. [7, Definition 1.8, (ii)]]. In Remark 3.9.1, we discuss an example of a *Tango function of level* 1 on a projective smooth curve of genus ≥ 2 whose *logarithmic differential is regular everywhere*. In particular, it follows from Theorem C that if $p = 3$, then the product of a primitive fourth root of unity and this logarithmic differential is a *strongly spiked* global differential on the curve.

Acknowledgments

This research was supported by JSPS KAKENHI Grant Number 18K03239 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1. Square Hasse Invariants of Indigenous Bundles and the Schwarz Torsor

In the present §1, we prove a result concerning a relationship between the *square Hasse invariants* of indigenous bundles and the torsor structure of the *Schwarz torsor* in characteristic 3 [cf. Theorem 1.7 below]. One immediate consequence of this result is that the isomorphism class of an indigenous bundle in characteristic 3 is *completely determined* by the associated *square Hasse invariant* [cf. Remark 1.7.1 below].

Throughout the present paper, let p be an *odd* prime number and (g, r) a pair of nonnegative integers such that $2g-2+r$ is *positive*; moreover, we shall use the notations "ω", "τ" to denote the *relative cotangent, tangent sheaves*, respectively.

Definition 1.1. — We shall write

$$\mathcal{M}$$

for the moduli stack of hyperbolic curves of type (g, r) in characteristic p and

$$(\mathcal{C}, \mathcal{D})$$

for the universal hyperbolic curve of type (g, r) over $\mathcal{M}$. In particular, $\mathcal{C}$ is a stack over $\mathcal{M}$, and $\mathcal{D}$ is a closed substack of $\mathcal{C}$ such that

- the stack $\mathcal{C}$ is smooth, proper, geometrically connected, and of relative dimension 1 over $\mathcal{M}$,
- each geometric fiber of $\mathcal{C}$ over $\mathcal{M}$ is [a necessarily smooth projective curve] of genus g, and
- the stack $\mathcal{D}$ is finite, étale, and of degree r over $\mathcal{M}$.

Moreover, we shall write

$$\pi\colon \mathcal{C} \longrightarrow \mathcal{M}$$

for the structure morphism of $\mathcal{C}$ over $\mathcal{M}$.

Definition 1.2. — We shall write

$$\mathcal{S} \longrightarrow \mathcal{M}$$

for the *Schwarz torsor* over $\mathcal{M}$ [cf. [10, Introduction, §0.4]], i.e., the moduli stack of hyperbolic curves of type (g, r) in characteristic p equipped with *indigenous bundles* [cf. [9, Chapter I, Definition 2.2]].

Definition 1.3. — We shall write

$$\mathcal{G} \stackrel{\text{def}}{=} \pi_*\big(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2}(\mathcal{D})\big)$$

and

$$\mathcal{V} \longrightarrow \mathcal{M}$$

for the vector bundle over $\mathcal{M}$ associated to the locally free coherent $\mathcal{O}_{\mathcal{M}}$-module obtained by forming the $\mathcal{O}_{\mathcal{M}}$-dual of the [necessarily locally free coherent] $\mathcal{O}_{\mathcal{M}}$-module $\pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes(p-1)})$, i.e., the vector bundle over $\mathcal{M}$ such that, for each scheme S over $\mathcal{M}$, there exists a natural bijection between the set of splittings of the morphism $\mathcal{V}|_S \to S$ and the module $\Gamma(S, \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes(p-1)})|_S)$.

Remark 1.3.1.

(i) Let us recall from [10, Introduction, §0.4] [cf. also [9, Chapter I, Corollary 2.9]] that the Schwarz torsor $\mathcal{S}$ admits a *natural structure of* $\mathcal{G}$*-torsor over* $\mathcal{M}$.

(ii) Suppose that $p = 3$. Then the natural inclusion $\mathcal{O}_{\mathcal{C}} \hookrightarrow \mathcal{O}_{\mathcal{C}}(\mathcal{D})$ determines an injective homomorphism $\mathcal{G} = \pi_*(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2}(\mathcal{D})) \hookrightarrow \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes 2}) = \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes(p-1)})$. In particular, one obtains an action of $\mathcal{G}$ on the vector bundle $\mathcal{V}$ over $\mathcal{M}$.

Definition 1.4. — By considering the *square Hasse invariants* [cf. [9, Chapter II, Proposition 2.6, (1)]] of the indigenous bundles parametrized by the Schwarz torsor, we obtain a morphism $\mathcal{S} \to \mathcal{V}$ of stacks over $\mathcal{M}$. We shall write

$$\text{sq-Hss}\colon \mathcal{S} \longrightarrow \mathcal{V}$$

for this morphism of stacks over $\mathcal{M}$.

Lemma 1.5. — *Suppose that $p = 3$. Let S be a noetherian scheme over $\mathbb{F}_p$, (X, D) a hyperbolic curve of type (g, r) over S, $(\mathcal{E}, \nabla_{\mathcal{E}})$ an* ***indigenous vector bundle*** *on (X, D) [cf.* [9, Chapter I, Definition 2.2]]*, and $F^0(\mathcal{E}) \subseteq \mathcal{E}$ an $\mathcal{O}_X$-submodule of rank 1 as in the discussion following* [9, Chapter I, Definition 2.2]*, i.e., an $\mathcal{O}_X$-submodule of rank* 1 *such that if one writes*

$$\mathcal{Q} \stackrel{\text{def}}{=} \mathcal{E}/F^0(\mathcal{E}),$$

then the composite

$$F^0(\mathcal{E}) \hookrightarrow \mathcal{E} \xrightarrow{\nabla_{\mathcal{E}}} \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{E} \twoheadrightarrow \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{Q}$$

is an ***isomorphism*** *of $\mathcal{O}_X$-modules, by means of which let us* ***identify*** *$\omega_{X/S}(D)$ with $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{Q}, F^0(\mathcal{E}))$:*

$$\omega_{X/S}(D) = \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{Q}, F^0(\mathcal{E})\big).$$

Moreover, let $x \in X \setminus D$ be a closed point of $X \setminus D$, $t \in \mathcal{O}_X$ a local parameter of X/S at x [which thus determines local trivializations $dt \in \omega_{X/S}(D)$, $d/dt \in \tau_{X/S}(-D)$ of the invertible sheaves $\omega_{X/S}(D)$, $\tau_{X/S}(-D)$ at x, respectively], and $e_F \in F^0(\mathcal{E})$ a local trivialization of the invertible sheaf $F^0(\mathcal{E})$ at x. Thus, we have a local section of $\mathcal{E}$ at x

$$e_{\mathcal{E}} \stackrel{\text{def}}{=} \nabla_{\mathcal{E}}\Big(\frac{d}{dt}\Big)(e_F) \in \mathcal{E}$$

such that the pair $(e_F, e_{\mathcal{E}})$ *forms a local trivialization of the* $\mathcal{O}_X$*-module* $\mathcal{E}$ *at* x [*cf. the above condition imposed on the* $\mathcal{O}_X$*-submodule* $F^0(\mathcal{E}) \subseteq \mathcal{E}$]. *Write* $e_{\mathcal{Q}} \in \mathcal{Q}$ *for the local trivialization of the invertible sheaf* $\mathcal{Q}$ *at* x *obtained by forming the image of* $e_{\mathcal{E}}$ *in the quotient* $\mathcal{Q}$ *and* f_F, $f_{\mathcal{E}} \in \mathcal{O}_X$ *for the local functions on* X *at* x *such that the local section* $\nabla_{\mathcal{E}}(e_{\mathcal{E}}) \in \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{E}$ *of* $\omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{E}$ *at* x *is given by*

$$dt \otimes (f_F \cdot e_F + f_{\mathcal{E}} \cdot e_{\mathcal{E}}).$$

Then the ***square Hasse invariant*** *— that is a global section of the invertible sheaf*

$$\omega_{X/S}(D)^{\otimes(p-1)} = \omega_{X/S}(D)^{\otimes 2} = \mathcal{H}om_{\mathcal{O}_X}\Big(\tau_{X/S}(-D)^{\otimes 3}, \mathcal{H}om_{\mathcal{O}_X}\big(F^0(\mathcal{E}), \mathcal{Q}\big)\Big)$$

— of the indigenous bundle obtained by forming the projectivization of the indigenous vector bundle $(\mathcal{E}, \nabla_{\mathcal{E}})$ *is given by*

$$\frac{d}{dt} \otimes \frac{d}{dt} \otimes \frac{d}{dt} \quad \mapsto \quad \Big(e_F \mapsto \Big(f_F + \frac{df_{\mathcal{E}}}{dt} + f_{\mathcal{E}}^2\Big) \cdot e_{\mathcal{Q}}\Big)$$

at x.

