On Intrinsic Hodge-Tate-ness of Galois Representations of Dimension Two

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Abstract. — In the present paper, we first prove that, for an arbitrary reducible Hodge-Tate p-adic representation of dimension two of the absolute Galois group of a p-adic local field and an arbitrary continuous automorphism of the absolute Galois group, the p-adic Galois representation obtained by pulling back the given p-adic Galois representation by the given continuous automorphism is Hodge-Tate. Next, we also prove the existence of an irreducible Hodge-Tate p-adic representation of dimension two of the absolute Galois group of a p-adic local field and a continuous automorphism of the absolute Galois group such that the p-adic Galois representation obtained by pulling back the given p-adic Galois representation by the given continuous automorphism is not Hodge-Tate.

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Introduction

In the present paper, we study the *intrinsic Hodge-Tate-ness* of p-adic representations of the absolute Galois group of a p-adic local field. In the present Introduction, let p be a prime number, k a finite extension of $\mathbb{Q}_p$, and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\overline{k}$. For a given $\mathbb{Q}_p$-vector space V of finite dimension and a given continuous representation $\rho\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ of G_k, we shall say that ρ is *Aut-intrinsically Hodge-Tate* if, for an arbitrary continuous automorphism α of G_k, the composite $\rho \circ \alpha\colon G_k \xrightarrow{\sim} G_k \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ is Hodge-Tate [cf. Definition 1.3].

Let us first recall that the author of the present paper proved that

> if p is *odd*, and $k = \mathbb{Q}_p$, then there exists a p-adic representation of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate* [cf. [1, Remark 3.3.1]].

Moreover, in the present paper, we establish a refinement of this result. That is to say, we verify that

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there exists a p-adic representation of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate* whenever p is *odd*, i.e., without the assumption that $k = \mathbb{Q}_p$ [cf. Corollary 1.5].

On the other hand, let us also observe that it is likely to be well-known that

an arbitrary *Hodge-Tate* p-adic representation *of dimension* 1 of G_k is *Aut-intrinsically Hodge-Tate* [cf. Theorem 2.7].

In this state of affairs, one may have the following question:

Is there a p-adic representation *of dimension* 2 of G_k that is *Hodge-Tate* but *not Aut-intrinsically Hodge-Tate*?

In the present paper, we give an answer to this question.

First, we consider the case where a given continuous representation is *reducible*. The first main result of the present paper is as follows [cf. Theorem 2.10]:

Theorem A. — *Let V be a $\mathbb{Q}_p$-vector space* ***of dimension 2*** *and $\rho\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ a continuous representation. Suppose that the continuous representation ρ is* ***reducible****. Then ρ is* ***Hodge-Tate*** *if and only if ρ is* ***Aut-intrinsically Hodge-Tate****.*

Next, we consider the case where a given continuous representation is *irreducible*. The second main result of the present paper is as follows [cf. Corollary 3.4]:

Theorem B. — *Let p be an* ***odd*** *prime number. Then there exist a finite extension K of $\mathbb{Q}_p$, an algebraic closure $\overline{K}$ of K, a $\mathbb{Q}_p$-vector space V* ***of dimension 2****, and a continuous representation $\rho\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ that is* ***irreducible****,* ***abelian****,* ***crystalline*** *[hence also* ***Hodge-Tate****], but* ***not Aut-intrinsically Hodge-Tate****.*

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1. Aut-intrinsic Hodge-Tate-ness of Representations

In the present §1, we introduce the notion of *Aut-intrinsic Hodge-Tate-ness* of p-adic representations [cf. Definition 1.3 below]. Moreover, we prove the existence of a p-adic representation that is *potentially crystalline* [hence also *Hodge-Tate*] but *not Aut-intrinsically Hodge-Tate* [cf. Corollary 1.5 below]. Finally, we also recall some basic facts concerning *abelian Hodge-Tate* p-adic representations [cf. Lemma 1.8 below and Lemma 1.9 below].

Definition 1.1. — We shall refer to a field isomorphic to a finite extension of $\mathbb{Q}_p$, for some prime number p, as an *MLF*. Here, "MLF" is to be understood as an abbreviation for "mixed-characteristic local field".

In the remainder of the present §1, let k be an MLF and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Definition 1.2. — We shall write

• $k^{(d=1)} \subseteq k$ for the [unique] minimal MLF contained in k [i.e., the unique subfield of k isomorphic to $\mathbb{Q}_p$, for some prime number p],

• $\mathcal{O}_k \subseteq k$ for the ring of integers of k,

• p_k for the characteristic of the residue field of $\mathcal{O}_k$,

• d_k for the extension degree of the [necessarily finite] extension $k/k^{(d=1)}$,

• $(k^\times)^\wedge$ for the profinite completion of the multiplicative module $k^\times$ of k,

• G_k^{ab} for the topological abelianization of G_k, i.e., the quotient of G_k by the closure of the commutator subgroup of G_k, and

• $\mathrm{rec}_k\colon (k^\times)^\wedge \xrightarrow{\sim} G_k^{\mathrm{ab}}$ for the isomorphism induced by the reciprocity homomorphism $k^\times \hookrightarrow G_k^{\mathrm{ab}}$ in local class field theory.

