Survey on the Combinatorial Anabelian Geometry of Hyperbolic Curves

Yuichiro Hoshi (RIMS, Kyoto University)

The 3rd MSJ-SI:
Development of Galois-Teichmüller theory and anabelian geometry at RIMS

October 30, 2010.

§1: A Combinatorial Version of the Grothendieck Conjecture

§2: Combinatorial Cuspidalization

§3: Injectivity of the Outer Galois Representations of Hyperbolic Curves

§4: A Version of the Grothendieck Conjecture for Universal Curves

§5: A Generalization of a Result due to Y. André

§6: Differences between $\mathrm{Out}^{\mathrm{FC}} \subseteq \mathrm{Out}^{\mathrm{F}} \subseteq \mathrm{Out}$

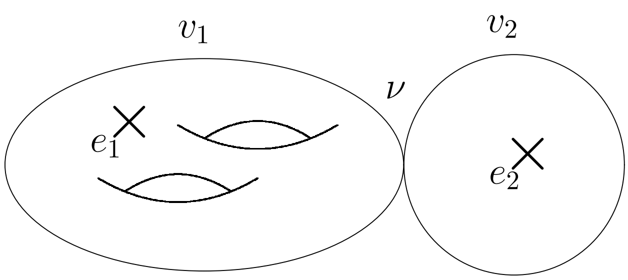
§1: A Combinatorial Version of the Grothendieck Conjecture

<u>semi-graphs of anabelioids of PSC-type</u>

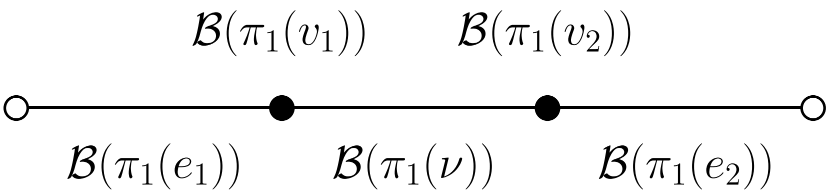
pointed stable curve

$\rightsquigarrow$

semi-graph of anabelioids of PSC-type



$\rightsquigarrow$



- irreducible component $\leftrightarrow$ vertex
- node $\leftrightarrow$ closed edge
- cusp $\leftrightarrow$ open edge

($\mathcal{B}(G)$: connected anabelioid [= Galois category] associated to G)

Comb. Groth. Conj. (CombGC)

$\mathcal{G}$: semi-graph of anabelioids of PSC-type

I: profinite group

Σ ($\neq \emptyset$): set of prime numbers

$\rho\colon I \to \mathrm{Out} \stackrel{\mathrm{def}}{=} \mathrm{Out}(\pi_1(\mathcal{G})^{\Sigma})$:

cont. hom. satisfying certain conditions

$\Longrightarrow$ Any element of $Z_{\mathrm{Out}}(\mathrm{Im}(\rho))$ is graphic,
i.e., arises from an automorphism of $\mathcal{G}$.

Note: Original Grothendieck conjecture

k: field satisfying certain conditions

X/k: hyperbolic curve

$\rho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Out} \stackrel{\mathrm{def}}{=} \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$:

outer Galois rep. ass. to X/k

$\Longrightarrow$ Any element of
$Z_{\mathrm{Out}}(\mathrm{Im}(\rho))$ ($\simeq \mathrm{Isom}_{G_k}(\pi_1(X))/\text{geom. inner}$)
is geometric,
i.e., arises from an automorphism of X over k.

Results of CombGC:

Theorem A (Mochizuki)

For $\phi \in Z_{\mathrm{Out}}(\mathrm{Im}(\rho))$,

- ρ: IPSC-type
- ϕ: C-admissible,

  i.e., preserves the set of cuspidal inertia subgroups of $\pi_1(\mathcal{G})^{\Sigma}$.

$\Longrightarrow \phi$: graphic

Theorem B (Mochizuki-H)

For $\phi \in Z_{\mathrm{Out}}(\mathrm{Im}(\rho))$,

- ρ: NN-type
- ϕ: C-admissible
- $\mathcal{G}$ has at least one cusp, i.e., $\mathcal{G}$ is not proper.