Proof. — Write $\mathcal{P}\colon \tau_{X/S}(-D)^{\otimes 3} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ for the *p-curvature* of the connection $\nabla_{\mathcal{E}}$. Then it follows from the "straightforward computation"

$$e_F \overset{\nabla_{\mathcal{E}}\left(\frac{d}{dt}\right)}{\mapsto} e_{\mathcal{E}} \overset{\nabla_{\mathcal{E}}\left(\frac{d}{dt}\right)}{\mapsto} f_F \cdot e_F + f_{\mathcal{E}} \cdot e_{\mathcal{E}}$$

$$\overset{\nabla_{\mathcal{E}}\left(\frac{d}{dt}\right)}{\mapsto} \Big(\frac{df_F}{dt} + f_F f_{\mathcal{E}}\Big) \cdot e_F + \Big(f_F + \frac{df_{\mathcal{E}}}{dt} + f_{\mathcal{E}}^2\Big) \cdot e_{\mathcal{E}}$$

that

$$\mathcal{P}\Big(\frac{d}{dt} \otimes \frac{d}{dt} \otimes \frac{d}{dt}\Big)(e_F) = \Big(\frac{df_F}{dt} + f_F f_{\mathcal{E}}\Big) \cdot e_F + \Big(f_F + \frac{df_{\mathcal{E}}}{dt} + f_{\mathcal{E}}^2\Big) \cdot e_{\mathcal{E}}.$$

Thus, since the square Hasse invariant of the indigenous bundle obtained by forming the projectivization of the indigenous vector bundle $(\mathcal{E}, \nabla_{\mathcal{E}})$ is defined to be the homomorphism $\tau_{X/S}(-D)^{\otimes 3} \to \mathcal{H}om_{\mathcal{O}_X}(F^0(\mathcal{E}), \mathcal{Q})$ that maps a local section $\partial \in \tau_{X/S}(-D)^{\otimes 3}$ of $\tau_{X/S}(-D)^{\otimes 3}$ to the composite

$$F^0(\mathcal{E}) \hookrightarrow \mathcal{E} \xrightarrow{\mathcal{P}(\partial)} \mathcal{E} \twoheadrightarrow \mathcal{Q},$$

we conclude that Lemma 1.5 holds. This completes the proof of Lemma 1.5. □

Lemma 1.6. — *In the situation of Lemma* 1.5, *let* $\theta \in \Gamma(X, \omega_{X/S}^{\otimes 2}(D))$ *be a global section of the invertible sheaf*

$$\omega_{X/S}^{\otimes 2}(D) = \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{Q}, \omega_{X/S} \otimes_{\mathcal{O}_X} F^0(\mathcal{E})\big).$$

Let us ***identify*** *the global section* θ *with the homomorphism* $\mathcal{E} \to \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{E}$ *of* $\mathcal{O}_X$*-modules obtained by forming the composite*

$$\mathcal{E} \twoheadrightarrow \mathcal{Q} \xrightarrow{\theta} \omega_{X/S} \otimes_{\mathcal{O}_X} F^0(\mathcal{E}) \hookrightarrow \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{E}.$$

Write $\nabla_{\mathcal{E}}^{\theta}$ *for the connection on* $\mathcal{E}$ [*necessarily relative to* $(X, D)/S$] *such that the homomorphism* $\mathcal{E} \to \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{E}$ *of* $\mathcal{O}_X$*-modules given by "*$e \mapsto \nabla_{\mathcal{E}}^{\theta}(e) - \nabla_{\mathcal{E}}(e)$*" coincides with the homomorphism* $\theta\colon \mathcal{E} \to \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{E}$ *of* $\mathcal{O}_X$*-modules, i.e., the connection on* $\mathcal{E}$

such that the pair $(\mathcal{E}, \nabla^{\theta}_{\mathcal{E}})$ forms an indigenous vector bundle, and, moreover, the indigenous bundle on (X, D) obtained by forming the projectivization of $(\mathcal{E}, \nabla^{\theta}_{\mathcal{E}})$ coincides with the indigenous bundle obtained by forming the result of the action [*cf. Remark* 1.3.1, (i)] *of θ on the indigenous bundle obtained by forming the projectivization of $(\mathcal{E}, \nabla_{\mathcal{E}})$. Then the* ***difference between the square Hasse invariants*** *— that is a global section of the invertible sheaf*

$$\omega_{X/S}(D)^{\otimes(p-1)} = \omega_{X/S}(D)^{\otimes 2}$$

— of the indigenous bundles obtained by forming the projectivizations of $(\mathcal{E}, \nabla^{\theta}_{\mathcal{E}})$, $(\mathcal{E}, \nabla_{\mathcal{E}})$ ***coincides*** *with the global section θ of $\omega^{\otimes 2}_{X/S}(D)$ $(\subseteq \omega_{X/S}(D)^{\otimes 2})$.*

Proof. — Let us first observe that since the homomorphism $\theta \colon \mathcal{E} \to \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{E}$ of $\mathcal{O}_X$-modules *annihilates* the $\mathcal{O}_X$-submodule $F^0(\mathcal{E}) \subseteq \mathcal{E}$, it follows from the definition of $\nabla^{\theta}_{\mathcal{E}}$ that $\nabla^{\theta}_{\mathcal{E}}(e_F) - \nabla_{\mathcal{E}}(e_F) = 0$, which thus implies that the local section "$e_{\mathcal{E}}$" of $\mathcal{E}$ at x of Lemma 1.5 in the case where we take the "$(\mathcal{E}, \nabla_{\mathcal{E}})$" of Lemma 1.5 to be the indigenous vector bundle $(\mathcal{E}, \nabla^{\theta}_{\mathcal{E}})$ is given by $e_{\mathcal{E}}$.

Write $\phi \in \mathcal{O}_X$ for the local function on X at x such that the global section θ of the invertible sheaf

$$\omega^{\otimes 2}_{X/S}(D) = \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{Q}, \omega_{X/S} \otimes_{\mathcal{O}_X} F^0(\mathcal{E})\big)$$

is given by

$$e_{\mathcal{Q}} \mapsto \phi \cdot dt \otimes e_F$$

at x. Then it is immediate from the definition of $\nabla^{\theta}_{\mathcal{E}}$ that

$$\nabla^{\theta}_{\mathcal{E}}(e_{\mathcal{E}}) - \nabla_{\mathcal{E}}(e_{\mathcal{E}}) = \phi \cdot dt \otimes e_F.$$

Thus, we conclude that the pair "$(f_F, f_{\mathcal{E}})$" of Lemma 1.5 in the case where we take the "$(\mathcal{E}, \nabla_{\mathcal{E}})$" of Lemma 1.5 to be the indigenous vector bundle $(\mathcal{E}, \nabla^{\theta}_{\mathcal{E}})$ is given by $(f_F + \phi, f_{\mathcal{E}})$. In particular, Lemma 1.6 follows immediately from Lemma 1.5. This completes the proof of Lemma 1.6. ☐

The following theorem is the first main result of the present paper.

Theorem 1.7. — *Suppose that $p = 3$. Then the morphism of stacks over $\mathcal{M}$*

$$\text{sq-Hss} \colon \mathcal{S} \longrightarrow \mathcal{V}$$

is ***compatible*** *with the respective actions of $\mathcal{G}$* [*cf. Remark* 1.3.1, (i), (ii)]. *In particular, this morphism is a* ***closed immersion****.*

Proof. — This assertion follows immediately, in light of [9, Chapter I, Proposition 2.6], from Lemma 1.6. ☐

Remark 1.7.1. — One immediate consequence of Theorem 1.7 is that the isomorphism class of an indigenous bundle in characteristic 3 is *completely determined* by the associated *square Hasse invariant.*

Remark 1.7.2. — Suppose that $(p, r) = (3, 0)$. Then it is immediate that the injective homomorphism $\mathcal{G} = \pi_*(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2}(\mathcal{D})) \hookrightarrow \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes 2}) = \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes(p-1)})$ of Remark 1.3.1, (ii), is an *isomorphism*. Thus, it follows from Theorem 1.7 that the morphism of stacks over $\mathcal{M}$

$$\text{sq-Hss}: \mathcal{S} \longrightarrow \mathcal{V}$$

is an *isomorphism*. In particular, we conclude that [1, Theorem A] may also be derived from Theorem 1.7.

