Classical p-adic Teichmüller theory in characteristic three

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References:

• S. Mochizuki, A theory of ordinary p-adic curves, *Publ. Res. Inst. Math. Sci.* **32** (1996).

• Y. Hoshi, Nilpotent admissible indigenous bundles via Cartier operators in characteristic three, to appear in *Kodai Math. J.*

§1: Review

$f\colon X \to S$: a projective smooth curve of genus ≥ 2

$P \to X$: a $\mathbb{P}^1$-bundle

∇: a connection on $P \to X$ (relative to X/S)

σ: a section of $P \to X$

By differentiating σ by means of ∇,
we obtain a homomorphism $\sigma^* \omega_{P/X} \to \omega_{X/S}$,
i.e., the *Kodaira-Spencer map* at σ relative to ∇.

(This measures the failure of σ to be horizontal.)

<u>Definition</u> [Gunning (1967)]

$(P \to X, \nabla, \sigma)$: an *indigenous bundle* over X/S

$\overset{\mathrm{def}}{\Leftrightarrow}$ the KS map $\sigma^* \omega_{P/X} \to \omega_{X/S}$ is an isomorphism

An important example

(but neither projective nor a scheme...)

M: the moduli stack of elliptic curves

$E \to M$: the universal elliptic curve

$\mathcal{L}$: the Hodge bundle of $E \to M$

$\Rightarrow$

- $\mathcal{H}_{\mathrm{dR}} \stackrel{\mathrm{def}}{=} \mathcal{H}^1_{\mathrm{dR}}(E/M)$: locally free of rank 2
- ∇_{GM}: the Gauß-Manin connection on $\mathcal{H}_{\mathrm{dR}}$
- $0 \to \mathcal{L} \to \mathcal{H}_{\mathrm{dR}} \to \mathcal{L}^\vee \to 0$: the Hodge filtration

$\Rightarrow$

- $\mathbb{P}(\mathcal{H}_{\mathrm{dR}}) \to M$: the $\mathbb{P}^1$-bundle determined by $\mathcal{H}_{\mathrm{dR}}$
- $\mathbb{P}(\nabla_{\mathrm{GM}})$: the connection determined by ∇_{GM}
- σ_{Hdg}: the section determined by the Hodge fil'n

$\Rightarrow \quad (\mathbb{P}(\mathcal{H}_{\mathrm{dR}}) \to M,\ \mathbb{P}(\nabla_{\mathrm{GM}}),\ \sigma_{\mathrm{Hdg}})$: an I.B. over M

$$\mathcal{L} \hookrightarrow \mathcal{H}_{\mathrm{dR}} \xrightarrow{\nabla_{\mathrm{GM}}} \mathcal{H}_{\mathrm{dR}} \otimes_{\mathcal{O}_M} \omega_M \twoheadrightarrow \mathcal{L}^\vee \otimes_{\mathcal{O}_M} \omega_M$$

(In particular, ∇_{GM} determines $\mathcal{L}^{\otimes 2} \xrightarrow{\sim} \omega_M$.)

In the remainder, suppose:

$S = \mathrm{Spec}(k)$, where $k = \overline{k}$, $\mathrm{char}(k) = p > 2$

$$\begin{array}{ccc} X^F & \longrightarrow & X \\ {\scriptstyle f^F}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathrm{Spec}(k) & \xrightarrow{\mathrm{Frob}_k} & \mathrm{Spec}(k) \end{array}$$

$\Phi \stackrel{\mathrm{def}}{=} (\mathrm{Frob}_X, f)\colon X \to X \times_{k,\mathrm{Frob}_k} k = X^F$:

the relative Frobenius of X/k

$$\begin{tikzcd}
X \arrow[r, "\Phi"] & X^F \arrow[r] \arrow[d, "f^F"'] & X \arrow[d, "f"] \\
& \mathrm{Spec}(k) \arrow[r, "\mathrm{Frob}_k"] & \mathrm{Spec}(k)
\end{tikzcd}$$

$P = (\pi\colon P \to X, \nabla, \sigma)$: an indigenous bundle over X/k

$\mathcal{A} \stackrel{\text{def}}{=} \pi_* \tau_{P/X}$: locally free of rank 3, $\mathcal{A} \cong \mathcal{A}^\vee$

$\psi\colon \Phi^* \tau_{X^F} \to \mathcal{A}$: the p-curvature map of P

(This measures the failure of p-power structure of derivations to be horizontal.)

