Mono-anabelian Transport

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Notation and Terminology

$$
\begin{array}{cccccccc}
 & \mathcal{O}^{\boldsymbol{\mu}} & \stackrel{\mathrm{def}}{=} & (\mathcal{O}^{\times})_{\mathrm{tor}} & \subseteq & \mathcal{O}^{\times} & \stackrel{\mathrm{def}}{=} & \{|z| = 1\} \\
\subseteq & \mathcal{O}^{\triangleright} & \stackrel{\mathrm{def}}{=} & \{0 < |z| \leq 1\} & \subseteq & \mathcal{O} & \stackrel{\mathrm{def}}{=} & \{|z| \leq 1\}
\end{array}
$$

an *isomorph* of A $\stackrel{\mathrm{def}}{\Leftrightarrow}$ an object which is isomorphic to A

a *poly-*(*iso*)*morphism* $A \to B$

$\stackrel{\mathrm{def}}{\Leftrightarrow}$ a set consisting of (iso)morphisms $A \to B$

For a topological group G,

$$
{}_{\infty}H^{i}(G, A) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G\text{: open subgps of finite index}} H^{i}(H, A)
$$

In inter-universal Teichmüller theory,

(mono-)anabelian geometry is applied so as to establish

"mono-anabelian transport"

of various Frobenius-like objects.

Mono-anabelian Transport

- Suppose that we are given a Frobenioid which consists of an étale-like portion and a Frobenius-like portion.
- One relates the Frobenius-like portion to the étale-like portion via a Kummer isomorphism which arises from a cyclotomic rigidity.
- Then one can establish a relationship between the Frobenius-like portions of (independent) Frobenioids once one has a connection between the étale-like portions of the Frobenioids.

an étale-like connection $\underset{\text{isomorphism}}{\overset{\text{Kummer}}{\Longrightarrow}}$ a Frobenius-like connection

In the following, let us observe some classical/typical examples of

"mono-anabelian transport".

Mono-anabelian Transport

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k: a p-adic local field, i.e., $[k:\mathbb{Q}_p] < \infty$

$G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$

Thus, we have (multiplicative) G_k-monoids

$$G_k \curvearrowright \quad (\mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}} \subseteq \mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\triangleright}_{\overline{k}} \subseteq \overline{k}^{\times}).$$

$G \curvearrowright M$: an isomorph of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ (resp. $\mathcal{O}^{\times}_{\overline{k}}$; $\overline{k}^{\times}$)

(Such a data consists of essentially the same data as a Frobenioid.)

For such a "Frobenioid":

- G: the étale-like portion of $G \curvearrowright M$
- M: the Frobenius-like portion of $G \curvearrowright M$

Here, let us recall that, by mono-anabelian geometry, one can reconstruct, from an isomorph H of G_k (e.g., G), isomorphs

$$H \curvearrowright \quad (\mathcal{O}^{\boldsymbol{\mu}}(H) \subseteq \mathcal{O}^{\times}(H) \subseteq \mathcal{O}^{\triangleright}(H) \subseteq \overline{k}^{\times}(H))$$

of the G_k-monoids

$$G_k \curvearrowright \quad (\mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}} \subseteq \mathcal{O}^{\times}_{\overline{k}} \subseteq \mathcal{O}^{\triangleright}_{\overline{k}} \subseteq \overline{k}^{\times}).$$

Mono-anabelian Transport

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$G \curvearrowright M$: an isomorph of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ (resp. $\mathcal{O}^{\times}_{\overline{k}}$; $\overline{k}^{\times}$)

$\Rightarrow G \curvearrowright M_{\mathrm{tor}}$: an isomorph of $G_k \curvearrowright \mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}}$

$\Rightarrow G \curvearrowright \Lambda(M) \stackrel{\mathrm{def}}{=} \varprojlim_n (M_{\mathrm{tor}}[n])$: an isomorph of $G_k \curvearrowright \widehat{\mathbb{Z}}(1)$,

i.e., a <u>cyclotome</u> $\stackrel{\mathrm{def}}{\Leftrightarrow}$ an isomorph of "$\widehat{\mathbb{Z}}(1)$"

On the other hand, by mono-anabelian geometry:

$G \quad \Rightarrow \quad G \curvearrowright \mathcal{O}^{\boldsymbol{\mu}}(G)$: an isomorph of $G_k \curvearrowright \mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}}$

$\Rightarrow G \curvearrowright \Lambda(G) \stackrel{\mathrm{def}}{=} \varprojlim_n (\mathcal{O}^{\boldsymbol{\mu}}(G)[n])$: a cyclotome

Thus:

- $G \curvearrowright M \Rightarrow$ the Frobenius-like cyclotome $G \curvearrowright \Lambda(M)$
- $G \curvearrowright M \Rightarrow$ ($G \Rightarrow$) the étale-like cyclotome $G \curvearrowright \Lambda(G)$

Theorem (Cyclotomic Rigidity via Local Class Field Theory)

$G \curvearrowright M$: an isomorph of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ (resp. $\mathcal{O}^{\times}_{\overline{k}}$; $\overline{k}^{\times}$)

$\Gamma \stackrel{\mathrm{def}}{=} \{1\}$ (resp. $\widehat{\mathbb{Z}}^{\times}$; $\{\pm 1\}$) $\quad \subseteq \widehat{\mathbb{Z}}^{\times}$ $(= \mathrm{Aut}_{G_k}(\widehat{\mathbb{Z}}(1)))$

$\Rightarrow$ $\exists$a functorial algorithm of reconstructing, from $G \curvearrowright M$, a Γ-orbit of G-equivariant isomorphisms

$$\Lambda(M) \xrightarrow{\sim} \Lambda(G)$$

(i.e., a G-eq. isom. $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$ well-defined up to the action of Γ)

Terminology

a <u>cyclotomic rigidity isomorphism</u> (<u>cyclotomic synchronization isom.</u>)

$\stackrel{\mathrm{def}}{\Leftrightarrow}$ a “suitable” (poly-)isomorphism between cyclotomes

Mono-anabelian Transport

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- Then one can establish a relationship between the Frobenius-like portions of (independent) Frobenioids once one has a connection between the étale-like portions of the Frobenioids.

$$\text{an étale-like connection} \xRightarrow[\text{isomorphism}]{\text{Kummer}} \text{a Frobenius-like connection}$$

$G \curvearrowright M$: an isomorph of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ (resp. $\mathcal{O}^{\times}_{\overline{k}}$; $\overline{k}^{\times}$)

$$\Rightarrow \quad G \curvearrowright \quad (1 \to M_{\mathrm{tor}} \to M^{\mathrm{gp}} \xrightarrow{n} M^{\mathrm{gp}} \to 1)$$

$$\overset{H^*(G,-)}{\Rightarrow} \quad M^G \hookrightarrow (M^{\mathrm{gp}})^G = H^0(G, M^{\mathrm{gp}}) \to H^1(G, M_{\mathrm{tor}}[n])$$

$$\overset{\varprojlim_n}{\Rightarrow} \quad M^G \hookrightarrow H^1(G, \Lambda(M))$$

$$\overset{\varinjlim_{H \subseteq G:\ \text{open subgps}}}{\Rightarrow} \quad G \curvearrowright \quad (M \hookrightarrow {}_\infty H^1(G, \Lambda(M)))$$

On the other hand, by mono-anabelian geometry:

$$G \quad \Rightarrow \quad G \curvearrowright M(G) \overset{\mathrm{def}}{=} \mathcal{O}^{\triangleright}(G) \text{ (resp. } \mathcal{O}^{\times}(G);\ \overline{k}^{\times}(G))$$

Thus, by a similar procedure to the above procedure, we obtain:

$$G \curvearrowright \quad (M(G) \hookrightarrow {}_\infty H^1(G, \Lambda(G)))$$

- $G \curvearrowright \quad (M \hookrightarrow {}_\infty H^1(G, \Lambda(M)))$
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Theorem

$G \curvearrowright M$: an isomorph of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ (resp. $\mathcal{O}^{\times}_{\overline{k}}$; $\overline{k}^{\times}$)

Then the Γ-orbit of G-equivariant isomorphisms

$$ {}_{\infty}H^1(G, \Lambda(M)) \xrightarrow{\sim} {}_{\infty}H^1(G, \Lambda(G)) $$

induced by the Γ-orbit of G-equivariant isomorphisms $\Lambda(M) \xrightarrow{\sim} \Lambda(G)$ of the preceding theorem determines a Γ-orbit of G-equivariant isomorphisms of G-submonoids

$$ M \xrightarrow{\sim} M(G). $$

By the above theorem, one can reconstruct, from $G \curvearrowright M$:

- a (single) G-equiv. isom. $M \xrightarrow{\sim} \mathcal{O}^{\triangleright}(G)$ if $M \cong \mathcal{O}^{\triangleright}_{\overline{k}}$
- a $\widehat{\mathbb{Z}}^{\times}$-orbit of G-equiv. isom. $M \xrightarrow{\sim} \mathcal{O}^{\times}(G)$ if $M \cong \mathcal{O}^{\times}_{\overline{k}}$
- a $\{\pm 1\}$-orbit of G-equiv. isom. $M \xrightarrow{\sim} \overline{k}^{\times}(G)$ if $M \cong \overline{k}^{\times}$

Terminology

a <u>Kummer isomorphism</u>

$\overset{\text{def}}{\Leftrightarrow}$ a "suitable" (poly-)isomorphism between the Frobenius-like and étale-like objects

Mono-anabelian Transport

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$$\text{an étale-like connection} \ \underset{\text{isomorphism}}{\overset{\text{Kummer}}{\Longrightarrow}} \ \text{a Frobenius-like connection}$$

${}^{\dagger}G \curvearrowright {}^{\dagger}M$, ${}^{\ddagger}G \curvearrowright {}^{\ddagger}M$: isomorphs of $G_k \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}}$ (resp. $\mathcal{O}^{\times}_{\overline{k}}$; $\overline{k}^{\times}$)

Suppose that we are given a connection between the étale-like portions of these two data ${}^{\dagger}G \curvearrowright {}^{\dagger}M$, ${}^{\ddagger}G \curvearrowright {}^{\ddagger}M$,

e.g., an isomorphism $\alpha\colon {}^{\dagger}G \xrightarrow{\sim} {}^{\ddagger}G$.

Then, by means of this "étale link" α and the Kummer isomorphisms, we obtain an poly-isomorphism between the Frobenius-like portions

$$^{\dagger}M \overset{\text{Kummer}}{\xrightarrow{\sim}} M(^{\dagger}G) \overset{M(\alpha)}{\xrightarrow{\sim}} M(^{\ddagger}G) \overset{\text{Kummer}^{-1}}{\xrightarrow{\sim}} {}^{\ddagger}M.$$

In summary, roughly speaking, by means of

- a connection between the étale-like portions and
- a Kummer isomorphism (which arises from a cyclotomic rigidity),

one can establish a relationship between the Frobenius-like portions.

In our example, by the "étale link" $\alpha\colon {}^{\dagger}G \xrightarrow{\sim} {}^{\ddagger}G$, we obtain:

- a (single) $({}^{\dagger}G, {}^{\ddagger}G)$-equiv. isom. ${}^{\dagger}M \xrightarrow{\sim} {}^{\ddagger}M$ if "M" $\cong \mathcal{O}^{\triangleright}_{\overline{k}}$
- a $\widehat{\mathbb{Z}}^{\times}$-orbit of $({}^{\dagger}G, {}^{\ddagger}G)$-equiv. isom. ${}^{\dagger}M \xrightarrow{\sim} {}^{\ddagger}M$ if "M" $\cong \mathcal{O}^{\times}_{\overline{k}}$
- a $\{\pm 1\}$-orbit of $({}^{\dagger}G, {}^{\ddagger}G)$-equiv. isom. ${}^{\dagger}M \xrightarrow{\sim} {}^{\ddagger}M$ if "M" $\cong \overline{k}^{\times}$

In particular, we can "transport", via an étale-like connection "α", Frobenius-like objects from the "† side" to the "‡ side".

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Indeterminacy

The "output isomorphism" of a mono-anabelian transport is often subject to a certain indeterminacy.

Example

- the $\widehat{\mathbb{Z}}^\times$-indeterminacies on ${}^\dagger M \xrightarrow{\sim} {}^\ddagger M$ in the case of "$\mathcal{O}_{\overline{k}}^\times$"
- the $\{\pm 1\}$-indeterminacies on ${}^\dagger M \xrightarrow{\sim} {}^\ddagger M$ in the case of "$\overline{k}^\times$"

- a <u>Kummer-detachment indeterminacy</u>

  $\overset{\text{def}}{\Leftrightarrow}$ an indeterminacy that occurs in the passage from Frobenius-like str. to etale-like str. via Kummer isomorphisms

- an <u>étale-transport indeterminacy</u>

  $\overset{\text{def}}{\Leftrightarrow}$ an indeterminacy that occurs in the transport of the resulting étale-like objects via étale-like connections

$$^{\dagger}M \quad \overset{\text{Kummer}}{\xrightarrow{\sim}} \quad M(^{\dagger}G) \overset{M(\alpha)}{\xrightarrow{\sim}} M(^{\ddagger}G) \quad \overset{\text{Kummer}^{-1}}{\xrightarrow{\sim}} \quad ^{\ddagger}M$$

(Thus, the above two examples (i.e., $\widehat{\mathbb{Z}}^{\times}$ and $\{\pm 1\}$) are examples of a *Kummer-detachment indeterminacy*.)

