TRIPOD-DEGREES

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Abstract. Let p, l be distinct prime numbers. A tripod-degree over p at l is defined to be an l-adic unit obtained by forming the image, by the l-adic cyclotomic character, of some continuous automorphism of the geometrically pro-l fundamental group of a split tripod over a finite field of characteristic p. The notion of a tripod-degree plays an important role in the study of the geometrically pro-l anabelian geometry of hyperbolic curves over finite fields, e.g., in the theory of cuspidalizations of the geometrically pro-l fundamental groups of hyperbolic curves over finite fields. In the present paper, we study the tripod-degrees. In particular, we prove that, under a certain condition, the group of tripod-degrees over p at l coincides with the closed subgroup of the group of l-adic units topologically generated by p. As an application of this result, we also conclude that, under a certain condition, the natural homomorphism from the group of automorphisms of the split tripod to the group of outer continuous automorphisms of the geometrically pro-l fundamental group of the split tripod that lie over the identity automorphism of the absolute Galois group of the basefield is surjective.

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Introduction

Let p, l be distinct prime numbers, and let $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p \stackrel{\mathrm{def}}{=} \mathbb{Z}/p\mathbb{Z}$ with p elements. Write T for the hyperbolic curve over $\mathbb{F}_p$ defined by

$$T \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\mathbb{F}_p} \setminus \{0, 1, \infty\},$$

which is a *split tripod* over $\mathbb{F}_p$. Write, moreover, $\Gamma_{\mathbb{F}_p} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ for the absolute Galois group of $\mathbb{F}_p$ determined by the algebraic closure $\overline{\mathbb{F}}_p$, Δ_T for the *pro-l geometric fundamental group* of T [i.e., the maximal pro-l quotient of the étale fundamental group $\pi_1^{\text{ét}}(T \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p)$ of $T \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$], and Π_T for the *geometrically pro-l fundamental group* of T [i.e., the quotient of the étale fundamental group $\pi_1^{\text{ét}}(T)$ of T by the kernel of the natural surjective continuous homomorphism $\pi_1^{\text{ét}}(T \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \twoheadrightarrow \Delta_T$ — cf., e.g., the discussion

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entitled “Fundamental groups” in [1], §0]. Thus, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_T \longrightarrow \Pi_T \longrightarrow \Gamma_{\mathbb{F}_p} \longrightarrow 1.$$

Then one may prove [cf., e.g., Lemma 1.2, (iii)] that every continuous automorphism of Π_T maps a cuspidal inertia subgroup of Δ_T bijectively to a cuspidal inertia subgroup of Δ_T. Write

$$\mathrm{Aut}^*_{\Gamma_{\mathbb{F}_p}}(\Pi_T) \subseteq \mathrm{Aut}_{\Gamma_{\mathbb{F}_p}}(\Pi_T)$$

for the subgroup consisting of continuous automorphisms of Π_T over $\Gamma_{\mathbb{F}_p}$ that determine the identity automorphism of the set of conjugacy classes of cuspidal inertia subgroups of Δ_T. Moreover, one may also prove [cf., e.g., Remark 4.2.2] that, for each $\alpha \in \mathrm{Aut}^*_{\Gamma_{\mathbb{F}_p}}(\Pi_T)$, there exists a unique element $\mathrm{Cyc}_T(\alpha) \in \mathbb{Z}_l^\times$ of $\mathbb{Z}_l^\times$ such that the continuous action of α on the topological abelianization of Δ_T is given by the multiplication by $\mathrm{Cyc}_T(\alpha)$. Thus, we have a homomorphism

$$\mathrm{Cyc}_T \colon \mathrm{Aut}^*_{\Gamma_{\mathbb{F}_p}}(\Pi_T) \longrightarrow \mathbb{Z}_l^\times,$$

i.e., the *l-adic cyclotomic character* associated to Π_T [cf. Definition 1.3, (ii); also Remark 4.2.2]. We shall write

$$\mathfrak{Tpd}_{p,l} \stackrel{\mathrm{def}}{=} \mathrm{Cyc}_T\big(\mathrm{Aut}^*_{\Gamma_{\mathbb{F}_p}}(\Pi_T)\big) \subseteq \mathbb{Z}_l^\times$$

for the image of $\mathrm{Aut}^*_{\Gamma_{\mathbb{F}_p}}(\Pi_T)$ by Cyc_T [cf. Definition 4.2] and refer to an element of $\mathfrak{Tpd}_{p,l}$ as a *tripod-degree* over p at l [cf. [1], Definition 3.1]. The notion of a tripod-degree plays an important role in the study of the geometrically pro-l anabelian geometry of hyperbolic curves over finite fields, e.g., in the theory of cuspidalizations of the geometrically pro-l fundamental groups of hyperbolic curves over finite fields [cf. [1], [5], [9]]. In the present paper, we study the tripod-degrees. More precisely, in the present paper, we completely determine the set of tripod-degrees under a certain condition. The main result of the present paper is as follows [cf. Theorem 4.6]. Write

$$\langle p \rangle \subseteq \mathbb{Z}_l^\times$$

for the closed subgroup topologically generated by $p \in \mathbb{Z}_l^\times$:

Theorem A. *Suppose that one of the following two conditions is satisfied:*

- *The equality $\langle p \rangle = \mathbb{Z}_l^\times$ holds, or, equivalently, the group $(\mathbb{Z}/l^2\mathbb{Z})^\times$ (respectively, $(\mathbb{Z}/8\mathbb{Z})^\times$) is generated by the image of p if $l \neq 2$ (respectively, $l = 2$).*
- *The element $-1 \in \mathbb{Z}_l^\times$ is not contained in the closed subgroup $\langle p \rangle \subseteq \mathbb{Z}_l^\times$.*

Then the equality

$$\mathfrak{Tpd}_{p,l} = \langle p \rangle$$

holds.

One main application of this main result of the present paper is the following result concerning the geometrically pro-l anabelian geometry of split tripods over finite fields [cf. Corollary 4.7]. Write

$$\mathrm{Aut}(T)$$

for the group of automorphisms of T and

$$\mathrm{Out}_{\Gamma_{\mathbb{F}_p}}(\Pi_T) \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\Gamma_{\mathbb{F}_p}}(\Pi_T)/\mathrm{Inn}(\Pi_T)$$

for the group of outer continuous automorphisms of Π_T that lie over the identity automorphism of the abelian profinite group $\Gamma_{\mathbb{F}_p}$.

Theorem B. *Suppose that one of the following two conditions is satisfied:*

- *The equality* $\langle p \rangle = \mathbb{Z}_l^\times$ *holds, or, equivalently, the group* $(\mathbb{Z}/l^2\mathbb{Z})^\times$ *(respectively,* $(\mathbb{Z}/8\mathbb{Z})^\times$*) is generated by the image of* p *if* $l \neq 2$ *(respectively,* $l = 2$*).*
- *The element* $-1 \in \mathbb{Z}_l^\times$ *is not contained in the closed subgroup* $\langle p \rangle \subseteq \mathbb{Z}_l^\times$*.*

Then the natural homomorphism

$$\mathrm{Aut}(T) \longrightarrow \mathrm{Out}_{\Gamma_{\mathbb{F}_p}}(\Pi_T)$$

is an isomorphism.

Note that a similar bijectivity to the bijectivity discussed in this main application in the case of étale fundamental groups (respectively, of geometrically pro-Σ fundamental groups for suitable sets Σ of prime numbers) [i.e., as opposed to the case of geometrically pro-l fundamental groups discussed in this main application] has been discussed in [8], Theorem 0.6 (respectively, [7], Theorem D).

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1. Actions on Cusps and Cyclotomes

In the present §1, we discuss the continuous actions of a continuous automorphism of the geometrically pro-l fundamental group of a hyperbolic curve over a finite field on the set of cusps [cf. Definition 1.3, (i), below] and on the associated cyclotome [cf. Definition 1.3, (ii), below].

