Introduction to Inter-universal Teichmüller Theory III

— Globalization of Local Theories —

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Notation and Terminology

For an odd prime number l,

an $\mathbb{F}_l^{\pm}$-torsor $\overset{\text{def}}{\Leftrightarrow}$ a set S equipped with an $\mathbb{F}_l^{\rtimes\pm}$-orbit of $S \xrightarrow{\sim} \mathbb{F}_l$

For an $\mathbb{F}_l^{\pm}$-group T,

$T^{\times} \overset{\text{def}}{=} T \setminus \{0\}$,

$|T| \overset{\text{def}}{=} T/\{\pm 1\}$,

$T^{*} \overset{\text{def}}{=} |T| \setminus \{\bar{0}\} = T^{\times}/\{\pm 1\}$

F: a number field, i.e., $[F : \mathbb{Q}] < \infty$, s.t. $\sqrt{-1} \in F$

E: an elliptic curve over F which has

either good or split multiplicative reduction at $\forall v \in \mathbb{V}(F)$

$F_{\mathrm{mod}} \subseteq F$: the field of moduli of E

$l \geq 5$: a prime number $\quad K \stackrel{\mathrm{def}}{=} F(E[l](\overline{F}))$

$X \stackrel{\mathrm{def}}{=} E \setminus \{o\}$: the hyperbolic curve ass'd to $E \quad C \stackrel{\mathrm{def}}{=} [X/\{\pm 1\}]$

These satisfy some assumptions, e.g.,

- $E[6](\overline{F}) = E[6](F) \quad (\Rightarrow E[2l](\overline{F}) = E[2l](K))$
- C_K is a K-*core* (cf. the assumptions of p.5 of II)
- F/F_{mod} is *Galois* $\quad (\Rightarrow K/F_{\mathrm{mod}}$ is *Galois*)
- $\mathrm{Im}(G_F \to \mathrm{Aut}(E[l](\overline{F})) \stackrel{\mathrm{out}}{\cong} \mathrm{GL}_2(\mathbb{F}_l)) \supseteq \mathrm{SL}_2(\mathbb{F}_l)$

$\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$: the image of a splitting of $\mathbb{V}(K) \twoheadrightarrow \mathbb{V}_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_{\mathrm{mod}})$

$\underline{\mathbb{V}}^{\mathrm{bad}} \subseteq \underline{\mathbb{V}}$: a nonempty subset which satisfies some assumptions, e.g.:

- E has *bad reduction* at $\forall \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$
- $\forall \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ does *not lie* over 2 and l

$$\mathcal{O}^{\triangleright}_{K_{\underline{v}}} \ni \underline{\underline{q}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \begin{cases} \text{a } 2l\text{-th root of } q_{\underline{v}} & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} \\ 1 & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{V}} \setminus \underline{\mathbb{V}}^{\mathrm{bad}} \end{cases}$$

$\underline{\underline{\mathfrak{q}}} \stackrel{\mathrm{def}}{=} (\underline{\underline{q}}_{\underline{v}})_{\underline{v} \in \underline{\mathbb{V}}} \in \prod_{\underline{v} \in \underline{\mathbb{V}}} \mathcal{O}^{\triangleright}_{K_{\underline{v}}}$

An Approximate Statement of the Main Theorem of IUT

For a "general E/F",

$\exists$a suitable multiradial algorithm whose output data consist of

the following three objects $\curvearrowleft$ mild indeterminacies (cf. p.23)

- the collection of log-shells $\{\mathcal{I}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$
- the theta values $(= \{\underline{\underline{\mathfrak{q}}}^{j^2}\}_{1\leq j\leq l^*})$ $\curvearrowright$ $\prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{I}_{\underline{v}}$
- F_{mod} via κ-coric functions $\curvearrowright$ $\prod_{\underline{v}\in\underline{\mathbb{V}}} \big((K_{\underline{v}})_+$ "via $\mathcal{I}_{\underline{v}}$"$\big)$

Moreover, this alg'm is *compatible* w/ the Θ-link (more precisely, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link) "${}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$"; "†theta values $\mapsto {}^{\ddagger}\underline{\underline{\mathfrak{q}}}$".

