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A NOTE ON STABLE REDUCTION OF SMOOTH CURVES WHOSE JACOBIANS ADMIT STABLE REDUCTION

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ABSTRACT. P. Deligne and D. Mumford proved that, for a smooth curve over the field of fractions of a discrete valuation ring whose residue field is perfect, if the associated Jacobian has stable reduction over the discrete valuation ring, then the smooth curve has stable reduction over the discrete valuation ring. Recently, I. Nagamachi proved a similar result over a connected normal Noetherian scheme of dimension one. In the present paper, we prove a similar result over a Prüfer domain, i.e., a domain whose localization at each of the prime ideals is a valuation ring. Moreover, we also give a counter-example in a situation over a higher dimensional base case. More precisely, we construct an example of a smooth curve over the field of fractions of a complete strictly Henselian normal Noetherian local domain of equal characteristic zero such that the associated Jacobian has good reduction over the local domain, but the smooth curve does not have stable reduction over the local domain.

INTRODUCTION

Let $g \geq 2$ be an integer. In the present paper, a *smooth curve of genus g over a scheme B* is defined to be a stable curve of genus g over B in the sense of [2], Definition 1.1, whose structure morphism is smooth. Let S be a connected normal scheme, and let X be a smooth curve of genus g over the function field of S. Write $J(X)$ for the Jacobian of X [cf., e.g., the discussion at the beginning of [1], §9.2]. Then the present paper investigates the following question concerning the existence of stable models of curves:

Question: Are the following two conditions equivalent?

(1) The smooth curve X has stable reduction over S, i.e., extends to a stable curve over S.

(2) The abelian variety $J(X)$ has stable reduction over S, i.e., extends to a semi-abelian scheme over S.

Here, let us first recall that Deligne proved the implication (1) $\Rightarrow$ (2) [cf., e.g., [1], §9.4, Theorem 1]. Moreover, let us also recall that Deligne and Mumford proved the implication (2) $\Rightarrow$ (1) in the case where S is the spectrum of a discrete valuation ring whose residue field is perfect [cf. [2], Theorem 2.4]. Recently, Nagamachi developed the theory of minimal log regular models of curves and proved, as an application of this theory, the implication (2) $\Rightarrow$ (1) in the case where S is Noetherian and of dimension one [cf. [9], Corollary 0.3].

The first main result of the present paper is as follows [cf. §1]:

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Theorem A. *Suppose that S is the spectrum of a Prüfer domain [i.e., a domain whose localization at each of the prime ideals is a valuation ring]. Then the implication* $(2) \Rightarrow (1)$*, hence also the equivalence* $(1) \Leftrightarrow (2)$*, holds.*

Note that Theorem A generalizes the implication $(2) \Rightarrow (1)$ in the case where S is the spectrum of a Dedekind domain [i.e., a Noetherian Prüfer domain] proved by Deligne, Mumford, and Nagamachi.

On the other hand, in a higher dimensional base case, one may construct a counter-example of the implication $(2) \Rightarrow (1)$. The second main result of the present paper is as follows [cf. §2]:

Theorem B. *There exist a complete strictly Henselian normal Noetherian local domain of equal characteristic zero and a smooth curve over the field of fractions of this local domain such that the Jacobian of the smooth curve extends to an abelian scheme over the local domain, but the smooth curve does not extend to a stable curve over the local domain. In particular, the implication* $(2) \Rightarrow (1)$ *in the case where S is the spectrum of this local domain does not hold.*

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1. Equivalence over Arbitrary Prüfer Domains

In the present §1, we give a proof of Theorem A. Let $g \geq 2$ be an integer, and let R be a Prüfer domain. Write K for the field of fractions of R. Let X be a smooth curve over K of genus g, and let $\overline{K}$ be an algebraic closure of K. Write $J(X)$ for the Jacobian of X, $\widetilde{K} \subseteq \overline{K}$ for the separable closure of K in $\overline{K}$, and $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{K}/K)$ for the absolute Galois group of K determined by the separable closure $\widetilde{K}$. In the present §1, to prove Theorem A, suppose that

the abelian variety $J(X)$ extends to a semi-abelian scheme over R.

