

RIMS-1960

**On Intrinsic Hodge-Tate-ness of
Galois Representations of Dimension Two**

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April 2022



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ABSTRACT. — In the present paper, we first prove that, for an arbitrary reducible Hodge-Tate p -adic representation of dimension two of the absolute Galois group of a p -adic local field and an arbitrary continuous automorphism of the absolute Galois group, the p -adic Galois representation obtained by pulling back the given p -adic Galois representation by the given continuous automorphism is Hodge-Tate. Next, we also prove the existence of an irreducible Hodge-Tate p -adic representation of dimension two of the absolute Galois group of a p -adic local field and a continuous automorphism of the absolute Galois group such that the p -adic Galois representation obtained by pulling back the given p -adic Galois representation by the given continuous automorphism is not Hodge-Tate.

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INTRODUCTION

In the present paper, we study the *intrinsic Hodge-Tate-ness* of p -adic representations of the absolute Galois group of a p -adic local field. In the present Introduction, let p be a prime number, k a finite extension of \mathbb{Q}_p , and \bar{k} an algebraic closure of k . Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$ for the absolute Galois group of k determined by the algebraic closure \bar{k} . For a given \mathbb{Q}_p -vector space V of finite dimension and a given continuous representation $\rho: G_k \rightarrow \text{Aut}_{\mathbb{Q}_p}(V)$ of G_k , we shall say that ρ is *Aut-intrinsically Hodge-Tate* if, for an arbitrary continuous automorphism α of G_k , the composite $\rho \circ \alpha: G_k \xrightarrow{\sim} G_k \rightarrow \text{Aut}_{\mathbb{Q}_p}(V)$ is Hodge-Tate [cf. Definition 1.3].

Let us first recall that the author of the present paper proved that

if p is *odd*, and $k = \mathbb{Q}_p$, then there exists a p -adic representation of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate* [cf. [1, Remark 3.3.1]].

Moreover, in the present paper, we establish a refinement of this result. That is to say, we verify that

2020 MATHEMATICS SUBJECT CLASSIFICATION. — 11S20.

KEY WORDS AND PHRASES. — p -adic local field, p -adic Galois representation, Hodge-Tate, intrinsically Hodge-Tate, Aut-intrinsically Hodge-Tate, group of MLF-type.

there exists a p -adic representation of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate* whenever p is *odd*, i.e., without the assumption that $k = \mathbb{Q}_p$ [cf. Corollary 1.5].

On the other hand, let us also observe that it is likely to be well-known that

an arbitrary *Hodge-Tate* p -adic representation of *dimension* 1 of G_k is *Aut-intrinsically Hodge-Tate* [cf. Theorem 2.7].

In this state of affairs, one may have the following question:

Is there a p -adic representation of *dimension* 2 of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate*?

In the present paper, we give an answer to this question.

First, we consider the case where a given continuous representation is *reducible*. The first main result of the present paper is as follows [cf. Theorem 2.10]:

THEOREM A. — *Let V be a \mathbb{Q}_p -vector space of dimension 2 and $\rho: G_k \rightarrow \text{Aut}_{\mathbb{Q}_p}(V)$ a continuous representation. Suppose that the continuous representation ρ is **reducible**. Then ρ is **Hodge-Tate** if and only if ρ is **Aut-intrinsically Hodge-Tate**.*

Next, we consider the case where a given continuous representation is *irreducible*. The second main result of the present paper is as follows [cf. Corollary 3.4]:

THEOREM B. — *Let p be an **odd** prime number. Then there exist a finite extension K of \mathbb{Q}_p , an algebraic closure \overline{K} of K , a \mathbb{Q}_p -vector space V of **dimension 2**, and a continuous representation $\rho: \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}_{\mathbb{Q}_p}(V)$ that is **irreducible, abelian, crystalline** [hence also **Hodge-Tate**], but **not Aut-intrinsically Hodge-Tate**.*

ACKNOWLEDGMENTS

This research was supported by JSPS KAKENHI Grant Number 21K03162 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1. AUT-INTRINSIC HODGE-TATE-NESS OF REPRESENTATIONS

In the present §1, we introduce the notion of *Aut-intrinsic Hodge-Tate-ness* of p -adic representations [cf. Definition 1.3 below]. Moreover, we prove the existence of a p -adic representation that is *potentially crystalline* [hence also *Hodge-Tate*] but *not Aut-intrinsically Hodge-Tate* [cf. Corollary 1.5 below]. Finally, we also recall some basic facts concerning *abelian Hodge-Tate* p -adic representations [cf. Lemma 1.8 below and Lemma 1.9 below].

DEFINITION 1.1. — We shall refer to a field isomorphic to a finite extension of \mathbb{Q}_p , for some prime number p , as an *MLF*. Here, “MLF” is to be understood as an abbreviation for “mixed-characteristic local field”.