Remark 1.2.1. — Let us recall the *functorial* assignment "$\mathcal{O}^\times(-)$" of [4, Definition 3.10, (i)]. Observe that it follows from the *functoriality* of the assignment "$\mathcal{O}^\times(-)$" that each continuous automorphism of G_k naturally induces a continuous automorphism of $\mathcal{O}^\times(G_k)$. In particular, by conjugating this continuous automorphism of $\mathcal{O}^\times(G_k)$ by the continuous isomorphism $\mathcal{O}_k^\times \xrightarrow{\sim} \mathcal{O}^\times(G_k)$ of [4, Proposition 3.11, (i)], one concludes that each continuous automorphism of G_k naturally induces a continuous automorphism of $\mathcal{O}_k^\times$.

Definition 1.3. — Let V be a $\mathbb{Q}_{p_k}$-vector space of finite dimension and $\rho\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then we shall say that ρ is *Aut-intrinsically Hodge-Tate* if, for an arbitrary continuous automorphism α of G_k, the composite $\rho \circ \alpha\colon G_k \xrightarrow{\sim} G_k \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ is Hodge-Tate.

The following result is a formal consequence of the main result of [1].

Theorem 1.4. — *For each $\square \in \{\circ, \bullet\}$, let $k_\square$ be an MLF and $\overline{k}_\square$ an algebraic closure of $k_\square$. Let $\alpha\colon \mathrm{Gal}(\overline{k}_\circ/k_\circ) \to \mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$ be an* ***open*** *continuous homomorphism [which thus implies that $p_{k_\circ} = p_{k_\bullet}$ — cf., e.g.,* [3, Proposition 3.4, (iii)] *and* [4, Proposition 3.6]*]. Then the following two conditions are equivalent:*

(1) *There exists an isomorphism $\overline{k}_\bullet \xrightarrow{\sim} \overline{k}_\circ$ of fields that is* ***compatible*** *with the respective natural actions of $\mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$, $\mathrm{Gal}(\overline{k}_\circ/k_\circ)$ on $\overline{k}_\bullet$, $\overline{k}_\circ$ relative to the given open continuous homomorphism $\alpha\colon \mathrm{Gal}(\overline{k}_\circ/k_\circ) \to \mathrm{Gal}(\overline{k}_\bullet/k_\bullet)$.*

(2) *For an arbitrary $\mathbb{Q}_{p_{k_\bullet}}$-vector space $V_\bullet$ of finite dimension and an arbitrary continuous representation $\rho_\bullet\colon \mathrm{Gal}(\overline{k}_\bullet/k_\bullet) \to \mathrm{Aut}_{\mathbb{Q}_{p_{k_\bullet}}}(V_\bullet)$, if $\rho_\bullet$ is* ***potentially crystalline****, then the composite $\rho_\bullet \circ \alpha\colon \mathrm{Gal}(\overline{k}_\circ/k_\circ) \to \mathrm{Gal}(\overline{k}_\bullet/k_\bullet) \to \mathrm{Aut}_{\mathbb{Q}_{p_{k_\bullet}}}(V_\bullet) = \mathrm{Aut}_{\mathbb{Q}_{p_{k_\circ}}}(V_\bullet)$ is* ***Hodge-Tate****.*

Proof. — The implication (1) $\Rightarrow$ (2) is immediate. To verify the implication (2) $\Rightarrow$ (1), suppose that condition (2) is satisfied. Then it follows immediately from [9, Chapter III, §A.4, Proposition 5], together with a similar argument to the argument applied in the proof of [1, Lemma 1.4], that the open continuous homomorphism α is *of HT-qLT-type* [cf. [1, Definition 1.3, (ii)]]. Thus, it follows from [1, Theorem 3.3] [cf. also Remark 1.4.1 below] that condition (1) is satisfied, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Theorem 1.4. $\square$

Remark 1.4.1. — Unfortunately, the proof of [1, Theorem 3.3], which was applied in the proof of Theorem 1.4 of the present paper, contains an *inessential inaccuracy* [cf. (i) below]. In light of the importance of [1, Theorem 3.3] in the present paper, we thus pause to discuss how this inaccuracy may be amended.

(i) In the final portion of the proof of [1, Claim 3.3.A], the author of the present paper has claimed that $\beta_{k_\bullet,k_\circ}$ is *inertially compatible* with α. However, it is not clear that $\beta_{k_\bullet,k_\circ}$ is inertially compatible with α.

(ii) Thus, the statement of [1, Claim 3.3.A] should be replaced by the following text:

($*$) Suppose that $k_\circ$ is *Galois* over $\mathbb{Q}_p$. Then the field $k_\circ$ is *isomorphic* to the field $k_\bullet$.

Here, let us observe that the argument given in the proof of [1, Claim 3.3.A] proves this assertion.