In the present §1, let p, l be distinct prime numbers, $\mathbb{F}$ a finite field of characteristic p, and X a hyperbolic curve over $\mathbb{F}$. Write

- X^+ for the *smooth compactification* of X [so X^+ is a projective smooth curve over $\mathbb{F}$],
- $g_X \stackrel{\mathrm{def}}{=} \dim_{\mathbb{F}}(H^1(X^+, \mathcal{O}_{X^+}))$ for the *genus* of X^+,
- Π_X for the *geometrically pro-l fundamental group* of X relative to some choice of basepoint,
- Π_{X^+} for the quotient of Π_X by the normal closed subgroup normally topologically generated by the cuspidal inertia subgroups of Π_X [so Π_{X^+} is none other than the *geometrically pro-l fundamental group* of X^+ relative to an appropriate choice of basepoint],
- $\overline{\mathbb{F}}$ for the algebraic closure of $\mathbb{F}$ determined by the basepoint used so as to define Π_X,

- $\Gamma_X \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ for the *absolute Galois group* of $\mathbb{F}$ determined by the algebraic closure $\overline{\mathbb{F}}$,
- $S_X \stackrel{\text{def}}{=} X^+(\overline{\mathbb{F}}) \setminus X(\overline{\mathbb{F}})$ for the set of [$\overline{\mathbb{F}}$-valued] *cusps* of X,
- $r_X \stackrel{\text{def}}{=} \# S_X$ for the number of cusps of X [so $2 - 2g_X - r_X < 0$],
- $\Delta_X \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_X \twoheadrightarrow \Gamma_X)$, $\Delta_{X^+} \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_{X^+} \twoheadrightarrow \Gamma_X)$ [so Δ_X, Δ_{X^+} are none other than the *pro-l geometric fundamental groups* of X, X^+ relative to appropriate choices of basepoints, respectively], and
- $\Lambda_{\overline{\mathbb{F}}}$ for the *pro-l cyclotome* associated to $\overline{\mathbb{F}}$, i.e., $\Lambda_{\overline{\mathbb{F}}} \stackrel{\text{def}}{=} \varprojlim_N \boldsymbol{\mu}_{l^N}(\overline{\mathbb{F}})$ — where the projective limit is taken over the positive integers N.

Thus, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & \Gamma_X & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_{X^+} & \longrightarrow & \Pi_{X^+} & \longrightarrow & \Gamma_X & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are exact, and the vertical arrows are the natural surjective continuous homomorphisms. Write, moreover,

- $\rho_X \colon \Gamma_X \to \mathrm{Out}(\Delta_X)$ for the *outer action* determined by the upper horizontal sequence of the above diagram and
- Λ_X for the *pro-l cyclotome* associated to X, i.e., the cyclotome [cf. [3], Definition 3.8, (i)] associated to the semi-graph of anabelioids of pro-l PSC-type [with no nodes] that arises from the hyperbolic curve $X \times_{\mathbb{F}} \overline{\mathbb{F}}$ over $\overline{\mathbb{F}}$.

In particular:

(a) If $r_X = 0$, then

$$\Lambda_X \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}_l}\big(H^2(\Delta_X, \mathbb{Z}_l), \mathbb{Z}_l\big)$$

[cf. [3], Theorem 3.7, (i)].

(b) The outer action $\rho_X \colon \Gamma_X \to \mathrm{Out}(\Delta_X)$ determines a natural structure of Γ_X-module on Λ_X. Moreover, the cyclotome Λ_X is isomorphic, as an abstract Γ_X-module, to the cyclotome $\Lambda_{\overline{\mathbb{F}}}$ [cf. [3], Corollary 3.9, (ii), (iii); also (a)], which thus implies that the cyclotome Λ_X is isomorphic, as an abstract module, to $\mathbb{Z}_l$.

Definition 1.1. We shall say that the hyperbolic curve X over $\mathbb{F}$ is *split* if the natural inclusion $X^+(\mathbb{F}) \setminus X(\mathbb{F}) \hookrightarrow S_X$ is bijective, i.e., every cusp of X is $\mathbb{F}$-rational.

Lemma 1.2. *The following assertions hold:*

(i) *The natural [necessarily surjective] map from S_X to the set of Δ_X-conjugacy classes of cuspidal inertia subgroups of Δ_X is bijective.*

(ii) *Every continuous automorphism of Π_X restricts to a continuous automorphism of the closed subgroup $\Delta_X \subseteq \Pi_X$. In particular, every continuous automorphism of Π_X is an automorphism over some continuous automorphism of Γ_X.*

(iii) *Every continuous automorphism of Π_X determines an automorphism of the set of cuspidal inertia subgroups of Δ_X.*

Proof. Assertion (i) follows immediately from the well-known structure of the pro-l geometric fundamental group of a hyperbolic curve over a field of characteristic $\neq l$ [cf. also the exact sequence (1-5) given in the discussion preceding [8], Corollary 1.4]. Assertion (ii) follows from a similar argument to the argument given in the discussion preceding [1], Remark 5, i.e., from the fact that the quotient $\Pi_X \twoheadrightarrow \Gamma_X$ of Π_X may be characterized as the [uniquely determined] maximal abelian torsion-free quotient of Π_X. Assertion (iii) follows — in light of the [easily verified] openness of the image, in $\mathbb{Z}_l^\times$, of the l-adic cyclotomic character associated to a finite field of characteristic $\neq l$ — from [4], Corollary 2.7, (i), together with assertion (ii). This completes the proof of Lemma 1.2. $\square$

Definition 1.3.

(i) It follows from Lemma 1.2, (i), (ii), (iii), that we obtain a homomorphism

$$\mathrm{Aut}(\Pi_X) \longrightarrow \mathrm{Aut}(S_X).$$

We shall write

$$\mathrm{Csp}_X$$

for this homomorphism.

(ii) It follows from Lemma 1.2, (ii), (iii), that we obtain a homomorphism

$$\mathrm{Aut}(\Pi_X) \longrightarrow \mathrm{Aut}(\Lambda_X) = \mathbb{Z}_l^\times$$

[cf. (b) in the discussion preceding Definition 1.1]. We shall write

$$\mathrm{Cyc}_X$$

for this homomorphism and refer to Cyc_X as the *l-adic cyclotomic character* associated to Π_X.

Remark 1.3.1. One verifies easily that each of the two homomorphisms Csp_X, Cyc_X factors through the quotient $\mathrm{Aut}(\Pi_X)/\mathrm{Inn}(\Delta_X)$.

Definition 1.4.

(i) We shall write

$$\mathrm{Aut}^*(\Pi_X) \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathrm{Csp}_X)$$

$$\supseteq \mathrm{Aut}^*_{\Gamma_X}(\Pi_X) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^*(\Pi_X) \cap \mathrm{Aut}_{\Gamma_X}(\Pi_X).$$

(ii) We shall write

$$\mathrm{Out}^*(\Delta_X) \subseteq \mathrm{Out}(\Delta_X)$$

for the subgroup of $\mathrm{Out}(\Delta_X)$ consisting of outer continuous automorphisms of Δ_X which fix each of the Δ_X-conjugacy classes of cuspidal inertia subgroups of Δ_X.

Remark 1.4.1.

(i) It is well-known [cf., e.g., [8], Corollary 1.4, (ii); [8], Proposition 1.11] that Δ_X is center-free. Thus, it is also well-known [cf., e.g., [6], Corollary 1.5.7] that the natural homomorphism

$$\mathrm{Aut}_{\Gamma_X}(\Pi_X)/\mathrm{Inn}(\Delta_X) \longrightarrow \mathrm{Out}(\Delta_X)$$

determines an isomorphism

$$\mathrm{Aut}_{\Gamma_X}(\Pi_X)/\mathrm{Inn}(\Delta_X) \xrightarrow{\sim} Z_{\mathrm{Out}(\Delta_X)}\big(\mathrm{Im}(\rho_X)\big).$$

(ii) It is immediate that the isomorphism of (i) restricts to an isomorphism

$$\mathrm{Aut}^*_{\Gamma_X}(\Pi_X)/\mathrm{Inn}(\Delta_X) \xrightarrow{\sim} Z_{\mathrm{Out}(\Delta_X)}\big(\mathrm{Im}(\rho_X)\big) \cap \mathrm{Out}^*(\Delta_X).$$

Remark 1.4.2.

(i) Since Γ_X is abelian, the subgroup $\mathrm{Inn}(\Pi_X) \subseteq \mathrm{Aut}(\Pi_X)$ is contained in the subgroup $\mathrm{Aut}_{\Gamma_X}(\Pi_X) \subseteq \mathrm{Aut}(\Pi_X)$:

$$\mathrm{Inn}(\Pi_X) \subseteq \mathrm{Aut}_{\Gamma_X}(\Pi_X).$$

(ii) It follows from Remark 1.3.1 that the composite

$$\Pi_X \twoheadrightarrow \mathrm{Inn}(\Pi_X) \hookrightarrow \mathrm{Aut}_{\Gamma_X}(\Pi_X) \xrightarrow{\mathrm{Csp}_X} \mathrm{Aut}(S_X)$$

— where the first arrow is the natural surjective homomorphism, and the second arrow is the inclusion discussed in (i) — factors through the natural surjective continuous homomorphism $\Pi_X \twoheadrightarrow \Gamma_X$. Moreover, one verifies immediately that the following three conditions are equivalent:

- The resulting homomorphism $\Gamma_X \to \mathrm{Aut}(S_X)$ is trivial.
- The image $\mathrm{Im}(\rho_X) \subseteq \mathrm{Out}(\Delta_X)$ is contained in $\mathrm{Out}^*(\Delta_X)$.
- The hyperbolic curve X is split.