In the remainder of the present §1, let k be an MLF and \bar{k} an algebraic closure of k . Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$.

DEFINITION 1.2. — We shall write

- $k^{(d=1)} \subseteq k$ for the [unique] minimal MLF contained in k [i.e., the unique subfield of k isomorphic to \mathbb{Q}_p , for some prime number p],
- $\mathcal{O}_k \subseteq k$ for the ring of integers of k ,
- p_k for the characteristic of the residue field of \mathcal{O}_k ,
- d_k for the extension degree of the [necessarily finite] extension $k/k^{(d=1)}$,
- $(k^\times)^\wedge$ for the profinite completion of the multiplicative module k^\times of k ,
- G_k^{ab} for the topological abelianization of G_k , i.e., the quotient of G_k by the closure of the commutator subgroup of G_k , and
- $\text{rec}_k: (k^\times)^\wedge \xrightarrow{\sim} G_k^{\text{ab}}$ for the isomorphism induced by the reciprocity homomorphism $k^\times \hookrightarrow G_k^{\text{ab}}$ in local class field theory.

DEFINITION 1.3. — Let V be a \mathbb{Q}_{p_k} -vector space of finite dimension and $\rho: G_k \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then we shall say that ρ is *Aut-intrinsically Hodge-Tate* if, for an arbitrary continuous automorphism α of G_k , the composite $\rho \circ \alpha: G_k \xrightarrow{\sim} G_k \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ is Hodge-Tate.

The following result is a formal consequence of the main result of [1].

THEOREM 1.4. — For each $\square \in \{\circ, \bullet\}$, let k_\square be an MLF and \bar{k}_\square an algebraic closure of k_\square . Let $\alpha: \text{Gal}(\bar{k}_\circ/k_\circ) \rightarrow \text{Gal}(\bar{k}_\bullet/k_\bullet)$ be an **open** continuous homomorphism [which thus implies that $p_{k_\circ} = p_{k_\bullet}$ — cf., e.g., [3, Proposition 3.4, (iii)] and [4, Proposition 3.6]]. Then the following two conditions are equivalent:

(1) There exists an isomorphism $\bar{k}_\bullet \xrightarrow{\sim} \bar{k}_\circ$ of fields that is **compatible** with the respective natural actions of $\text{Gal}(\bar{k}_\bullet/k_\bullet)$, $\text{Gal}(\bar{k}_\circ/k_\circ)$ on \bar{k}_\bullet , \bar{k}_\circ relative to the given open continuous homomorphism $\alpha: \text{Gal}(\bar{k}_\circ/k_\circ) \rightarrow \text{Gal}(\bar{k}_\bullet/k_\bullet)$.

(2) For an arbitrary $\mathbb{Q}_{p_{k_\bullet}}$ -vector space V_\bullet of finite dimension and an arbitrary continuous representation $\rho_\bullet: \text{Gal}(\bar{k}_\bullet/k_\bullet) \rightarrow \text{Aut}_{\mathbb{Q}_{p_{k_\bullet}}}(V_\bullet)$, if ρ_\bullet is **potentially crystalline**, then the composite $\rho_\bullet \circ \alpha: \text{Gal}(\bar{k}_\circ/k_\circ) \xrightarrow{\sim} \text{Gal}(\bar{k}_\bullet/k_\bullet) \rightarrow \text{Aut}_{\mathbb{Q}_{p_{k_\bullet}}}(V_\bullet) = \text{Aut}_{\mathbb{Q}_{p_{k_\circ}}}(V_\bullet)$ is **Hodge-Tate**.

PROOF. — The implication (1) \Rightarrow (2) is immediate. To verify the implication (2) \Rightarrow (1), suppose that condition (2) is satisfied. Then it follows immediately from [9, Chapter III, §A.4, Proposition 5], together with a similar argument to the argument applied in the proof of [1, Lemma 1.4], that the open continuous homomorphism α is of *HT-qLT-type* [cf. [1, Definition 1.3, (ii)]]. Thus, it follows from [1, Theorem 3.3] [cf. also Remark 1.4.1 below] that condition (1) is satisfied, as desired. This completes the proof of the implication (2) \Rightarrow (1), hence also of Theorem 1.4. \square

REMARK 1.4.1. — Unfortunately, the proof of [1, Theorem 3.3], which was applied in the proof of Theorem 1.4 of the present paper, contains an *inessential inaccuracy* [cf. (i) below]. In light of the importance of [1, Theorem 3.3] in the present paper, we thus pause to discuss how this inaccuracy may be amended.

(i) In the final portion of the proof of [1, Claim 3.3.A], the author of the present paper has claimed that $\beta_{k_\bullet, k_\circ}$ is *inertially compatible* with α . However, it is not clear that $\beta_{k_\bullet, k_\circ}$ is inertially compatible with α .

