Hodge-Arakelov-theoretic Evaluation I

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Notation and Terminology

"Δ": the geometric portion of "Π",

i.e., the kernel of the outer surjection from "Π" to the arithmetic quotient of "Π"

$\widehat{(-)}$: the profinite completion of $(-)$

For a topological group G,

$${}_{\infty}H^i(G, A) \stackrel{\mathrm{def}}{=} \varinjlim_{H \subseteq G\text{: open subgps of finite index}} H^i(H, A)$$

§2 Galois-theoretic Theta Evaluation

In $\S 2$, $\S 2\frac{1}{2}$, and $\S 3$: Fix a $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

$$
\begin{array}{ccccccc}
 & & \Pi^{\mathrm{tp}}_{\ddot{\underline{\underline{Y}}}_{\underline{v}}} & \longrightarrow & \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}_{\underline{v}}} & \longrightarrow & \Pi_{\underline{v}} = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\Pi_{\underline{v}} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} & \underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Longrightarrow}} & \Pi^{\mathrm{tp}}_{\ddot{Y}_{\underline{v}}} & \longrightarrow & \Pi^{\mathrm{tp}}_{Y_{\underline{v}}} & \longrightarrow & \Pi^{\pm}_{\underline{v}} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{\underline{X}_{\underline{v}}} \\
 & & & & & & \downarrow \\
 & & & & & & \Pi^{\mathrm{cor}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{C_{\underline{v}}}
\end{array}
$$

$\mathbb{M}^{\Theta}_{*} = (\mathbb{M}^{\Theta}_{M})_{M}$: a projective system of mono-theta environments

$\underline{\underline{\mathcal{F}}}_{\underline{v}}$: the tempered Frobenioid determined by $\underline{\underline{X}}_{\underline{v}}$

Goal: $\underline{\underline{\Theta}}_{\underline{v}} \overset{\text{evaluation}}{\underset{\text{labeled by } t}{\Rightarrow}} \underline{\underline{q}}_{\underline{v}}^{t^2} \quad (t \in \text{LabCusp}^{\pm} \text{ at } \underline{v})$

Problem 1 A conjugacy indeterminacy in a situation related to the theta value "$\underline{\underline{q}}_{\underline{v}}^{t^2}$" at $|t| \in |\mathbb{F}_l|$ depends, a priori, on the label $|t| \in |\mathbb{F}_l|$. That is to say, various objects at $|t| \in |\mathbb{F}_l|$ is well-defined up to conjugation which is, a priori, independent of the label $|t| \in |\mathbb{F}_l|$. On the other hand, we want to establish a suitable Kummer theory for such theta values.

$\Rightarrow$ We have to synchronizes conjugacy indeterminacies in situations related to the theta values at various $|t| \in |\mathbb{F}_l|$.

Problem 2 By using the structure of a Hodge-theater, we synchronized globally the various "$\mathrm{LabCusp}^{\pm}_{\underline{v}}$" by means of — relative to ${}^{\dagger}\mathcal{D}_{\succ,\underline{v}} \overset{\text{through } 0}{\to} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ $(\overset{\text{through } 0}{\leftarrow} {}^{\dagger}\mathcal{D}_{\succ,\underline{w}})$ — the var. bij.

$$\{\text{cuspidal inertia subgps of } \pi_1^{\mathrm{tp}}(\underline{\underline{X}}_{\underline{v}})\}/\underline{\mathrm{Inn}(\pi_1^{\mathrm{tp}}(\underline{\underline{X}}_{\underline{v}}))}$$
$$\overset{\sim}{\to} \{\text{cuspidal inertia subgps of } \pi_1^{\text{ét}}(\underline{X}_K)\}/\underline{\mathrm{Inn}(\pi_1^{\text{ét}}(\underline{X}_K))}.$$

Moreover, the crucial global $\mathbb{F}_l^{\rtimes\pm}$-symmetry arises from a <u>profinite conjugation</u> (cf. $\pi_1^{\text{ét}}(C_{\overline{F}})/\pi_1^{\text{ét}}(\underline{X}_{\overline{F}}) \overset{\sim}{\to} \pi_1^{\text{ét}}(C_K)/\pi_1^{\text{ét}}(\underline{X}_K) \cong \mathbb{F}_l^{\rtimes\pm}$).