An Example of an Étale-transport Indeterminacy

In the case where "M" $\cong \mathcal{O}^{\rhd}_{\overline{k}}$, let us consider, as a connection between the étale-like portions,

the *full poly-isomorphism* $\phi\colon {}^{\dagger}G \xrightarrow{\sim} {}^{\ddagger}G$

(i.e., the poly-isomorphism obtained by forming $\mathrm{Isom}({}^{\dagger}G, {}^{\ddagger}G)$).

$\Rightarrow$

$$
{}^{\dagger}M \underset{\text{no indet.}}{\xrightarrow[\sim]{\text{Kummer}}} \mathcal{O}^{\rhd}_{\overline{k}}({}^{\dagger}G) \underset{\mathrm{Aut}({}^{\dagger}G)\cong\mathrm{Aut}({}^{\ddagger}G)\curvearrowright}{\xrightarrow[\sim]{M(\phi)}} \mathcal{O}^{\rhd}_{\overline{k}}({}^{\ddagger}G) \underset{\text{no indet.}}{\xrightarrow[\sim]{\text{Kummer}^{-1}}} {}^{\ddagger}M
$$

Thus, our resulting isomorphism ${}^{\dagger}M \xrightarrow{\sim} {}^{\ddagger}M$ is subject to indeterminacies which arise from the action of $\mathrm{Aut}({}^{\dagger}G) \cong \mathrm{Aut}({}^{\ddagger}G)$.

Let us recall (cf. the preceding talk):

An Approximate Statement of the Main Theorem of IUT

$\exists$A suitable multiradial algorithm whose output data consist of the following three objects $\mathfrak{R} \curvearrowleft$ mild indeterminacies

- the collection of log-shells $\{\mathcal{I}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$
- the theta values $\left(= \{\mathfrak{q}^{j^2/2l}\}_{1\leq j\leq l^{*}\stackrel{\mathrm{def}}{=}\frac{l-1}{2}}\right) \curvearrowright \prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{I}_{\underline{v}}$
- F_{mod} via κ-coric functions $\curvearrowright \prod_{\underline{v}} \left((K_{\underline{v}})_{+}\text{ “via } \mathcal{I}_{\underline{v}}\text{”}\right)$

Moreover, this alg'm is *compatible* w/ the Θ-link (more precisely, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link) “${}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}} \stackrel{\sim}{\to} {}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$”; “†theta values $\mapsto {}^{\ddagger}\mathfrak{q}^{1/2l}$”.

That is to say, we want to establish, relative to the link

$$\text{“}^{\dagger}\text{theta values } (= \{\mathfrak{q}^{j^2/2l}\}_{1 \le j \le l^*}) \quad \mapsto \quad {}^{\ddagger}\mathfrak{q}^{1/2l}\text{”},$$

a suitable relationship between “$^{\dagger}\mathfrak{R}$” and “$^{\ddagger}\mathfrak{R}$”.

Thus, to obtain the main theorem of IUT, we have to consider

- the operation of “multiradial Kummer-detachment” — i.e., the passage from a Frobenius-like structure to an étale-like structure via a multiradial Kummer isomorphism — for theta functions and κ-coric functions and
- the operation of multiradial passage from such functions to special values (which belong to the log-shells).

- "multiradial Kmm-detach. for theta": closely related to elliptic cuspidalization and the "additive" symm. of Hodge-theaters
- "multiradial Kmm-detach. for κ-coric": closely related to Belyi cuspidalization and the "multiplicative" symm. of Hodge-theaters
- "multiradial passage into special values": given by the operation of "Galois evaluation" — i.e., the operation of restricting Kummer classes of functions to decomposition subgroups associated to suitable closed points of (orbi)curves

(cf. "animation video concerning IUTeich")