Introduction to Inter-universal Teichmüller Theory II

— An Étale Aspect of the Theory of Étale Theta Functions —

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Notation and Terminology

For an odd prime number l,

$$\mathbb{F}_l^{\rtimes\pm} \stackrel{\mathrm{def}}{=} (\mathbb{F}_l)_+ \rtimes \{\pm 1\}, \quad \mathbb{F}_l^{*} \stackrel{\mathrm{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}, \quad l^{*} \stackrel{\mathrm{def}}{=} \sharp\mathbb{F}_l^{*} = \frac{l-1}{2}$$

an $\mathbb{F}_l^{\pm}$-group $\stackrel{\mathrm{def}}{\Leftrightarrow}$ a set S equipped with a $\{\pm 1\}$-orbit of $S \xrightarrow{\sim} \mathbb{F}_l$

For a topological group G,

$${}_{\infty}H^i(G, A) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G\text{: open subgps of finite index}} H^i(H, A)$$

For a p-adic local field k, the $\times\boldsymbol{\mu}$-*Kummer structure* of $G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\times\boldsymbol{\mu}}$

$$\stackrel{\mathrm{def}}{\Leftrightarrow} \{\mathrm{Im}\big((\mathcal{O}_{\overline{k}}^{\times})^H = \mathcal{O}_{\overline{k}^H}^{\times} \hookrightarrow \mathcal{O}_{\overline{k}}^{\times} \twoheadrightarrow \mathcal{O}_{\overline{k}}^{\times\boldsymbol{\mu}}\big)\}_{H \subseteq G_k\text{: open subgps}}$$

a *poly-*(*iso*)*morphism* $A \to B$

$\stackrel{\mathrm{def}}{\Leftrightarrow}$ a set consisting of (iso)morphisms $A \to B$

the *full poly-isomorphism* $A \xrightarrow{\sim} B \stackrel{\mathrm{def}}{\Leftrightarrow}$ the poly-isom. $\mathrm{Isom}(A, B)$

Fundamental Strategy (cf. p.23 of I)

$\square$ is, for instance, a log-shell, a theta function, or a κ-coric function.

- Start with a usual/existing $\square$ (i.e., a *Frobenius-like* $\square$).
- Construct *links* by means of such Frobenius-like objects.
- Take an *étale-like* object closely related to $\square$

  (e.g., "$\pi_1^{\text{temp}}(\underline{\underline{X}}_{\underline{v}})$" for a theta function — cf. II and III).
- Give a multiradial mono-anabelian algorithm of reconstructing $\square$ from the étale-like object, i.e., construct a suitable *étale-like* $\square$.
- Establish "multiradial Kummer-detachment" of $\square$, i.e., a suitable Kummer isomorphism "Frob.-like $\square \stackrel{\sim}{\to}$ étale-like $\square$".

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p, l: prime numbers

k: a p-adic local field, i.e., $[k : \mathbb{Q}_p] < \infty$, s.t. $\sqrt{-1} \in k$

E: an elliptic curve$/k$ which has split multiplicative reduction$/\mathcal{O}_k$

$q \in \mathcal{O}_k^{\triangleright}$: the q-parameter of E

$X^{\log} \stackrel{\text{def}}{=} (E, \{o\} \subseteq E)$: the smooth log curve$/k$ determined by E

$\{\pm 1\} \curvearrowright E$, hence also $\curvearrowright X^{\log} \quad \Rightarrow \quad X^{\log} \to C^{\log} \stackrel{\text{def}}{=} [X^{\log}/\{\pm 1\}]$

Assumptions

- 2, p, and l are distinct prime numbers
- $E[2l](\overline{k}) = E[2l](k) \quad (\Leftrightarrow \boldsymbol{\mu}_l(\overline{k}) \subseteq k$ and $\underline{\underline{q}} \stackrel{\text{def}}{=} q^{1/2l} \in k)$
- $C^{\log}$ is a k-*core* $\quad$ ($\Rightarrow$ one may apply *elliptic cuspidalization*)

One obtains a comm. diagram of conn'd log étale tempered coverings

$$
\begin{array}{ccccc}
\underline{\underline{\ddot{Y}}}^{\log} & \xrightarrow[\boldsymbol{\mu}_l]{(1)} & \ddot{Y}^{\log} & & \\
(2)\downarrow \boldsymbol{\mu}_2 & & (3)\downarrow \boldsymbol{\mu}_2 & & \\
\underline{\underline{Y}}^{\log} & \xrightarrow[\boldsymbol{\mu}_l]{(4)} & Y^{\log} & & \\
(5)\downarrow l\cdot\underline{\mathbb{Z}} & & (6)\downarrow l\cdot\underline{\mathbb{Z}} & & \\
\underline{\underline{X}}^{\log} & \xrightarrow[\boldsymbol{\mu}_l]{(7)} & \underline{X}^{\log} & \xrightarrow[\underline{\mathbb{F}}_l]{(8)} & X^{\log} \\
 & & (9)\downarrow \{\pm 1\} & & (10)\downarrow \{\pm 1\} \\
 & & \underline{C}^{\log} & \xrightarrow{(11),\ \deg=l} & C^{\log}
\end{array}
$$

$\mathrm{Csp}(-) \stackrel{\mathrm{def}}{=}$ the set of cusps of "$(-)$"

$\mathrm{Irr}(-) \stackrel{\mathrm{def}}{=}$ the set of irreducible comp. of the special fiber of "$(-)$"

- The three squares are *cartesian*.
- The composite $Y^{\log} \overset{(6)}{\to} \underline{X}^{\log} \overset{(8)}{\to} X^{\log}$: the covering determined by the dual semi-graph of the special fiber of $X^{\log}$.

  $\Rightarrow \underline{\mathbb{Z}} \overset{\text{def}}{=} \mathrm{Gal}(Y^{\log}/X^{\log}) \cong \mathbb{Z}$; $\mathrm{Csp}(Y^{\log})$, $\mathrm{Irr}(Y^{\log})$: $\underline{\mathbb{Z}}$-torsors
- $\underline{X}^{\log} \overset{(8)}{\to} X^{\log}$: the intermediate covering corresp'g to $l \cdot \underline{\mathbb{Z}} \subseteq \underline{\mathbb{Z}}$

  $\Rightarrow \underline{\mathbb{F}}_l \overset{\text{def}}{=} \mathrm{Gal}(\underline{X}^{\log}/X^{\log}) \cong \mathbb{F}_l$; $\mathrm{Csp}(\underline{X}^{\log})$, $\mathrm{Irr}(\underline{X}^{\log})$: $\underline{\mathbb{F}}_l$-torsors

Fix an $\in \mathrm{Csp}(\underline{X}^{\log})$, i.e., the *zero cusp* of $\underline{X}^{\log}$.

$\Rightarrow$ a structure of elliptic curve on the underlying scheme of $\underline{X}^{\log}$

- $\underline{X}^{\log} \overset{(9)}{\to} \underline{C}^{\log} \overset{\text{def}}{=} [\underline{X}^{\log}/\{\pm 1\}]$

  zero cusp of $\underline{X}^{\log} \Rightarrow$ a cusp of $\underline{C}^{\log}$, i.e., the *zero cusp* of $\underline{C}^{\log}$

- $\underline{\underline{X}}^{\log} \overset{(7)}{\to} \underline{X}^{\log}$ totally ramifies at $\forall \in \mathrm{Csp}(\underline{X}^{\log})$, $\mathrm{Gal} \cong \boldsymbol{\mu}_l$
  $\Rightarrow \mathrm{Csp}(\underline{\underline{X}}^{\log}) \overset{\sim}{\to} \mathrm{Csp}(\underline{X}^{\log})$, $\mathrm{Irr}(\underline{\underline{X}}^{\log}) \overset{\sim}{\to} \mathrm{Irr}(\underline{X}^{\log})$
  zero cusp of $\underline{X}^{\log} \Rightarrow$ a cusp of $\underline{\underline{X}}^{\log}$, i.e., the *zero cusp* of $\underline{\underline{X}}^{\log}$
- $\ddot{Y}^{\log} \overset{(3)}{\to} Y^{\log}$: the double covering determined by "$\ddot{u} = u^{1/2}$"
  $\Rightarrow \mathrm{Csp}(\ddot{Y}^{\log}) \overset{2:1}{\twoheadrightarrow} \mathrm{Csp}(Y^{\log})$, $\mathrm{Irr}(\ddot{Y}^{\log}) \overset{\sim}{\to} \mathrm{Irr}(Y^{\log})$
- $\mathrm{Aut}_k(\underline{C}^{\log}) = \mathrm{Aut}_k(C^{\log}) = \{\mathrm{id}\}$
- $\mathrm{Aut}_k(X^{\log}) = \mathrm{Gal}(X^{\log}/C^{\log}) = \{\pm 1\}$
- $\mathrm{Aut}_k(\underline{X}^{\log}) = \mathrm{Gal}(\underline{X}^{\log}/X^{\log}) \rtimes \mathrm{Gal}(X^{\log}/C^{\log}) = \underline{\mathbb{F}}_l \rtimes \{\pm 1\}$
- $\mathrm{Aut}_k(\underline{\underline{X}}^{\log}) = \mathrm{Gal}(\underline{\underline{X}}^{\log}/\underline{X}^{\log}) \times \mathrm{Gal}(\underline{X}^{\log}/\underline{C}^{\log}) = \boldsymbol{\mu}_l \times \{\pm 1\}$
  ($\Rightarrow \mathrm{Csp}(\underline{\underline{X}}^{\log})^{\mathrm{Aut}_k(\underline{\underline{X}}^{\log})} = \{$the zero cusp$\}$)

Labels of Cusps and Components

$\Rightarrow$ geometry of $\underline{\underline{X}}^{\log}/k$ det. a natural str. of $\mathbb{F}_l^{\pm}$-gp on $\mathrm{Csp}(\underline{\underline{X}}^{\log})$,

hence also on $\mathrm{Irr}(\underline{\underline{X}}^{\log})$, $\mathrm{Csp}(\underline{X}^{\log})$, $\mathrm{Irr}(\underline{X}^{\log})$,

i.e., each element of these sets is *labeled by* an $\in \mathbb{F}_l$ up to $\{\pm 1\}$.

$$\mathrm{LabCusp}^{\pm} \overset{\mathrm{def}}{=} \mathrm{Csp}(\underline{X}^{\log}) \overset{\{\pm 1\}\curvearrowright}{\overset{\sim}{\to}} \mathbb{F}_l \quad (\curvearrowleft \ (\mathbb{F}_l^{\rtimes \pm} \ \cong) \ \mathrm{Aut}_k(\underline{X}^{\log}))$$

Fix a lifting $\in \mathrm{Irr}(Y^{\log})$ of $0 \in \mathrm{Irr}(\underline{X}^{\log})$.

(Such liftings form an $(l \cdot \underline{\mathbb{Z}})$-torsor.)

$\Rightarrow$ Such a lifting det. $\underline{\mathbb{Z}} \overset{\sim}{\to} \mathrm{Irr}(Y^{\log}) \overset{\sim}{\leftarrow} \mathrm{Irr}$ of $\underline{\underline{\ddot{Y}}}^{\log}$, $\ddot{Y}^{\log}$, $\underline{\underline{Y}}^{\log}$,

i.e., each $\in \mathrm{Irr}$ of $Y^{\log}$, $\underline{\underline{\ddot{Y}}}^{\log}$, $\ddot{Y}^{\log}$, $\underline{\underline{Y}}^{\log}$ is *labeled by* an $\in \underline{\mathbb{Z}}$.

Evaluation Points

$\Rightarrow \exists ! \mu_- \in \underline{X}(k)$: a 2-torsion whose closure intersects $0 \in \mathrm{Irr}(\underline{X}^{\log})$

$\mu_-^Y \in Y(k)$: a unique lifting of μ_- whose closure inter. $0 \in \mathrm{Irr}(Y^{\log})$

$\xi_a^Y \in Y(k)$: the image of μ_-^Y by $a \in \underline{\mathbb{Z}} = \mathrm{Gal}(Y^{\log}/X^{\log})$

Definition

- an *evaluation point* of $\underline{\underline{\ddot{Y}}}^{\log}$ (resp. $\ddot{Y}^{\log}$) labeled by $a \in \underline{\mathbb{Z}}$

  $\overset{\text{def}}{\Leftrightarrow}$ a (necessarily k-rat'l) lifting $\in \underline{\underline{\ddot{Y}}}$ (resp. $\ddot{Y}$) of $\xi_a^Y \in Y(k)$

- an *evaluation point* of $\underline{\underline{X}}^{\log}$ labeled by $a \in \mathrm{LabCusp}^{\pm}$

  $\overset{\text{def}}{\Leftrightarrow}$ the image $\in \underline{\underline{X}}$ of an evaluation point of $\in \underline{\underline{\ddot{Y}}}^{\log}$ labeled by a lifting $\in \underline{\mathbb{Z}}$ of $a \in \mathrm{LabCusp}^{\pm}$

Theta Functions

The function

$$\ddot{\Theta}(\ddot{u}) \;=\; q^{-\frac{1}{8}} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot q^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot \ddot{u}^{2n+1}$$

on $0 \in \mathrm{Irr}(\ddot{Y}^{\log})$ uniquely extends to a meromorphic function $\ddot{\Theta}$ on the stable model of $\ddot{Y}$.

- the zero divisor of $\ddot{\Theta} = \sum_{c \in \mathrm{Csp}(\ddot{Y}^{\log})} [c]$
- the pole divisor of $\ddot{\Theta} = \sum_{a \in \underline{\mathbb{Z}} \cong \mathrm{Irr}(\ddot{Y}^{\log})} \frac{a^2 \cdot \mathrm{ord}_k(q)}{2} \cdot [a]$

$\Rightarrow$ $\exists$An "l-th root" $\underline{\underline{\Theta}}$ on $\underline{\underline{\ddot{Y}}}$ of $\ddot{\Theta}$(an evaluation pt labeled by 0) $\cdot\, \ddot{\Theta}^{-1}$

(Note: $\ddot{\Theta}$(an ev. pt labeled by 0) $= -\ddot{\Theta}$(another ev. pt labeled by 0))

Special Values

$\underline{\underline{\Theta}}$ on $\underline{\underline{\ddot{Y}}}$: $\quad \underline{\underline{\Theta}}$(an evaluation point labeled by j) $\in \boldsymbol{\mu}_{2l} \cdot \underline{\underline{q}}^{j^2}$

$$
\begin{array}{ccccccccccccccc}
 & & -2 & & -1 & & 0 & & 1 & & 2 & & & j & \\
\cdots & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \cdots & \bullet & \cdots
\end{array}
$$

$$\text{ev.} \ \Downarrow \ \text{pt}$$

$$
\begin{array}{cccccc}
\underline{\underline{q}}^{4} & \underline{\underline{q}} & 1 & \underline{\underline{q}} & \underline{\underline{q}}^{4} & \underline{\underline{q}}^{j^2} \\
 & & & & & \text{mod } \boldsymbol{\mu}_{2l}
\end{array}
$$

$$\underbrace{-l^* \quad -l^*+1 \quad \cdots \quad -1 \quad 0 \quad 1 \quad \cdots \quad l^*-1 \quad l^*}_{\mathrm{LabCusp}^{\pm} \overset{\{\pm 1\} \curvearrowright}{\cong} \mathbb{F}_l}$$

$\Pi_{(-)}$: the log étale π_1 of "$(-)^{\log}$"

$\Delta_{(-)} \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_{(-)} \twoheadrightarrow G_k)$, i.e., the geom. log étale π_1 of "$(-)^{\log}$"

$\Pi^{\text{tp}}_{(-)}$: the log tempered π_1 of "$(-)^{\log}$"

$\Delta^{\text{tp}}_{(-)} \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi^{\text{tp}}_{(-)} \twoheadrightarrow G_k)$, i.e., the geom. log temp'd π_1 of "$(-)^{\log}$"

For $N \geq 1$, if "$J \stackrel{\exists}{\twoheadrightarrow} G_k$", then $J[\boldsymbol{\mu}_N] \stackrel{\text{def}}{=} \boldsymbol{\mu}_N(\overline{k}) \rtimes J$.

- a tautological splitting $s_J \colon J \to J[\boldsymbol{\mu}_N]$ of $J[\boldsymbol{\mu}_N] \twoheadrightarrow J$
- a natural homomorphism $H^1(J, \boldsymbol{\mu}_N(\overline{k})) \to \mathrm{Out}(J[\boldsymbol{\mu}_N])$

Thus, by the Kummer theory, we have:

$$k^\times \twoheadrightarrow k^\times/(k^\times)^N \hookrightarrow H^1(\Pi^{\text{tp}}_{\underline{\underline{Y}}}, \boldsymbol{\mu}_N(\overline{k})) \to \mathrm{Out}(\Pi^{\text{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$$

$$\mathcal{D}_{\underline{\underline{Y}}} \stackrel{\text{def}}{=} \langle \mathrm{Im}(k^\times),\ \mathrm{Gal}(\underline{\underline{Y}}^{\log}/\underline{\underline{X}}^{\log}) \cong l \cdot \mathbb{Z} \rangle \subseteq \mathrm{Out}(\Pi^{\text{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$$

$\Delta_\Theta \stackrel{\text{def}}{=} [\Delta_X, \Delta_X]/[\Delta_X, [\Delta_X, \Delta_X]]$

$\Rightarrow \quad \Delta_\Theta \cong \Delta_X^{\text{ab}} \wedge \Delta_X^{\text{ab}}$, i.e., "$\cong \widehat{\mathbb{Z}}(1)$ ($\stackrel{\text{def}}{=} \varprojlim_n \boldsymbol{\mu}_n(\overline{k})$)"

$\Rightarrow \quad l \cdot \Delta_\Theta \cong \widehat{\mathbb{Z}}(1)$

$\ddot{\eta}^\Theta \ \in \ H^1(\Pi^{\text{tp}}_{\ddot{Y}}, \Delta_\Theta)$: the Kummer class of a suitable $\in \mathcal{O}_k^\times \cdot \ddot{\Theta}$

$\Rightarrow \exists \underline{\underline{\ddot{\eta}}}^\Theta \ \in \ H^1(\Pi^{\text{tp}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \Delta_\Theta)$ s.t. $\ddot{\eta}^\Theta|_{\underline{\underline{\ddot{Y}}}} = \text{Im}(\underline{\underline{\ddot{\eta}}}^\Theta)$ in $H^1(\Pi^{\text{tp}}_{\underline{\underline{\ddot{Y}}}}, \ \Delta_\Theta)$,

i.e., the Kummer class of an $\in \mathcal{O}_k^\times \cdot \underline{\underline{\Theta}}^{-1}$

$\underline{\underline{\ddot{\eta}}}^{\Theta, l\cdot\underline{\mathbb{Z}}\times\boldsymbol{\mu}_2} \ \subseteq \ H^1(\Pi^{\text{tp}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \Delta_\Theta)$: the orbit of $\underline{\underline{\ddot{\eta}}}^\Theta$ by

$\text{Gal}(\underline{\underline{\ddot{Y}}}^{\log}/\underline{\underline{X}}^{\log}) \ = \ \Pi^{\text{tp}}_{\underline{\underline{X}}}/\Pi^{\text{tp}}_{\underline{\underline{\ddot{Y}}}} \ = \ l \cdot \underline{\mathbb{Z}} \times \boldsymbol{\mu}_2$

($\Rightarrow$ *indep.* of the choice of a lifting $\in \text{Irr}(Y^{\log})$ of $0 \in \text{Irr}(\underline{X}^{\log})$)

Thus, relative to $(l \cdot \Delta_\Theta) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \underset{\text{theory}}{\overset{\text{by scheme}}{\overset{\sim}{\to}}} \boldsymbol{\mu}_N(\overline{k})$,

each $\in \underline{\underline{\ddot{\eta}}}^{\Theta, l\cdot\underline{\mathbb{Z}}\times\boldsymbol{\mu}_2}$ mod $N \subseteq H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}, (l \cdot \Delta_\Theta) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z})$ can be obtained as "$\mathfrak{s}^{\Theta}_{\underline{\underline{\ddot{Y}}}} - s_{\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}}$" for some $\mathfrak{s}^{\Theta}_{\underline{\underline{\ddot{Y}}}}$:

$$1 \longrightarrow \boldsymbol{\mu}_N(\overline{k}) \longrightarrow \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}[\boldsymbol{\mu}_N] \overset{\mathfrak{s}^{\Theta}_{\underline{\underline{\ddot{Y}}}},\ s_{\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}}}{\underset{\curvearrowleft}{\longrightarrow}} \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}} \longrightarrow 1$$

$s^{\Theta}_{\underline{\underline{\ddot{Y}}}}\colon\ \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}} \overset{\mathfrak{s}^{\Theta}_{\underline{\underline{\ddot{Y}}}}}{\hookrightarrow} \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}[\boldsymbol{\mu}_N] \hookrightarrow \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$: a (mod N) *theta section*

Mono-theta Environments and Associated Cyclotomes

Definition

- A (mod N) *model mono-theta environment* $\overset{\text{def}}{\Leftrightarrow}$ a triple $(\Pi^{\text{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N],\ \mathcal{D}_{\underline{\underline{Y}}} \subseteq \text{Out}(\Pi^{\text{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]),\ \{\gamma \cdot \text{Im}(s^{\Theta}_{\underline{\underline{\ddot{Y}}}}) \cdot \gamma^{-1}\}_{\gamma \in \boldsymbol{\mu}_N(\overline{k})})$
- A (mod N) *mono-theta environment* $\overset{\text{def}}{\Leftrightarrow}$ an isomorph $\mathbb{M}^{\Theta}_N = (\Pi,\ \mathcal{D}_{\Pi} \subseteq \text{Out}(\Pi),\ s^{\Theta}_{\Pi})$ of a mod N model mono-theta env.
- The subgroup of Π (of $\mathbb{M}^{\Theta}_N$) corresp'g to "$\boldsymbol{\mu}_N(\overline{k}) \subseteq \Pi^{\text{tp}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$" is *group-theoretic*. $\Rightarrow$ $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_N)$: the exterior cyclotome
- The subquotient of Π (of $\mathbb{M}^{\Theta}_N$) corresponding to "$l \cdot \Delta_{\Theta}$" is *group-theoretic*. $\Rightarrow$ $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_N)$: the interior cyclotome

Algorithmic Reconstruction

$\Pi_\bullet$: an isomorph of $\Pi^{\rm tp}_{\underline{\underline{X}}}$

$\mathbb{M}^{\Theta}_N$: a mod N mono-theta environment

- $\Pi_\bullet \overset{\exists\text{func'l}}{\underset{\text{algorithm}}{\Rightarrow}}$ topological gp corresponding to the topological gp

$$\Pi^{\rm tp}_{\underline{\underline{\ddot{Y}}}},\ \Pi^{\rm tp}_{\ddot{Y}},\ \Pi^{\rm tp}_{\underline{\underline{Y}}},\ \Pi^{\rm tp}_{Y},\ \Pi^{\rm tp}_{\underline{X}},\ \Pi^{\rm tp}_{X},\ \Pi^{\rm tp}_{\underline{C}},\ \Pi^{\rm tp}_{C},\ G_k,\ l\cdot\Delta_\Theta$$

- $\Pi_\bullet \overset{\exists\text{func'l}}{\underset{\text{algorithm}}{\Rightarrow}}$ a subset corresponding to the subset

$$(l\cdot\underline{\mathbb{Z}}\times\boldsymbol{\mu}_2)\text{-orbit of } \mathcal{O}_k^\times\cdot\underline{\underline{\Theta}}\ \subseteq\ H^1(\Pi^{\rm tp}_{\underline{\underline{\ddot{Y}}}}, l\cdot\Delta_\Theta)$$

- $\Pi_\bullet \overset{\exists\text{func'l}}{\underset{\text{algorithm}}{\Rightarrow}}$ a mod N mono-theta environment
- $\mathbb{M}^{\Theta}_N \overset{\exists\text{func'l}}{\underset{\text{algorithm}}{\Rightarrow}}$ a topological group corresponding to $\Pi^{\rm tp}_{\underline{\underline{X}}}$

Rigidity Properties of Mono-theta Environments

- Cyclotomic Rigidity
- Discrete Rigidity
- Constant Multiple Rigidity
- Isomorphism Class Compatibility
- Frobenioid Structure Compatibility

Cyclotomic Rigidity

$\mathbb{M}_N^{\Theta} \underset{\text{algorithm}}{\overset{\exists\text{func'l}}{\Rightarrow}}$ a canonical $(l \cdot \Delta_{\Theta})(\mathbb{M}_N^{\Theta}) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \overset{\sim}{\to} \Pi_{\boldsymbol{\mu}}(\mathbb{M}_N^{\Theta})$,

i.e., "$(l \cdot \Delta_{\Theta}) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \overset{\sim}{\to} \boldsymbol{\mu}_N(\overline{k})$ by scheme theory"

(cf. "a suitable Kmm isom. Frob.-like $\square \overset{\sim}{\to}$ étale-like $\square$" of p.3)

Discrete Rigidity

By means of the var. surj. $\boldsymbol{\mu}_N(\overline{k}) \twoheadrightarrow \boldsymbol{\mu}_M(\overline{k})$ $(M|N)$, one may define the notion of a "projective system of mono-theta env. $\{\mathbb{M}_N^{\Theta}\}_{N\geq 1}$".

$\forall$proj. system $\cong$ the natural proj. system of model mono-theta env.

Constant Multiple Rigidity

$\mathbb{M}_N^{\Theta} \overset{\exists\text{func'l}}{\underset{\text{algorithm}}{\Rightarrow}}$ the subset corresponding to the subset

$$\underline{\underline{\theta}} \overset{\text{def}}{=} (l \cdot \underline{\mathbb{Z}} \times \boldsymbol{\mu}_2)\text{-orbit of } \boldsymbol{\mu}_l \cdot \underline{\underline{\Theta}} \ \subseteq \ H^1(\Pi^{\text{tp}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \Delta_{\Theta})$$

of

$$(l \cdot \underline{\mathbb{Z}} \times \boldsymbol{\mu}_2)\text{-orbit of } \mathcal{O}_k^{\times} \cdot \underline{\underline{\Theta}} \ \subseteq \ H^1(\Pi^{\text{tp}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \Delta_{\Theta})$$

Pointed Inversion Automorphisms

Consider a pair $(\iota^{\underline{\underline{X}}}, \mu_-^{\underline{\underline{X}}})$ of

- a unique $\iota^{\underline{\underline{X}}} \in \mathrm{Aut}_k(\underline{\underline{X}})$ *of order two*, i.e., "$-1 \curvearrowright \mathrm{LabCusp}^{\pm}$"
- an evaluation point $\mu_-^{\underline{\underline{X}}}$ of $\underline{\underline{X}}^{\log}$ labeled by $0 \in \mathrm{LabCusp}^{\pm}$

Definition

- A *pointed inversion automorphism* of $\underline{\underline{\ddot{Y}}}^{\log}$ $\overset{\mathrm{def}}{\Leftrightarrow}$ a lifting $(\iota^{\underline{\underline{\ddot{Y}}}}, \mu_-^{\underline{\underline{\ddot{Y}}}})$ on $\underline{\underline{\ddot{Y}}}^{\log}$ of $(\iota^{\underline{\underline{X}}}, \mu_-^{\underline{\underline{X}}})$ s.t. $\iota^2 = \mathrm{id}$, $\iota(\mu) = \mu$
- A *group-theoretic pointed inversion automorphism* $\overset{\mathrm{def}}{\Leftrightarrow}$ a "group-theoretic pair (ι, D)" associated to a $(\iota^{\underline{\underline{\ddot{Y}}}}, \mu_-^{\underline{\underline{\ddot{Y}}}})$

Thus, such a $(\iota,\ D)$ det. a "lifting $\in \mathrm{Irr}(Y^{\log})$ of $0 \in \mathrm{Irr}(\underline{X}^{\log})$".

In particular:

$\underline{\underline{\theta}} = (l \cdot \underline{\underline{\mathbb{Z}}} \times \boldsymbol{\mu}_2)$-orbit of $\boldsymbol{\mu}_l \cdot \underline{\underline{\Theta}} \quad \supseteq \quad \underline{\underline{\theta}}^{\iota} = \boldsymbol{\mu}_{2l}$-multiples of a "$\underline{\underline{\Theta}}$"

(where $(-)^{\iota}$ is the "set of ι-invariants"), which thus implies that

$$\begin{array}{ccc} H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \Delta_{\Theta}) & \overset{\text{restriction to } D}{\longrightarrow} & H^1(D, l \cdot \Delta_{\Theta}) \\ \underline{\underline{\theta}}^{\iota} & \overset{\sim}{\to} & \text{(Kummer class of) } \boldsymbol{\mu}_{2l}, \end{array}$$

as well as, for a decomp. subgp $D_j \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}$ labeled by j (for (ι, D)),

$$\begin{array}{ccc} H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \Delta_{\Theta}) & \overset{\text{restriction to } D_j}{\longrightarrow} & H^1(D_j, l \cdot \Delta_{\Theta}) \\ \underline{\underline{\theta}}^{\iota} & \overset{\sim}{\to} & \text{(Kummer class of) } \boldsymbol{\mu}_{2l} \cdot \underline{\underline{q}}^{j^2}. \end{array}$$

(The operation of *Galois evaluation* w.r.t. D, as well as D_j)

Reconstructions of Étale Theta Functions via Mono-theta Env.

$\mathbb{M}^{\Theta}_{*} = \{\mathbb{M}^{\Theta}_{N}\}_{N}$: a projective system of mono-theta environments

$(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*}) \stackrel{\mathrm{def}}{=} \varprojlim(\cdots \stackrel{\sim}{\to} (l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{N}) \stackrel{\sim}{\to} (l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{M}) \stackrel{\sim}{\to} \cdots)$

$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\mathrm{def}}{=} \varprojlim_{N} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{N})$

By the cycl. rig.: $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*}) \stackrel{\sim}{\to} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*})$

By the cons. mult. rig.: $\underline{\underline{\theta}}(\mathbb{M}^{\Theta}_{*}) \subseteq H^{1}(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*}), (l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*}))$

$(\underline{\underline{\theta}}(\mathbb{M}^{\Theta}_{*}) \subseteq)\ {}_{\infty}\underline{\underline{\theta}}(\mathbb{M}^{\Theta}_{*}) \subseteq {}_{\infty}H^{1}(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*}), (l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*}))$

defined by $\{\, \eta \in {}_{\infty}H^{1} \mid n \cdot \eta \in \underline{\underline{\theta}}$ for some $n \geq 1 \,\}$

By $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*}) \stackrel{\sim}{\to} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*})$:

$$\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \subseteq {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \subseteq {}_{\infty}H^{1}(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*}), \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}))$$

Reconstructions of Constant Portions via Mono-theta Environments

$\Pi_\bullet$: an isomorph of $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$ $\overset{\text{func'l}}{\underset{\text{algorithm}}{\Rightarrow}}$ $\mathbb{M} \overset{\mathrm{def}}{=} \mathbb{M}^{\Theta}_{*}(\Pi_\bullet)$: a proj. system

Since $\underline{\underline{X}}^{\log}$ is *of strictly Belyi type*, by *Belyi cuspidalization*,

$\exists$a functorial algorithm for reconstructing, from $\Pi_\bullet$, an isomorph

$$\Pi_\bullet \twoheadrightarrow G_\bullet \overset{\mathrm{def}}{=} G_k(\Pi_\bullet) \curvearrowright \overline{k}(\Pi_\bullet) \supseteq \overline{k}(\Pi_\bullet)^\times \hookrightarrow {}_\infty H^1(G_\bullet, (l \cdot \Delta_\Theta)(\Pi_\bullet))$$

of the "$\Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \twoheadrightarrow G_k \curvearrowright \overline{k} \supseteq \overline{k}^\times \overset{\text{Kummer}}{\hookrightarrow} {}_\infty H^1(G_k, l \cdot \Delta_\Theta)$"

$$\mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(\Pi_\bullet)} \subseteq \mathcal{O}^{\times}_{\overline{k}(\Pi_\bullet)} \subseteq \mathcal{O}^{\rhd}_{\overline{k}(\Pi_\bullet)} \subseteq \overline{k}(\Pi_\bullet)^\times \subseteq {}_\infty H^1(G_\bullet, (l \cdot \Delta_\Theta)(\Pi_\bullet))$$