$\Longrightarrow \phi$: graphic

“IPSC” (inertial pointed stable curve)

“NN” (nodally nondegenerate)

$\rho\colon I \to \mathrm{Out}(\pi_1(\mathcal{G})^{\Sigma})$: IPSC-type

$\overset{\text{def}}{\Leftrightarrow}$ ρ arises from a stable log curve, i.e.,

$\exists\ X^{\log} \to S^{\log} \overset{\text{def}}{=} \mathrm{Spec}\,(\mathbb{N} \to k : n \mapsto 0^n)$:

stable log curve (where $k = \overline{k}$ of char.$\notin \Sigma$)

s.t. ρ “is”

$$\rho_{X^{\log}/S^{\log}} \colon \pi_1(S^{\log})^{\Sigma} \longrightarrow \mathrm{Out}(\pi_1(X^{\log}/S^{\log})^{\Sigma})$$

$$[\pi_1(X^{\log}/S^{\log}) \overset{\text{def}}{=} \mathrm{Ker}(\pi_1(X^{\log}) \twoheadrightarrow \pi_1(S^{\log}))].$$

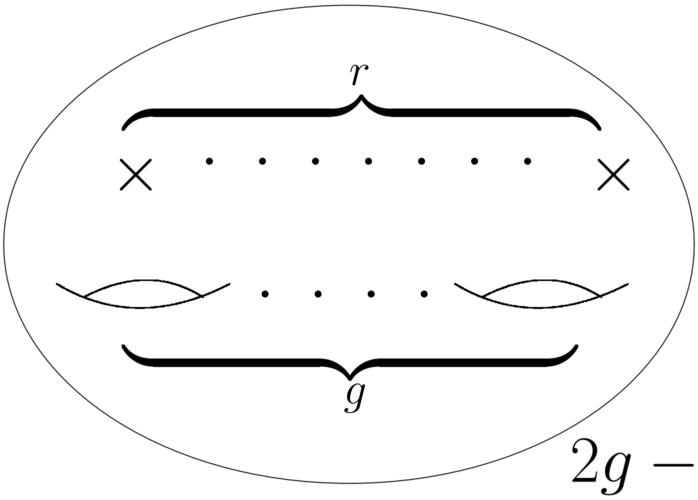
$\rho\colon I \to \mathrm{Out}(\pi_1(\mathcal{G})^{\Sigma})$: NN-type $\overset{\text{def}}{\Leftrightarrow} \cdots$

Remark

- “NN” is a purely group-theoretic condition.
- “IPSC” $\Longrightarrow$ “NN”

§2: Combinatorial Cuspidalization

X: hyperbolic Riemann surface of type (g, r), i.e.,



$$2g - 2 + r > 0$$

X_n: n-th configuration space of X, i.e.,

$$X_n \stackrel{\text{def}}{=} \overbrace{X \times \cdots \times X}^{n} \setminus \text{various diagonals}$$

$\dagger \in \{\text{discrete, profinite, pro-}l\}$

$\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\mathrm{top}}(X_n)^\dagger) \subseteq \mathrm{Out}(\pi_1^{\mathrm{top}}(X_n)^\dagger)$:

group of <u>F-admissible</u> and <u>C-admissible</u> outer automorphisms of $\pi_1^{\mathrm{top}}(X_n)^\dagger$, i.e.,

- induce "id" on the set of <u>F</u>iber subgroups.
- preserve the set of <u>C</u>uspidal inertia subgroups.

Theorem C (Mochizuki-H)

The homomorphism

$$\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\mathrm{top}}(X_{n+1})^{\dagger}) \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\mathrm{top}}(X_n)^{\dagger})$$

induced by the projection $X_{n+1} \to X_n$ is

- injective if $n > 0$;
- surjective if either
  $\dagger =$"discrete", $n > 3$, or $n > 2$ and $r > 0$.

Remark

The injectivity and surjectivity of similar homomorphisms have been studied by various reseachers:

e.g., D. Harbater; Y. Ihara; M. Kaneko; M. Matsumoto; H. Nakamura; L. Schneps; N. Takao; H. Tsunogai; R. Ueno ...

§3: Injectivity of the Outer Galois Representations of Hyperbolic Curves

Theorem D (Mochizuki-H)

Either $[k:\mathbb{Q}] < \infty$ or $[k:\mathbb{Q}_p] < \infty$

X/k: hyperbolic curve

$\rho_{X/k}: \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$:

outer Galois rep. ass. to X/k

$\Longrightarrow \rho_{X/k}$: injective

Remark

- If X is a tripod, i.e., $\simeq \mathbb{P}^1_k \setminus \{0, 1, \infty\}$, then this was proven by G. V. Belyi.
- If X is affine, then this was proven by M. Matsumoto.

Outline of proof of Thm D:

- By Thm C, it suffices to verify the injectivity of

$\rho_{X_3/k}\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Out}^{\mathrm{FC}}(\pi_1(X_3 \otimes_k \overline{k}))\,.$

- By considering a "tripod in X_3",

$$\mathrm{Ker}(\rho_{X_3/k}) \subseteq \mathrm{Ker}(\rho_{\mathrm{tripod}/k})\,.$$

- By the above result of Belyi,

$$\mathrm{Ker}(\rho_{\mathrm{tripod}/k}) = \{1\}\,.$$

§4: A Version of the Grothendieck Conjecture for Universal Curves

(g, r) s.t. $2g - 2 + r > 0$

$\mathcal{M}_{g,r}/\mathbb{C}$: moduli stack of (g, r)-curves over $\mathbb{C}$

$(\mathcal{C}^{\mathrm{cpt}}_{g,r} \to \mathcal{M}_{g,r};\ \ s_1, \cdots, s_r \colon \mathcal{M}_{g,r} \to \mathcal{C}^{\mathrm{cpt}}_{g,r})$: universal curve over $\mathcal{M}_{g,r}$

$\mathcal{C}_{g,r} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\mathrm{cpt}}_{g,r} \setminus \bigcup_{i=1}^{r} \mathrm{Im}(s_i) \quad (\simeq \mathcal{M}_{g,r+1})$

<u>Theorem E (Mochizuki-H)</u>

ϕ: outer automorphism of $\pi_1(\mathcal{C}_{g,r})$ over $\pi_1(\mathcal{M}_{g,r})$

- $2g - 2 + r > 2$
- ϕ preserves the set of cuspidal inertia subgroups associated to the s_i's.

$\Longrightarrow$

ϕ arises from an automorphism of $\mathcal{C}_{g,r}$ over $\mathcal{M}_{g,r}$, i.e., $\phi = \mathrm{id}$.

$\Longrightarrow$

The image of the universal outer monodromy representation is <u>center-free</u>.

Outline of proof of Thm E:

- If $r > 0$, then the left-hand vertical arrow of

$$\begin{array}{ccccc} \text{fiber product} & \longrightarrow & \pi_1(\mathcal{C}_{g,r}) & & \\ \downarrow & & \downarrow & & \\ \text{Ker(proj.)} & \longrightarrow & \pi_1(\mathcal{M}_{g,r}) & \xrightarrow{\text{proj.}} & \pi_1(\mathcal{M}_{g,r-1}) \end{array}$$

is isomorphic to "$\pi_1(X_2) \to \pi_1(X)$" for a $(g, r-1)$-curve X.

Thus, Thm C and Thm H $\Rightarrow$ Thm E.

- If $r = 0$, then by considering the various irreducible components of the divisor at infinity of $\mathcal{M}_{g,r}$, Thm E in the case where $r > 0$ and Thm B $\Rightarrow$ α is a profinite Dehn twist,

i.e., graphic outer automorphism of

$\pi_1($semi-graph of anab. of PSC-type$)$

that induces "id" on the underlying graph and on any irreducible component.

Thus, the consideration of the various degenerations of the fiber $\Rightarrow$ Thm E.

§5: A Generalization of a Result due to Y. André

$[k : \mathbb{Q}] < \infty$

$\mathfrak{p}$: nonarchimedean prime of k

X/k: hyperbolic curve

$\pi_1^{\mathrm{temp}}(X_{\mathfrak{p}})$: tempered π_1 of $X \otimes_k (\overline{k}_{\mathfrak{p}})^\wedge$

$\mathrm{Out}^{\mathrm{temp}} \stackrel{\mathrm{def}}{=} \mathrm{Out}(\pi_1^{\mathrm{temp}}(X_{\mathfrak{p}}))$

$\Longrightarrow$

$$\begin{array}{ccc}
\mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) & \xrightarrow{\rho^{\mathrm{temp}}_{X/k:\mathfrak{p}}} & \mathrm{Out}^{\mathrm{temp}} \\
\cap\big\downarrow & & \big\downarrow\cap \\
\mathrm{Gal}(\overline{k}/k) & \xrightarrow[\rho_{X/k}]{} & \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))
\end{array}$$

<u>Theorem F (André)</u>

$X \stackrel{\exists\,\text{fét}}{\longleftarrow} Y_{/\overline{k}} \stackrel{\exists\,\text{nonconstant}}{\longrightarrow} \text{tripod}_{/\overline{k}}$

$\Longrightarrow$

$$\mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) = \mathrm{Gal}(\overline{k}/k) \ \cap \ \mathrm{Out}^{\mathrm{temp}}$$

$$\mathrm{Out}^{\mathrm{M}} \subseteq \ (\mathrm{Out}^{\mathrm{temp}} \subseteq) \ \mathrm{Out}(\pi_1(X \otimes_k \overline{k})) :$$

group of <u>isometric</u> outer automorphisms,

i.e., preserve the metrics of nodes of the various coverings.

(The metric of "$\mathfrak{o}_{\mathfrak{p}}[[x,y]]/(xy-a)$" is $v_{\mathfrak{p}}(a)$.)

$\Longrightarrow$

$$\begin{array}{ccc}
\mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) & \longrightarrow & \mathrm{Out}^{\mathrm{M}} \\
\| & & \downarrow \cap \\
\mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) & \xrightarrow[\rho^{\mathrm{temp}}_{X/k:\mathfrak{p}}]{} & \mathrm{Out}^{\mathrm{temp}} \\
\cap \downarrow & & \downarrow \cap \\
\mathrm{Gal}(\overline{k}/k) & \xrightarrow[\rho_{X/k}]{} & \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))
\end{array}$$

<u>Theorem G (Mochizuki-H)</u>

$$\mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) = \mathrm{Gal}(\overline{k}/k) \ \cap \ \mathrm{Out}^{\mathrm{M}}$$

Outline of proof of Thm G:

By (almost pro-l) Thm C, the map

$$\mathrm{Out}^{\mathrm{FC}}(\pi_1(X_3 \otimes_k \overline{k})) \to \mathrm{Out}(\pi_1(\mathrm{tripod}))$$

obtained by considering a "tripod in X_3" induces

$$\mathrm{Out}^{\mathrm{FC}}(\pi_1(X_3 \otimes_k \overline{k})) \cap \mathrm{Out}^{\mathrm{M}}$$

$$\stackrel{(*)}{\longrightarrow} \mathrm{Out}^{\mathrm{M}}(\pi_1(\mathrm{tripod}))\,.$$

Therefore,

$$\mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$$

$$\subseteq \mathrm{Gal}(\overline{k}/k) \cap \mathrm{Out}^{\mathrm{M}}$$

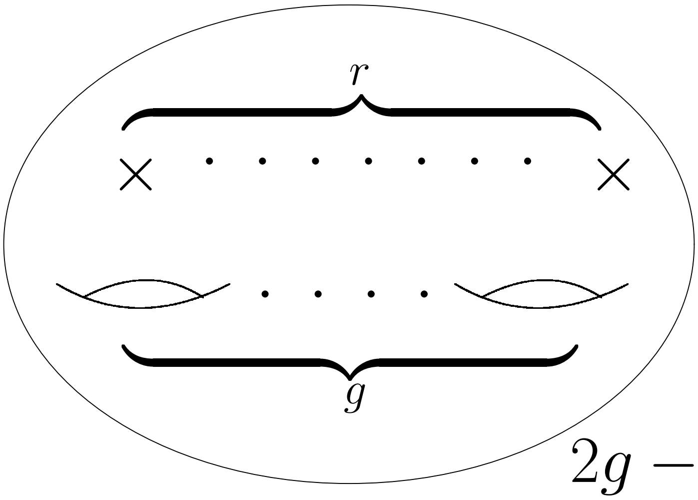
$$= \mathrm{Gal}(\overline{k}/k) \cap \mathrm{Out}^{\mathrm{FC}}(\pi_1(X_3 \otimes_k \overline{k})) \cap \mathrm{Out}^{\mathrm{M}}$$

$$\stackrel{(*)+\mathrm{ThmD}}{\subseteq} \mathrm{Gal}(\overline{k}/k) \cap \mathrm{Out}^{\mathrm{M}}(\pi_1(\mathrm{tripod}))$$

$$\stackrel{\text{André}}{\subseteq} \mathrm{Gal}(\overline{k}_{\mathfrak{p}}/k_{\mathfrak{p}})\,.$$