2. Ordinariness of Nilpotent Admissible Indigenous Bundles

In the present §2, we prove a result concerning the *ordinariness* of nilpotent admissible indigenous bundles in characteristic 3. More precisely, we prove that, for a given nilpotent admissible indigenous bundle in characteristic 3, the indigenous bundle is *ordinary* if and only if the associated *Hasse defect* is *parabolically ordinary* [cf. Theorem 2.4 below].

In the present §2, let S be a noetherian scheme over $\mathbb{F}_p$, (X, D) a hyperbolic curve of type (g, r) over S, and (P, ∇_P) an *indigenous bundle* on (X, D). Suppose that the indigenous bundle (P, ∇_P) is *nilpotent* [cf. [9, Chapter II, Definition 2.4]] and *admissible* [cf. [9, Chapter II, Definition 2.4]].

Definition 2.1. — Let us recall from [9, Chapter II, Proposition 2.6, (3)] that there exist a unique, up to isomorphism [cf. [9, Chapter II, Proposition 2.6, (4)]], invertible sheaf $\mathcal{H}$ on X and a global section χ of $\mathcal{H}$ [i.e., the *Hasse invariant* of (P, ∇_P)] such that the global section $\chi^{\otimes 2}$ of the square $\mathcal{H}^{\otimes 2}$ coincides with the square Hasse invariant of (P, ∇_P). We shall refer to the invertible sheaf on X

$$\mathcal{H}om_{\mathcal{O}_X}\big(\omega_{X/S}(D)^{\otimes(p-1)/2}, \mathcal{H}\big)$$

as the *Hasse defect* of (P, ∇_P).

Remark 2.1.1. — Suppose that $r = 0$. Then one verifies immediately from [1, Proposition B.4] that the *Hasse defect* of (P, ∇_P) in the sense of Definition 2.1 *coincides* with the *Hasse defect* of (P, ∇_P) in the sense of [1, Definition B.2].

Proposition 2.2. — *The* ***square*** *of the Hasse defect of (P, ∇_P) is* ***trivial****.*

Proof. — This assertion follows from the fact that the square Hasse invariant of an indigenous bundle on (X, D) is a global section of the invertible sheaf $\omega_{X/S}(D)^{\otimes(p-1)}$. □

Lemma 2.3. — *Suppose that S is the spectrum of an algebraically closed field [i.e., of characteristic p]. Let $x \in X \setminus D$ be a closed point of $X \setminus D$ and $t \in \mathcal{O}_X$ a local parameter of X/S at x [which thus determines a local trivialization $dt \in \omega_{X/S}(D)$ of the invertible sheaf $\omega_{X/S}(D)$ at x]. Write $\phi \in \mathcal{O}_X$ for the local function on X at x such that the square Hasse invariant of (P, ∇_P) [that is a global section of the invertible sheaf $\omega_{X/S}(D)^{\otimes(p-1)}$] is given by*

$$\phi \cdot dt^{\otimes(p-1)}$$

at x. Then the following two conditions are equivalent:

(1) *The nilpotent admissible indigenous bundle (P, ∇_P) is* ***ordinary*** [*cf.* [9, Chapter II, Definition 3.1]].

(2) *For every nonzero global section of the invertible sheaf $\omega_{X/S}^{\otimes 2}(D)$, if $\psi \in \mathcal{O}_X$ is the local function on X at x such that the global section of $\omega_{X/S}^{\otimes 2}(D)$ is given by*

$$\psi \cdot dt \otimes dt$$

at x, then the equality

$$\frac{d^{p-1}}{dt^{p-1}}(\phi\psi) = 0$$

does ***not hold****.*

Proof. — This assertion follows immediately from [9, Chapter II, Lemma 2.11] and [9, Chapter II, Proposition 2.12] [cf. also the discussion concerning the *Cartier operator* given in [8, §2.1] — especially, the equality (2.1.13) in [8, §2.1]]. □

The following theorem is the second main result of the present paper.

Theorem 2.4. — *Suppose that $p = 3$. Then the following two conditions are equivalent:*

(1) *The nilpotent admissible indigenous bundle (P, ∇_P) is* ***ordinary****.*

(2) *The* ***Hasse defect*** *of (P, ∇_P) is* ***parabolically ordinary*** [*cf. Proposition* 2.2*;* [1, Definition A.7]].

Proof. — Let us first observe that it follows immediately from the various definitions involved [cf. also the proof of [9, Chapter II, Proposition 3.4]] that, to verify Theorem 2.4, we may assume without loss of generality, by replacing X by a geometric fiber of X/S, that S is the spectrum of an algebraically closed field [i.e., of characteristic 3].

Write $\mathcal{L}$ for the *Hasse defect* of (P, ∇_P), $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)^{(p-1)/2}) = \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ for the *Hasse invariant* of (P, ∇_P), and E_{ss} for the *supersingular divisor* of (P, ∇_P) [i.e., the divisor obtained by forming the zero locus of the Hasse invariant χ — cf. [9, Chapter II, Proposition 2.6, (3)]]. Fix a global trivialization Θ of the square of $\mathcal{L}$ [cf. Proposition 2.2]. Let $x \in X \setminus D$ be a closed point of $X \setminus D$, $t \in \mathcal{O}_X$ a local parameter of X/S at x [which thus determines a local trivialization $dt \in \omega_{X/S}(D)$ of the invertible sheaf $\omega_{X/S}(D)$ at x], and $l \in \mathcal{L}$ a local trivialization of $\mathcal{L}$ at x. Write $\phi \in \mathcal{O}_X$ for the local function on X at x such that the Hasse invariant χ is given by

$$\phi \cdot l \otimes dt$$

at x and $\delta \stackrel{\mathrm{def}}{=} \Theta(l \otimes l) \in \mathcal{O}_X^{\times}$ for the local unit on X at x determined by the global trivialization Θ and the local trivialization l. Thus, it follows from Lemma 2.3 that, to verify Theorem 2.4, it suffices to verify that condition (2) in the statement of Theorem 2.4 is *equivalent* to the following condition:

(1′) For every nonzero global section of the invertible sheaf $\omega_{X/S}^{\otimes 2}(D)$, if $\psi \in \mathcal{O}_X$ is the local function on X at x such that the global section of $\omega_{X/S}^{\otimes 2}(D)$ is given by

$$\psi \cdot dt \otimes dt$$

at x, then the equality

$$\frac{d^2}{dt^2}(\phi^2\delta\psi) = 0$$

does *not hold*.

Now let us verify the following assertion:

> Claim 2.4.A: Let η be a global section of the invertible sheaf $\omega_{X/S}^{\otimes 2}(D)$. Write $\psi \in \mathcal{O}_X$ for the local function on X at x such that the global section η is given by
>
> $$\psi \cdot dt \otimes dt$$
>
> at x. Suppose that the equality
>
> $$\frac{d^2}{dt^2}(\phi^2\delta\psi) = 0$$
>
> holds. Then η is *contained* in the subspace $\Gamma(X, \omega_{X/S}^{\otimes 2}(D - E_{\mathrm{ss}})) \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2}(D))$.