By $(l \cdot \Delta_\Theta)(\Pi_\bullet) \overset{\sim}{\to} (l \cdot \Delta_\Theta)(\mathbb{M}) \overset{\sim}{\to} \Pi_{\boldsymbol{\mu}}(\mathbb{M})$:

$$\begin{array}{ccccccccc} & & & & & & \overline{k}(\mathbb{M}) & & \\ & & & & & & \cup & & \\ \mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(\mathbb{M})} & \subseteq & \mathcal{O}^{\times}_{\overline{k}(\mathbb{M})} & \subseteq & \mathcal{O}^{\rhd}_{\overline{k}(\mathbb{M})} & \subseteq & \overline{k}(\mathbb{M})^\times & \subseteq & {}_\infty H^1(G_\bullet, \Pi_{\boldsymbol{\mu}}(\mathbb{M})) \end{array}$$

Reconstructions of Splittings via Mono-theta Environments

$$(\mathcal{O}^\times \cdot {}_\infty\underline{\underline{\theta}}_{\mathrm{env}})(\mathbb{M}) \stackrel{\mathrm{def}}{=} \mathcal{O}^\times_{\overline{k}(\mathbb{M})} + {}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}) \subseteq {}_\infty H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}), \Pi_{\boldsymbol{\mu}}(\mathbb{M}))$$

In particular, for a gp-th'c pt'd inv. aut. (ι, D) for $\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M})$,

$$\begin{array}{ccc} {}_\infty H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}), \Pi_{\boldsymbol{\mu}}(\mathbb{M})) & \xrightarrow{\text{Gal. ev. w.r.t. } D} & {}_\infty H^1(D, \Pi_{\boldsymbol{\mu}}(\mathbb{M})) \\ (\mathcal{O}^\times \cdot {}_\infty\underline{\underline{\theta}}_{\mathrm{env}})(\mathbb{M})^\iota & \twoheadrightarrow & \mathcal{O}^\times_{\overline{k}(\mathbb{M})} \end{array}$$

(i.e., *Gal. ev. labeled by* $0 \in \mathrm{LabCusp}^\pm$) determines a splitting

$$(\mathcal{O}^\times \cdot {}_\infty\underline{\underline{\theta}}_{\mathrm{env}})(\mathbb{M})^\iota / \mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(\mathbb{M})} = \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}(\mathbb{M})} \times \big({}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M})^\iota / \mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(\mathbb{M})}\big).$$

Thus, in summary, we obtain:

A Local Multiradial Algorithm Related to Étale Theta Functions

$\exists$A multiradial algorithm as follows:

coric data: an isomorph $(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}, \times\boldsymbol{\mu}\text{-Kmm})$ of $G_k \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$

radial data: $(\Pi_\bullet \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi_\bullet)),$ a coric data, $\alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}})$

for an isomorph $\Pi_\bullet$ of $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$,

where $\alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}}$ is the pair of the *full poly-isomorphism* $G_\bullet \overset{\sim}{\to}$ "G"

and $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi_\bullet)) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \overset{\text{zero}}{\to}$ "$\mathcal{O}^{\times\boldsymbol{\mu}}$"

output: the radial data and:

A Local Multiradial Algorithm Related to Étale Theta Functions

- The proj. system of mono-theta environments $\mathbb{M} \stackrel{\text{def}}{=} \mathbb{M}^{\Theta}_{*}(\Pi_{\bullet})$
- The subsets

$$\mathcal{O}^{\times}_{\overline{k}(\mathbb{M})} \ \cup \ (\mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}_{\text{env}})(\mathbb{M}) \ \subseteq \ {}_{\infty}H^{1}(\Pi^{\text{tp}}_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}), \Pi_{\boldsymbol{\mu}}(\mathbb{M}))$$

- The set of group-th'c pointed inversion automorphisms $\{(\iota, D)\}$
- The splittings for the various "(ι, D)"

$$(\mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}_{\text{env}})(\mathbb{M})^{\iota}/\mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(\mathbb{M})} \ = \ \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}(\mathbb{M})} \times \left({}_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M})^{\iota}/\mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(\mathbb{M})}\right)$$

via the operation of Galois evaluation w.r.t. "D"

- The diagram

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \stackrel{\text{nat'l}}{\stackrel{\sim}{\to}} \mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(\mathbb{M})} \stackrel{\text{nat'l}}{\stackrel{\sim}{\to}} \mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(\Pi_{\bullet})} \stackrel{\text{zero}}{\to} \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}(\Pi_{\bullet})} \underset{\text{poly}}{\overset{\text{full}}{\stackrel{\sim}{\to}}} \text{``}\mathcal{O}^{\times\boldsymbol{\mu}}\text{''}$$