Definition

(1) P: *nilpotent* $\overset{\text{def}}{\Leftrightarrow} (\Phi^*\tau_{X^F} \xrightarrow{\psi} \mathcal{A} \cong \mathcal{A}^\vee \xrightarrow{\psi^\vee} \Phi^*\omega_{X^F}) = 0$

(2) P: *admissible* $\overset{\text{def}}{\Leftrightarrow} V(\psi) = \emptyset$,
where $V(-) \overset{\text{def}}{=}$ the zero locus of $(-)$,
($\Leftrightarrow \psi$: nowhere vanishing, i.e., a locally split injection)

(3) $H \in f_*(\omega_X^{\otimes p-1})$: the *square Hasse invariant* of P
$\overset{\text{def}}{=} \Phi^*\tau_{X^F} \xrightarrow{\psi} \mathcal{A} \twoheadrightarrow \mathcal{A}/\pi_*(\tau_{P/X}(-\sigma)) \xrightarrow{\sim} \tau_X$

(3) $H \in f_*(\omega_X^{\otimes p-1})$: the *square Hasse invariant* of P

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(4) $\mathbb{R}^1 f^F_* \tau_{X^F} \to \mathbb{R}^1 f^F_* \Phi_* \Phi^* \tau_{X^F} \xrightarrow{\sim} \mathbb{R}^1 f_* \Phi^* \tau_{X^F}$

$\xrightarrow{H} \mathbb{R}^1 f_* \tau_X$: the *Frobenius* induced by P

(5) P: *ordinary* $\stackrel{\mathrm{def}}{\Leftrightarrow}$ the Frobenius is an isomorphism

(Remark: ordinary $\Rightarrow$ admissible $\Rightarrow$ $H \neq 0$)

(6) If P is nilpotent and admissible,

then $\exists \mathcal{H}$ and $\exists \chi \in f_* \mathcal{H}$ s.t.

$\mathcal{H}^{\otimes 2} \cong \omega_X^{\otimes p-1}$ and $\chi \otimes \chi = H$

χ: the *Hasse invariant* of P

$V(\chi)$: the *supersingular divisor* of P (Note: $H \neq 0$)

Example

The above example $(\mathbb{P}(\mathcal{H}_{\mathrm{dR}}),\ \mathbb{P}(\nabla_{\mathrm{GM}}),\ \sigma_{\mathrm{Hdg}})$ is nilpotent and ordinary.

"$\mathcal{H}$" $= \mathcal{L}^{\otimes p-1}$

(Recall: ∇_{GM} determines $\mathcal{L}^{\otimes 2} \overset{\sim}{\to} \omega_M$.)

"χ" $=$ the classical Hasse invariant $(\mathcal{L}_{E^F/M^F})^\vee \to \mathcal{L}^\vee$

"$V(\chi)$" $=$ the locus of M parametrizing supersingular elliptic curves

A part of classical p-adic Teichmüller theory

a nilpotent ordinary indigenous bundle on X/k

$\Rightarrow$

- a canonical lifting $\mathbb{X}$ of X over $W(k)$
- a canonical lifting of $\Phi|_{X^{\mathrm{ord}}}$ over $W(k)$
- a canonical coordinate of $\mathbb{X}^{\mathrm{ord}}/W(k)$
- a canonical rep'n $\pi_1(\mathbb{X}[1/p]) \to GL_2(\mathbb{Z}_p)/\{\pm 1\}$

....

When does X/k admit a nilpotent ordinary indigenous bundle?

Theorem [Mochizuki]

A general X/k is *hyperbolically ordinary*, i.e., admits a nilpotent ordinary indigenous bundle.

Remark

One of basic questions of p-adic Teichmüller theory is: Is every X/k hyperbolically ordinary?

Is there an easier object that helps us understand/classify nilpotent ordinary indigenous bundles?

§2: Results

$\mathcal{L}$: an invertible sheaf on X s.t. $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$

$\Rightarrow \quad C_{\mathcal{L}} \overset{\text{up to } k^\times}{\curvearrowright} f(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_X)$:

the *Cartier operator* associated to $\mathcal{L}$

- $1/p$-linear, i.e., $C_{\mathcal{L}}(c \cdot v) = c^{1/p} \cdot C_{\mathcal{L}}(v)$
- $C_{\mathcal{O}_X} =$ the usual Cartier operator on $f_*\omega_X$

D: an effective divisor on X

Definition

D: *of CE-type*

$\overset{\text{def}}{\Leftrightarrow} \exists \mathcal{L}$ and $\exists \chi \in f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_X)$ s.t.

(1) D: reduced

(2) $D = V(\chi)$

(3) $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$

(4) χ is a Cartier eigenform, i.e., $\chi \neq 0$, $C_{\mathcal{L}}(\chi) \in k^{\times} \cdot \chi$

Definition

D: *of CEO-type*
$\overset{\text{def}}{\Leftrightarrow} \exists(\mathcal{L}, \chi \in f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_X))$ as in the above definition s.t. either

- $\mathcal{L} \cong \mathcal{O}_X$ and J_X is ordinary

or

- $\mathcal{L} \ncong \mathcal{O}_X$ and $J_{X_\mathcal{L}}/\mathrm{Im}(J_X)$ is ordinary,
  where $X_\mathcal{L} \to X$: the finite étale double covering which trivializes $\mathcal{L}$

Theorem

If $p = 3$, then, by considering the supersingular divisors,

$$
\begin{tikzcd}
\{\text{nilp. adm. I.B.}\}/\cong \arrow[r, "\sim"] & \{\text{of CE-type}\} \\
\{\text{nilp. ord. I.B.}\}/\cong \arrow[r, "\sim"] \arrow[u, hook, "\cup"] & \{\text{of CEO-type}\}. \arrow[u, hook, "\cup"]
\end{tikzcd}
$$

Consequence To find a nilp. adm. I.B.:

(1) Take an invertible sheaf $\mathcal{L}$ of order at most 2.
($\sharp$ of such invertible sheaves $= 2^{2g}$)

(2) Find an "eigenvector" of $C_{\mathcal{L}} \curvearrowright f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_X)$.
(semi-linear algebra,
the essential $\sharp$ of such vectors $= \sharp\mathbb{P}^{p\text{-rank}-1}(\mathbb{F}_3)$)

(3) Consider the issue of whether or not the zero locus of the eigenvector of (2) is reduced. (algebraic geometry)

To find a nilp. ord. I.B.:
(4) Consider the issue of whether or not the abelian variety under consideration is ordinary.

Remark

(1) The injectivity follows from general theory.

(2) A priori, there is no relationship between the hyperbolic ordinariness of curves and the usual ordinariness of abelian varieties. On the other hand, by the above theorem, we have such a relationship.

Corollary

$p = 3$, P: a nilpotent ordinary I.B. over X/k
$\Rightarrow \quad \exists\, Y \to X$: a connected finite étale covering s.t.
$P|_Y$: not ordinary

Remark

One of basic questions of p-adic Teichmüller theory is:
Is the pull-back, via a con'd fét cov'g, of nilp. ord. I.B. still ordinary?

Key ingredient in the proof

A theorem of Raynaud: For every X, there exists a con'd fét cov'g whose Jacobian variety is nonordinary.

Corollary

$(p, g) = (3, 2)$

$\Rightarrow$ Every X/k is hyperbolically ordinary,

i.e., admits a nilp. ord. I.B.

Remark

One of basic questions of p-adic Teichmüller theory is:

Is every X/k hyperbolically ordinary?

Proof

Explicit computation.

§3: Outline

(1) $\mathcal{B} \stackrel{\mathrm{def}}{=} \mathrm{Coker}(\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X)$: loc'y free of rank $p-1$
$P_0 \stackrel{\mathrm{def}}{=} \mathbb{P}(\mathcal{B})$: a $\mathbb{P}^1$-bundle (by $p=3$) equipped with a "dormant" connection ∇_0 (by a theorem of Cartier)
Moreover: $P_0 \stackrel{\mathrm{def}}{=} (P_0, \nabla_0, \exists!\text{a section})$: an I.B.

(2) $\{$I.B.$\}/\cong$ forms a $f_*(\omega_X^{\otimes 2})$-torsor (by general theory)
Thus, by P_0 of (1),

$$\begin{array}{ccc} f_*(\omega_X^{\otimes 2}) & \stackrel{\sim}{\longrightarrow} & \{\text{I.B.}\}/\cong \\ \theta & \mapsto & P_\theta \stackrel{\mathrm{def}}{=} \text{"}P_0 + \theta\text{"} \end{array}$$

(3) By a local computation, if $\theta = a(t)dt \otimes dt$,

- $\theta \in k^\times \cdot$ (the square Hasse inv't of P_θ) ($\subseteq f_*(\omega_X^{p-1})$)
- P_θ: nilpotent $\Leftrightarrow$ $(a')^2 + aa'' + a^3 = 0$
- P_θ: admissible $\Leftrightarrow$ $\mathrm{ord}_{\forall x}(a) \leq 2$

(4) (For simplicity, suppose that "$\mathcal{H}$" $= \omega_X$.)
(Recall: $\mathcal{H}^{\otimes 2} \cong \omega_X^{\otimes p-1}$)
By a local computation, if $\theta = \chi \otimes \chi$, and $\chi = b(t)dt$,

- P_θ: nilp. $\Leftrightarrow$ $b'' = b^3$,i.e.,χ is a Cartier eigenform
- P_θ: adm. $\Leftrightarrow$ $\mathrm{ord}_{\forall x}(b) \leq 1$, i.e., $V(\chi)$ is reduced