Definition 1.1. Let K_1 be an algebraic extension field of K contained in $\overline{K}$. Then we shall say that K_1 is *admissible* if the smooth curve $X \times_K K_1$ over K_1 extends to a stable curve over the normalization of R in K_1.

Thus, to verify Theorem A, it suffices to verify that the trivial extension field K of K is admissible. Now observe that one verifies immediately that, to verify the admissibility of K, we may assume without loss of generality, by replacing R by a strict Henselization of the localization of R at a prime ideal and applying étale descent, that

the ring R is a strictly Henselian valuation ring.

Definition 1.2. We shall write $\mathscr{M}_g \subseteq \overline{\mathscr{M}}_g$ for the moduli stacks of smooth, stable curves of genus g over R, respectively.

Lemma 1.3. *Let K_1, K_2 be algebraic extension fields of K contained in $\overline{K}$. Suppose that both K_1 and K_2 are admissible. Then the algebraic extension field of K obtained by forming the intersection $K_1 \cap K_2$ is admissible.*

Proof. Let i be an element of $\{1,2\}$. Write $R_i \subseteq K_i$ for the normalization of R in K_i. Then since K_i is admissible, it follows that the composite $\mathrm{Spec}(K_i) \to \mathrm{Spec}(K) \to \overline{\mathscr{M}}_g$ — where the second arrow is the K-valued point that classifies the smooth curve X over K — factors through the natural morphism $\mathrm{Spec}(K_i) \to \mathrm{Spec}(R_i)$. Next, observe that it follows immediately from [2], Lemma

1.12, that the image of the closed point of $\mathrm{Spec}(R_1)$ by the resulting morphism $\mathrm{Spec}(R_1) \to \overline{\mathscr{M}}_g$ coincides with the image of the closed point of $\mathrm{Spec}(R_2)$ by the resulting morphism $\mathrm{Spec}(R_2) \to \overline{\mathscr{M}}_g$. Let $\mathrm{Spec}(A) \to \overline{\mathscr{M}}_g$ be an affine étale neighborhood of this image of the closed points. Then since R_i is strictly Henselian, by pulling-back the resulting morphism $\mathrm{Spec}(R_i) \to \overline{\mathscr{M}}_g$ by this affine étale neighborhood, we obtain a factorization

$$\mathrm{Spec}(K_i) \longrightarrow \mathrm{Spec}(R_i) \longrightarrow \mathrm{Spec}(A) \longrightarrow \overline{\mathscr{M}}_g$$

of the composite $\mathrm{Spec}(K_i) \to \mathrm{Spec}(K) \to \overline{\mathscr{M}}_g$. Thus, one may conclude that the image of A in R_1 [i.e., by the homomorphism of rings induced by the second arrow of the above display in the case where we take the "i" to be 1] is contained in $R_1 \cap R_2$, which is the normalization of R in $K_1 \cap K_2$. In particular, one may conclude that the composite $\mathrm{Spec}(K_1 \cap K_2) \to \mathrm{Spec}(K) \to \overline{\mathscr{M}}_g$ factors through the natural morphism $\mathrm{Spec}(K_1 \cap K_2) \to \mathrm{Spec}(R_1 \cap R_2)$, as desired. This completes the proof of Lemma 1.3. $\square$

If R is a discrete valuation ring, then Theorem A may be proved only essentially by means of some results of [2] and some arguments concerning weakly unramified algebraic extension fields of K. To this end, let us introduce some notions.

Definition 1.4. Write k for the residue field of R. [So k is separably closed.] Suppose that R is a discrete valuation ring, and that k is of positive characteristic $p > 0$.

(i) Let $S \subseteq R^{\times}$ be a subset of $R^{\times}$. Then we shall say that S is a *p-basis-lifting* of R if $S \subseteq R^{\times}$ maps bijectively onto a p-basis of $k^{\times}$.

(ii) Let $S \subseteq R^{\times}$ be a subset of $R^{\times}$. Then we shall say that S is a *sub-p-basis-lifting* of R if S is contained in a p-basis-lifting of R.

(iii) Let n be a positive integer, and let S be a sub-p-basis-lifting of R. Then we shall say that an algebraic extension field of K is *of type* (n,S) (respectively, *of type* (∞,S)) if there exist, for each $s \in S$, a p^n-th root $s_n \in \overline{K}$ of $s \in S$ (respectively, a sequence $(s_n)_{n\geq 0} \subseteq \overline{K}$ that satisfies $s = s_0$ and $s_{n+1}^p = s_n$ for each $n \geq 0$) such that the extension field coincides with the algebraic extension field obtained by adjoining, to K, the subset $\{s_n\}_{s\in S} \subseteq \overline{K}$ (respectively, the subset $\{s_n\}_{s\in S, n\geq 0} \subseteq \overline{K}$). Note that one verifies easily that an arbitrary algebraic extension field of K of type (n,S) or of type (∞,S) is weakly unramified over K. Note also that if S is finite, then an arbitrary algebraic extension field of K of type (n,S) is finite over K.

(iv) Let S be a sub-p-basis-lifting of R. Then S is *admissible* if an arbitrary algebraic extension field of K of type (∞,S) contained in $\overline{K}$ is admissible.

Lemma 1.5. *In the situation of Definition 1.4, the following assertions hold:*

(i) *An arbitrary p-basis-lifting is admissible.*

(ii) *There exist a positive integer n, a finite sub-p-basis-lifting S, and an algebraic extension field of K of type (n,S) contained in $\overline{K}$ that is admissible.*

(iii) *Suppose that K contains a primitive p-th root of unity [which thus implies that K is of characteristic zero]. Let S be a p-basis-lifting of R, and let S_1, S_2 be subsets of S such that $S_1 \cap S_2 = \emptyset$. For each $i \in \{1,2\}$, let K_i be an algebraic extension field of K of type (∞,S_i) contained in $\overline{K}$. Then the equality $K = K_1 \cap K_2$ holds.*

Proof. First, we verify assertion (i). One verifies easily that if S is a p-basis-lifting of R, then the residue field of the normalization of R in an arbitrary algebraic extension field of K of type (∞,S)

is algebraically closed. Thus, it follows from [2], Theorem 2.4, that such an extension field is admissible. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let T be a p-basis-lifting of R, and let K_T be an algebraic extension field of K of type (∞, T) contained in $\overline{K}$. Then observe that it follows immediately from [11], Théorème 3.4.1, that an arbitrary stable curve over a Henselian local ring is of finite presentation over the Henselian local ring. In particular, by descending, to the normalization of R in a suitable intermediate field of K_T/K, the stable curve over the normalization of R in K_T that extends the smooth curve $X \times_K K_T$ over K_T [cf. assertion (i)], one may conclude that assertion (ii) holds. This completes the proof of assertion (ii). Assertion (iii) follows immediately from Kummer theory, together with the various definitions involved. This completes the proof of Lemma 1.5. $\square$

Now let us give a proof of Theorem A in the case where R is a discrete valuation ring only essentially by means of some results of [2] and some arguments concerning weakly unramified algebraic extension fields of K discussed in Lemma 1.5:

Proof of Theorem A in the case where R is a discrete valuation ring. Suppose that R is a discrete valuation ring. Write k for the residue field of R and p for the characteristic of k. Thus, since [we have assumed that] R is strictly Henselian, the field k is separably closed. If $p = 0$, then Theorem A follows from [2], Theorem 2.4. Suppose that $p > 0$. Let S be a p-basis-lifting of R. For each sub-p-basis-lifting U of R, let K_U be an algebraic extension field of K of type (∞, U) contained in $\overline{K}$. Recall from Lemma 1.5, (i), that K_S is admissible.

First, suppose that K is of positive characteristic. Then it is well-known [cf., e.g., [6], Chapter 10, Lemma 3.32] that there exists a separable algebraic extension field of K contained in $\overline{K}$ such that the residue field of the normalization of R in this extension field is algebraically closed, which thus [cf. [2], Theorem 2.4] implies that this extension field of K is admissible. Thus, since [it is immediate that] the extension field K_S is purely inseparable over K, it follows from Lemma 1.3 that K is admissible, as desired.

Next, suppose that K is of characteristic zero. Let us first observe that it follows immediately from Lemma 1.3 and Lemma 1.5, (ii), that, by considering the intersection of the admissible algebraic extension fields of K contained in $\overline{K}$, one may conclude that there exists a unique minimal admissible algebraic extension field of K contained in $\overline{K}$, which is necessarily finite and weakly unramified over K. In particular, it follows from Lemma 1.3 that, to verify the admissibility of K, we may assume without loss of generality, by replacing K by the finite extension field of K obtained by adjoining a primitive p-th root of unity in $\overline{K}$, that K contains a primitive p-th root of unity.

Let π be a prime element of R, U a sub-p-basis-lifting of R, and u an element of U. Write $F_1 \overset{\text{def}}{=} \{u\}$, $F_2 \overset{\text{def}}{=} \{(1+\pi) \cdot u\}$, $V \overset{\text{def}}{=} U \setminus F_1$, and $W \overset{\text{def}}{=} V \cup F_2$. Then it is immediate that F_1, F_2, V, and W are sub-p-basis-liftings of R. Write R_V for the normalization of R in K_V. [So $\pi \in R \subseteq R_V$ is a prime element of R_V.] Now we claim that

> if the sub-p-basis-liftings U and W are admissible, then the sub-p-basis-lifting V is admissible.

Indeed, observe that, to verify the admissibility of K_V, we may assume without loss of generality that the extension fields K_U, K_W are obtained by adjoining, to K_V, subsets $\{u_n\}_{n \geq 0}$, $\{w_m\}_{m \geq 0} \subseteq \overline{K}$ that satisfy the conditions that $u_0 = u$, $u_{n+1}^p = u_n$, $w_0 = (1+\pi) \cdot u$, $w_{m+1}^p = w_m$ for each n, $m \geq 0$, respectively. Next, let us observe that it follows from Lemma 1.3 that $K_U \cap K_W$ is admissible. Assume that $K_V \neq K_U \cap K_W$. Then it follows immediately from Kummer theory — together with our assumption that K, hence also K_V, contains a primitive p-th root of unity — that the element $1+\pi \in R_V^\times$ is contained in $(K_V^\times)^p$. On the other hand, it follows immediately from the [easily verified] injectivity of the p-th power endomorphism of the residue field of R_V and the [easily verified] fact that the module $(1+\pi R_V)/(1+\pi^2 R_V)$ is annihilated by p that $1+\pi \notin (K_V^\times)^p$. In particular, we obtain that $K_V = K_U \cap K_W$, which thus implies that K_V is admissible, as desired.

Next, we claim that

if F is a finite subset of S, then the sub-p-basis-lifting $S \setminus F$ is admissible.

Indeed, this claim follows immediately, by induction on $\#F$, from Lemma 1.5, (i), and the claim of the preceding paragraph, together with the observation that, in the preceding paragraph, if $U = S$, then W is a p-basis-lifting of R.

Next, let us observe that it follows from Lemma 1.5, (ii), that there exists a finite subset $S_0 \subseteq S$ of S such that K_{S_0} is admissible. On the other hand, it follows from the claim of the preceding paragraph that $K_{S\setminus S_0}$ is admissible. In particular, it follows from Lemma 1.3 that $K_{S_0} \cap K_{S\setminus S_0}$, hence [cf. Lemma 1.5, (iii)] also K, is admissible, as desired. This completes the proof of Theorem A in the case where R is a discrete valuation ring. □

Let us return to our discussion of Theorem A in the general situation.

Proposition 1.6. *Let R_0 be an excellent Henselian normal Noetherian local domain. Write K_0 for the field of fractions of R_0. Let $\widetilde{K}_0$ be a separable closure of K_0. Write $G_{K_0} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{K}_0/K_0)$ for the absolute Galois group of K_0 determined by the separable closure $\widetilde{K}_0$ and $I_{K_0} \subseteq G_{K_0}$ for the inertia subgroup of G_{K_0}. Let A_0 be an abelian variety over K_0, and let l be a prime number invertible in R_0. Suppose that A_0 extends to a semi-abelian scheme over R_0. Then the natural continuous action of I_{K_0} on the group of l-torsion points $A_0[l](\widetilde{K}_0)$ of A_0 is unipotent.*

Proof. Write $\widehat{R}_0$ for the completion of R_0. Then since R_0 is excellent, it follows from [4], Scholie 7.8.3, (iii), (v), that $\widehat{R}_0$ is a [necessarily excellent] complete normal Noetherian local domain. Write $\widehat{K}_0$ for the field of fractions of $\widehat{R}_0$, $G_{\widehat{K}_0}$ for the absolute Galois group of $\widehat{K}_0$ determined by some separable closure of $\widehat{K}_0$ that contains $\widetilde{K}_0$, and $I_{\widehat{K}_0} \subseteq G_{\widehat{K}_0}$ for the inertia subgroup of $G_{\widehat{K}_0}$. Now observe that since the field K_0 is separably closed in the extension field $\widehat{K}_0$ [cf. our assumption that R_0 is Henselian], the natural homomorphism $G_{\widehat{K}_0} \to G_{K_0}$, hence [cf. our assumption that R_0 is Henselian] also the natural homomorphism $I_{\widehat{K}_0} \to I_{K_0}$, is surjective. Thus, to verify Proposition 1.6, we may assume without loss of generality, by replacing R_0 by $\widehat{R}_0$, that R_0 is complete. Then the desired unipotency follows immediately from the theory of Raynaud extensions [cf., e.g., [3], Chapter II, §1; [3], Chapter III, Corollary 7.3]. This completes the proof of Proposition 1.6. □

Proposition 1.7. *Let A be an abelian variety over K, and let l be a prime number invertible in R. Suppose that A extends to a semi-abelian scheme over R. Then the natural continuous action of G_K on the group of l-torsion points $A[l](\widetilde{K})$ of A is unipotent. In particular, this natural continuous action factors thorough a finite l-group of G_K.*

Proof. Let us first observe that it follows immediately from [11], Théorème 3.4.1, that an arbitrary semi-abelian scheme over a Henselian local ring is of finite presentation over the Henselian local ring. Thus, one verifies immediately [cf., e.g., [4], Scholie 7.8.3, (ii), (iii)] that there exist an excellent Noetherian domain R_0, an injective homomorphism $R_0 \hookrightarrow R$ of rings, a semi-abelian scheme B_0 over R_0, and an isomorphism $A \xrightarrow{\sim} B_0 \times_{R_0} K$ over K. Since R is normal, and R_0 is excellent, we may assume without loss of generality [cf. [4], Scholie 7.8.3, (ii), (vi)], by replacing R_0 by the normalization of R_0, that R_0 is normal. Moreover, we may also assume without loss of generality [cf. [4], Scholie 7.8.3, (ii)], by replacing R_0 by the localization of R_0 at the prime ideal determined by the maximal ideal of R, that the ring R_0 is local, and the homomorphism $R_0 \hookrightarrow R$ is local. In particular, since R is Henselian, we may also assume without loss of generality [cf. [5], Théorème 18.6.6, (v); [5], Théorème 18.6.9, (i); [5], Corollaire 18.7.6], by replacing R_0 by the Henselization of R_0, that R_0 is Henselian. Then the desired unipotency follows immediately from Proposition 1.6. This completes the proof of Proposition 1.7. ☐

Definition 1.8. If l is an odd prime number invertible in R, then we shall write $\mathscr{M}_g[l]$ for the moduli stack of smooth curves of genus g over R equipped with Teichmüller structures of level l and $\overline{\mathscr{M}}_g[l]$ for the normalization of $\overline{\mathscr{M}}_g$ in the function field of $\mathscr{M}_g[l]$. Recall that it is well-known [cf., e.g., [10], Remark 2.3.7] that the stack $\overline{\mathscr{M}}_g[l]$ is a scheme and is proper over R.

We complete the proof of Theorem A as follows:

Proof of Theorem A. Let l_1, l_2 be distinct odd prime numbers invertible in R. Let i be an element of $\{1,2\}$. Write $J(X)[l_i]$ for the finite étale group scheme over K of l_i-torsion points of $J(X)$. Then it follows from Proposition 1.7 that there exists a finite extension field K_i of K of degree a power of l_i contained in $\overline{K}$ such that $J(X)[l_i] \times_K K_i$ is a constant group scheme over K_i. In particular, we obtain a K_i-valued point of $\mathscr{M}_g[l_i]$ that lifts the K_i-valued point of $\mathscr{M}_g$ that classifies the smooth curve $X \times_K K_i$ over K_i. Write R_i for the normalization of R in K_i. Then since $\overline{\mathscr{M}}_g[l_i]$ is a proper scheme over R, it follows from the valuative criterion for properness that this resulting K_i-valued point of $\mathscr{M}_g[l_i]$ extends to an R_i-valued point of $\overline{\mathscr{M}}_g[l_i]$, which thus implies that the finite extension field K_i of K is admissible. Now since [it is immediate from the fact that the extension degree of K_i/K is a power of l_i that] the equality $K = K_1 \cap K_2$ holds, we conclude from Lemma 1.3 that K is admissible, as desired. This completes the proof of Theorem A. ☐

2. Counter-example in a Higher Dimensional Base Case

In the present §2, we give a proof of Theorem B. Let $g \geq 3$ be an integer, and let k be an algebraically closed field of characteristic zero.

Definition 2.1. We shall write

- $\overline{\mathscr{M}}_g$ for the moduli stack of stable curves of genus g over k,
- $\mathscr{M}_g \subseteq \overline{\mathscr{M}}_g$ for the open substack of $\overline{\mathscr{M}}_g$ that classifies smooth curves of genus g over k,
- $\mathscr{X} \subseteq \overline{\mathscr{M}}_g$ for the open substack of $\overline{\mathscr{M}}_g$ that classifies stable curves of genus g over k whose dual graphs are trees,
- $\mathscr{A}_g$ for the moduli stack of principally polarized abelian varieties of dimension g over k, and
- $\mathscr{T}\colon \mathscr{X} \to \mathscr{A}_g$ for the [extended] Torelli map [cf., e.g., [7], §1.3].

We shall also write

- A for the moduli stack of principally polarized abelian varieties of dimension g over k equipped with level three structures,
- $X \stackrel{\text{def}}{=} \mathscr{X} \times_{\mathscr{A}_g} A$ for the fiber product of the Torelli map $\mathscr{X} \to \mathscr{A}_g$ and the natural finite étale covering $A \to \mathscr{A}_g$,
- $T\colon X \to A$ for the base-change of $\mathscr{T}$ by the natural finite étale covering $A \to \mathscr{A}_g$, and

- $\overline{M}$ for the normalization of $\overline{\mathscr{M}}_g$ in the function field of X.

In particular, we have a commutative diagram of stacks over k

$$\begin{tikzcd} A \arrow[d] & X \arrow[l, "T"'] \arrow[r, hook] \arrow[d] & \overline{M} \arrow[d] \\ \mathscr{A}_g & \mathscr{X} \arrow[l, "\mathscr{T}"] \arrow[r, hook] & \overline{\mathscr{M}}_g \end{tikzcd}$$

— where the vertical arrows are finite, and the right-hand horizontal arrows are the natural open immersions. Observe that the stacks A and $\overline{M}$ are schemes [cf., e.g., [3], Chapter IV, Remarks 6.2, (c); [10], Remark 2.3.7], which thus implies that the stack X is a scheme.

Let us recall the following well-known facts:

Lemma 2.2. *For a k-valued point $x \in \overline{\mathscr{M}}_g(k)$ of $\overline{\mathscr{M}}_g$, write $C(x)$ for the stable curve classified by $x \in \overline{\mathscr{M}}_g(k)$ and $J(x)$ for the Jacobian of the stable curve $C(x)$ [cf., e.g., the discussion at the beginning of* [1], *§9.2]. Then the following assertions hold:*

(i) *Let x be a k-valued point of $\overline{\mathscr{M}}_g$. Then the Jacobian $J(x)$ is an abelian variety over k if and only if x is a k-valued point of $\mathscr{X}$. Moreover, in this situation, the Jacobian $J(x)$ is isomorphic, as a principally polarized abelian variety over k, to the fiber product over k of the Jacobians of the irreducible components [each of which is necessarily a smooth curve over k] of the stable curve $C(x)$.*

(ii) *Let x_1, x_2 be k-valued points of $\mathscr{X}$. Suppose that the normalization of $C(x_1)$ is isomorphic to the normalization of $C(x_2)$ over k. Then $J(x_1)$ is isomorphic, as a principally polarized abelian variety over k, to $J(x_2)$.*

(iii) *The Torelli map $\mathscr{T}\colon \mathscr{X} \to \mathscr{A}_g$ is proper and restricts to a quasi-finite morphism $\mathscr{M}_g \to \mathscr{A}_g$. In particular, the morphism $T\colon X \to A$ is proper and generically quasi-finite.*

(iv) *There exists a k-valued point of $\mathscr{A}_g$ at which the fiber of the Torelli map $\mathscr{T}\colon \mathscr{X} \to \mathscr{A}_g$ is of positive dimension.*

Proof. Assertion (i) follows from [1], §9.2, Example 8. Assertion (ii) is an immediate consequence of assertion (i). Assertion (iii) follows from [7], §1.3, and the Torelli theorem.

Finally, we verify assertion (iv). Let us recall that we have assumed that $g \geq 3$. Thus, by considering various stable curves of genus g over K [necessarily classified by k-valued points of $\mathscr{X}$] obtained by glueing two fixed smooth curves of genus 1, $g-1$ over K, one may conclude that this assertion follows immediately from assertion (ii), together with the well-known [cf., e.g., [2], Theorem 1.11] finiteness of the automorphism groups of smooth curves of genus ≥ 2. This completes the proof of assertion (iv), hence also of Lemma 2.2. □

We complete the proof of Theorem B as follows:

Proof of Theorem B. Write $Z \to A$ for the finite morphism obtained by forming the normalization in the function field of X of the scheme-theoretic image of $T\colon X \to A$. In particular, the proper morphism $T\colon X \to A$ admits a factorization

$$X \xrightarrow{T_Z} Z \longrightarrow A$$

— where the first arrow T_Z is proper and birational [cf. Lemma 2.2, (iii)]. Thus, it follows from Lemma 2.2, (iv), that there exists a closed point $z \in Z$ at which the fiber of T_Z is of positive dimension. Write $R \stackrel{\mathrm{def}}{=} \mathscr{O}_{Z,z}$, $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R)$, η for the generic point of S, $\widehat{R}$ for the completion of R, $\widehat{S} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\widehat{R})$, and $\widehat{\eta}$ for the generic point of $\widehat{S}$. Then it follows immediately from the fact that T_Z is birational, together with the various definitions involved, that there exists a commutative diagram of schemes

$$\begin{tikzcd} \widehat{\eta} \arrow[r] \arrow[d] & \eta \arrow[r] \arrow[d] & X \arrow[d, "T_Z"'] \arrow[dr, "T"] & \\ \widehat{S} \arrow[r] & S \arrow[r] & Z \arrow[r] & A. \end{tikzcd}$$

Write

$$C \longrightarrow \widehat{\eta}$$

for the smooth curve classified by the composite $\widehat{\eta} \to \eta \to X$ $(\to \mathscr{X} \subseteq \overline{\mathscr{M}}_g)$. Then one verifies easily from the commutativity of the above diagram that the abelian scheme over $\widehat{S}$ classified by the composite $\widehat{S} \to S \to A$ $(\to \mathscr{A}_g)$ restricts to the Jacobian of this smooth curve $C \to \widehat{\eta}$. In particular, to verify Theorem B, it suffices to verify that the smooth curve C over $\widehat{\eta}$ does not extend to a stable curve over $\widehat{S}$. To this end, in the remainder of the present proof, assume that

the smooth curve C over $\widehat{\eta}$ extends to a stable curve over $\widehat{S}$.

Next, let us observe that it follows immediately from Lemma 2.2, (i), that the resulting stable curve over $\widehat{S}$ [i.e., that extends the smooth curve C over $\widehat{\eta}$] determines a morphism $\widehat{S} \to X$ over Z, hence also a splitting $s\colon \widehat{S} \to \widehat{Y}$ of the left-hand vertical arrow of the following commutative diagram of schemes

$$\begin{tikzcd} \widehat{Y} \stackrel{\mathrm{def}}{=} \widehat{S} \times_Z X \arrow[r] \arrow[d] & Y \stackrel{\mathrm{def}}{=} S \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "T_Z"] \\ \widehat{S} \arrow[r] & S \arrow[r] & Z \end{tikzcd}$$

— where the squares are cartesian. Write $\underline{Y}$ for the fiber of T_Z at $z \in Z$. Then $\underline{Y}$ is of positive dimension [cf. our choice of $z \in Z$] and may be identified with the fibers at the closed points of S, $\widehat{S}$ of the morphisms $Y \to S$, $\widehat{Y} \to \widehat{S}$ [i.e., that appear in the above diagram], respectively. Write, moreover, $y_0 \in \underline{Y}$ for the image of the closed point of $\widehat{S}$ by the splitting s. [So $\mathrm{Im}(s) \cap \underline{Y} = \{y_0\}$.] Fix a closed point y_1 of $\underline{Y} \setminus \{y_0\}$ [which is nonempty — cf. the fact that $\underline{Y}$ is of positive dimension].

Next, let us observe that since T_Z is birational, there exists a nonempty open subscheme $V \subseteq Z$ such that T_Z induces an isomorphism $T_Z^{-1}(V) \xrightarrow{\sim} V$. Fix such an open subscheme $V \subseteq Z$ and a closed point $x \in T_Z^{-1}(V)$. Then since X is irreducible [cf. [2], Theorem 5.2], it follows immediately from a similar argument to the argument applied in the proof of [8], p.56, Lemma [i.e., essentially proved by Bertini's theorem], that there exists an irreducible closed subscheme $D \subseteq X$ of dimension one such that $y_0 \notin D$, $y_1 \in D$, and $x \in D$. Write η_D for the generic point of D. Now we have the following commutative diagram of schemes

$$\begin{tikzcd}
 & D_{\widehat{S}} \stackrel{\text{def}}{=} \widehat{S} \times_Z D \arrow[r] \arrow[d, hook] & D_S \stackrel{\text{def}}{=} S \times_Z D \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\
\underline{Y} \arrow[r, hook] \arrow[d] & \widehat{Y} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d, "T_Z"] \\
\mathrm{Spec}(k(z)) \arrow[r, hook] & \widehat{S} \arrow[r] & S \arrow[r] & Z
\end{tikzcd}$$

— where we write $k(z)$ for the residue field of Z at z, and the squares are cartesian, and the upper vertical and left-hand horizontal arrows are the natural closed immersions.

Next, let us observe that since $y_1 \in D$, the inclusion $\eta_D \in D_S$ $(\subseteq D)$ holds. On the other hand, since $x \in D$, the inclusion $\eta_D \in T_Z^{-1}(V)$ holds. In particular, since the morphism T_Z induces an isomorphism $T_Z^{-1}(V) \xrightarrow{\sim} V$, one verifies immediately, by considering the base-change of the above diagram by the natural open immersion $V \hookrightarrow Z$, that every point $\eta' \in D_{\widehat{S}}$ $(\subseteq \widehat{Y})$ that maps to $\eta_D \in D_S$ by the morphism $D_{\widehat{S}} \to D_S$ is contained in the image of the splitting $s \colon \widehat{S} \to \widehat{Y}$. Fix such a point $\eta' \in D_{\widehat{S}}$ and write $F \subseteq D_{\widehat{S}}$ for the closure of η'. Note that since T_Z is proper, the morphism $\widehat{Y} \to \widehat{S}$ is proper. Thus, the image $\mathrm{Im}(s) \subseteq \widehat{Y}$ is a closed subset, which thus [cf. the inclusion $\eta' \in \mathrm{Im}(s)$] implies the inclusion $F \subseteq \mathrm{Im}(s)$. Moreover, the properness of the morphism $\widehat{Y} \to \widehat{S}$ also implies that $F \cap \underline{Y} \neq \emptyset$. In particular, we obtain that $\emptyset \neq F \cap \underline{Y} \subseteq \mathrm{Im}(s) \cap \underline{Y} = \{y_0\}$, which thus implies that $y_0 \in F \cap \underline{Y} \subseteq D_{\widehat{S}} \cap \underline{Y} = D \cap \underline{Y} \subseteq D$. However, this contradicts our choice of D. Therefore, we conclude that the smooth curve C over $\widehat{\eta}$ does not extend to a stable curve over $\widehat{S}$. This completes the proof of Theorem B. $\square$

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