Recall $\{\underline{\underline{\mathfrak{q}}}^{j^2}\}_{1\leq j\leq l^*} \in$ output

$$\begin{array}{ccccccccc} \mathrm{LabCusp}^{\pm} \overset{\{\pm 1\}\curvearrowright}{\cong} \mathbb{F}_l & \ni & \pm 1 & \pm 2 & \cdots & \pm j & \cdots & \pm l^* \\ \underline{\underline{\Theta}}_{\underline{v}} & \mapsto & \underline{\underline{q}}_{\underline{v}} & \underline{\underline{q}}_{\underline{v}}^{4} & \cdots & \underline{\underline{q}}_{\underline{v}}^{j^2} & \cdots & \underline{\underline{q}}_{\underline{v}}^{(l^*)^2} \\ \underline{\underline{\Theta}}_{\underline{w}} & \mapsto & \underline{\underline{q}}_{\underline{w}} & \underline{\underline{q}}_{\underline{w}}^{4} & \cdots & \underline{\underline{q}}_{\underline{w}}^{j^2} & \cdots & \underline{\underline{q}}_{\underline{w}}^{(l^*)^2} \\ \vdots & & & \vdots & & & & \end{array}$$

$\Rightarrow$ We have to synchronize globally the various sets of local labels of evaluation points, i.e., the various "$\mathrm{LabCusp}^{\pm}$" for the $\underline{v}$'s.

$\Rightarrow$ Use of the global data

Recall $(\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{1 \leq j \leq l^*} \curvearrowright \mathcal{I}_{\underline{v}}) \in$ output

- $\underline{\underline{\Theta}}_{\underline{v}}$ =(Galois ev. at $j \in \mathrm{LabCusp}^{\pm} \setminus \{0\}$)⇒ $\underline{\underline{q}}_{\underline{v}}^{j^2}$
- $\underline{\underline{\Theta}}_{\underline{v}} \cdot \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$ =(Galois eval. at $0 \in \mathrm{LabCusp}^{\pm}$)⇒ $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$ $(\overset{\log}{\Rightarrow} \mathcal{I}_{\underline{v}})$
- $\underline{\underline{q}}_{\underline{v}}^{j^2}$ (obtained by "$j \neq 0$") $\curvearrowright$ $\mathcal{I}_{\underline{v}}$ (obtained by "0")

(cf. II)

⇒ We have to relate "$\mathbb{F}_l^{\times}$" to "$\{0\}$".

⇒ Use of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry

"$\mathrm{Aut}_k(\underline{X}^{\log})$ $(\cong \mathbb{F}_l^{\rtimes\pm}) \curvearrowright (\mathbb{F}_l \cong)$ $\mathrm{LabCusp}^{\pm}$" (cf. p.9 of II)

(⇒ "Label-independent" objects)

- Use of the global data to synchronize "$\mathrm{LabCusp}^{\pm} \stackrel{\mathrm{def}}{=} \mathrm{Csp}(\underline{X}^{\log})$"
- Use of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry "$\mathrm{Aut}_k(\underline{X}^{\log}) \curvearrowright \mathrm{LabCusp}^{\pm}$"

Suppose: $\exists$a conn'd fét Galois covering $\underline{X}_K \to X_K$ of degree l s.t.

$\underline{X}_{\underline{v}} \to X_{\underline{v}}$ is "$\underline{X}^{\log} \to X^{\log}$" of II for $\forall \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

(In particular, the special fiber of $\underline{X}_{\underline{v}}$ has precisely l components.)

(cf. our restriction to $\underline{\mathbb{V}}^{\mathrm{bad}} \subseteq \underline{\mathbb{V}}$ $(\subseteq \mathbb{V}(K))$)

Fix a cusp of $\underline{X}_K$ among the l cusps, i.e., the *zero cusp* of $\underline{X}_K$.

$\Rightarrow$ a structure of elliptic curve on the compactification of $\underline{X}_K$

$\underline{C}_K \stackrel{\mathrm{def}}{=} [\underline{X}_K/\{\pm 1\}] \quad \leftarrow \underline{X}_K$

$\Rightarrow$ The zero cusp of $\underline{X}_K$ det. a cusp of $\underline{C}_K$, i.e., the *zero cusp* of $\underline{C}_K$.

$$\begin{array}{ccc} \underline{X}_K & \xrightarrow{\ \mathbb{F}_l\ } & X_K \\ {\scriptstyle \{\pm 1\}}\Big\downarrow & & \Big\downarrow{\scriptstyle \{\pm 1\}} \\ \underline{C}_K & \xrightarrow[\deg = l]{} & C_K \end{array}$$

C_K: a K-core $\quad\Rightarrow\quad \mathrm{Aut}_K(\underline{X}_K) \cong \mathbb{F}_l^{\rtimes\pm}$, $\mathrm{Aut}_K(\underline{C}_K) = \{\mathrm{id}\}$