(iii) It follows from Remark 1.3.1 that the composite

$$\Pi_X \twoheadrightarrow \mathrm{Inn}(\Pi_X) \hookrightarrow \mathrm{Aut}_{\Gamma_X}(\Pi_X) \xrightarrow{\mathrm{Cyc}_X} \mathbb{Z}_l^\times$$

— where the first arrow is the natural surjective homomorphism, and the second arrow is the inclusion discussed in (i) — factors through the natural surjective continuous homomorphism $\Pi_X \twoheadrightarrow \Gamma_X$. Moreover, as discussed in (b) in the discussion preceding Definition 1.1, the resulting homomorphism $\Gamma_X \to \mathbb{Z}_l^\times$ is none other than the usual *l-adic cyclotomic character* associated to $\mathbb{F}$, i.e., the unique continuous character that maps the $\#\mathbb{F}$-th power Frobenius element of Γ_X to $\#\mathbb{F} \in \mathbb{Z}_l^\times$. In particular, the inclusions

$$\langle \#\mathbb{F} \rangle \subseteq \mathrm{Cyc}_X\big(\mathrm{Aut}_{\Gamma_X}(\Pi_X)\big) \subseteq \mathbb{Z}_l^\times$$

— where we write $\langle \#\mathbb{F} \rangle \subseteq \mathbb{Z}_l^\times$ for the closed subgroup topologically generated by $\#\mathbb{F} \in \mathbb{Z}_l^\times$ — hold.

Remark 1.4.3. It follows from [8], Remark 6.4, that every continuous automorphism of the étale fundamental group $\pi_1^{\text{ét}}(X)$ of X [i.e., as opposed to the geometrically pro-l fundamental group Π_X of X] lies over the identity automorphism of the "arithmetic quotient", i.e., over the quotient Γ_X. On the other hand, it follows from [1], Remark 10, (ii), that, in general, the equality $\mathrm{Aut}(\Pi_X) = \mathrm{Aut}_{\Gamma_X}(\Pi_X)$ does not hold.

2. Cuspidally Normalized and Cuspidally Quasi-normalized Functions

In the present §2, we introduce and consider the notion of a *cuspidally normalized* function [cf. Definition 2.2, (i), below] and the notion of a *cuspidally quasi-normalized* function [cf. Definition 2.2, (ii), below].

In the present §2, we maintain the notational conventions introduced at the beginning of the preceding §1. In particular, we have distinct prime numbers p, l and a hyperbolic curve X over a finite field $\mathbb{F}$. Suppose, moreover, that

(a) the hyperbolic curve X is split [cf. Definition 1.1].

For each cusp $x \in S_X$,

(b) let us fix a cuspidal inertia subgroup $I_x \subseteq \Delta_X$ associated to $x \in S_X$.

Moreover,

(c) let us fix a(n) [necessarily Γ_X-equivariant] isomorphism

$$\iota \colon \Lambda_{\overline{\mathbb{F}}} \xrightarrow{\sim} \Lambda_X$$

[cf. (b) in the discussion preceding Definition 1.1].

Definition 2.1.

(i) Let S and T be sets, and let $\phi \colon S \to T$ be a map. Suppose that S is finite. Then we shall write

$$[\mathrm{Im}](\phi) \overset{\mathrm{def}}{=} \left(\#\phi^{-1}(\{t\})\right)_{t \in T} \in \prod_T \mathbb{Z}.$$

Note that one verifies easily that if we write $[\mathrm{Im}](\phi) = (n_t)_{t \in T}$, then the equality $\mathrm{Im}(\phi) = \{\, t \in T \mid n_t \neq 0 \,\}$ holds.

(ii) Let G be a finite abelian group. Then we shall write

$$G(l)$$

for the [uniquely determined] *maximal quotient of* G *of order a power of* l.

Definition 2.2. Let f be a rational function on X, and let $S \subseteq S_X$ be a subset of S_X.

(i) We shall say that f is S-*cuspidally normalized* if the support of the principal divisor determined by f is contained in S [which thus implies that the rational function f is invertible on X — i.e., is contained in $\mathcal{O}_X^\times(X)$], and, moreover, there exists a cusp $x \in S$ contained in S such that the equality $f(x) = 1$ holds. We shall say that f is *cuspidally normalized* if f is S_X-cuspidally normalized.

(ii) We shall say that f is S-*cuspidally quasi-normalized* if f is obtained by forming the product of finitely many S-cuspidally normalized functions. We shall say that f is *cuspidally quasi-normalized* if f is S_X-cuspidally quasi-normalized.

Lemma 2.3. *The following assertions hold:*

(i) *The natural surjective continuous homomorphism* $\Pi_X \twoheadrightarrow \Gamma_X$ *and the natural inclusions* $I_x \hookrightarrow \Pi_X$ [*cf.* (b) *in the discussion preceding Definition* 2.1] — *where* x *ranges over the elements of* S_X — *determine an exact sequence of modules*

$$0 \longrightarrow H^1(\Gamma_X, \Lambda_X) \longrightarrow H^1(\Pi_X, \Lambda_X) \longrightarrow \bigoplus_{x \in S_X} \mathrm{Hom}_{\mathbb{Z}_l}(I_x, \Lambda_X).$$

(ii) *Let* $x_0 \in S_X$ *be a cusp of* X*. Write* $D_{x_0} \overset{\mathrm{def}}{=} N_{\Pi_X}(I_{x_0})$ *for the cuspidal decomposition subgroup of* Π_X *associated to* x_0 *determined by* I_{x_0}*. Then the natural continuous homomorphisms* $I_{x_0} \hookrightarrow D_{x_0} \hookrightarrow \Pi_X \twoheadrightarrow \Gamma_X$ *determine an exact sequence of modules*

$$0 \longrightarrow H^1(\Gamma_X, \Lambda_X) \longrightarrow H^1(D_{x_0}, \Lambda_X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l}(I_{x_0}, \Lambda_X).$$

(iii) *In the situation of* (ii)*, the synchronization isomorphism* $I_{x_0} \overset{\sim}{\to} \Lambda_X$ *discussed in* [3]*, Corollary* 3.9*,* (v)*, determines an isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}_l}(I_{x_0}, \Lambda_X) \xrightarrow{\sim} \mathbb{Z}_l.$$

In particular, by (i) *and* (ii)*, we obtain a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\Gamma_X, \Lambda_X) & \longrightarrow & H^1(\Pi_X, \Lambda_X) & \longrightarrow & \bigoplus_{x \in S_X} \mathbb{Z}_l \\ & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(\Gamma_X, \Lambda_X) & \longrightarrow & H^1(D_{x_0}, \Lambda_X) & \longrightarrow & \mathbb{Z}_l \end{array}$$

— where the middle vertical arrow is the homomorphism induced by the natural inclusion $D_{x_0} \hookrightarrow \Pi_X$*, and the right-hand vertical arrow is the projection homomorphism onto the factor labeled by* $x_0 \in S_X$*.*

Proof. Assertion (i) follows immediately from [5], Proposition 2.1, (ii). Assertions (ii), (iii) are immediate. □

Definition 2.4. Let $x_0 \in S_X$ be a cusp of X.

(i) We shall write

$$\mathrm{ord}_{x_0} \colon H^1(\Pi_X, \Lambda_X) \longrightarrow \mathbb{Z}_l$$

for the composite of the middle vertical arrow $H^1(\Pi_X, \Lambda_X) \to H^1(D_{x_0}, \Lambda_X)$ of the diagram of the second display of Lemma 2.3, (iii), and the third arrow $H^1(D_{x_0}, \Lambda_X) \to \mathbb{Z}_l$ of the lower sequence of the diagram of the second display of Lemma 2.3, (iii).

(ii) We shall write

$$\mathrm{ev}_{x_0} \colon \mathrm{Ker}(\mathrm{ord}_{x_0}) \longrightarrow H^1(\Gamma_X, \Lambda_X)$$

for the homomorphism determined by the middle vertical arrow $H^1(\Pi_X, \Lambda_X) \to H^1(D_{x_0}, \Lambda_X)$ of the diagram of the second display of Lemma 2.3, (iii).