(iii) Next, suppose that we are in the situation of [1, Theorem 3.3]. Thus, we have a continuous isomorphism $\alpha\colon G_{k_\circ} \xrightarrow{\sim} G_{k_\bullet}$ [cf. the first paragraph of the proof of [1, Theorem 3.3]]. For each $\square \in \{\circ, \bullet\}$ and each positive real number ν, write $G^\nu_{k_\square} \subseteq G_{k_\square}$ for the *higher ramification subgroup* of $G_{k_\square}$ associated to ν in the "upper numbering". Then let us observe that one verifies immediately from the various definitions involved that, for each $\square \in \{\circ, \bullet\}$ and each positive real number ν, if the MLF $k_\square$ is *Galois* over $\mathbb{Q}_p$, then the fixed field $(\overline{k}_\square)^{G^\nu_{k_\square}}$ of $G^\nu_{k_\square}$ is *Galois* over $\mathbb{Q}_p$, which thus implies that

$$G^\nu_{k_\square} = \bigcap_{K_\square} \mathrm{Gal}(\overline{k}_\square/K_\square)$$

— where the intersection is taken over the finite extensions $K_\square \subseteq \overline{k}_\square$ of $k_\square$ that are *Galois* over $\mathbb{Q}_p$ and *contained* in the fixed field $(\overline{k}_\square)^{G^\nu_{k_\square}}$. In particular, if the MLF $k_\circ$, hence also the MLF $k_\bullet$ [cf. the assertion ($*$) of (ii)], is *Galois* over $\mathbb{Q}_p$, then, by applying the assertion ($*$) of (ii) [cf. also [1, Lemma 1.4]], one may conclude immediately that

$$\alpha(G^\nu_{k_\circ}) = \alpha\big(\bigcap_{K_\circ} \mathrm{Gal}(\overline{k}_\circ/K_\circ)\big) = \bigcap_{K_\circ} \alpha\big(\mathrm{Gal}(\overline{k}_\circ/K_\circ)\big) = \bigcap_{K_\circ} \mathrm{Gal}(\overline{k}_\bullet/(K_\circ)^\dagger) = G^\nu_{k_\bullet}$$

— where the intersections are taken over the finite extensions $K_\circ \subseteq \overline{k}_\circ$ of $k_\circ$ that are *Galois* over $\mathbb{Q}_p$ and *contained* in the fixed field $(\overline{k}_\circ)^{G^\nu_{k_\circ}}$; for each such an extension $K_\circ$, we write $(K_\circ)^\dagger$ for the *unique* [cf. the assumption that the extension $K_\circ/\mathbb{Q}_p$ is *Galois*] subfield of $\overline{k}_\bullet$ *isomorphic* to $K_\circ$. Thus, the conclusion of [1, Theorem 3.3] in the case where $k_\circ$ is *Galois* over $\mathbb{Q}_p$, hence also the conclusion of [1, Theorem 3.3] for an *arbitrary* $k_\circ$, follows immediately from [6, Theorem].

Corollary 1.5. — *Let k be an MLF and $\overline{k}$ an algebraic closure of k. Suppose that p_k is* ***odd****. Then there exist a $\mathbb{Q}_{p_k}$-vector space V of finite dimension and a continuous representation $\rho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ that is* ***potentially crystalline*** *[hence also* ***Hodge-Tate****] but* ***not Aut-intrinsically Hodge-Tate****.*

Proof. — Let us first recall that if $d_k = 1$ (respectively, $d_k \neq 1$), then it follows from, for instance, the discussion given at the final portion of [7, Chapter VII, §5] (respectively, [4, Proposition 3.6] and [5, Corollary 1.6, (iv)]) that we have a continuous automorphism of $\mathrm{Gal}(\overline{k}/k)$ such that an arbitrary automorphism of the field $\overline{k}$ is *not compatible* with the natural action of $\mathrm{Gal}(\overline{k}/k)$ on $\overline{k}$ relative to the continuous automorphism of $\mathrm{Gal}(\overline{k}/k)$. Thus, Corollary 1.5 follows from Theorem 1.4. This completes the proof of Corollary 1.5. □

Remark 1.5.1. — The content of Corollary 1.5 in the case where $d_k = 1$ is essentially contained in [1, Remark 3.3.1].

In the remainder of the present §1, let us recall some basic facts concerning *abelian Hodge-Tate* p-adic representations. Let E be either k or $k^{(d=1)}$. Suppose that E is *absolutely Galois*, i.e., that the finite extension $E/k^{(d=1)}$ is Galois [cf. [3, Definition 4.2, (i)]].

Definition 1.6. — We shall write E_+ for the $\mathbb{Q}_{p_k}$-vector space [necessarily of finite dimension] obtained by forming the underlying additive module of the MLF E. Thus, we have a natural injective continuous homomorphism $\mathcal{O}_E^\times \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_{p_k}}(E_+)$, i.e., by multiplication, by means of which we regard $\mathcal{O}_E^\times$ as a [necessarily closed] subgroup of the topological group $\mathrm{Aut}_{\mathbb{Q}_{p_k}}(E_+)$:

$$\mathcal{O}_E^\times \subseteq \mathrm{Aut}_{\mathbb{Q}_{p_k}}(E_+).$$

Definition 1.7. — Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$ and σ an element of $\mathrm{Gal}(E/k^{(d=1)})$. If $E = k$ (respectively, $E = k^{(d=1)}$), then we shall write