To this end, let us first recall from [2, Proposition A.4] that the supersingular divisor E_{ss} is *reduced*, i.e., that ϕ is *of order* ≤ 1 at x. Thus, since

$$0 = \frac{d^2}{dt^2}(\phi^2\delta\psi) = -\delta\psi\Big(\frac{d\phi}{dt}\Big)^2 - \phi\delta\psi\frac{d^2\phi}{dt^2} + \phi^2\psi\frac{d^2\delta}{dt^2} + \phi^2\delta\frac{d^2\psi}{dt^2}$$

$$+\phi\psi\frac{d\phi}{dt}\frac{d\delta}{dt} - \phi^2\frac{d\delta}{dt}\frac{d\psi}{dt} + \phi\delta\frac{d\phi}{dt}\frac{d\psi}{dt},$$

if $x \in \mathrm{Supp}(E_{\mathrm{ss}})$ [i.e., $\phi(x) = 0$], then x is *contained* in the zero locus of η [i.e., $\psi(x) = 0$]. Thus, since $\mathrm{Supp}(E_{\mathrm{ss}})$ does *not intersect* the closed subscheme D [cf. [2, Proposition A.4]], by varying "x" and again by applying the *reducedness* of the divisor E_{ss}, we conclude that Claim 2.4.A holds. This completes the proof of Claim 2.4.A.

Next, let us observe that it follows immediately from Claim 2.4.A, together with the definition of the supersingular divisor E_{ss}, that condition $(1')$ is *equivalent* to the following condition:

$(1'')$ For every nonzero global section of the invertible sheaf $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$, if $\psi \in \mathcal{O}_X$ is the local function on X at x such that the global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is given by

$$\psi \cdot l \otimes dt$$

at x, then the equality

$$\frac{d^2}{dt^2}(\phi^3\delta^2\psi) = 0$$

does *not hold*.

On the other hand, in the situation of condition $(1'')$, we have an equality

$$\frac{d^2}{dt^2}(\phi^3\delta^2\psi) = \phi^3\delta^3\frac{d^2}{dt^2}(\delta^{-1}\psi).$$

Thus, it follows from [1, Lemma A.9, (i)] that condition $(1'')$ is *equivalent* to condition (2) in the statement of Theorem 2.4, as desired. This completes the proof of Theorem 2.4. □

REMARK 2.4.1. — Note that Theorem 2.4 *generalizes* [1, Proposition 4.4].

REMARK 2.4.2. — Theorem 2.4 yields the following example of a *nilpotent admissible indigenous bundle* that arises from the universal elliptic curve over a *modular curve* but is *not ordinary*: Write $(Y(32), D(32))$ for the hyperbolic curve over $\overline{\mathbb{F}}_3$ obtained by considering the *modular curve* over $\overline{\mathbb{F}}_3$ associated to the congruence subgroup $\Gamma(32) \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then the projectivization of the *relative first de Rham cohomology module*, equipped with the *Gauss-Manin connection*, of the universal stable curve of type $(1, 1)$ over $Y(32)$ forms a *nilpotent admissible indigenous bundle* $(P(32), \nabla_{P(32)})$ on $(Y(32), D(32))$ [cf., e.g., [2, Lemma 2,8] and the discussion preceding [9, Chapter II, Proposition 3.5] in which "$\overline{\mathcal{M}}_{1,1}^{\log}[2]$" appears]. Write $\mathcal{H}(32)$ for the *Hodge bundle* on $Y(32)$. Then let us recall that the [classical] Hasse invariant associated to the modular curve $(Y(32), D(32))$ [i.e., the Hasse invariant of the nilpotent admissible indigenous bundle $(P(32), \nabla_{P(32)})$] is a modular form of weight $p-1 = 2$ [i.e., is a global section of the invertible sheaf $\mathcal{H}(32)^{\otimes 2}$]. Thus, since the Kodaira-Spencer map determines an *isomorphism* $\mathcal{H}(32)^{\otimes 2} \xrightarrow{\sim} \omega_{Y(32)/\overline{\mathbb{F}}_3}(D(32))$ of $\mathcal{O}_{Y(32)}$-modules, we conclude that

(a) the Hasse defect of $(P(32), \nabla_{P(32)})$ is *trivial*.

Next, let us recall that it is well-known that the elliptic curve "$y^2 = x^3 - x$" over $\mathbb{Q}$ is *of conductor* 32 and *supersingular* at 3. Thus, we conclude that the Jacobian variety of $Y(32)$ is *not ordinary*, or, alternatively,

(b) the structure sheaf $\mathcal{O}_{Y(32)}$ is *not parabolically ordinary*.

Thus, it follows from Theorem 2.4, together with (a), (b), that the nilpotent admissible indigenous bundle $(P(32), \nabla_{P(32)})$ is *not ordinary*.

COROLLARY 2.5. — *Suppose that $p = 3$. Suppose, moreover, that one of the following three conditions is satisfied:*

(1) *The **equality** $g = 0$ holds.*

(2) *The **equality** $g = 1$ holds, and the family of elliptic curves over S obtained by forming the Jacobian variety of X/S is **ordinary**.*

(3) *The **equality** $g = 1$ holds, and the Hasse defect of (P, ∇_P) is **nontrivial**.*

*Then the nilpotent admissible indigenous bundle (P, ∇_P) is **ordinary**. In particular, if the hyperbolic curve (X, D) **satisfies** either (1) or (2) and, moreover, admits a **nilpotent admissible indigenous bundle**, then the hyperbolic curve (X, D) is **hyperbolically ordinary** [cf.* [9, Chapter II, Definition 3.3]].

PROOF. — This assertion follows from Theorem 2.4. □

REMARK 2.5.1. — Let us recall the following basic question in p-adic Teichmüller theory [cf. [10, Introduction, §2.1, (1)]]:

> $(*)$ Is an arbitrary hyperbolic curve over an algebraically closed field of odd characteristic *hyperbolically ordinary*? Put another way, does an arbitrary hyperbolic curve over an algebraically closed field of odd characteristic admit a *nilpotent ordinary indigenous bundle*?

(i) One may easily find that Corollary 2.5 is closely related to this question $(*)$. Now let us also recall that some results on this question $(*)$ may be found in, for instance, [2, Theorem C] and [3, Theorem A] [cf. also the discussion following [2, Theorem C]].

(ii) Let us observe that since every nilpotent *ordinary* indigenous bundle is a nilpotent *admissible* indigenous bundle [cf. [9, Chapter II, Proposition 3.2]], it follows from Theorem 2.4 that an *affirmative* answer to this question $(*)$ implies the following assertion:

> $(**)$ An arbitrary projective smooth curve over an algebraically closed field of characteristic 3 admits an invertible sheaf that is [*of order* ≤ 2 and] *parabolically ordinary.* That is to say, for an arbitrary projective smooth curve over an algebraically closed field of characteristic 3, either
>
> - the *Jacobian* variety of the curve is an *ordinary* abelian variety, or
> - there exists a connected finite étale covering of the curve *of degree* 2 whose *Prym* variety is an *ordinary* abelian variety.

Here, let us also recall that the author of the present paper already gave a proof of this assertion $(**)$ [cf. [6, Theorem 2.7, (ii)]]. In fact, this "implication" is one of the main motivations for studying the assertion $(**)$ in [6].

Corollary 2.6. — *Suppose that S is the spectrum of an algebraically closed field of characteristic* 3. *Then there exist a hyperbolic curve (Y, E) over S and a finite flat tamely ramified covering $(Y, E) \to (X, D)$ over S such that the* [*necessarily nilpotent admissible — cf.* [2, Lemma 2,8]] *indigenous bundle $(P, \nabla_P)|_{(Y,E)}$ on (Y, E) obtained by forming the pull-back of (P, ∇_P) by the covering $(Y, E) \to (X, D)$ is* ***not ordinary****.*

Proof. — Let us first observe that we may assume without loss of generality, by replacing X by the connected finite étale covering of X [i.e., of degree 1 or 2] that trivializes the Hasse defect of (P, ∇_P), that the Hasse defect of (P, ∇_P) is *trivial.* Moreover, one verifies easily that we may assume without loss of generality, by replacing (X, D) by a suitable connected finite flat tamely ramified covering of (X, D), that $g \geq 2$. Then it follows from [11, Théorème 2] that we may assume without loss of generality, by replacing X by a suitable connected finite étale covering of X, that the Jacobian variety of X is *not ordinary.* Then it follows from Theorem 2.4 that the indigenous bundle (P, ∇_P) is *not ordinary*, as desired. This completes the proof of Corollary 2.6. □

Remark 2.6.1.

(i) Note that Corollary 2.6 *generalizes* [1, Theorem C].

(ii) Note that Corollary 2.6 yields a *negative* answer to the basic question in p-adic Teichmüller theory given as [10, Introduction, §2.1, (2)].

