GALOIS-THEORETIC CHARACTERIZATION OF GEOMETRIC ISOMORPHISM CLASSES OF QUASI-MONODROMICALLY FULL HYPERBOLIC CURVES WITH SMALL NUMERICAL INVARIANTS

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ABSTRACT. Let l be a prime number. In the present paper, we prove that the geometric isomorphism class of a quasi-l-monodromically full hyperbolic curve with small numerical invariants over a sub-l-adic field is completely determined by the commensurability class of the kernel of the associated pro-l outer Galois action.

CONTENTS

INTRODUCTION

In the present paper, we establish a Galois-theoretic characterization of geometric isomorphism classes of *quasi-monodromically full hyperbolic curves* with small numerical invariants.

First, let us review the notion of [*quasi-*]*monodromic fullness* as follows: Let g and r be nonnegative integers such that $2g-2+r>0$, k a field of characteristic zero, $\overline{k}$ an algebraic closure of k, l a prime number, and X a *hyperbolic curve* of type (g,r) over k [cf. Definition 1.3, (iii)]. Write $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by $\overline{k}$ and Δ_X^l for the maximal pro-l quotient of the étale fundamental group $\pi_1(X \times_k \overline{k})$ of $X \times_k \overline{k}$. In particular, we have an exact sequence

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of profinite groups

$$1 \longrightarrow \pi_1(X \times_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1,$$

which thus determines an outer continuous action

$$\rho^l_X : G_k \longrightarrow \mathrm{Out}(\Delta^l_X).$$

Write, moreover, $\mathcal{M}_{g,r}$ for the moduli stack [cf. [4], [14]] of [ordered] r-pointed smooth proper curves of genus g over k [cf. Definition 1.3, (ii)], $\mathcal{M}_{g,[r]}$ ($\leftarrow \mathcal{M}_{g,r}$) for the moduli stack of hyperbolic curves of type (g, r) over k, $\mathcal{C}_{g,[r]} \to \mathcal{M}_{g,[r]}$ for the universal hyperbolic curve of type (g, r) over $\mathcal{M}_{g,[r]}$, $\Delta_{g,r}$ for the kernel of the outer continuous surjective homomorphism $\pi_1(\mathcal{C}_{g,[r]}) \twoheadrightarrow \pi_1(\mathcal{M}_{g,[r]})$ induced by the structure morphism $\mathcal{C}_{g,[r]} \to \mathcal{M}_{g,[r]}$, and $\Delta^l_{g,r}$ for the maximal pro-l quotient of $\Delta_{g,r}$. In particular, we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_{g,r} \longrightarrow \pi_1(\mathcal{C}_{g,[r]}) \longrightarrow \pi_1(\mathcal{M}_{g,[r]}) \longrightarrow 1,$$

which thus determines an outer continuous action

$$\rho^l_{g,r} : \pi_1(\mathcal{M}_{g,[r]}) \longrightarrow \mathrm{Out}(\Delta^l_{g,r}).$$

Since ρ^l_X factors, via the outer continuous homomorphism $G_k \to \pi_1(\mathcal{M}_{g,[r]})$ induced by the k-rational point of $\mathcal{M}_{g,[r]}$ that classifies X/k, through $\rho^l_{g,r}$, we have the following natural inclusions

$$\rho^l_X(G_k) \subseteq \rho^l_{g,r}\big(\pi_1(\mathcal{M}_{g,[r]})\big) \supseteq \rho^l_{g,r}\big(\pi_1(\mathcal{M}_{g,r})\big).$$

We shall say that X is *l-monodromically full* [cf. [6, Definition 2.2, (i)]] (respectively, *quasi-l-monodromically full* [cf. [6, Definition 2.2, (iii)]]) if the closed subgroup $\rho^l_X(G_k)$ of $\rho^l_{g,r}(\pi_1(\mathcal{M}_{g,[r]}))$ contains the [normal open] subgroup $\rho^l_{g,r}(\pi_1(\mathcal{M}_{g,r}))$ of $\rho^l_{g,r}(\pi_1(\mathcal{M}_{g,[r]}))$ (respectively, is open in $\rho^l_{g,r}(\pi_1(\mathcal{M}_{g,[r]}))$). The property of being quasi-monodromically full may be regarded as an analogue for hyperbolic curves of the property of not admitting complex multiplication for elliptic curves [cf., e.g., [6, Introduction], [8, Introduction]]. Moreover, one may prove that, as in the case of elliptic curves that do not admit complex multiplication, for an arbitrary prime number l, "many" hyperbolic curves are in fact *quasi-l-monodromically full* [cf., e.g., Proposition 1.8]. The study of [quasi-]monodromically full hyperbolic curves was initiated by *M. Matsumoto* and *A. Tamagawa* in [15]. Moreover, some results concerning [quasi-]monodromically full hyperbolic curves have been established in [6], [7], [8] by the first author of the present paper.

Next, let us recall that *S. Mochizuki* proved the following theorem [cf. [17, Theorem 1.1]].

Theorem A. *Suppose that k is finite over the minimal subfield of k. For each $i \in \{1, 2\}$, let E_i be an elliptic curve over k such that*

$E_i \times_k \overline{k}$ does not have complex multiplication. Then the following two conditions are equivalent:

- *The elliptic curve E_1 over k is isomorphic to the elliptic curve E_2 over k.*
- *For each positive integer n, if one writes $E_1(\overline{k})[n] \subseteq E_1(\overline{k})$, $E_2(\overline{k})[n] \subseteq E_2(\overline{k})$ for the submodules of $E_1(\overline{k})$, $E_2(\overline{k})$ consisting of n-torsion elements, respectively, then the kernel of the natural continuous action $G_k \to \mathrm{Aut}(E_1(\overline{k})[n])$ coincides with the kernel of the natural continuous action $G_k \to \mathrm{Aut}(E_2(\overline{k})[n])$.*

Moreover, let us also recall that the first author of the present paper proved the following theorem [cf. [6, Theorem A]].

Theorem B. *Suppose that k is finitely generated over the minimal subfield of k. For each $i \in \{1, 2\}$, let $r_i \geq 3$ be an integer and X_i an l-monodromically full hyperbolic curve of type $(0, r_i)$ over k. Suppose that, for each $i \in \{1, 2\}$, every cusp of X_i is rational over k. Then the following two conditions are equivalent:*

- *The hyperbolic curve X_1 over k is isomorphic to the hyperbolic curve X_2 over k.*
- *The equality $\mathrm{Ker}(\rho^l_{X_1}) = \mathrm{Ker}(\rho^l_{X_2})$ holds.*

Let us observe that these two theorems assert that, roughly speaking, for a given object, if the "monodromy action" associated to the object has a "*big image*", then the isomorphism class of the given object is *completely determined* by the *kernel* of the associated "monodromy action". In the present paper, we will give proofs of some assertions in this direction. One main theorem of the present paper is as follows [cf. Corollary 4.4 for more details]. Here, let us recall that we shall say that two closed subgroups H_1, $H_2 \subseteq G$ of a profinite group G are *commensurable* if the intersection $H_1 \cap H_2$ is open both in H_1 and in H_2.

Theorem C. *Suppose that k is sub-l-adic [cf. Definition 1.1, (i)]. Let g and r be nonnegative integers such that*

$$(g, r) \in \{(0, 3), (0, 5), (0, 6), (0, 7), (1, 3), (1, 4), (2, 0), (2, 1)\}.$$

For each $i \in \{1, 2\}$, let X_i be a quasi-l-monodromically full hyperbolic curve of type (g, r) over k. Suppose that $l = 2$ whenever $g \neq 0$. Then the following two conditions are equivalent:

- *The hyperbolic curve $X_1 \times_k \overline{k}$ over $\overline{k}$ is isomorphic to the hyperbolic curve $X_2 \times_k \overline{k}$ over $\overline{k}$.*
- *The closed subgroups $\mathrm{Ker}(\rho^l_{X_1})$, $\mathrm{Ker}(\rho^l_{X_2}) \subseteq G_k$ of G_k are commensurable.*

The present paper is organized as follows. In §1, we recall some basic notational conventions that appear in the present paper. In §2, we prove a technical anabelian lemma [cf. Lemma 2.2], which will be applied in the proof of the main result of the present paper. In §3, we discuss the geometry of certain finite étale coverings of the moduli stacks of hyperbolic curves [cf. Lemma 3.5], which play important roles in the proof of the main result of the present paper. In §4, we give proofs of the main results of the present paper [cf. Theorem 4.2, Corollary 4.4, and Theorem 4.6]. In §5, we give some complements to the main results of the present paper [cf. Remark 5.3].

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1. Preliminaries

In the present §1, we recall some basic notational conventions that appear in the present paper. In the present §1, let g and r be nonnegative integers such that $2g-2+r>0$, k a field of characteristic zero, $\overline{k}$ an algebraic closure of k, and l a prime number; write $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by $\overline{k}$.

Definition 1.1.

(i) We shall say that a field is *sub-l-adic* if the field is isomorphic to a subfield of a field finitely generated over the l-adic completion of the field of rational numbers.

(ii) Let G be a profinite group. Then we shall say that two closed subgroups H_1, $H_2 \subseteq G$ of G are *commensurable* if the intersection $H_1 \cap H_2$ is open both in H_1 and in H_2.

Definition 1.2. Let X be a scheme geometrically connected and of finite type over k.

(i) We shall write Δ^l_X for the maximal pro-l quotient of the étale fundamental group $\pi_1(X \times_k \overline{k})$ of $X \times_k \overline{k}$. Note that one has to fix a basepoint of "$(-)$" to define the étale fundamental group of "$(-)$"; however, since the étale fundamental group is, in an immediate sense, independent, up to inner automorphisms, of the choice of basepoint, we shall omit mention of basepoint throughout the present paper.

(ii) We shall write

$$\rho^l_X \colon G_k \longrightarrow \text{Out}(\Delta^l_X)$$

for the outer continuous action determined by the well-known exact sequence of profinite groups

$$1 \longrightarrow \pi_1(X \times_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1.$$

We shall refer to ρ^l_X as the *pro-l outer Galois action* associated to X/k.

(iii) Let $Y \to X$ be a connected finite étale covering of X. Then we shall say that $Y \to X$ is a *connected finite étale l-special covering* if the following condition is satisfied: If one writes $Z \to X$ for the Galois closure of the finite étale covering $Y \to X$, then the restriction of the base-change $Z \times_k \overline{k} \to X \times_k \overline{k}$ to some [or, alternatively, an arbitrary] connected component of $Z \times_k \overline{k}$ is of degree a power of l.

Definition 1.3. Let S be a scheme and C a scheme over S.

(i) We shall say that C is a *smooth proper curve of genus g* over S if C is smooth and proper over S, and, moreover, every geometric fiber of C over S is a [necessarily smooth and proper] connected curve of genus g.

(ii) Let $s_1, \ldots, s_r \colon S \to C$ be r sections of the structure morphism $C \to S$. Then we shall say that $(C, s_1, \ldots, s_r)$ is an *r-pointed smooth proper curve of genus g* over S if C is a smooth proper curve of genus g over S, and, moreover, the image of s_i does not intersect the image of s_j whenever $i \neq j$.

(iii) We shall say that C is a *hyperbolic curve of type (g, r)* over S if there exist a smooth proper curve C^+ of genus g over S and a [possibly empty] closed subscheme $D \subseteq C^+$ of C^+ such that the composite $D \hookrightarrow C^+ \to S$ is finite, étale, and of degree r, and, moreover, the complement $C^+ \setminus D$ of D in C^+ is isomorphic to C over S.

Definition 1.4.

(i) We shall write $\mathcal{M}_{g,r}$ for the moduli stack [cf. [4], [14]] of r-pointed smooth proper curves of genus g over k [cf. Definition 1.3, (ii)], $(\mathcal{C}^+_{g,r}, s^{\mathcal{M}}_1, \ldots, s^{\mathcal{M}}_r)$ for the universal r-pointed smooth proper curve of genus g over $\mathcal{M}_{g,r}$, and $\mathcal{C}_{g,r} \subseteq \mathcal{C}^+_{g,r}$ for the open substack of $\mathcal{C}^+_{g,r}$ obtained by forming the complement of the images of the r sections $s^{\mathcal{M}}_1, \ldots, s^{\mathcal{M}}_r$.

(ii) We shall write $\mathcal{M}_{g,[r]}$ for the moduli stack of hyperbolic curves of type (g, r) over k [cf. Definition 1.3, (iii)] and $\mathcal{C}_{g,[r]} \to \mathcal{M}_{g,[r]}$ for the universal hyperbolic curve of type (g, r) over $\mathcal{M}_{g,[r]}$.

In particular, it follows immediately from the various definitions involved that we have an isomorphism over $\mathcal{M}_{g,r}$

$$\mathcal{C}_{g,r} \xrightarrow{\sim} \mathcal{M}_{g,r+1}$$

— where we regard $\mathcal{M}_{g,r+1}$ as a stack over $\mathcal{M}_{g,r}$ by considering the morphism obtained by forgetting the $(r+1)$-st splitting — and a cartesian diagram of stacks

$$\begin{array}{ccc} \mathcal{C}_{g,r} & \longrightarrow & \mathcal{M}_{g,r} \\ \downarrow & & \downarrow \\ \mathcal{C}_{g,[r]} & \longrightarrow & \mathcal{M}_{g,[r]} \end{array}$$

— where each of the vertical arrows is the natural finite étale Galois covering whose Galois group is isomorphic to the symmetric group on r letters.

Definition 1.5.

(i) We shall write $\Delta_{g,r}$ for the kernel of the outer continuous surjective homomorphism $\pi_1(\mathcal{C}_{g,[r]}) \twoheadrightarrow \pi_1(\mathcal{M}_{g,[r]})$ induced by the structure morphism $\mathcal{C}_{g,[r]} \to \mathcal{M}_{g,[r]}$ and $\Delta^l_{g,r}$ for the maximal pro-l quotient of $\Delta_{g,r}$. [Here, we define an *outer continuous homomorphism* $G \to H$ between topological groups G and H is defined to be an equivalence class of continuous homomorphisms $G \to H$, where we say that two continuous homomorphisms $f,\ g\colon G \to H$ are equivalent if f coincides with the composite of g with an inner automorphism of H.] Thus, the cartesian diagram of stacks of Definition 1.4 induces a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{g,r} & \longrightarrow & \pi_1(\mathcal{C}_{g,r}) & \longrightarrow & \pi_1(\mathcal{M}_{g,r}) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_{g,r} & \longrightarrow & \pi_1(\mathcal{C}_{g,[r]}) & \longrightarrow & \pi_1(\mathcal{M}_{g,[r]}) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are exact, and the vertical arrows are open injective.

(ii) We shall write

$$\rho^l_{g,r}\colon \pi_1(\mathcal{M}_{g,[r]}) \longrightarrow \mathrm{Out}(\Delta^l_{g,r})$$

for the outer continuous action determined by the lower horizontal exact sequence of the commutative diagram of (i).

(iii) We shall write

$$\Delta^{l,\mathrm{ab},+}_{g,r} \twoheadleftarrow \Delta^l_{g,r}$$

for the quotient of $\Delta^l_{g,r}$ by the normal closed subgroup normally topologically generated by the commutator subgroup of $\Delta^l_{g,r}$ and the inertia subgroups associated to the irreducible components of the complement $\mathcal{C}^+_{g,r} \setminus \mathcal{C}_{g,r}$ [cf. the left-hand vertical equality of the commutative diagram of (i)].

(iv) Let n be a positive integer. Then we shall write

$$\mathcal{M}_{g,r}(l^n) \longrightarrow \mathcal{M}_{g,r}$$

for the finite étale Galois covering of $\mathcal{M}_{g,r}$ that corresponds to the normal open subgroup of $\pi_1(\mathcal{M}_{g,r})$ obtained by forming the kernel of the continuous action of $\pi_1(\mathcal{M}_{g,r})$ on $\Delta_{g,r}^{l,\mathrm{ab},+} \otimes_{\mathbf{Z}_l} (\mathbf{Z}/l^n\mathbf{Z})$ [i.e., determined by $\rho_{g,r}^l$].

One famous result concerning the outer continuous action defined in Definition 1.5, (ii), is the following result, which asserts that *Oda's problem* concerning the universal pro-l outer monodromy representation [cf., e.g., [15, §4.2], [20, §0]] has an *affirmative* answer.

Proposition 1.6. *The image of* $\mathrm{Ker}(\rho_{g,r}^l) \subseteq \pi_1(\mathcal{M}_{g,[r]})$ *by the outer continuous surjective homomorphism* $\pi_1(\mathcal{M}_{g,[r]}) \twoheadrightarrow G_k$ *induced by the structure morphism* $\mathcal{M}_{g,[r]} \to \mathrm{Spec}(k)$ *does not depend on* (g,r)*.*

Proof. This assertion is [20, Theorem 0.5, (2)]. [Note that although [20, Theorem 0.5, (2)] works in a situation where one takes the base field "k" to be the field of rational numbers, since the commutative diagram of Definition 1.5, (i), is compatible, in an immediate sense, with base field extensions, one may conclude immediately the desired assertion from [20, Theorem 0.5, (2)].] ☐

Definition 1.7. Let X be a hyperbolic curve of type (g,r) over k. Thus, the k-rational point of $\mathcal{M}_{g,[r]}$ that classifies X/k induces an outer continuous homomorphism $G_k \to \pi_1(\mathcal{M}_{g,[r]})$ that fits, relative to a suitable continuous isomorphism $\iota\colon \Delta_X^l \xrightarrow{\sim} \Delta_{g,r}^l$, into the following commutative diagram of groups

$$\begin{array}{ccc} G_k & \xrightarrow{\rho_X^l} & \mathrm{Out}(\Delta_X^l) \\ \downarrow & & \wr\downarrow \mathrm{Out}(\iota) \\ \pi_1(\mathcal{M}_{g,[r]}) & \xrightarrow[\rho_{g,r}^l]{} & \mathrm{Out}(\Delta_{g,r}^l). \end{array}$$

(i) We shall say that X is *l-monodromically full* [cf. [6, Definition 2.2, (i)]] if the image of the composite $\mathrm{Out}(\iota) \circ \rho_X^l$ contains $\rho_{g,r}^l(\pi_1(\mathcal{M}_{g,r}))$.

(ii) We shall say that X is *quasi-l-monodromically full* [cf. [6, Definition 2.2, (iii)]] if the image of the composite $\mathrm{Out}(\iota) \circ \rho_X^l$ is open in the image of $\rho_{g,r}^l$.

Note that one verifies easily that the issue of whether or not each of these two conditions is satisfied does not depend on the choice of ι as above.

One fundamental result concerning the notions defined in Definition 1.7 is as follows.

Proposition 1.8. *Suppose that k is finitely generated over the minimal subfield of k. Fix an inclusion $k \hookrightarrow \mathbf{C}$ of fields. Then the subset of $\mathcal{M}_{g,[r]}(\mathbf{C})$ consisting of $\mathbf{C}$-valued points $s \in \mathcal{M}_{g,[r]}(\mathbf{C})$ that satisfy the following condition $(*^{\mathrm{MF}})$ is dense with respect to the complex topology of $\mathcal{M}_{g,[r]}(\mathbf{C})$:*

- $(*^{\mathrm{MF}})$ *There exist a finite extension K of k in $\mathbf{C}$ and a K-valued point of $\mathcal{M}_{g,[r]}$ that maps to $s \in \mathcal{M}_{g,[r]}(\mathbf{C})$ and classifies an l-monodromically full hyperbolic curve of type (g, r) over K.*

Proof. This assertion follows from [6, Theorem 2.3] [whose proof is essentially the same as the proof of [15, Theorem 1.2]]. □

Definition 1.9. Let X_1 and X_2 be hyperbolic curves of type (g, r) over k. Then we shall say that X_1 is *moduli-correspondence-equivalent* to X_2 if there exist two connected finite étale coverings $f_1,\ f_2 \colon \mathcal{N} \to \mathcal{M}_{g,[r]}$ of $\mathcal{M}_{g,[r]}$ and a k-rational point $s \in \mathcal{N}(k)$ of $\mathcal{N}$ such that, for each $i \in \{1, 2\}$, the k-rational point of $\mathcal{M}_{g,[r]}$ obtained by forming the image of $s \in \mathcal{N}(k)$ by f_i classifies X_i/k.

Proposition 1.10. *Suppose that $2g - 2 + r > 2$. Let X_1 and X_2 be hyperbolic curves of type (g, r) over $\overline{k}$. Then X_1 is moduli-correspondence-equivalent to X_2 if and only if X_1 is isomorphic to X_2 over $\overline{k}$.*

Proof. This assertion is a formal consequence of [16, Theorem C], which is a trivial consequence of a famous theorem of Royden. □

Definition 1.11. Let G be a profinite group. Then we shall define

$$\text{sb-rk}(G) \in \mathbf{Z} \cup \{\infty\}$$

to be the maximal integer n — or to be ∞ when such a maximal integer does not exist — such that there exist a prime number p and a continuous injective homomorphism from the direct product of n copies of $\mathbf{Z}_p$ into G.

Definition 1.12. Let d be a positive integer and X a hyperbolic polycurve [cf. [9, Definition 2.1, (ii)]] over k of dimension d. Then we shall say that the hyperbolic polycurve X over k satisfies *condition* $(*)_l$ if the following condition is satisfied [cf. [19, Definition 3.10]]: There exists a sequence of parametrizing morphisms

$$X = X_d \longrightarrow X_{d-1} \longrightarrow \dots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = \mathrm{Spec}(k)$$

[cf. [9, Definition 2.1, (ii)]] such that, for each integers $0 \leq a \leq b \leq d$, the natural continuous outer homomorphism

$$\mathrm{Ker}\big(\pi_1(X_b \times_k \overline{k}) \to \pi_1(X_a \times_k \overline{k})\big) \longrightarrow \mathrm{Ker}\big(\Delta^l_{X_b} \to \Delta^l_{X_a}\big)$$

determines an outer continuous isomorphism of the maximal pro-l quotient of the kernel $\mathrm{Ker}(\pi_1(X_b \times_k \overline{k}) \to \pi_1(X_a \times_k \overline{k}))$ with the kernel $\mathrm{Ker}(\Delta^l_{X_b} \to \Delta^l_{X_a})$.

Lemma 1.13. *The following assertions hold:*

(i) *Let d be a positive integer and X a hyperbolic polycurve over k of dimension d that satisfies condition $(*)_l$ [cf. Definition 1.12]. Then* $\text{sb-rk}(\Delta^l_X) \not\geq d+1$.

(ii) *Suppose that k is algebraically closed. Let G be an open subgroup of the image of $\rho^l_{g,r}$. Then* $\text{sb-rk}(G) \geq 3g-3+r$.

(iii) *Let G be a profinite group and $H \subseteq G$ an open subgroup of G. Then the equality* $\text{sb-rk}(G) = \text{sb-rk}(H)$ *holds.*

Proof. Assertion (i) follows immediately from a similar argument to the argument applied in the proof of [13, Lemma 1.5]. Assertion (ii) follows immediately, by considering a point of $\mathcal{M}_{g,[r]}$ that classifies a "totally degenerate pointed stable curve of type (g,r)" [which has *precisely* $3g-3+r$ *nodes*], from [12, Lemma 5.4, (ii)] and [12, Proposition 5.6, (ii)]. Assertion (iii) is immediate. ☐

2. A Technical Lemma

In the present §2, we prove a technical anabelian lemma [cf. Lemma 2.2 below], which will be applied in the proof of the main result of the present paper. In the present §2, we maintain the notational conventions introduced in the preceding §1.

Lemma 2.1. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Pi & \xrightarrow{\mathrm{pr}_\Pi} & G & \longrightarrow & 1 \\ & & \downarrow & & {\scriptstyle p_\Pi}\downarrow & & \downarrow{\scriptstyle p_G} & & \\ 1 & \longrightarrow & \Delta_\Gamma & \longrightarrow & \Gamma & \xrightarrow[\mathrm{pr}_\Gamma]{} & G_\Gamma & \longrightarrow & 1 \end{array}$$

be a commutative diagram of profinite groups, where the horizontal sequences are exact, and the vertical arrows are surjective. Write $\Pi \twoheadrightarrow \underline{\Pi}$ for the quotient of Π by the normal closed subgroup $\Delta \cap \mathrm{Ker}(p_\Pi)$. Thus, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Pi & \xrightarrow{\mathrm{pr}_\Pi} & G & \longrightarrow & 1 \\ & & \downarrow & & {\scriptstyle \underline{p}_\Pi}\downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_\Gamma & \longrightarrow & \underline{\Pi} & \xrightarrow{\mathrm{pr}_{\underline{\Pi}}} & G & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle p_G} & & \\ 1 & \longrightarrow & \Delta_\Gamma & \longrightarrow & \Gamma & \xrightarrow[\mathrm{pr}_\Gamma]{} & G_\Gamma & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are exact, and the vertical arrows are surjective. For each open subgroup $\underline{\Pi}^\circ$ of $\underline{\Pi}$ and each open subgroup G° of G, write $\underline{\Pi}^\circ|_{G^\circ} \overset{\mathrm{def}}{=} \underline{\Pi}^\circ \cap \mathrm{pr}_{\underline{\Pi}}^{-1}(G^\circ) \subseteq \underline{\Pi}$.

Let s_1 and s_2 be continuous splittings of the continuous surjective homomorphism $\mathrm{pr}_\Pi \colon \Pi \twoheadrightarrow G$. For each $i \in \{1,2\}$, write $s_i^\Gamma \overset{\mathrm{def}}{=} p_\Pi \circ$

$s_i\colon G \to \Gamma$ and $\underline{s}_i \stackrel{\text{def}}{=} p_{\underline{\Pi}} \circ s_i\colon G \to \underline{\Pi}$. Suppose that the following two conditions are satisfied:

(1) *The closed subgroups $\mathrm{Ker}(s_1^{\Gamma})$, $\mathrm{Ker}(s_2^{\Gamma})$ of G are commensurable.*

(2) *For each $i \in \{1, 2\}$, the continuous homomorphism s_i^{Γ} is open.*

Then there exist

- *an open subgroup $\underline{\Pi}^{\circ}$ of $\underline{\Pi}$,*
- *an open subgroup G° of G, and*
- *a continuous open injective homomorphism $\phi\colon \underline{\Pi}^{\circ}|_{G^{\circ}} \hookrightarrow \underline{\Pi}|_{G^{\circ}}$ over G°*

such that the restriction $\underline{s}_1|_{G^{\circ}}\colon G^{\circ} \to \underline{\Pi}|_{G^{\circ}}$ factors through the open embedding $\underline{\Pi}^{\circ}|_{G^{\circ}} \hookrightarrow \underline{\Pi}|_{G^{\circ}}$, and, moreover, the resulting continuous homomorphism $\underline{s}_1|_{G^{\circ}}\colon G^{\circ} \to \underline{\Pi}^{\circ}|_{G^{\circ}}$ fits into the following commutative diagram of profinite groups

$$\begin{array}{ccccc} & & G^{\circ} & & \\ & \swarrow^{\underline{s}_1|_{G^{\circ}}} & & \searrow^{\underline{s}_2|_{G^{\circ}}} & \\ \underline{\Pi}^{\circ}|_{G^{\circ}} & & \xrightarrow[\phi]{} & & \underline{\Pi}|_{G^{\circ}}. \end{array}$$

If, moreover, conditions (1), (2) may be replaced by the following two conditions (1′), (2′), then one may take the above "$(\underline{\Pi}^{\circ}, G^{\circ})$" to be $(\underline{\Pi}, G)$ and the above "ϕ" to be an isomorphism:

(1′) *The equality $\mathrm{Ker}(s_1^{\Gamma}) = \mathrm{Ker}(s_2^{\Gamma})$ holds.*

(2′) *For each $i \in \{1, 2\}$, the continuous homomorphism s_i^{Γ} is surjective.*

Proof. Let us first observe that, to verify Lemma 2.1, it follows from condition (1) that we may assume without loss of generality, after replacing G by a suitable open subgroup of G if necessary [cf. also condition (1′)], that

(a) the *equality* $\mathrm{Ker}(s_1^{\Gamma}) = \mathrm{Ker}(s_2^{\Gamma})$ holds.

Write

- $N \stackrel{\text{def}}{=} \mathrm{Ker}(s_1^{\Gamma}) = \mathrm{Ker}(s_2^{\Gamma})$ [cf. (a)],
- $\Gamma_1 \stackrel{\text{def}}{=} \mathrm{Im}(s_1^{\Gamma}) \cap \mathrm{Im}(s_2^{\Gamma}) \subseteq \Gamma$, and
- $\Gamma_2 \stackrel{\text{def}}{=} \mathrm{Im}(s_2^{\Gamma})$ $(\supseteq \Gamma_1)$.

Moreover, for each $i \in \{1, 2\}$, write

- $G_i \subseteq G$ for the inverse image of $\Gamma_i \subseteq \Gamma$ by $s_i^{\Gamma}\colon G \to \Gamma$,
- $t_i^{\Gamma}\colon G_i \twoheadrightarrow \Gamma_i$ for the [necessarily surjective] continuous homomorphism induced by $s_i^{\Gamma}\colon G \to \Gamma$, and
- $\iota_i\colon G_i/(N \cap G_i) \xrightarrow{\sim} \Gamma_i$ for the continuous isomorphism induced by $t_i^{\Gamma}\colon G_i \twoheadrightarrow \Gamma_i$.

Then it follows from condition (2) that

(b) the subgroups $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma$ of Γ and the subgroups $G_1 \subseteq G_2 \subseteq G$ of G are *open.*

(b′) If, moreover, conditions (1′), (2′) are satisfied, then the *equalities* $\Gamma_1 = \Gamma_2 = \Gamma$, $G_1 = G_2 = G$ hold [cf. condition (2′)].

Moreover, since both s_1 and s_2 are *splittings* of $\mathrm{pr}_\Pi \colon \Pi \twoheadrightarrow G$, one verifies easily that,

(c) for each $i \in \{1, 2\}$, the diagram of profinite groups

$$\begin{array}{ccc} G_i/(N \cap G_i) & \xrightarrow[\sim]{\iota_i} & \Gamma_i \\ & \searrow \quad \swarrow & \\ & G_\Gamma & \end{array}$$

— where the left-hand diagonal arrow is the continuous homomorphism induced by $p_G \colon G \twoheadrightarrow G_\Gamma$, and the right-hand diagonal arrow is the continuous homomorphism induced by $\mathrm{pr}_\Gamma \colon \Gamma \twoheadrightarrow G_\Gamma$ — *commutes.*

Write

$$\iota \colon \Gamma_1 \xrightarrow[\sim]{\iota_1^{-1}} G_1/(N \cap G_1) \hookrightarrow G_2/(N \cap G_2) \xrightarrow[\sim]{\iota_2} \Gamma_2$$

— where the second arrow is the [necessarily injective] continuous homomorphism induced by the natural inclusion $G_1 \hookrightarrow G_2$. Then it is immediate that

(d) the diagram of profinite groups

$$\begin{array}{ccc} G_1 & \hookrightarrow & G_2 \\ {\scriptstyle t_1^\Gamma}\downarrow & & \downarrow{\scriptstyle t_2^\Gamma} \\ \Gamma_1 & \underset{\iota}{\hookrightarrow} & \Gamma_2 \end{array}$$

— where the upper horizontal arrow is the natural inclusion — *commutes.*

Thus, it follows from (c), (d) that

(e) the diagram of profinite groups

$$\begin{array}{ccc} \Gamma_1 & \overset{\iota}{\hookrightarrow} & \Gamma_2 \\ & \searrow \quad \swarrow & \\ & G_\Gamma & \end{array}$$

— where the diagonal arrows are the continuous homomorphisms induced by $\mathrm{pr}_\Gamma \colon \Gamma \twoheadrightarrow G_\Gamma$ — *commutes.*

For each $i \in \{1, 2\}$, write $\underline{\Pi}_i \overset{\mathrm{def}}{=} \Gamma_i \times_{G_\Gamma} G_i \subseteq \underline{\Pi}$ $(= \Gamma \times_{G_\Gamma} G)$ for the fiber product of the composite $\Gamma_i \hookrightarrow \Gamma \overset{\mathrm{pr}_\Gamma}{\twoheadrightarrow} G_\Gamma$ and the composite $G_i \hookrightarrow G \overset{p_G}{\twoheadrightarrow} G_\Gamma$. Then it follows from (b) that

(f) the subgroups $\underline{\Pi}_1 \subseteq \underline{\Pi}_2 \subseteq \underline{\Pi}$ of $\underline{\Pi}$ are *open*.
(f′) If, moreover, conditions (1′), (2′) are satisfied, then the *equalities* $\underline{\Pi}_1 = \underline{\Pi}_2 = \underline{\Pi}$ hold [cf. (b′)].

Moreover, it follows from (e) that the continuous injective homomorphism $\iota\colon \Gamma_1 \hookrightarrow \Gamma_2$ and the natural inclusion $G_1 \hookrightarrow G_2$ determine a [necessarily injective] continuous homomorphism

$$\psi\colon \underline{\Pi}_1 \ (= \Gamma_1 \times_{G_\Gamma} G_1) \hookrightarrow \underline{\Pi}_2 \ (= \Gamma_2 \times_{G_\Gamma} G_2)$$

over G, i.e., relative to the second projections $\underline{\Pi}_1 \twoheadrightarrow G_1$ $(\subseteq G)$, $\underline{\Pi}_2 \twoheadrightarrow G_2$ $(\subseteq G)$. [Note that the surjectivity of these second projections $\underline{\Pi}_1 \twoheadrightarrow G_1$, $\underline{\Pi}_2 \twoheadrightarrow G_2$ may be verified easily from the definitions of G_i, Γ_i.] Now observe that it follows from (d), together with the various definitions involved, that,

(g) for each $i \in \{1, 2\}$, the restriction $\underline{s}_i|_{G_i}\colon G_i \to \underline{\Pi}$ *factors* through the subgroup $\underline{\Pi}_i \subseteq \underline{\Pi}$ of $\underline{\Pi}$, and, moreover, the resulting continuous homomorphism $\underline{s}_i|_{G_i}\colon G_i \to \underline{\Pi}_i$ fits into a *commutative* diagram of profinite groups

$$\begin{array}{ccc} G_1 & \hookrightarrow & G_2 \\ {\scriptstyle \underline{s}_1|_{G_1}} \downarrow & & \downarrow {\scriptstyle \underline{s}_2|_{G_2}} \\ \underline{\Pi}_1 & \underset{\psi}{\hookrightarrow} & \underline{\Pi}_2 \end{array}$$

— where the upper horizontal arrow is the natural inclusion.

Moreover, it follows from (b) that

(h) the continuous homomorphism ψ is *open*.
(h′) If, moreover, conditions (1′), (2′) are satisfied, then the continuous homomorphism ψ is an *isomorphism* [cf. condition (b′)].

Thus, if one writes

- $\underline{\Pi}^\circ \stackrel{\text{def}}{=} \underline{\Pi}_1$,
- $G^\circ \stackrel{\text{def}}{=} G_1$, and
- $\phi\colon \underline{\Pi}^\circ|_{G^\circ} \to \underline{\Pi}|_{G^\circ}$ for the composite of the continuous homomorphism $\underline{\Pi}^\circ|_{G^\circ} = \underline{\Pi}_1 \to \underline{\Pi}_2|_{G^\circ}$ induced by $\psi\colon \underline{\Pi}^\circ|_{G^\circ} = \underline{\Pi}_1 \to \underline{\Pi}_2$ and the natural inclusion $\underline{\Pi}_2|_{G^\circ} \hookrightarrow \underline{\Pi}|_{G^\circ}$,

then it follows from (b), (b′), (f), (f′), (g), (h), (h′) that this collection $(\underline{\Pi}^\circ, G^\circ, \phi)$ of data satisfies the desired condition. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Suppose that k is sub-l-adic. Let X be a hyperbolic polycurve over k and $\pi_1(X) \twoheadrightarrow \Gamma$ a topological quotient of $\pi_1(X)$. For each $i \in \{1, 2\}$, let $x_i \in X(k)$ be a k-rational point of X and $s_i\colon G_k \to \pi_1(X)$ a continuous splitting that arises from $x_i \in X(k)$ of the natural continuous surjective homomorphism $\pi_1(X) \twoheadrightarrow G_k$. For each $i \in \{1, 2\}$,* *write* $s_i^\Gamma \colon G_k \to \Gamma$ *for the composite of* $s_i \colon G_k \to \pi_1(X)$ *and the natural continuous surjective homomorphism* $\pi_1(X) \twoheadrightarrow \Gamma$*. Suppose that the following four conditions are satisfied:*

(1) *The closed subgroups* $\mathrm{Ker}(s_1^\Gamma)$, $\mathrm{Ker}(s_2^\Gamma) \subseteq G_k$ *of* G_k *are commensurable.*
(2) *For each* $i \in \{1, 2\}$*, the continuous homomorphism* s_i^Γ *is open.*
(3) *The quotient of* $\pi_1(X \times_k \overline{k})$ *determined by the quotient* $\pi_1(X) \twoheadrightarrow \Gamma$ *coincides with the maximal pro-*l *quotient* Δ_X^l *of* $\pi_1(X \times_k \overline{k})$ [*cf. Definition* 1.2, (i)].
(4) *The hyperbolic polycurve* X *over* k *is of dimension* ≤ 4 *and satisfies condition* $(*)_l$ [*cf. Definition* 1.12].

Then there exist

- *a finite extension* K *of* k *in* $\overline{k}$*,*
- *a hyperbolic polycurve* Y *over* K*,*
- *two connected finite étale* l*-special coverings* [*cf. Definition* 1.2, (iii)] $f_1, f_2 \colon Y \to X_K \stackrel{\mathrm{def}}{=} X \times_k K$ *over* K*, and*
- *a* K*-rational point* $y \in Y(K)$ *of* Y

such that, for each $i \in \{1, 2\}$*, the image of* $y \in Y(K)$ *by* $f_i \colon Y \to X_K$ *coincides with the* K*-rational point of* X_K *determined by* $x_i \in X(k)$*.*

Proof. If F is a finite extension of k in $\overline{k}$, and V is a scheme geometrically connected and of finite type over F, then write $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/F)$ for the absolute Galois group of F determined by $\overline{k}$ and Π_V^l for the quotient of the étale fundamental group $\pi_1(V)$ of V by the kernel of the natural continuous surjective homomorphism $\pi_1(V \times_F \overline{k}) \twoheadrightarrow \Delta_V^l$. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_V^l \longrightarrow \Pi_V^l \longrightarrow G_F \longrightarrow 1.$$

Moreover, for each $i \in \{1, 2\}$, write $\underline{s}_i \colon G_k \to \Pi_X^l$ for the composite of $s_i \colon G_k \to \pi_1(X)$ and the natural continuous surjective homomorphism $\pi_1(X) \twoheadrightarrow \Pi_X^l$. Then it follows from conditions (1), (2), (3) that, by applying Lemma 2.1 [i.e., in the case where we take the "$(1 \to \Delta \to \Pi \to G \to 1, \Pi \twoheadrightarrow \Gamma, s_1, s_2)$" of Lemma 2.1 to be $(1 \to \pi_1(X \times_k \overline{k}) \to \pi_1(X) \to G_k \to 1, \pi_1(X) \twoheadrightarrow \Gamma, s_1, s_2)$], we obtain

- a finite extension K of k in $\overline{k}$,
- a hyperbolic polycurve Y over K [cf. also [9, Proposition 2.3]],
- a connected finite étale l-special covering $f_1 \colon Y \to X_K \stackrel{\mathrm{def}}{=} X \times_k K$ over K [which thus implies that Π_Y^l may be regarded as an open subgroup of $\Pi_{X_K}^l$], and
- a continuous open injective homomorphism $\phi \colon \Pi_Y^l \hookrightarrow \Pi_{X_K}^l$ over G_K

such that the restriction $\underline{s}_1|_{G_K} \colon G_K \to \Pi_{X_K}^l = \Pi_X^l \times_{G_k} G_K$ factors through the open subgroup $\Pi_Y^l \subseteq \Pi_{X_K}^l$ [i.e., determined by f_1], and, moreover, the resulting continuous homomorphism $\underline{s}_1|_{G_K} : G_K \to \Pi^l_Y$ fits into the following *commutative* diagram of profinite groups

$$\begin{array}{ccc} & G_K & \\ {}^{\underline{s}_1|_{G_K}}\swarrow & & \searrow^{\underline{s}_2|_{G_K}} \\ \Pi^l_Y & \underset{\phi}{\hookrightarrow} & \Pi^l_{X_K}. \end{array}$$

Now let us observe that since the connected finite étale covering $f_1 : Y \to X$ arises from an open subgroup of $\Pi^l_{X_K}$, it follows from [19, Lemma 3.22, (i)], together with condition (4), that the hyperbolic polycurve Y over K is *of dimension* ≤ 4 and satisfies *condition* $(*)_l$ [cf. Definition 1.12]. Thus, it follows from [19, Theorem 1.1], together with condition (4), that the continuous open injective homomorphism $\phi \colon \Pi^l_Y \hookrightarrow \Pi^l_{X_K}$ over G_K *arises* from a connected finite étale l-special covering $f_2 \colon Y \to X_K$ over K. Then one verifies immediately from [19, Proposition 4.2, (ii)], together with the various definitions involved, that the conclusion of Lemma 2.2 holds, as desired. This completes the proof of Lemma 2.2. □

Lemma 2.3. *In the situation of Lemma* 2.2*, suppose, moreover, that the following two conditions are satisfied:*

(1′) *The equality* $\mathrm{Ker}(s^\Gamma_1) = \mathrm{Ker}(s^\Gamma_2)$ *holds.*
(2′) *For each* $i \in \{1, 2\}$*, the continuous homomorphism* s^Γ_i *is surjective.*

Then one may take the "(K, Y)*" of Lemma* 2.2 *to be* (k, X) *and, moreover, the two connected finite étale* l*-special coverings "*f_1*" and "*f_2*" of Lemma* 2.2 *to be isomorphisms.*

Proof. This assertion follows immediately from a similar argument to the argument applied in the proof of Lemma 2.2, together with the final portion of Lemma 2.1. □

3. Certain Finite Étale Coverings of Moduli Spaces of Curves

In the present §3, we discuss the geometry of certain finite étale coverings of the moduli stacks of hyperbolic curves [cf. Lemma 3.5 below], which play important roles in the proof of the main result of the present paper. In the present §3, we maintain the notational conventions introduced in §1.

Definition 3.1. We shall write $r_0(g) \overset{\text{def}}{=} 3$ (respectively, 1; 0) if $g = 0$ (respectively, $= 1$; ≥ 2). Moreover, we shall write $r_1(g) \overset{\text{def}}{=} 3g + r_0(g)$.

Lemma 3.2. *Let $r \geq 3$ be an integer. Write*

$$A_{0,r} \stackrel{\text{def}}{=} k\Big[x_1, \dots, x_{r-1}, \prod_{1 \leq i < j \leq r-1} \frac{1}{x_i - x_j}\Big]/(x_{r-2}, x_{r-1} - 1)$$

— where $x_1, \dots, x_{r-1}$ are indeterminates. Then there exists an isomorphism over k

$$\mathcal{M}_{0,r} \xrightarrow{\sim} \mathrm{Spec}(A_{0,r}).$$

In particular, the stack $\mathcal{M}_{0,3}$ is isomorphic to $\mathrm{Spec}(k)$*.*

Proof. This assertion follows from the well-known structure of the stack $\mathcal{M}_{0,r}$ [cf., e.g., [6, Lemma 4.1, (i)]]. □

In the remainder of the present paper, let us fix an isomorphism as in Lemma 3.2.

Definition 3.3.

(i) We shall write

$$\mathcal{L}_{0,3} \stackrel{\text{def}}{=} \mathcal{C}_{0,3}.$$

Thus, it is obvious that the natural morphism

$$\mathcal{L}_{0,3} \longrightarrow \mathcal{M}_{0,3}$$

determines a structure on $\mathcal{L}_{0,3}$ of hyperbolic curve of type $(0, 3)$ over $\mathcal{M}_{0,3}$.

(ii) We shall write

$$\mathcal{L}_{1,1} \stackrel{\text{def}}{=} \mathrm{Spec}\Big(A_{0,4}[s, t]/\big(s^2 - t(t-1)(t - x_1)\big)\Big)$$

— where s and t are indeterminates. Thus, one verifies easily that the natural morphism

$$\mathcal{L}_{1,1} \longrightarrow \mathcal{M}_{0,4}$$

determines a structure on $\mathcal{L}_{1,1}$ of hyperbolic curve of type $(1, 1)$ over $\mathcal{M}_{0,4}$.

(iii) We shall write

$$\mathcal{L}_{2,6} \stackrel{\text{def}}{=} \mathrm{Spec}\Big(A_{0,6}\Big[s, t, \frac{1}{s}\Big]/\big(s^2 - t(t-1)(t-x_1)(t-x_2)(t-x_3)\big)\Big)$$

— where s and t are indeterminates. Thus, one verifies easily that the natural morphism

$$\mathcal{L}_{2,6} \longrightarrow \mathcal{M}_{0,6}$$

determines a structure on $\mathcal{L}_{2,6}$ of hyperbolic curve of type $(2, 6)$ over $\mathcal{M}_{0,6}$. Moreover, we shall write

$$\mathcal{L}_{2,0} \longrightarrow \mathcal{M}_{0,6}$$

for the hyperbolic curve of type $(2, 0)$ over $\mathcal{M}_{0,6}$ obtained by forming the smooth compactification of the hyperbolic curve $\mathcal{L}_{2,6}$ of type (2,6) over $\mathcal{M}_{0,6}$.

In particular, for each $g \in \{0, 1, 2\}$, we have a hyperbolic curve of type $(g, r_0(g))$ over $\mathcal{M}_{0,r_1(g)}$

$$\mathcal{L}_{g,r_0(g)} \longrightarrow \mathcal{M}_{0,r_1(g)}.$$

Let us observe that it is obvious that the morphism $\mathcal{M}_{0,r_1(g)} \to \mathcal{M}_{g,[r_0(g)]}$ that classifies this hyperbolic curve of type $(g, r_0(g))$ over $\mathcal{M}_{0,r_1(g)}$ uniquely lifts to a morphism over k

$$\mathcal{M}_{0,r_1(g)} \longrightarrow \mathcal{M}_{g,r_0(g)},$$

i.e., relative to the natural finite étale Galois covering $\mathcal{M}_{g,r_0(g)} \to \mathcal{M}_{g,[r_0(g)]}$.

Definition 3.4. Suppose that $g \in \{0, 1, 2\}$. Let us recall that $r_0(g) \stackrel{\text{def}}{=} 3$ (respectively, 1; 0) if $g = 0$ (respectively, $= 1$; ≥ 2); $r_1(g) \stackrel{\text{def}}{=} 3g+r_0(g)$ [cf. Definition 3.1].

(i) We shall write $\underline{\mathcal{M}}_{g,r_0(g)} \stackrel{\text{def}}{=} \mathcal{M}_{0,r_1(g)}$.

(ii) We shall write

$$\Phi_{g,r_0(g)} \colon \underline{\mathcal{M}}_{g,r_0(g)} \longrightarrow \mathcal{M}_{g,r_0(g)}$$

for the unique lifting [cf. Definition 3.3] of the morphism $\underline{\mathcal{M}}_{g,r_0(g)} = \mathcal{M}_{0,r_1(g)} \to \mathcal{M}_{g,[r_0(g)]}$ that classifies the hyperbolic curve $\mathcal{L}_{g,r_0(g)} \to \mathcal{M}_{0,r_1(g)}$ of type $(g, r_0(g))$ over $\mathcal{M}_{0,r_1(g)}$.

(iii) We shall write

$$\Phi_{g,r} \colon \underline{\mathcal{M}}_{g,r} \longrightarrow \mathcal{M}_{g,r}$$

for the base-change of the morphism $\Phi_{g,r_0(g)} \colon \underline{\mathcal{M}}_{g,r_0(g)} \to \mathcal{M}_{g,r_0(g)}$ by the morphism $\mathcal{M}_{g,r} \to \mathcal{M}_{g,r_0(g)}$ obtained by forgetting the last $r - r_0(g)$ splitting(s):

$$\begin{array}{ccc} \underline{\mathcal{M}}_{g,r} & \longrightarrow & \underline{\mathcal{M}}_{g,r_0(g)} \\ {\scriptstyle \Phi_{g,r}}\downarrow & & \downarrow{\scriptstyle \Phi_{g,r_0(g)}} \\ \mathcal{M}_{g,r} & \longrightarrow & \mathcal{M}_{g,r_0(g)}. \end{array}$$

[Note that since [we have assumed that] the inequality $2g - 2 + r > 0$ holds, the integer $r - r_0(g)$ is always nonnegative.]

Lemma 3.5. *Suppose that $g \in \{0, 1, 2\}$. Then the following assertions hold:*

(i) *The morphism* $\Phi_{g,r}\colon \underline{\mathcal{M}}_{g,r} \to \mathcal{M}_{g,r}$ *and the finite étale coverings* $\mathcal{M}_{g,r}(4) \to \mathcal{M}_{g,r}(2) \to \mathcal{M}_{g,r}$ [*cf. Definition* 1.5, (iv)] *fit into a sequence of finite étale coverings*

$$\mathcal{M}_{g,r}(4) \longrightarrow \underline{\mathcal{M}}_{g,r} \longrightarrow \mathcal{M}_{g,r}(2) \longrightarrow \mathcal{M}_{g,r}.$$

(ii) *The stack* $\underline{\mathcal{M}}_{g,r_0(g)}$ *is isomorphic to the* $(r_1(g)-3)$*-rd configuration space* [*cf., e.g.,* [18, Definition 2.1, (i)]] *of the hyperbolic curve* $\mathcal{C}_{0,3}$ *of type* $(0,3)$ *over* $\mathcal{M}_{0,3} = \mathrm{Spec}(k)$ [*cf. Lemma* 3.2].

(iii) *Write*

$$\Phi^*_{g,r}\mathcal{C}_{g,r} \longrightarrow \underline{\mathcal{M}}_{g,r}$$

for the base-change of $\mathcal{C}_{g,r} \to \mathcal{M}_{g,r}$ *by* $\Phi_{g,r}\colon \underline{\mathcal{M}}_{g,r} \to \mathcal{M}_{g,r}$. *Then there exists an isomorphism over* $\underline{\mathcal{M}}_{g,r}$

$$\Phi^*_{g,r}\mathcal{C}_{g,r} \xrightarrow{\sim} \underline{\mathcal{M}}_{g,r+1}.$$

In particular, the natural morphism

$$\underline{\mathcal{M}}_{g,r+1} \longrightarrow \underline{\mathcal{M}}_{g,r}$$

determines a structure on $\underline{\mathcal{M}}_{g,r+1}$ *of hyperbolic curve of type* (g,r) *over* $\underline{\mathcal{M}}_{g,r}$.

(iv) *The stack* $\underline{\mathcal{M}}_{g,r}$ *is isomorphic to the* $(r-r_0(g))$*-th configuration space of the hyperbolic curve* $\underline{\mathcal{M}}_{g,r_0(g)+1}$ *of type* $(g,r_0(g))$ *over* $\underline{\mathcal{M}}_{g,r_0(g)}$ [*cf.* (iii)].

(v) *The sequence of schemes* [*cf.* (iv), *Lemma* 3.2]

$$\underline{\mathcal{M}}_{g,r} \longrightarrow \underline{\mathcal{M}}_{g,r-1} \longrightarrow \dots \longrightarrow \underline{\mathcal{M}}_{g,r_0(g)}$$

$$= \mathcal{M}_{0,r_1(g)} \longrightarrow \mathcal{M}_{0,r_1(g)-1} \longrightarrow \dots \longrightarrow \mathcal{M}_{0,3} = \mathrm{Spec}(k)$$

determines a structure on $\underline{\mathcal{M}}_{g,r}$ *of hyperbolic polycurve of dimension* $3g-3+r$ *over* k.

Proof. First, we verify assertion (i). If $g = 0$, then assertion (i) is immediate. Suppose that $g \neq 0$. Next, let us observe that it follows immediately from the various definitions involved that, to verify assertion (i), it suffices to verify assertion (i) in the case where $r = r_0(g)$. On the other hand, assertion (i) in the case where $r = r_0(g)$ follows immediately from [2, Proposition 3.3] and [2, Remark 3.4], together with the explicit construction in Definition 3.3 of the hyperbolic curve $\mathcal{L}_{g,r_0(g)} \to \mathcal{M}_{0,r_1(g)}$ of type $(g,r_0(g))$ over $\mathcal{M}_{0,r_1(g)}$ [cf. also the discussion following [2, Remark 3.4]]. This completes the proof of assertion (i).

Assertion (ii) is well-known [cf., e.g., [6, Lemma 4.1, (i)]]. Assertions (iii), (iv) follow immediately from the definition of the stack $\underline{\mathcal{M}}_{g,r}$. Assertion (v) follows from Lemma 3.2 and assertion (iii). This completes the proof of Lemma 3.5. □

Proposition 3.6. *Let X_1 and X_2 be hyperbolic curves of type* $(2, 0)$ *over* $\overline{k}$*. Then X_1 is moduli-correspondence-equivalent to X_2 if and only if X_1 is isomorphic to X_2 over* $\overline{k}$*.*

Proof. The sufficiency is immediate. We verify the necessity. Suppose that X_1 is *moduli-correspondence-equivalent* to X_2. For each $i \in \{1, 2\}$, write P_i for the scheme-theoretic quotient of X_i by the hyperelliptic involution of X_i and $D_i \subseteq P_i$ for the branch locus of the resulting finite flat covering $X_i \to P_i$ [necessarily of degree two].

Next, let us observe that it follows immediately from [16, Theorem C], together with Lemma 3.5, (i), that there exists a finite étale covering $\mathcal{M}_{2,0} \times_k \overline{k} \to \mathcal{M}_{0,[6]} \times_k \overline{k}$ over $\overline{k}$ that fits into a *commutative* diagram of stacks over $\overline{k}$

$$\begin{array}{ccc} \underline{\mathcal{M}}_{2,0} \times_k \overline{k} & = & \mathcal{M}_{0,6} \times_k \overline{k} \\ {\scriptstyle \Phi_{2,0}} \downarrow & & \downarrow \\ \mathcal{M}_{2,0} \times_k \overline{k} & \longrightarrow & \mathcal{M}_{0,[6]} \times_k \overline{k} \end{array}$$

— where the right-hand vertical arrow is the natural finite étale Galois covering. In particular, again by [16, Theorem C], it follows immediately from the definition of the terminology "moduli-correspondence-equivalent" and the definition of $\Phi_{2,0}$ that there exists an isomorphism $P_1 \xrightarrow{\sim} P_2$ over $\overline{k}$ that maps D_1 to D_2. Thus, one may conclude that X_1 is *isomorphic* to X_2 over $\overline{k}$, as desired. This completes the proof of the necessity, hence also of Proposition 3.6. ☐

4. Galois-theoretic Characterization

In the present §4, we establish a Galois-theoretic characterization of geometric isomorphism classes of quasi-monodromically full hyperbolic curves with small numerical invariants [cf. Corollary 4.4 below]. In the present §4, we maintain the notational conventions introduced in §1.

Lemma 4.1. *Suppose that the following two conditions are satisfied:*

- *The inclusion* $g \in \{0, 1, 2\}$ *holds.*
- *The equality* $l = 2$ *holds whenever* $g \neq 0$*.*

Then the following assertions hold:

(i) *Suppose that k is algebraically closed. Then the quotient of $\pi_1(\underline{\mathcal{M}}_{g,r})$ determined by the composite*

$$\pi_1(\underline{\mathcal{M}}_{g,r}) \xhookrightarrow{\pi_1(\Phi_{g,r})} \pi_1(\mathcal{M}_{g,r}) \hookrightarrow \pi_1(\mathcal{M}_{g,[r]}) \xrightarrow{\rho^l_{g,r}} \mathrm{Out}(\Delta^l_{g,r})$$

coincides with the maximal pro-l quotient of $\pi_1(\underline{\mathcal{M}}_{g,r})$.

(ii) *The hyperbolic polycurve $\underline{\mathcal{M}}_{g,r}$ over k* [*cf. Lemma* 3.5*,* (v)] *satisfies condition* $(*)_l$ [*cf. Definition* 1.12].

(iii) *Suppose that k is algebraically closed. Let G be an open subgroup of the image of $\rho^l_{g,r}$. Then* $\text{sb-rk}(G) = 3g - 3 + r$ [*cf. Definition* 1.11].

Proof. First, we verify assertion (i). Assertion (i) in the case where $g = 0$ follows from Remark at the end of [1, §1]. Next, let us consider assertion (i) in the case where $g \in \{1, 2\}$ [which implies that $l = 2$]. Then since [one verifies easily that] the kernel of the natural surjective homomorphism $\mathrm{GL}_{2g}(\mathbf{Z}/4\mathbf{Z}) \twoheadrightarrow \mathrm{GL}_{2g}(\mathbf{Z}/2\mathbf{Z})$ is of order a *power of* 2, it follows immediately from Lemma 3.5, (i), that, to verify assertion (i) in the case where $g \in \{1, 2\}$, it suffices to verify that the quotient of $\pi_1(\mathcal{M}_{g,r}(2))$ determined by the composite

$$\pi_1\big(\mathcal{M}_{g,r}(2)\big) \hookrightarrow \pi_1(\mathcal{M}_{g,r}) \hookrightarrow \pi_1(\mathcal{M}_{g,[r]}) \xrightarrow{\rho^2_{g,r}} \mathrm{Out}(\Delta^2_{g,r})$$

coincides with the *maximal pro-*2 *quotient* of $\pi_1(\mathcal{M}_{g,r}(2))$. On the other hand, if $g = 1$ (respectively, 2), then this assertion follows from [10, Theorem A, (i)] (respectively, [3, Theorem 1.4, (ii)]). This completes the proof of assertion (i) in the case where $g \in \{1, 2\}$, hence also of assertion (i).

Next, we verify assertion (ii). Let us first observe that, to verify assertion (ii), we may assume without loss of generality, by replacing k by $\overline{k}$, that k is *algebraically closed*. Write

$$\pi_1(\underline{\mathcal{M}}_{g,r}; \underline{\mathcal{M}}_{g,r_0(g)}) \subseteq \pi_1(\underline{\mathcal{M}}_{g,r})$$

for the kernel of the outer continuous surjective homomorphism $\pi_1(\underline{\mathcal{M}}_{g,r}) \twoheadrightarrow \pi_1(\underline{\mathcal{M}}_{g,r_0(g)})$ induced by the natural morphism $\underline{\mathcal{M}}_{g,r} \to \underline{\mathcal{M}}_{g,r_0(g)}$ [i.e., determined by the morphism $\mathcal{M}_{g,r} \to \mathcal{M}_{g,r_0(g)}$ obtained by forgetting the last $r - r_0(g)$ splitting(s)],

$$Q_r \twoheadleftarrow \pi_1(\underline{\mathcal{M}}_{g,r}; \underline{\mathcal{M}}_{g,r_0(g)})$$

for the maximal pro-l quotient of $\pi_1(\underline{\mathcal{M}}_{g,r}; \underline{\mathcal{M}}_{g,r_0(g)})$, and

$$\underline{Q}_r \twoheadleftarrow \pi_1(\underline{\mathcal{M}}_{g,r})$$

for the quotient of $\pi_1(\underline{\mathcal{M}}_{g,r})$ by the kernel of the natural continuous surjective homomorphism $\pi_1(\underline{\mathcal{M}}_{g,r}; \underline{\mathcal{M}}_{g,r_0(g)}) \twoheadrightarrow Q_r$. Thus, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\underline{\mathcal{M}}_{g,r}; \underline{\mathcal{M}}_{g,r_0(g)}) & \longrightarrow & \pi_1(\underline{\mathcal{M}}_{g,r}) & \longrightarrow & \pi_1(\underline{\mathcal{M}}_{g,r_0(g)}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & Q_r & \longrightarrow & \underline{Q}_r & \longrightarrow & \pi_1(\underline{\mathcal{M}}_{g,r_0(g)}) & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*. Write

$$\rho_r^Q \colon \pi_1(\underline{\mathcal{M}}_{g,r_0(g)}) \longrightarrow \mathrm{Out}(Q_r)$$

for the outer continuous action determined by the lower horizontal exact sequence of this commutative diagram. Now let us recall from [19, Example 3.13] that a hyperbolic polycurve isomorphic to the configuration space of a hyperbolic curve over a field of characteristic zero satisfies *condition* $(*)_l$ [cf. Definition 1.12]. Thus, by applying a similar argument to the argument applied in the proof of [5, Proposition 1.2], one concludes immediately from Lemma 3.5, (ii), (iv), together with the various definitions involved, that, to verify assertion (ii), it suffices to verify that the outer continuous action ρ_r^Q factors through a *pro-l quotient of* $\pi_1(\underline{\mathcal{M}}_{g,r_0(g)})$.

Next, let us observe that it follows immediately from Lemma 3.5, (iii), that

(1) there exists a continuous isomorphism $\iota \colon \Delta^l_{g,r_0(g)} \xrightarrow{\sim} Q_{r_0(g)+1}$ such that the diagram of groups

$$\begin{array}{ccc} & & \mathrm{Out}(\Delta^l_{g,r_0(g)}) \\ & \overset{\rho^l_{g,r_0(g)}}{\nearrow} & \\ \pi_1(\underline{\mathcal{M}}_{g,r_0(g)}) & & \wr \downarrow \mathrm{Out}(\iota) \\ & \underset{\rho^Q_{r_0(g)+1}}{\searrow} & \\ & & \mathrm{Out}(Q_{r_0(g)+1}) \end{array}$$

commutes.

Moreover, one also verifies immediately from Lemma 3.5, (iv), that

(2) the profinite group Q_r has a natural structure of *pro-l configuration space group* [cf. [18, Definition 2.3, (i)]] with respect to which, for each integer $r_0(g) \leq r' \leq r$, the kernel of the natural $Q_{r'}$-conjugacy class of continuous surjective homomorphisms $Q_r \twoheadrightarrow Q_{r'}$ [i.e., induced by the natural morphism $\underline{\mathcal{M}}_{g,r} \to \underline{\mathcal{M}}_{g,r'}$] is a *fiber subgroup of length* $r - r'$ [cf. [18, Definition 2.3, (iii)]].

In particular, it follows from (1) and assertion (i) that the outer continuous action $\rho^Q_{r_0(g)+1}$ factors through a *pro-l quotient of* $\pi_1(\underline{\mathcal{M}}_{g,r_0(g)})$. Thus, we conclude immediately from (2) and [20, Theorem 0.1] [cf. also [11, Theorem B]] that the outer continuous action ρ_r^Q factors through a *pro-l quotient of* $\pi_1(\underline{\mathcal{M}}_{g,r_0(g)})$, as desired. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). It follows from Lemma 1.13, (ii), that $\text{sb-rk}(G) \geq 3g - 3 + r$. Next, let us observe that, it follows from

Lemma 1.13, (iii), that, to verify assertion (iii), we may assume without loss of generality that G is the image of the composite in the statement of assertion (i). Then it follows from assertions (i), (ii), together with Lemma 1.13, (i), and Lemma 3.5, (v), that $\text{sb-rk}(G) \not\geq 3g-2+r$. This completes the proof of assertion (iii). □

One main result of the present paper is as follows.

Theorem 4.2. *Let l be a prime number, k a sub-l-adic field* [*cf. Definition* 1.1, (i)], *and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by $\overline{k}$. For each $i \in \{1,2\}$, let g_i and r_i be nonnegative integers such that $2g_i - 2 + r_i > 0$ and X_i a hyperbolic curve of type (g_i, r_i) over k* [*cf. Definition* 1.3, (iii)]; *write $\rho^l_{X_i}$ for the pro-l outer Galois action associated to X_i/k* [*cf. Definition* 1.2, (ii)]. *Suppose that the following five conditions are satisfied:*

(1) *The closed subgroups $\text{Ker}(\rho^l_{X_1})$, $\text{Ker}(\rho^l_{X_2}) \subseteq G_k$ of G_k are commensurable* [*cf. Definition* 1.1, (ii)].
(2) *For each $i \in \{1,2\}$, the hyperbolic curve X_i over k is quasi-l-monodromically full* [*cf. Definition* 1.7, (ii)].
(3) *Either the inequality $3g_1 - 3 + r_1 < 5$ or the inequality $3g_2 - 3 + r_2 < 5$ holds.*
(4) *The equality $l = 2$ holds whenever $g_1 g_2 \neq 0$.*
(5) *Either the equality $g_1 = g_2$ or the equality $r_1 = r_2$ holds.*

Then the equality $(g_1, r_1) = (g_2, r_2)$ holds. Moreover, the hyperbolic curve $X_1 \times_k \overline{k}$ over $\overline{k}$ is moduli-correspondence-equivalent [*cf. Definition* 1.9] *to the hyperbolic curve $X_2 \times_k \overline{k}$ over $\overline{k}$.*

Proof. Let us first observe that, to verify Theorem 4.2, it follows from condition (1) that we may assume without loss of generality, by replacing k by a suitable finite extension of k in $\overline{k}$, that the *equality* $\text{Ker}(\rho^l_{X_1}) = \text{Ker}(\rho^l_{X_2})$ holds. In particular, one may conclude immediately from Proposition 1.6, together with condition (2), that

(a) an open subgroup of $\rho^l_{g_1,r_1}(\pi_1(\mathcal{M}_{g_1,[r_1]} \times_k \overline{k}))$ is *isomorphic* to an open subgroup of $\rho^l_{g_2,r_2}(\pi_1(\mathcal{M}_{g_2,[r_2]} \times_k \overline{k}))$.

Thus, if $g_i = 0$ for some $i \in \{1,2\}$, then it follows from Lemma 1.13, (ii), and Lemma 4.1, (iii), that, for the unique $j \in \{1,2\} \setminus \{i\}$, the *inequality* $3g_j - 3 + r_j \leq 3g_i - 3 + r_i$, hence also [cf. condition (5)] the *inequality* $g_j \leq g_i = 0$, holds. In particular, one may conclude that

(b) the *equality* $g_1 g_2 = 0$ is *equivalent* to the *equality* $(g_1, g_2) = (0,0)$.

Next, let us also observe that, to verify Theorem 4.2, it follows from condition (3) that we may assume without loss of generality, by replacing (X_1, X_2) by (X_2, X_1) if necessary, that

(c) the *inequality* $3g_1 - 3 + r_1 < 5$, hence also the *inclusion* $g_1 \in \{0,1,2\}$, holds.

Thus, it follows from Lemma 1.13, (ii), and Lemma 4.1, (iii), together with (a), (b) [cf. also condition (4)], that the *inequality* $3g_2 - 3 + r_2 \leq 3g_1 - 3 + r_1$ holds. In particular, it follows from (c) that

(d) the *inequality* $3g_2 - 3 + r_2 < 5$, hence also the *inclusion* $g_2 \in \{0, 1, 2\}$, holds.

Now one may conclude from Lemma 4.1, (iii), together with (a), (b), (c), (d) [cf. also condition (4)], that the *equality* $3g_1-3+r_1 = 3g_2-3+r_2$, hence also [cf. condition (5)] the *equality* $(g_1, r_1) = (g_2, r_2)$, holds. In the remainder of the present proof, write $(g, r) \stackrel{\text{def}}{=} (g_1, r_1) = (g_2, r_2)$.

Next, let us observe that, to verify Theorem 4.2, we may assume without loss of generality, by replacing k by a suitable finite extension of k in $\overline{k}$, that, for each $i \in \{1, 2\}$, the k-rational point of $\mathcal{M}_{g,[r]}$ that classifies X_i/k *lifts* to a k-rational point x_i of $\underline{\mathcal{M}}_{g,r}$, relative to the composite $\underline{\mathcal{M}}_{g,r} \stackrel{\Phi_{g,r}}{\to} \mathcal{M}_{g,r} \to \mathcal{M}_{g,[r]}$ of the *finite étale coverings* [cf. Lemma 3.5, (i)]. For each $i \in \{1, 2\}$, let $s_i \colon G_k \to \pi_1(\underline{\mathcal{M}}_{g,r})$ be a continuous splitting that arises from $x_i \in \underline{\mathcal{M}}_{g,r}(k)$ of the natural continuous surjective homomorphism $\pi_1(\underline{\mathcal{M}}_{g,r}) \twoheadrightarrow G_k$. Then it follows from Lemma 3.5, (v), and Lemma 4.1, (i), (ii), together with conditions (1), (2), (3), (4), that, by applying Lemma 2.2 [i.e., in the case where we take the "(X, x_1, x_2, s_1, s_2)" of Lemma 2.2 to be $(\underline{\mathcal{M}}_{g,r}, x_1, x_2, s_1, s_2)$ and the "$\pi_1(X) \twoheadrightarrow \Gamma$" of Lemma 2.2 to be the quotient of $\pi_1(\underline{\mathcal{M}}_{g,r})$ determined by the restriction of $\rho^l_{g,r} \colon \pi_1(\mathcal{M}_{g,[r]}) \to \mathrm{Out}(\Delta^l_{g,r})$ to $\pi_1(\underline{\mathcal{M}}_{g,r}) \subseteq \pi_1(\mathcal{M}_{g,[r]})$], we obtain

- a hyperbolic polycurve Y over $\overline{k}$,
- two connected finite étale coverings f_1, $f_2 \colon Y \to (\underline{\mathcal{M}}_{g,r})_{\overline{k}} \stackrel{\text{def}}{=} \underline{\mathcal{M}}_{g,r} \times_k \overline{k}$ over $\overline{k}$, and
- a $\overline{k}$-rational point $y \in Y(\overline{k})$ of Y

such that, for each $i \in \{1, 2\}$, the image of $y \in Y(\overline{k})$ by $f_i \colon Y \to (\underline{\mathcal{M}}_{g,r})_{\overline{k}}$ *coincides* with the $\overline{k}$-rational point of $(\underline{\mathcal{M}}_{g,r})_{\overline{k}}$ determined by $x_i \in \underline{\mathcal{M}}_{g,r}(k)$. In particular, we conclude that the hyperbolic curve $X_1 \times_k \overline{k}$ over $\overline{k}$ is *moduli-correspondence-equivalent* to the hyperbolic curve $X_2 \times_k \overline{k}$ over $\overline{k}$, as desired. This completes the proof of Theorem 4.2. □

Remark 4.3. Let us recall that the following fact [i.e., proved in Lemma 4.1, (i)] plays an important role in the proof of Theorem 4.2:

> Suppose that k is *algebraically closed*. Then, for each positive integer r, the quotient of $\pi_1(\underline{\mathcal{M}}_{1,r})$ determined by the composite
>
> $$\pi_1(\underline{\mathcal{M}}_{1,r}) \xhookrightarrow{\pi_1(\Phi_{1,r})} \pi_1(\mathcal{M}_{1,r}) \hookrightarrow \pi_1(\mathcal{M}_{1,[r]}) \xrightarrow{\rho^2_{1,r}} \mathrm{Out}(\Delta^2_{1,r})$$
>
> coincides with the *maximal pro-2 quotient* of $\pi_1(\underline{\mathcal{M}}_{1,r})$.

On the other hand, let us also recall that the following assertion may be regarded as an immediate consequence of [10, Theorem A, (ii)]:

> Suppose that k is *algebraically closed*, and that the *inequality* $l > 7$ holds. Then, for an arbitrary positive integer r and an arbitrary connected finite étale covering $\mathcal{N} \to \mathcal{M}_{1,[r]}$ of $\mathcal{M}_{1,[r]}$, the quotient of $\pi_1(\mathcal{N})$ determined by the composite
>
> $$\pi_1(\mathcal{N}) \hookrightarrow \pi_1(\mathcal{M}_{1,[r]}) \xrightarrow{\rho^l_{1,r}} \mathrm{Out}(\Delta^l_{1,r})$$
>
> does *not coincide with the maximal pro-l quotient* of $\pi_1(\mathcal{N})$.

In particular, one may conclude that at least a "naive imitation" of the above proof of Theorem 4.2 in the case where $g = 1$ and $l > 7$ does *not work*.

Corollary 4.4. *Let l be a prime number, k a sub-l-adic field* [*cf. Definition* 1.1*,* (i)]*, and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by $\overline{k}$. For each $i \in \{1, 2\}$, let g_i and r_i be nonnegative integers such that $2g_i - 2 + r_i > 0$ and X_i a hyperbolic curve of type (g_i, r_i) over k* [*cf. Definition* 1.3*,* (iii)]*; write $\rho^l_{X_i}$ for the pro-l outer Galois action associated to X_i/k* [*cf. Definition* 1.2*,* (ii)]*. Suppose that the following four conditions are satisfied:*

(1) *For each $i \in \{1, 2\}$, the hyperbolic curve X_i over k is quasi-l-monodromically full* [*cf. Definition* 1.7*,* (ii)]*.*
(2) *The intersection*

$$\{(g_1, r_1), (g_2, r_2)\} \cap \{(0, 3), (0, 5), (0, 6), (0, 7), (1, 3), (1, 4), (2, 0), (2, 1)\}$$

is nonempty.
(3) *The equality $l = 2$ holds whenever $g_1 g_2 \neq 0$.*
(4) *Either the equality $g_1 = g_2$ or the equality $r_1 = r_2$ holds.*

Then the following two conditions are equivalent:

(a) *The hyperbolic curve $X_1 \times_k \overline{k}$ over $\overline{k}$ is isomorphic to the hyperbolic curve $X_2 \times_k \overline{k}$ over $\overline{k}$.*
(b) *The closed subgroups $\mathrm{Ker}(\rho^l_{X_1})$, $\mathrm{Ker}(\rho^l_{X_2}) \subseteq G_k$ of G_k are commensurable* [*cf. Definition* 1.1*,* (ii)]*.*

Proof. The implication (a) ⇒ (b) is immediate. Next, we verify the implication (b) ⇒ (a). Suppose that condition (b) is satisfied. Then it follows from Theorem 4.2, together with conditions (1), (2), (3), (4), that the *equality* $(g_1, r_1) = (g_2, r_2)$ holds, and, moreover, the hyperbolic curve $X_1 \times_k \overline{k}$ over $\overline{k}$ is *moduli-correspondence-equivalent* to the hyperbolic curve $X_2 \times_k \overline{k}$ over $\overline{k}$. Now observe that the implication (b) ⇒ (a) in the case where (g_1, r_1) $(= (g_2, r_2)) = (0, 3)$ follows from

Lemma 3.2. Suppose that (g_1, r_1) $(= (g_2, r_2)) \neq (0, 3)$. Then it follows from Proposition 1.10 and Proposition 3.6 that condition (a) is satisfied. This completes the proof of Corollary 4.4. $\square$

A similar result to Corollary 4.4 for [once-punctured] elliptic curves may be established as follows.

Definition 4.5. Let S be a connected scheme of finite type over k, $s \in S(k)$ a k-rational point of S, and X a hyperbolic curve of type (g, r) over S. Thus, the k-rational point $s \in S(k)$ and the S-valued point of $\mathcal{M}_{g,[r]}$ that classifies X/S induce outer continuous homomorphisms $G_k \to \pi_1(S) \to \pi_1(\mathcal{M}_{g,[r]})$. Then we shall say that $s \in S(k)$ is *l-monodromically full with respect to* X/S [cf. [6, Definition 2.1, (i)]] if the image of the composite $G_k \to \pi_1(S) \to \pi_1(\mathcal{M}_{g,[r]}) \stackrel{\rho^l_{g,r}}{\to} \mathrm{Out}(\Delta^l_{g,r})$ contains the image of the composite $\pi_1(T) \hookrightarrow \pi_1(S) \to \pi_1(\mathcal{M}_{g,[r]}) \stackrel{\rho^l_{g,r}}{\to} \mathrm{Out}(\Delta^l_{g,r})$ — where $T \to S$ is a connected finite étale covering of S obtained by considering a connected component of the fiber product of the S-valued point $S \to \mathcal{M}_{g,[r]}$ that classifies X/S and the natural finite étale Galois covering $\mathcal{M}_{g,r} \to \mathcal{M}_{g,[r]}$.

Theorem 4.6. *Let k be a sub-2-adic field* [*cf. Definition* 1.1, (i)] *and x_1, x_2 two k-rational points of $\underline{\mathcal{M}}_{1,1}$* [*cf. Definition* 3.4, (i)]. *For each $i \in \{1, 2\}$, write X_i for the hyperbolic curve of type* $(1, 1)$ *over k* [*cf. Definition* 1.3, (iii)] *obtained by forming the fiber of the hyperbolic curve $\mathcal{L}_{1,1} \to \underline{\mathcal{M}}_{1,1}$ of type* $(1, 1)$ *over $\underline{\mathcal{M}}_{1,1}$* [*cf. Definition* 3.3, (ii)] *at $x_i \in \underline{\mathcal{M}}_{1,1}(k)$, $\rho^2_{X_i}$ for the pro-*2 *outer Galois action associated to X_i/k* [*cf. Definition* 1.2, (ii)], *and E_i for the elliptic curve over k associated to X_i. Suppose that, for each $i \in \{1, 2\}$, the k-rational point x_i of $\underline{\mathcal{M}}_{1,1}$ is* 2*-monodromically full with respect to $\mathcal{L}_{1,1}/\underline{\mathcal{M}}_{1,1}$* [*cf. Definition* 4.5]. *Then the following two conditions are equivalent:*

(1) *The elliptic curve E_1 over k is isomorphic to the elliptic curve E_2 over k.*
(2) *The equality* $\mathrm{Ker}(\rho^2_{X_1}) = \mathrm{Ker}(\rho^2_{X_2})$ *holds.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. Let us first observe that it follows immediately from a similar argument to the argument applied in the proof of Theorem 4.2, together with Lemma 2.3, that there exists an automorphism f of $\underline{\mathcal{M}}_{1,1}$ over k such that the *equality* $f \circ x_1 = x_2$ holds. Thus, it follows from [6, Lemma 4.1, (ii)] and the explicit construction in Definition 3.3, (ii), of the hyperbolic curve $\mathcal{L}_{1,1} \to \mathcal{M}_{0,4} = \underline{\mathcal{M}}_{1,1}$ of type $(1, 1)$ over $\mathcal{M}_{0,4} = \underline{\mathcal{M}}_{1,1}$ that the elliptic curve E_1 over k is *isomorphic* to the elliptic curve E_2 over k, as desired. This completes the proof of Theorem 4.6. $\square$

5. Complements

In the present §5, we give some complements to the contents of Corollary 4.4 and Theorem 4.6. In the present §5, we maintain the notational conventions introduced in §1. The main purpose of the present §5 is to prove the following two propositions, i.e., Proposition 5.1 below and Proposition 5.2 below [cf. also Remark 5.3 below].

Proposition 5.1. *Suppose that k is finitely generated over the minimal subfield of k. Then there exist a finite extension K of k in $\overline{k}$ and hyperbolic curves X_1 and X_2 of type $(1, 1)$ over K that satisfy the following three conditions:*

- *For each $i \in \{1, 2\}$, the hyperbolic curve X_i over K is quasi-l-monodromically full.*
- *The closed subgroups $\mathrm{Ker}(\rho^l_{X_1})$, $\mathrm{Ker}(\rho^l_{X_2})$ of $\mathrm{Gal}(\overline{k}/K)$ are commensurable.*
- *The hyperbolic curve $X_1 \times_K \overline{k}$ over $\overline{k}$ is not isomorphic to the hyperbolic curve $X_2 \times_K \overline{k}$ over $\overline{k}$.*

Proposition 5.2. *Suppose that k is finitely generated over the minimal subfield of k. Then there exist a finite extension K of k in $\overline{k}$ and hyperbolic curves X_1 and X_2 of type $(0, 4)$ over K that satisfy the following three conditions:*

- *For each $i \in \{1, 2\}$, the hyperbolic curve X_i over K is quasi-2-monodromically full.*
- *The closed subgroups $\mathrm{Ker}(\rho^2_{X_1})$, $\mathrm{Ker}(\rho^2_{X_2})$ of $\mathrm{Gal}(\overline{k}/K)$ are commensurable.*
- *The hyperbolic curve $X_1 \times_K \overline{k}$ over $\overline{k}$ is not isomorphic to the hyperbolic curve $X_2 \times_K \overline{k}$ over $\overline{k}$.*

Remark 5.3.

(i) Let us observe that the objects discussed in Proposition 5.1 yield a "counter-example" of the assertion obtained by replacing condition (2) in the statement of Corollary 4.4 by the condition that $(g_1, r_1) = (g_2, r_2) = (1, 1)$. Moreover, let us observe that the objects discussed in Proposition 5.1 also yield a "counter-example" of a "quasi-monodromically full-version" of Theorem 4.6 [cf. also the implication (2) $\Rightarrow$ (4) of Lemma 5.10 below].

(ii) Let us observe that the objects discussed in Proposition 5.2 yield a "counter-example" of the assertion obtained by replacing condition (2) in the statement of Corollary 4.4 by the condition that $(g_1, r_1) = (g_2, r_2) = (0, 4)$. Moreover, let us observe that the objects discussed in Proposition 5.2 also yield a "counter-example" of a "quasi-monodromically full-version" of Theorem B.

Lemma 5.4. *Let X_1 and X_2 be hyperbolic curves of type $(1,1)$ over k. Write E_1 and E_2 for the respective elliptic curves over k associated to X_1 and X_2. Suppose that there exists an isogeny $E_1 \to E_2$ over k of degree a power of l. Then the closed subgroups $\mathrm{Ker}(\rho^l_{X_1})$, $\mathrm{Ker}(\rho^l_{X_2}) \subseteq G_k$ of G_k are commensurable.*

Proof. Let us observe that an isogeny $f\colon E_1 \to E_2$ over k of degree a *power of l* determines a sequence of hyperbolic curves over k

$$X_1 \hookleftarrow E_1 \setminus \mathrm{Ker}(f) \xrightarrow{f|_{E_1\setminus \mathrm{Ker}(f)}} X_2$$

— where the first arrow is the natural open immersion, and the second arrow is a *connected finite étale l-special covering* over k. Thus, it follows immediately from [7, Lemma 23, (i), (ii), (iii)], together with [18, Remark 1.2.2] and [18, Proposition 1.4], that we may assume without loss of generality, by replacing G_k by a suitable open subgroup of G_k, that $\mathrm{Ker}(\rho^l_{X_2}) \subseteq \mathrm{Ker}(\rho^l_{X_1})$. In particular, by applying a similar argument to this argument to the dual isogeny $E_2 \to E_1$ of f, one may conclude that the closed subgroups $\mathrm{Ker}(\rho^l_{X_1})$, $\mathrm{Ker}(\rho^l_{X_2}) \subseteq G_k$ of G_k are *commensurable*, as desired. This completes the proof of Lemma 5.4. □

Definition 5.5. Let $\mathcal{N} \to \mathcal{M}_{1,1}(l)$ be a connected finite étale covering of $\mathcal{M}_{1,1}(l)$, hence also of $\mathcal{M}_{1,1}$, such that the stack $\mathcal{N}$ has a natural structure of scheme. Write $\mathcal{C}_{1,1}|_{\mathcal{N}} \to \mathcal{N}$ (respectively, $\mathcal{C}^+_{1,1}|_{\mathcal{N}} \to \mathcal{N}$) for the base-change of the natural morphism $\mathcal{C}_{1,1} \to \mathcal{M}_{1,1}$ (respectively, $\mathcal{C}^+_{1,1} \to \mathcal{M}_{1,1}$) by the finite étale covering $\mathcal{N} \to \mathcal{M}_{1,1}$. [So the stack $\mathcal{C}^+_{1,1}|_{\mathcal{N}}$ over $\mathcal{N}$, equipped with the splitting determined by the splitting $s_1^{\mathcal{M}}$ of Definition 1.4, (i), has a natural structure of elliptic curve over $\mathcal{N}$.] Let $s(l)$ be a nontrivial l-torsion splitting of the elliptic curve $\mathcal{C}^+_{1,1}|_{\mathcal{N}} \to \mathcal{N}$. [Note that such a splitting always exists by the definition of $\mathcal{M}_{1,1}(l) \to \mathcal{M}_{1,1}$.] Write $\mathcal{Q}^+ \to \mathcal{N}$ for the elliptic curve over $\mathcal{N}$ obtained by forming the quotient of the elliptic curve $\mathcal{C}^+_{1,1}|_{\mathcal{N}} \to \mathcal{N}$ by the subgroup scheme generated by the torsion splitting $s(l)$ and $\mathcal{Q} \subseteq \mathcal{Q}^+$ for the open subscheme of $\mathcal{Q}^+$ obtained by forming the complement in $\mathcal{Q}^+$ of the image of the composite $\mathcal{N} \overset{s(l)}{\to} \mathcal{C}^+_{1,1}|_{\mathcal{N}} \twoheadrightarrow \mathcal{Q}^+$. [So we have a natural isogeny $\mathcal{C}^+_{1,1}|_{\mathcal{N}} \twoheadrightarrow \mathcal{Q}^+$ over $\mathcal{N}$ of degree l; moreover, the natural morphism $\mathcal{Q} \to \mathcal{N}$ determines a structure on $\mathcal{Q}$ of hyperbolic curve of type $(1,1)$ over $\mathcal{N}$.]

Lemma 5.6. *In the situation of Definition 5.5, suppose that k is finitely generated over the minimal subfield of k, i.e., over the unique subfield of k isomorphic to the field of rational numbers. Fix an inclusion $\overline{k} \hookrightarrow \mathbf{C}$ of fields. Then the subset of $\mathcal{N}(\mathbf{C})$ consisting of $\mathbf{C}$-valued points $s \in \mathcal{N}(\mathbf{C})$ that satisfy the following condition $(*^{\mathrm{MF}})$ is dense with respect to the complex topology of $\mathcal{N}(\mathbf{C})$:*

$(*^{\mathrm{MF}})$ *There exist a finite extension K of k in $\overline{k}$ $(\subseteq \mathbf{C})$ and a K-valued point of $\mathcal{N}$ that maps to $s \in \mathcal{N}(\mathbf{C})$ and is l-monodromically* *full* [*cf. Definition* 4.5] *both with respect to* $\mathcal{C}_{1,1}|_{\mathcal{N}}/\mathcal{N}$ *and with respect to* $\mathcal{Q}/\mathcal{N}$.

Proof. Write $\Delta_{\mathcal{Q}/\mathcal{N}}$ for the kernel of the outer continuous surjective homomorphism $\pi_1(\mathcal{Q}) \twoheadrightarrow \pi_1(\mathcal{N})$ induced by the structure morphism $\mathcal{Q} \to \mathcal{N}$ and $\Delta^l_{\mathcal{Q}/\mathcal{N}}$ for the maximal pro-l quotient of $\Delta_{\mathcal{Q}/\mathcal{N}}$. Thus, we have natural *exact* sequences of profinite groups

$$1 \longrightarrow \Delta_{1,1} \longrightarrow \pi_1(\mathcal{C}_{1,1}|_{\mathcal{N}}) \longrightarrow \pi_1(\mathcal{N}) \longrightarrow 1,$$

$$1 \longrightarrow \Delta_{\mathcal{Q}/\mathcal{N}} \longrightarrow \pi_1(\mathcal{Q}) \longrightarrow \pi_1(\mathcal{N}) \longrightarrow 1.$$

Write

$$\rho^l_{\mathcal{C}} \colon \pi_1(\mathcal{N}) \longrightarrow \mathrm{Out}(\Delta^l_{1,1}), \qquad \rho^l_{\mathcal{Q}} \colon \pi_1(\mathcal{N}) \longrightarrow \mathrm{Out}(\Delta^l_{\mathcal{Q}/\mathcal{N}})$$

for the respective outer continuous actions determined by the above upper, lower exact sequences. Then it follows from [6, Lemma 2.5], together with the *separatedness* and the *smoothness* of $\mathcal{M}_{1,1}$ over k [cf. [14, Theorem 2.7]], that the respective images of $\rho^l_{\mathcal{C}}$, $\rho^l_{\mathcal{Q}}$ are *topologically finitely generated*. Thus, it follows from [15, Proposition 3.1] that there exists a finite étale Galois covering $\mathcal{N}' \to \mathcal{N}$ of $\mathcal{N}$ that satisfies the following condition: For a given finite extension K of k and a given K-valued point s of $\mathcal{N}$, if the fiber of $\mathcal{N}' \to \mathcal{N}$ at $s \in \mathcal{N}(K)$ is *connected*, then the K-valued point $s \in \mathcal{N}(K)$ is *l-monodromically full* both with respect to $\mathcal{C}_{1,1}|_{\mathcal{N}}/\mathcal{N}$ and with respect to $\mathcal{Q}/\mathcal{N}$. In particular, since [it is well-known that] k is *Hilbertian*, Lemma 5.6 follows immediately from a similar argument to the argument applied in the proof of [15, Theorem 1.2]. This completes the proof of Lemma 5.6. □

Lemma 5.7. *Suppose that k is finitely generated over the minimal subfield of k. Then there exist a finite extension K of k in $\overline{k}$ and hyperbolic curves X_1 and X_2 of type* $(1,1)$ *over K such that if one writes E_1 and E_2 for the elliptic curves over K associated to X_1 and X_2, respectively, then the following three conditions are satisfied:*

1. *For each $i \in \{1,2\}$, the hyperbolic curve X_i over K is quasi-l-monodromically full.*
2. *There exists an isogeny $E_1 \to E_2$ over K of degree l.*
3. *For each $i \in \{1,2\}$, the elliptic curve $E_i \times_K \overline{k}$ over $\overline{k}$ does not have complex multiplication.*

Proof. Now let us recall that it follows from [6, Remark 3.7.1] and [6, Proposition 3.8] that condition (1) implies condition (3). Thus, this assertion is an immediate consequence of Lemma 5.6. □

Proof of Proposition 5.1. Let us observe that, for two elliptic curves E_1 and E_2 over $\overline{k}$ that admit an isogeny $E_1 \to E_2$ over $\overline{k}$ *of degree l*, one may conclude immediately, by considering the ring of endomorphisms of E_1 over $\overline{k}$, that E_1 is *not isomorphic* to E_2 over $\overline{k}$ under the assumption that E_1 does *not have complex multiplication*. Thus, Proposition 5.1 follows from Lemma 5.4 and Lemma 5.7. This completes the proof of Proposition 5.1. $\square$

Remark 5.8. The authors would like to thank *Akio Tamagawa* for explaining to them the arguments applied to prove Proposition 5.1.

Lemma 5.9. *The kernel of the composite*

$$\pi_1(\underline{\mathcal{M}}_{1,1}) \xhookrightarrow{\pi_1(\Phi_{1,1})} \pi_1(\mathcal{M}_{1,1}) \xrightarrow{\sim} \pi_1(\mathcal{M}_{1,[1]}) \xrightarrow{\rho^2_{1,1}} \mathrm{Out}(\Delta^2_{1,1})$$

coincides with the kernel of the composite

$$\pi_1(\underline{\mathcal{M}}_{1,1}) = \pi_1(\mathcal{M}_{0,4}) \hookrightarrow \pi_1(\mathcal{M}_{0,[4]}) \xrightarrow{\rho^2_{0,4}} \mathrm{Out}(\Delta^2_{0,4}).$$

Proof. Write $N_1,\ N_2 \subseteq \pi_1(\underline{\mathcal{M}}_{1,1})$ for the kernels of the composites of the first, second displays of the statement of Lemma 5.9, respectively. Write, moreover, $J_1,\ J_2$ for the quotients of G_k by the normal closed subgroups of G_k obtained by forming the images of $N_1,\ N_2 \subseteq \pi_1(\underline{\mathcal{M}}_{1,1})$ by the outer continuous surjective homomorphism $\pi_1(\underline{\mathcal{M}}_{1,1}) \twoheadrightarrow G_k$ induced by the structure morphism $\underline{\mathcal{M}}_{1,1} \to \mathrm{Spec}(k)$, respectively.

First, we verify the assertion that the quotient J_1 of G_k *coincides* with the quotient J_2 of G_k. To this end, let us first recall from Proposition 1.6 that the image of $\mathrm{Ker}(\rho^2_{1,1}) \subseteq \pi_1(\mathcal{M}_{1,[1]})$ in G_k *coincides* with the image of $\mathrm{Ker}(\rho^2_{0,4}) \subseteq \pi_1(\mathcal{M}_{0,[4]})$ in G_k. Thus, the desired assertion follows from Lemma 3.5, (i), together with the [easily verified] fact that, for each integer $r \geq r_0(g)$, the closed subgroup $\mathrm{Ker}(\rho^2_{g,r}) \subseteq \pi_1(\mathcal{M}_{g,[r]})$ of $\pi_1(\mathcal{M}_{g,[r]})$ is *contained* in the normal open subgroup $\pi_1(\mathcal{M}_{g,r}(4)) \subseteq \pi_1(\mathcal{M}_{g,[r]})$ of $\pi_1(\mathcal{M}_{g,[r]})$. This completes the proof of the desired assertion.

Next, we verify the *inclusion* $N_1 \subseteq N_2$ (respectively, $N_2 \subseteq N_1$). Let us first observe that it follows immediately from a similar argument to the argument applied in the proof of [10, Theorem 2.10] that the image of $N_1 \subseteq \pi_1(\underline{\mathcal{M}}_{1,1})$ (respectively, $N_2 \subseteq \pi_1(\underline{\mathcal{M}}_{1,1})$) by the composite of the second (respectively, first) display of the statement of Lemma 5.9 is *finite* [cf. also the commutative diagram of k-algebras which appear in Lemma 3.2 and Definition 3.3, (ii),

$$\begin{array}{ccc} A_{0,4} = k\Big[x_1, \dfrac{1}{x_1(x_1-1)}\Big] & \hookrightarrow & A_{0,5} = A_{0,4}\Big[x_2, \dfrac{1}{x_2(x_2-1)(x_1-x_2)}\Big] \\ \big\downarrow & & \big\downarrow \\ B \stackrel{\mathrm{def}}{=} A_{0,4}[s,t]/\big(s^2 - t(t-1)(t-x_1)\big) & \hookrightarrow & B\Big[\dfrac{1}{s}\Big] \end{array}$$

— where the right-hand vertical arrow is the homomorphism of $A_{0,4}$-algebras that maps x_2 to t]. Thus, since the image of $\pi_1(\mathcal{M}_{0,4} \times_k \overline{k}) \subseteq \pi_1(\mathcal{M}_{0,4}) = \pi_1(\underline{\mathcal{M}}_{1,1})$ by the composite of the second (respectively, first) display of the statement of Lemma 5.9 is *torsion-free* [cf. [18, Remark 1.2.2], Lemma 3.2, and Lemma 4.1, (i)], to verify the desired *inclusion*, it suffices to verify that the image of $N_1 \subseteq \pi_1(\underline{\mathcal{M}}_{1,1})$ (respectively, $N_2 \subseteq \pi_1(\underline{\mathcal{M}}_{1,1})$) in J_2 (respectively, J_1) is *trivial.* On the other hand, the desired *triviality* is a formal consequence of the assertion that the quotient J_1 of G_k *coincides* with the quotient J_2 of G_k, already verified above. This completes the proof of Lemma 5.9. □

Lemma 5.10. *Let* λ *be an element of* $k \setminus \{0, 1\}$. *Consider the following four conditions:*

(1) *The hyperbolic curve of type* $(0, 4)$ *over* k

$$\mathrm{Spec}\Big(k\Big[x, \frac{1}{x}, \frac{1}{x-1}, \frac{1}{x-\lambda}\Big]\Big)$$

— where x *is an indeterminate — is* 2*-monodromically full.*

(2) *The* k*-rational point of* $\underline{\mathcal{M}}_{1,1} = \mathcal{M}_{0,4}$ *given by "*$x_1 = \lambda$*"* [*cf. Lemma* 3.2] *is* 2*-monodromically full with respect to* $\mathcal{L}_{1,1}/\underline{\mathcal{M}}_{1,1}$.

(3) *The hyperbolic curve of type* $(0, 4)$ *over* k

$$\mathrm{Spec}\Big(k\Big[x, \frac{1}{x}, \frac{1}{x-1}, \frac{1}{x-\lambda}\Big]\Big)$$

— where x *is an indeterminate — is quasi-*2*-monodromically full.*

(4) *The hyperbolic curve of type* $(1, 1)$ *over* k

$$\mathrm{Spec}\Big(k[s, t]/\big(s^2 - t(t-1)(t-\lambda)\big)\Big)$$

— where s *and* t *are indeterminates — is quasi-*2*-monodromically full.*

Then the following implications hold:

$$(1) \Longleftrightarrow (2) \Longrightarrow (3) \Longleftrightarrow (4).$$

Proof. The implication (1) ⇒ (3) is immediate. The equivalences (1) ⇔ (2) and (3) ⇔ (4) follow immediately from Lemma 5.9 [cf. also Lemma 3.5, (i)]. This completes the proof of Lemma 5.10. □

Proof of Proposition 5.2. Proposition 5.2 follows immediately from Proposition 5.1, Lemma 5.9, and the implication (4) ⇒ (3) of Lemma 5.10. □

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