$\Rightarrow$ We have to discuss the comparison between <u>tempered conjugation</u> and <u>profinite conjugation</u>.

$\Gamma_{(-)}$: the dual graph of the special fiber of $(-)_{\underline{v}}$

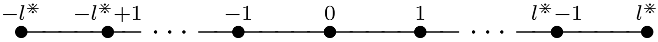
$\mathrm{Irr}(-)$: the set of irreducible components of the special fiber of $(-)_{\underline{v}}$

Fix an inversion automorphism $\iota \stackrel{\text{def}}{=} \iota_{\underline{\underline{\ddot{Y}}}}$ of $\underline{\underline{\ddot{Y}}}_{\underline{v}}$

$\Rightarrow \iota_{\underline{X}}$, $\iota_{\ddot{Y}}$, and $\underline{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Irr}(\underline{\underline{\ddot{Y}}})$ $(\cong \mathrm{Irr}(\underline{\underline{Y}}) \cong \mathrm{Irr}(\ddot{Y}) \cong \mathrm{Irr}(Y))$,

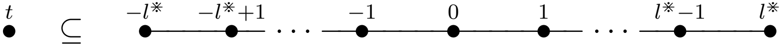
(Note: $\iota \curvearrowright \mathrm{Irr} \rightsquigarrow -1 \curvearrowright \underline{\mathbb{Z}}$)

$\Rightarrow$ a subgraph $\Gamma^{\blacktriangleright}_{(-)} \subseteq \Gamma_{(-)}$ for $(-) \in \{\underline{X}, \underline{\underline{X}}, \ddot{Y}, \underline{\underline{\ddot{Y}}}\}$, i.e.,



$t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$

$\Rightarrow$ a subgraph $\Gamma^{\bullet t}_{(-)} \subseteq \Gamma^{\blacktriangleright}_{(-)}$ for $(-) \in \{\underline{X}, \underline{\underline{X}}, \ddot{Y}, \underline{\underline{\ddot{Y}}}\}$, i.e.,



"$\bullet$" $\stackrel{\text{def}}{=}$ "$\bullet 0$" (i.e., for instance, $\Gamma^{\bullet}_{(-)} \stackrel{\text{def}}{=} \Gamma^{\bullet 0}_{(-)}$)

$\square \in \{\bullet t, \blacktriangleright\}$

$\Pi_{\underline{v}\square} \subseteq \Pi_{\underline{v}}$: a decomposition subgroup of $\Gamma^{\square}_{\underline{\underline{X}}} \subseteq \Gamma_{\underline{\underline{X}}}$, i.e., "$\Pi^{\mathrm{tp}}_{\underline{\underline{X}},\Gamma^{\square}_{\underline{\underline{X}}}}$"

$$\Pi^{\pm}_{\underline{v}\square} \stackrel{\mathrm{def}}{=} N_{\Pi^{\pm}_{\underline{v}}}(\Pi_{\underline{v}\square}) \subseteq \Pi^{\pm}_{\underline{v}}$$

$$\Pi_{\underline{v}\ddot{\square}} \stackrel{\mathrm{def}}{=} \Pi_{\underline{v}\square} \cap \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$$

Thus, for instance:

$$\Pi^{\pm}_{\underline{v}\square}/\Pi_{\underline{v}\square} \stackrel{\sim}{\to} \Pi^{\pm}_{\underline{v}}/\Pi_{\underline{v}} \stackrel{\sim}{\to} \mathrm{Gal}(\underline{\underline{X}}_{\underline{v}}/\underline{X}_{\underline{v}})\ (\cong \boldsymbol{\mu}_l), \quad \Pi^{\pm}_{\underline{v}\square} \cap \Pi_{\underline{v}} = \Pi_{\underline{v}\square}$$

$$[\Pi_{\underline{v}\square} : \Pi_{\underline{v}\ddot{\square}}] = 2, \quad [\Pi^{\pm}_{\underline{v}\square} : \Pi_{\underline{v}\square}] = l, \quad \ldots$$

Then:

$$\Pi_{\underline{v}} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} (\Pi_{\underline{v}\bullet} \subseteq \Pi_{\underline{v}\blacktriangleright} \subseteq \Pi_{\underline{v}},\ \iota) \text{ well-defined up to } \Pi_{\underline{v}}\text{-conj.}$$