$\mathrm{LabCusp}^{\pm}_{\circledcirc} \stackrel{\mathrm{def}}{=} \{$cusps of $\underline{X}_K\}$ $\quad$ ($\Rightarrow \sharp\mathrm{LabCusp}^{\pm}_{\circledcirc} = l$)

$\mathrm{LabCusp}_{\circledcirc} \stackrel{\mathrm{def}}{=} \{$nonzero cusps of $\underline{C}_K\}$ $\quad$ ($\Rightarrow \sharp\mathrm{LabCusp}_{\circledcirc} = l^{*}$)

Suppose: $\exists \underline{\epsilon} \in \mathrm{LabCusp}_{\circledcirc}$ s.t.

$\underline{\epsilon}_{\underline{v}} =$ "$\overline{1} \in \mathbb{F}_l/\{\pm 1\} \cong \mathrm{Csp}(\underline{C}^{\log})$" of II for $\forall \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

(cf. our restriction to $\underline{\mathbb{V}}^{\mathrm{bad}} \subseteq \underline{\mathbb{V}}$ ($\subseteq \mathbb{V}(K)$))

$\Rightarrow$ an *initial* Θ*-data* $(\overline{F}/F,\ E,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\mathbb{V}}^{\mathrm{bad}},\ \underline{\epsilon})$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} \Rightarrow \exists \underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}}$, i.e., "$\underline{\underline{X}}^{\log} \to \underline{X}^{\log}$" of II

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \Rightarrow \exists$a suitable connected fét covering $\underrightarrow{X}_{\underline{v}} \to \underline{X}_{\underline{v}}$

$$\Pi_{\underline{v}} \stackrel{\mathrm{def}}{=} \begin{cases} \pi_1^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} \\ \pi_1^{\text{ét}}(\underrightarrow{X}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}, \text{ finite} \end{cases}$$

$$\mathcal{D}_{\underline{v}} \stackrel{\mathrm{def}}{=} \begin{cases} \Pi_{\underline{v}} \quad \text{(well-defined up to conjugation)} & \underline{v}\text{: finite} \\ \text{Aut-holomorphic space ass'd to } \underrightarrow{X}_{\underline{v}} & \underline{v}\text{: infinite} \end{cases}$$

$\mathfrak{D} \stackrel{\mathrm{def}}{=} \{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$

a $\mathcal{D}$*-prime-strip* $\stackrel{\mathrm{def}}{\Leftrightarrow}$ an "isomorph" of $\mathfrak{D}$,

i.e., more precisely, $\{$an isomorph of $\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$

(Note: A $\mathcal{D}$-p.-s. may be reg'd as a coll. of local "étale-like hol. str.".)

- Use of the global data to synchronize "$\mathrm{LabCusp}^{\pm} \stackrel{\mathrm{def}}{=} \mathrm{Csp}(\underline{X}^{\log})$"
- Use of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry "$\mathrm{Aut}_k(\underline{X}^{\log}) \curvearrowright \mathrm{LabCusp}^{\pm}$"

$$\mathrm{Aut}(\mathcal{D}_{\underline{v}}) \curvearrowright \mathrm{LabCusp}^{\pm}_{\underline{v}} \stackrel{\sim}{\leftarrow} \mathrm{LabCusp}^{\pm}_{\circledcirc} \curvearrowleft \mathrm{Aut}_K(\underline{X}_K) \ (\cong \mathbb{F}_l^{\rtimes\pm} \supseteq \mathbb{F}_l)$$

$$\Rightarrow \quad 1 \to \mathrm{Aut}_+(\mathcal{D}_{\underline{v}}) \to \mathrm{Aut}(\mathcal{D}_{\underline{v}}) \to \mathbb{F}_l^{\rtimes\pm}/\mathbb{F}_l \ (\cong \{\pm 1\}) \to 1$$

$\mathcal{D}^{\circledcirc\pm} \stackrel{\mathrm{def}}{=} \pi_1^{\text{ét}}(\underline{X}_K)$ (well-defined up to conjugation)

$$1 \to \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \to \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \to \mathrm{Aut}(\mathrm{LabCusp}^{\pm}_{\circledcirc})$$