(ii) Thus, the statement of [1, Claim 3.3.A] should be replaced by the following text:

(*) Suppose that k_\circ is *Galois* over \mathbb{Q}_p . Then the field k_\circ is *isomorphic* to the field k_\bullet .

Here, let us observe that the argument given in the proof of [1, Claim 3.3.A] proves this assertion.

(iii) Next, let us observe that one verifies immediately from the various definitions involved that if the MLF k is *Galois* over $k^{(d=1)}$, then, for an arbitrary positive real number ν , the algebraic extension of k in \bar{k} that corresponds to the higher ramification subgroup of G_k associated to ν in the “upper numbering” is *Galois* over $k^{(d=1)}$. In particular, in the situation of [1, Theorem 3.3], one may conclude — by applying the assertion (*) of (ii) to the various restrictions of the given continuous isomorphism [cf. the first paragraph of the proof of [1, Theorem 3.3]] α (respectively, of the inverse of the given continuous isomorphism α) to the open subgroups of G_{k_\circ} (respectively, of G_{k_\bullet}) that correspond to the finite extensions of k_\circ in \bar{k}_\circ (respectively, of k_\bullet in \bar{k}_\bullet) *Galois* over \mathbb{Q}_p — that if k_\circ is *Galois* over \mathbb{Q}_p , then the continuous isomorphism [cf. the first paragraph of the proof of [1, Theorem 3.3]] α is *compatible* with the respective *higher ramification subgroups* of $G_{k_\circ}, G_{k_\bullet}$ associated to the positive real numbers in the “upper numbering”. Thus, the conclusion of [1, Theorem 3.3] in the case where k_\circ is *Galois* over \mathbb{Q}_p , hence also the conclusion of [1, Theorem 3.3] for an *arbitrary* k_\circ , follows immediately from [6, Theorem].

COROLLARY 1.5. — *Let k be an MLF and \bar{k} an algebraic closure of k . Suppose that p_k is **odd**. Then there exist a \mathbb{Q}_{p_k} -vector space V of finite dimension and a continuous representation $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ that is **potentially crystalline** [hence also **Hodge-Tate**] but **not Aut-intrinsically Hodge-Tate**.*

PROOF. — Let us first recall that if $d_k = 1$ (respectively, $d_k \neq 1$), then it follows from, for instance, the discussion given at the final portion of [7, Chapter VII, §5] (respectively, [4, Proposition 3.6] and [5, Corollary 1.6, (iv)]) that we have a continuous automorphism of $\text{Gal}(\bar{k}/k)$ such that an arbitrary automorphism of the field \bar{k} is *not compatible* with the natural action of $\text{Gal}(\bar{k}/k)$ on \bar{k} relative to the continuous automorphism of $\text{Gal}(\bar{k}/k)$. Thus, Corollary 1.5 follows from Theorem 1.4. This completes the proof of Corollary 1.5. \square

REMARK 1.5.1. — The content of Corollary 1.5 in the case where $d_k = 1$ is essentially contained in [1, Remark 3.3.1].

In the remainder of the present §1, let us recall some basic facts concerning *abelian Hodge-Tate p -adic representations*. Let E be either k or $k^{(d=1)}$. Suppose that E is

absolutely Galois, i.e., that the finite extension $E/k^{(d=1)}$ is Galois [cf. [3, Definition 4.2, (i)]]].

DEFINITION 1.6. — We shall write E_+ for the \mathbb{Q}_{p_k} -vector space [necessarily of finite dimension] obtained by forming the underlying additive module of the MLF E . Thus, we have a natural injective continuous homomorphism $\mathcal{O}_E^\times \hookrightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(E_+)$, i.e., by multiplication, by means of which we regard \mathcal{O}_E^\times as a [necessarily closed] subgroup of the topological group $\text{Aut}_{\mathbb{Q}_{p_k}}(E_+)$:

$$\mathcal{O}_E^\times \subseteq \text{Aut}_{\mathbb{Q}_{p_k}}(E_+).$$

DEFINITION 1.7. — Let $\pi \in \mathcal{O}_k$ be a uniformizer of \mathcal{O}_k and σ an element of $\text{Gal}(E/k^{(d=1)})$. If $E = k$ (respectively, $E = k^{(d=1)}$), then we shall write

$$\Phi_\sigma: \mathcal{O}_k^\times \longrightarrow \mathcal{O}_E^\times$$

for the continuous automorphism of \mathcal{O}_k^\times determined by σ (respectively, the continuous homomorphism $\mathcal{O}_k^\times \rightarrow \mathcal{O}_{k^{(d=1)}}^\times$ determined by the norm map with respect to the finite extension $k/k^{(d=1)}$). Moreover, we shall write

$$\chi_{\pi,\sigma}: G_k^{\text{ab}} \xrightarrow[\sim]{\text{rec}_k^{-1}} (k^\times)^\wedge \twoheadrightarrow \mathcal{O}_k^\times \xrightarrow{\Phi_\sigma} \mathcal{O}_E^\times$$

— where the second arrow is the surjective continuous homomorphism obtained by considering the quotient by the closed submodule of the topological module $(k^\times)^\wedge$ topologically generated by $\pi \in k^\times$.

