A Note on Open Homomorphisms Between Global Solvably Closed Galois Groups

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Abstract. — In the present paper, we study continuous open homomorphisms between the Galois groups of solvably closed Galois field extensions of number fields. In particular, we discuss Uchida's conjecture that asserts that an arbitrary continuous open homomorphism between the Galois groups of solvably closed Galois field extensions of number fields arises from a homomorphism between the given Galois field extensions. In the present paper, we prove that this conjecture is equivalent to the assertion that if the Galois group of a Galois field extension of a number field is isomorphic to an open subgroup of the maximal prosolvable quotient of the absolute Galois group of the field of rational numbers, then, for all prime numbers l and all but finitely many prime numbers p, the given Galois extension field contains l roots of the polynomial $t^l - p$. Moreover, we prove that this conjecture is also equivalent to the assertion that if the Galois group of a Galois field extension of an absolutely Galois number field is isomorphic to an open subgroup of the maximal prosolvable quotient of the absolute Galois group of the field of rational numbers, then the given Galois extension field is absolutely Galois.

Contents

Introduction

In the present paper, we study continuous open homomorphisms between the Galois groups of solvably closed Galois field extensions of number fields. We shall define

- a *number field* [cf. Definition 2.2, (i)] to be a field that is of characteristic zero and is finite over the minimal subfield [i.e., the prime subfield] of the field,

- a *solvably closed* field [cf. Definition 2.2, (ii)] to be a field that admits no nontrivial abelian field extension, and

- an *absolutely Galois* field [cf. Definition 3.3] to be a field that is [algebraic and] Galois over the minimal subfield of the field.

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In the present paper, we discuss the following conjecture posed by *K. Uchida* [cf. [8, Conjecture, p.595]]:

Conjecture A (Uchida). — *Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be* ***Galois*** *extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are* ***solvably closed****. Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous ***open*** *homomorphism. Then there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings from which the homomorphism α* ***arises****. Put another way, there exists a homomorphism $\alpha_{\widetilde{F}}\colon \widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings that is* ***compatible*** *with the respective actions of* $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$*, relative to the homomorphism α, i.e., such that, for each $\gamma \in \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, the* ***equality*** *$\gamma \circ \alpha_{\widetilde{F}} = \alpha_{\widetilde{F}} \circ \alpha(\gamma)$ holds.*

Let us first recall that Uchida *solved affirmatively* the assertion obtained by replacing "a continuous *open* homomorphism" in the statement of Conjecture A by "a continuous *open injective* homomorphism" [cf. [7, Theorem]]. Moreover, Uchida also gave, in [8], some important results concerning Conjecture A. For instance, Uchida proved, in the situation of Conjecture A,

- the existence of a homomorphism "$\alpha_{\widetilde{F}}$" as in the statement of Conjecture A in the case where the number field $F_\circ$ is *isomorphic to the field of rational numbers* [cf. [8, Theorem 1]],
- the existence of a homomorphism "$\alpha_{\widetilde{F}}$" as in the statement of Conjecture A in the case where the homomorphism α *satisfies a certain condition concerning decomposition subgroups of nonarchimedean primes* [cf. [8, Theorem 2]], and
- the *uniqueness* of a homomorphism "$\alpha_{\widetilde{F}}$" as in the statement of Conjecture A [cf. [8, Proposition 2]].

Moreover, the author of the present paper

- studied Conjecture A from a "group-theoretic algorithmic" point of view [cf. [2], [4]] and
- proved the existence of a homomorphism "$\alpha_{\widetilde{F}}$" as in the statement of Conjecture A in the case where the homomorphism α is *compatible with the cyclotomic characters* [cf. [5, Theorem]].

In the present paper, we give some *necessary and sufficient conditions* for a homomorphism "α" as in Conjecture A to arise from a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings from the point of view of the *kernel* of the homomorphism "α". Suppose that we are given a homomorphism α as in Conjecture A. Write ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ for the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of α. Then one immediate observation with respect to Conjecture A is that if ${}^{\alpha}F_\circ$ is *solvably closed*, then one concludes immediately from [7, Theorem] [i.e., an affirmative solution to the assertion obtained by replacing "a continuous *open* homomorphism" in the statement of Conjecture A by "a continuous *open injective* homomorphism"] that α arises from a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings, as desired. In the present paper, we give results related to this observation, i.e., the relationship between the *kernel* of α and the "*field-theoreticity/geometricity*" of α [cf. Theorem 2.7, Theorem 3.4]. Moreover, as applications of these results, we conclude the following result, which is the main result of the present paper:

Theorem B. — *Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Write $\mathbb{Q}^{\mathrm{slv}} \subseteq \overline{\mathbb{Q}}$ for the maximal prosolvable extension field of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. Then the following three assertions are equivalent:*

(1) *Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be* ***Galois*** *extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are* ***solvably closed****. Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous ***open*** *homomorphism. Then there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings* ***compatible*** *with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α.*

(2) *Let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ such that the field extension $F/\mathbb{Q}$ is* ***finite****, and, moreover, the field extension K/F is* ***Galois****. Suppose that the topological group $\mathrm{Gal}(K/F)$ is* ***isomorphic*** *to an open subgroup of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$. Then, for all prime numbers l and all but finitely many prime numbers p,* ***every l-th power root of*** *p in $\overline{\mathbb{Q}}$ is* ***contained*** *in $K \subseteq \overline{\mathbb{Q}}$.*

(3) *Let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ such that the field extension $F/\mathbb{Q}$ is* ***finite*** *and* ***Galois****, and, moreover, the field extension K/F is* ***Galois****. Suppose that the topological group $\mathrm{Gal}(K/F)$ is* ***isomorphic*** *to an open subgroup of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$. Then the field extension $K/\mathbb{Q}$ is* ***Galois****.*

In §1 of the present paper, we prove a technical lemma concerning continuous homomorphisms between topological groups *of MLF-type* [cf. Lemma 1.3]. This technical lemma may be regarded as a *partial generalization* of a result that was obtained in the study of the anabelian geometry of mixed-characteristic local fields [cf. Remark 1.3.1].

In §2 of the present paper, we prove the equivalence (1) ⇔ (2) of Theorem B. To explain one main observation in the proof of the implication (2) ⇒ (1) of Theorem B, suppose that we are given a homomorphism α as in Conjecture A, and write ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ for the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of α. Then one main observation in the proof of the implication (2) ⇒ (1) of Theorem B is that, under some mild assumptions, if, for all prime numbers l and all but finitely many prime numbers p, every l-th power root of p in $\widetilde{F}_\circ$ is *contained* in ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$, then the homomorphism $\mathrm{Gal}(F_\circ/F_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to \mathrm{Gal}(F_\bullet/F_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ determined by α is *Frobenius-preserving* [cf. [5, Definition 2.7]]. This observation will be essentially verified in Lemma 2.6.