Definition 2.7. — We shall write

$$\mathcal{N} \subseteq \mathcal{S}$$

[cf. Definition 1.2] for the moduli stack of *smooth nilcurves* [cf. the discussion preceding [10, Introduction, Theorem 0.1]] of type (g, r) in characteristic p, i.e., the moduli stack of hyperbolic curves of type (g, r) in characteristic p equipped with *nilpotent* indigenous bundles;

$$\mathcal{N}^{\mathrm{adm}} \subseteq \mathcal{N}$$

for the *admissible locus* of $\mathcal{N}$, i.e., the [necessarily open] substack of $\mathcal{N}$ that parametrizes hyperbolic curves of type (g, r) in characteristic p equipped with nilpotent *admissible* indigenous bundles;

$$\mathcal{N}^{\mathrm{ord}} \subseteq \mathcal{N}^{\mathrm{adm}}$$

for the *ordinary locus* of $\mathcal{N}$, i.e., the [necessarily open] substack of $\mathcal{N}$ that parametrizes hyperbolic curves of type (g, r) in characteristic p equipped with nilpotent *ordinary* indigenous bundles.

Definition 2.8. — We shall write

$$\mathcal{J} \longrightarrow \mathcal{M}$$

for the Jacobian variety of $\mathcal{C}/\mathcal{M}$ and

$$\mathcal{M}^{\text{pb-ord}} \subseteq \mathcal{M}$$

for the *parabolically ordinary locus* of $\mathcal{M}$ [cf. the discussion following [9, Chapter II, Definition 3.3]], i.e., the [unique] maximal open substack of $\mathcal{M}$ such that the geometric fiber of $\mathcal{J} \to \mathcal{M}$ at each geometric point of $\mathcal{M}^{\text{pb-ord}}$ is an ordinary abelian variety. For a positive integer n, we shall write

$$\mathcal{J}[n] \subseteq \mathcal{J}$$

for the [necessarily closed] substack of $\mathcal{J}$ obtained by forming the kernel of the endomorphism of $\mathcal{J}$ over $\mathcal{M}$ given by multiplication by n. Moreover, we shall write

$$\mathcal{J}[2]^{\text{pb-ord}} \subseteq \mathcal{J}[2]$$

for the *parabolically ordinary locus* of $\mathcal{J}[2]$, i.e., the open substack of $\mathcal{J}[2]$ defined to be the union of the open substack $\mathcal{J}[2] \setminus \mathcal{J}[1]$ and the open substack $(\mathcal{J}[2] \hookrightarrow \mathcal{J} \to \mathcal{M})^{-1}(\mathcal{M}^{\text{pb-ord}})$ (respectively, to be the open substack "$\mathcal{J}_g[2]^{\text{pb-ord}}$" of [1, Definition C.4]) if $g \leq 1$ (respectively, ≥ 2).

Definition 2.9. — By considering the *Hasse defects* of the nilpotent admissible indigenous bundles parametrized by the admissible locus $\mathcal{N}^{\mathrm{adm}}$, we obtain a morphism $\mathcal{N}^{\mathrm{adm}} \to \mathcal{J}[2]$ of stacks over $\mathcal{M}$ [cf. Proposition 2.2]. We shall write

$$\text{Hss-df}\colon \mathcal{N}^{\mathrm{adm}} \longrightarrow \mathcal{J}[2]$$

for this morphism of stacks over $\mathcal{M}$.

Remark 2.9.1. — It is immediate [cf. also Remark 2.1.1] that if $r = 0$, then the morphism $\text{Hss-df}\colon \mathcal{N}^{\mathrm{adm}} \to \mathcal{J}[2]$ of Definition 2.9 *coincides* with the *Hasse defect morphism* defined in [1, Definition C.1].

Corollary 2.10. — *Suppose that $p = 3$. Then we have a **cartesian** diagram of stacks over $\mathcal{M}$*

$$\begin{array}{ccc} \mathcal{N}^{\mathrm{ord}} & \longrightarrow & \mathcal{J}[2]^{\text{pb-ord}} \\ \downarrow & & \downarrow \\ \mathcal{N}^{\mathrm{adm}} & \xrightarrow[\text{Hss-df}]{} & \mathcal{J}[2] \end{array}$$

— *where the vertical arrows are the natural open immersions.*

Proof. — This assertion follows from Theorem 2.4. □

Remark 2.10.1. — Note that Corollary 2.10 *generalizes* [1, Corollary 5.5].

3. Strongly Spiked Indigenous Bundles and Tango Curves

In the present §3, we prove a result concerning a relationship between *strongly spiked* indigenous bundles and *Tango curves* in characteristic 3 [cf. Theorem 3.9 below]. One immediate consequence of this result is that if a projective hyperbolic curve in characteristic 3 admits a global differential whose square coincides with the square Hasse invariant of a *strongly spiked* indigenous bundles, then the curve is a *Tango curve* [cf. Corollary 3.10 below].

In the present §3, let k be an algebraically closed field of characteristic p and (X, D) a hyperbolic curve of type (g, r) over k.

Definition 3.1. — Let $(\mathbb{P})$ be a property of an indigenous bundle [e.g., nilpotent, admissible, or ordinary]. Then we shall say that a global section s of the invertible sheaf $\omega_{X/k}(D)$ *satisfies the property to be* $(\mathbb{P})$, or, for simplicity, *is* $(\mathbb{P})$, if the global section $s^{\otimes(p-1)}$ of the invertible sheaf $\omega_{X/k}(D)^{\otimes(p-1)}$ coincides with the square Hasse invariant of an indigenous bundle on (X, D) that satisfies the property to be $(\mathbb{P})$.

Remark 3.1.1. — Suppose that $p \neq 3$. Then since [it is immediate that] an *admissible* indigenous bundle on (X, D) is *active* [cf. [10, Chapter II, Definition 1.1]], it follows from [2, Proposition A.5] that a global section of the invertible sheaf $\omega_{X/k}(D)$ is *never nilpotent admissible* [i.e., never satisfies the property to be nilpotent admissible — cf. Definition 3.1].

Some results proved in [1, §2, §3, §4] may be summarized as follows.

Proposition 3.2. — *Suppose that $(p, r) = (3, 0)$ [which thus implies that $D = \emptyset$]. Then the following assertions hold:*

(i) *A global section of the invertible sheaf $\omega_{X/k}(D) = \omega_{X/k}$ is **active** (respectively, **dormant** [cf.* [10, Chapter II, Definition 1.1]]) *if and only if the global section is **nonzero** (respectively, **zero**).*

(ii) *A global section of the invertible sheaf $\omega_{X/k}(D) = \omega_{X/k}$ is **nilpotent** if and only if the global section may be written as the product of a primitive fourth root of unity and* *the **logarithmic differential of a** [**possibly constant**] **nonzero rational function** on X.*

(iii) *A global section of the invertible sheaf $\omega_{X/k}(D) = \omega_{X/k}$ is **admissible** if and only if the zero locus of the global section is **reduced**.*

(iv) *A global section of the invertible sheaf $\omega_{X/k}(D) = \omega_{X/k}$ is **nilpotent ordinary** if and only if the global section is **nilpotent admissible** [cf.* (ii), (iii)], *and, moreover, the Jacobian variety of X is an **ordinary** abelian variety over k.*

PROOF. — Assertion (i) follows from [1, Corollary 2.4] [cf. also Theorem 1.7 of the present paper]. Assertion (iii) (respectively, (iv)) follows, in light of [1, Corollary 2.4] [cf. also Theorem 1.7 of the present paper], from [1, Proposition 3.1, (ii)] (respectively, [1, Proposition 4.4]).

Finally, we verify assertion (ii). Let us first recall that it follows, in light of [1, Corollary 2.4] [cf. also Theorem 1.7 of the present paper], from [1, Proposition 4.1] [cf. also assertion (i)] that a global section of the invertible sheaf $\omega_{X/k}(D) = \omega_{X/k}$ is *nilpotent* if and only if

($*$) the global section is either *zero* or a *normalized Cartier eigenform* [cf. [1, Definition A.8, (i)]] associated to the square-trivialized invertible sheaf [cf. [1, Definition A.3]] on X obtained by forming the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$.

On the other hand, it is well-known [cf., e.g., [8, Théorème 2.1.17]] that condition ($*$) is *equivalent* to the condition that the global section may be written as the product of a primitive fourth root of unity and the *logarithmic differential of a* [*possibly constant*] *nonzero rational function* on X. This completes the proof of assertion (ii), hence also of Proposition 3.2. □

PROPOSITION 3.3. — *Let (P, ∇_P) be an indigenous bundle on (X, D). Suppose that the indigenous bundle (P, ∇_P) is **nilpotent** and **active**. Then the following four conditions are equivalent:*

(1) *The indigenous bundle (P, ∇_P) is **mildly spiked of strength $(p-1)(2g-2+r)$*** [*cf.* [10, Chapter II, Definition 3.1]].

(2) *The divisor on X obtained by forming the zero locus of the **square Hasse invariant** of (P, ∇_P) **coincides** with the **spiked locus** of (P, ∇_P)* [*cf.* [10, Chapter II, Definition 3.1]].

(3) *The **generalized supersingular divisor** of (P, ∇_P)* [*cf.* [2, Definition A.2, (iii)]] *is **zero**.*

(4) *The image of the **Hodge section** of (P, ∇_P)* [*cf.* [9, Chapter I, Proposition 2.4]] *does **not intersect** the image of the **conjugate section** of (P, ∇_P)* [*cf.* [2, Definition A.2, (ii)]].

PROOF. — Write E_{sH}, E_{gss}, E_{spk} for the divisors on X obtained by forming the zero locus of the *square Hasse invariant* of (P, ∇_P), the *generalized supersingular divisor* of (P, ∇_P), and the *spiked locus* of (P, ∇_P), respectively. Then since

$$E_{\mathrm{spk}} \leq E_{\mathrm{sH}}, \qquad 2\deg(E_{\mathrm{gss}}) + \deg(E_{\mathrm{spk}}) = \deg(E_{\mathrm{sH}}) = (p-1)(2g-2+r)$$

[cf. [2, Proposition A.3, (iii)], [2, Lemma A.7, (i)]], the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold. Moreover, it follows from the definition of E_{gss} that the equivalence (3) $\Leftrightarrow$ (4) holds. This completes the proof of Proposition 3.3. $\square$

The following notion is one central notion of the discussion of the present §3.

Definition 3.4. — We shall say that an indigenous bundle on (X, D) is *strongly spiked* if the indigenous bundle is nilpotent and active, and, moreover, one of the four conditions in the statement of Proposition 3.3 is satisfied.

Remark 3.4.1. — It follows from condition (2) in the statement of Proposition 3.3, [2, Proposition A.3, (ii)], and [2, Lemma A.7, (i)] that if (X, D) has a *strongly spiked* indigenous bundle, then $2g - 2 + r$ is *divisible* by p.

Lemma 3.5. — *Let*

$$\phi = \sum_{-\infty < i < \infty} a_i t^i \in k((t))$$

be an element of the field $k((t))$. *Write*

$$\operatorname{ord}(\phi) \stackrel{\text{def}}{=} \inf\{\, i \in \mathbb{Z} \mid a_i \neq 0 \,\}$$

$$\leq \operatorname{ord}^{\notin p\mathbb{Z}}(\phi) \stackrel{\text{def}}{=} \inf\{\, i \in \mathbb{Z} \mid a_i \neq 0 \text{ and } i \notin p\mathbb{Z} \,\} \quad (\in \mathbb{Z} \cup \{\infty\}).$$

[*So the assignment* "ord" *coincides with the* t*-adic valuation on* $k((t))$ *that maps* $t \in k((t))$ *to* $1 \in \mathbb{Z}$.] *Suppose that* $\phi \neq 0$ [*which implies that* $\operatorname{ord}(\phi)$ *is an integer*]. *Then the following four conditions are equivalent:*

(1) *The integer* $\operatorname{ord}(\phi)$ *is* ***divisible*** *by* p.

(2) *The* ***inequality*** $\operatorname{ord}(\phi) < \operatorname{ord}^{\notin p\mathbb{Z}}(\phi)$ *holds.*

(3) *The* ***inequality*** $\operatorname{ord}(\phi) \leq \operatorname{ord}(d\phi/dt)$ *holds.*

(4) *The logarithmic derivative* $(d\phi/dt)/\phi$ *is* ***contained*** *in the subring* $k[[t]] \subseteq k((t))$.

Proof. — Let us observe that the equality $\operatorname{ord}(d\phi/dt) = \operatorname{ord}^{\notin p\mathbb{Z}}(\phi) - 1$ holds. Thus, Lemma 3.5 is immediate. $\square$

Lemma 3.6. — *Suppose that* $(p, r) = (3, 0)$ [*which thus implies that* $D = \emptyset$]. *Let* (P, ∇_P) *be an indigenous bundle on* (X, D). *Suppose, moreover, that the indigenous bundle* (P, ∇_P) *is* ***nilpotent*** *and* ***active***. *Write* E_{sH}, E_{gss}, E_{spk} *for the divisors on* X *obtained by forming the zero locus of the* ***square Hasse invariant*** *of* (P, ∇_P), *the* ***generalized supersingular divisor*** *of* (P, ∇_P), *and the* ***spiked locus*** *of* (P, ∇_P), *respectively. Let* $x \in X$ *be a closed point of* X *and* $t \in \mathcal{O}_X$ *a local parameter of* X/k *at* x [*which thus determines local trivializations* $dt \in \omega_{X/k}$, $d/dt \in \tau_{X/k}$ *of the invertible sheaves* $\omega_{X/k}$, $\tau_{X/k}$ *at* x, *respectively*]. *Write* $\theta \in \Gamma(X, \omega_{X/k}^{\otimes(p-1)}(D)) = \Gamma(X, \omega_{X/k}^{\otimes 2})$ *for the* ***square Hasse invariant*** *of the indigenous bundle* (P, ∇_P) *and* $\phi \in \mathcal{O}_X$ *for the* [*necessarily* ***nonzero***

— *cf.* [1, Proposition 3.2]] *local function on* X *at* x *such that the global section* θ *of the invertible sheaf* $\omega_{X/k}^{\otimes(p-1)}(D) = \omega_{X/k}^{\otimes 2}$ *is given by*

$$\phi \cdot dt \otimes dt$$

at x [*which thus implies that the equality*

$$\mathrm{ord}_x(E_{\mathrm{sH}}) = \mathrm{ord}_x(\phi)$$

holds.] *Then the following assertions hold:*

(i) *The following three conditions are equivalent:*

(i-1) *The closed point* $x \in X$ *is* ***contained*** *in* $\mathrm{Supp}(E_{\mathrm{sH}}) \subseteq X$*.*

(i-2) *The* ***inequality*** $\mathrm{ord}_x(\phi) > 0$ *holds.*

(i-3) *The* ***inequality*** $\mathrm{ord}_x(\phi) > 1$ *holds.*

(ii) *The following two conditions are equivalent:*

(ii-1) *The closed point* $x \in X$ *is* ***contained*** *in* $\mathrm{Supp}(E_{\mathrm{spk}}) \subseteq X$*.*

(ii-2) *The* ***inequality*** $\mathrm{ord}_x(\phi) > 2$ *holds.*

(iii) *The following three conditions are equivalent:*

(iii-1) *The closed point* $x \in X$ *is* ***contained*** *in* $\mathrm{Supp}(E_{\mathrm{gss}}) \subseteq X$*.*

(iii-2) *The* ***inequalities*** $0 < \mathrm{ord}_x(d\phi/dt) < \mathrm{ord}_x(\phi)$ *hold.*

(iii-3) *The integer* $\mathrm{ord}_x(\phi) - 2$ *is* ***divisible*** *by* 3*.*

Proof. — Let us first recall from [9, Chapter I, Proposition 2.6] that there exists an *indigenous vector bundle* $(\mathcal{E}, \nabla_{\mathcal{E}})$ on (X, D) whose projectivization is (P, ∇_P). Let $F^0(\mathcal{E}) \subseteq \mathcal{E}$ be an $\mathcal{O}_X$-submodule of rank 1 as in the discussion following [9, Chapter I, Definition 2.2], i.e., an $\mathcal{O}_X$-submodule of rank 1 such that the composite

$$F^0(\mathcal{E}) \hookrightarrow \mathcal{E} \xrightarrow{\nabla_{\mathcal{E}}} \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E} \twoheadrightarrow \omega_{X/k} \otimes_{\mathcal{O}_X} \big(\mathcal{E}/F^0(\mathcal{E})\big)$$

is an *isomorphism* of $\mathcal{O}_X$-modules. Moreover, let us also recall that it follows immediately from the discussion at the beginning of [1, §3] [cf. also [1, Remark 2.4.1]], together with [1, Proposition 3.2], that we may assume without loss of generality that there exists a *local trivialization* $(e_F, e_{\mathcal{E}})$ of the $\mathcal{O}_X$-module $\mathcal{E}$ at x such that

(a) the local section $e_F \in \mathcal{E}$ of $\mathcal{E}$ at x is *contained* in the $\mathcal{O}_X$-submodule $F^0(\mathcal{E}) \subseteq \mathcal{E}$ and forms a *local trivialization* of the invertible sheaf $F^0(\mathcal{E})$ at x, and, moreover,

(b) the *p-curvature* $\tau_{X/S}^{\otimes 3} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ of the connection $\nabla_{\mathcal{E}}$ is given by

$$\frac{d}{dt} \otimes \frac{d}{dt} \otimes \frac{d}{dt} \quad \mapsto \quad \Big((e_F, e_{\mathcal{E}}) \mapsto \Big(\frac{d\phi}{dt} \cdot e_F + \phi \cdot e_{\mathcal{E}}, \Big(\phi^2 + \frac{d^2\phi}{dt^2}\Big) \cdot e_F - \frac{d\phi}{dt} \cdot e_{\mathcal{E}}\Big)\Big)$$

at x.

First, we verify assertion (i). The equivalence (i-1) $\Leftrightarrow$ (i-2) is immediate from the definition of the local function ϕ. Moreover, since $\mathrm{ord}_x(\phi) \neq 1$ [cf. [1, Lemma 3.5]], the equivalence (i-2) $\Leftrightarrow$ (i-3) holds. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first recall from [2, Lemma A.11, (ii)] that condition (ii-1) is *equivalent* to the condition that the image of the restriction of the

p-curvature of the connection $\nabla_{\mathcal{E}}$ to [the spectrum of the residue field at] x is *zero*. Thus, it follows from (b) that condition (ii-1) is *equivalent* to the condition that

$$\phi|_x = \frac{d\phi}{dt}\Big|_x = \frac{d^2\phi}{dt^2}\Big|_x = 0.$$

In particular, the equivalence (ii-1) $\Leftrightarrow$ (ii-2) holds. This completes the proof of assertion (ii).

Next, we verify the equivalence (iii-1) $\Leftrightarrow$ (iii-2). Let us first recall from [2, Lemma A.10, (ii)] that it follows from (a) that condition (iii-1) is *equivalent* to the condition that the restriction to [the spectrum of the residue field at] x of the local section e_F is *contained* in the restriction to [the spectrum of the residue field at] x of the *conjugate filtration* of $(\mathcal{E}, \nabla_{\mathcal{E}})$, i.e., the unique maximal horizontal invertible subsheaf of $\mathcal{E}$ [cf. the discussion following [2, Lemma A.7]]. Write $d_0 \stackrel{\text{def}}{=} \min\{\mathrm{ord}_x(\phi), \mathrm{ord}_x(d\phi/dt)\}$. Then one verifies immediately — by considering the unique maximal invertible subsheaf of $\mathcal{E}$ "*annihilated by the p-curvature*" [cf. our assumption that the endomorphism of the $\mathcal{O}_X$-module $\mathcal{E}$ given by the image by the p-curvature of an arbitrary element of $\tau_{X/S}^{\otimes 3}$ is *nilpotent*] — from (b) that the conjugate filtration of $(\mathcal{E}, \nabla_{\mathcal{E}})$ is generated by the local section at x

$$t^{-d_0} \cdot \Big(\frac{d\phi}{dt} \cdot e_F + \phi \cdot e_{\mathcal{E}}\Big) \in \mathcal{E}.$$

Thus, we conclude that condition (iii-1) is *equivalent* to the condition that

$$\Big(t^{-d_0} \cdot \frac{d\phi}{dt}\Big)\Big|_x \neq 0, \qquad (t^{-d_0} \cdot \phi)|_x = 0,$$

or, alternatively, the condition that $\mathrm{ord}_x(d\phi/dt) < \mathrm{ord}_x(\phi)$. On the other hand, since $\mathrm{ord}_x(\phi) \neq 1$ [cf. [1, Lemma 3.5]], the inequality 0 $(\leq \mathrm{ord}(d\phi/dt)) < \mathrm{ord}_x(\phi)$ implies the inequality $0 < \mathrm{ord}_x(d\phi/dt)$. This completes the proof of the equivalence (iii-1) $\Leftrightarrow$ (iii-2).

Finally, we verify the equivalence (iii-2) $\Leftrightarrow$ (iii-3). First, suppose that condition (iii-2) is satisfied. Then it follows from Lemma 3.5 that $\mathrm{ord}_x(\phi)$ is *not divisible* by 3. Thus, since $\mathrm{ord}_x(\phi)-1$ is *not divisible* by 3 [cf. [1, Lemma 3.5]], condition (iii-3) holds. Next, suppose that condition (iii-3) is satisfied. Then it follows from Lemma 3.5 that $\mathrm{ord}_x(d\phi/dt) < \mathrm{ord}_x(\phi)$. Thus, since $\mathrm{ord}_x(\phi) \neq 1$ [cf. [1, Lemma 3.5]], the inequality 0 $(\leq \mathrm{ord}_x(d\phi/dt))$ $< \mathrm{ord}_x(\phi)$ implies the inequality $0 < \mathrm{ord}_x(d\phi/dt)$, hence also condition (iii-2). This completes the proof of assertion (iii), hence also of Lemma 3.6. □

PROPOSITION 3.7. — *Suppose that $(p, r) = (3, 0)$. Let (P, ∇_P) be an indigenous bundle on (X, D). Suppose, moreover, that the indigenous bundle (P, ∇_P) is* ***nilpotent*** *and* ***active****. Write E_{sH}, E_{gss}, E_{spk} for the divisors on X obtained by forming the zero locus of the* ***square Hasse invariant*** *of (P, ∇_P), the* ***generalized supersingular divisor*** *of (P, ∇_P), and the* ***spiked locus*** *of (P, ∇_P), respectively. Then the* ***equality***

$$E_{\mathrm{sH}} = 2E_{\mathrm{gss}} + E_{\mathrm{spk}}$$

holds.

PROOF. — Let us first observe that it follows from [2, Proposition A.3, (iii)] that, to verify Proposition 3.7, it suffices to verify that, for each closed point $x \in X$ of X, the inequality

$$\mathrm{ord}_x(E_{\mathrm{spk}}) \leq \mathrm{ord}_x(E_{\mathrm{sH}}) - 2\mathrm{ord}_x(E_{\mathrm{gss}})$$

holds. Moreover, let us also observe that, again by [2, Proposition A.3, (iii)], if $x \in X$ is *not contained* in $\mathrm{Supp}(E_{\mathrm{gss}})$, then the desired inequality holds.

Let $x \in \mathrm{Supp}(E_{\mathrm{gss}})$ be a closed point of X contained in the support of E_{gss}. Thus, it follows from Lemma 3.6, (iii), that

(a) the integer $\mathrm{ord}_x(E_{\mathrm{sH}}) - 2$ is *divisible* by 3.

Now let us recall from [2, Proposition A.3, (ii)] that

(b) the integer $\mathrm{ord}_x(E_{\mathrm{spk}})$ is *divisible* by 3,

which thus [cf. (a), [2, Proposition A.3, (iii)]] implies that

(c) the inequality $\mathrm{ord}_x(E_{\mathrm{spk}}) < \mathrm{ord}_x(E_{\mathrm{sH}})$ holds.

Thus, we conclude immediately from (a), (b), (c) that $\mathrm{ord}_x(E_{\mathrm{spk}}) \leq \mathrm{ord}_x(E_{\mathrm{sH}}) - 2 = \mathrm{ord}_x(E_{\mathrm{sH}}) - 2\mathrm{ord}_x(E_{\mathrm{gss}})$ [cf. [2, Proposition A.3, (i)]], as desired. This completes the proof of Proposition 3.7. □

PROPOSITION 3.8. — *In the situation of Proposition* 3.7*, the following three conditions are equivalent:*

(1) *The indigenous bundle* (P, ∇_P) *is* ***strongly spiked****.*

(2) *There exists a divisor* $\underline{E}_{\mathrm{sH}}$ *on* X *such that the* ***equality*** $E_{\mathrm{sH}} = 3\underline{E}_{\mathrm{sH}}$ *holds.*

(3) *The order of* E_{sH} *is* ***divisible*** *by* 3 *at each closed point of* X*.*

PROOF. — The implication (1) ⇒ (2) follows from [2, Proposition A.3, (ii)] and condition (2) of Proposition 3.3. The implication (2) ⇒ (3) is immediate. The implication (3) ⇒ (1) follows from [2, Proposition A.3, (iii)], condition (3) of Proposition 3.3, and Lemma 3.6, (iii). This completes the proof of Proposition 3.8. □

The following theorem is the third main result of the present paper.

THEOREM 3.9. — *Suppose that* $(p, r) = (3, 0)$ [*which thus implies that* $D = \emptyset$]*. Then a global section of the invertible sheaf* $\omega_{X/k}(D) = \omega_{X/k}$ *is* ***strongly spiked*** *if and only if the global section may be written as the product of a primitive fourth root of unity and the* ***logarithmic differential of a Tango function of level*** **1** [*cf.* [7, Definition 1.3]] *on* X*.*

PROOF. — Let us first observe that it follows from Proposition 3.2, (i), (ii), and Proposition 3.8 that, to verify Theorem 3.9, it suffices to verify that, for a nonzero rational function f on X such that

($*$) the logarithmic differential of f is *nonzero* and *contained* in $\Gamma(X, \omega_{X/k})$,

the following two conditions are *equivalent*:

(1) The order of the logarithmic differential of f is *divisible* by 3 at each closed point of X.

(2) The rational function f is a *Tango function of level* 1.

Write $\mathrm{div}(f)$, $\mathrm{div}(df)$ for the divisors on X associated to the rational function f, the rational differential df, respectively. Then it follows from Lemma 3.5 that condition ($*$)

implies that the integer $\mathrm{ord}_x(\mathrm{div}(f))$ is *divisible* by 3 at each closed point $x \in X$ of X. Thus, condition (1) is *equivalent* to the condition that

(1′) the integer $\mathrm{ord}_x(\mathrm{div}(df))$ is *divisible* by 3 at each closed point $x \in X$ of X.

On the other hand, it follows immediately from [7, Theorem 1.9, (ii)] that condition (1′) is *equivalent* to condition (2), as desired. This completes the proof of Theorem 3.9. □

REMARK 3.9.1. — Suppose that we are in the proof of [5, Theorem 3]. Suppose, moreover, that the integer "N" is *equal* to 1, and that the integer "n" is *positive*. Then it follows from [5, Lemma 9] and [5, Remark 10] that the rational function "f" on the projective smooth curve "C" gives an example of a *Tango function of level* 1 on a projective smooth curve of genus ≥ 2 whose *logarithmic differential is regular everywhere*. In particular, it follows from Theorem 3.9 that if $p = 3$, then the product of a primitive fourth root of unity and this logarithmic differential is a *strongly spiked* global differential on the curve.

REMARK 3.9.2. — Suppose that $r = 0$. Let us first recall from [7, Theorem B] that

- giving a certain class of *Tango functions of level* 1 on X is *equivalent* to giving a *Frobenius-affine-indigenous structure of level* 1 on X [cf. [7, Definition 3.3]].

Next, let us recall that

- a *Frobenius-affine-indigenous structure of level* 1 on X is defined to be a pair consisting of a *dormant indigenous bundle* (P, ∇_P) on (X, D) and a *splitting* of the $\mathbb{P}^1$-bundle $P \to X$ which is *horizontal* [i.e., with respect to ∇_P] and whose image does *not intersect* the image of the Hodge section of (P, ∇_P) [cf. [7, Lemma 3.4], [7, Remark 3.4.2], and [4, Remark 4.4.1, (ii)]].

In particular, one may conclude that

> the notion of a *Tango function of level* 1 is closely related to the notion of a *dormant indigenous bundle* on (X, D).

On the other hand, one may also conclude from Theorem 3.9 [cf. also Theorem 1.7] that

> a suitable *Tango function of level* 1 naturally yields a *strongly spiked indigenous bundle* on (X, D) whenever $p = 3$.

[Now let us observe that it is immediate that

> a *strongly spiked* indigenous bundle is *never dormant*.]

COROLLARY 3.10. — *Suppose that* $(p, r) = (3, 0)$. *If the hyperbolic curve* (X, D) *has a* ***strongly spiked*** *global section of the invertible sheaf* $\omega_{X/k}(D)$, *then the projective smooth curve* X *is a* ***Tango curve*** [*cf.* [7, Definition 1.8, (ii)]].

PROOF. — This assertion follows from Theorem 3.9 and [7, Theorem A]. □

COROLLARY 3.11. — *Suppose that* $r = 0$. *Then the following two conditions are equivalent:*

(1) *The moduli stack of hyperbolic curves of type* (g, r) *in characteristic* 3 *equipped with* ***strongly spiked indigenous bundles*** *is* ***of dimension 3g − 3***, ***smooth*** *over* $\mathbb{F}_3$, *and* ***flat*** *and* ***quasi-finite*** *over* $\mathcal{M}$ [*cf. Definition* 1.1].

(2) *The integer* $g-1$ *is* ***divisible*** *by* 3.

Proof. — The implication (1) $\Rightarrow$ (2) follows from Remark 3.4.1. Next, to verify the implication (2) $\Rightarrow$ (1), suppose that condition (2) is satisfied. Then it follows from [5, Lemma 9] and [5, Remark 10] that there exists a projective smooth curve of genus g in characteristic 3 that admits a *Tango function of level* 1 whose *logarithmic differential is regular everywhere*. Thus, condition (1) follows immediately from Theorem 3.9 and [10, Chapter II, Theorem 3.9] [cf. also condition (1) in the statement of Proposition 3.3]. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Corollary 3.11. □

References

[1] Y. Hoshi: Nilpotent admissible indigenous bundles via Cartier operators in characteristic three. *Kodai Math. J.* **38** (2015), no. **3**, 690-731.

[2] Y. Hoshi: On the supersingular divisors of nilpotent admissible indigenous bundles. *Kodai Math. J.* **42** (2019), no. **1**, 1-47.

[3] Y. Hoshi: Hyperbolic ordinariness of hyperelliptic curves of lower genus in characteristic three. *Kyushu J. Math.* **73** (2019), no. **2**, 317-335.

[4] Y. Hoshi: Frobenius-projective structures on curves in positive characteristic. *Publ. Res. Inst. Math. Sci.* **56** (2020), no. **2**, 401-430.

[5] Y. Hoshi: A note on the existence of Tango curves. *Kodai Math. J.* **44** (2021), no. 1, 77-80.

[6] Y. Hoshi: A note on torsion points on ample divisors on abelian varieties. *Math. J. Okayama Univ.* **64** (2022), 1-11.

[7] Y. Hoshi: *Frobenius-affine structures and Tango curves.* RIMS Preprint **1913** (April 2020).

[8] L. Illusie: Complexe de de Rham-Witt et cohomologie cristalline. *Ann. Sci. École Norm. Sup.* (**4**) **12** (1979), no. **4**, 501-661.

[9] S. Mochizuki: A theory of ordinary p-adic curves. *Publ. Res. Inst. Math. Sci.* **32** (1996), no. **6**, 957-1152.

[10] S. Mochizuki: Foundations of p-adic Teichmüller theory. AMS/IP Studies in Advanced Mathematics, **11**. *American Mathematical Society, Providence, RI; International Press, Cambridge, MA,* 1999.

[11] M. Raynaud: Revêtements des courbes en caractéristique $p > 0$ et ordinarité. *Compositio Math.* **123** (2000), no. **1**, 73-88.

(Yuichiro Hoshi) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN

Email address: yuichiro@kurims.kyoto-u.ac.jp