Definition 2.5. Let s be an element of $H^1(\Pi_X, \Lambda_X)$, and let $S \subseteq S_X$ be a subset of S_X.

(i) We shall say that s is *S-cuspidally normalized* if the following two conditions are satisfied:
- The inclusion $s \in \mathrm{Ker}(\mathrm{ord}_x)$ holds for each $x \in S_X \setminus S$.
- There exists a cusp $x_0 \in S_X$ contained in S such that the inclusions $s \in \mathrm{Ker}(\mathrm{ord}_{x_0})$ and $s \in \mathrm{Ker}(\mathrm{ev}_{x_0})$ hold.

We shall say that s is *cuspidally normalized* if s is S_X-cuspidally normalized.

(ii) We shall say that s is *S-cuspidally quasi-normalized* if s is obtained by forming the product [i.e., if the module operation of $H^1(\Pi_X, \Lambda_X)$ is written multiplicatively] of finitely many S-cuspidally normalized cohomology classes in $H^1(\Pi_X, \Lambda_X)$. We shall say that s is *cuspidally quasi-normalized* if s is S_X-cuspidally quasi-normalized.

Lemma 2.6. *Write*

$$\kappa_X(\iota) \colon \mathcal{O}_X^\times(X) \longrightarrow H^1_{\text{ét}}(X, \Lambda_{\overline{\mathbb{F}}}) \xrightarrow{\sim} H^1(\Pi_X, \Lambda_X)$$

for the composite of the Kummer homomorphism $\mathcal{O}_X^\times(X) \to H^1_{\text{ét}}(X, \Lambda_{\overline{\mathbb{F}}})$ *— i.e., the homomorphism that arises from the Kummer exact sequences on* X *— and the isomorphism* $H^1_{\text{ét}}(X, \Lambda_{\overline{\mathbb{F}}}) \xrightarrow{\sim} H^1(\Pi_X, \Lambda_X)$ *induced by* ι [*cf.* (c) *in the discussion preceding Definition* 2.1]*. Let* $f \in \mathcal{O}_X^\times(X)$ *be an invertible regular function on* X*, and let* $x_0 \in S_X$ *be a cusp of* X*. Then the following assertions hold:*

(i) *The composite*

$$\mathbb{F}^{\times} \hookrightarrow \mathcal{O}_X^{\times}(X) \xrightarrow{\kappa_X(\iota)} H^1(\Pi_X, \Lambda_X)$$

factors through the natural surjective homomorphism $\mathbb{F}^{\times} \twoheadrightarrow \mathbb{F}^{\times}(l)$ [*cf.* *Definition* 2.1, (ii)] *and the second arrow* $H^1(\Gamma_X, \Lambda_X) \hookrightarrow H^1(\Pi_X, \Lambda_X)$ *of the upper horizontal sequence of the diagram of the second display of Lemma* 2.3, (iii). *Moreover, the resulting homomorphism*

$$\mathbb{F}^{\times}(l) \longrightarrow H^1(\Gamma_X, \Lambda_X)$$

is an isomorphism.

(ii) *The diagram*

$$\begin{array}{ccc} \mathcal{O}_X^{\times}(X) & \longrightarrow & \mathbb{Z} \\ {\scriptstyle \kappa_X(\iota)}\downarrow & & \downarrow \\ H^1(\Pi_X, \Lambda_X) & \xrightarrow[\mathrm{ord}_{x_0}]{} & \mathbb{Z}_l \end{array}$$

— where the upper horizontal arrow is the order homomorphism at x_0, *and the right-hand vertical arrow is the natural inclusion — commutes up to multiplication by an element of* $\mathbb{Z}_l^{\times}$.

(iii) *Suppose that the* [*rational function determined by the invertible regular*] *function* f *is of order zero at* x_0 [*i.e., that* $f(x_0) \in \mathbb{F}^{\times}$], *which thus implies* [*cf.* (ii)] *that* $\mathrm{ord}_{x_0}(\kappa_X(\iota)(f)) = 0$. *Then, relative to the isomorphism of the second display of* (i), *the equality*

$$\kappa_X(\iota)\big(f(x_0)\big) = \mathrm{ev}_{x_0}\big(\kappa_X(\iota)(f)\big)$$

holds.

(iv) *Let* $S \subseteq S_X$ *be a subset of* S_X. *Suppose that the* [*rational function determined by the invertible regular*] *function* $f \in \mathcal{O}_X^{\times}(X)$ *is* S*-cuspidally normalized* (*respectively,* S*-cuspidally quasi-normalized*). *Then the image* $\kappa_X(\iota)(f) \in H^1(\Pi_X, \Lambda_X)$ *of* f *by* $\kappa_X(\iota)$ *is an* S*-cuspidally normalized* (*respectively,* S*-cuspidally quasi-normalized*) *cohomology class.*

Proof. Assertion (i) is a formal consequence of the Kummer theory for the finite field $\mathbb{F}$. Assertion (ii) follows immediately from the definition of "ord_{x_0}" given in Definition 2.4, (i). Assertion (iii) follows immediately from the functoriality of Kummer classes. Finally, we verify assertion (iv). Let us first observe that it is immediate that, to verify assertion (iv), it suffices to verify the "non-resp'd portion" of assertion (iv). On the other hand, this "non-resp'd portion" is a formal consequence of assertions (ii), (iii). This completes the proof of assertion (iv), hence also of Lemma 2.6. □

Definition 2.7. Let $S \subseteq S_X$ be a subset of S_X. Then we shall say that an element of the set $\prod_{\mathbb{F}^{\times}(l)} \mathbb{Z}$ is *S-cuspidally quasi-normalized* if the element coincides with "[Im]" [cf. Definition 2.1, (i)] of either

- the map given by the composite

$$X(\mathbb{F}) \xrightarrow{f} \mathbb{F}^{\times} \twoheadrightarrow \mathbb{F}^{\times}(l)$$

for some S-cuspidally quasi-normalized function f on X or

- the map given by the composite

$$S_X \setminus S \xrightarrow{f} \mathbb{F}^\times \twoheadrightarrow \mathbb{F}^\times(l)$$

for some S-cuspidally quasi-normalized function f on X.

We shall say that an element of the set $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ is *cuspidally quasi-normalized* if the element is S_X-cuspidally quasi-normalized.

Remark 2.7.1. Let us observe that $\mathbb{F}^\times(l)$ has a natural structure of $\mathbb{Z}_l$-module. Thus, $\mathbb{Z}_l^\times$ naturally acts on $\mathbb{F}^\times(l)$, as well as the set $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ [i.e., that appears in Definition 2.7].

The following result is the main result of the present §2.

Theorem 2.8. *Let $S \subseteq S_X$ be a subset of S_X, and let $\alpha \in \mathrm{Aut}_{\Gamma_X}(\Pi_X)$ be a continuous automorphism of Π_X over Γ_X. Suppose that the automorphism $\mathrm{Csp}_X(\alpha) \in \mathrm{Aut}(S_X)$ of S_X induces the identity automorphism of the subset $S \subseteq S_X$. Then every S-cuspidally quasi-normalized element of $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ is fixed by the natural action* [*cf. Remark* 2.7.1] *of $\mathrm{Cyc}_X(\alpha) \in \mathbb{Z}_l^\times$ on the set $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$.*

Proof. We begin the proof of Theorem 2.8 with the following claim:

> Claim A: Every S-cuspidally quasi-normalized cohomology class in $H^1(\Pi_X, \Lambda_X)$ is fixed by the natural action of α on $H^1(\Pi_X, \Lambda_X)$.

Indeed, let us first observe that it follows immediately from the various definitions involved that, to verify Claim A, it suffices to verify that every S-cuspidally normalized cohomology class in $H^1(\Pi_X, \Lambda_X)$ is fixed by the natural action of α on $H^1(\Pi_X, \Lambda_X)$. Let $s \in H^1(\Pi_X, \Lambda_X)$ be an S-cuspidally normalized cohomology class. Thus,

- the inclusion $s \in \mathrm{Ker}(\mathrm{ord}_x)$ holds for each $x \in S_X \setminus S$, and
- there exists a cusp $x_0 \in S_X$ contained in S such that the inclusions $s \in \mathrm{Ker}(\mathrm{ord}_{x_0})$ and $s \in \mathrm{Ker}(\mathrm{ev}_{x_0})$ hold.

Next, let us observe that it follows from the exactness of the upper horizontal sequence of the diagram of the second display of Lemma 2.3, (iii), that the homomorphism

$$\bigcap_{x \in (S_X \setminus S) \cup \{x_0\}} \mathrm{Ker}(\mathrm{ord}_x) \xrightarrow{(\mathrm{ev}_{x_0}, (\mathrm{ord}_x)_{x \in S \setminus \{x_0\}})} H^1(\Gamma_X, \Lambda_X) \times \bigoplus_{x \in S \setminus \{x_0\}} \mathbb{Z}_l$$

is injective. In particular, it is immediate that, to verify Claim A, the image of $s \in H^1(\Pi_X, \Lambda_X)$ by this injective homomorphism is fixed by the natural action of α, i.e., on the codomain of this injective homomorphism. On the other hand, since $s \in \mathrm{Ker}(\mathrm{ev}_{x_0})$, the desired assertion follows formally from our assumption that the automorphism $\mathrm{Csp}_X(\alpha) \in \mathrm{Aut}(S_X)$ of S_X induces the identity automorphism of the subset $S \subseteq S_X$. This completes the proof of Claim A.

Write $X^\circ \subseteq X$ for the open subscheme of X obtained by forming the complement in X of [the closed subset of X determined by the finite subset] $X(\mathbb{F})$, i.e.,

$$X^\circ = X \setminus X(\mathbb{F}).$$

Then since X is split [cf. (a) in the discussion preceding Definition 2.1], one verifies easily that X° coincides with $X^+ \setminus X^+(\mathbb{F})$, which thus implies that X° is a split hyperbolic curve over $\mathbb{F}$ with $S_{X^\circ} = X^+(\mathbb{F})$. Moreover, since α is a continuous automorphism over Γ_X, it follows immediately from [9], Corollary 4.5, that there exists a continuous automorphism α° of Π_{X° which fits into a commutative diagram of profinite groups

$$\begin{array}{ccc} \Pi_{X^\circ} & \xrightarrow[\sim]{\alpha^\circ} & \Pi_{X^\circ} \\ \downarrow & & \downarrow \\ \Pi_X & \xrightarrow[\alpha]{\sim} & \Pi_X \end{array}$$

— where each of the vertical arrows is the Δ_X-conjugacy class of the surjective continuous homomorphisms $\Pi_{X^\circ} \twoheadrightarrow \Pi_X$ induced by the open immersion $X^\circ \hookrightarrow X$ over $\mathbb{F}$. In particular, this diagram, together with the synchronization isomorphism $\Lambda_{X^\circ} \xrightarrow{\sim} \Lambda_X$ discussed in [3], Corollary 3.9, (ii), induces a commutative diagram of modules

$$\begin{array}{ccc} H^1(\Pi_{X^\circ}, \Lambda_X) & \xleftarrow[\sim]{H_{\alpha^\circ}} & H^1(\Pi_{X^\circ}, \Lambda_X) \\ \uparrow & & \uparrow \\ H^1(\Pi_X, \Lambda_X) & \xleftarrow[H_\alpha]{\sim} & H^1(\Pi_X, \Lambda_X) \end{array}$$

— where the upper, lower horizontal arrows are the natural actions of α°, α on $H^1(\Pi_{X^\circ}, \Lambda_X)$, $H^1(\Pi_X, \Lambda_X)$, respectively, and the vertical arrows are injective [cf. Lemma 2.3, (i)].

Let z be an S-cuspidally quasi-normalized element of the set $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$. Thus, there exists an S-cuspidally quasi-normalized function f on X such that the element z is given by "[Im]" of either

- the map given by the composite

$$X(\mathbb{F}) \xrightarrow{f} \mathbb{F}^\times \twoheadrightarrow \mathbb{F}^\times(l)$$

  or

- the map given by the composite

$$S_X \setminus S \xrightarrow{f} \mathbb{F}^\times \twoheadrightarrow \mathbb{F}^\times(l).$$

Now observe that one verifies immediately from the functoriality of Kummer classes that the image of $\kappa_X(\iota)(f) \in H^1(\Pi_X, \Lambda_X)$ in $H^1(\Pi_{X^\circ}, \Lambda_X)$ coincides with $\kappa_{X^\circ}(\iota^\circ)(f|_{X^\circ})$ — where we write $\iota^\circ \colon \Lambda_{\overline{\mathbb{F}}} \xrightarrow{\sim} \Lambda_{X^\circ}$ for the composite of the fixed isomorphism $\iota \colon \Lambda_{\overline{\mathbb{F}}} \xrightarrow{\sim} \Lambda_X$ and the inverse of the synchronization isomorphism $\Lambda_{X^\circ} \xrightarrow{\sim} \Lambda_X$ discussed in [3], Corollary 3.9, (ii). Moreover, since f is S-cuspidally quasi-normalized, it follows from Lemma 2.6, (iv), that the image $\kappa_X(\iota)(f) \in H^1(\Pi_X, \Lambda_X)$ is S-cuspidally quasi-normalized. Thus, it follows from Claim A that $\kappa_X(\iota)(f) \in H^1(\Pi_X, \Lambda_X)$ is fixed by H_α, which thus [cf. the above diagram of cohomology modules] implies that $\kappa_{X^\circ}(\iota^\circ)(f|_{X^\circ}) \in H^1(\Pi_{X^\circ}, \Lambda_X)$ is fixed by H_{α°:

$$H_{\alpha^\circ}\big(\kappa_{X^\circ}(\iota^\circ)(f|_{X^\circ})\big) = \kappa_{X^\circ}(\iota^\circ)(f|_{X^\circ}).$$

Moreover, let us also observe that since α is a continuous automorphism over Γ_X [which thus implies that α° is a continuous automorphism over $\Gamma_{X^\circ} = \Gamma_X$], for each $x \in X(\mathbb{F}) \subseteq X^+(\mathbb{F}) = S_{X^\circ}$, the diagram of modules

$$\begin{array}{ccc} \mathrm{Ker}(\mathrm{ord}_{\mathrm{Csp}_{X^\circ}(\alpha)(x)}) & \xrightarrow[\sim]{H_{\alpha^\circ}} & \mathrm{Ker}(\mathrm{ord}_x) \\ \downarrow{\scriptstyle \mathrm{ev}_{\mathrm{Csp}_{X^\circ}(\alpha)(x)}} & & \downarrow{\scriptstyle \mathrm{ev}_x} \\ H^1(\Gamma_X, \Lambda_X) & \xrightarrow[\mathrm{Cyc}_X(\alpha)]{\sim} & H^1(\Gamma_X, \Lambda_X) \end{array}$$

commutes. In particular, it follows immediately from Lemma 2.6, (iii), together with the various definitions involved, that $z \in \prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ is fixed by the natural action of $\mathrm{Cyc}_X(\alpha) \in \mathbb{Z}_l^\times$ on the set $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$, as desired. This completes the proof of Theorem 2.8. $\square$

Corollary 2.9. *Let $\alpha \in \mathrm{Aut}_{\Gamma_X}(\Pi_X)$ be a continuous automorphism of Π_X over Γ_X. Suppose that $\mathrm{Csp}_X(\alpha) \in \mathrm{Aut}(S_X)$ is trivial. Then every cuspidally quasi-normalized element of $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ is fixed by the natural action of $\mathrm{Cyc}_X(\alpha) \in \mathbb{Z}_l^\times$ on the set $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$.*

Proof. This assertion is none other than Theorem 2.8 in the case where we take the "S" to be S_X. $\square$

3. Jacobi Sums

In the present §3, we recall a result concerning the field obtained by adjoining, to the field of rational numbers, various Jacobi sums [cf. Theorem 3.2 below].

In the present §3, we maintain the notational conventions introduced at the beginning of the preceding §2. In particular, we have distinct prime numbers p, l and a finite field $\mathbb{F}$ of characteristic p. Write

- N for the [uniquely determined] nonnegative integer such that $\#\mathbb{F}^\times(l) = l^N$,
- K for the finite Galois extension of the field $\mathbb{Q}$ of rational numbers obtained by adjoining, to $\mathbb{Q}$, a primitive l^N-th root of unity,
- $G \overset{\mathrm{def}}{=} \mathrm{Gal}(K/\mathbb{Q})$ for the *Galois group* of the finite Galois extension $K/\mathbb{Q}$, and
- $D \subseteq G$ for the *decomposition subgroup* associated to p.

For $t \in (\mathbb{Z}/l^N\mathbb{Z})^\times$, we shall write

- $\sigma_t \in G$ for the [uniquely determined] element that induces the t-th power map on $\boldsymbol{\mu}_{l^N}(K)$.

In particular:

(a) The assignment "$t \mapsto \sigma_t$" determines an isomorphism $(\mathbb{Z}/l^N\mathbb{Z})^\times \xrightarrow{\sim} G$ of groups.
(b) The subgroup $D \subseteq G$ coincides with the subgroup $\langle \sigma_p \rangle \subseteq G$ generated by σ_p, i.e., corresponds, via the isomorphism of (a), to the subgroup $\langle p \rangle \subseteq (\mathbb{Z}/l^N\mathbb{Z})^\times$ generated by the image of p.

Moreover, let us fix a homomorphism

$$\chi\colon \mathbb{F}^\times \longrightarrow K^\times$$

whose image coincides with $\boldsymbol{\mu}_{l^N}(K) \subseteq K^\times$, i.e., which factors through the natural surjective homomorphism $\mathbb{F}^\times \twoheadrightarrow \mathbb{F}^\times(l)$ and an injective homomorphism $\mathbb{F}^\times(l) \hookrightarrow K^\times$.

Following [10], let us define the notion of a *Jacobi sum* as follows.

Definition 3.1. Let $a = (a_1, a_2)$ be a pair of integers. Then we shall write

$$\mathbf{j}_a \overset{\mathrm{def}}{=} - \sum_{x \in \mathbb{F} \setminus \{0,-1\}} \chi(x)^{a_1} \cdot \chi(-1-x)^{a_2}$$

$$= -\chi(-1)^{a_1+a_2} \sum_{x \in \mathbb{F} \setminus \{0,1\}} \chi(x)^{a_1} \cdot \chi(1-x)^{a_2} \in K$$

for the *Jacobi sum* associated to χ and $a = (a_1, a_2)$ [cf. [10], (I)].

The following result will play an important role in the next §4.

Theorem 3.2. *Suppose that $-1 \in (\mathbb{Z}/l^N\mathbb{Z})^\times$ is not contained in the subgroup generated by the image of p. Suppose, moreover, that, for every proper subfield $\mathbb{F}' \subsetneq \mathbb{F}$, the inequality $\#(\mathbb{F}')^\times(l) < l^N$ holds [or, equivalently, the field $\mathbb{F}$ is isomorphic to the residue field of the ring of integers of K by a maximal ideal of residue characteristic p]. Then the equality*

$$\mathbb{Q}(\{\mathbf{j}_{(a_1,a_2)}\}_{a_1,\, a_2 \in \mathbb{Z}}) = K^D$$

holds. Put another way, for each $t \in (\mathbb{Z}/l^N\mathbb{Z})^\times$, the following two conditions are equivalent:

(1) *The element $t \in (\mathbb{Z}/l^N\mathbb{Z})^\times$ is contained in the subgroup generated by the image of p.*
(2) *For each pair $a = (a_1, a_2)$ of integers, the automorphism $\sigma_t \in G$ of the field K fixes the element*

$$\sum_{x \in \mathbb{F} \setminus \{0,1\}} \chi(x)^{a_1} \cdot \chi(1-x)^{a_2} \in K.$$

Proof. This assertion is the content of [2], Theorem A, (ii). □

4. Tripod-degrees

In the present §4, we prove the main result of the present paper concerning *tripod-degrees* [cf. Theorem 4.6 below]. Moreover, we also prove an application of this main result to the study of geometrically pro-l anabelian geometry for tripods over finite fields [cf. Corollary 4.7 below].

In the present §4, we maintain the notational conventions introduced at the beginning of the preceding §3. In particular, we have distinct prime numbers p, l and a hyperbolic curve X over a finite field $\mathbb{F}$. Write

- $\mathbb{F}_0 \subseteq \mathbb{F}$ for the [uniquely determined] *minimal subfield* of $\mathbb{F}$,
- T for the [necessarily split] *tripod* over $\mathbb{F}_0$ defined by

$$T \stackrel{\text{def}}{=} \mathbb{P}^1_{\mathbb{F}_0} \setminus \{0, 1, \infty\},$$

- Π_T for the *geometrically pro-l fundamental group* of T relative to some choice of basepoint,
- $\Gamma_T \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_0)$ for the *absolute Galois group* of $\mathbb{F}_0$ determined by the algebraic closure $\overline{\mathbb{F}}$, and
- $\Delta_T \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_T \twoheadrightarrow \Gamma_T)$ [so Δ_T is none other than the *pro-l geometric fundamental group* of T relative to an appropriate choice of basepoint].

Suppose, moreover, that the hyperbolic curve X over $\mathbb{F}$ is given by $T \times_{\mathbb{F}_0} \mathbb{F}$, i.e.,

$$X = T \times_{\mathbb{F}_0} \mathbb{F},$$

which thus implies that the equality $(X^+, g_X, r_X) = (\mathbb{P}^1_{\mathbb{F}}, 0, 3)$ holds. In particular, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & \Gamma_X & \longrightarrow & 1 \\ & & \wr\downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_T & \longrightarrow & \Pi_T & \longrightarrow & \Gamma_T & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are exact, the vertical arrows are open injective, and the right-hand square is cartesian. Let us identify Δ_X with Δ_T by the natural isomorphism, i.e., the left-hand vertical arrow of this diagram:

$$\Delta_X = \Delta_T.$$

Write, moreover,

- $\rho_T \colon \Gamma_T \to \mathrm{Out}(\Delta_T) = \mathrm{Out}(\Delta_X)$ for the *outer action* determined by the lower horizontal sequence of the above diagram.

Proposition 4.1. *The following assertions hold:*

(i) *The homomorphism*

$$\mathrm{Csp}_X \colon \mathrm{Aut}_{\Gamma_X}(\Pi_X) \longrightarrow \mathrm{Aut}(S_X)$$

is surjective.

(ii) *The homomorphism*

$$\mathrm{Aut}^*_{\Gamma_X}(\Pi_X)/\mathrm{Inn}(\Delta_X) \longrightarrow \mathbb{Z}_l^\times$$

determined by Cyc_X [*cf. Remark* 1.3.1] *is injective.*

(iii) *The group* $\mathrm{Aut}^*_{\Gamma_X}(\Pi_X)/\mathrm{Inn}(\Delta_X)$ *is abelian.*

(iv) *The restriction homomorphism*

$$\mathrm{Aut}_{\Gamma_T}(\Pi_T) \longrightarrow \mathrm{Aut}_{\Gamma_X}(\Pi_X)$$

[*cf. the diagram in the discussion preceding Proposition* 4.1] *is an isomorphism.*

Proof. Assertion (i) follows immediately from the well-known fact concerning automorphisms of $X = \mathbb{P}^1_{\mathbb{F}} \setminus \{0, 1, \infty\}$ over $\mathbb{F}$. Assertion (ii) is the content of [1], Remark 6, (iv) [cf. also the proof of [6], Lemma 2.2.4]. Assertion (iii) is an immediate consequence of assertion (ii).

Finally, we verify assertion (iv). Let us first observe that it follows from assertion (i) that, to verify assertion (iv), it suffices to verify the bijectivity of the restriction homomorphism $\mathrm{Aut}^*_{\Gamma_T}(\Pi_T) \to \mathrm{Aut}^*_{\Gamma_X}(\Pi_X)$. Thus, it follows from Remark 1.4.1, (ii), that, to verify assertion (iv), it suffices to verify that the immediate inclusion

$$Z_{\mathrm{Out}^*(\Delta_X)}\big(\mathrm{Im}(\rho_T)\big) \subseteq Z_{\mathrm{Out}^*(\Delta_X)}\big(\mathrm{Im}(\rho_X)\big)$$

[cf. also Remark 1.4.2, (ii)] is in fact an equality. On the other hand, it follows from assertion (iii), together with Remark 1.4.1, (ii), that $Z_{\mathrm{Out}^*(\Delta_X)}(\mathrm{Im}(\rho_X))$ is abelian. Moreover, since Γ_T, hence also $\mathrm{Im}(\rho_T)$, is abelian, the inclusion

$$\mathrm{Im}(\rho_T) \subseteq Z_{\mathrm{Out}^*(\Delta_X)}\big(\mathrm{Im}(\rho_X)\big)$$

holds. Thus, we conclude that the above inclusion $Z_{\mathrm{Out}^*(\Delta_X)}\big(\mathrm{Im}(\rho_T)\big) \subseteq Z_{\mathrm{Out}^*(\Delta_X)}\big(\mathrm{Im}(\rho_X)\big)$ is an equality, as desired. This completes the proof of assertion (iv), hence also of Proposition 4.1. ☐

Definition 4.2. We shall write

$$\mathfrak{Tpd} = \mathfrak{Tpd}_{p,l} \stackrel{\mathrm{def}}{=} \mathrm{Cyc}_T\big(\mathrm{Aut}^*_{\Gamma_T}(\Pi_T)\big) \subseteq \mathbb{Z}_l^\times$$

for the image of $\mathrm{Aut}^*_{\Gamma_T}(\Pi_T)$ via Cyc_T. We shall refer to an element of $\mathfrak{Tpd}$ as a *tripod-degree* [over p at l] [cf. [1], Definition 3.1].

Remark 4.2.1. It follows immediately from Proposition 4.1, (iv), that the equality

$$\mathfrak{Tpd} = \mathrm{Cyc}_X\bigl(\mathrm{Aut}^*_{\Gamma_X}(\Pi_X)\bigr)$$

holds.

Remark 4.2.2. One verifies immediately from [3], Corollary 3.9, (v), together with the the well-known structure of the pro-l geometric fundamental group Δ_T of the split tripod T [cf. also the exact sequence (1-5) given in the discussion preceding [8], Corollary 1.4], that, for each $\alpha \in \mathrm{Aut}^*_{\Gamma_T}(\Pi_T)$, the tripod-degree $\mathrm{Cyc}_T(\alpha) \in \mathfrak{Tpd}$ is the unique element of $\mathbb{Z}_l^\times$ such that the continuous action of α on the topological abelianization of Δ_T is given by the multiplication by $\mathrm{Cyc}_T(\alpha)$.

Proposition 4.3. *Let C be a(n) [arbitrary] hyperbolic curve over a finite field $\mathbb{F}_C$ of characteristic p. Write $\mathbb{F}_C$ (respectively, Π_C; Γ_C; Cyc_C) for the "$\mathbb{F}_X$" (respectively, "Π_X"; "Γ_X"; "Cyc_X") that occurs in the case where we take the "X" to be C. Then the inclusions*

$$\begin{array}{ccccc} \langle \#\mathbb{F}_C\rangle & \hookrightarrow & \mathrm{Cyc}_C\bigl(\mathrm{Aut}_{\Gamma_C}(\Pi_C)\bigr) & & \\ \downarrow & & \downarrow & & \\ \langle p\rangle & \hookrightarrow & \mathfrak{Tpd} & \hookrightarrow & \mathbb{Z}_l^\times \end{array}$$

— where we write $\langle \#\mathbb{F}_C\rangle$, $\langle p\rangle \subseteq \mathbb{Z}_l^\times$ for the closed subgroups topologically generated by $\#\mathbb{F}_C$, $p \in \mathbb{Z}_l^\times$, respectively — hold.

Proof. The inclusions $\langle \#\mathbb{F}_C\rangle \subseteq \langle p\rangle$, $\mathfrak{Tpd} \subseteq \mathbb{Z}_l^\times$ are immediate. The inclusions $\langle \#\mathbb{F}_C\rangle \subseteq \mathrm{Cyc}_C(\mathrm{Aut}_{\Gamma_C}(\Pi_C))$, $\langle p\rangle \subseteq \mathfrak{Tpd}$ follow from Remark 1.4.2, (iii). Thus, to verify Proposition 4.3, it suffices to verify the inclusion

$$\mathrm{Cyc}_C\bigl(\mathrm{Aut}_{\Gamma_C}(\Pi_C)\bigr) \subseteq \mathfrak{Tpd}.$$

Let us observe that since Π_C is topologically finitely generated [cf., e.g., [8], Proposition 1.1, (ii)], one verifies immediately that there exists a characteristic open subgroup of Π_C such that the "g_X" for the connected finite étale covering of C that corresponds to the characteristic open subgroup is ≥ 2. Thus, it follows immediately from [3], Corollary 3.9, (iii), that, to verify the inclusion $\mathrm{Cyc}_C(\mathrm{Aut}_{\Gamma_C}(\Pi_C)) \subseteq \mathfrak{Tpd}$, we may assume without loss of generality, by replacing C by the connected finite étale covering of C that corresponds to such a characteristic open subgroup of Π_C, that the "g_X" for C is ≥ 2. Next, let us observe that it follows from [3], Corollary 3.9, (ii), together with Lemma 1.2, (iii), that, to verify the inclusion $\mathrm{Cyc}_C(\mathrm{Aut}_{\Gamma_C}(\Pi_C)) \subseteq \mathfrak{Tpd}$, we may assume without loss of generality, by replacing C by the "X^+" for C, that the "r_X" for C is $= 0$. On the other hand, the inclusion $\mathrm{Cyc}_C(\mathrm{Aut}_{\Gamma_C}(\Pi_C)) \subseteq \mathfrak{Tpd}$ is then — in light of Remark 4.2.1 — a formal consequence of [1], Lemma 4.17. This completes the proof of the inclusion $\mathrm{Cyc}_C(\mathrm{Aut}_{\Gamma_C}(\Pi_C)) \subseteq \mathfrak{Tpd}$, hence also of Proposition 4.3. □

Definition 4.4. We shall write

$$\mathrm{Aut}(X)$$

for the group of automorphisms of X [as an abstract scheme, i.e., not necessarily over $\mathbb{F}$] and

$$\mathrm{Out}_{\Gamma_X}(\Pi_X) \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\Gamma_X}(\Pi_X)/\mathrm{Inn}(\Pi_X)$$

[cf. Remark 1.4.2, (i)]. Thus, we have a natural homomorphism

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Out}_{\Gamma_X}(\Pi_X).$$

Proposition 4.5. *The following assertions hold:*

(i) *The homomorphism*

$$\mathrm{Aut}_{\Gamma_X}(\Pi_X)/\mathrm{Inn}(\Delta_X) \longrightarrow \mathrm{Aut}(S_X) \times \mathfrak{Tpd}$$

determined by Csp_X *and* Cyc_X [*cf. Remark* 1.3.1] *is an isomorphism. In particular, we have an isomorphism*

$$\mathrm{Out}_{\Gamma_X}(\Pi_X) \xrightarrow{\sim} \mathrm{Aut}(S_X) \times (\mathfrak{Tpd}/\langle \#\mathbb{F}\rangle)$$

— where we write $\langle \#\mathbb{F}\rangle \subseteq \mathfrak{Tpd}$ *for the closed subgroup topologically generated by* $\#\mathbb{F} \in \mathfrak{Tpd}$ [*cf. Proposition* 4.3].

(ii) *The natural homomorphism* $\mathrm{Aut}(X) \to \mathrm{Out}_{\Gamma_X}(\Pi_X)$ *fits into an exact sequence of finite groups*

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Out}_{\Gamma_X}(\Pi_X) \longrightarrow \mathfrak{Tpd}/\langle p\rangle \longrightarrow 1$$

— where we write $\langle p\rangle \subseteq \mathfrak{Tpd}$ *for the closed subgroup topologically generated by* $p \in \mathfrak{Tpd}$ [*cf. Proposition* 4.3].

Proof. Assertion (i) follows immediately from Proposition 4.1, (i), (ii), and Remark 4.2.1. Assertion (ii) follows immediately from assertion (i), together with the well-known fact concerning automorphisms of the abstract scheme $X = \mathbb{P}^1_{\mathbb{F}} \setminus \{0, 1, \infty\}$. ☐

Remark 4.5.1. One verifies easily from Proposition 4.5, (i), that, in general, the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{Out}_{\Gamma_X}(\Pi_X)$ is not injective.

The following result is the main result of the present paper.

Theorem 4.6. *Write* $\langle p\rangle \subseteq \mathbb{Z}_l^\times$ *for the closed subgroup topologically generated by* $p \in \mathbb{Z}_l^\times$. *Suppose that one of the following two conditions is satisfied:*

(1) *The equality* $\langle p\rangle = \mathbb{Z}_l^\times$ *holds, or, equivalently, the group* $(\mathbb{Z}/l^2\mathbb{Z})^\times$ (*respectively,* $(\mathbb{Z}/8\mathbb{Z})^\times$) *is generated by the image of* p *if* $l \neq 2$ (*respectively,* $l = 2$).

(2) *The element* $-1 \in \mathbb{Z}_l^\times$ *is not contained in the closed subgroup* $\langle p\rangle \subseteq \mathbb{Z}_l^\times$.

Then the equality

$$\mathfrak{Tpd}_{p,l} = \langle p\rangle$$

holds.

Proof. Let us first observe that if condition (1) is satisfied, then the desired equality follows from Proposition 4.3. In the remainder of the present proof, suppose that condition (2) is satisfied.

Next, let us observe that it follows from Remark 4.2.1 that, to verify the desired equality, it suffices to verify the equality

$$\mathrm{Cyc}_X\bigl(\mathrm{Aut}^*_{\Gamma_X}(\Pi_X)\bigr) = \langle p\rangle.$$

In the remainder of the present proof, we verify this equality.