In §3 of the present paper, we prove the equivalence (1) ⇔ (3) of Theorem B. To explain one main observation in the proof of the implication (3) ⇒ (1) of Theorem B, suppose that we are given a homomorphism α as in Conjecture A, and write ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ for the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of α. Then one main observation in the proof of the implication (3) ⇒ (1) of Theorem B is that if $\widetilde{F}_\circ$ is *algebraically closed*, and ${}^{\alpha}F_\circ$ is *Galois* over the minimal subfield of ${}^{\alpha}F_\circ$, then the homomorphism α *extends* to a homomorphism from the absolute Galois group of the minimal subfield of ${}^{\alpha}F_\circ$. This observation will be verified in the proof of Theorem 3.4.

Finally, let us observe that assertions (2), (3) that appear in the statement of Theorem B may be considered to be *purely "field-theoretic"*, hence also be *independent* of the study of anabelian geometry. Moreover, at least the author of the present paper does *not have any immediate proof* of the equivalence (2) $\Leftrightarrow$ (3) of Theorem B. In particular, the equivalence (2) $\Leftrightarrow$ (3) of Theorem B may be regarded as an *application* to the purely "field-theoretic" study of number fields, i.e., of the study of the anabelian geometry of number fields.

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1. Homomorphisms Between Topological Groups of MLF-type

In the present §1, we prove a technical lemma concerning continuous homomorphisms between topological groups *of MLF-type* [cf. Lemma 1.3 below]. This technical lemma may be regarded as a *partial generalization* of a result that was obtained in the study of the anabelian geometry of mixed-characteristic local fields [cf. Remark 1.3.1 below].

Definition 1.1. — Let D be a topological group of MLF-type [cf. [2, Definition 1.1], [2, Proposition 1.2, (i), (ii)]], i.e., a topological group such that there exist

- a prime number p,
- a finite extension field k of $\mathbb{Q}_p$,
- an algebraic closure $\overline{k}$ of k, and
- an isomorphism $\alpha_D\colon \mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} D$ of topological groups.

(i) Let us recall the positive integers

$$p(D), \qquad f(D)$$

defined in [2, Theorem 1.4, (1), (2)]. In particular, it follows from [2, Theorem 1.4, (i)] that the existence of the above isomorphism α_D implies that

(i-a) the positive integer $p(D)$ coincides with the prime number p, and that

(i-b) the positive integer $f(D)$ coincides with the extension degree of the residue field of k over the minimal subfield of the residue field of k.

(ii) Let us recall the closed subgroups

$$P(D) \subseteq I(D) \subseteq D$$

of D defined in [2, Theorem 1.4, (3)]. In particular, it follows from [2, Theorem 1.4, (ii)] that

(ii-a) the above isomorphism α_D restricts to a continuous isomorphism of the inertia subgroup of $\mathrm{Gal}(\overline{k}/k)$ with the closed subgroup $I(D)$ of D, and that

(ii-b) the above isomorphism α_D restricts to a continuous isomorphism of the wild inertia subgroup of $\mathrm{Gal}(\overline{k}/k)$ with the closed subgroup $P(D)$ of D.

(iii) Let us recall the closed subgroup

$$\mathcal{O}^\times(D) \stackrel{\text{def}}{=} \mathrm{Im}\big(I(D) \hookrightarrow D \twoheadrightarrow D^{\mathrm{ab}}\big) \subseteq D^{\mathrm{ab}}$$

of D^{ab} defined in [2, Theorem 1.4, (5)]. In particular, it follows from [2, Theorem 1.4, (iii)] that the existence of the above isomorphism α_D implies that

(iii-a) the topological group of units of the normalization of $\mathbb{Z}_p$ in k is isomorphic to the topological group $\mathcal{O}^\times(D)$.

Lemma 1.2. — *Let D be a topological group* ***of MLF-type****, and let l be a prime number* ***not equal*** *to $p(D)$. Then every pro-l-Sylow subgroup of $I(D)$ is* ***isomorphic*** *to the topological group $\mathbb{Z}_l$.*

Proof. — This assertion is well-known [cf., e.g., [3, Lemma 1.5, (ii)] and Definition 1.1, (i-a), (ii-a), (ii-b)]. □

Lemma 1.3. — *Let $D_\circ$, $D_\bullet$ be topological groups* ***of MLF-type****, and let $\alpha\colon D_\circ \to D_\bullet$ be a continuous homomorphism. Suppose that the following two conditions are satisfied:*

(1) *The* ***equality*** *$p(D_\circ) = p(D_\bullet)$ holds.*

(2) *Let l be a prime number* ***not equal*** *to $p(D_\circ) = p(D_\bullet)$ [cf. (1)]. Then there exist a pro-l-Sylow subgroup ${}_lI(D_\circ)$ of $I(D_\circ)$ and a normal open subgroup N of $D_\bullet$ such that the image of the composite*

$$ {}_lI(D_\circ) \hookrightarrow I(D_\circ) \hookrightarrow D_\circ \xrightarrow{\alpha} D_\bullet \twoheadrightarrow D_\bullet/N$$

— where the first and second arrows are the natural inclusions, and the fourth arrow is the natural continuous surjective homomorphism — is a ***nontrivial l-Sylow*** *subgroup of the finite group $D_\bullet/N$.*

Then the following assertions hold:

(i) *Let l be a prime number* ***not equal*** *to $p(D_\circ) = p(D_\bullet)$ [cf. (1)], and let ${}_lI(D_\circ) \subseteq I(D_\circ)$ be a pro-l-Sylow subgroup of $I(D_\circ)$. Then the homomorphism α restricts to an* ***isomorphism*** *of ${}_lI(D_\circ)$ with a pro-l-Sylow subgroup of $I(D_\bullet)$.*

(ii) *The integer $f(D_\circ)$ is* ***divisible*** *by the integer $f(D_\bullet)$.*

Proof. — We begin the proof of Lemma 1.3 with the following claim:

> **Claim 1.3.A.** — Let l be a prime number *not equal* to $p(D_\circ) = p(D_\bullet)$ [cf. condition (1)], and let ${}_lI(D_\circ) \subseteq I(D_\circ)$ be a pro-l-Sylow subgroup of $I(D_\circ)$. Then the image of the composite
>
> $$ {}_lI(D_\circ) \hookrightarrow I(D_\circ) \hookrightarrow D_\circ \xrightarrow{\alpha} D_\bullet$$
>
> — where the first and second arrows are the natural inclusions — is *contained* in the subgroup $I(D_\bullet)$ of $D_\bullet$.

To this end, let us first observe that it is well-known [cf., e.g., [3, Lemma 1.5, (i)] and Definition 1.1, (ii-a)] that the quotient $D_\bullet/I(D_\bullet)$ is *abelian* and *torsion-free.* In particular, to verify Claim 1.3.A, it suffices to verify the *triviality* of the image of ${}_lI(D_\circ)$ in the maximal abelian torsion-free quotient of $D_\circ$. On the other hand, since [we have assumed that] $l \neq p(D_\circ)$, this *triviality* is well-known [cf., e.g., [3, Lemma 1.2, (i)], [3, Lemma 1.7, (i)], and Definition 1.1, (i-a), (ii-a)]. This completes the proof of Claim 1.3.A.

First, we verify assertion (i). Let $N \subseteq D_\bullet$ be as in condition (2). Let us first observe that it follows from Claim 1.3.A that there exists a pro-l-Sylow subgroup ${}_lI(D_\bullet)$ of $I(D_\bullet)$ that *contains* the image of the composite discussed in Claim 1.3.A. Let ${}_l(D_\bullet/N) \subseteq D_\bullet/N$ be an l-Sylow subgroup of $D_\bullet/N$ that *contains* the image of ${}_lI(D_\bullet) \subseteq I(D_\bullet)$ in $D_\bullet/N$. Then it follows from condition (2) that

- the group ${}_l(D_\bullet/N)$ is *nontrivial*, and that
- the composite

$$ {}_lI(D_\circ) \longrightarrow {}_lI(D_\bullet) \longrightarrow {}_l(D_\bullet/N) $$

— where the first arrow is the homomorphism induced by α, and the second arrow is the homomorphism induced by the natural continuous surjective homomorphism $D_\bullet \twoheadrightarrow D_\bullet/N$ — is *surjective*.

In particular, one concludes immediately from Lemma 1.2 that the homomorphism ${}_lI(D_\circ) \to {}_lI(D_\bullet)$ induced by α is an *isomorphism*, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). It follows immediately from assertion (i) that, for each prime number l *not equal* to $p(D_\circ) = p(D_\bullet)$ [cf. condition (1)], the homomorphism α determines a *surjective* homomorphism from the [unique] pro-l Sylow subgroup of $\mathcal{O}^\times(D_\circ)$ to the [unique] pro-l Sylow subgroup of $\mathcal{O}^\times(D_\bullet)$. In particular, one concludes immediately from [3, Lemma 1.2, (i)] and Definition 1.1, (i-a), (i-b), (iii-a), that $p(D_\circ)^{f(D_\circ)} - 1$ is *divisible* by $p(D_\bullet)^{f(D_\bullet)} - 1$, which thus implies [cf. condition (1)] that $f(D_\circ)$ is *divisible* by $f(D_\bullet)$, as desired. This completes the proof of assertion (ii), hence also of Lemma 1.3. ☐

Remark 1.3.1. — Let $D_\circ$, $D_\bullet$ be topological groups *of MLF-type*, and let $\alpha\colon D_\circ \to D_\bullet$ be a continuous homomorphism. Suppose that the homomorphism α is *surjective*. Then one verifies easily from [1, Proposition 3.4, (i), (iii)] [cf. also [3, Lemma 1.5, (ii)] and Definition 1.1, (i-a), (ii-a), (ii-b)] that conditions (1), (2) in the statement of Lemma 1.3 are satisfied. Moreover, it follows from the final assertion of [1, Proposition 3.4, (iii)] that the *equality* $f(D_\circ) = f(D_\bullet)$ holds. Thus, Lemma 1.3, (ii), may be regarded as a *partial generalization* of the final assertion of [1, Proposition 3.4, (iii)].

2. The First Equivalence

In the present §2, we give a proof of the first main result of the present paper.

Lemma 2.1. — *Let p, l be* ***distinct*** *prime numbers, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$, $\zeta_l \in \overline{\mathbb{Q}}_p$ a primitive l-th root of unity, $p^{1/l} \in \overline{\mathbb{Q}}_p$ an l-th power root of $p \in \overline{\mathbb{Q}}_p$, and $L \subseteq \mathbb{Q}_p$ a subfield of $\mathbb{Q}_p$. Write $D \overset{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for the absolute Galois group of $\mathbb{Q}_p$ determined by the algebraic closure $\overline{\mathbb{Q}}_p$ and $I \subseteq D$ for the inertia subgroup of D. Let ${}_lI \subseteq I$ be a pro-l-Sylow subgroup of I. Then the following assertions hold:*

(i) *The subgroup $\mathrm{Gal}(L(\zeta_l, p^{1/l})/L(\zeta_l)) \subseteq \mathrm{Gal}(L(\zeta_l, p^{1/l})/L)$ is a* ***unique nontrivial*** ***l-Sylow*** *subgroup of $\mathrm{Gal}(L(\zeta_l, p^{1/l})/L)$.*

(ii) *The continuous homomorphism* $D \to \mathrm{Gal}(L(\zeta_l, p^{1/l})/L)$ *induced by the natural inclusion* $L(\zeta_l, p^{1/l}) \hookrightarrow \overline{\mathbb{Q}}_p$ *restricts to a continuous* ***surjective*** *homomorphism*

$${}_lI \twoheadrightarrow \mathrm{Gal}\big(L(\zeta_l, p^{1/l})/L(\zeta_l)\big).$$

PROOF. — These assertions are immediate. □

DEFINITION 2.2.

(i) We shall say that a field is a *number field* if the field is of characteristic zero and finite over the minimal subfield of the field.

(ii) We shall say that a field is *solvably closed* if the field admits no nontrivial abelian field extension.

LEMMA 2.3. — *Let F be a number field, $\widetilde{F}$ a* ***Galois*** *extension field of F that is* ***solvably closed****, D a topological group* ***of MLF-type****,*

$$\alpha \colon D \longrightarrow \mathrm{Gal}(\widetilde{F}/F)$$

a continuous homomorphism, and l a prime number ***not equal*** *to $p(D)$. Suppose that the following two conditions are satisfied:*

(1) *The number field F is* ***totally imaginary****.*

(2) *The image of a pro-l-Sylow subgroup of $I(D)$ by α is* ***nontrivial****.*

Then there exist a ***unique*** *nonarchimedean prime $\mathfrak{p}$ of F and a* ***unique*** *decomposition subgroup $D_{\mathfrak{p}}$ of $\mathrm{Gal}(\widetilde{F}/F)$ at $\mathfrak{p}$ such that the image of α is* ***contained*** *in $D_{\mathfrak{p}} \subseteq \mathrm{Gal}(\widetilde{F}/F)$. Moreover, in this situation, the* ***residue characteristic*** *of $\mathfrak{p}$ is* ***not equal*** *to l.*

PROOF. — Let ${}_lI(D) \subseteq I(D)$ be a pro-l-Sylow subgroup of $I(D)$. Let us first observe that since [we have assumed that — cf. condition (1)] the number field F is *totally imaginary*, the group $\mathrm{Gal}(\widetilde{F}/F)$ has *no nontrivial torsion element* [cf., e.g., the argument given in [8, pp.596-597]]. Thus, since [we have assumed that — cf. condition (2)] the image of ${}_lI(D)$ by α is *nontrivial*, it follows from Lemma 1.2 that the restriction of α to ${}_lI(D)$ is *injective*. In particular, it follows immediately from the well-known structure of a pro-l-Sylow subgroup of D [cf., e.g., the classification of the topological quotients of "$G_{\mathfrak{p}_1,l}$" given in [8, p.596]; also Definition 1.1, (i-a), (ii-a)] that the restriction of α to a pro-l-Sylow subgroup of D is *injective*. Thus, it follows immediately from a similar argument to the argument given in [8, pp.595-596] [cf. also [6, Proposition 2.3, (iv)]] that there exist a *unique* nonarchimedean prime $\mathfrak{p}$ of F and a *unique* decomposition subgroup $D_{\mathfrak{p}}$ of $\mathrm{Gal}(\widetilde{F}/F)$ at $\mathfrak{p}$ that satisfy the desired conditions. This completes the proof of Lemma 2.3. □

DEFINITION 2.4. — Let F be a number field, and let $\mathfrak{p}$ be a nonarchimedean prime of F.

(i) We shall say that $\mathfrak{p}$ is *of absolute degree one* if the completion of F at $\mathfrak{p}$ is isomorphic to $\mathbb{Q}_p$, where we write p for the residue characteristic of $\mathfrak{p}$.

(ii) We shall say that $\mathfrak{p}$ is *of absolute residue degree one* if the residue field of F at $\mathfrak{p}$ is isomorphic to $\mathbb{F}_p$, where we write p for the residue characteristic of $\mathfrak{p}$.

Definition 2.5. — Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be Galois extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are solvably closed. Let

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous open homomorphism. Then we shall write

$${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$$

for the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of the continuous homomorphism α.

Lemma 2.6. — *In the situation of Definition* 2.5, *suppose that* α *is* ***surjective***. *Let* p *be a prime number,* $\mathfrak{p}_\circ$ *a nonarchimedean prime of* $F_\circ$ ***of residue characteristic*** p, *and* $D_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ *a decomposition subgroup of* $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ *at* $\mathfrak{p}_\circ$. *Suppose, moreover, that the following three conditions are satisfied:*

(1) *The number field* $F_\bullet$ *is* ***totally imaginary***.

(2) *The nonarchimedean prime* $\mathfrak{p}_\circ$ *is* ***of absolute degree one***.

(3) *For all prime numbers* l ***not equal*** *to* p, ***every*** l***-th power root of*** p *in* $\widetilde{F}_\circ$ *is* ***contained*** *in* ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$.

Then the following assertions hold:

(i) *There exist a* ***unique*** *nonarchimedean prime* $\mathfrak{p}_\bullet$ *of* $F_\bullet$ *and a* ***unique*** *decomposition subgroup* $D_\bullet$ *of* $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ *at* $\mathfrak{p}_\bullet$ *that satisfy the following four conditions:*

(a) *The image of* $D_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ *by* α *is* ***contained*** *in* $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$.

(b) *The nonarchimedean prime* $\mathfrak{p}_\bullet$ *is* ***of residue characteristic*** p.

(c) *Let* l *be a prime number* ***not equal*** *to* p. *Then the homomorphism* α *restricts to an* ***isomorphism*** *of a pro-*l*-Sylow subgroup of the inertia subgroup of* $D_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ *with a pro-*l*-Sylow subgroup of the inertia subgroup of* $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$.

(d) *The nonarchimedean prime* $\mathfrak{p}_\bullet$ *is* ***of absolute residue degree one***.

(ii) *Suppose, moreover, that the following two conditions are satisfied:*

(4) *The prime number* p *is* ***odd***.

(5) *There exists a finite set* S *of prime numbers such that if* q *is a prime number* ***not contained*** *in* S, *then* ***every square root of*** q *in* $\widetilde{F}_\circ$ *is* ***contained*** *in* ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$.

Then the homomorphism $D_\circ^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to D_\bullet^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ *induced by* α [*cf.* (a)] *is an* ***isomorphism***.

(iii) *In the situation of* (ii), *write* $F_\circ^{\mathrm{ab}} \subseteq \widetilde{F}_\circ$, $F_\bullet^{\mathrm{ab}} \subseteq \widetilde{F}_\bullet$ *for the respective maximal abelian extension fields of* $F_\circ$, $F_\bullet$ *in* $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$. *Then the continuous homomorphism* $\mathrm{Gal}(F_\circ^{\mathrm{ab}}/F_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to \mathrm{Gal}(F_\bullet^{\mathrm{ab}}/F_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ *determined by* α *restricts to a* ***bijection*** *between the subset* $\mathrm{FL}_2(\mathfrak{p}_\circ) \subseteq \mathrm{Gal}(F_\circ^{\mathrm{ab}}/F_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ [*i.e., consisting of the elements of the decomposition subgroup of* $\mathrm{Gal}(F_\circ^{\mathrm{ab}}/F_\circ) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ *at* $\mathfrak{p}_\circ$ *whose natural actions on the residue field of the valuation ring in the algebraic extension of the completion of* $F_\circ$ *determined by the pair* $(F_\circ^{\mathrm{ab}}, \mathfrak{p}_\circ)$ *are given by the* p*-th power Frobenius map — cf.* [5, Definition 2.1, (ii)], (2)] *and the subset* $\mathrm{FL}_2(\mathfrak{p}_\bullet) \subseteq \mathrm{Gal}(F_\bullet^{\mathrm{ab}}/F_\bullet) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ [*cf.* (d)].

PROOF. — We begin the proof of Lemma 2.6 with the following claim:

CLAIM 2.6.A. — Let l be a prime number *not equal* to p, ${}_lI_\circ \subseteq D_\circ$ a pro-l-Sylow subgroup of the inertia subgroup of $D_\circ$, $\zeta_l \in \widetilde{F}_\circ$ a primitive l-th root of unity, and $p^{1/l} \in \widetilde{F}_\circ$ an l-th power root of $p \in \widetilde{F}_\circ$. Then the image of the composite

$${}_lI_\circ \hookrightarrow \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \twoheadrightarrow \mathrm{Gal}({}^\alpha F_\circ/F_\circ) \twoheadrightarrow \mathrm{Gal}\big(F_\circ(\zeta_l, p^{1/l})/F_\circ\big)$$

— where the first arrow is the natural inclusion, and the second, third arrows are the continuous surjective homomorphisms determined by the natural inclusions ${}^\alpha F_\circ \hookrightarrow \widetilde{F}_\circ$, $F_\circ(\zeta_l, p^{1/l}) \hookrightarrow {}^\alpha F_\circ$ [cf. condition (3)], respectively — is a *unique nontrivial l-Sylow* subgroup of $\mathrm{Gal}(F(\zeta_l, p^{1/l})/F_\circ)$.

To this end, let us first recall that [we have assumed that — cf. condition (2)] the nonarchimedean prime $\mathfrak{p}_\circ$ is *of absolute degree one*. Thus, Claim 2.6.A follows immediately form Lemma 2.1, (i), (ii). This completes the proof of Claim 2.6.A.

First, we verify assertion (i). Observe that it is immediate from Claim 2.6.A that, for each prime number l *not equal* to p and each pro-l-Sylow subgroup ${}_lI_\circ \subseteq D_\circ$ of the inertia subgroup of $D_\circ$, the image of the composite

$${}_lI_\circ \hookrightarrow \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \overset{\alpha}{\twoheadrightarrow} \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

— where the first arrow is the natural inclusion — is *nontrivial*. Thus, one concludes immediately from Lemma 2.3 [cf. also condition (1)] that there exist a *unique* nonarchimedean prime $\mathfrak{p}_\bullet$ of $F_\bullet$ and a *unique* decomposition subgroup $D_\bullet$ of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ at $\mathfrak{p}_\bullet$ that satisfy conditions (a), (b). Moreover, since [we have assumed that — cf. condition (2)] the nonarchimedean prime $\mathfrak{p}_\circ$ is *of absolute degree one*, hence also *of absolute residue degree one*, by applying Lemma 1.3, (i), (ii) [cf. also Definition 1.1, (i-a), (i-b), (ii-a)], to the homomorphism $D_\circ \to D_\bullet$ induced by α [cf. condition (a)], one also concludes immediately from condition (b) and Claim 2.6.A that conditions (c), (d) are satisfied. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $I_\circ \subseteq D_\circ$, $I_\bullet \subseteq D_\bullet$ for the respective inertia subgroups of $D_\circ$, $D_\bullet$. Now recall that [we have assumed that — cf. condition (4)] the prime number p is *odd*. Thus, it is well-known [cf., e.g., [3, Lemma 1.5, (ii)]] that if ${}_2I_\circ \subseteq I_\circ$, ${}_2I_\bullet \subseteq I_\bullet$ are pro-2-Sylow subgroups of $I_\circ$, $I_\bullet$, respectively, then the natural homomorphisms ${}_2I_\circ^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to I_\circ^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$, ${}_2I_\bullet^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to I_\bullet^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ are *isomorphisms* [cf. also condition (b)]. In particular, it follows from conditions (a), (c) that the homomorphism α induces a homomorphism

$$(D_\circ/I_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \longrightarrow (D_\bullet/I_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z},$$

and, moreover, to verify assertion (ii), it suffices to verify that this homomorphism is an *isomorphism*. Thus, since [it is well-known — cf., e.g., [3, Lemma 1.5, (i)] — that] both $(D_\circ/I_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ and $(D_\bullet/I_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ are *of order two*, one concludes [cf. also condition (c)] that, to verify assertion (ii), it suffices to verify that there exists a homomorphism $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \to \mathbb{Z}/2\mathbb{Z}$ such that the composite

$$D_\circ \hookrightarrow \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \overset{\alpha}{\longrightarrow} \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

— where the first arrow is the natural inclusion, and the third arrow is the homomorphism under consideration — determines an *isomorphism* $(D_\circ/I_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \overset{\sim}{\to} \mathbb{Z}/2\mathbb{Z}$. On the other hand, such a homomorphism $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \to \mathbb{Z}/2\mathbb{Z}$ may be obtained by pulling back, by the inverse of the isomorphism $\mathrm{Gal}({}^\alpha F_\circ/F_\circ) \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ determined by α, the continuous surjective homomorphism $\mathrm{Gal}({}^\alpha F_\circ/F_\circ) \twoheadrightarrow \mathrm{Gal}(F_\circ(q^{1/2})/F_\circ)$ determined by the natural inclusion $F_\circ(q^{1/2}) \hookrightarrow {}^\alpha F_\circ$, where q is a prime number *not contained* in the finite set S of condition (5) such that the image of q in $\mathbb{F}_p$ is *not contained* in $\mathbb{F}_p^2$ ($\stackrel{\mathrm{def}}{=} \{ a^2 \in \mathbb{F}_p \mid a \in \mathbb{F}_p \}$), and $q^{1/2} \in {}^\alpha F_\circ$ is a *square root* of q [cf. condition (5)]. [Note that it follows from *Dirichlet's theorem on primes in arithmetic progressions* that the set consisting of prime numbers whose images in $\mathbb{F}_p$ are *not contained* in $\mathbb{F}_p^2$ is *infinite.*] This completes the proof of assertion (ii).

Next, we verify assertion (iii). Observe that it is immediate that, since p is *odd* [cf. condition (4)], the subsets $\mathrm{FL}_2(\mathfrak{p}_\circ) \subseteq D_\circ^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$, $\mathrm{FL}_2(\mathfrak{p}_\bullet) \subseteq D_\bullet^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ *coincide* with the complements in $D_\circ^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$, $D_\bullet^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ of the images of pro-2-Sylow subgroups of the inertia subgroups of $D_\circ$, $D_\bullet$, respectively. Thus, assertion (iii) follows immediately from assertion (ii), together with condition (c). This completes the proof of assertion (iii), hence also of Lemma 2.6. □

Theorem 2.7. — *In the situation of Definition* 2.5*, the following two conditions are equivalent:*

(1) *There exists a homomorphism* $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ *of rings* ***compatible*** *with the respective actions of* $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$*, relative to the homomorphism* α*.*

(2) *For all prime numbers* l *and all but finitely many prime numbers* p*,* ***every*** l***-th power root of*** p *in* $\widetilde{F}_\circ$ *is* ***contained*** *in* ${}^\alpha F_\circ \subseteq \widetilde{F}_\circ$*.*

Proof. — First, we verify the implication (1) ⇒ (2). Suppose that condition (1) is satisfied. Then it is immediate that the field ${}^\alpha F_\circ$ contains the field *isomorphic* to $\widetilde{F}_\bullet$. Thus, since [we have assumed that] the field $\widetilde{F}_\bullet$ is *solvably closed*, it is immediate that, for all prime numbers l, p, *every* l*-th power root of* p in $\widetilde{F}_\circ$ is *contained* in ${}^\alpha F_\circ \subseteq \widetilde{F}_\circ$, as desired. This completes the proof of the implication (1) ⇒ (2).

Next, we verify the implication (2) ⇒ (1). Suppose that condition (2) is satisfied. Let us first observe that since [we have assumed that] the continuous homomorphism α is *open*, to verify condition (1), we may assume without loss of generality, by replacing $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ by the image of α, that α is *surjective*.

Let $K_\bullet \subseteq \widetilde{F}_\bullet$ be a finite Galois extension field of $F_\bullet$ contained in $\widetilde{F}_\bullet$ that is *totally imaginary.* Write $K_\circ \subseteq \widetilde{F}_\circ$ for the finite Galois extension field of $F_\circ$ contained in $\widetilde{F}_\circ$ that corresponds to the normal open subgroup of $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ obtained by forming the inverse image by the continuous surjective homomorphism α of $\mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet) \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$. Thus, we have a commutative diagram of topological groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\circ/K_\circ) & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) & \longrightarrow & \mathrm{Gal}(K_\circ/F_\circ) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \beta_K} & & \downarrow{\scriptstyle \alpha} & & \wr\downarrow{\scriptstyle \alpha_K} & & \\ 1 & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet) & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) & \longrightarrow & \mathrm{Gal}(K_\bullet/F_\bullet) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, the vertical arrows are *surjective*, and the right-hand vertical arrow is an *isomorphism.* Write $\overline{\beta}_K \colon \mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ for the continuous surjective homomorphism determined by the left-hand vertical arrow of this diagram. Then one concludes immediately from Lemma 2.6, (i), (iii) [cf. also condition (2)], that the homomorphism $\overline{\beta}_K$ is *Frobenius-preserving* [cf. [5, Definition 2.7]]. Thus, it follows from [5, Corollary 2.8] that the homomorphism $\overline{\beta}_K$ *arises* from a *uniquely determined* homomorphism of rings $\iota_K \colon K_\bullet \hookrightarrow K_\circ$.

Next, observe that the above diagram determines a commutative diagram of groups

$$\begin{array}{ccc} \mathrm{Gal}(K_\circ/F_\circ) & \longrightarrow & \mathrm{Aut}^*\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \\ {\scriptstyle \alpha_K} \downarrow \wr & & \downarrow \\ \mathrm{Gal}(K_\bullet/F_\bullet) & \longrightarrow & \mathrm{Aut}\big(\mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \end{array}$$

— where we write

$$\mathrm{Aut}^*\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \subseteq \mathrm{Aut}\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big)$$

for the subgroup consisting of the continuous automorphisms of the topological group $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ that *preserve* the kernel of $\overline{\beta}_K$, the horizontal arrows are the respective natural continuous actions, i.e., determined by the horizontal sequences of the above diagram, and the right-hand vertical arrow is the homomorphism induced by $\overline{\beta}_K$, i.e., by ι_K. In particular, one concludes immediately from the *commutativity* of this diagram, together with the *faithfulness* portion of [5, Theorem 2.6], that, for each $\gamma \in \mathrm{Gal}(K_\circ/F_\circ)$, the *equality* $\gamma \circ \iota_K = \iota_K \circ \alpha_K(\gamma)$ holds, i.e., that the homomorphism $\iota_K \colon K_\bullet \hookrightarrow K_\circ$ of rings is *compatible* with the respective actions of $\mathrm{Gal}(K_\bullet/F_\bullet)$, $\mathrm{Gal}(K_\circ/F_\circ)$, relative to the isomorphism $\alpha_K \colon \mathrm{Gal}(K_\circ/F_\circ) \xrightarrow{\sim} \mathrm{Gal}(K_\bullet/F_\bullet)$. Thus, by allowing "$K_\bullet$" to vary, one also concludes that there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α, as desired. This completes the proof of the implication (2) ⇒ (1), hence also of Theorem 2.7. □

Proof of the Equivalence (1) ⇔ (2) of Theorem B. — First, we verify the implication (1) ⇒ (2). Suppose that assertion (1) is satisfied. Let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ as in assertion (2), which thus implies that there exists a continuous *open injective* homomorphism $\mathrm{Gal}(K/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$. Then, by applying assertion (1) to the composite

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F) \twoheadrightarrow \mathrm{Gal}(K/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$$

— where the first arrow is the continuous surjective homomorphism determined by the natural inclusion $K \hookrightarrow \overline{\mathbb{Q}}$, and the second arrow is a continuous *open injective* homomorphism — one concludes immediately that the field K *contains* $\mathbb{Q}^{\mathrm{slv}}$. In particular, for all prime numbers l, p, *every* l*-th power root of* p in $\overline{\mathbb{Q}}$ is *contained* in $K \subseteq \overline{\mathbb{Q}}$, as desired. This completes the proof of the implication (1) ⇒ (2).

Next, we verify the implication (2) ⇒ (1). Suppose that assertion (2) is satisfied. Let $F_\circ$, $F_\bullet$, $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$, $\alpha \colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \to \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ be as in assertion (1). Write $(F_\bullet)_0 \subseteq F_\bullet$ for the minimal subfield of $F_\bullet$, $(F_\bullet)_0^\sim \subseteq \widetilde{F}_\bullet$ for the maximal prosolvable extension field of $(F_\bullet)_0$ in $\widetilde{F}_\bullet$,

$$\pi \colon \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \longrightarrow \mathrm{Gal}\big((F_\bullet)_0^\sim/(F_\bullet)_0\big)$$

for the continuous open homomorphism determined by the natural inclusion $(F_\bullet)_0^\sim \hookrightarrow \widetilde{F}_\bullet$, and ${}^{\pi\circ\alpha}F_\circ \subseteq {}^{\alpha}F_\circ$ [cf. Definition 2.5] for the subfield of ${}^{\alpha}F_\circ$ that corresponds to the kernel of the continuous homomorphism $\pi \circ \alpha$. Then since [it is immediate that] the topological group $\mathrm{Gal}({}^{\pi\circ\alpha}F_\circ/F_\circ)$ is *isomorphic* to an open subgroup of $\mathrm{Gal}((F_\bullet)_0^{\sim}/(F_\bullet)_0)$, it follows immediately from assertion (2) that, for all prime numbers l and all but finitely many prime numbers p, *every l-th power root of p* in $\widetilde{F}_\circ$ is *contained* in ${}^{\pi\circ\alpha}F_\circ \subseteq \widetilde{F}_\circ$, hence also in ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$. In particular, it follows from the implication (2) $\Rightarrow$ (1) of Theorem 2.7 that there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of the equivalence (1) $\Leftrightarrow$ (2) of Theorem B. $\square$

3. The Second Equivalence

In the present §3, we give a proof of the second main result of the present paper.

Lemma 3.1. — *Let F be a number field, $\widetilde{F}$ a* ***Galois*** *extension field of F that is* ***solvably closed****, and F' a* ***finite Galois*** *extension field of F in $\widetilde{F}$. Write $\mathrm{Aut}(\mathrm{Gal}(\widetilde{F}/F'))$ for the group of continuous automorphisms of the topological group $\mathrm{Gal}(\widetilde{F}/F')$. Then there exist a subfield F_0 of F over which $\widetilde{F}$ is* ***Galois*** *and a commutative diagram of groups*

$$\begin{array}{ccc} & & \mathrm{Aut}\big(\mathrm{Gal}(\widetilde{F}/F')\big) \\ & \nearrow & \\ \mathrm{Gal}(\widetilde{F}/F) & & \downarrow \wr \\ & \searrow & \\ & & \mathrm{Gal}(\widetilde{F}/F_0) \end{array}$$

— where the upper diagonal arrow is the continuous action by conjugation, the lower diagonal arrow is the continuous open injective homomorphism determined by the natural inclusion $F_0 \hookrightarrow F$, and the right-hand vertical arrow is an ***isomorphism****.*

Proof. — This assertion is a formal consequence of [7, Theorem]. $\square$

Lemma 3.2. — *Let F be a number field, $\widetilde{F}$ a* ***Galois*** *extension field of F that is* ***solvably closed****, Γ a topological group, and $\iota\colon \mathrm{Gal}(\widetilde{F}/F) \hookrightarrow \Gamma$ a continuous* ***injective*** *homomorphism. Suppose that the image of ι is either* ***normal*** *or* ***of finite index*** *in Γ. Then there exist a subfield F_0 of F over which $\widetilde{F}$ is* ***Galois*** *and a commutative diagram of groups*

$$\begin{array}{ccc} & & \Gamma \\ & \overset{\iota}{\nearrow} & \\ \mathrm{Gal}(\widetilde{F}/F) & & \downarrow \\ & \searrow & \\ & & \mathrm{Gal}(\widetilde{F}/F_0) \end{array}$$

— *where the lower diagonal arrow is the continuous open injective homomorphism determined by the natural inclusion* $F_0 \hookrightarrow F$, *and the image of the right-hand vertical arrow is* ***open***.

PROOF. — Observe that since [we have assumed that] the image of ι is either *normal* or *of finite index* in Γ, one verifies easily that there exists a *finite* [necessarily *Galois*] extension field F' of F in $\widetilde{F}$ such that the image of $\mathrm{Gal}(\widetilde{F}/F')$ by ι is *normal* in Γ. Then the assertion follows immediately from Lemma 3.1 by considering the continuous action of Γ on [the image by ι of] $\mathrm{Gal}(\widetilde{F}/F')$ by conjugation. $\square$

DEFINITION 3.3. — We shall say that a field is *absolutely Galois* if the field is [algebraic and] Galois over the minimal subfield of the field.

THEOREM 3.4. — *In the situation of Definition* 2.5, *suppose that the subfield* ${}^{\alpha}F_{\circ}$ *of* $\widetilde{F}_{\circ}$ *is* ***absolutely Galois***. *Then there exists a homomorphism* $\widetilde{F}_{\bullet} \hookrightarrow \widetilde{F}_{\circ}$ *of rings* ***compatible*** *with the respective actions of* $\mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet})$, $\mathrm{Gal}(\widetilde{F}_{\circ}/F_{\circ})$, *relative to the homomorphism* α.

PROOF. — Write $(F_{\circ})_0 \subseteq F_{\circ}$ for the minimal subfield of $F_{\circ}$. Let us first observe that it is immediate that, to verify Theorem 3.4, we may assume without loss of generality, by replacing $\widetilde{F}_{\circ}$ by an algebraic closure of $\widetilde{F}_{\circ}$, that $\widetilde{F}_{\circ}$ is *algebraically closed*. Moreover, since [we have assumed that] the continuous homomorphism α is *open*, to verify Theorem 3.4, we may assume without loss of generality, by replacing $\mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet})$ by the image of α, that α is *surjective*.

Next, observe that it follows immediately from Lemma 3.2 — i.e., in the case where we take the "$(F, \widetilde{F}, \Gamma)$" of Lemma 3.2 to be $(F_{\bullet}, \widetilde{F}_{\bullet}, \mathrm{Gal}({}^{\alpha}F_{\circ}/(F_{\circ})_0))$ [cf. our assumption that ${}^{\alpha}F_{\circ}$ is *absolutely Galois*] and the "ι" of Lemma 3.2 to be the [necessarily *open*] homomorphism obtained by forming the composite of $\alpha^{-1}\colon \mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet}) \xrightarrow{\sim} \mathrm{Gal}({}^{\alpha}F_{\circ}/F_{\circ})$ with the natural inclusion $\mathrm{Gal}({}^{\alpha}F_{\circ}/F_{\circ}) \hookrightarrow \mathrm{Gal}({}^{\alpha}F_{\circ}/(F_{\circ})_0)$ — that there exist a subfield $(F_{\bullet})_0$ of $F_{\bullet}$ over which $\widetilde{F}_{\bullet}$ is *Galois* and a commutative diagram of topological groups

$$\begin{array}{ccc} \mathrm{Gal}({}^{\alpha}F_{\circ}/F_{\circ}) & \hookrightarrow & \mathrm{Gal}\big({}^{\alpha}F_{\circ}/(F_{\circ})_0\big) \\ \alpha \downarrow \wr & & \downarrow \\ \mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet}) & \hookrightarrow & \mathrm{Gal}\big(\widetilde{F}_{\bullet}/(F_{\bullet})_0\big) \end{array}$$

— where the upper, lower horizontal arrows are the respective continuous open injective homomorphisms determined by the natural inclusions $(F_{\circ})_0 \hookrightarrow F_{\circ}$, $(F_{\bullet})_0 \hookrightarrow F_{\bullet}$, and the vertical arrows are *open*. In particular, to verify Theorem 3.4, we may assume without loss of generality, by replacing α by the continuous surjective homomorphism determined by the composite of the right-hand vertical arrow of this diagram with the natural surjective homomorphism $\mathrm{Gal}(\widetilde{F}_{\circ}/(F_{\circ})_0) \twoheadrightarrow \mathrm{Gal}({}^{\alpha}F_{\circ}/(F_{\circ})_0)$, that $F_{\circ}$ is *isomorphic to the field of rational numbers*. On the other hand, it follows from [8, Theorem 1] that there exists a homomorphism $\widetilde{F}_{\bullet} \hookrightarrow \widetilde{F}_{\circ}$ of rings *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet})$, $\mathrm{Gal}(\widetilde{F}_{\circ}/F_{\circ})$, relative to the homomorphism α whenever $F_{\circ}$ is *isomorphic to the field of rational numbers*. This completes the proof of Theorem 3.4. $\square$

Lemma 3.5. — *Let G be a group, $H \subseteq G$ a* ***normal*** *subgroup of G, and α an automorphism of G. Suppose that the following three conditions are satisfied:*

(1) *The* ***centralizer*** *of H in G is* ***trivial****.*

(2) *The automorphism α* ***preserves*** *the subgroup H.*

(3) *The automorphism of H induced by α [cf. (2)] is the* ***identity automorphism*** *of H.*

Then the automorphism α is the ***identity automorphism*** *of G.*

Proof. — Observe that it follows from condition (1) that the conjugation action $G \to \mathrm{Aut}(H)$ is *faithful*. On the other hand, it is immediate that this *injective* homomorphism $G \hookrightarrow \mathrm{Aut}(H)$ is *compatible* with the respective natural actions of α on G and $\mathrm{Aut}(H)$ [cf. condition (2)]. Thus, it follows from condition (3) that the automorphism α is the *identity automorphism* of G, as desired. This completes the proof of Lemma 3.5. $\square$

Lemma 3.6. — *In the situation of Definition 2.5, let $K_\circ$ be a finite extension field of $F_\circ$ in $\widetilde{F}_\circ$. Suppose that there exists a homomorphism $\alpha_{\widetilde{F}} \colon \widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings* ***compatible*** *with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)$, relative to the restriction to $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ) \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ of the homomorphism α. Then the homomorphism $\alpha_{\widetilde{F}} \colon \widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings is* ***compatible*** *with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α.*

Proof. — Observe that it is immediate that, to verify Lemma 3.6, we may assume without loss of generality, by replacing $K_\circ$ by a suitable finite extension field of $K_\circ$ in $\widetilde{F}_\circ$, that $K_\circ$ is *Galois* over $F_\circ$.

Next, observe that one verifies easily that the diagram of groups

$$\begin{array}{ccc} \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) & \longrightarrow & \mathrm{Aut}^*\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)\big) \\ \alpha\downarrow & & \downarrow \\ \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) & \longrightarrow & \mathrm{Aut}\big(\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)\big) \end{array}$$

— where we write

$$\mathrm{Aut}^*\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)\big) \subseteq \mathrm{Aut}\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)\big)$$

for the subgroup consisting of the continuous automorphisms of the topological group $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)$ that *preserve* the *normal* closed subgroup $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ) \cap \mathrm{Ker}(\alpha)$, and, moreover, the induced automorphisms of $\alpha \colon \mathrm{Gal}(\widetilde{F}_\circ/K_\circ)/(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ) \cap \mathrm{Ker}(\alpha)) \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)$ *uniquely* [cf. [6, Corollary 1.3], Lemma 3.5] *extend* to automorphisms of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ ($\supseteq \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)$), the horizontal arrows are the respective conjugation actions, and the right-hand vertical arrow is the homomorphism determined by the definition of the subgroup $\mathrm{Aut}^*(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ))$ — *commutes.* Next, observe that it follows from our assumption [i.e., that the homomorphism $\alpha_{\widetilde{F}} \colon \widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings is *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)$, relative to the restriction to $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ) \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ of the homomorphism α] that the right-hand vertical arrow of this diagram *coincides* with the homomorphism induced by the homomorphism $\alpha_{\widetilde{F}} \colon \widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings. In particular, one concludes immediately from the *commutativity* of the above diagram, together with the *uniqueness* portion of [7, Theorem], that, for each $\gamma \in \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, the *equality* $\gamma \circ \alpha_{\widetilde{F}} = \alpha_{\widetilde{F}} \circ \alpha(\gamma)$ holds, as desired. This completes the proof of Lemma 3.6. □

PROOF OF THE EQUIVALENCE (1) ⇔ (3) OF THEOREM B. — First, we verify the implication (1) ⇒ (3). Suppose that assertion (1) is satisfied. Let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ as in assertion (3), which thus implies that there exists a continuous *open injective* homomorphism $\mathrm{Gal}(K/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$. Then, by applying assertion (1) to the composite

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F) \twoheadrightarrow \mathrm{Gal}(K/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$$

— where the first arrow is the continuous surjective homomorphism determined by the natural inclusion $K \hookrightarrow \overline{\mathbb{Q}}$, and the second arrow is a continuous *open injective* homomorphism — one concludes immediately that the *equality* $K = F \cdot \mathbb{Q}^{\mathrm{slv}}$ in $\overline{\mathbb{Q}}$ holds. In particular, the field extension $K/\mathbb{Q}$ is *Galois*, as desired. This completes the proof of the implication (1) ⇒ (3).

Next, we verify the implication (3) ⇒ (1). Suppose that assertion (3) is satisfied. Let $F_\circ$, $F_\bullet$, $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$, $\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \to \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ be as in assertion (1). Let us first observe that it is immediate that, to verify the existence of a homomorphism "$\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$" as in assertion (1), we may assume without loss of generality, by replacing $\widetilde{F}_\circ$ by an algebraic closure of $\widetilde{F}_\circ$, that $\widetilde{F}_\circ$ is *algebraically closed*. Moreover, it follows from Lemma 3.6 that, to verify the existence of a homomorphism "$\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$" as in assertion (1), we may assume without loss of generality, by replacing $F_\circ$ by a suitable finite extension field of $F_\circ$ in $\widetilde{F}_\circ$, that $F_\circ$ is *absolutely Galois*. Write $(F_\bullet)_0 \subseteq F_\bullet$ for the minimal subfield of $F_\bullet$, $(F_\bullet)_0^{\sim} \subseteq \widetilde{F}_\bullet$ for the maximal prosolvable extension field of $(F_\bullet)_0$ in $\widetilde{F}_\bullet$, and

$$\pi\colon \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \longrightarrow \mathrm{Gal}\big((F_\bullet)_0^{\sim}/(F_\bullet)_0\big)$$

for the continuous open homomorphism determined by the natural inclusion $(F_\bullet)_0^{\sim} \hookrightarrow \widetilde{F}_\bullet$. Then, to verify the existence of a homomorphism "$\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$" as in assertion (1) [i.e., to verify condition (2) of Theorem 2.7 — cf. Theorem 2.7], we may assume without loss of generality, by replacing α by $\pi \circ \alpha$, that $F_\bullet$ is *isomorphic to the field of rational numbers*, and $\widetilde{F}_\bullet$ is *isomorphic to a maximal prosolvable extension field of the field of rational numbers*. Then since [it is immediate that] the topological group $\mathrm{Gal}({}^{\alpha}F_\circ/F_\circ)$ is *isomorphic* to an open subgroup of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, it follows from assertion (3) that the field ${}^{\alpha}F_\circ$ is *absolutely Galois*. In particular, it follows from Theorem 3.4 that there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α, as desired. This completes the proof of the implication (3) ⇒ (1), hence also of the equivalence (1) ⇔ (3) of Theorem B. □

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