LEMMA 1.8. — Let $\pi \in \mathcal{O}_k$ be a uniformizer of \mathcal{O}_k and $\phi: G_k^{\text{ab}} \rightarrow \mathcal{O}_E^\times$ a continuous homomorphism. Then the following two conditions are equivalent:

- (1) The continuous representation obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{\text{ab}} \xrightarrow{\phi} \mathcal{O}_E^\times \hookrightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(E_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion — is **Hodge-Tate**.

- (2) There exist an integer i_σ for each $\sigma \in \text{Gal}(E/k^{(d=1)})$ and an open subgroup J of the inertia subgroup of G_k such that

- the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\text{ab}}$ and the given homomorphism $\phi: G_k^{\text{ab}} \rightarrow \mathcal{O}_E^\times$

coincides with

- the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\text{ab}}$ and the homomorphism

$$\prod_{\sigma \in \text{Gal}(E/k^{(d=1)})} \chi_{\pi,\sigma}^{i_\sigma}: G_k^{\text{ab}} \longrightarrow \mathcal{O}_E^\times.$$

PROOF. — This assertion follows from [9, Chapter III, §A.5, Corollary]. □

LEMMA 1.9. — Let $\phi: G_k^{\text{ab}} \rightarrow \mathcal{O}_k^\times$ be a continuous homomorphism. Suppose that the continuous representation obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{\text{ab}} \xrightarrow{\phi} \mathcal{O}_k^\times \hookrightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(k_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion — is **Hodge-Tate**. Then the image of some open submodule of $\mathcal{O}_{k(d=1)}^\times$ by the composite

$$\mathcal{O}_{k(d=1)}^\times \hookrightarrow \mathcal{O}_k^\times \xrightarrow{\text{rec}_k} G_k^{\text{ab}} \xrightarrow{\phi} \mathcal{O}_k^\times$$

— where the first arrow is the natural inclusion — is **contained** in the submodule $\mathcal{O}_{k(d=1)}^\times \subseteq \mathcal{O}_k^\times$.

PROOF. — This assertion follows immediately from Lemma 1.8. \square

2. THE CASE OF REDUCIBLE REPRESENTATIONS OF DIMENSION TWO

In the present §2, we introduce the notion of *intrinsic Hodge-Tate-ness* of p -adic representations [cf. Definition 2.2 below]. Moreover, we prove that an arbitrary *reducible Hodge-Tate* p -adic representation of *dimension 2* is *Aut-intrinsically Hodge-Tate* [cf. Theorem 2.10 below].

DEFINITION 2.1. — We shall refer to a group isomorphic to the absolute Galois group of an MLF as a group of *MLF-type* [cf. [2, Definition 1.1]]. Here, “MLF” is to be understood as an abbreviation for “mixed-characteristic local field”. Let us always regard a group of MLF-type as a profinite group by means of the profinite topology discussed in [2, Proposition 1.2, (i)].

In the remainder of the present §2, let G be a group of MLF-type. Thus, by applying various functorial group-theoretic reconstruction algorithms established in the study of mono-anabelian geometry to the group G of MLF-type, we obtain

- a prime number $p(G)$ [cf. [4, Definition 3.5, (i)]],
- a positive integer $d(G)$ [cf. [4, Definition 3.5, (ii)]],
- a normal closed subgroup $I(G) \subseteq G$ of G [cf. [4, Definition 3.5, (iii)]],
- topological modules $\mathcal{O}^\times(G) \subseteq k^\times(G)$ [cf. [4, Definition 3.10, (i), (iv)]], and
- a topological field $\mathbb{Q}_p(G)$ [cf. [3, Definition 4.5, (iii)] and [3, Lemma 4.6, (i)]].

Moreover, in the remainder of the present §2, let V be a $\mathbb{Q}_p(G)$ -vector space of finite dimension and $\rho: G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation.

DEFINITION 2.2. — We shall say that the given continuous representation ρ is *intrinsically Hodge-Tate* if, for an arbitrary MLF-envelope $(k, \bar{k}, \alpha: \text{Gal}(\bar{k}/k) \xrightarrow{\sim} G)$ of G [cf. [2, Definition 1.1]], the continuous [cf. [2, Proposition 1.2, (ii)]] representation obtained by forming the composite $\rho \circ \alpha: \text{Gal}(\bar{k}/k) \xrightarrow{\sim} G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V) = \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ [cf. [3, Lemma 4.6, (i)] and [4, Proposition 3.6]] is Hodge-Tate.

REMARK 2.2.1. — In the situation of Definition 1.3, it is immediate that the implications

$$\begin{aligned} \rho \text{ is intrinsically Hodge-Tate} &\implies \rho \text{ is Aut-intrinsically Hodge-Tate} \\ &\implies \rho \text{ is Hodge-Tate} \end{aligned}$$

hold.

DEFINITION 2.3. — Let V' be a $\mathbb{Q}_p(G)$ -vector space of finite dimension and $\rho': G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V')$ a continuous representation. Then we shall say that ρ is *inertially isomorphic* to ρ' if there exists an open subgroup $J \subseteq I(G)$ of $I(G)$ such that the restriction of ρ to $J \subseteq (I(G) \subseteq) G$ is isomorphic to the restriction of ρ' to $J \subseteq (I(G) \subseteq) G$.

DEFINITION 2.4. — Let w be an integer. Then we shall say that the continuous representation ρ is *w-cyclotomic* if ρ is isomorphic to the continuous representation of dimension 1 obtained by considering the w -th power of the character $G \rightarrow \mathbb{Q}_p(G)^\times$ determined by the maximal pro- $p(G)$ quotient of the cyclotome $\Lambda(G)$ associated to G [cf. [4, Definition 4.1, (iii)]]].

REMARK 2.4.1. — Let k be an MLF and \bar{k} an algebraic closure of k . Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$.

(i) Let us recall from [4, Proposition 3.6] that the normal closed subgroup $I(G_k) \subseteq G_k$ of G_k coincides with the *inertia subgroup* of G_k .

(ii) Let us recall from [4, Proposition 4.2, (iv)] that the character $G_k \rightarrow \mathbb{Q}_p(G_k)^\times \xleftarrow{\sim} \mathbb{Q}_{p_k}^\times$ [cf. [3, Lemma 4.6, (i)] and [4, Proposition 3.6]] determined by the maximal pro- $p(G_k)$, i.e., pro- p_k [cf. [4, Proposition 3.6]], quotient of the cyclotome $\Lambda(G_k)$ associated to G_k coincides with the *p_k -adic cyclotomic character* of G_k .

LEMMA 2.5. — Let k be an MLF, \bar{k} an algebraic closure of k , V a \mathbb{Q}_{p_k} -vector space **of dimension 1**, and $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then the following two conditions are equivalent:

- (1) The continuous representation ρ is **Hodge-Tate**.
- (2) The continuous representation ρ is **inertially isomorphic** to the **w-cyclotomic** representation of $\text{Gal}(\bar{k}/k)$ for some integer w .

PROOF. — This assertion follows — in light of Remark 2.4.1, (i), (ii) — from Lemma 1.8, together with [9, Chapter III, §A.4, Corollary]. \square

THEOREM 2.6. — Let G be a group of MLF-type, V a $\mathbb{Q}_p(G)$ -vector space **of dimension 1**, and $\rho: G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation. Then the following two conditions are equivalent:

- (1) The continuous representation ρ is **intrinsically Hodge-Tate**.
- (2) The continuous representation ρ is **inertially isomorphic** to the **w-cyclotomic** representation of G for some integer w .

PROOF. — This assertion follows from Lemma 2.5. \square

THEOREM 2.7. — *Let k be an MLF, \bar{k} an algebraic closure of k , V a \mathbb{Q}_{p_k} -vector space of dimension 1, and $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then the following three conditions are equivalent:*

- (1) *The continuous representation ρ is **Hodge-Tate**.*
- (2) *The continuous representation ρ is **intrinsically Hodge-Tate**.*
- (3) *The continuous representation ρ is **Aut-intrinsically Hodge-Tate**.*

PROOF. — It follows from Remark 2.2.1 that, to verify Theorem 2.7, it suffices to verify the implication (1) \Rightarrow (2). On the other hand, the implication (1) \Rightarrow (2) follows from Lemma 2.5 and Theorem 2.6. This completes the proof of Theorem 2.7. \square

LEMMA 2.8. — *Let k be an MLF, \bar{k} an algebraic closure of k , V a \mathbb{Q}_{p_k} -vector space of dimension 2, and $\rho: G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Suppose that the continuous representation ρ is **reducible**. Then the continuous representation ρ is **Hodge-Tate** if and only if there exist integers w, w' and a G_k -stable \mathbb{Q}_{p_k} -subspace $W \subseteq V$ of V of dimension 1 such that the continuous representations $G_k \rightarrow \text{Aut}(W)$, $G_k \rightarrow \text{Aut}(V/W)$ determined by ρ are, respectively, **inertially isomorphic** to the **w -cyclotomic**, **w' -cyclotomic** representations of G_k , and, moreover, one of the following two conditions is satisfied:*

- (1) *There exists an open subgroup J of the inertia subgroup of G_k such that the natural surjective homomorphism $V \twoheadrightarrow V/W$ has a **J -equivariant splitting**.*
- (2) *The equality $w = w'$ does **not hold**.*

PROOF. — First, we verify sufficiency. Suppose that there exist w, w', W as in the statement of Lemma 2.8. If condition (1) is satisfied, then it follows immediately — in light of Remark 2.4.1, (i), (ii), and [9, Chapter III, §A.1, Corollary 2] — from Lemma 2.5 that the continuous representation ρ is *Hodge-Tate*. If condition (2) is satisfied, then it follows immediately — in light of Remark 2.4.1, (i), (ii), and [9, Chapter III, §A.1, Corollary 2] — from [10, Proposition 8, (b)] that the continuous representation ρ is *Hodge-Tate*. This completes the proof of sufficiency.

Next, we verify necessity. Suppose that the continuous representation ρ is *Hodge-Tate*. Then since [we have assumed that] the continuous representation ρ is *reducible* and of dimension 2, there exists a G_k -stable \mathbb{Q}_{p_k} -subspace $W \subseteq V$ of V of dimension 1. Now since ρ is *Hodge-Tate*, and both W and V/W are of dimension 1, it follows from Lemma 2.5 that there exist integers w, w' such that the continuous representations $G_k \rightarrow \text{Aut}(W)$, $G_k \rightarrow \text{Aut}(V/W)$ determined by ρ are, respectively, *inertially isomorphic* to the *w -cyclotomic*, *w' -cyclotomic* representations of G_k . Now suppose that condition (2) is *not satisfied*. Then it follows immediately from [8, Corollary 1] that condition (1) is satisfied, as desired. This completes the proof of necessity, hence also of Lemma 2.8. \square

THEOREM 2.9. — *Let G be a group of MLF-type, V a $\mathbb{Q}_p(G)$ -vector space of dimension 2, and $\rho: G \rightarrow \text{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation. Suppose that ρ is **reducible**. Then the continuous representation ρ is **intrinsically Hodge-Tate** if and only if there*

exist integers w, w' and a G -stable $\mathbb{Q}_p(G)$ -subspace $W \subseteq V$ of V **of dimension 1** such that the continuous representations $G \rightarrow \text{Aut}(W)$, $G \rightarrow \text{Aut}(V/W)$ determined by ρ are, respectively, **inertially isomorphic** to the **w -cyclotomic**, **w' -cyclotomic** representations of G , and, moreover, one of the following two conditions is satisfied:

- (1) There exists an open subgroup $J \subseteq I(G)$ of $I(G) (\subseteq G)$ such that the natural surjective homomorphism $V \twoheadrightarrow V/W$ has a **J -equivariant splitting**.
- (2) The equality $w = w'$ does **not hold**.

PROOF. — This assertion follows — in light of Remark 2.4.1, (i) — from Lemma 2.8. \square

THEOREM 2.10. — Let k be an MLF, \bar{k} an algebraic closure of k , V a \mathbb{Q}_{p_k} -vector space **of dimension 2**, and $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Suppose that the continuous representation ρ is **reducible**. Then the following three conditions are equivalent:

- (1) The continuous representation ρ is **Hodge-Tate**.
- (2) The continuous representation ρ is **intrinsically Hodge-Tate**.
- (3) The continuous representation ρ is **Aut-intrinsically Hodge-Tate**.

PROOF. — It follows from Remark 2.2.1 that, to verify Theorem 2.10, it suffices to verify the implication (1) \Rightarrow (2). On the other hand, the implication (1) \Rightarrow (2) follows — in light of Remark 2.4.1, (i) — from Lemma 2.8 and Theorem 2.9. This completes the proof of Theorem 2.10. \square

3. THE CASE OF IRREDUCIBLE REPRESENTATIONS OF DIMENSION TWO

In the present §3, we prove the existence of an *irreducible crystalline* [hence also *Hodge-Tate*] p -adic representation of dimension 2 that is *not Aut-intrinsically Hodge-Tate* [cf. Corollary 3.4 below].

In the present §3, let k be an MLF and \bar{k} an algebraic closure of k . Write $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$. We shall also apply the notational conventions introduced in Definition 1.2.

LEMMA 3.1. — Suppose that p_k is **odd**, and that $d_k = 2$. Write $\text{Nm}: k^\times \rightarrow (k^{(d=1)})^\times$ for the norm map with respect to the finite extension $k/k^{(d=1)}$. Then the following assertions hold:

- (i) There exists an **open** submodule $U \subseteq \mathcal{O}_k^\times$ of \mathcal{O}_k^\times such that
 - (1) the topological module U has a natural structure of **free \mathbb{Z}_{p_k} -module of rank 2**, and, moreover,
 - (2) the submodule $U \subseteq \mathcal{O}_k^\times$ is **preserved** by an arbitrary continuous automorphism of \mathcal{O}_k^\times .
- (ii) Let $U \subseteq \mathcal{O}_k^\times$ be as in (i). Then the topological modules $U \cap \mathcal{O}_{k^{(d=1)}}^\times$, $U \cap \text{Ker}(\text{Nm})$ have natural structures of **free \mathbb{Z}_{p_k} -modules of rank 1**, respectively.
- (iii) Let $U \subseteq \mathcal{O}_k^\times$ be as in (i). Then the **equality** $U \cap \mathcal{O}_{k^{(d=1)}}^\times \cap \text{Ker}(\text{Nm}) = \{1\}$ holds.

(iv) Let $U \subseteq \mathcal{O}_k^\times$ be as in (i). Then the closed submodule of U topologically generated by the closed submodules $U \cap \mathcal{O}_{k^{(d=1)}}^\times$ and $U \cap \text{Ker}(\text{Nm})$ is **open**.

(v) There exists a continuous automorphism α of G_k such that, for an arbitrary nonzero integer n , if one writes α_\times^n for the continuous automorphism of \mathcal{O}_k^\times induced by α^n [cf. [4, Proposition 3.11, (iv)]], then the intersection $\alpha_\times^n(\mathcal{O}_{k^{(d=1)}}^\times) \cap \mathcal{O}_{k^{(d=1)}}^\times$ is **not open** in $\mathcal{O}_{k^{(d=1)}}^\times$. In particular, the continuous automorphism α_\times^n of \mathcal{O}_k^\times does **not preserve** the submodule $\mathcal{O}_{k^{(d=1)}}^\times \subseteq \mathcal{O}_k^\times$.

PROOF. — Assertions (i), (ii) follow from [4, Lemma 1.2, (i)] [cf. also our assumption that $d_k = 2$]. Assertion (iii) is immediate [cf. the fact that U is *torsion-free* — cf. condition (1) of assertion (i)]. Assertion (iv) follows from assertions (ii), (iii), together with condition (1) of assertion (i).

Finally, we verify assertion (v). Let α be a continuous automorphism of G_k as in the discussion preceding [5, Theorem 1.5] [cf. also [4, Proposition 3.6]]. Write β for the continuous automorphism of the submodule $U \subseteq \mathcal{O}_k^\times$ obtained by forming the restriction of α_\times^n [cf. condition (2) of assertion (i)]. Thus, it follows immediately from [5, Theorem 1.5] [cf. also [4, Definition 3.10, (vi)]] that

- (a) the continuous automorphism β is *not the identity automorphism* of U , but
- (b) the image of the *square* of the endomorphism of U given by “ $a \mapsto \beta(a) \cdot a^{-1}$ ” consists of the identity element of U .

Moreover, it follows immediately from [5, Lemma 2.3, (i)] that

- (c) the continuous automorphism β *preserves* the submodule $U \cap \text{Ker}(\text{Nm})$ of U .

Thus, it follows immediately from assertion (iv) [cf. also condition (1) of assertion (i)], together with (b) and (c), that if the continuous automorphism β *preserves* some open submodule of the submodule $U \cap \mathcal{O}_{k^{(d=1)}}^\times$, then β is the *identity automorphism* of U — in contradiction to (a). In particular, the continuous automorphism β does *not preserve* any open submodule of the submodule $U \cap \mathcal{O}_{k^{(d=1)}}^\times$, which thus implies [cf. assertion (ii)] that $\beta(U \cap \mathcal{O}_{k^{(d=1)}}^\times) \cap U \cap \mathcal{O}_{k^{(d=1)}}^\times = \{1\}$. Thus, it follows immediately from [4, Lemma 1.2, (i)] that $\alpha_\times^n(\mathcal{O}_{k^{(d=1)}}^\times) \cap \mathcal{O}_{k^{(d=1)}}^\times$ is *not open* in $\mathcal{O}_{k^{(d=1)}}^\times$, as desired. This completes the proof of assertion (v), hence also of Lemma 3.1. \square

REMARK 3.1.1. — One may conclude from the final portion of Lemma 3.1, (v), that it is *impossible* to establish a functorial group-theoretic reconstruction algorithm for constructing, from an *arbitrary* group H of MLF-type, a closed submodule of the topological module $\mathcal{O}^\times(H)$ which “*corresponds*” to the closed submodule $\mathcal{O}_{k^{(d=1)}}^\times \subseteq \mathcal{O}_k^\times$ of the topological module \mathcal{O}_k^\times . Put another way, one may conclude from the final portion of Lemma 3.1, (v), that the closed submodule $\mathcal{O}_{k^{(d=1)}}^\times \subseteq \mathcal{O}_k^\times$ should be considered to be “*not group-theoretic*”.

PROPOSITION 3.2. — Suppose that p_k is **odd**, and that d_k is **even**. Suppose, moreover, that k is **absolutely abelian**, i.e., that k is *absolutely Galois*, and the Galois group $\text{Gal}(k/k^{(d=1)})$ is *abelian* [cf. [3, Definition 4.2, (ii)]]. Then there exists a continuous automorphism α of G_k such that, for an arbitrary nonzero integer n , if one writes α_\times^n for the continuous automorphism of \mathcal{O}_k^\times induced by α^n [cf. [4, Proposition 3.11, (iv)]], then the intersection $\alpha_\times^n(\mathcal{O}_{k^{(d=1)}}^\times) \cap \mathcal{O}_{k^{(d=1)}}^\times$ is **not open** in $\mathcal{O}_{k^{(d=1)}}^\times$.

PROOF. — Let us first observe that since d_k is *even*, and k is *absolutely abelian*, one verifies easily that there exists a *quadratic* extension of $k^{(d=1)}$ contained in k . Moreover, since k is *absolutely abelian*, it follows immediately from the implication (1) \Rightarrow (2) of [3, Theorem F, (i)] that G_k is a *characteristic* subgroup of the absolute Galois group of the quadratic extension of $k^{(d=1)}$ determined by the algebraic closure \bar{k} . Thus, one may conclude that we may assume without loss of generality, by applying a similar argument to the argument applied in the proof of [5, Lemma 2.6, (ii)] and replacing k by the quadratic extension of $k^{(d=1)}$, that $d_k = 2$. On the other hand, if $d_k = 2$, then the desired conclusion follows from Lemma 3.1, (v). This completes the proof of Proposition 3.2. \square

THEOREM 3.3. — *Let k be an MLF and \bar{k} an algebraic closure of k . Suppose that p_k is **odd**, that d_k is **even**, and that k is **absolutely abelian**. Then there exist a \mathbb{Q}_{p_k} -vector space V of **dimension d_k** and a continuous representation $\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(V)$ that is **irreducible, abelian, crystalline** [hence also **Hodge-Tate**], but **not Aut-intrinsically Hodge-Tate**.*

PROOF. — Let $\pi \in \mathcal{O}_k$ be a uniformizer of \mathcal{O}_k . Write ρ for the continuous representation of $G_k \stackrel{\text{def}}{=} \text{Gal}(\bar{k}/k)$ [necessarily of dimension d_k] obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{\text{ab}} \xrightarrow{\chi_{\pi, \text{id}_k}} \mathcal{O}_k^\times \hookrightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(k_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion. Then one verifies easily that this continuous representation ρ is *irreducible* and *abelian*. Moreover, it follows immediately from [9, Chapter III, §A.4, Proposition 5] that this continuous representation ρ is *crystalline*.

Next, to verify that the continuous representation ρ is *not Aut-intrinsically Hodge-Tate*, let us recall that it follows immediately from the various definitions involved that the composite

$$\mathcal{O}_k^\times \xrightarrow{\text{rec}_k} G_k^{\text{ab}} \xrightarrow{\chi_{\pi, \text{id}_k}} \mathcal{O}_k^\times$$

is an *automorphism* that restricts to an *automorphism* of the submodule $\mathcal{O}_{k^{(d=1)}}^\times \subseteq \mathcal{O}_k^\times$. In particular, if α is a continuous automorphism of G_k as in Proposition 3.2, then it follows immediately from Lemma 1.9, together with the various definitions involved, that the composite $\rho \circ \alpha: G_k \xrightarrow{\sim} G_k \rightarrow \text{Aut}_{\mathbb{Q}_{p_k}}(k_+)$ is *not Hodge-Tate*, which thus implies that the continuous representation ρ is *not Aut-intrinsically Hodge-Tate*, as desired. This completes the proof of Theorem 3.3. \square

COROLLARY 3.4. — *Let p be an **odd** prime number. Then there exist an MLF K such that $p_K = p$, an algebraic closure \bar{K} of K , a \mathbb{Q}_{p_K} -vector space V of **dimension 2**, and a continuous representation $\rho: \text{Gal}(\bar{K}/K) \rightarrow \text{Aut}_{\mathbb{Q}_{p_K}}(V)$ that is **irreducible, abelian, crystalline** [hence also **Hodge-Tate**], but **not Aut-intrinsically Hodge-Tate**.*

PROOF. — This assertion is a formal consequence of Theorem 3.3. \square

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