$$\Phi_\sigma\colon \mathcal{O}_k^\times \longrightarrow \mathcal{O}_E^\times$$

for the continuous automorphism of $\mathcal{O}_k^\times$ determined by σ (respectively, the continuous homomorphism $\mathcal{O}_k^\times \to \mathcal{O}_{k^{(d=1)}}^\times$ determined by the norm map with respect to the finite extension $k/k^{(d=1)}$). Moreover, we shall write

$$\chi_{\pi,\sigma}\colon G_k^{\mathrm{ab}} \xrightarrow[\sim]{\mathrm{rec}_k^{-1}} (k^\times)^\wedge \twoheadrightarrow \mathcal{O}_k^\times \xrightarrow{\Phi_\sigma} \mathcal{O}_E^\times$$

— where the second arrow is the surjective continuous homomorphism obtained by considering the quotient by the closed submodule of the topological module $(k^\times)^\wedge$ topologically generated by $\pi \in k^\times$.

Lemma 1.8. — *Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$ and $\phi\colon G_k^{\mathrm{ab}} \to \mathcal{O}_E^\times$ a continuous homomorphism. Then the following two conditions are equivalent:*

(1) *The continuous representation obtained by forming the composite*

$$G_k \twoheadrightarrow G_k^{\mathrm{ab}} \xrightarrow{\phi} \mathcal{O}_E^\times \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_{p_k}}(E_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion — is ***Hodge-Tate****.*

(2) *There exist an integer i_σ for each $\sigma \in \mathrm{Gal}(E/k^{(d=1)})$ and an open subgroup J of the inertia subgroup of G_k such that*

- *the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\mathrm{ab}}$ and the given homomorphism $\phi\colon G_k^{\mathrm{ab}} \to \mathcal{O}_E^\times$*

coincides *with*

- *the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\mathrm{ab}}$ and the homomorphism*

$$\prod_{\sigma \in \mathrm{Gal}(E/k^{(d=1)})} \chi_{\pi,\sigma}^{i_\sigma}\colon G_k^{\mathrm{ab}} \longrightarrow \mathcal{O}_E^\times.$$

Proof. — This assertion follows from [9, Chapter III, §A.5, Corollary]. □

Lemma 1.9. — *Let $\phi\colon G_k^{\mathrm{ab}} \to \mathcal{O}_k^\times$ be a continuous homomorphism. Suppose that the continuous representation obtained by forming the composite*

$$G_k \twoheadrightarrow G_k^{\mathrm{ab}} \xrightarrow{\phi} \mathcal{O}_k^\times \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion — is ***Hodge-Tate****. Then the image of some open submodule of $\mathcal{O}_{k^{(d=1)}}^\times$ by the composite*

$$\mathcal{O}_{k^{(d=1)}}^\times \hookrightarrow \mathcal{O}_k^\times \xhookrightarrow{\mathrm{rec}_k} G_k^{\mathrm{ab}} \xrightarrow{\phi} \mathcal{O}_k^\times$$

— where the first arrow is the natural inclusion — is ***contained*** *in the submodule $\mathcal{O}_{k^{(d=1)}}^\times \subseteq \mathcal{O}_k^\times$.*

Proof. — This assertion follows immediately from Lemma 1.8. □

2. The Case of Reducible Representations of Dimension Two

In the present §2, we introduce the notion of *intrinsic Hodge-Tate-ness* of p-adic representations [cf. Definition 2.2 below]. Moreover, we prove that an arbitrary *reducible Hodge-Tate* p-adic representation *of dimension* 2 is *Aut-intrinsically Hodge-Tate* [cf. Theorem 2.10 below].

Definition 2.1. — We shall refer to a group isomorphic to the absolute Galois group of an MLF as a group *of MLF-type* [cf. [2, Definition 1.1]]. Here, "MLF" is to be understood as an abbreviation for "mixed-characteristic local field". Let us always regard a group

of MLF-type as a profinite group by means of the profinite topology discussed in [2, Proposition 1.2, (i)].

In the remainder of the present §2, let G be a group of MLF-type. Thus, by applying various functorial group-theoretic reconstruction algorithms established in the study of mono-anabelian geometry to the group G of MLF-type, we obtain

- a prime number $p(G)$ [cf. [4, Definition 3.5, (i)]],
- a positive integer $d(G)$ [cf. [4, Definition 3.5, (ii)]],
- a normal closed subgroup $I(G) \subseteq G$ of G [cf. [4, Definition 3.5, (iii)]],
- topological modules $\mathcal{O}^\times(G) \subseteq k^\times(G)$ [cf. [4, Definition 3.10, (i), (iv)]], and
- a topological field $\mathbb{Q}_p(G)$ [cf. [3, Definition 4.5, (iii)] and [3, Lemma 4.6, (i)]].

Moreover, in the remainder of the present §2, let V be a $\mathbb{Q}_p(G)$-vector space of finite dimension and $\rho\colon G \to \mathrm{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation.

