Homomorphisms of Global Solvably Closed Galois Groups Compatible with Cyclotomic Characters

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Abstract. — In the present paper, we study a continuous open homomorphism between the Galois groups of solvably closed Galois extensions of number fields. We prove that a continuous open homomorphism between the Galois groups of solvably closed Galois extensions of number fields arises from a homomorphism between the given solvably closed Galois extensions if, and only if, the continuous open homomorphism is compatible with the cyclotomic characters.

Contents

Introduction

For $\square \in \{\circ, \bullet\}$, let $F_\square$ be a *number field* [i.e., a field that is of characteristic zero and finite over the minimal subfield of F — cf. Definition 1.2, (ii)] and $\widetilde{F}_\square$ a *Galois* extension of $F_\square$ that is *solvably closed* [i.e., that does not have nontrivial abelian extension — cf. Definition 3.2]. Let

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous *open* homomorphism of profinite groups. *K. Uchida* posed the following conjecture [cf. [9, Conjecture in p.595]].

Conjecture (Uchida). — *There exists a* ***uniquely determined*** *homomorphism* $\alpha_F\colon \widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ *of fields* ***compatible*** *with the respective actions of* $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ *relative to the continuous open homomorphism* α, *i.e., such that, for each* $\gamma \in \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, *the* ***equality*** $\gamma \circ \alpha_F = \alpha_F \circ \alpha(\gamma)$ *holds.*

Uchida *solved affirmatively* this conjecture in the case where α is *injective* [cf. [8, Theorem in p.359]]. Moreover, Uchida also gave, in [9], some interesting results concerning this conjecture. For instance, Uchida proved

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• Conjecture in the case where the number field $F_\circ$ is *isomorphic to the field of rational numbers* [cf. [9, Theorem 1]],

• Conjecture in the case where the homomorphism α *satisfies a certain condition concerning decomposition subgroups at nonarchimedean primes* [cf. [9, Theorem 2]], and

• the *uniqueness* of a homomorphism "$\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$" as in the statement of Conjecture [cf. [9, Proposition 2]].

The present paper discusses this conjecture. One main result of the present paper is as follows [cf. Theorem 3.4].

THEOREM. — *The following two conditions are equivalent:*

• *There exists a [necessarily* ***unique*** *— cf.* [9, Proposition 2]] *homomorphism* $\alpha_F\colon \widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ *of fields* ***compatible*** *with the respective actions of* $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ *relative to the continuous open homomorphism* α*, i.e., such that, for each* $\gamma \in \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$*, the* ***equality*** $\gamma \circ \alpha_F = \alpha_F \circ \alpha(\gamma)$ *holds.*

• *For each* $\Box \in \{\circ, \bullet\}$*, write* $\rho_\Box^{\mathrm{cycl}}\colon \mathrm{Gal}(\widetilde{F}_\Box/F_\Box) \to \widehat{\mathbb{Z}}^\times$ *for the* ***cyclotomic character*** *on* $\mathrm{Gal}(\widetilde{F}_\Box/F_\Box)$*. Then the* ***equality***

$$\rho_\circ^{\mathrm{cycl}} = \rho_\bullet^{\mathrm{cycl}} \circ \alpha$$

holds.

Now let us recall that, for a continuous open homomorphism between the absolute Galois groups of *p-adic local fields* [i.e., fields isomorphic to finite extensions of $\mathbb{Q}_p$], it holds that the continuous open homomorphism *arises* from a homomorphism between the given p-adic local fields if and only if the continuous open homomorphism preserves the *Hodge-Tate-ness* of the p-adic representations [cf. [1, Theorem in pp.285-286]]. Observe that the above theorem may be regarded as an *analogue for number fields* of this result for p-adic local fields. We refer to Remark 3.4.1 for more details.

Finally, let us also recall that conditional results in the study of the *function field version* of the above conjecture may be found in [6].

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1. A Sufficient Condition to be Fully Faithful

In the present §1, we establish a sufficient condition for a functor whose domain is the category of number fields to be *fully faithful* [cf. Theorem 1.9 below]. Note that the content of the present §1 is inspired by the work of *H. Smit* in [7].

DEFINITION 1.1.

(i) We shall write $\mathfrak{Primes}$ for the set of prime numbers.

(ii) Let S, T be sets; $f, g \colon S \to T$ maps of sets. Then we shall write $\mathrm{Eq}(f,g) \subseteq S$ for the *equalizer* of f and g, i.e., the subset of S consisting of the elements s of S such that $f(s) = g(s)$.

(iii) We shall say that a subset of a set is *cofinite* if the complement in the set of the subset is finite.

(iv) Let S be a set; S_1, S_2 subsets of S. Then we shall write $S_1 \overset{\bullet}{\subseteq} S_2$ if S_2 contains a cofinite subset of S_1.

(v) Let S be a set. Then we shall write $\mathcal{S}et_S$ for the category defined as follows:

- An object of $\mathcal{S}et_S$ is defined to be an S-set [i.e., a set equipped with a map to S].
- A morphism in $\mathcal{S}et_S$ is defined to be a map of S-sets [i.e., a map of sets over S relative to the structure maps].

DEFINITION 1.2.

(i) Let F be a field. Then we shall write $F^{\mathrm{prm}} \subseteq F$ for the *prime field* contained in F, i.e., the [unique] minimal subfield of F.

(ii) We shall say that a field F is a *number field* if F is of characteristic zero and finite over the subfield F^{prm}.

(iii) We shall write $\mathcal{NF}$ for the category defined as follows:

- An object of $\mathcal{NF}$ is defined to be a number field.
- A morphism in $\mathcal{NF}$ is defined to be a homomorphism of fields.

DEFINITION 1.3. — Let F be a number field.

(i) Let $\mathfrak{p}$ be a nonarchimedean prime of F. Then we shall write $\mathrm{char}(\mathfrak{p}) \in \mathfrak{Primes}$ for the *residue characteristic* of $\mathfrak{p}$.

(ii) Let $\mathfrak{p}$ be a nonarchimedean prime of F. Then we shall say that $\mathfrak{p}$ is *of absolute degree one* if the completion of F at $\mathfrak{p}$ is isomorphic, as an abstract field, to the completion of F^{prm} at $\mathfrak{p}|_{F^{\mathrm{prm}}}$. Moreover, we shall say that $\mathfrak{p}$ is *of absolute residue degree one* if the residue field at $\mathfrak{p}$ is of cardinality $\mathrm{char}(\mathfrak{p})$.

(iii) We shall write $\mathcal{V}(F)$ for the set of nonarchimedean primes of F of absolute residue degree one. Let us observe that one verifies easily that we have a natural identification between $\mathcal{V}(F^{\mathrm{prm}})$ and $\mathfrak{Primes}$ [i.e., determined by "char"]. In the remainder of the present paper, let us identify $\mathcal{V}(F^{\mathrm{prm}})$ with $\mathfrak{Primes}$ by means of this natural identification:

$$\mathcal{V}(F^{\mathrm{prm}}) = \mathfrak{Primes}$$

(iv) Let S be a subset of $\mathfrak{Primes}$. Then we shall write $\mathcal{V}|_S(F) \subseteq \mathcal{V}(F)$ for the subset consisting of the elements of $\mathcal{V}(F)$ of residue characteristic $\in S$. Let us observe that one verifies immediately that the assignments "$F \rightsquigarrow \mathcal{V}(F)$", "$F \rightsquigarrow \mathcal{V}|_S(F)$" naturally determine contravariant functors

$$\mathcal{V} \colon \mathcal{NF} \longrightarrow \mathcal{S}et_{\mathfrak{Primes}}, \qquad \mathcal{V}|_S \colon \mathcal{NF} \longrightarrow \mathcal{S}et_S,$$

respectively.

(v) Let K be a number field that contains F. Then we shall write $\mathcal{V}_*(K/F) \subseteq \mathcal{V}(F)$ for the subset consisting of the elements of $\mathcal{V}(F)$ that split completely in K and $\mathcal{V}^*(K/F) \stackrel{\text{def}}{=} \mathcal{V}(F \hookrightarrow K)^{-1}(\mathcal{V}_*(K/F)) \subseteq \mathcal{V}(K)$.

LEMMA 1.4. — *Let F, K be number fields; ϕ, $\psi\colon F \hookrightarrow K$ homomorphisms of fields. Then it holds that the* ***equality*** *$\phi = \psi$ holds if and only if the intersection $\mathrm{Eq}(\mathcal{V}(\phi), \mathcal{V}(\psi)) \cap \mathcal{V}^*(K/K^{\mathrm{prm}})$ is* ***nonempty****.*

PROOF. — It follows from *Chebotarev's density theorem* [cf., e.g., [5, Chapter VII, Theorem 13.4]] that $\mathcal{V}^*(K/K^{\mathrm{prm}})$ is *nonempty*. In particular, the necessity is immediate. Next, we verify the sufficiency. Suppose that $\mathrm{Eq}(\mathcal{V}(\phi), \mathcal{V}(\psi)) \cap \mathcal{V}^*(K/K^{\mathrm{prm}}) \neq \emptyset$. Let $\mathfrak{p}$ be an element of $\mathrm{Eq}(\mathcal{V}(\phi), \mathcal{V}(\psi)) \cap \mathcal{V}^*(K/K^{\mathrm{prm}})$. Write $F_{\mathfrak{p}}$, $K_{\mathfrak{p}}$ for the completions of F, K at $\mathcal{V}(\phi)(\mathfrak{p}) = \mathcal{V}(\psi)(\mathfrak{p})$, $\mathfrak{p}$, respectively. Write, moreover, $\phi_{\mathfrak{p}}$, $\psi_{\mathfrak{p}}\colon F_{\mathfrak{p}} \hookrightarrow K_{\mathfrak{p}}$ for the homomorphisms of fields induced by ϕ, ψ, respectively. Then since $\mathfrak{p} \in \mathcal{V}^*(K/K^{\mathrm{prm}})$, one verifies easily that $\phi_{\mathfrak{p}} = \psi_{\mathfrak{p}}$. Thus, one may conclude immediately from the various definitions involved that $\phi = \psi$, as desired. This completes the proof of Lemma 1.4. □

DEFINITION 1.5. — We shall say that a field F is *absolutely Galois* if F is Galois over the subfield F^{prm}.

LEMMA 1.6. — *Let F, K, and L be number fields; $\phi_F\colon F \hookrightarrow L$, $\phi_K\colon K \hookrightarrow L$ homomorphisms of fields; $\mathfrak{p}_L$ an element of $\mathcal{V}^*(L/L^{\mathrm{prm}})$; $\iota \in \mathrm{Aut}_{\mathcal{NF}}(L)$ an automorphism of the field L. Write $p \stackrel{\text{def}}{=} \mathrm{char}(\mathfrak{p}_L) \in \mathfrak{Primes}$ for the residue characteristic of $\mathfrak{p}_L$. Suppose that L is* ***absolutely Galois****. Then the following two conditions are equivalent:*

(1) *The composite $F \stackrel{\phi_F}{\hookrightarrow} L \stackrel{\iota}{\xrightarrow{\sim}} L$* ***factors*** *through the homomorphism $\phi_K\colon K \hookrightarrow L$.*

(2) *There exists an element $\mathfrak{p}_F$ of $\mathcal{V}|_{\{p\}}(F)$ such that the image of $\mathcal{V}(\phi_K)^{-1}(\{\mathcal{V}(\phi_K)(\mathfrak{p}_L)\}) \subseteq \mathcal{V}(L)$ by $\mathcal{V}(\iota)$ is* ***contained*** *in $\mathcal{V}(\phi_F)^{-1}(\{\mathfrak{p}_F\}) \subseteq \mathcal{V}(L)$.*

PROOF. — The implication (1) ⇒ (2) follows from the [easily verified] fact that the assignment "$F \rightsquigarrow \mathcal{V}(F)$" naturally determines a contravariant functor. Next, we verify the implication (2) ⇒ (1). Suppose that condition (2) is satisfied. Then it is immediate that $\mathcal{V}(\iota)(\mathfrak{p}_L) \in \mathcal{V}(\phi_F)^{-1}(\{\mathfrak{p}_F\})$. Thus, since L is *absolutely Galois*, condition (2) implies the inclusion

$$\{\, \mathcal{V}(\gamma_K \circ \iota)(\mathfrak{p}_L) \mid \gamma_K \in \mathrm{Gal}(\phi_K) \,\} \subseteq \{\, \mathcal{V}(\iota \circ \gamma_F)(\mathfrak{p}_L) \mid \gamma_F \in \mathrm{Gal}(\phi_F) \,\}$$

— where we write "$\mathrm{Gal}(-)$" for the Galois group of the finite Galois extension determined by "$(-)$". Thus, since $\mathfrak{p}_L \in \mathcal{V}^*(L/L^{\mathrm{prm}})$, it follows from Lemma 1.4 that we have an inclusion $\mathrm{Gal}(\phi_K) \subseteq \iota \circ \mathrm{Gal}(\phi_F) \circ \iota^{-1}$, which thus implies condition (1). This completes the proof of the implication (2) ⇒ (1), hence also of Lemma 1.6. □

DEFINITION 1.7. — Let F be a number field and S a subset of $\mathcal{V}(F)$. Then we shall say that S is *thin* if there exists an infinite sequence $F = F_1 \hookrightarrow F_2 \hookrightarrow \cdots$ of number fields such that, for each positive integer n, the extension F_n of F is Galois, the homomorphism $F_n \hookrightarrow F_{n+1}$ is not an isomorphism, and, moreover, $S \stackrel{\bullet}{\subseteq} \mathcal{V}_*(F_n/F)$.

Lemma 1.8. — *Let F be a number field. Then every subset of the union of finitely many **thin** subsets of $\mathcal{V}(F)$ is **not of positive Dirichlet density**.*

Proof. — Let n be a positive integer; $S_1, \ldots, S_n \subseteq \mathcal{V}(F)$ *thin* subsets of $\mathcal{V}(F)$. Thus, it follows from the definition of a thin subset that, for each $i \in \{1, \ldots, n\}$ and each positive real number δ, there exists a finite Galois extension K_i/F *of degree* $> n/\delta$ such that $S_i \overset{\bullet}{\subseteq} \mathcal{V}_*(K_i/F)$. Now let us observe that it follows immediately from *Chebotarev's density theorem* [cf., e.g., [5, Chapter VII, Theorem 13.4]] [cf. also the easily verified *equality* $\mathcal{V}_*(K/F) \cap \mathcal{V}_*(L/F) = \mathcal{V}_*((K \cdot L)/F)$ for two finite Galois extensions K, L of F in a fixed algebraic closure of F] that the union $\bigcup_{i=1}^n \mathcal{V}_*(K_i/F)$ [*admits a Dirichlet density* and] is *of Dirichlet density* $\leq \sum_{i=1}^n 1/[K_i : F] < \delta$. Thus, one may conclude that every subset of $\bigcup_{i=1}^n S_i$ ($\overset{\bullet}{\subseteq} \bigcup_{i=1}^n \mathcal{V}_*(K_i/F)$) is *not of positive Dirichlet density*, as desired. This completes the proof of Lemma 1.8. $\square$

Theorem 1.9. — *Let $\mathcal{C}$ be a category and*

$$\mathcal{O} \colon \mathcal{NF} \longrightarrow \mathcal{C}$$

a contravariant functor. Suppose that there exist

- *a **cofinite** subset S of $\mathfrak{Primes}$,*
- *a covariant functor*

$$\underline{\mathcal{V}} \colon \mathcal{C} \longrightarrow \mathcal{S}et_S,$$

and

- *a **natural equivalence***

$$\theta \colon \mathcal{V}|_S \xrightarrow{\sim} \underline{\mathcal{V}} \circ \mathcal{O}$$

*that satisfy the following condition: For number fields F, K and morphisms $\alpha, \beta \colon \mathcal{O}(K) \to \mathcal{O}(F)$ in $\mathcal{C}$, it holds that the **equality** $\alpha = \beta$ holds if and only if the equalizer*

$$\mathrm{Eq}\big(\underline{\mathcal{V}}(\alpha), \underline{\mathcal{V}}(\beta)\big) \subseteq \underline{\mathcal{V}}\big(\mathcal{O}(K)\big) \xleftarrow[\sim]{\theta(K)} \mathcal{V}|_S(K)$$

*of the two maps $\underline{\mathcal{V}}(\alpha)$, $\underline{\mathcal{V}}(\beta) \colon \underline{\mathcal{V}}(\mathcal{O}(K)) \to \underline{\mathcal{V}}(\mathcal{O}(F))$ is **not thin** [i.e., as a subset of $\mathcal{V}(K)$]. Then the functor $\mathcal{O}$ is **fully faithful**.*

Proof. — Let F, K be number fields. First, we verify the *faithfulness* of the functor $\mathcal{O}$. Let us observe that we have a sequence of sets

$$\mathrm{Hom}_{\mathcal{NF}}(F, K) \xrightarrow{\mathcal{O}} \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{O}(K), \mathcal{O}(F)\big) \longrightarrow \mathrm{Hom}_{\mathcal{S}et_S}\big(\mathcal{V}|_S(K), \mathcal{V}|_S(F)\big)$$

— where the second arrow is the map of sets determined by $\underline{\mathcal{V}}$, $\theta(K)$, and $\theta(F)$. Thus, since this composite is *injective* [cf. Lemma 1.4], one may conclude that the functor $\mathcal{O}$ is *faithful*, as desired.

Next, we verify the *fullness* of the functor $\mathcal{O}$. Let $\alpha \colon \mathcal{O}(K) \to \mathcal{O}(F)$ be a morphism in $\mathcal{C}$. Now let us observe that it is immediate that there exist a number field L that is *absolutely Galois* and homomorphisms $\phi_F \colon F \hookrightarrow L$, $\phi_K \colon K \hookrightarrow L$ of fields. For each $\iota \in \mathrm{Aut}_{\mathcal{NF}}(L)$, write $E(\iota) \subseteq \mathcal{V}|_S(L)$ for the equalizer of the two maps of sets

$$\mathcal{V}|_S(L) \xrightarrow[\sim]{\mathcal{V}|_S(\iota)} \mathcal{V}|_S(L) \xrightarrow{\mathcal{V}|_S(\phi_F)} \mathcal{V}|_S(F),$$

$$\mathcal{V}|_S(L) \xrightarrow{\mathcal{V}|_S(\phi_K)} \mathcal{V}|_S(K) \xrightarrow[\sim]{\theta(K)} \underline{\mathcal{V}}\big(\mathcal{O}(K)\big) \xrightarrow{\underline{\mathcal{V}}(\alpha)} \underline{\mathcal{V}}\big(\mathcal{O}(F)\big) \xrightarrow[\sim]{\theta(F)^{-1}} \mathcal{V}|_S(F).$$

Now let us observe that since L is *absolutely Galois*, one verifies easily that, for each $p \in \mathcal{V}_*(L/L^{\mathrm{prm}})$, the action of $\mathrm{Aut}_{\mathcal{NF}}(L)$ on $\mathcal{V}(L)$ determines a *transitive* action of $\mathrm{Aut}_{\mathcal{NF}}(L)$ on $\mathcal{V}|_{\{p\}}(L)$. Let us also observe that one verifies easily that, for each $p \in \mathcal{V}_*(L/L^{\mathrm{prm}})$, the map $\mathcal{V}|_{\{p\}}(L) \to \mathcal{V}|_{\{p\}}(F)$ induced by $\mathcal{V}|_S(\phi_F)$ is *surjective*. In particular, one verifies immediately that the *inclusion*

$$\mathcal{V}|_S(L) \cap \mathcal{V}^*(L/L^{\mathrm{prm}}) \subseteq \bigcup_{\iota \in \mathrm{Aut}_{\mathcal{NF}}(L)} E(\iota)$$

holds. Thus, since $\mathrm{Aut}_{\mathcal{NF}}(L)$ is *finite*, it follows from Lemma 1.8, together with the well-known fact that the subset $\mathcal{V}|_S(L) \cap \mathcal{V}^*(L/L^{\mathrm{prm}})$ of $\mathcal{V}(L)$ is *of Dirichlet density one* [cf., e.g., the discussion of [5, p.543] preceding the definition of the natural density], that there exists an automorphism $\iota_0 \in \mathrm{Aut}_{\mathcal{NF}}(L)$ of L such that $E(\iota_0)$ is *not thin*. In particular, it follows from the condition in the statement of Theorem 1.9 that the diagram in $\mathcal{C}$

$$\begin{array}{ccc}
\mathcal{O}(L) & \xrightarrow{\mathcal{O}(\phi_K)} & \mathcal{O}(K) \\
{\scriptstyle \mathcal{O}(\iota_0)}\downarrow\wr & & \downarrow{\scriptstyle \alpha} \\
\mathcal{O}(L) & \xrightarrow[\mathcal{O}(\phi_F)]{} & \mathcal{O}(F),
\end{array}$$

hence also the diagram in $\mathcal{S}et_S$

$$\begin{array}{ccccc}
\mathcal{V}|_S(L) & \xrightarrow{\mathcal{V}|_S(\phi_K)} & \mathcal{V}|_S(K) & \xrightarrow[\sim]{\theta(K)} & \underline{\mathcal{V}}\big(\mathcal{O}(K)\big) \\
{\scriptstyle \mathcal{V}|_S(\iota_0)}\downarrow\wr & & & & \downarrow{\scriptstyle \underline{\mathcal{V}}(\alpha)} \\
\mathcal{V}|_S(L) & \xrightarrow[\mathcal{V}|_S(\phi_F)]{} & \mathcal{V}|_S(F) & \xrightarrow[\theta(F)]{\sim} & \underline{\mathcal{V}}\big(\mathcal{O}(F)\big),
\end{array}$$

commutes. Thus, it follows immediately from Lemma 1.6 that the composite $F \overset{\phi_F}{\hookrightarrow} L \overset{\iota_0}{\overset{\sim}{\to}} L$ *factors* through the homomorphism $\phi_K \colon K \hookrightarrow L$, which thus implies that the diagram in $\mathcal{S}et_S$

$$\begin{array}{ccc}
\mathcal{V}|_S(L) & \xrightarrow{\mathcal{V}|_S(\phi_K)} & \mathcal{V}|_S(K) \\
{\scriptstyle \mathcal{V}|_S(\iota_0)}\downarrow\wr & & \downarrow{\scriptstyle \mathcal{V}|_S(\psi)} \\
\mathcal{V}|_S(L) & \xrightarrow[\mathcal{V}|_S(\phi_F)]{} & \mathcal{V}|_S(F)
\end{array}$$

— where we write $\psi \colon F \hookrightarrow K$ for the resulting homomorphism of fields — *commutes*. In particular, it follows from the *commutativity* of the last two diagrams that the equalizer of the two maps

$$\mathcal{V}|_S(K) \xrightarrow[\sim]{\theta(K)} \underline{\mathcal{V}}\big(\mathcal{O}(K)\big) \xrightarrow{\underline{\mathcal{V}}(\alpha)} \underline{\mathcal{V}}\big(\mathcal{O}(F)\big) \xrightarrow[\sim]{\theta(F)^{-1}} \mathcal{V}|_S(F), \qquad \mathcal{V}|_S(K) \xrightarrow{\mathcal{V}|_S(\psi)} \mathcal{V}|_S(F)$$

contains the image of $\mathcal{V}|_S(\phi_K)$. Now observe that it follows from *Chebotarev's density theorem* [cf., e.g., [5, Chapter VII, Theorem 13.4]] that the image of $\mathcal{V}|_S(\phi_K)$ is *of Dirichlet density* $1/[L : K]$. In particular, it follows from Lemma 1.8 and the condition in the statement of Theorem 1.9 that the *equality* $\alpha = \mathcal{O}(\psi)$ holds. This completes the proof of the *fullness* of the functor $\mathcal{O}$, hence also of Theorem 1.9. $\square$

2. A Fully Faithful Functor

In the present §2, we construct a *fully faithful* functor whose domain is the category of number fields by means of Theorem 1.9 [cf. Theorem 2.6 below].

In the present §2, let l be a prime number and S a *cofinite* subset of $\mathfrak{Primes}$.

Definition 2.1. — Let F be a number field.

(i) We shall write $A_l(F) \stackrel{\text{def}}{=} \mathrm{Gal}(F^{\mathrm{ab}}/F) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/l\mathbb{Z}$ — where F^{ab} is a maximal abelian extension of F.

(ii) Let $\mathfrak{p}$ be an element of $\mathcal{V}(F)$. Then we shall write $\mathrm{FL}_l(\mathfrak{p}) \subseteq A_l(F)$ for the subset consisting of the liftings of the char($\mathfrak{p}$)-th power Frobenius element at $\mathfrak{p}$ — i.e., the subset consisting of the elements of the decomposition subgroup of $A_l(F)$ at $\mathfrak{p}$ whose natural actions on the residue field of the valuation ring in the algebraic extension of the completion of F determined by $(F^{\mathrm{ab}}, \mathfrak{p})$ are given by the char($\mathfrak{p}$)-th power Frobenius maps.

(iii) Let $\mathfrak{p}$ be an element of $\mathcal{V}(F)$. Then we shall write $\Phi_{\mathfrak{p}}$ for the set of continuous [necessarily surjective] homomorphisms $\chi\colon A_l(F) \to \mathbb{Z}/l\mathbb{Z}$ of profinite modules such that, for each element $\mathfrak{q}$ of $\mathcal{V}(F)$ with $\mathrm{char}(\mathfrak{q}) \leq \mathrm{char}(\mathfrak{p})$, the following two conditions are satisfied:

- The image by χ of the inertia subgroup of $A_l(F)$ at $\mathfrak{q}$ is trivial.
- It holds that the image by χ of the decomposition subgroup of $A_l(F)$ at $\mathfrak{q}$ is nontrivial if and only if the equality $\mathfrak{q} = \mathfrak{p}$ holds.

Lemma 2.2. — *Let F be a number field. Then the following assertions hold:*

(i) *Let $\mathfrak{p}$ be an element of $\mathcal{V}(F)$. Then the set $\Phi_{\mathfrak{p}}$ is **nonempty**.*

(ii) *Let $\mathfrak{p}$, $\mathfrak{q}$ be **distinct** elements of $\mathcal{V}(F)$. Then the intersection $\mathrm{FL}_l(\mathfrak{p}) \cap \mathrm{FL}_l(\mathfrak{q})$ is **empty**.*

(iii) *Let $H \subseteq A_l(F)$ be a **finite** closed submodule of $A_l(F)$ [i.e., with respect to the natural profinite topology of $A_l(F)$]. Then the set consisting of the elements $\mathfrak{p}$ of $\mathcal{V}(F)$ that satisfy the following condition is **finite**: For an arbitrary $\chi \in \Phi_{\mathfrak{p}}$, the submodule H is **not annihilated** by χ.*

Proof. — Assertion (i) is an immediate consequence of the *Grunwald-Wang theorem* [cf., e.g., [5, Chapter VI, §6, Exercise 7]]. Assertion (ii) is a formal consequence of assertion (i). Next, we verify assertion (iii). Assume that the set consisting of the elements of $\mathcal{V}(F)$ that satisfy the condition in the statement of assertion (iii) is *infinite*. Thus, we have an *infinite* sequence $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$ of elements of $\mathcal{V}(F)$ such that, for each positive integer n, the element $\mathfrak{p}_n$ satisfies the condition in the statement of assertion (iii), and, moreover, the *inequality* $\mathrm{char}(\mathfrak{p}_n) < \mathrm{char}(\mathfrak{p}_{n+1})$ holds. For each positive integer n, fix an element χ_n of $\Phi_{\mathfrak{p}_n}$ [cf. assertion (i)]. Then one verifies easily from the definition of "$\Phi_{\mathfrak{p}}$" that, for positive integers $m < n$, the product $\chi_m \cdot \chi_n^{-1}$ is *contained* in $\Phi_{\mathfrak{p}_m}$, which thus implies that the submodule H is *not annihilated* by $\chi_m \cdot \chi_n^{-1}$. In particular, one may conclude that, for positive integers $m \neq n$, the restriction $\chi_m|_H$ does *not coincide* with the restriction $\chi_n|_H$. However, this *contradicts* the *finiteness* of H. This completes the proof of assertion (iii), hence also of Lemma 2.2. □

REMARK 2.2.1. — The proof of Lemma 2.2, (iii), is essentially the same as the proof of [7, Lemma 6.15].

DEFINITION 2.3. — We shall write $\mathcal{C}_{A,S}$ for the category defined as follows:

- An object of $\mathcal{C}_{A,S}$ is defined to be a collection $(A, V, (\mathrm{FL}(v))_{v\in V})$ of data consisting of a topological module A, an S-set V, and, for each $v \in V$, a subset $\mathrm{FL}(v) \subseteq A$ of A.

- Let $(A_1, V_1, (\mathrm{FL}(v_1)_1)_{v_1\in V_1})$, $(A_2, V_2, (\mathrm{FL}(v_2)_2)_{v_2\in V_2})$ be objects of $\mathcal{C}_{A,S}$. Then a morphism $(A_1, V_1, (\mathrm{FL}(v_1)_1)_{v_1\in V_1}) \to (A_2, V_2, (\mathrm{FL}(v_2)_2)_{v_2\in V_2})$ in $\mathcal{C}_{A,S}$ is defined to be a pair (α_A, α_V) consisting of a continuous homomorphism $\alpha_A \colon A_1 \to A_2$ of topological modules and a map $\alpha_V \colon V_1 \to V_2$ of S-sets such that, for each $v_1 \in V_1$, the inclusion $\alpha_A(\mathrm{FL}(v_1)_1) \subseteq \mathrm{FL}(\alpha_V(v_1))_2$ holds.

DEFINITION 2.4. — We shall write

$$\mathcal{O}_{A_l,S} \colon \mathcal{NF} \longrightarrow \mathcal{C}_{A,S}$$

for the contravariant functor defined as follows:

- Let F be a number field. Then the object $\mathcal{O}_{A_l,S}(F)$ of $\mathcal{C}_{A,S}$ is defined to be the collection $(A_l(F), \mathcal{V}|_S(F), (\mathrm{FL}_l(\mathfrak{p}))_{\mathfrak{p}\in\mathcal{V}|_S(F)})$ of data.

- Let F, K be number fields; $\phi \colon F \hookrightarrow K$ a homomorphism of fields. Then the morphism $\mathcal{O}_{A_l,S}(\phi) \colon \mathcal{O}_{A_l,S}(K) \to \mathcal{O}_{A_l,S}(F)$ in $\mathcal{C}_{A,S}$ is defined to be the pair consisting of the continuous homomorphism $A_l(K) \to A_l(F)$ and the map $\mathcal{V}|_S(K) \to \mathcal{V}|_S(F)$ induced by the homomorphism $\phi \colon F \hookrightarrow K$.

LEMMA 2.5. — *Let F, K be number fields; (α_A, α_V), $(\beta_A, \beta_V) \colon \mathcal{O}_{A_l,S}(K) \to \mathcal{O}_{A_l,S}(F)$ morphisms in $\mathcal{C}_{A,S}$. Write $H \overset{\mathrm{def}}{=} \mathrm{Im}(\alpha_A \cdot \beta_A^{-1}) \subseteq A_l(F)$ for the closed submodule of $A_l(F)$ obtained by forming the image of the product $\alpha_A \cdot \beta_A^{-1} \colon A_l(K) \to A_l(F)$. Let $H \twoheadrightarrow Q$ be a **finite** topological quotient of H. Write $K_Q \subseteq K^{\mathrm{ab}}$ for the finite Galois extension of K in K^{ab} that corresponds to the finite topological quotient $A_l(K) \twoheadrightarrow (H \twoheadrightarrow)\ Q$. Then $\mathrm{Eq}(\alpha_V, \beta_V) \overset{\bullet}{\subseteq} \mathcal{V}_*(K_Q/K)$.*

PROOF. — Since $H \twoheadrightarrow Q$ is a finite topological quotient of H, it is immediate that there exists an open submodule $U \subseteq A_l(F)$ of $A_l(F)$ such that $H \cap U$ is *contained* in the kernel of the natural continuous surjective homomorphism $H \twoheadrightarrow Q$. In particular, since [it is immediate that] the quotient $A_l(F)/U$ has a natural structure of linear space over $\mathbb{F}_l$ of finite dimension, by considering the composite of the natural continuous surjective homomorphism $A_l(F) \twoheadrightarrow A_l(F)/U$ and a splitting of the natural inclusion $H/(H \cap U) \hookrightarrow A_l(F)/U$, one may conclude that there exist a finite topological quotient $A_l(F) \twoheadrightarrow J$ of $A_l(F)$ and an isomorphism $Q \xrightarrow{\sim} J$ of modules that fits into the following commutative diagram

$$\begin{array}{ccc} H & \hookrightarrow & A_l(F) \\ \downarrow & & \downarrow \\ Q & \xrightarrow[\sim]{} & J \end{array}$$

— where the upper horizontal arrow is the natural inclusion, and the vertical arrows are the natural continuous surjective homomorphisms. Write $F_J \subseteq F^{\mathrm{ab}}$ for the finite Galois extension of F in F^{ab} that corresponds to the finite topological quotient $A_l(F) \twoheadrightarrow J$. Next, observe that it follows from the various definitions involved that, for each $\mathfrak{p} \in \mathcal{V}(K)$, if the nonarchimedean prime $\alpha_V(\mathfrak{p})$ (respectively, $\beta_V(\mathfrak{p})$) of F is *unramified* in the finite Galois extension F_J/F, then the image in J via α_A (respectively, β_A) of $\mathrm{FL}_l(\mathfrak{p}) \subseteq A_l(K)$ *coincides* with the subset [necessarily *of cardinality one*] consisting of the unique lifting of the $\mathrm{char}(\mathfrak{p})$-th power Frobenius element in $J = \mathrm{Gal}(F_J/F)$ at $\alpha_V(\mathfrak{p})$ (respectively, $\beta_V(\mathfrak{p})$). In particular, since [it is well-known that] the respective sets of nonarchimedean primes of F, K *ramified* in the finite Galois extensions F_J/F, K_Q/K are *finite*, there exists a *cofinite* subset T of S such that, for each $\mathfrak{p} \in \mathcal{V}|_T(K) \cap \mathrm{Eq}(\alpha_V, \beta_V)$, the image in J via the product $\alpha_A \cdot \beta_A^{-1}$ of $\mathrm{FL}_l(\mathfrak{p}) \subseteq A_l(K)$ is *contained in the trivial submodule* of J, and, moreover, the nonarchimedean prime $\mathfrak{p}$ is *unramified* in the finite Galois extension K_Q/K. Thus, we conclude that the *inclusion* $\mathcal{V}|_T(K) \cap \mathrm{Eq}(\alpha_V, \beta_V) \subseteq \mathcal{V}_*(K_Q/K)$, hence also $\mathrm{Eq}(\alpha_V, \beta_V) \overset{\bullet}{\subseteq} \mathcal{V}_*(K_Q/K)$, holds, as desired. This completes the proof of Lemma 2.5. □

Theorem 2.6. — *The functor $\mathcal{O}_{A_l,S}$ is **fully faithful**.*

Proof. — Let us first observe that it follows from Theorem 1.9 that, to verify Theorem 2.6, it suffices to verify the following assertion: In the situation of Lemma 2.5, if $(\alpha_A, \alpha_V) \neq (\beta_A, \beta_V)$, then the equalizer $\mathrm{Eq}(\alpha_V, \beta_V) \subseteq \mathcal{V}|_S(K)$ is *thin*. To this end, suppose that we are in the situation of Lemma 2.5, and that $(\alpha_A, \alpha_V) \neq (\beta_A, \beta_V)$.

First, we consider the case where H is *infinite*. Then it is immediate that we have an infinite sequence of finite topological quotients of H

$$H \twoheadrightarrow \ldots \twoheadrightarrow H_{n+1} \twoheadrightarrow H_n \twoheadrightarrow \ldots \twoheadrightarrow H_2 \twoheadrightarrow H_1$$

such that, for each positive integer n, the module H_n is *of order l^n*. For each positive integer n, write $K_n \subseteq K^{\mathrm{ab}}$ for the finite Galois extension of K in K^{ab} that corresponds to the finite topological quotient $A_l(K) \twoheadrightarrow H_n$. Then it follows from Lemma 2.5 that, for each positive integer n, $\mathrm{Eq}(\alpha_V, \beta_V) \overset{\bullet}{\subseteq} \mathcal{V}_*(K_n/K)$. Thus, one may conclude that the subset $\mathrm{Eq}(\alpha_V, \beta_V) \subseteq \mathcal{V}|_S(K)$ is *thin*, as desired.

Next, we consider the case where H is *finite*. Write $K_H \subseteq K^{\mathrm{ab}}$ for the finite Galois extension of K in K^{ab} that corresponds to the finite topological quotient $A_l(K) \twoheadrightarrow H$. Then it follows from Lemma 2.2, (iii), that there exists a *cofinite* subset T of S such that, for each $\mathfrak{p} \in \mathcal{V}|_T(F)$, the set $\Phi_{\mathfrak{p}}$ has an element $\chi_{\mathfrak{p}}$ that *annihilates* the submodule H. Let $\mathfrak{q}$ be an element of $\mathcal{V}|_T(K)$ and $\chi_{\alpha_V(\mathfrak{q})}$ an element of $\Phi_{\alpha_V(\mathfrak{q})}$ that *annihilates* the submodule H. Now observe that it follows immediately from the various definitions involved that

- the image of the subset $\mathrm{FL}_l(\mathfrak{q}) \subseteq A_l(K)$ by the composite of $\alpha_A \colon A_l(K) \to A_l(F)$ and $\chi_{\alpha_V(\mathfrak{q})} \colon A_l(F) \to \mathbb{Z}/l\mathbb{Z}$ is *nontrivial*, and
- it holds that the image of the subset $\mathrm{FL}_l(\mathfrak{q}) \subseteq A_l(K)$ by the composite of $\beta_A \colon A_l(K) \to A_l(F)$ and $\chi_{\alpha_V(\mathfrak{q})} \colon A_l(F) \to \mathbb{Z}/l\mathbb{Z}$ is *nontrivial* if and only if the *equality* $\alpha_V(\mathfrak{q}) = \beta_V(\mathfrak{q})$ holds.

Thus, one may conclude immediately, by considering the composite

$$A_l(K) \xrightarrow{\alpha_A \cdot \beta_A^{-1}} A_l(F) \xrightarrow{\chi_{\alpha_V(\mathfrak{q})}} \mathbb{Z}/l\mathbb{Z}$$

— whose image is *trivial* [cf. our choice of $\chi_{\alpha_V(\mathfrak{q})}$] — that the *inclusion* $\mathcal{V}|_T(K) \subseteq \mathrm{Eq}(\alpha_V, \beta_V)$ holds. Thus, since [it is well-known that] $\mathcal{V}|_T(K)$ is *of Dirichlet density one*, and $\mathrm{Eq}(\alpha_V, \beta_V) \overset{\bullet}{\subseteq} \mathcal{V}_*(K_H/K)$ [cf. Lemma 2.5], it follows from *Chebotarev's density theorem* [cf., e.g., [5, Chapter VII, Theorem 13.4]] that $K = K_H$, i.e., that $\alpha_A = \beta_A$. On the other hand, it follows from Lemma 2.2, (ii), that this equality $\alpha_A = \beta_A$ implies the equality $\alpha_V = \beta_V$, which *contradicts* our assumption that $(\alpha_A, \alpha_V) \neq (\beta_A, \beta_V)$. This completes the proof of Theorem 2.6. $\square$

Definition 2.7. — For $\square \in \{\circ, \bullet\}$, let $F_\square$ be a number field and $F_\square^{\mathrm{ab}}$ a maximal abelian extension of $F_\square$. Let

$$\alpha\colon \mathrm{Gal}(F_\circ^{\mathrm{ab}}/F_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/l\mathbb{Z} \longrightarrow \mathrm{Gal}(F_\bullet^{\mathrm{ab}}/F_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/l\mathbb{Z}$$

be a continuous homomorphism of profinite modules. Then we shall say that α is *Frobenius-preserving* if there exist a cofinite subset T of $\mathfrak{Primes}$ and a map $\alpha_V\colon \mathcal{V}|_T(F_\circ) \to \mathcal{V}|_T(F_\bullet)$ of sets over T such that, for each $\mathfrak{p} \in \mathcal{V}|_T(F_\circ)$, the inclusion $\alpha(\mathrm{FL}_l(\mathfrak{p})) \subseteq \mathrm{FL}_l(\alpha_V(\mathfrak{p}))$ holds.

Corollary 2.8. — *For $\square \in \{\circ, \bullet\}$, let $F_\square$ be a number field and $F_\square^{\mathrm{ab}}$ a maximal abelian extension of $F_\square$. Let*

$$\alpha\colon \mathrm{Gal}(F_\circ^{\mathrm{ab}}/F_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/l\mathbb{Z} \longrightarrow \mathrm{Gal}(F_\bullet^{\mathrm{ab}}/F_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/l\mathbb{Z}$$

be a continuous homomorphism of profinite modules. Suppose that α is ***Frobenius-preserving****. Then there exists a* ***uniquely determined*** *homomorphism $F_\bullet \hookrightarrow F_\circ$ of fields from which α* ***arises****.*

Proof. — This assertion is a formal consequence of Theorem 2.6 [cf. also Lemma 2.2, (ii)]. $\square$

3. Homomorphisms Between Global Solvably Closed Galois Groups

In the present §3, we study the conjecture by Uchida stated in Introduction of the present paper.

Lemma 3.1. — *Let F be a number field and $\mathfrak{p}$ a nonarchimedean prime of F. Write $p \overset{\mathrm{def}}{=} \mathrm{char}(\mathfrak{p}) \in \mathfrak{Primes}$ for the residue characteristic of $\mathfrak{p}$ and k for the completion of F at $\mathfrak{p}$. Let $\overline{k}$ be an algebraic closure of k. Write*

$$\rho_k^{\mathrm{cycl}}\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow \widehat{\mathbb{Z}}^\times$$

for the ***cyclotomic character*** *on $\mathrm{Gal}(\overline{k}/k)$. Then the following assertions hold:*

(i) *Suppose that $\mathfrak{p}$ is* ***of absolute degree one****. Then the image by ρ_k^{cycl} of every pro-2 Sylow subgroup of the inertia subgroup of $\mathrm{Gal}(\overline{k}/k)$ is* ***nontrivial****.*

(ii) *Let l be a prime number. Then it holds that the* ***equality*** *$l = p$ holds if and only if the image of ρ_k^{cycl} contains a closed subgroup isomorphic to the* ***direct product of two copies of*** *$\mathbb{Z}_l$.*

(iii) *It holds that* $\mathfrak{p}$ *is* ***of absolute residue degree one*** *if and only if the image of the composite*

$$\mathrm{Gal}(\overline{k}/k) \xrightarrow{\rho_k^{\mathrm{cycl}}} \widehat{\mathbb{Z}}^\times = \prod_{l \in \mathfrak{Primes}} \mathbb{Z}_l^\times \twoheadrightarrow \prod_{l \in \mathfrak{Primes}\setminus\{p\}} \mathbb{Z}_l^\times$$

— where the second arrow is the natural continuous surjective homomorphism — ***contains*** *the image of* $p \in \mathbb{Z} \subseteq \widehat{\mathbb{Z}}$.

(iv) *Suppose that* p *is* ***odd****, and that* $\mathfrak{p}$ *is* ***of absolute residue degree one****. Write* $k^{\mathrm{ab}} \subseteq \overline{k}$ *for the maximal abelian extension of* k *in* $\overline{k}$*. Then the subset of* $\mathrm{Gal}(k^{\mathrm{ab}}/k) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ *consisting of the liftings of the* p*-th power Frobenius element* ***coincides*** *with the subset of* $\mathrm{Gal}(k^{\mathrm{ab}}/k) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ *consisting of the elements whose images by the homomorphism induced by* ρ_k^{cycl}

$$\mathrm{Gal}(k^{\mathrm{ab}}/k) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \Big(\prod_{l \in \mathfrak{Primes}\setminus\{p\}} \mathbb{Z}_l^\times \Big) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$$

are ***nontrivial****.*

Proof. — Let us first recall that it is well-known [cf., e.g., [5, Chapter V, §1]] that if $\mathfrak{p}$ is *of absolute degree one*, then there exists a commutative diagram of profinite modules

$$\begin{array}{ccc} \mathrm{Gal}(k^{\mathrm{ab}}/k) & \longrightarrow & \widehat{\mathbb{Z}}^\times \\ \wr\downarrow & & \downarrow\wr \\ \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}} & \hookrightarrow & \mathbb{Z}_p^\times \times \Big(\prod_{l \in \mathfrak{Primes}\setminus\{p\}} \mathbb{Z}_l^\times \Big) \end{array}$$

— where the upper horizontal arrow is the continuous homomorphism induced by ρ_k^{cycl}, the lower horizontal arrow is the continuous *injective* homomorphism given by "$(a, n) \mapsto (a, p^n)$", and the vertical arrows are continuous *isomorphisms* — such that the image in the left-hand lower profinite module $\mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$ of the inertia subgroup of $\mathrm{Gal}(\overline{k}/k)$ is given by $\mathbb{Z}_p^\times \times \{0\} \subseteq \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$. Thus, assertions (i), (ii), (iii) are immediate. Moreover, assertion (iv) follows immediately from the well-known fact that there exists a prime number l such that p is *not a quadratic residue modulo* l. This completes the proof of Lemma 3.1. $\square$

Definition 3.2. — We shall say that a field F is *solvably closed* if there is no nontrivial abelian extension of F.

Lemma 3.3. — *For* $\square \in \{\circ, \bullet\}$*, let* $F_\square$ *be a number field and* $\widetilde{F}_\square$ *a* ***Galois*** *extension of* $F_\square$ *that is* ***solvably closed****; write*

$$\rho_\square^{\mathrm{cycl}} \colon \mathrm{Gal}(\widetilde{F}_\square/F_\square) \longrightarrow \widehat{\mathbb{Z}}^\times$$

for the ***cyclotomic character*** *on* $\mathrm{Gal}(\widetilde{F}_\square/F_\square)$*. Let*

$$\alpha \colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous homomorphism of profinite groups. Suppose that the following two conditions are satisfied:

(1) *The* ***equality***

$$\rho_\circ^{\mathrm{cycl}} = \rho_\bullet^{\mathrm{cycl}} \circ \alpha$$

holds:

$$\begin{array}{ccc} \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) & \xrightarrow{\ \alpha\ } & \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \\ & {\scriptstyle \rho_\circ^{\mathrm{cycl}}} \searrow \quad \swarrow {\scriptstyle \rho_\bullet^{\mathrm{cycl}}} & \\ & \widehat{\mathbb{Z}}^\times & \end{array}$$

(2) *The number field $F_\bullet$ is* ***totally imaginary****.*

Then the following assertions hold:

(i) *Let $\mathfrak{p}_\circ$ be an element of $\mathcal{V}(F_\circ)$ and $D_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ a decomposition subgroup of $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ at $\mathfrak{p}_\circ$. Suppose that $\mathfrak{p}_\circ$ is* ***of odd residue characteristic*** *and* ***of absolute degree one****. Then there exist a* ***unique*** *nonarchimedean prime $\mathfrak{p}_\bullet$ of $F_\bullet$ and a* ***unique*** *decomposition subgroup $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ at $\mathfrak{p}_\bullet$ that satisfy the following three conditions:*

(a) *The image of $D_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ by α is* ***contained*** *in $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$.*

(b) *The* ***equality*** $\mathrm{char}(\mathfrak{p}_\circ) = \mathrm{char}(\mathfrak{p}_\bullet)$ *holds.*

(c) *The nonarchimedean prime $\mathfrak{p}_\bullet$ is* ***of absolute residue degree one****, i.e., is* ***contained*** *in $\mathcal{V}(F_\bullet)$.*

(ii) *For $\Box \in \{\circ, \bullet\}$, write $F_\Box^{\mathrm{ab}} \subseteq \widetilde{F}_\Box$ for the maximal abelian extension of $F_\Box$ in $\widetilde{F}_\Box$. Then the continuous homomorphism of profinite modules induced by α*

$$\mathrm{Gal}(F_\circ^{\mathrm{ab}}/F_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Gal}(F_\bullet^{\mathrm{ab}}/F_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$$

is ***Frobenius-preserving****.*

Proof. — First, we verify assertion (i). Let $T_\circ \subseteq D_\circ$ be a pro-2 Sylow subgroup of $D_\circ$. Write $I_\circ \subseteq D_\circ$ for the inertia subgroup of $D_\circ$ and $T_\circ^I \stackrel{\mathrm{def}}{=} I_\circ \cap T_\circ \subseteq I_\circ$ for the pro-2 Sylow subgroup of $I_\circ$ determined by the pro-2 Sylow subgroup $T_\circ$ [cf., e.g., the well-known structure of $T_\circ$ explained in [9, p.596]]. Then [since the profinite group $D_\circ$ may be naturally identified with the Galois group of a suitable Galois extension of the completion of $F_\circ$ at $\mathfrak{p}_\circ$] it follows from Lemma 3.1, (i), that the image by $\rho_\circ^{\mathrm{cycl}}$ of $T_\circ^I \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ is *nontrivial.* Thus, it follows from condition (1) that the image by α of $T_\circ^I \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ is *nontrivial.* Now observe that since [we have assumed that] $F_\bullet$ is *totally imaginary* [cf. condition (2)], the profinite group $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ has *no element of order two* [cf., e.g., the argument given in [9, pp.596-597]]. Thus, since [it is well-known that] $T_\circ^I$ is *isomorphic*, as an abstract profinite group, to $\mathbb{Z}_2$, it follows from the *nontriviality* of the image by α of $T_\circ^I \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ already verified above that the restriction of α to $T_\circ^I \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ is *injective.* In particular, it follows immediately from the well-known structure of $T_\circ$ [cf., e.g., the classification of the topological quotients of $T_\circ$ given in [9, p.596]] that the restriction of α to $T_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ is *injective.* Thus, it follows from the argument given in [9, pp.595-596] that there exist a *unique* nonarchimedean prime $\mathfrak{p}_\bullet$ of $F_\bullet$ and a *unique* decomposition subgroup $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ at $\mathfrak{p}_\bullet$ that satisfy condition (a). In particular, [since the profinite group $D_\bullet$ may be naturally identified with the

Galois group of a suitable Galois extension of the completion of $F_\bullet$ at $\mathfrak{p}_\bullet$] it follows from Lemma 3.1, (ii), together with conditions (1), (a), that condition (b) is satisfied. Thus, since [we have assumed that] $\mathfrak{p}_\circ$ is *of absolute degree one* [hence also *of absolute residue degree one*], it follows from Lemma 3.1, (iii), together with conditions (1), (a), (b), that condition (c) is satisfied. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that one verifies easily from assertion (i), together with the various definitions involved [cf. also the well-known *finiteness* of the set of nonarchimedean primes of $F_\circ^{\mathrm{prm}}$ *ramified* in the finite extension $F_\circ/F_\circ^{\mathrm{prm}}$], that, to verify assertion (ii), it suffices to verify the following assertion: In the situation of assertion (i), the image by the composite

$$D_\circ \hookrightarrow \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \xrightarrow{\alpha} \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \twoheadrightarrow \mathrm{Gal}(F_\bullet^{\mathrm{ab}}/F_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$$

— where the first arrow is the natural inclusion, and the third arrow is the natural continuous surjective homomorphism — of a lifting of the char($\mathfrak{p}_\circ$)-th power Frobenius element in $D_\circ$ is a lifting of the char($\mathfrak{p}_\bullet$)-th power Frobenius element in $\mathrm{Gal}(F_\bullet^{\mathrm{ab}}/F_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ at $\mathfrak{p}_\bullet$. On the other hand, this assertion follows immediately from Lemma 3.1, (iv), and assertion (i), together with condition (1). This completes the proof of assertion (ii), hence also of Lemma 3.3. □

THEOREM 3.4. — *For $\square \in \{\circ, \bullet\}$, let $F_\square$ be a* ***number field*** *and $\widetilde{F}_\square$ a* ***Galois*** *extension of $F_\square$ that is* ***solvably closed****. Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous ***open*** *homomorphism of profinite groups. Then the following two conditions are equivalent:*

(1) *There exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of fields* ***compatible*** *with the respective actions of* $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ *relative to the continuous open homomorphism α.*

(2) *For each $\square \in \{\circ, \bullet\}$, write $\rho_\square^{\mathrm{cycl}}\colon \mathrm{Gal}(\widetilde{F}_\square/F_\square) \to \widehat{\mathbb{Z}}^\times$ for the* ***cyclotomic character*** *on* $\mathrm{Gal}(\widetilde{F}_\square/F_\square)$. *Then the* ***equality***

$$\rho_\circ^{\mathrm{cycl}} = \rho_\bullet^{\mathrm{cycl}} \circ \alpha$$

holds.

PROOF. — The implication (1) ⇒ (2) is immediate. Next, we verify the implication (2) ⇒ (1). Suppose that condition (2) is satisfied. Now let us observe that since [we have assumed that] the continuous homomorphism α is *open*, to verify condition (1), we may assume without loss of generality, by replacing $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ by the image of α, that α is *surjective*.

Let $K_\bullet \subseteq \widetilde{F}_\bullet$ be a finite Galois extension of $F_\bullet$ contained in $\widetilde{F}_\bullet$ that is *totally imaginary*. Write $K_\circ \subseteq \widetilde{F}_\circ$ for the finite Galois extension of $F_\circ$ contained in $\widetilde{F}_\circ$ that corresponds to the normal open subgroup of $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ obtained by forming the inverse image by the continuous *surjective* homomorphism α of $\mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet) \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$. Thus, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\circ/K_\circ) & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) & \longrightarrow & \mathrm{Gal}(K_\circ/F_\circ) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \alpha & & \wr\downarrow \alpha_K & & \\ 1 & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet) & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) & \longrightarrow & \mathrm{Gal}(K_\bullet/F_\bullet) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, the vertical arrows are *surjective*, and the right-hand vertical arrow is an *isomorphism*. Now, for each $\Box \in \{\circ, \bullet\}$, write $K_\Box^{\mathrm{ab}} \subseteq \widetilde{F}_\Box$ for the maximal abelian extension of $K_\Box$ in $\widetilde{F}_\Box$. Write, moreover,

$$\mathrm{Aut}^*\big(\mathrm{Gal}(K_\circ^{\mathrm{ab}}/K_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \subseteq \mathrm{Aut}\big(\mathrm{Gal}(K_\circ^{\mathrm{ab}}/K_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big)$$

for the subgroup consisting of the continuous automorphisms of $\mathrm{Gal}(K_\circ^{\mathrm{ab}}/K_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ that preserve the kernel of the continuous surjective homomorphism $\mathrm{Gal}(K_\circ^{\mathrm{ab}}/K_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \mathrm{Gal}(K_\bullet^{\mathrm{ab}}/K_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ induced by the left-hand vertical arrow of the above diagram. Then the above diagram determines a *commutative* diagram of groups

$$\begin{array}{ccc} \mathrm{Gal}(K_\circ/F_\circ) & \longrightarrow & \mathrm{Aut}^*\big(\mathrm{Gal}(K_\circ^{\mathrm{ab}}/K_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \\ \alpha_K \downarrow \wr & & \downarrow \\ \mathrm{Gal}(K_\bullet/F_\bullet) & \longrightarrow & \mathrm{Aut}\big(\mathrm{Gal}(K_\bullet^{\mathrm{ab}}/K_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \end{array}$$

— where the horizontal arrows are the respective natural continuous actions, and the right-hand vertical arrow is the homomorphism induced by the left-hand vertical arrow of the above diagram. In particular, since [it is immediate that] the left-hand vertical arrow $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ) \twoheadrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)$ of the above diagram satisfies a similar condition to condition (2), one may conclude immediately from Corollary 2.8 and Lemma 3.3, (ii), together with the *commutativity* of this diagram, that the isomorphism $\alpha_K \colon \mathrm{Gal}(K_\circ/F_\circ) \xrightarrow{\sim} \mathrm{Gal}(K_\bullet/F_\bullet)$ *arises* from a *uniquely determined* homomorphism $K_\bullet \hookrightarrow K_\circ$ of fields. Thus, by allowing "$K_\bullet$" to vary, it follows that the continuous open homomorphism α *arises* from a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of fields, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Theorem 3.4. $\Box$

REMARK 3.4.1.

(i) Let p be a prime number. For $\Box \in \{\circ, \bullet\}$, let $k_\Box$ be a *p-adic local field* [i.e., a field isomorphic to a finite extension of $\mathbb{Q}_p$] and $\overline{k}_\Box$ an algebraic closure of $k_\Box$. Let

$$\alpha \colon \mathrm{Gal}(\overline{k}_\circ/k_\circ) \longrightarrow \mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$$

be a continuous *open* homomorphism of profinite groups. Then it follows from [1, Corollary 3.4] that the following three conditions are equivalent:

(1) There exists an isomorphism $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields *compatible* with the respective actions of $\mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$, $\mathrm{Gal}(\overline{k}_\circ/k_\circ)$ relative to the continuous open homomorphism α.

(2) Let n be a positive integer and $\rho \colon \mathrm{Gal}(\overline{k}_\bullet/k_\bullet) \to \mathrm{GL}_n(\mathbb{Q}_p)$ a *Hodge-Tate* representation of $\mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$. Then the representation of $\mathrm{Gal}(\overline{k}_\circ/k_\circ)$ obtained by forming the composite $\mathrm{Gal}(\overline{k}_\circ/k_\circ) \xrightarrow{\alpha} \mathrm{Gal}(\overline{k}_\bullet/k_\bullet) \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{Q}_p)$ is *Hodge-Tate*.

(3) The continuous open homomorphism α is *of HT-qLT-type*, i.e., roughly speaking, for each open subgroups $H_\circ \subseteq \mathrm{Gal}(\overline{k}_\circ/k_\circ)$, $H_\bullet \subseteq \mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$ such that $\alpha(H_\circ) \subseteq H_\bullet$ and each character χ on $H_\bullet$, if χ satisfies a certain condition, then the character $\chi \circ \alpha$ on $H_\circ$ is Hodge-Tate [cf. [1, Definition 1.3, (ii)]].

Moreover, let us observe that a key step in the proof of the implication (3) $\Rightarrow$ (1) is to prove that α is "*compatible*" with various open subgroups of $\mathrm{Gal}(\overline{k}_\circ/k_\circ)$, $\mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$ that correspond to finite extensions of $k_\circ$, $k_\bullet$ which are *Galois over the minimal p-adic local fields* contained in $k_\circ$, $k_\bullet$, respectively [cf. the proof of [1, Theorem 3.3]; also [4, Remark 1.4.1]].

(ii) Let us also recall that a key step in the proof of the implication (2) $\Rightarrow$ (1) of Theorem 3.4 of the present paper is to prove that α is "*compatible*" with various decomposition subgroups of $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ at nonarchimedean primes of $F_\circ$, $F_\bullet$, respectively [cf. the proof of Lemma 3.3, (i), (ii)].

(iii) By the discussions of (i) and (ii), the proof of the implication (3) $\Rightarrow$ (1) of (i) and the proof of the implication (2) $\Rightarrow$ (1) of Theorem 3.4 may be summarized that

> the *compatibility* with suitable *characters* that arise from arithmetic of the fields under consideration implies the *compatibility* with suitable *closed subgroups* that arise from arithmetic of the fields under consideration.

From this point of view, Theorem 3.4 may be regarded as an *analogue for number fields* of the equivalences of (i) for p-adic local fields.

Finally, we give an interpretation of Theorem 3.4 from the point of view of *mono-anabelian reconstruction algorithms established in* [2], [3] as follows.

Corollary 3.5. — *Let $G_\circ$, $G_\bullet$ be profinite groups **of GSC-type** [cf.* [2, Definition 3.2]*]; $\alpha\colon G_\circ \to G_\bullet$ a continuous **open** homomorphism. Then the following two conditions are equivalent:*

- *There exists a homomorphism $\widetilde{F}(G_\bullet) \hookrightarrow \widetilde{F}(G_\circ)$ of fields [cf.* [3, Definition 3.7], [3, Theorem 3.8, (i)]*] **compatible** with the respective actions of $G_\bullet$, $G_\circ$ relative to the continuous open homomorphism α.*

- *The topological $G_\circ$-module obtained by forming the **cyclotome** associated to $G_\circ$* [2, Proposition 3.7, (4)] *is **isomorphic** to the topological $G_\circ$-module obtained by regarding the **cyclotome** associated to $G_\bullet$ as a topological $G_\circ$-module by α.*

Proof. — This assertion is a formal consequence of [2, Proposition 3.7, (iii)], [3, Theorem 3.8, (i), (ii), (iii)], and Theorem 3.4. □

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