(Recall: $\Pi_{\underline{v}} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} \underline{\underline{\theta}}(\Pi_{\underline{v}}) \subseteq {}_{\infty}\underline{\underline{\theta}}(\Pi_{\underline{v}}) \subseteq {}_{\infty}H^1(\Pi^{\text{tp}}_{\underline{\underline{\ddot{Y}}}}(\Pi_{\underline{v}}), (l \cdot \Delta_{\Theta})(\Pi_{\underline{v}})))$

Thus:

$$\Pi_{\underline{v}} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} \underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}}) \subseteq \underline{\underline{\theta}}(\Pi_{\underline{v}}),\ {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}}) \subseteq {}_{\infty}\underline{\underline{\theta}}(\Pi_{\underline{v}}):\ \boldsymbol{\mu}_{2l}\text{-, } \boldsymbol{\mu}\text{-torsors}$$

Moreover: $\Pi_{\underline{v}} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}}$

$(l \cdot \Delta_{\Theta})(\Pi_{\underline{v}\ddot{\blacktriangleright}})$: the subquotient of $\Pi_{\underline{v}\ddot{\blacktriangleright}}$ det'd by $(l \cdot \Delta_{\Theta})(\Pi_{\underline{v}})$

$\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}})$: the arithmetic quotient of $\Pi_{\underline{v}}$

$\Pi_{\underline{v}\ddot{\blacktriangleright}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$: the arithmetic quotient of $\Pi_{\underline{v}\ddot{\blacktriangleright}}$

Key Lemma (Comparison Between Temp'd Conj. and Prof. Conj.)

$I_t \subseteq \Pi_{\underline{v}}$: an inertia subgp ass'd to the cusp lab'd by t s.t. $I_t \subseteq \Delta_{\underline{v}\square}$

γ, $\gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}}$

Then the following three conditions are equivalent:

- $\gamma' \in \Delta^{\pm}_{\underline{v}\square}$
- $I_t^{(\gamma\cdot\gamma')} \subseteq \Pi^{\gamma}_{\underline{v}\square}$
- $I_t^{(\gamma\cdot\gamma')} \subseteq (\Pi^{\pm}_{\underline{v}\square})^{\gamma}$

where $(-)^{\gamma} \stackrel{\text{def}}{=} \gamma \cdot (-) \cdot \gamma^{-1}$

Key Lemma follows from the theory of semi-graphs of anabelioids.

In the situation of Key Lemma, write $\delta \stackrel{\text{def}}{=} \gamma \cdot \gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}}$.

$(\stackrel{\text{Lemma}}{\Rightarrow} I_t^{\delta} = I_t^{(\gamma\cdot\gamma')} \subseteq \Pi^{\gamma}_{\underline{v}\square} = \Pi^{\delta}_{\underline{v}\square})$

By the theory of semi-graphs of anabelioids,

one can construct, from the inclusions $I_t^\delta = I_t^{(\gamma\cdot\gamma')} \subseteq \Pi_{\underline{v}\Box}^\gamma = \Pi_{\underline{v}\Box}^\delta$:

(a) the dec. gp $D_t^\delta \stackrel{\mathrm{def}}{=} N_{\Pi_{\underline{v}}^\delta}(I_t^\delta) \subseteq \Pi_{\underline{v}}^\delta$ that contains I_t^δ

(b) a dec. gp $D_{\mu_-}^\delta \subseteq \Pi_{\underline{v}\ddot{\blacktriangleright}}^\delta$, well-def'd up to $(\Pi_{\underline{v}\ddot{\blacktriangleright}}^\pm)^\delta$-conj., ass'd to μ_-

(c) a dec. gp $D_{t,\mu_-}^\delta \subseteq \Pi_{\underline{v}\ddot{\Box}}^\delta$, well-def'd up to $(\Pi_{\underline{v}\ddot{\Box}}^\pm)^\delta$-conj., ass'd to the μ_--translation of the cusp that det. I_t^δ (i.e., an <u>ev. pt lab'd by t</u>)

Moreover, this construction is compatible w/:

- the $\widehat{\Delta}_{\underline{v}}^\pm$-conjugation
- the inclusion $\Pi_{\underline{v}\bullet t} \subseteq \Pi_{\underline{v}\blacktriangleright}$

If, moreover, $\Box = \bullet t$, then the construction of (a) and (c) is compatible w/ the $\widehat{\Pi}_{\underline{v}}^{\mathrm{cor}}$-conjugation.

(Recall: the $\mathbb{F}_l^{\rtimes\pm}$-symmetry arises from the $\widehat{\Pi}_{\underline{v}}^{\mathrm{cor}}$-conjugation.)

$$I_t^{\delta} = I_t^{(\gamma \cdot \gamma')} \subseteq \Pi^{\delta}_{\underline{v}\ddot{\blacktriangleright}} \subseteq \Pi^{\gamma}_{\underline{v}\blacktriangleright} = \Pi^{\delta}_{\underline{v}\blacktriangleright}$$

By restricting $\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}}) \subseteq {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}})$ to $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}} \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\Pi^{\gamma}_{\underline{v}})$,

we obtain $\boldsymbol{\mu}_{2l}$-, $\boldsymbol{\mu}$-torsors

$$\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \ \subseteq \ {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \ \subseteq \ {}_{\infty}H^{1}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}, (l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})).$$

Thus, by restricting them to $(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \overset{\sim}{\leftarrow}) \ D^{\delta}_{t,\mu_{-}} \subseteq \Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$, we obtain

$$\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \ \subseteq \ {}_{\infty}\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \ \subseteq \ {}_{\infty}H^{1}(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}), (l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})),$$

i.e., "$\boldsymbol{\mu}_{2l} \cdot \underline{\underline{q}}_{\underline{v}}^{\underline{t}^2} \subseteq \boldsymbol{\mu} \cdot \underline{\underline{q}}_{\underline{v}}^{\underline{t}^2}$", where $-l^{*} \leq \underline{t} \leq l^{*}$ is det'd by t.

$${}_{(\infty)}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \ \stackrel{\mathrm{def}}{=} \ {}_{(\infty)}\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \ (= \ {}_{(\infty)}\underline{\underline{\theta}}^{-t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}))$$

In summary:

$\Pi_{\underline{v}} \underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} \{\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}})\}_{|t|\in|\mathbb{F}_l|}$, $\{{}_{\infty}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}})\}_{|t|\in|\mathbb{F}_l|}$ arising from

- $\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}$: an arbitrary $\widehat{\Delta}^{\pm}_{\underline{v}}$-conjugate of $\Pi_{\underline{v}\blacktriangleright\ddot{}}$
- I^{δ}_{t}: an arbitrary $\widehat{\Delta}^{\pm}_{\underline{v}}$-conjugate of I_t s.t. $I^{\delta}_{t} \subseteq \Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}$

(t ranges over the elements of $\mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \overset{\gamma\text{-conj.}}{\xrightarrow{\sim}} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$)

Moreover, this alg'm is compatible w/ the <u>independent</u> conj. actions of $\widehat{\Delta}^{\pm}_{\underline{v}}$ on the sets of (not temp'd but) prof. conj. $\{\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}\}_{\gamma}$ and $\{I^{\delta}_{t}\}_{\delta}$.

Remark

A conjugacy indeterminacy in a situation related to the theta value "$_{(\infty)}\underline{\underline{\theta}}^{|t|}$" at $|t| \in |\mathbb{F}_l|$ depends, a priori, on the label $|t| \in |\mathbb{F}_l|$.

That is to say, various objects at $|t| \in |\mathbb{F}_l|$ is well-defined up to conjugation which is, a priori, independent of the label $|t| \in |\mathbb{F}_l|$.

However, our resulting theta values

$$_{(\infty)}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}^{\blacktriangleright}}) \subseteq {}_{\infty}H^1(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}^{\blacktriangleright}}), (l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}^{\blacktriangleright}}))$$

for various $|t| \in |\mathbb{F}_l|$ are computed relative to "label-independent" $G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}^{\blacktriangleright}})$ and $(l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}^{\blacktriangleright}})$. conjugate synchronization

($\Rightarrow$ One may apply Kummer theory related to theta values.)

Recall: $\Pi_{\underline{v}} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} M^{\times}_{\mathbb{TM}}(\Pi_{\underline{v}}) \subseteq (M^{\times}_{\mathbb{TM}} \cdot \underline{\underline{\theta}}^{\iota})(\Pi_{\underline{v}}) \subseteq (M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota})(\Pi_{\underline{v}})$
in ${}_{\infty}H^1(\Pi_{\underline{v}}, (l \cdot \Delta_{\Theta})(\Pi_{\underline{v}}))$

(Recall: "$M^{\times}_{\mathbb{TM}}$" is an isomorph of "$\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$".)

By restricting them to $\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}} \subseteq \Pi_{\underline{v}}$, we obtain

$M^{\times}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}) \subseteq (M^{\times}_{\mathbb{TM}} \cdot \underline{\underline{\theta}}^{\iota})(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}) \subseteq (M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota})(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}})$
in ${}_{\infty}H^1(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}, (l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}))$

Thus, by the nat'l "${}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}) \twoheadrightarrow {}_{\infty}\underline{\underline{\theta}}^{0}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}})$", we obtain a splitting

$$(M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota})(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}) = M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}}) \times \left({}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright\ddot{}})\right).$$

(Recall: "$M^{\boldsymbol{\mu}}_{\mathbb{TM}}$" (resp. "$M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}$") is an isomorph of "$\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" (resp. "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$").)

In the remainder of §2, suppose: $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) = \Pi_{\underline{v}}$

$\Rightarrow 1 \to \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}) \to \Pi_{\mathbb{M}^{\Theta}_{*}} \to \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$

By base-chan'g $\Pi_{\underline{v}\ddot{\blacktriangleright}} \subseteq \Pi_{\underline{v}\blacktriangleright} \subseteq \Pi_{\underline{v}} = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ via $\Pi_{\mathbb{M}^{\Theta}_{*}} \to \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$,

we obtain closed subgroups $\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}} \subseteq \Pi_{\mathbb{M}^{\Theta}_{*\blacktriangleright}} \subseteq \Pi_{\mathbb{M}^{\Theta}_{*}}$.

$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$, $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$, $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$, $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$:

the respective "corresponding subquotients" of $\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}}$

$\Rightarrow$ $\exists$a cyclotomic rigidity isomorphism $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \xrightarrow{\sim} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$

By applying the cycl. rig. isom. $(l \cdot \Delta_{\Theta})((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \xrightarrow{\sim} \Pi_{\boldsymbol{\mu}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$ arising from the "γ-conjugate" $(\mathbb{M}^{\Theta}_{*})^{\gamma}$ of $\mathbb{M}^{\Theta}_{*}$ (where $\gamma \in \widehat{\Delta}^{\pm}_{\underline{v}}$) to

$$\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \subseteq {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \subseteq {}_{\infty}H^{1}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}, (l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})),$$

$$\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \subseteq {}_{\infty}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \subseteq {}_{\infty}H^{1}(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}), (l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})), \text{ and}$$

$$(M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota})(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) = M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \times ({}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})),$$

we obtain

$$\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \subseteq {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \subseteq {}_{\infty}H^{1}(\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}), \Pi_{\boldsymbol{\mu}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}),$$

$$\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \subseteq {}_{\infty}\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \subseteq {}_{\infty}H^{1}(G_{\underline{v}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}), \Pi_{\boldsymbol{\mu}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})),$$

$$\begin{aligned}(M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}})((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \\ = M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \times \Big({}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})\Big).\end{aligned}$$

In a similar vein, by applying the cycl. rig. isom.

$$(l \cdot \Delta_\Theta)((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \stackrel{\sim}{\to} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})),$$

we obtain:

$$\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \subseteq {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$$

$$\subseteq {}_{\infty}H^1(\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}), \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}))),$$

$$\underline{\underline{\theta}}^{|t|}_{\mathrm{bs}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \subseteq {}_{\infty}\underline{\underline{\theta}}^{|t|}_{\mathrm{bs}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$$

$$\subseteq {}_{\infty}H^1(G_{\underline{v}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}), \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}))),$$

$$(M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}})((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) / M^{\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})_{\mathrm{bs}}$$

$$= M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})_{\mathrm{bs}} \times \Big({}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{bs}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) / M^{\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})_{\mathrm{bs}}\Big).$$

$\S 2\frac{1}{2}$ Multiradial Kummer-detachment of Theta Monoids

Goal: "multiradial Kmm-detach." of theta monoids, i.e., "$\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}_{\underline{v}}$"

Strategy: By the final assertion of §1, we have:

$$\Pi_{\underline{v}} \quad \underset{\text{alg'm}}{\overset{\text{multiradial}}{\rightsquigarrow}} \quad \underline{\text{étale-like}}\ \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}_{\underline{v}}$$

$$\underset{\text{cycl. rig.}}{\overset{\text{via multiradial}}{\rightsquigarrow}} \quad \underline{\text{mono-theta}}\ \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}_{\underline{v}}, \text{ i.e., labeled by } \underline{\text{"env"}}$$

Thus, by applying the Kummer theory for theta functions:

$$\underline{\text{Frobenius-like}}\ \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}_{\underline{v}} \underset{\text{theory}}{\overset{\text{Kummer}}{\rightsquigarrow}} (\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}_{\underline{v}})_{\text{env}}$$

- Recall: $\underline{\underline{\mathcal{F}}}_{\underline{v}} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} \boldsymbol{\mu}_{2l}(\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}^{\div}) \cdot \underline{\underline{\Theta}}_{\underline{v}} \subseteq \mathcal{O}^{\times}(\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}^{\div})$

  (which determines the monoid $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) = \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{N}}|_{A^{\Theta}_{\infty}}$)

$$\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} = \left\{ \Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \overset{\text{def}}{=} \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot (\underline{\underline{\Theta}}_{\underline{v}}^{\alpha})^{\mathbb{N}}|_{A^{\Theta}_{\infty}} \right\}_{\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})}$$

$${}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} = \left\{ {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \overset{\text{def}}{=} \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot (\underline{\underline{\Theta}}_{\underline{v}}^{\alpha})^{\mathbb{Q}_{\geq 0}}|_{A^{\Theta}_{\infty}} \right\}_{\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})}$$

- Recall: $\underline{\underline{\mathcal{F}}}_{\underline{v}} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}}$ the base-theoretic hull $\mathcal{C}_{\underline{v}} \subseteq \underline{\underline{\mathcal{F}}}_{\underline{v}}$

$(\Pi_{\underline{v}} \curvearrowright)\ \Psi_{\mathcal{C}_{\underline{v}}} \overset{\text{def}}{=} \mathcal{O}^{\rhd}_{\mathcal{C}_{\underline{v}}}(A^{\Theta}_{\infty})$ (well-defined up to $\Pi_{\underline{v}}$-conjugation)

${}_{(\infty)}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}}$: the Frobenius-like theta monoid

$\Psi_{\mathcal{C}_{\underline{v}}}$: the Frobenius-like constant monoid

- Recall: $\mathbb{M}_*^{\Theta} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} M_{\mathbb{TM}}^{\times}(\mathbb{M}_*^{\Theta})$, $\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}_*^{\Theta}) \subseteq {}_{\infty}\underline{\underline{\theta}}_{\text{env}}(\mathbb{M}_*^{\Theta})$
in ${}_{\infty}H^1(\Pi_{\underline{\underline{\ddot{Y}}}}^{\text{tp}}(\mathbb{M}_*^{\Theta}), \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}))$

$$\Psi_{\text{env}}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} \left\{ \Psi_{\text{env}}^{\iota}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} M_{\mathbb{TM}}^{\times}(\mathbb{M}_*^{\Theta}) \cdot \underline{\underline{\theta}}_{\text{env}}^{\iota}(\mathbb{M}_*^{\Theta})^{\mathbb{N}} \right\}_{\iota:\ \text{inv. autom.}}$$

$$ {}_{\infty}\Psi_{\text{env}}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} \left\{ {}_{\infty}\Psi_{\text{env}}^{\iota}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} M_{\mathbb{TM}}^{\times}(\mathbb{M}_*^{\Theta}) \cdot {}_{\infty}\underline{\underline{\theta}}_{\text{env}}^{\iota}(\mathbb{M}_*^{\Theta})^{\mathbb{Q}_{\geq 0}} \right\}_{\iota:\ \text{inv. aut.}}$$

- Recall: $\mathbb{M}_*^{\Theta} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} G_{\underline{v}}(\mathbb{M}_*^{\Theta})$, $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\mathbb{M}_*^{\Theta})) \xrightarrow{\sim} \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta})$
$\overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} M_{\mathbb{TM}}(\mathbb{M}_*^{\Theta}) \subseteq {}_{\infty}H^1(\Pi_{\underline{\underline{\ddot{Y}}}}^{\text{tp}}(\mathbb{M}_*^{\Theta}), \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}))$

(Recall: "$M_{\mathbb{TM}}$" is an isomorph of "$\mathcal{O}_{\overline{F}_{\underline{v}}}^{\triangleright}$".)

$$(\Pi_{\underline{\underline{X}}}^{\text{tp}}(\mathbb{M}_*^{\Theta}) \curvearrowright)\ \Psi_{\text{cns}}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} M_{\mathbb{TM}}(\mathbb{M}_*^{\Theta})$$

${}_{(\infty)}\Psi_{\text{env}}(\mathbb{M}_*^{\Theta})$: the <u>mono-theta-theoretic theta monoid</u>

$\Psi_{\text{cns}}(\mathbb{M}_*^{\Theta})$: the <u>mono-theta-theoretic constant monoid</u>

In particular, by applying the above algorithm to $\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}})$:

$_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}}))$: the étale-like theta monoid

$\Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}}))$: the étale-like constant monoid

In order to obtain "multiradial Kummer-detachment" of "$\underline{\underline{\Theta}}_{\underline{v}}$",

let us relate

Frobenius-like/mono-theta-theoretic/étale-like theta monoids.

In the remainder of $\S 2\frac{1}{2}$, suppose: $\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) = \mathbb{M}^{\Theta}_{*}$

Then, by applying the Kummer theory, relative to a suitable assign't "$\iota \mapsto \alpha$", we obtain an isomorphism ${}_{(\infty)}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \stackrel{\sim}{\to} {}_{(\infty)}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$ (cf. the Kummer theory of theta functions in tempered Frobenioids).

Write

$$ {}_{(\infty)}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} \stackrel{\sim}{\longrightarrow} {}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) $$

for the collection of the above isomorphisms.

Moreover, again by applying the Kummer theory, we obtain an isomorphism

$$ \Psi_{\mathcal{C}_{\underline{v}}} \stackrel{\sim}{\longrightarrow} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}). $$

Thus, every isomorphism $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$ determines:

$$\begin{array}{ccccc}
\text{(étale)} & & \text{(mono-theta)} & & \text{(Frobenius)} \\
\Pi_{\underline{v}} = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) & = & \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
{}_{\infty}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & {}_{\infty}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \xleftarrow{\sim} & {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} \\
\bigcup & & \bigcup & & \bigcup \\
\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \xleftarrow{\sim} & \Psi_{\mathcal{F}^{\Theta}_{\underline{v}}}
\end{array}$$

$$\begin{array}{ccccc}
G_{\underline{v}} = G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) & = & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} & \xrightarrow{\sim} & \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})^{\times} & \xleftarrow{\sim} & \Psi^{\times}_{\mathcal{F}^{\Theta}_{\underline{v}}}
\end{array}$$

Moreover, every isom. $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$ also determines:

$$
\begin{array}{ccccc}
\text{(étale)} & & \text{(mono-theta)} & & \text{(Frobenius)} \\
\Pi_{\underline{v}} = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) & = & \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}) & \xleftarrow{\sim} & \Psi_{\mathcal{C}_{\underline{v}}}
\end{array}
$$

$$
\begin{array}{ccccc}
G_{\underline{v}} = G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) & = & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} & \xrightarrow{\sim} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})^{\times} & \xleftarrow{\sim} & \Psi^{\times}_{\mathcal{C}_{\underline{v}}}
\end{array}
$$

That is to say, we obtain various Kummer isomorphisms.

Thus, by the final assertion of §1, we obtain

"multiradial Kummer-detachment" of theta monoids:

$$\Pi_{\underline{v}} \underset{\text{alg'm (cf. §1)}}{\overset{\text{multiradial}}{\rightsquigarrow}} {}_{(\infty)}\Psi_{\text{env}}(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}})) \underset{\text{cycl. rig.}}{\overset{\text{via multiradial}}{\overset{\sim}{\leftarrow}}} {}_{(\infty)}\Psi_{\mathcal{F}_{\underline{v}}^{\Theta}}$$

Remark

On the other hand, the above discussion only gives

"uniradial Kummer-detachment" of constant monoids.