For $t \in \mathbb{F}_l$, $\quad \phi^{\Theta^{\mathrm{ell}}}_{t,\underline{v}} : \mathcal{D}_{t,\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{D}_{\underline{v}} \underset{\text{poly}}{\overset{\mathrm{Aut}_+}{\xrightarrow{\sim}}} \mathcal{D}_{\underline{v}} \overset{\text{nat'l}}{\to} \mathcal{D}^{\circledcirc\pm} \underset{\text{poly}}{\overset{\mathrm{Aut}_{\mathrm{csp}}}{\xrightarrow{\sim}}} \mathcal{D}^{\circledcirc\pm} \overset{t\curvearrowright}{\xrightarrow{\sim}} \mathcal{D}^{\circledcirc\pm}$

$\Rightarrow \phi^{\Theta^{\mathrm{ell}}}_{t} : \mathfrak{D}_t \stackrel{\mathrm{def}}{=} \{\mathcal{D}_{t,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \to \mathcal{D}^{\circledcirc\pm}$

$\Rightarrow \phi^{\Theta^{\mathrm{ell}}}_{\pm} : \mathfrak{D}_{\pm} \stackrel{\mathrm{def}}{=} \{\mathfrak{D}_t\}_{t\in\mathbb{F}_l} \to \mathcal{D}^{\circledcirc\pm}$

a $\mathcal{D}$-Θ^{ell}-*bridge* $\stackrel{\mathrm{def}}{\Leftrightarrow}$ an "isomorph" ${}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm} : {}^{\dagger}\mathfrak{D}_T \stackrel{\mathrm{def}}{=} \{{}^{\dagger}\mathfrak{D}_t\}_{t\in T} \to {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ of $\phi^{\Theta^{\mathrm{ell}}}_{\pm}$, where ${}^{\dagger}\mathfrak{D}_t$ is a $\mathcal{D}$-prime-strip, and T is an $\mathbb{F}_l^{\pm}$-torsor

$$\{\underline{\underline{X}}_{t,\underline{v}},\ \underset{\to}{X}_{t,\underline{v}}\}_{\underline{v}} \xrightarrow[\mathrm{Aut}_{\mathrm{csp}}]{\mathbb{F}_l\ni t\curvearrowright} \underline{X}_K$$

$$\left\{\{\dots\}_{\underline{v}}\right\}_t$$

For $t \in \mathbb{F}_l$, $\quad \phi^{\Theta^\pm}_{t,\underline{v}} : \mathcal{D}_{t,\underline{v}} \overset{\text{def}}{=} \mathcal{D}_{\underline{v}} \underset{\text{poly}}{\overset{\text{Aut}_+}{\overset{\sim}{\to}}} \mathcal{D}_{\succ,\underline{v}} \overset{\text{def}}{=} \mathcal{D}_{\underline{v}}$

$\Rightarrow \phi^{\Theta^\pm}_t : \mathfrak{D}_t \overset{\text{def}}{=} \{\mathcal{D}_{t,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \to \mathfrak{D}_\succ \overset{\text{def}}{=} \{\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

$\Rightarrow \phi^{\Theta^\pm}_\pm : \mathfrak{D}_\pm \overset{\text{def}}{=} \{\mathfrak{D}_t\}_{t\in\mathbb{F}_l} \to \mathfrak{D}_\succ$

a $\mathcal{D}$-$\Theta^\pm$-*bridge* $\overset{\text{def}}{\Leftrightarrow}$ an "isomorph" ${}^\dagger\phi^{\Theta^\pm}_\pm : {}^\dagger\mathfrak{D}_T \overset{\text{def}}{=} \{{}^\dagger\mathfrak{D}_t\}_{t\in T} \to {}^\dagger\mathfrak{D}_\succ$ of $\phi^{\Theta^\pm}_\pm$, where ${}^\dagger\mathfrak{D}_\square$ is a $\mathcal{D}$-prime-strip, and T is an $\mathbb{F}_l^\pm$-group

$$\{\underline{\underline{X}}_{t,\underline{v}},\ \underset{\to}{X}_{t,\underline{v}}\}_{\underline{v}} \xrightarrow[\text{Aut}_+]{\sim} \{\underline{\underline{X}}_{\succ,\underline{v}},\ \underset{\to}{X}_{\succ,\underline{v}}\}_{\underline{v}}$$

$$\Big\{\{\ldots\}_{\underline{v}}\Big\}_t$$

${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathfrak{D}_{\succ} \overset{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}{\leftarrow} {}^{\dagger}\mathfrak{D}_{T} \overset{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\rightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$: a $\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}$-*Hodge theater*

${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \Rightarrow$ bijections of $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$, $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\square})$, T