§6: Differences between $\mathrm{Out}^{\mathrm{FC}} \subseteq \mathrm{Out}^{\mathrm{F}} \subseteq \mathrm{Out}$

X: hyperbolic Riemann surface of type (g, r), i.e.,



$$2g - 2 + r > 0$$

X_n: n-th configuration space of X, i.e.,

$$X_n \stackrel{\text{def}}{=} \overbrace{X \times \cdots \times X}^{n} \setminus \text{various diagonals}$$

$\dagger \in \{\text{discrete, profinite, pro-}l\}$

$\Pi_n \stackrel{\text{def}}{=} \pi_1^{\text{top}}(X_n)^{\dagger}$

$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$:

- induce "id" on the set of Fiber subgroups.
- preserve the set of Cuspidal inertia subgroups.

$\sim \mathrm{Out}^{\mathrm{F}}$ v.s. Out $\sim$

<u>Theorem H (Mochizuki-Tamagawa)</u>

$2g - 2 + r > 1 \Longrightarrow$

Any element of $\mathrm{Out}(\Pi_n)$ preserves the set of fiber subgroups, i.e., $\exists$ split exact sequence

$$1 \longrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n) \longrightarrow \mathrm{Out}(\Pi_n) \longrightarrow \mathfrak{S}_n \longrightarrow 1\,.$$

$\sim \mathrm{Out}^{\mathrm{FC}}$ v.s. $\mathrm{Out}^{\mathrm{F}} \sim$

<u>Theorem I (Mochizuki-H)</u>

$$\mathrm{Im}\Big(\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)\Big) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)\,.$$

Outline of proof of Thm I:

For simplicity, $n = 1$, $\dagger =$ "pro-l", $g > 0$.

$$\begin{array}{ccc} \mathrm{Out}^{\mathrm{F}}(\Pi_2) & \longrightarrow & \mathrm{Out}^{\mathrm{F}}(\Pi_1) = \mathrm{Out}(\Pi_1) \\ \alpha_2 & \mapsto & \alpha \end{array}$$

$$\begin{array}{ccccccccc} & & & & \alpha_2 \curvearrowright & & \alpha_2 \curvearrowright & & \\ & & & & \Pi_2 & = & \Pi_2 & & \\ & & & & \downarrow & & \downarrow {\scriptstyle (\mathrm{pr}_1, \mathrm{pr}_2)} & & \\ 1 & \longrightarrow & \mathbb{Z}_l(1) & \longrightarrow & \Pi_2^{\text{c-cn}} & \longrightarrow & \Pi_1 \times \Pi_1 & \longrightarrow & 1 \\ & & & & \alpha_2^{\text{c-cn}} \curvearrowright & & \alpha \times \beta \curvearrowright & & \end{array}$$

$$\begin{array}{ccc} H^2(\Pi_1 \times \Pi_1, \mathbb{Z}_l(1)) & \simeq & H^2(X \times X, \mathbb{Z}_l(1)) \\ \Pi_2^{\text{c-cn}} & & \mapsto c_1(\text{diagonal in } X \times X) \end{array}$$

$$\begin{array}{l} \to H^1(\Pi_1, \mathbb{Z}_l)^{\otimes 2} \otimes \mathbb{Z}_l(1) \simeq \mathrm{Hom}((\Pi_1^{\mathrm{ab}})^{\otimes 2}, \mathbb{Z}_l(1)) \\ \mapsto \qquad\qquad\qquad\qquad\qquad \text{Poincaré duality} \end{array}$$

P.D. factors through $(\Pi_1^{\mathrm{ab}})^{\otimes 2} \twoheadrightarrow \pi_1^{\mathrm{ab}}(X^{\mathrm{cpt}})^{\otimes 2}$

$\Rightarrow \alpha$ preserves $\mathrm{Ker}(\Pi_1^{\mathrm{ab}} = \pi_1^{l\text{-ab}}(X) \twoheadrightarrow \pi_1^{l\text{-ab}}(X^{\mathrm{cpt}}))$

... Apply this argument to various fét of X_2 ...