Next, let us observe that one verifies easily from Proposition 4.1, (iv), that, to verify the equality $\mathrm{Cyc}_X(\mathrm{Aut}^*_{\Gamma_X}(\Pi_X)) = \langle p\rangle$, we may assume without loss of generality, by replacing $\mathbb{F}$ by a suitable finite extension field of $\mathbb{F}$ in $\overline{\mathbb{F}}$, that the nonnegative integer N introduced in the discussion at the beginning of the preceding §3 [i.e., the nonnegative integer N such that $\#\mathbb{F}^\times(l) = l^N$] satisfies the condition that

$$\mathrm{Ker}\bigl(\mathbb{Z}_l^\times \twoheadrightarrow (\mathbb{Z}/l^N\mathbb{Z})^\times\bigr) \subseteq \langle p\rangle.$$

Moreover, let us also observe that one verifies easily from Proposition 4.1, (iv), that, to verify the equality $\mathrm{Cyc}_X(\mathrm{Aut}^*_{\Gamma_X}(\Pi_X)) = \langle p \rangle$, we may assume without loss of generality, by replacing $\mathbb{F}$ by a suitable subfield of $\mathbb{F}$, that, for every proper subfield $\mathbb{F}' \subsetneq \mathbb{F}$, the inequality $\#(\mathbb{F}')^\times(l) < l^N$ holds.

Next, let us observe that since $\langle p \rangle \subseteq \mathrm{Cyc}_X(\mathrm{Aut}^*_{\Gamma_X}(\Pi_X))$ [cf. Remark 4.2.1 and Proposition 4.3], one verifies easily that, to verify the equality $\mathrm{Cyc}_X(\mathrm{Aut}^*_{\Gamma_X}(\Pi_X)) = \langle p \rangle$, it suffices to verify the following claim:

> Claim A: For each $\alpha \in \mathrm{Aut}^*_{\Gamma_X}(\Pi_X)$, the image of $\mathrm{Cyc}_X(\alpha) \in \mathbb{Z}_l^\times$ in $(\mathbb{Z}/l^N\mathbb{Z})^\times \xrightarrow{\sim} G$ [cf. (a) in the discussion preceding Definition 3.1] is contained in the subgroup of G generated by the image of p in $(\mathbb{Z}/l^N\mathbb{Z})^\times \xrightarrow{\sim} G$, i.e., in $D \subseteq G$ [cf. (b) in the discussion preceding Definition 3.1].

Recall the fixed homomorphism $\chi \colon \mathbb{F}^\times \to K^\times$. Write

$$\chi(l) \colon \mathbb{F}^\times(l) \xrightarrow{\sim} \boldsymbol{\mu}_{l^N}(K)$$

for the isomorphism induced by χ [cf. the discussion preceding Definition 3.1]. Then since [we have assumed that] condition (2) is satisfied, it follows from Theorem 3.2 that, to verify Claim A, it suffices to verify the following claim:

> Claim B: For each $\alpha \in \mathrm{Aut}^*_{\Gamma_X}(\Pi_X)$ and each pair (a_1, a_2) of integers, the automorphism of the field K given by the image of $\mathrm{Cyc}_X(\alpha) \in \mathbb{Z}_l^\times$ in $(\mathbb{Z}/l^N\mathbb{Z})^\times \xrightarrow{\sim} G = \mathrm{Gal}(K/\mathbb{Q})$ fixes the element
>
> $$\sum_{x \in \mathbb{F} \setminus \{0,1\}} \chi(x)^{a_1} \cdot \chi(1-x)^{a_2} \in K.$$

In order to verify Claim B, let us fix a continuous automorphism $\alpha \in \mathrm{Aut}^*_{\Gamma_X}(\Pi_X)$ and a pair (a_1, a_2) of integers. Moreover, let us also fix a regular function $t \in \mathcal{O}_X(X)$ on X which determines an isomorphism of schemes over $\mathbb{F}$

$$X \xrightarrow{\sim} \mathrm{Spec}\Big(\mathbb{F}\Big[t, \frac{1}{t}, \frac{1}{1-t}\Big]\Big).$$

Then one verifies easily that the [rational function determined by the] invertible regular functions $t, 1-t \in \mathcal{O}_X^\times(X)$ are cuspidally normalized [cf. Definition 2.2, (i)]. In particular, the [rational function determined by the] invertible regular function $t^{a_1}(1-t)^{a_2} \in \mathcal{O}_X^\times(X)$ is cuspidally quasi-normalized [cf. Definition 2.2, (ii)]. Thus, it follows from Corollary 2.9 that

(a) the natural action of $\mathrm{Cyc}_X(\alpha) \in \mathbb{Z}_l^\times$ on $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ fixes the element $[\mathrm{Im}](\phi) \in \prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ — where we write ϕ for the map given by

$$X(\mathbb{F}) = \mathbb{F} \setminus \{0,1\} \xrightarrow{t^{a_1}(1-t)^{a_2}} \mathbb{F}^\times \twoheadrightarrow \mathbb{F}^\times(l).$$

Now let us consider the map

$$\prod_{\mathbb{F}^\times(l)} \mathbb{Z} \longrightarrow K$$

$$(n_a)_{a \in \mathbb{F}^\times(l)} \longmapsto \sum_{a \in \mathbb{F}^\times(l)} n_a \cdot \chi(l)(a).$$

Then one verifies easily that

(b) this map is compatible — relative to the natural surjective homomorphisms $\mathbb{Z}_l^\times \twoheadrightarrow (\mathbb{Z}/l^N\mathbb{Z})^\times \xrightarrow{\sim} G$ — with the natural action of $\mathbb{Z}_l^\times$ on $\prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ and the natural action of G on K, and that

(c) the image of the element $[\mathrm{Im}](\phi) \in \prod_{\mathbb{F}^\times(l)} \mathbb{Z}$ via this map is given by

$$\sum_{x \in \mathbb{F} \setminus \{0,1\}} \chi(x)^{a_1} \cdot \chi(1-x)^{a_2} \in K.$$

In particular, Claim B follows from (a), (b), (c). This completes the proof of Theorem 4.6. □

Remark 4.6.1. Observe that Theorem 4.6 yields infinitely many examples of pairs "(p, l)" such that

$$\mathfrak{Tpd}_{p,l} \neq \mathbb{Z}_l^\times$$

[cf. [1], Remark 6, (iii)]. For instance, if $p \equiv 2 \bmod 7$, then $\mathfrak{Tpd}_{p,7} \neq \mathbb{Z}_7^\times$.

Corollary 4.7. *Suppose that one of conditions* (1), (2) *in the statement of Theorem* 4.6 *is satisfied. Then the natural homomorphism*

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Out}_{\Gamma_X}(\Pi_X)$$

is surjective [*cf. also Remark* 4.5.1]. *If, moreover,* $\mathbb{F} = \mathbb{F}_0$*, then this natural homomorphism is an isomorphism.*

Proof. This assertion follows from Theorem 4.6, together with Proposition 4.5, (i), (ii), together with the well-known fact concerning automorphisms of the abstract scheme $T = \mathbb{P}^1_{\mathbb{F}_0} \setminus \{0, 1, \infty\}$. □

Remark 4.7.1. A similar surjectivity to the surjectivity discussed in Corollary 4.7 in the case of étale fundamental groups (respectively, of geometrically pro-Σ fundamental groups for suitable sets Σ of prime numbers) [i.e., as opposed to the case of geometrically pro-l fundamental groups discussed in Corollary 4.7] has been discussed in [8], Theorem 0.6 (respectively, [7], Theorem D).

Corollary 4.8. *Suppose that one of conditions* (1), (2) *in the statement of Theorem* 4.6 *is satisfied. Let C be a*(*n*) [*arbitrary*] *hyperbolic curve over the finite field with p elements. Write Π_C* (*respectively,* Γ_C*;* Cyc_C) *for the "Π_X"* (*respectively, "Γ_X"; "Cyc_X"*) *that occurs in the case where we take the "X" to be C. Then the equality*

$$\mathrm{Cyc}_C\big(\mathrm{Aut}_{\Gamma_C}(\Pi_C)\big) = \langle p \rangle$$

— where we write $\langle p \rangle \subseteq \mathbb{Z}_l^\times$ for the closed subgroup topologically generated by $p \in \mathbb{Z}_l^\times$ — holds.

Proof. This assertion follows from Theorem 4.6, together with Proposition 4.3. □

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