Moreover: $\mathrm{Aut}({}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}) \cong \mathbb{F}_l^{\rtimes\pm}$, $\mathrm{Aut}({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}) \cong \{\pm 1\}$

$$\left(\begin{array}{ccccc} \{\underline{\underline{X}}_{\succ,\underline{v}}, \ \underset{\rightarrow}{X}_{\succ,\underline{v}}\}_{\underline{v}} & \xleftarrow[\mathrm{Aut}_+]{\sim} & \{\underline{\underline{X}}_{t,\underline{v}}, \ \underset{\rightarrow}{X}_{t,\underline{v}}\}_{\underline{v}} & \xrightarrow[\mathrm{Aut}_{\mathrm{csp}}]{\mathbb{F}_l \ni t \curvearrowright} & \underline{X}_K \\ & & \left\{ \{\ldots\}_{\underline{v}} \right\}_t & & \curvearrowleft \mathbb{F}_l^{\rtimes\pm} \end{array} \right) \curvearrowleft \{\pm 1\}$$

Recall $F_{\mathrm{mod}} \in$ output

Various objects (e.g., $\underline{X}_K$, $\underline{C}_K$) are defined over (not F_{mod} but) K.

$\Rightarrow$ We have to discuss the representation of F_{mod} while relating the descent data of K/F_{mod}.

On the other hand:

$\mathrm{Aut}_K(\underline{C}_K) = \{\mathrm{id}\}$, F_{mod}: the field of moduli of E

$\Rightarrow \mathrm{Aut}(\underline{C}_K) \to \mathrm{Aut}(K)$ induces an *injection*

$$\mathrm{Aut}(\underline{C}_K) \hookrightarrow \mathrm{Gal}(K/F_{\mathrm{mod}}),$$

i.e., $\mathrm{Aut}(\underline{C}_K)$ is naturally related to the descent data of K/F_{mod}.

$\mathcal{D}^{\circledcirc} \stackrel{\text{def}}{=} \pi_1^{\text{ét}}(\underline{C}_K)$ (well-defined up to conjugation)

$\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \stackrel{\text{def}}{=}$ the stabilizer of $\underline{\epsilon} \in \mathrm{LabCusp}_{\circledcirc}$ w.r.t.

$$\mathrm{Aut}(\mathcal{D}^{\circledcirc}) \curvearrowright \mathrm{LabCusp}_{\circledcirc} = \{\text{nonzero cusps of } \underline{C}_K\}$$

Then since $\mathrm{Im}(G_F \to \mathrm{Aut}(E[l](\overline{F})) \stackrel{\text{out}}{\cong} \mathrm{GL}_2(\mathbb{F}_l)) \supseteq \mathrm{SL}_2(\mathbb{F}_l)$,

by considering $\mathrm{Aut}(\mathcal{D}^{\circledcirc}) \stackrel{\text{mod } \{\pm 1\}}{\curvearrowright} \pi_1^{\text{ét}}(X_K)/\pi_1^{\text{ét}}(\underline{X}_K)$ $(\cong \mathbb{F}_l)$,

we obtain a natural isomorphism $\mathrm{Aut}(\mathcal{D}^{\circledcirc})/\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathbb{F}_l^{*}$.

(In particular, $\mathbb{F}_l^{*}$ may be naturally regarded as a subquotient of the Galois group $\mathrm{Gal}(K/F_{\mathrm{mod}})$.)

$\Rightarrow \mathbb{F}_l^{*} \curvearrowright \mathcal{D}^{\circledcirc}$ modulo $\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc})$

For $j \in \mathbb{F}_l^{\circledast}$, $\quad \phi^{\mathrm{NF}}_{j,\underline{v}} : \mathcal{D}_{j,\underline{v}} \overset{\mathrm{def}}{=} \mathcal{D}_{\underline{v}} \underset{\text{poly}}{\overset{\mathrm{Aut}}{\xrightarrow{\sim}}} \mathcal{D}_{\underline{v}} \overset{\text{nat'l}}{\to} \mathcal{D}^{\circledcirc} \underset{\text{poly}}{\overset{\mathrm{Aut}_{\underline{\epsilon}}}{\xrightarrow{\sim}}} \mathcal{D}^{\circledcirc} \overset{j\curvearrowright}{\xrightarrow{\sim}} \mathcal{D}^{\circledcirc}$

$\Rightarrow \phi^{\mathrm{NF}}_{j} : \mathfrak{D}_j \overset{\mathrm{def}}{=} \{\mathcal{D}_{j,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \to \mathcal{D}^{\circledcirc}$

$\Rightarrow \phi^{\mathrm{NF}}_{\circledast} : \mathfrak{D}_{\circledast} \overset{\mathrm{def}}{=} \{\mathfrak{D}_j\}_{j\in\mathbb{F}_l^{\circledast}} \to \mathcal{D}^{\circledcirc}$

a $\mathcal{D}$-*NF-bridge* $\overset{\mathrm{def}}{\Leftrightarrow}$ an "isomorph" ${}^{\dagger}\phi^{\mathrm{NF}}_{\circledast} : {}^{\dagger}\mathfrak{D}_J \overset{\mathrm{def}}{=} \{{}^{\dagger}\mathfrak{D}_j\}_{j\in J} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ of $\phi^{\mathrm{NF}}_{\circledast}$, where ${}^{\dagger}\mathfrak{D}_j$ is a $\mathcal{D}$-prime-strip, and J is a set of l^{*} elements

$$\{\underline{\underline{X}}_{j,\underline{v}},\ \underset{\to}{X}_{j,\underline{v}}\}_{\underline{v}} \xrightarrow[\mathrm{Aut}_{\underline{\epsilon}}]{\mathbb{F}_l^{\circledast}\ni j\curvearrowright} \underline{C}_K$$

$$\left\{\{\dots\}_{\underline{v}}\right\}_j$$

For $j \in \mathbb{F}_l^{\circledast}$, $\quad \phi^{\Theta}_{j,\underline{v}} \colon \mathcal{D}_{j,\underline{v}} \overset{\text{def}}{=} \mathcal{D}_{\underline{v}} \underset{\text{poly}}{\overset{\text{Aut}}{\overset{\sim}{\to}}} \mathcal{D}_{\underline{v}} \overset{(*)}{\to} \mathcal{D}_{>,\underline{v}} \overset{\text{def}}{=} \mathcal{D}_{\underline{v}} \underset{\text{poly}}{\overset{\text{Aut}}{\overset{\sim}{\to}}} \mathcal{D}_{\underline{v}}$

$$(*) = \begin{cases} \Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}} \underset{\text{(a lifting } \in \mathbb{F}_l^{\times} \text{ of) } j}{\overset{\text{ev. pt lab'd by}}{\hookrightarrow}} \Pi_{\underline{v}} & \underline{v} \in \underline{\mathbb{V}}^{\text{bad}} \\ \mathcal{D}_{\underline{v}} \overset{\sim}{\to} \mathcal{D}_{\underline{v}} & \underline{v} \in \underline{\mathbb{V}}^{\text{good}} \end{cases}$$

$\Rightarrow \phi^{\Theta}_{j} \colon \mathfrak{D}_j \overset{\text{def}}{=} \{\mathcal{D}_{j,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}} \to \mathfrak{D}_{>} \overset{\text{def}}{=} \{\mathcal{D}_{>,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$

$\Rightarrow \phi^{\Theta}_{\circledast} \colon \mathfrak{D}_{\circledast} \overset{\text{def}}{=} \{\mathfrak{D}_j\}_{j \in \mathbb{F}_l^{\circledast}} \to \mathfrak{D}_{>}$

a $\mathcal{D}$-Θ-*bridge* $\overset{\text{def}}{\Leftrightarrow}$ an "isomorph" ${}^{\dagger}\phi^{\Theta}_{\circledast} \colon {}^{\dagger}\mathfrak{D}_J \overset{\text{def}}{=} \{{}^{\dagger}\mathfrak{D}_j\}_{j \in J} \to {}^{\dagger}\mathfrak{D}_{>}$ of $\phi^{\Theta}_{\circledast}$, where ${}^{\dagger}\mathfrak{D}_{\square}$ is a $\mathcal{D}$-prime-strip, and J is a set of l^{*} elements

$$\{\underline{\underline{X}}_{j,\underline{v}},\ \underset{\rightarrow}{X}_{j,\underline{v}}\}_{\underline{v}} \xrightarrow[\text{Aut}]{\text{ev. pt lab'd by } j} \{\underline{\underline{X}}_{>,\underline{v}},\ \underset{\rightarrow}{X}_{>,\underline{v}}\}_{\underline{v}}$$

$$\left\{\{\ldots\}_{\underline{v}}\right\}_j$$

${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^\dagger\mathcal{D}^{\circledcirc} \overset{{}^\dagger\phi^{\mathrm{NF}}_{\circledast}}{\leftarrow} {}^\dagger\mathfrak{D}_J \overset{{}^\dagger\phi^{\Theta}_{\circledast}}{\rightarrow} {}^\dagger\mathfrak{D}_{>})$: a $\mathcal{D}$-$\Theta\mathrm{NF}$-*Hodge theater*