Definition 2.2. — We shall say that the given continuous representation ρ is *intrinsically Hodge-Tate* if, for an arbitrary MLF-envelope $(k, \overline{k}, \alpha\colon \mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} G)$ of G [cf. [2, Definition 1.1]], the continuous [cf. [2, Proposition 1.2, (ii)]] representation obtained by forming the composite $\rho \circ \alpha\colon \mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} G \to \mathrm{Aut}_{\mathbb{Q}_p(G)}(V) = \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ [cf. [3, Lemma 4.6, (i)] and [4, Proposition 3.6]] is Hodge-Tate.

Remark 2.2.1. — In the situation of Definition 1.3, it is immediate that the implications

$$\begin{aligned} \rho \text{ is intrinsically Hodge-Tate} \quad &\Longrightarrow \quad \rho \text{ is Aut-intrinsically Hodge-Tate} \\ &\Longrightarrow \quad \rho \text{ is Hodge-Tate} \end{aligned}$$

hold.

Definition 2.3. — Let V' be a $\mathbb{Q}_p(G)$-vector space of finite dimension and $\rho'\colon G \to \mathrm{Aut}_{\mathbb{Q}_p(G)}(V')$ a continuous representation. Then we shall say that ρ is *inertially isomorphic* to ρ' if there exists an open subgroup $J \subseteq I(G)$ of $I(G)$ such that the restriction of ρ to $J \subseteq (I(G) \subseteq)\ G$ is isomorphic to the restriction of ρ' to $J \subseteq (I(G) \subseteq)\ G$.

Definition 2.4. — Let w be an integer. Then we shall say that the continuous representation ρ is *w-cyclotomic* if ρ is isomorphic to the continuous representation of dimension 1 obtained by considering the w-th power of the character $G \to \mathbb{Q}_p(G)^\times$ determined by the maximal pro-$p(G)$ quotient of the cyclotome $\Lambda(G)$ associated to G [cf. [4, Definition 4.1, (iii)]].

Remark 2.4.1. — Let k be an MLF and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

(i) Let us recall from [4, Proposition 3.6] that the normal closed subgroup $I(G_k) \subseteq G_k$ of G_k coincides with the *inertia subgroup* of G_k.

(ii) Let us recall from [4, Proposition 4.2, (iv)] that the character $G_k \to \mathbb{Q}_p(G_k)^\times \xleftarrow{\sim} \mathbb{Q}_{p_k}^\times$ [cf. [3, Lemma 4.6, (i)] and [4, Proposition 3.6]] determined by the maximal pro-$p(G_k)$, i.e., pro-p_k [cf. [4, Proposition 3.6]], quotient of the cyclotome $\Lambda(G_k)$ associated to G_k coincides with the *p_k-adic cyclotomic character* of G_k.

Lemma 2.5. — *Let k be an MLF, $\overline{k}$ an algebraic closure of k, V a $\mathbb{Q}_{p_k}$-vector space* ***of dimension*** **1**, *and $\rho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then the following two conditions are equivalent:*

(1) *The continuous representation ρ is* ***Hodge-Tate****.*

(2) *The continuous representation ρ is* ***inertially isomorphic*** *to the* ***w-cyclotomic*** *representation of $\mathrm{Gal}(\overline{k}/k)$ for some integer w.*

Proof. — This assertion follows — in light of Remark 2.4.1, (i), (ii) — from Lemma 1.8, together with [9, Chapter III, §A.4, Corollary]. □

Theorem 2.6. — *Let G be a group of MLF-type, V a $\mathbb{Q}_p(G)$-vector space* ***of dimension*** **1**, *and $\rho\colon G \to \mathrm{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation. Then the following two conditions are equivalent:*

(1) *The continuous representation ρ is* ***intrinsically Hodge-Tate****.*

(2) *The continuous representation ρ is* ***inertially isomorphic*** *to the* ***w-cyclotomic*** *representation of G for some integer w.*

Proof. — This assertion follows from Lemma 2.5. □

Theorem 2.7. — *Let k be an MLF, $\overline{k}$ an algebraic closure of k, V a $\mathbb{Q}_{p_k}$-vector space* ***of dimension*** **1**, *and $\rho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Then the following three conditions are equivalent:*

(1) *The continuous representation ρ is* ***Hodge-Tate****.*

(2) *The continuous representation ρ is* ***intrinsically Hodge-Tate****.*

(3) *The continuous representation ρ is* ***Aut-intrinsically Hodge-Tate****.*

Proof. — It follows from Remark 2.2.1 that, to verify Theorem 2.7, it suffices to verify the implication (1) $\Rightarrow$ (2). On the other hand, the implication (1) $\Rightarrow$ (2) follows from Lemma 2.5 and Theorem 2.6. This completes the proof of Theorem 2.7. □

Lemma 2.8. — *Let k be an MLF, $\overline{k}$ an algebraic closure of k, V a $\mathbb{Q}_{p_k}$-vector space* ***of dimension*** **2**, *and $\rho\colon G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Suppose that the continuous representation ρ is* ***reducible****. Then the continuous representation ρ is* ***Hodge-Tate*** *if and only if there exist integers w, w' and a G_k-stable $\mathbb{Q}_{p_k}$-subspace $W \subseteq V$ of V* ***of dimension*** **1** *such that the continuous representations $G_k \to \mathrm{Aut}(W)$, $G_k \to \mathrm{Aut}(V/W)$ determined by ρ are, respectively,* ***inertially isomorphic*** *to the* ***w-cyclotomic****,* ***w'-cyclotomic*** *representations of G_k, and, moreover, one of the following two conditions is satisfied:*

(1) *There exists an open subgroup J of the inertia subgroup of G_k such that the natural surjective homomorphism $V \twoheadrightarrow V/W$ has a* ***J-equivariant splitting****.*

(2) *The equality $w = w'$ does* ***not hold****.*

Proof. — First, we verify sufficiency. Suppose that there exist w, w', W as in the statement of Lemma 2.8. If condition (1) is satisfied, then it follows immediately — in light of Remark 2.4.1, (i), (ii), and [9, Chapter III, §A.1, Corollary 2] — from Lemma 2.5 that the continuous representation ρ is *Hodge-Tate*. If condition (2) is satisfied, then it follows immediately — in light of Remark 2.4.1, (i), (ii), and [9, Chapter III, §A.1, Corollary 2] — from [10, Proposition 8, (b)] that the continuous representation ρ is *Hodge-Tate*. This completes the proof of sufficiency.

Next, we verify necessity. Suppose that the continuous representation ρ is *Hodge-Tate*. Then since [we have assumed that] the continuous representation ρ is *reducible* and *of dimension* 2, there exists a G_k-stable $\mathbb{Q}_{p_k}$-subspace $W \subseteq V$ of V *of dimension* 1. Now since ρ is *Hodge-Tate*, and both W and V/W are *of dimension* 1, it follows from Lemma 2.5 that there exist integers w, w' such that the continuous representations $G_k \to \mathrm{Aut}(W)$, $G_k \to \mathrm{Aut}(V/W)$ determined by ρ are, respectively, *inertially isomorphic* to the *w-cyclotomic*, *w'-cyclotomic* representations of G_k. Now suppose that condition (2) is *not satisfied*. Then it follows immediately from [8, Corollary 1] that condition (1) is satisfied, as desired. This completes the proof of necessity, hence also of Lemma 2.8. □

Theorem 2.9. — *Let G be a group of MLF-type, V a $\mathbb{Q}_p(G)$-vector space* ***of dimension 2****, and $\rho\colon G \to \mathrm{Aut}_{\mathbb{Q}_p(G)}(V)$ a continuous representation. Suppose that ρ is* ***reducible****. Then the continuous representation ρ is* ***intrinsically Hodge-Tate*** *if and only if there exist integers w, w' and a G-stable $\mathbb{Q}_p(G)$-subspace $W \subseteq V$ of V* ***of dimension 1*** *such that the continuous representations $G \to \mathrm{Aut}(W)$, $G \to \mathrm{Aut}(V/W)$ determined by ρ are, respectively,* ***inertially isomorphic*** *to the* ***w-cyclotomic****,* ***w'-cyclotomic*** *representations of G, and, moreover, one of the following two conditions is satisfied:*

(1) *There exists an open subgroup $J \subseteq I(G)$ of $I(G)$ $(\subseteq G)$ such that the natural surjective homomorphism $V \twoheadrightarrow V/W$ has a* ***J-equivariant splitting****.*

(2) *The equality $w = w'$ does* ***not hold****.*

Proof. — This assertion follows — in light of Remark 2.4.1, (i) — from Lemma 2.8. □

Theorem 2.10. — *Let k be an MLF, $\overline{k}$ an algebraic closure of k, V a $\mathbb{Q}_{p_k}$-vector space* ***of dimension 2****, and $\rho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ a continuous representation. Suppose that the continuous representation ρ is* ***reducible****. Then the following three conditions are equivalent:*

(1) *The continuous representation ρ is* ***Hodge-Tate****.*

(2) *The continuous representation ρ is* ***intrinsically Hodge-Tate****.*

(3) *The continuous representation ρ is* ***Aut-intrinsically Hodge-Tate****.*

Proof. — It follows from Remark 2.2.1 that, to verify Theorem 2.10, it suffices to verify the implication (1) ⇒ (2). On the other hand, the implication (1) ⇒ (2) follows — in light of Remark 2.4.1, (i) — from Lemma 2.8 and Theorem 2.9. This completes the proof of Theorem 2.10. □

3. The Case of Irreducible Representations of Dimension Two

In the present §3, we prove the existence of an *irreducible crystalline* [hence also *Hodge-Tate*] p-adic representation *of dimension* 2 that is *not Aut-intrinsically Hodge-Tate* [cf. Corollary 3.4 below].

In the present §3, let k be an MLF and $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$. We shall also apply the notational conventions introduced in Definition 1.2.

Lemma 3.1. — *Suppose that p_k is* ***odd****, and that $d_k = 2$. Write* $\mathrm{Nm}\colon k^\times \to (k^{(d=1)})^\times$ *for the norm map with respect to the finite extension $k/k^{(d=1)}$. Then the following assertions hold:*

(i) *There exists an* ***open*** *submodule $U \subseteq \mathcal{O}_k^\times$ of $\mathcal{O}_k^\times$ such that*

(1) *the topological module U has a natural structure of* ***free $\mathbb{Z}_{p_k}$-module of rank 2****, and, moreover,*

(2) *the submodule $U \subseteq \mathcal{O}_k^\times$ is* ***preserved*** *by an arbitrary continuous automorphism of $\mathcal{O}_k^\times$.*

(ii) *Let $U \subseteq \mathcal{O}_k^\times$ be as in* (i)*. Then the topological modules $U \cap \mathcal{O}_{k^{(d=1)}}^\times$, $U \cap \mathrm{Ker}(\mathrm{Nm})$ have natural structures of* ***free $\mathbb{Z}_{p_k}$-modules of rank 1****, respectively.*

(iii) *Let $U \subseteq \mathcal{O}_k^\times$ be as in* (i)*. Then the* ***equality*** $U \cap \mathcal{O}_{k^{(d=1)}}^\times \cap \mathrm{Ker}(\mathrm{Nm}) = \{1\}$ *holds.*

(iv) *Let $U \subseteq \mathcal{O}_k^\times$ be as in* (i)*. Then the closed submodule of U topologically generated by the closed submodules $U \cap \mathcal{O}_{k^{(d=1)}}^\times$ and $U \cap \mathrm{Ker}(\mathrm{Nm})$ is* ***open****.*

(v) *There exists a continuous automorphism α of G_k such that, for an arbitrary nonzero integer n, if one writes $\alpha_\times^n$ for the continuous automorphism of $\mathcal{O}_k^\times$ induced by α^n [cf. Remark* 1.2.1*], then the intersection $\alpha_\times^n(\mathcal{O}_{k^{(d=1)}}^\times) \cap \mathcal{O}_{k^{(d=1)}}^\times$ is* ***not open*** *in $\mathcal{O}_{k^{(d=1)}}^\times$. In particular, the continuous automorphism $\alpha_\times^n$ of $\mathcal{O}_k^\times$ does* ***not preserve*** *the submodule $\mathcal{O}_{k^{(d=1)}}^\times \subseteq \mathcal{O}_k^\times$.*

Proof. — Assertions (i), (ii) follow from [4, Lemma 1.2, (i)] [cf. also our assumption that $d_k = 2$]. Assertion (iii) is immediate [cf. the fact that U is *torsion-free* — cf. condition (1) of assertion (i)]. Assertion (iv) follows from assertions (ii), (iii), together with condition (1) of assertion (i).

Finally, we verify assertion (v). Let α be a continuous automorphism of G_k as in the discussion preceding [5, Theorem 1.5]. [Note that since $p(G_k) = p_k \neq 2$, $d(G_k) = d_k = 2 > 1$ — cf. [4, Proposition 3.6] — we are in the situation of the discussion preceding [5, Theorem 1.5].] Write β for the continuous automorphism of the submodule $U \subseteq \mathcal{O}_k^\times$ obtained by forming the restriction of $\alpha_\times^n$ [cf. condition (2) of assertion (i)]. Thus, since it follows from [5, Theorem 1.5] [cf. also [4, Definition 3.10, (vi)]] that

($a^\dagger$) the automorphism of $\mathcal{O}_k^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by $\alpha_\times^n$ is *not the identity automorphism*, but

($b^\dagger$) the image of the *square* of the endomorphism of $\mathcal{O}_k^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by the endomorphism of $\mathcal{O}_k^\times$ given by "$a \mapsto \alpha_\times^n(a) \cdot a^{-1}$" consists of the identity element of $\mathcal{O}_k^\times \otimes_{\mathbb{Z}} \mathbb{Q}$,

one concludes that

(a) the continuous automorphism β is *not the identity automorphism* of U, but

(b) the image of the *square* of the endomorphism of U given by "$a \mapsto \beta(a) \cdot a^{-1}$" consists of the identity element of U.

Moreover, it follows immediately from [5, Lemma 2.3, (i)] that

(c) the continuous automorphism β *preserves* the submodule $U \cap \mathrm{Ker}(\mathrm{Nm})$ of U.

Thus, it follows immediately from assertion (iv) [cf. also condition (1) of assertion (i)], together with (b) and (c), that if the continuous automorphism β *preserves* some open submodule of the submodule $U \cap \mathcal{O}^{\times}_{k^{(d=1)}}$, then β is the *identity automorphism* of U — in contradiction to (a). In particular, the continuous automorphism β does *not preserve* any open submodule of the submodule $U \cap \mathcal{O}^{\times}_{k^{(d=1)}}$, which thus implies [cf. assertion (ii)] that $\beta(U \cap \mathcal{O}^{\times}_{k^{(d=1)}}) \cap U \cap \mathcal{O}^{\times}_{k^{(d=1)}} = \{1\}$. Thus, it follows immediately from [4, Lemma 1.2, (i)] that $\alpha^n_{\times}(\mathcal{O}^{\times}_{k^{(d=1)}}) \cap \mathcal{O}^{\times}_{k^{(d=1)}}$ is *not open* in $\mathcal{O}^{\times}_{k^{(d=1)}}$, as desired. This completes the proof of assertion (v), hence also of Lemma 3.1. □

Remark 3.1.1. — One may conclude from the final portion of Lemma 3.1, (v), that it is *impossible* to establish a functorial group-theoretic reconstruction algorithm for constructing, from an *arbitrary* group H of MLF-type, a closed submodule of the topological module $\mathcal{O}^{\times}(H)$ which "*corresponds*" to the closed submodule $\mathcal{O}^{\times}_{k^{(d=1)}} \subseteq \mathcal{O}^{\times}_k$ of the topological module $\mathcal{O}^{\times}_k$. Put another way, one may conclude from the final portion of Lemma 3.1, (v), that the closed submodule $\mathcal{O}^{\times}_{k^{(d=1)}} \subseteq \mathcal{O}^{\times}_k$ should be considered to be "*not group-theoretic*".

Proposition 3.2. — *Suppose that p_k is* ***odd****, and that d_k is* ***even****. Suppose, moreover, that k is* ***absolutely abelian****, i.e., that k is absolutely Galois, and the Galois group* $\mathrm{Gal}(k/k^{(d=1)})$ *is abelian* [*cf.* [3, Definition 4.2, (ii)]]. *Then there exists a continuous automorphism α of G_k such that, for an arbitrary nonzero integer n, if one writes $\alpha^n_{\times}$ for the continuous automorphism of $\mathcal{O}^{\times}_k$ induced by α^n* [*cf. Remark* 1.2.1], *then the intersection $\alpha^n_{\times}(\mathcal{O}^{\times}_{k^{(d=1)}}) \cap \mathcal{O}^{\times}_{k^{(d=1)}}$ is* ***not open*** *in $\mathcal{O}^{\times}_{k^{(d=1)}}$.*

Proof. — Let us first observe that since d_k is *even*, and k is *absolutely abelian*, one verifies easily that there exists a *quadratic* extension of $k^{(d=1)}$ *contained* in k. Moreover, since k is *absolutely abelian*, it follows immediately from the implication (1) ⇒ (2) of [3, Theorem F, (i)] that G_k is a *characteristic* subgroup of the absolute Galois group of the quadratic extension of $k^{(d=1)}$ determined by the algebraic closure $\overline{k}$. Thus, one may conclude that we may assume without loss of generality, by applying a similar argument to the argument applied in the proof of [5, Lemma 2.6, (ii)] and replacing k by the quadratic extension of $k^{(d=1)}$, that $d_k = 2$. On the other hand, if $d_k = 2$, then the desired conclusion follows form Lemma 3.1, (v). This completes the proof of Proposition 3.2. □

Theorem 3.3. — *Let k be an MLF and $\overline{k}$ an algebraic closure of k. Suppose that p_k is* ***odd****, that d_k is* ***even****, and that k is* ***absolutely abelian****. Then there exist a $\mathbb{Q}_{p_k}$-vector space V* ***of dimension d_k*** *and a continuous representation $\rho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(V)$ that is* ***irreducible****,* ***abelian****,* ***crystalline*** [*hence also* ***Hodge-Tate***], *but* ***not Aut-intrinsically Hodge-Tate****.*

Proof. — Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$. Write ρ for the continuous representation of $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ [necessarily of dimension d_k] obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{\mathrm{ab}} \xrightarrow{\chi_{\pi,\mathrm{id}_k}} \mathcal{O}_k^\times \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion. Then one verifies easily that this continuous representation ρ is *irreducible* and *abelian*. Moreover, it follows immediately from [9, Chapter III, §A.4, Proposition 5] that this continuous representation ρ is *crystalline*.

Next, to verify that the continuous representation ρ is *not Aut-intrinsically Hodge-Tate*, let us recall that it follows immediately from the various definitions involved that the composite

$$\mathcal{O}_k^\times \xhookrightarrow{\mathrm{rec}_k} G_k^{\mathrm{ab}} \xrightarrow{\chi_{\pi,\mathrm{id}_k}} \mathcal{O}_k^\times$$

is an *automorphism* that restricts to an *automorphism* of the submodule $\mathcal{O}_{k^{(d=1)}}^\times \subseteq \mathcal{O}_k^\times$. In particular, if α is a continuous automorphism of G_k as in Proposition 3.2, then it follows immediately from Lemma 1.9, together with the various definitions involved, that the composite $\rho \circ \alpha \colon G_k \xrightarrow{\sim} G_k \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+)$ is *not Hodge-Tate*, which thus implies that the continuous representation ρ is *not Aut-intrinsically Hodge-Tate*, as desired. This completes the proof of Theorem 3.3. □

Corollary 3.4. — *Let p be an* ***odd*** *prime number. Then there exist an MLF K such that $p_K = p$, an algebraic closure $\overline{K}$ of K, a $\mathbb{Q}_{p_K}$-vector space V* ***of dimension 2****, and a continuous representation $\rho \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}_{\mathbb{Q}_{p_K}}(V)$ that is* ***irreducible****,* ***abelian****,* ***crystalline*** *[hence also* ***Hodge-Tate****], but* ***not Aut-intrinsically Hodge-Tate****.*

Proof. — This assertion is a formal consequence of Theorem 3.3. □

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