(cf. "$(\Pi_{\underline{v}} \curvearrowright \Psi_{\mathcal{C}_{\underline{v}}}) \rightsquigarrow (G_{\underline{v}}(\mathbb{M}_*^{\Theta}) \curvearrowright \Psi_{\mathcal{C}_{\underline{v}}}^{\times})$": a uniradial environment)

($\Rightarrow$ the theory of log-shells)

$\S 2\frac{3}{4}$ Definition (used in $\S 4$)

(1) $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \Rightarrow \mathcal{F}^{\vdash\times}_{\underline{v}}$: the Fro'd "corresponding to" $G_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$

(omit the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

an $\mathcal{F}^{\vdash\times}$-prime-strip $\overset{\mathrm{def}}{\Leftrightarrow}$ $\{$an isomorph of $\mathcal{F}^{\vdash\times}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

(2) $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \Rightarrow \mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$: the $\times\boldsymbol{\mu}$-Kummer Frobenioid "corresponding to" $G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$ equipped with the $\times\boldsymbol{\mu}$-*Kummer structure*, i.e., $\{\mathrm{Im}\big((\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}})^{H} = \mathcal{O}^{\times}_{\overline{F}^{H}_{\underline{v}}} \hookrightarrow \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}\big)\}_{H\subseteq G_{\underline{v}}\text{: open subgps}}$

(omit the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

an $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip $\overset{\mathrm{def}}{\Leftrightarrow}$ $\{$an isomorph of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

(3) ${}^{\dagger}\mathfrak{F}^{\vdash} = \{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: an $\mathcal{F}^{\vdash}$-prime-strip

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} \Rightarrow {}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$ "corresponds to" $G_{\underline{v}} \curvearrowright (\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}} \overset{\text{mod } \boldsymbol{\mu}_{2l}}{\hookleftarrow} \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}})$

${}^{\dagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{v}}$: the split-$\times\boldsymbol{\mu}$-Kummer Frobenioid "corresponding to"

$G_{\underline{v}} \curvearrowright (\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \times (\boldsymbol{\mu}_{2l} \cdot \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}/\boldsymbol{\mu}_{2l}) \hookleftarrow (\boldsymbol{\mu}_{2l} \cdot \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}/\boldsymbol{\mu}_{2l}))$ w/ $\times\boldsymbol{\mu}$-Kmm str.

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}} \Rightarrow {}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$ "corresponds to" $G_{\underline{v}} \curvearrowright (\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \times p_{\underline{v}}^{\mathbb{N}} \hookleftarrow p_{\underline{v}}^{\mathbb{N}})$

${}^{\dagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{v}}$: the split-$\times\boldsymbol{\mu}$-Kummer Frobenioid "corresponding to"

$G_{\underline{v}} \curvearrowright (\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \times p_{\underline{v}}^{\mathbb{N}} \hookleftarrow p_{\underline{v}}^{\mathbb{N}})$ w/ $\times\boldsymbol{\mu}$-Kmm str.

(omit the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

an $\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip $\overset{\mathrm{def}}{\Leftrightarrow}$ {an isomorph of $\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

(4) an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime strip $\overset{\text{def}}{\Leftrightarrow}$ a suitable collection of data

$$({}^{\dagger}\mathcal{C}^{\Vdash}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}) \overset{\sim}{\to} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}, \{{}^{\dagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}),$$

i.e., a collection of data obtained by replacing the "${}^{\dagger}\mathfrak{F}^{\vdash}$" of an $\mathcal{F}^{\Vdash}$-prime strip "$({}^{\dagger}\mathcal{C}^{\Vdash}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}) \overset{\sim}{\to} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash}, \{{}^{\dagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$" by an $\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$

Thus:

${}^{\dagger}\mathcal{F}^{\vdash}$-prime-strip $\underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}}$ ${}^{\dagger}\mathcal{F}^{\vdash\times}$-prime-strip $\underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}}$ ${}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip

${}^{\dagger}\mathcal{F}^{\vdash}$-prime-strip $\underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}}$ ${}^{\dagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip

${}^{\dagger}\mathcal{F}^{\Vdash}$-prime-strip $\underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}}$ ${}^{\dagger}\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip