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ABSTRACT. In the present paper, we construct abundant explicit nongeometric pro-p Galois sections of certain punctured projective lines. Moreover, we also obtain an application to the theory of Massey products.

INTRODUCTION

In the present Introduction, let p be a prime number, F a finite extension either of the field $\mathbb{Q}$ of rational numbers or of the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$, and $\overline{F}$ an algebraic closure of F. Write $G_F \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/F)$ for the absolute Galois group [determined by the algebraic closure $\overline{F}$] of F, X_F for the punctured projective line over F obtained by forming the spectrum of the F-algebra

$$F\left[t_X, \frac{1}{t_X}, \frac{1}{t_X - 1}\right]$$

— where t_X is an indeterminate — and J_F for the affine scheme over F obtained by forming the spectrum of the F-algebra

$$F\left[t_{(J,1)}, \frac{1}{t_{(J,1)}}, t_{(J,2)}, \frac{1}{t_{(J,2)}}\right]$$

— where $t_{(J,1)}$, $t_{(J,2)}$ are indeterminates. Thus, we have a closed immersion over F

$$X_F \hookrightarrow J_F$$

determined by "$(t_{(J,1)}, t_{(J,2)}) \mapsto (t_X, 1 - t_X)$". For each $\Box \in \{X, J\}$, write, moreover, $\Box_{\overline{F}} \stackrel{\text{def}}{=} \Box_F \times_F \overline{F}$,

$$\Delta_\Box \ \left(\twoheadleftarrow \pi_1(\Box_{\overline{F}})\right)$$

for the *pro-p geometric étale fundamental group* of $\Box_F$, i.e., the maximal pro-p quotient of $\pi_1(\Box_{\overline{F}})$, and

$$\Pi_\Box \ \left(\twoheadleftarrow \pi_1(\Box_F)\right)$$

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for the *geometrically pro-p étale fundamental group* of $\Box_F$, i.e., the quotient of $\pi_1(\Box_F)$ by the kernel of the natural continuous surjective homomorphism $\pi_1(\Box_{\overline{F}}) \twoheadrightarrow \Delta_\Box$. Thus, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_J & \longrightarrow & \Pi_J & \longrightarrow & G_F & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the left-hand (respectively, middle) vertical arrow is the continuous outer homomorphism (respectively, Δ_J-conjugacy class of continuous homomorphisms over G_F) induced by the above closed immersion $X_F \hookrightarrow J_F$ over F. Now let us recall [cf. [2, Definition 1.1, (i)]] that a continuous section of the right-hand upper (respectively, lower) horizontal arrow of this diagram is called a *pro-p Galois section* of X_F (respectively, J_F). For each $\Box \in \{X, J\}$, write

$$\mathrm{GalSect}(\Box)$$

for the set of $\Delta_\Box$-conjugacy classes of pro-p Galois sections of $\Box_F$. Here, let us observe that one verifies immediately that, for each $\Box \in \{X, J\}$, by the functoriality of "π_1", an F-rational point of $\Box_F$ [i.e., a section of the structure morphism $\Box_F \to \mathrm{Spec}(F)$] gives rise to an element of $\mathrm{GalSect}(\Box)$. In particular, by the above diagram and this observation, we obtain a commutative diagram of sets

$$\begin{array}{ccccc} F \setminus \{0,1\} & \xleftarrow[\sim]{t_X} & X_F(F) & \longrightarrow & \mathrm{GalSect}(X) \\ {\scriptstyle a \mapsto (a, 1-a)} \downarrow & & & & \downarrow \\ F^\times \times F^\times & \xleftarrow[\sim]{(t_{(J,1)}, (t_{(J,2)})} & J_F(F) & \longrightarrow & \mathrm{GalSect}(J). \end{array}$$

One main problem discussed in the present paper is as follows:

> Find an element of $F^\times \times F^\times$, i.e., an F-rational point of J_F [cf. the above diagram], such that the associated Δ_J-conjugacy class of pro-p Galois sections of J_F *lifts*, relative to the right-hand vertical arrow of the above diagram, to a Δ_X-conjugacy class of pro-p Galois sections of X_F.

Write $\Omega_{p\text{-unr}} \subseteq \overline{F}$ for the [unique] maximal Galois extension of $\mathbb{Q}$ in $\overline{F}$ that satisfies the following two conditions:

- The extension $\Omega_{p\text{-unr}}/\mathbb{Q}$ is unramified outside p.
- If $\zeta_p \in \overline{F}$ is a primitive p-th root of unity, then $\zeta_p \in \Omega_{p\text{-unr}}$, and, moreover, the extension $\Omega_{p\text{-unr}}/\mathbb{Q}(\zeta_p)$ is pro-p.

In the remainder of the present Introduction, let

$$x_1, x_2 \in \Omega_{p\text{-unr}} \cap F^\times$$

be two elements of $\Omega_{p\text{-unr}} \cap F^\times$, which thus determine an F-rational point of J_F

$$(x_1, x_2) \in F^\times \times F^\times \xleftarrow[\sim]{(t_{(J,1)}, (t_{(J,2)})} J_F(F).$$

The first main result of the present paper is as follows [cf. Corollary 2.5]:

Theorem A. *Suppose that the following three conditions are satisfied:*

(1) *The prime number* p *is* [*odd and*] ***regular***.
(2) *The field* F *contains a* ***primitive*** p***-th root of unity***.
(3) *Each of* x_1, x_2 *is a* p***-unit***.

Then the following assertions hold:

(i) *There exists an element of* $\mathrm{GalSect}(X)$ *whose image in* $\mathrm{GalSect}(J)$ ***coincides*** *with the element of* $\mathrm{GalSect}(J)$ *determined by the pair* $(x_1, x_2) \in F^\times \times F^\times$.
(ii) *Suppose that there is* ***no*** *pair* (ζ_1, ζ_2) *of roots of unity of* F ***of order prime to*** p *such that* $1 = \zeta_1 x_1 + \zeta_2 x_2$. *Then an element of* $\mathrm{GalSect}(X)$ *as in* (i) *is* ***not contained*** *in the image of the map* $X_F(F) \to \mathrm{GalSect}(X)$.
(iii) *In the situation of* (ii), *suppose, moreover, that* ***neither*** x_1 ***nor*** x_2 *is a root of unity* ***of order prime to*** p. *Then an arbitrary pro-*p *Galois section of* X_F *contained in an element of* $\mathrm{GalSect}(X)$ *as in* (i) *is* ***not geometric*** [*cf.* [2, Definition 1.1, (iii)]], *i.e., the image of an arbitrary pro-*p *Galois section of* X_F *contained in an element of* $\mathrm{GalSect}(X)$ *as in* (i) *is* ***not contained*** *in a decomposition subgroup of* Π_X *associated to an* F*-rational point of the smooth compactification of* X_F.

By means of Theorem A, one may construct "abundant" and "explicit" nongeometric pro-p Galois sections of the punctured projective line X_F [cf., e.g., Remark 2.5.1].

By combining Theorem A with a result by *K. Wickelgren*, we obtain the following application to the theory of *Massey products* [cf., e.g., [6, §2]]. Let $r \geq 2$ be an integer *less than* p. Write $\mu_{p^i}(\overline{F}) \subseteq \overline{F}^\times$ for the continuous G_F-module of p^i-th roots of unity in $\overline{F}$ for each positive integer i,

$$\mathbb{Z}_p(1) \stackrel{\mathrm{def}}{=} \varprojlim_{i \geq 1} \mu_{p^i}(\overline{F})$$

— where the projective limit is taken over the positive integers i — $\mathbb{Z}_p(r)$ for the continuous G_F-module obtained by forming the tensor product over $\mathbb{Z}_p$ of r copies of $\mathbb{Z}_p(1)$, and

$$\kappa_F \colon F^\times \longrightarrow H^1\big(G_F, \mathbb{Z}_p(1)\big)$$

for the natural homomorphism in Kummer theory.

The second main result of the present paper is as follows [cf. Corollary 2.6]:

Theorem B. *Suppose that the three conditions* (1), (2), (3) *in the statement of Theorem A are satisfied. Then, for an arbitrary map* $f \colon \{1, \ldots, r\} \to \{1, 2\}$ *such that* $f^{-1}(\{2\})$ *is of cardinality one, the* r***-th order Massey product*** *in* $H^2(G_F, \mathbb{Z}_p(r))$ [*cf.* [6, Definition 2.3]]

$$\langle \kappa_F(x_{f(1)}), \ldots, \kappa_F(x_{f(r)}) \rangle$$

vanishes.

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1. PRO-p GALOIS SECTIONS OF PUNCTURED PROJECTIVE LINES

In the present §1, we discuss generalities on pro-p Galois sections of certain punctured projective lines. In the present §1, let

- $n \geq 2$ be an integer,
- p a prime number,
- A a *regular integral domain of dimension* ≤ 1 in which p is *invertible* and whose field of fractions is *of characteristic zero*, and
- $a_1, \ldots, a_n$ *distinct* n elements of A such that $a_1 = 0$, $a_2 = 1$, and, moreover, for each i, $j \in \{1, \ldots, n\}$, the difference $a_i - a_j \in A$ is *invertible* in A whenever $i \neq j$.

Write

- K for the field of fractions of A [necessarily *of characteristic zero*],
- X for the affine scheme over A obtained by forming the spectrum of the A-algebra
$$A\left[t_X, \frac{1}{t_X - a_1}, \ldots, \frac{1}{t_X - a_n}\right]$$
— where t_X is an indeterminate —
- J for the affine scheme over A obtained by forming the spectrum of the A-algebra
$$A\left[t_{(J,1)}, \frac{1}{t_{(J,1)}}, \ldots, t_{(J,n)}, \frac{1}{t_{(J,n)}}\right]$$
— where $t_{(J,1)}, \ldots, t_{(J,n)}$ are indeterminates — and
- $\iota_X \colon X \hookrightarrow J$ for the closed immersion over A determined by "$t_{(J,i)} \mapsto a_i - t_X$" for each $i \in \{1, \ldots, n\}$.

Let us fix an algebraic closure $\overline{K}$ of K. Write, moreover, $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group [determined by the algebraic closure $\overline{K}$] of K and, for each $\square \in \{X, J\}$,
$$\square_K \stackrel{\text{def}}{=} \square \times_A K, \qquad \square_{\overline{K}} \stackrel{\text{def}}{=} \square \times_A \overline{K}.$$
Thus, for each $\square \in \{X, J\}$, we have a natural exact sequence of profinite groups
$$1 \longrightarrow \pi_1(\square_{\overline{K}}) \longrightarrow \pi_1(\square_K) \longrightarrow G_K \longrightarrow 1$$
[cf. [7, Exposé IX, Théorème 6.1]].

Definition 1.1. Let $\square$ be either X or J. Then we shall write
$$\Delta_\square \ \big(\twoheadleftarrow \pi_1(\square_{\overline{K}})\big)$$
for the *pro-p geometric étale fundamental group* of $\square_K$, i.e., the maximal pro-p quotient of $\pi_1(\square_{\overline{K}})$, and
$$\Pi_\square \ \big(\twoheadleftarrow \pi_1(\square_K)\big)$$
for the *geometrically pro-p étale fundamental group* of $\square_K$, i.e., the quotient of $\pi_1(\square_K)$ by the kernel of the natural continuous surjective homomorphism $\pi_1(\square_{\overline{K}}) \twoheadrightarrow \Delta_\square$. Thus, the exact sequence preceding the present Definition 1.1 determines an exact sequence of profinite groups
$$1 \longrightarrow \Delta_\square \longrightarrow \Pi_\square \longrightarrow G_K \longrightarrow 1.$$

In particular, we have a continuous outer action of G_K on $\Delta_\square$, which we denote by

$$\rho_\square \colon G_K \longrightarrow \mathrm{Out}(\Delta_\square).$$

Moreover, we shall write

$$\iota_X^\Pi \colon \Pi_X \longrightarrow \Pi_J$$

for the Δ_J-conjugacy class of continuous homomorphisms over G_K induced by the closed immersion ι_X over A and

$$\iota_X^\Delta \colon \Delta_X \longrightarrow \Delta_J$$

for the continuous outer homomorphism induced by the closed immersion ι_X.

Lemma 1.2. *The following assertions hold:*

(i) *The profinite group Δ_X is **center-free**.*

(ii) *The profinite group Δ_J is **isomorphic**, as an abstract profinite group, to the direct product of n copies of $\mathbb{Z}_p$. Moreover, the continuous outer action $\rho_J \colon G_K \to \mathrm{Out}(\Delta_J) = \mathrm{GL}_n(\mathbb{Z}_p)$ coincides with the direct product of n copies of the **p-adic cyclotomic character** of G_K.*

(iii) *Write Δ_X^{ab} for the topological abelianization of the profinite group Δ_X. Then the continuous outer homomorphism $\iota_X^\Delta \colon \Delta_X \to \Delta_J$ determines a continuous **isomorphism** $\Delta_X^{\mathrm{ab}} \xrightarrow{\sim} \Delta_J$.*

Proof. These assertions follow immediately from the well-known structures of the étale fundamental groups of curves and tori over fields of characteristic zero. $\square$

Definition 1.3. Let $\square$ be either X or J. Then we shall write

$$\mathrm{GalSect}(\square)$$

for the set of conjugacy classes of *pro-p Galois sections* of $\square_K$ [cf. [2, Definition 1.1, (i)]], i.e., the set of $\Delta_\square$-conjugacy classes of continuous sections of the continuous surjective homomorphism $\Pi_\square \twoheadrightarrow G_K$,

$$\square(K)$$

for the set of K-rational points of $\square$, and

$$\mathrm{gs}_\square \colon \square(K) \longrightarrow \mathrm{GalSect}(\square)$$

for the map that sends a K-rational point of $\square$ to the associated conjugacy class of pro-p Galois sections of $\square_K$ [cf. [2, Definition 1.1, (ii)]].

Proposition 1.4. *The following assertions hold:*

(i) *Write*

$$(K^\times)^{(p)} \stackrel{\mathrm{def}}{=} \varprojlim_{i \geq 1} \big(K^\times \otimes_{\mathbb{Z}} (\mathbb{Z}/p^i\mathbb{Z})\big)$$

— where the projective limit is taken over the positive integers i. Then there exists a bijective map

$$\kappa_J \colon \mathrm{GalSect}(J) \xrightarrow{\sim} \prod_{i=1}^{n} (K^\times)^{(p)}$$

that satisfies the following condition: The bijective maps

$$t_X : X(K) \xrightarrow{\sim} K \setminus \{a_1, \ldots, a_n\}, \qquad (t_{(J,1)}, \ldots, t_{(J,n)}) : J(K) \xrightarrow{\sim} \prod_{i=1}^{n} K^{\times}$$

determined by the regular functions $t_X \in \Gamma(X, \mathscr{O}_X)$, $t_{(J,1)}, \ldots, t_{(J,n)} \in \Gamma(J, \mathscr{O}_J)$, *respectively, fit into a commutative diagram of sets*

$$\begin{array}{ccccccc}
K \setminus \{a_1, \ldots, a_n\} & \xleftarrow[\sim]{t_X} & X(K) & \xrightarrow{\mathrm{gs}_X} & \mathrm{GalSect}(X) & & \\
\Big\downarrow {\scriptstyle a \mapsto (a_i - a)_{i=1}^n} & & & & \Big\downarrow {\scriptstyle \iota_X^{\Pi}} & & \\
\prod_{i=1}^{n} K^{\times} & \xleftarrow[\sim]{(t_{(J,i)})_{i=1}^n} & J(K) & \xrightarrow{\mathrm{gs}_J} & \mathrm{GalSect}(J) & \xrightarrow[\sim]{\kappa_J} & \prod_{i=1}^{n} (K^{\times})^{(p)}
\end{array}$$

such that the composite of the three lower horizontal arrows ***coincides*** *with the natural homomorphism*

$$\prod_{i=1}^{n} K^{\times} \longrightarrow \prod_{i=1}^{n} (K^{\times})^{(p)}.$$

(ii) *Let* $x_1, \ldots, x_n$ *be* n *elements of* $K^{\times}$. *Write* $\sigma_J \in \mathrm{GalSect}(J)$ *for the conjugacy class of pro-*p *Galois sections of* J_K *that arises from the* K*-rational point of* J_K [*cf.* [2, Definition 1.1, (ii)]] *determined by "*$t_{(J,i)} \mapsto x_i$*" for each* $i \in \{1, \ldots, n\}$. *Consider the following four conditions:*

(1) *There exists a* K*-rational point of* X *whose image in* $\mathrm{GalSect}(J)$*, relative to the diagram of* (i), ***coincides*** *with the conjugacy class* $\sigma_J \in \mathrm{GalSect}(J)$.

(2) *There exists an element* $a \in K \setminus \{a_1, \ldots, a_n\}$ *such that, for each* $i \in \{1, \ldots, n\}$*, the element* $(a_i - a)/x_i \in K^{\times}$ *is* ***p-divisible***.

(3) *There exists a conjugacy class of* ***geometric*** [*cf.* [2, Definition 1.1, (iii)]] *pro-*p *Galois sections of* X_K *whose image in* $\mathrm{GalSect}(J)$*, relative to the diagram of* (i), ***coincides*** *with the conjugacy class* $\sigma_J \in \mathrm{GalSect}(J)$.

(4) *There exists an element* $i_0 \in \{1, \ldots, n\}$ *such that, for each* $i \in \{1, \ldots, n\} \setminus \{i_0\}$*, the element* $(a_i - a_{i_0})/x_i \in K^{\times}$ *is* ***p-divisible***.

Then we have the following implications:

$$(1) \Longleftrightarrow (2) \Longrightarrow (3) \Longleftrightarrow \textit{either } (2) \textit{ or } (4)$$

Proof. Assertion (i) follows immediately from [3, Lemma 5.28, (ii)], together with the various definitions involved. Next, we verify assertion (ii). The implications

$$(1) \Longleftrightarrow (2) \Longrightarrow (3) \Longrightarrow \text{either } (2) \text{ or } (4)$$

follow immediately from assertion (i), together with the [easily verified] fact that the kernel of the natural homomorphism $K^{\times} \to (K^{\times})^{(p)}$ is given by the subgroup of $K^{\times}$ consisting of *p-divisible* elements. Finally, we verify the implication (4) $\Rightarrow$ (3). Suppose that condition (4) is satisfied. Let $D_{a_{i_0}} \subseteq \Pi_X$ be a decomposition subgroup associated to the K-rational point of the smooth compactification of X_K determined by "$t_X \mapsto a_{i_0}$". Write $(J_{i_0})_K$ for the affine scheme over K obtained by forming the spectrum of the K-algebra

$$K\left[t_{(J,i_0)}, \frac{1}{t_{(J,i_0)}}\right]$$

— where $t_{(J,i_0)}$ is an indeterminate — and Q_{i_0} for the *geometrically pro-p étale fundamental group* [cf. Definition 1.1] of $(J_{i_0})_K$. Then one verifies immediately from the well-known structures of the étale fundamental groups of curves and tori over fields of characteristic zero that the composite

$$D_{a_{i_0}} \hookrightarrow \Pi_X \xrightarrow{\iota_X^{\Pi}} \Pi_J \longrightarrow Q_{i_0}$$

— where the third arrow is the $\mathrm{Ker}(Q_{i_0} \twoheadrightarrow G_K)$-conjugacy class of continuous homomorphisms over G_K induced by the natural inclusion of K-algebras

$$K\left[t_{(J,i_0)}, \frac{1}{t_{(J,i_0)}}\right] \hookrightarrow K\left[t_{(J,1)}, \frac{1}{t_{(J,1)}}, \ldots, t_{(J,n)}, \frac{1}{t_{(J,n)}}\right]$$

— is a $\mathrm{Ker}(Q_{i_0} \twoheadrightarrow G_K)$-conjugacy class of continuous *isomorphisms* over G_K. Thus, this composite and a pro-p Galois section of $(J_{i_0})_K$ that arises from the K-rational point of $(J_{i_0})_K$ determined by "$t_{(J,i_0)} \mapsto x_{i_0}$" give rise to a $\mathrm{Ker}(D_{a_{i_0}} \twoheadrightarrow G_K)$-conjugacy class of continuous sections of the continuous surjective homomorphism $D_{a_{i_0}} \twoheadrightarrow G_K$, hence also a conjugacy class of [necessarily *geometric*] pro-p Galois section of X_K. Moreover, one verifies immediately from condition (4) and assertion (i), together with the [easily verified] fact that the kernel of the natural homomorphism $K^\times \to (K^\times)^{(p)}$ is given by the subgroup of $K^\times$ consisting of *p-divisible* elements, that the resulting conjugacy class of pro-p Galois sections of X_K satisfies condition (3). This completes the proof of the implication (4) $\Rightarrow$ (3), hence also of assertion (ii). □

2. Proofs

In the present §2, we prove the main results of the present paper. In the present §2, we maintain the notational conventions introduced at the beginning of the preceding §1.

Definition 2.1. Let $\square$ be either X or J. Then we shall write

$$\Pi_\square^{p\text{-ét}} \ (\twoheadleftarrow \Pi_\square)$$

for the quotient [cf. [7, Exposé V, Proposition 8.2]] of $\Pi_\square$ obtained by forming the maximal pro-p quotient of $\pi_1(\square)$ and

$$G_K^{p\text{-unr}} \ (\twoheadleftarrow G_K)$$

for the quotient of G_K that corresponds to the [unique] maximal Galois extension of K in $\overline{K}$ that is pro-p over K and unramified at each of the discrete valuations of K associated to the nonzero maximal ideals of A, i.e., the quotient of G_K obtained by forming the maximal pro-p quotient of $\pi_1(\mathrm{Spec}(A))$. Moreover, we shall write

$$\iota_X^{\Pi^{p\text{-ét}}} : \Pi_X^{p\text{-ét}} \longrightarrow \Pi_J^{p\text{-ét}}$$

for the $\mathrm{Ker}(\Pi_J^{p\text{-ét}} \twoheadrightarrow G_K^{p\text{-unr}})$-conjugacy class of continuous homomorphisms over $G_K^{p\text{-unr}}$ induced by the closed immersion ι_X over A.

Lemma 2.2. *Suppose that K contains a* ***primitive p-th root of unity****. Write $Z_{\Pi_X}(\Delta_X)$ for the centralizer of Δ_X in Π_X. Then the following assertions hold:*

(i) *The quotient $\Pi_X \twoheadrightarrow \Pi_X/Z_{\Pi_X}(\Delta_X)$ is* ***pro-**p.

(ii) *The quotient $\Pi_X \twoheadrightarrow \Pi_X/Z_{\Pi_X}(\Delta_X)$* ***factors*** *through the quotient $\Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}}$:*

$$\Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}} \twoheadrightarrow \Pi_X/Z_{\Pi_X}(\Delta_X).$$

(iii) *The exact sequence of the third display of Definition 1.1 in the case where $\square = X$ and the natural continuous surjective homomorphisms $\Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}}$ and $G_K \twoheadrightarrow G_K^{p\text{-unr}}$ fit into a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X^{p\text{-ét}} & \longrightarrow & G_K^{p\text{-unr}} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are ***exact****.*

(iv) *The exact sequence of the third display of Definition 1.1 in the case where $\square = J$ and the natural continuous surjective homomorphisms $\Pi_J \twoheadrightarrow \Pi_J^{p\text{-ét}}$ and $G_K \twoheadrightarrow G_K^{p\text{-unr}}$ fit into a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_J & \longrightarrow & \Pi_J & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_J & \longrightarrow & \Pi_J^{p\text{-ét}} & \longrightarrow & G_K^{p\text{-unr}} & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are ***exact****.*

(v) *The two squares of the commutative diagram of profinite groups*

$$\begin{array}{ccccc} \Pi_X & \xrightarrow{\iota_X^{\Pi}} & \Pi_J & \twoheadrightarrow & G_K \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_X^{p\text{-ét}} & \underset{\iota_X^{\Pi^{p\text{-ét}}}}{\twoheadrightarrow} & \Pi_J^{p\text{-ét}} & \twoheadrightarrow & G_K^{p\text{-unr}} \end{array}$$

— where the vertical arrows are the natural continuous surjective homomorphisms — are ***cartesian****.*

Proof. First, we verify assertion (i). Let us first observe that it is immediate that, to verify assertion (i), it suffices to verify that the continuous action of Π_X on Δ_X by conjugation factors through a *pro-p* quotient of Π_X. Write Δ_X^{ab} for the topological abelianization of the profinite group Δ_X. Thus, since the kernel of the natural homomorphism $\mathrm{Aut}(\Delta_X) \to \mathrm{Aut}(\Delta_X^{\mathrm{ab}})$ is *pro-p* [cf. [1, Theorem 6]], to verify assertion (i), it suffices to verify that the continuous action of Π_X on Δ_X^{ab} by conjugation factors through a *pro-p* quotient of Π_X. On the other hand, since [we have assumed that] K contains a *primitive p-th root of unity*, this follows from Lemma 1.2, (ii), (iii). This completes the proof of assertion (i).

Next, we verify assertion (ii). If A is *of dimension zero*, then assertion (ii) is an immediate consequence of assertion (i). In the remainder of the proof of assertion (ii), suppose that A is *of* *dimension one*. Let us first observe that it follows from [7, Exposé V, Proposition 8.2] and [7, Exposé X, Théorème de pureté 3.1] that the kernel of the natural continuous surjective homomorphism $\Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}}$ is *topologically generated* by

- the *kernel* of the natural continuous surjective homomorphism from Π_X onto the maximal pro-p quotient of Π_X and
- the *inertia subgroups* of Π_X associated to the irreducible divisors on X obtained by forming the pull-backs of the closed points of the spectrum of A.

Thus, it follows from assertion (i) that the desired conclusion is *equivalent* to the assertion that each of such *inertia subgroups* of Π_X is *contained* in $Z_{\Pi_X}(\Delta_X)$. In particular, we may assume without loss of generality, by replacing A by the strict henselization of A with respect to a suitable geometric point of the spectrum of A, that A is *strictly henselian*. Then since [we have assumed that] p is *invertible* in A, it follows from the well-known theory of specialization homomorphisms of étale fundamental groups of [7, Exposé XIII] [cf. also our assumption imposed on the a_i's] that the composite $\Delta_X \hookrightarrow \Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}}$ is a continuous *isomorphism*, which thus implies that the natural inclusions from $\mathrm{Ker}(\Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}})$ and Δ_X into Π_X determine a continuous *isomorphism* $\mathrm{Ker}(\Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}}) \times \Delta_X \xrightarrow{\sim} \Pi_X$. In particular, it follows from the *center-free-ness* of Δ_X [cf. Lemma 1.2, (i)] that the centralizer of Δ_X in Π_X coincides with $\mathrm{Ker}(\Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}}) \subseteq \Pi_X$, which thus implies that the quotient $\Pi_X \twoheadrightarrow \Pi_X/Z_{\Pi_X}(\Delta_X)$ coincides with the quotient $\Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}}$, as desired. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Let us first observe that one verifies immediately from [7, Exposé V, Proposition 8.2] and [7, Exposé X, Théorème de pureté 3.1] that, to verify assertion (iii), it suffices to verify that the composite $\Delta_X \hookrightarrow \Pi_X \twoheadrightarrow \Pi_X^{p\text{-ét}}$ is *injective*. On the other hand, the desired injectivity follows from assertion (ii), together with the *center-free-ness* of Δ_X [cf. Lemma 1.2, (i)]. This completes the proof of assertion (iii). Assertion (iv) follows immediately from assertion (iii), together with Lemma 1.2, (iii). Assertion (v) follows from assertions (iii), (iv). This completes the proof of Lemma 2.2. ☐

Definition 2.3. If F is a field of characteristic zero, and $\overline{F}$ is an algebraic closure of F, then we shall write $F_0 \subseteq F$ for the [unique] minimal subfield of F and

$$\Omega_{p\text{-unr}} \subseteq \overline{F}$$

for the [unique] maximal Galois extension of F_0 in $\overline{F}$ that satisfies the following two conditions:

- The extension $\Omega_{p\text{-unr}}/F_0$ is unramified outside p.
- If $\zeta_p \in \overline{F}$ is a primitive p-th root of unity, then $\zeta_p \in \Omega_{p\text{-unr}}$, and, moreover, the extension $\Omega_{p\text{-unr}}/F_0(\zeta_p)$ is pro-p.

Theorem 2.4. *In the notational conventions introduced at the beginning of the preceding* §1, *let*

$$x_1, \ldots, x_n \in \Omega_{p\text{-unr}} \cap K$$

be n elements of $\Omega_{p\text{-unr}} \cap K$ [which thus implies that each of $x_1, \ldots, x_n$ is an algebraic number]. Suppose that the following four conditions are satisfied:

(1) *The prime number p is [odd and]* ***regular****.*
(2) *The field K contains a* ***primitive p-th root of unity****.*
(3) *Each of the algebraic numbers $x_1, \ldots, x_n$ is a p-**unit**.*

(4) *The elements* $a_1,\dots,a_n \in A$ *are* ***contained*** *in* $\Omega_{p\text{-unr}}$ [*which thus implies that each of* $a_1,\dots,a_n$ *is an algebraic number*], *and, moreover, for each* $i,\ j \in \{1,\dots,n\}$, *the algebraic number* $a_i - a_j$ *is a **p-unit** whenever* $i \neq j$.

Then an arbitrary pro-p Galois section of J_K *that arises from the* K*-rational point of* J_K *determined by "*$t_{(J,i)} \mapsto x_i$*" for each* $i \in \{1,\dots,n\}$ [*cf.* (3)] ***lifts****, relative to the* Δ_J*-conjugacy class* $\iota_X^{\Pi} : \Pi_X \to \Pi_J$ *of continuous homomorphisms over* G_K*, to a pro-p Galois section of* X_K.

Proof. Let us first observe that one verifies immediately that, to verify Theorem 2.4, we may assume without loss of generality — by replacing A by the normalization of the subring of K generated by $A \subseteq K$ and $x_1,\dots,x_n \in \Omega_{p\text{-unr}} \cap K$ — that $x_1,\dots,x_n \in A$. Moreover, one also verifies immediately that, to verify Theorem 2.4, we may assume without loss of generality — by replacing A by the normalization of the subring of A generated by $1/p \in A$, a primitive p-th root of unity in A [cf. condition (2)], and $a_1,\dots,a_n,x_1,\dots,x_n \in A$ — that K is a *number field* [i.e., finite over the minimal subfield of K] *contained in* $\Omega_{p\text{-unr}}$, and A is the subring of K generated by the *ring of integers of* K *and* $1/p \in K$ [cf. conditions (3), (4)]. In particular, one verifies easily [cf. also condition (2)] that the two quotients $G_K^{p\text{-unr}}$ and $\mathrm{Gal}(\Omega_{p\text{-unr}}/K)$ of G_K *coincide*.

Let σ_J be a pro-p Galois section of J_K that arises from the K-rational point of J_K determined by "$t_{(J,i)} \mapsto x_i$" for each $i \in \{1,\dots,n\}$ [cf. condition (3)]. Now let us observe that since p is *invertible* in A, it follows from condition (3) that $x_i \in A^\times$ for each $i \in \{1,\dots,n\}$. Thus, one verifies immediately from the various definitions involved that there exists a continuous section $\sigma_J^{p\text{-ét}}$ of the continuous surjective homomorphism $\Pi_J^{p\text{-ét}} \twoheadrightarrow G_K^{p\text{-unr}}$ such that the pull-back [cf. the right-hand *cartesian* square of the diagram of Lemma 2.2, (v)] of $\sigma_J^{p\text{-ét}}$ by the continuous surjective homomorphism $G_K \twoheadrightarrow G_K^{p\text{-unr}}$ *coincides* with σ_J. In particular, since $G_K^{p\text{-unr}} = \mathrm{Gal}(\Omega_{p\text{-unr}}/K)$ is *free pro-p* [cf., e.g., [2, Lemma 3.2], also condition (1)], the continuous section $\sigma_J^{p\text{-ét}}$ *lifts*, relative to the $\mathrm{Ker}(\Pi_J^{p\text{-ét}} \twoheadrightarrow G_K^{p\text{-unr}})$-conjugacy class $\iota_X^{\Pi^{p\text{-ét}}} : \Pi_X^{p\text{-ét}} \twoheadrightarrow \Pi_J^{p\text{-ét}}$ of continuous homomorphisms over $G_K^{p\text{-unr}}$, to a continuous section $\sigma_X^{p\text{-ét}}$ of the continuous surjective homomorphism $\Pi_X^{p\text{-ét}} \twoheadrightarrow G_K^{p\text{-unr}}$. Then it follows from the various definitions involved that the pro-p Galois section of X_K obtained by forming the pull-back [cf. the two *cartesian* squares of the diagram of Lemma 2.2, (v)] of $\sigma_X^{p\text{-ét}}$ by the continuous surjective homomorphism $G_K \twoheadrightarrow G_K^{p\text{-unr}}$ *lifts*, relative to the Δ_J-conjugacy class $\iota_X^{\Pi} : \Pi_X \to \Pi_J$ of continuous homomorphisms over G_K, the pro-p Galois section σ_J of J_K. This completes the proof of Theorem 2.4. $\square$

Corollary 2.5. *Let* p *be a prime number and* F *a finite extension either of the field* $\mathbb{Q}$ *of rational numbers or of the* p*-adic completion* $\mathbb{Q}_p$ *of* $\mathbb{Q}$*. Write* X_F *for the punctured projective line over* F *obtained by forming the spectrum of the* F*-algebra*

$$F\left[t_X, \frac{1}{t_X}, \frac{1}{t_X - 1}\right]$$

— where t_X *is an indeterminate — and* J_F *for the affine scheme over* F *obtained by forming the spectrum of the* F*-algebra*

$$F\left[t_{(J,1)}, \frac{1}{t_{(J,1)}}, t_{(J,2)}, \frac{1}{t_{(J,2)}}\right]$$

— where $t_{(J,1)}$, $t_{(J,2)}$ *are indeterminates. Let* x_1, x_2 *be elements of* $\Omega_{p\text{-unr}} \cap F$*. Suppose that the following three conditions are satisfied:*

(1) *The prime number p is* [*odd and*] ***regular***.
(2) *The field F contains a* ***primitive*** *p*-***th root of unity***.
(3) *Each of x_1, x_2 is a p*-***unit***.

Then the following assertions hold:

(i) *An arbitrary pro-p Galois section of J_F that arises from the F-rational point of J_F determined by "$t_{(J,i)} \mapsto x_i$" for each $i \in \{1,2\}$* [*cf.* (3)] ***lifts***, *relative to an arbitrary continuous homomorphism between the geometrically pro-p étale fundamental groups of X_F, J_F induced by the closed immersion $X_F \hookrightarrow J_F$ over F determined by "$(t_{(J,1)}, t_{(J,2)}) \mapsto (t_X, 1-t_X)$", to a pro-$p$ Galois section of X_F.*

(ii) *Suppose that there is* ***no*** *pair (ζ_1, ζ_2) of roots of unity of F* ***of order prime to*** *p such that $1 = \zeta_1 x_1 + \zeta_2 x_2$. Then the resulting pro-p Galois section of X_F of* (i) *does* ***not arise*** *from any F-rational point of X_F.*

(iii) *In the situation of* (ii), *suppose, moreover, that* ***neither*** *x_1* ***nor*** *x_2 is a root of unity* ***of order prime to*** *p. Then the resulting pro-p Galois section of X_F of* (i) *is* ***not geometric***.

Proof. One verifies immediately that we are in the situation of Theorem 2.4, i.e., by taking the "(n,A,K)" of Theorem 2.4 to be $(2,F,F)$. Thus, assertion (i) follows from Theorem 2.4. Assertions (ii), (iii) follow immediately from Proposition 1.4, (ii), together with the [easily verified] fact that, for each element of $F^\times$, it holds that the element is *p-divisible* if and only if the element is a root of unity *of order prime to p*. This completes the proof of Corollary 2.5. ☐

Remark 2.5.1. Suppose that we are in the situation of Corollary 2.5. Let n_1, n_2 be two *nonzero* integers. Then it is immediate that both p^{n_1} and p^{n_2} are *p-units*. In particular, it follows from Corollary 2.5, (i), that an arbitrary pro-p Galois section of J_F that arises from the F-rational point of J_F determined by "$t_{(J,i)} \mapsto p^{n_i}$" for each $i \in \{1,2\}$ *lifts* to a pro-p Galois section of X_F. Moreover, one verifies easily that the pair (p^{n_1}, p^{n_2}) satisfies conditions imposed on the pair "(x_1, x_2)" in the statements of Corollary 2.5, (ii), (iii). In particular, it follows from Corollary 2.5, (iii), that the resulting pro-p Galois section of X_F is *not geometric*.

Remark 2.5.2. Suppose that we are in the situation of Corollary 2.5. Let d be a positive integer and C a hyperbolic curve over F. Then it follows from [5, Theorem 0.4] [cf. also [4, Theorem C, (i)]] that the kernel of the continuous outer action "$\rho_\square$" defined in Definition 1.1 for the hyperbolic curve C over F *coincides* with the kernel of the continuous outer action "$\rho_\square$" defined in Definition 1.1 for the d-th configuration space of the hyperbolic curve C over F. One verifies immediately from this fact that the construction of *nongeometric* pro-p Galois sections given in Corollary 2.5, (iii) [and, moreover, the proof of the existence of *nongeometric* pro-p Galois sections given in [2, §4]], may also be applied to the case of the *d-th configuration space of suitable hyperbolic curves*. We leave the routine details to the interested reader.

Remark 2.5.3. One open problem concerning *nongeometric* Galois sections of hyperbolic curves is the following: Are there an *irregular* prime number l and a *nongeometric pro-l* Galois section of a hyperbolic curve over a finite extension of the field $\mathbb{Q}$ of rational numbers or of the l-adic completion $\mathbb{Q}_l$ of $\mathbb{Q}$? It is not clear to the authors at the time of writing whether or not this question may be answered in the affirmative.

Corollary 2.6. *Let $r \geq 2$ be an integer, p a prime number* ***greater than*** *r, F a finite extension either of the field $\mathbb{Q}$ of rational numbers or of the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$, and $\overline{F}$ an algebraic closure of F. Write $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$ for the absolute Galois group* [*determined by the algebraic closure*

$\overline{F}$] of F, $\mu_{p^i}(\overline{F}) \subseteq \overline{F}^{\times}$ for the continuous G_F-module of p^i-th roots of unity in $\overline{F}$ for each positive integer i,

$$\mathbb{Z}_p(1) \stackrel{\text{def}}{=} \varprojlim_{i\geq 1} \mu_{p^i}(\overline{F})$$

— where the projective limit is taken over the positive integers i — $\mathbb{Z}_p(r)$ for the continuous G_F-module obtained by forming the tensor product over $\mathbb{Z}_p$ of r copies of $\mathbb{Z}_p(1)$, and

$$\kappa_F \colon F^{\times} \longrightarrow H^1\big(G_F, \mathbb{Z}_p(1)\big)$$

for the natural homomorphism in Kummer theory. Let x_1, x_2 be elements of $\Omega_{p\text{-unr}} \cap F$. Suppose that the following three conditions are satisfied:

(1) *The prime number p is [odd and] **regular**.*
(2) *The field F contains a **primitive p-th root of unity**.*
(3) *Each of x_1, x_2 is a **p-unit**.*

*Then, for an arbitrary map $f\colon \{1,\dots,r\} \to \{1,2\}$ such that $f^{-1}(\{2\})$ is of cardinality one, the **r-th order Massey product** in $H^2(G_F, \mathbb{Z}_p(r))$ [cf. [6, Definition 2.3]]*

$$\langle \kappa_F(x_{f(1)}), \dots, \kappa_F(x_{f(r)}) \rangle$$

*[cf. (3)] **vanishes**.*

Proof. This assertion is a formal consequence of Corollary 2.5, (i), and [6, Corollary 7]. □

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