${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \Rightarrow$ bijections of $\mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc})$, $\mathrm{LabCusp}({}^\dagger\mathfrak{D}_{\square})$, J

Moreover: $\mathrm{Aut}({}^\dagger\phi^{\mathrm{NF}}_{\circledast}) \cong \mathbb{F}_l^{\circledast}$, $\mathrm{Aut}({}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}) \cong \{\mathrm{id}\}$

$$\left(\begin{array}{cccccc} \underline{C}_K & \xleftarrow[\mathrm{Aut}_{\epsilon}]{\mathbb{F}_l^{\circledast}\ni j\curvearrowright} & \{\underline{\underline{X}}_{j,\underline{v}},\ \underrightarrow{X}_{j,\underline{v}}\}_{\underline{v}} & \xrightarrow[\mathrm{Aut}]{\text{ev. by } j} & \{\underline{\underline{X}}_{>,\underline{v}},\ \underrightarrow{X}_{>,\underline{v}}\}_{\underline{v}} \\ \mathbb{F}_l^{\circledast} \curvearrowright & & \left\{\{\ldots\}_{\underline{v}}\right\}_j & & \end{array}\right) \curvearrowleft \{\mathrm{id}\}$$

By "gluing" a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater and a $\mathcal{D}$-$\Theta\mathrm{NF}$-Hodge theater,

i.e., by considering an isomorphism of ${}^{\dagger}\phi^{\Theta}_{\divideontimes}$ with the $\mathcal{D}$-Θ-bridge constructed from ${}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$,

we obtain a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-*Hodge theater*

$$ {}^{\dagger}\mathfrak{D}_{\succ} \xleftarrow{{}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}} {}^{\dagger}\mathfrak{D}_{T} \xrightarrow{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm} $$

$$ \downarrow \text{glue} $$

$$ {}^{\dagger}\mathfrak{D}_{>} \xleftarrow{{}^{\dagger}\phi^{\Theta}_{\divideontimes}} {}^{\dagger}\mathfrak{D}_{J} \xrightarrow{{}^{\dagger}\phi^{\mathrm{NF}}_{\divideontimes}} {}^{\dagger}\mathcal{D}^{\circledcirc} $$

$\mathbb{F}_l^{\rtimes\pm} \curvearrowright \mathbb{F}_l \cong T \quad \supseteq \mathbb{F}_l^{\times} \cong T^{\times} \twoheadrightarrow \quad T^{\divideontimes} = J \cong \mathbb{F}_l^{\divideontimes} \curvearrowleft \mathbb{F}_l^{\divideontimes}$

$\mathrm{Aut}_K(\underline{X}_K) \cong \mathbb{F}_l^{\rtimes\pm} \curvearrowright \mathrm{LabCusp}^{\pm}$: additive, geometric

$\mathrm{Gal}(K/F_{\mathrm{mod}}) \overset{\text{sub-}}{\underset{\text{quotient}}{\rightsquigarrow}} \mathbb{F}_l^{\divideontimes} \curvearrowright \mathrm{LabCusp}$: multiplicative, arithmetic

By considering a suitable structure whose underlying structure is a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater, we obtain a $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-*Hodge theater*

$$\begin{array}{ccccccccc} & & {}^\dagger\mathfrak{F}_{\succ} & \overset{{}^\dagger\psi^{\Theta^\pm}_{\pm}}{\longleftarrow} & {}^\dagger\mathfrak{F}_T & \overset{{}^\dagger\psi^{\Theta^{\mathrm{ell}}}_{\pm}}{\longrightarrow} & {}^\dagger\mathcal{D}^{\circledcirc\pm} & & \\ & & & \downarrow \text{glue} & & & & & \\ {}^\dagger\mathcal{HT}^{\Theta} & \dashleftarrow & {}^\dagger\mathfrak{F}_{>} & \overset{{}^\dagger\psi^{\Theta}_{\circledast}}{\longleftarrow} & {}^\dagger\mathfrak{F}_J & \overset{{}^\dagger\psi^{\mathrm{NF}}_{\circledast}}{\longrightarrow} & {}^\dagger\mathcal{F}^{\circledcirc} & \dashrightarrow & {}^\dagger\mathcal{F}^{\circledast} \end{array}$$

Here, "${}^\dagger\mathfrak{F}_\square$" ($\square \in \{\succ\} \cup \{>\} \cup T \cup J$) is an $\mathcal{F}$*-prime-strip*.

$\mathcal{F}_{\underline{v}} \overset{\mathrm{def}}{=} (\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}})$ (well-defined up to conjugation) if $\underline{v}$ is finite

(omit for an infinite $\underline{v}$)

$\mathfrak{F} \overset{\mathrm{def}}{=} \{\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

an $\mathcal{F}$*-prime-strip* $\overset{\mathrm{def}}{\Leftrightarrow}$ an "isomorph" of $\mathfrak{F}$

- $\mathfrak{log}$-*link* ${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{n+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, i.e.,

  for $\forall\Box \in \{\succ\} \cup \{>\} \cup T \cup J$; at, for instance, a finite $\underline{v} \in \underline{\mathbb{V}}$,

  $${}^{n}\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}} \supseteq {}^{n}\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \twoheadrightarrow {}^{n}\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \stackrel{\log}{\stackrel{\sim}{\to}} ({}^{n}\overline{F}_{\underline{v}})_{+} \text{``}{=}\text{''}\, {}^{n}\overline{F}_{\underline{v}} \supseteq \mathcal{O}^{\rhd}_{{}^{n}\overline{F}_{\underline{v}}} \stackrel{\sim}{\to} {}^{n+1}\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$$

  $$\Rightarrow {}^{n+1}\mathcal{O}^{\rhd}_{K_{\underline{v}}} \curvearrowright {}^{n}\mathcal{I}_{\underline{v}} \quad \text{(i.e., ``}\boxtimes \curvearrowright \boxplus\text{'')}$$

- $\Theta^{\pm\mathrm{ell}}$-Hodge theater (i.e., roughly speaking, $\mathbb{F}_l^{\rtimes\pm}$-symmetry)

  $\stackrel{\text{(cf. also II)}}{\Rightarrow}$ multiradial$\{\underline{\underline{\Theta}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}}$,coric/"label-indep." $\{G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

- $\Theta\mathrm{NF}$-Hodge theater (i.e., roughly speaking, $\mathbb{F}_l^{\circledast}$-symmetry)

  $\Rightarrow$ multiradial monoid of κ-coric functions

$\underline{\underline{\Theta}}_{\underline{v}}$		$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$		κ-coric
Gal. $\Downarrow$ eval.		$\mathfrak{log}$- $\Downarrow$ link		Gal. $\Downarrow$ eval.
theta values	$\curvearrowright$	(procession of) $\mathcal{I}_{\underline{v}}$	$\curvearrowleft$	F_{mod}

Indeterminacies

We want to obtain a "coric $(G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}})$".

$\Rightarrow$ We have to forget the holomorphic structure of "$(G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}})$".

$\Rightarrow$ $\underline{(\mathrm{Ind}_1)\colon\ \mathrm{Aut}(G_{\underline{v}}) \curvearrowright G_{\underline{v}}}$

$\underline{(\mathrm{Ind}_2)\colon\ \mathrm{Ism} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{G_{\underline{v}}, \times\boldsymbol{\mu}\text{-Kmm}}(\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}) \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}}$ (cf. also p.25 of II)

We want to apply the $\mathfrak{log}$-link; but "log" depends on the hol. str.

$\Rightarrow$ We have to consider the infinite chain of $\mathfrak{log}$-links

$$\cdots \stackrel{\mathfrak{log}}{\to} {}^{n-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{n+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} \cdots$$

and the associated "$\mathfrak{log}$-Kummer correspondence". $\Rightarrow$ $\underline{(\mathrm{Ind}_3) \curvearrowright \mathcal{I}_{\underline{v}}}$

"mild indeterminacies" in the main theorem $= (\mathrm{Ind}_1), (\mathrm{Ind}_2), (\mathrm{Ind}_3)$

Recall (p.25 of II):

A Local Multiradial Algorithm Related to Étale Theta Functions

$\exists$A multiradial algorithm as follows:

coric data: an isomorph $(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}, \times\boldsymbol{\mu}\text{-Kmm})$ of $G_k \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}}$

radial data: $(\Pi_\bullet \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi_\bullet)),$ a coric data, $\alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}})$

for an isomorph $\Pi_\bullet$ of $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$,

where $\alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}}$ is the pair of the *full poly-isomorphism* $G_\bullet \xrightarrow{\sim}$ "G"

and $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi_\bullet)) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \stackrel{\text{zero}}{\to}$ "$\mathcal{O}^{\times\boldsymbol{\mu}}$"

output: the radial data and: