Hodge-Arakelov-theoretic Evaluation II

Yuichiro Hoshi

RIMS, Kyoto University

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§3 Tempered Gaussian Frobenioids

Goal: Construction of Gaussian monoids,

$$\text{i.e., ``}\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \underline{\underline{q}}_{\underline{v}}^{2^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(j^*)^2})\text{''}$$

and the evaluation isomorphism,

$$\text{i.e., ``}\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}_{\underline{v}} \overset{\sim}{\to} \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \underline{\underline{q}}_{\underline{v}}^{2^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(j^*)^2})\text{''}$$

(On the other hand, as in the case of constant monoids, we obtain only "uniradial Kummer-detachment" of Gaussian monoids.

($\Rightarrow$ the theory of log-shells))

Recall: $\mathbb{M}^{\Theta}_{*} \overset{\exists\text{func'l}}{\underset{\text{algorithm}}{\Rightarrow}} \Pi^{\text{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \subseteq \Pi^{\text{tp}}_{\underline{X}}(\mathbb{M}^{\Theta}_{*}) \subseteq \Pi^{\text{tp}}_{C}(\mathbb{M}^{\Theta}_{*})$

$\overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} (\Delta^{\text{tp}}_{C}/\Delta^{\text{tp}}_{\underline{X}})(\mathbb{M}^{\Theta}_{*}) \curvearrowright \text{LabCusp}^{\pm}(\Pi^{\text{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$: the $\mathbb{F}^{\rtimes\pm}_{l}$-symmetry

For $t \in \text{LabCusp}^{\pm}(\Pi^{\text{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$,

$(-)_t$: a copy of $(-)$ labeled by t

$\Rightarrow$ The $\mathbb{F}^{\rtimes\pm}_{l}$-symmetry determines an isomorphism

$$(G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\blacktriangleright})_t \curvearrowright \Psi_{\text{cns}}(\mathbb{M}^{\Theta}_{*})_t) \xrightarrow{\sim} (G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\blacktriangleright})_{t'} \curvearrowright \Psi_{\text{cns}}(\mathbb{M}^{\Theta}_{*})_{t'}).$$

$(-)_{|t|} \overset{\text{def}}{=} (-)_t \ (\xrightarrow{\sim} (-)_{-t})$

$(-)_{\langle|\mathbb{F}_l|\rangle}$ (resp. $(-)_{\langle\mathbb{F}^{\circledast}_l\rangle}$) $\overset{\text{def}}{=}$ the "diagonal" $\subseteq \prod_{|t|\in|\mathbb{F}_l|(\text{resp.}\,\mathbb{F}^{\circledast}_l)}(-)_{|t|}$

$\Rightarrow$ By "$(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \overset{\sim}{\leftarrow})\ D^{\delta}_{t,\mu_-} \hookrightarrow \Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$" (cf. §2 (p.10))

(and taking $\gamma = 1$), we obtain a diagram

$$\begin{array}{ccccc} (\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) & \hookleftarrow) & \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & \twoheadrightarrow & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{\langle|\mathbb{F}_l|\rangle} \\ & & \curvearrowright & & \curvearrowright \\ & & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}) & \overset{\sim}{\to} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle|\mathbb{F}_l|\rangle}. \end{array}$$

By the submonoid $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle|\mathbb{F}_l|\rangle} \subseteq \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_0 \times \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{\circledast}_l\rangle}$,

we obtain a nat'l identification of $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_0$ with $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{\circledast}_l\rangle}$:

$$\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_0 \ = \ \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{\circledast}_l\rangle}$$

(i.e., "$0 \leftrightarrow \langle\mathbb{F}^{\circledast}_l\rangle$")

A value-profile (i.e., "$(\zeta_{2l}^{i_1} \cdot \underline{\underline{q}}_{v}, \ldots, \zeta_{2l}^{i_{l^*}} \cdot \underline{\underline{q}}_{v}^{(l^*)^2})$")

$\overset{\text{def}}{\Leftrightarrow}$ an element of

$$\underline{\underline{\theta}}_{\text{env}}^{\mathbb{F}_l^*}(\mathbb{M}_{*\blacktriangleright}^{\Theta}) \overset{\text{def}}{=} \prod_{|t| \in \mathbb{F}_l^*} \underline{\underline{\theta}}_{\text{env}}^{|t|}(\mathbb{M}_{*\blacktriangleright}^{\Theta}) \subseteq \prod_{|t| \in \mathbb{F}_l^*} \Psi_{\text{cns}}(\mathbb{M}_*^{\Theta})_{|t|}$$

$$\Psi_{\text{gau}}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} \left\{ \Psi_{\xi}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} \Psi_{\text{cns}}^{\times}(\mathbb{M}_*^{\Theta})_{\langle \mathbb{F}_l^* \rangle} \cdot \xi^{\mathbb{N}} \right\}_{\xi\text{: value-profiles}}$$

$$_{\infty}\Psi_{\text{gau}}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} \left\{ _{\infty}\Psi_{\xi}(\mathbb{M}_*^{\Theta}) \overset{\text{def}}{=} \Psi_{\text{cns}}^{\times}(\mathbb{M}_*^{\Theta})_{\langle \mathbb{F}_l^* \rangle} \cdot \xi^{\mathbb{Q}_{\geq 0}} \right\}_{\xi\text{: value-profiles}}$$

$_{(\infty)}\Psi_{\text{gau}}(\mathbb{M}_*^{\Theta})$: the mono-theta-theoretic Gaussian monoid

Then, by the nat'l "${}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \xrightarrow{\sim} {}_{\infty}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$", we obtain a diagram

$$\begin{array}{ccccc}
(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \hookleftarrow) & \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & \dashleftarrow & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|t|}\}_{|t|\in\langle|\mathbb{F}_l|\rangle} \\
& \curvearrowright & & \curvearrowright \\
& {}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \xrightarrow{\sim} & {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) \\
& \cup & & \cup \\
& \Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \xrightarrow{\sim} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}),
\end{array}$$

where "$\dashleftarrow$" denotes an "evident compatibility" (w.r.t "$\{D^{\delta}_{t,\mu_-}\}_t$").

Write

$$ {}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \xrightarrow{\sim} {}_{(\infty)}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}) $$

for the coll. of the above isom., i.e., the evaluation isomorphism:

$$ \text{``}\underline{\underline{\Theta}}_{\underline{v}} \mapsto (\zeta_{2l}^{i_1}\cdot \underline{\underline{q}}_{\underline{v}}, \ldots, \zeta_{2l}^{i_{l^*}}\cdot \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})\text{''} $$

In particular, by applying the above algorithm to $\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}})$,

$_{(\infty)}\Psi_{\mathrm{gau}}(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}}))$: the étale-like Gaussian monoid

$_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}})) \xrightarrow{\sim} {}_{(\infty)}\Psi_{\mathrm{gau}}(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}}))$: the étale-like eval. isom.

Next, let us consider the Frobenius-like Gaussian monoid.

In the remainder of §3, suppose: $\mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) = \mathbb{M}^{\Theta}_{*}$

Recall: By applying the Kummer theory, we have an isomorphism

$(G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) \curvearrowright \Psi_{\mathcal{C}_{\underline{v}}}) \xrightarrow{\sim} (G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}))$. (cf. §$2\frac{1}{2}$ (p.22))

$\Rightarrow$ For $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$,

$$(G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_t \curvearrowright (\Psi_{\mathcal{C}_{\underline{v}}})_t) \xrightarrow{\sim} (G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_t \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_t)$$

which is compatible with the $\mathbb{F}_l^{\rtimes\pm}$-symmetry.

(Note: This isom. is well-def'd up to conj. which is independent of t.)

$\Rightarrow$ For a value-profile $\xi \in \underline{\underline{\theta}}^{\mathbb{F}_l^{\circledast}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \subseteq \prod_{|t|\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|}$,

$${}_{(\infty)}\Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \mathrm{Im}\Big({}_{(\infty)}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) \hookrightarrow \prod_{\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|} \xleftarrow{\sim} \prod_{\mathbb{F}_l^{\circledast}} (\Psi_{\mathcal{C}_{\underline{v}}})_{|t|}\Big)$$

${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \left\{ {}_{(\infty)}\Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \right\}_{\xi}$: the Frob.-like Gaussian monoid

Thus, we have a diagram:

$$\begin{array}{ccccccc}
\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}_{*\ddot{\blacktriangleright}}) & = & \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}_{*\ddot{\blacktriangleright}}) & \dashleftarrow & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|t|}\}_{\langle \mathbb{F}^{\circledast}_l \rangle} & \stackrel{\sim}{\to} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_{|t|}\} \\
\curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\
{}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} & \stackrel{\sim}{\to} & {}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \stackrel{\sim}{\to} & {}_{\infty}\Psi^{\iota}_{\xi}(\mathbb{M}^{\Theta}_{*}) & \stackrel{\sim}{\to} & {}_{\infty}\Psi^{\iota}_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} & \stackrel{\sim}{\to} & \Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \stackrel{\sim}{\to} & \Psi^{\iota}_{\xi}(\mathbb{M}^{\Theta}_{*}) & \stackrel{\sim}{\to} & \Psi^{\iota}_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}})
\end{array}$$

Write

$${}_{(\infty)}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} \stackrel{\sim}{\longrightarrow} {}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\sim}{\longrightarrow} {}_{(\infty)}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\sim}{\longrightarrow} {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

for the coll. of the above isom., i.e., the (Frobenius-like) <u>eval. isom</u>:

$$\text{“}\underline{\underline{\Theta}}_{\underline{v}} \mapsto (\zeta_{2l}^{i_1} \cdot \underline{\underline{q}}_{\underline{v}}, \ldots, \zeta_{2l}^{i_{l^{\circledast}}} \cdot \underline{\underline{q}}_{\underline{v}}^{(l^{\circledast})^2})\text{”}$$

In summary, every isom. $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$ determines:

$$
\begin{array}{ccccc}
\Pi_{\underline{v}\ddot{\blacktriangleright}} & \dashleftarrow & \{G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})_{|t|}\}_{\mathbb{F}_l^{\circledast}} & & \\
\curvearrowright & & \curvearrowright & & \\
{}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & & \\
\bigcup & & \bigcup & & \\
\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & & \\
 & \xrightarrow{\sim} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\underline{v}\ddot{\blacktriangleright}})_{|t|}\}_{\mathbb{F}_l^{\circledast}} & \xrightarrow{\sim} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_{|t|}\}_{\mathbb{F}_l^{\circledast}} \\
 & & \curvearrowright & & \curvearrowright \\
 & \xrightarrow{\sim} & {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) & \xrightarrow{\sim} & {}_{\infty}\Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \\
 & & \bigcup & & \bigcup \\
 & \xrightarrow{\sim} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) & \xrightarrow{\sim} & \Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}})
\end{array}
$$

$$(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}}) \stackrel{\sim}{\to} \mathbb{M}_*^{\Theta} = \mathbb{M}_*^{\Theta}(\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

$$\begin{array}{ccc}
G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) & \stackrel{\sim}{\to} & G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})_{\langle \mathbb{F}_l^{\circledast} \rangle} \\
\curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}}))^{\times} & \stackrel{\sim}{\to} & \Psi_{\xi}(\mathbb{M}_*^{\Theta}(\Pi_{\underline{v}}))^{\times}
\end{array}$$

$$\begin{array}{ccccc}
\stackrel{\sim}{\to} & G_{\underline{v}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})_{\langle \mathbb{F}_l^{\circledast} \rangle} & \stackrel{\sim}{\to} & G_{\underline{v}}(\mathbb{M}_*^{\Theta})_{\langle \mathbb{F}_l^{\circledast} \rangle} \\
 & \curvearrowright & & \curvearrowright \\
\stackrel{\sim}{\to} & \Psi_{\xi}(\mathbb{M}_*^{\Theta})^{\times} & \stackrel{\sim}{\to} & \Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times}
\end{array}$$

Remark

As in the case of constant monoids, the above discussion only gives "uniradial Kummer-detachment" of Gaussian monoids.

($\Rightarrow$ the theory of log-shells)

§4 Global Gaussian Frobenioids

${}^{\dagger}\mathfrak{D}^{\vdash} = \{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: a $\mathcal{D}^{\vdash}$-prime-strip

Suppose: $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ $(\Rightarrow {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\text{ “}=\text{” } G_{\underline{v}})$

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \overset{\mathrm{def}}{=} \mathcal{O}^{\triangleright}(G_{\underline{v}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})) \subseteq {}_{\infty}H^1(G_{\underline{v}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}), \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})))$$

$$\Psi^{\mathbb{R}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \overset{\mathrm{def}}{=} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})^{\mathrm{rlf}} \ (\cong \mathbb{R}_{\geq 0})$$

Recall: ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} \log^{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}(p_{\underline{v}}) \in \Psi^{\mathbb{R}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$

$$(G_{\underline{v}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \curvearrowright)\ \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \overset{\mathrm{def}}{=} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})^{\times} \times \Psi^{\mathbb{R}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$$

(omit the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

$\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}^{\vdash}) \colon \underline{\mathbb{V}} \ni \underline{v} \mapsto \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$

Recall: ${}^{\dagger}\mathfrak{D}^{\vdash} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}}$

$\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash})$: an isomorph of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$

$\mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash})) \xrightarrow{\sim} \underline{\mathbb{V}}$

$\{{}^{\dagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}} \colon \Phi_{\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}),\underline{v}} \xrightarrow{\sim} \Phi^{\mathbb{R}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$

${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: a $\mathcal{D}$-prime-strip

Recall: ${}^\dagger\mathcal{D}_{\underline{v}} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} {}^\dagger\mathcal{D}^\vdash_{\underline{v}}, \ \mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}_{\underline{v}})$

Suppose: $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$ ($\Rightarrow {}^\dagger\mathcal{D}_{\underline{v}}$ "$=$" $\Pi_{\underline{v}}$)

$\Psi_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}}) \overset{\mathrm{def}}{=} \Psi_{\mathrm{cns}}({}^\dagger\mathcal{D}^\vdash_{\underline{v}})$

in ${}_\infty H^1(G_{\underline{v}}({}^\dagger\mathcal{D}^\vdash_{\underline{v}}), \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}({}^\dagger\mathcal{D}^\vdash_{\underline{v}}))) \hookrightarrow {}_\infty H^1(\Pi^{(\pm)}_{\underline{v}}({}^\dagger\mathcal{D}_{\underline{v}}), \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}({}^\dagger\mathcal{D}_{\underline{v}})))$

$\Psi^{\mathbb{R}}_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}}) \overset{\mathrm{def}}{=} \Psi^{\mathbb{R}}_{\mathrm{cns}}({}^\dagger\mathcal{D}^\vdash_{\underline{v}}) \quad \Rightarrow \log^{{}^\dagger\mathcal{D}_{\underline{v}}}(p_{\underline{v}}) \overset{\mathrm{def}}{=} \log^{{}^\dagger\mathcal{D}^\vdash_{\underline{v}}}(p_{\underline{v}}) \in \Psi^{\mathbb{R}}_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}})$

$\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}}) \overset{\mathrm{def}}{=} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\dagger\mathcal{D}^\vdash_{\underline{v}})$

$\mathbb{F}_l^{\rtimes\pm}$-symmetry $\Rightarrow \Psi_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}})_t \overset{\sim}{\to} \Psi_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}})_{t'}$ ($t,\ t' \in \mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}_{\underline{v}})$)

$\Rightarrow \Psi_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}})_{\langle|\mathbb{F}_l|\rangle},\ \Psi_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}})_{\langle\mathbb{F}_l^*\rangle},\ \Psi_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}})_0 \overset{\sim}{\to} \Psi_{\mathrm{cns}}({}^\dagger\mathcal{D}_{\underline{v}})_{\langle\mathbb{F}_l^*\rangle}$

$$ {}_{(\infty)}\Psi_{\mathrm{env}}({}^{\dagger}\mathcal{D}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}_{\underline{v}})^{\times} \times \mathbb{R}_{\geq 0} \cdot \underline{\log^{{}^{\dagger}\mathcal{D}_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{{}^{\dagger}\mathcal{D}_{\underline{v}}}(\underline{\underline{\Theta}})} \text{(formal symbol)} $$

$$ {}_{(\infty)}\Psi_{\mathrm{gau}}({}^{\dagger}\mathcal{D}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}_{\underline{v}})^{\times}_{\langle \mathbb{F}_l^{*} \rangle} \times \mathbb{R}_{\geq 0} \cdot (\cdots, j^2 \cdot \log^{{}^{\dagger}\mathcal{D}_{\underline{v}}}(p_{\underline{v}})), \cdots) $$

$$ \subseteq \prod_{j \in \mathbb{F}_l^{*}} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}_{\underline{v}})_j $$

$\Psi_{\mathrm{env}}({}^{\dagger}\mathcal{D}_{\underline{v}}) \stackrel{\sim}{\to} \Psi_{\mathrm{gau}}({}^{\dagger}\mathcal{D}_{\underline{v}})$ given by "$\log^{{}^{\dagger}\mathcal{D}_{\underline{v}}}(\underline{\underline{\Theta}}) \mapsto (\cdots, j^2, \cdots)$"

(omit the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

${}_{(\infty)}\Psi_{\square}({}^{\dagger}\mathfrak{D})\colon \underline{\mathbb{V}} \ni \underline{v} \mapsto {}_{(\infty)}\Psi_{\square}({}^{\dagger}\mathcal{D}_{\underline{v}})$, where $\square \in \{$cns, env, gau$\}$

$\Rightarrow {}_{(\infty)}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}) \stackrel{\sim}{\to} {}_{(\infty)}\Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D})$: the evaluation isomorphism

${}^{\dagger}\mathfrak{F} = \{{}^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: an $\mathcal{F}$-prime-strip

Recall: ${}^{\dagger}\mathfrak{F} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} {}^{\dagger}\mathfrak{D}$ Suppose: $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$

$$(\Pi_{\underline{v}}({}^{\dagger}\mathcal{D}_{\underline{v}}) \curvearrowright)\ \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \overset{\mathrm{def}}{=} \Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \quad (\Rightarrow \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \overset{\text{Kummer}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}_{\underline{v}}))$$

$\mathbb{F}_l^{\rtimes\pm}$-symmetry $\Rightarrow \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}})_t \overset{\sim}{\to} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}})_{t'}$ ($t,\ t' \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$)

$$\Rightarrow \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}})_{\langle|\mathbb{F}_l|\rangle},\ \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}})_{\langle\mathbb{F}_l^{*}\rangle},\ \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}})_0 \overset{\sim}{\to} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}})_{\langle\mathbb{F}_l^{*}\rangle}$$

(omit the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}) : \underline{\mathbb{V}} \ni \underline{v} \mapsto \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\underline{v}})$$

Recall: ${}^{\dagger}\mathfrak{F} \overset{\exists\text{func'l}}{\underset{\text{alg'm}}{\Rightarrow}} {}^{\dagger}\mathfrak{D}^{\vdash}$, ${}^{\dagger}\mathfrak{F}^{\Vdash} = ({}^{\dagger}\mathcal{C}^{\Vdash}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}) \overset{\sim}{\to} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash}, \{{}^{\dagger}\rho_{\underline{v}}\})$,

$$({}^{\dagger}\mathcal{C}^{\Vdash}, \mathrm{Prime} \overset{\sim}{\to} \underline{\mathbb{V}}, \{{}^{\dagger}\rho_{\underline{v}}\}) \overset{\text{Kummer}}{\overset{\sim}{\to}} (\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}), \mathrm{Prime} \overset{\sim}{\to} \underline{\mathbb{V}}, \{{}^{\dagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\})$$

${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$: a Θ-Hodge theater

Suppose: $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \cap \underline{\mathbb{V}}^{\mathrm{non}}$

Recall: ${}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$ (in ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$) $\underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}}$ ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$

$G_{\underline{v}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \curvearrowright \Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$

Recall: ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} \underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} {}^{\dagger}\mathcal{D}_{\underline{v}} \underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$

$\Pi_{\underline{v}}({}^{\dagger}\mathcal{D}_{\underline{v}}) \curvearrowright \Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}}$

$\mathbb{F}_l^{\rtimes\pm}$-symmetry $\Rightarrow (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_t \xrightarrow{\sim} (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_{t'}$ $(t,\ t' \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}}))$

$\Rightarrow (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_{\langle|\mathbb{F}_l|\rangle}, (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_{\langle\mathbb{F}_l^{*}\rangle}, (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_0 \xrightarrow{\sim} (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_{\langle\mathbb{F}_l^{*}\rangle}$

By “Mono-anabelian Transport”:

$\exists!$a ${}^{\dagger}\Pi_{\underline{v}}$-equivariant isomorphism $\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \overset{\text{Kummer}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}_{\underline{v}})$

$\exists!$a $\widehat{\mathbb{Z}}^{\times}$-orbit of ${}^{\dagger}G_{\underline{v}}$-equivariant isomorphisms $\Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \overset{\text{Kmm}}{\overset{\sim}{\to}} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})^{\times}$

${}_{(\infty)}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}$, ${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$: the monoids corresponding, via

$$\exists! \quad (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_t \overset{\sim}{\to} \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{D}_{\underline{v}})_t \qquad (t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})),$$

to ${}_{(\infty)}\Psi_{\mathrm{env}}({}^{\dagger}\mathcal{D}_{\underline{v}})$, ${}_{(\infty)}\Psi_{\mathrm{gau}}({}^{\dagger}\mathcal{D}_{\underline{v}})$, respectively.

(omit the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$)

${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})$: $\underline{\mathbb{V}} \ni \underline{v} \mapsto {}_{(\infty)}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}$

${}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})$: $\underline{\mathbb{V}} \ni \underline{v} \mapsto {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$

${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathfrak{D}_{\succ} \overset{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}{\leftarrow} {}^{\dagger}\mathfrak{D}_{T} \overset{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\rightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$: a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater

Recall: $\underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} {}^{\dagger}\zeta_{\succ} \overset{\mathrm{def}}{=} {}^{\dagger}\zeta_{\pm} \circ \zeta_{0}^{\Theta^{\mathrm{ell}}} \circ ({}^{\dagger}\zeta_{0}^{\Theta^{\pm}})^{-1} \colon \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ}) \overset{\sim}{\to} T$

$\mathbb{F}_{l}^{\rtimes\pm}$-symmetry $\Rightarrow \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{t} \overset{\sim}{\to} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{t'}$ $(t,\, t' \in T)$

$\Rightarrow \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle|\mathbb{F}_{l}|\rangle}$, $\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle\mathbb{F}_{l}^{*}\rangle}$, $\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{0} \overset{\sim}{\to} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle\mathbb{F}_{l}^{*}\rangle}$

Recall: ${}^{\dagger}\mathfrak{D}_{\succ} \underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} {}_{(\infty)}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ}) \overset{\sim}{\to} {}_{(\infty)}\Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ})$, ${}^{\dagger}\mathfrak{D}_{\succ}^{\vdash}$

$\underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} \mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}_{\succ}^{\vdash})$, $\mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}_{\succ}^{\vdash})) \overset{\sim}{\to} \underline{\mathbb{V}}$, $\{{}^{\dagger}\rho_{\mathcal{D}^{\Vdash}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

$$\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \overset{\mathrm{def}}{=} \text{“}\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \cdot \log^{{}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}}(\underline{\underline{\Theta}})\text{”}$$

$$\Rightarrow \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}),\underline{v}} \xrightarrow{\sim} \Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ})^{\mathrm{rlf}}_{\underline{v}}$$

$$\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \overset{\mathrm{def}}{=} \text{“}(\cdots, j^2, \cdots) \cdot \mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})\text{”} \subseteq \prod_{j \in \mathbb{F}^{*}_{l}} \mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})_j$$

$$\Rightarrow \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}),\underline{v}} \xrightarrow{\sim} \Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ})^{\mathrm{rlf}}_{\underline{v}}$$

Moreover: $\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \xrightarrow{\sim} \mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$: <u>evaluation isomorphism</u>

$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathfrak{F}_{\succ} \overset{{}^{\dagger}\psi^{\Theta^{\pm}}_{\pm}}{\leftarrow} {}^{\dagger}\mathfrak{F}_{T} \overset{{}^{\dagger}\psi^{\Theta^{\mathrm{ell}}}_{\pm}}{\rightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$: a $\Theta^{\pm\mathrm{ell}}$-Hodge theater

$({}^{\dagger}\mathfrak{F}_{J} \overset{{}^{\dagger}\psi^{\Theta}_{*}}{\rightarrow} {}^{\dagger}\mathfrak{F}_{>} \dashrightarrow {}^{\dagger}\mathcal{HT}^{\Theta})$: a Θ-bridge glued to ${}^{\dagger}\psi^{\Theta^{\pm}}_{\pm}$

Recall: ${}^{\dagger}\mathfrak{F}_{>} \underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} {}^{\dagger}\mathfrak{D}^{\vdash}_{>}$, ${}^{\dagger}\mathfrak{F}^{\Vdash}_{>} = ({}^{\dagger}\mathcal{C}^{\Vdash}_{>}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{>}) \overset{\sim}{\to} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash}_{>}, \{{}^{\dagger}\rho_{\underline{v}}\})$

$$({}^{\dagger}\mathcal{C}^{\Vdash}_{>}, \mathrm{Prime} \overset{\sim}{\to} \underline{\mathbb{V}}, \{{}^{\dagger}\rho_{\underline{v}}\}) \overset{\sim}{\to} (\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{>}), \mathrm{Prime} \overset{\sim}{\to} \underline{\mathbb{V}}, \{{}^{\dagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\})$$

$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \overset{\mathrm{def}}{=} \text{“}{}^{\dagger}\mathcal{C}^{\Vdash}_{>} \cdot \log^{{}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}}(\underline{\underline{\Theta}})\text{”}$

$\Rightarrow \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \overset{\sim}{\to} \Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathrm{rlf}}_{\underline{v}}$

$\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}) \overset{\mathrm{def}}{=} \text{“}(\cdots, j^2, \cdots) \cdot {}^{\dagger}\mathcal{C}^{\Vdash}_{>}\text{”} \subseteq \prod_{j \in \mathbb{F}^{*}_{l}} ({}^{\dagger}\mathcal{C}^{\Vdash}_{>})_j$

$\Rightarrow \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \overset{\sim}{\to} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathrm{rlf}}_{\underline{v}}$

Moreover: $\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \overset{\sim}{\to} \mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>}) \overset{\sim}{\to} \mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>}) \overset{\sim}{\to} \mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$

$${}^{\dagger}\mathfrak{F}^{\Vdash} = ({}^{\dagger}\mathcal{C}^{\Vdash}_{>}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{>}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash}_{>}, \{{}^{\dagger}\rho_{\underline{v}}\})$$

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \stackrel{\mathrm{def}}{=} \text{“}{}^{\dagger}\mathcal{C}^{\Vdash}_{>} \cdot \log^{{}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}}(\underline{\underline{\Theta}})\text{”}$$

$$\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}) \stackrel{\mathrm{def}}{=} \text{“}(\cdots, j^2, \cdots) \cdot {}^{\dagger}\mathcal{C}^{\Vdash}_{>}\text{”} \subseteq \prod_{j \in \mathbb{F}^{\circledast}_{l}} ({}^{\dagger}\mathcal{C}^{\Vdash}_{>})_j$$

By rep. $({}^{\dagger}\mathfrak{F}^{\vdash}_{>}, \{{}^{\dagger}\rho_{\underline{v}}\})$ by a suitable data det'd by $\{\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}\}$:

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}), \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}}, \{{}^{\dagger}\rho_{\mathrm{env},\underline{v}}\})$$

By rep. $({}^{\dagger}\mathfrak{F}^{\vdash}_{>}, \{{}^{\dagger}\rho_{\underline{v}}\})$ by a suitable data det'd by $\{\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}\}$:

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}), \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{gau}}, \{{}^{\dagger}\rho_{\mathrm{gau},\underline{v}}\})$$

$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_0 = \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_{\langle \mathbb{F}^{\circledast}_{l} \rangle}$ determines:

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\Delta} = ({}^{\dagger}\mathcal{C}^{\Vdash}_{\Delta}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\Delta}) \xrightarrow{\sim} \underline{\mathbb{V}}, {}^{\dagger}\mathfrak{F}^{\vdash}_{\Delta}, \{{}^{\dagger}\rho_{\Delta,\underline{v}}\})$$

$\Rightarrow {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$: eval. isom., ${}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\Delta} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\mathrm{env}} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\mathrm{gau}}$

${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater

Recall: $\underset{\text{alg'm}}{\overset{\exists\text{func'l}}{\Rightarrow}} {}^{\dagger}\zeta_{\circledast} \colon \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}^{\circledcirc}) \xrightarrow{\sim} J$, $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_{j}$

$\mathbb{F}_{l}^{\circledast}$-symmetry $\Rightarrow$ For j, $j' \in J$:

$$\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_{j} \xrightarrow{\sim} \mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_{j'}$$

$$\mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j} \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j'}$$

$$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j} \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j'}$$

$$(\pi_{1}^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}))_{j} \xrightarrow{\sim} (\pi_{1}^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}))_{j'}$$

$$(\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j} \to \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j}) \xrightarrow{\sim} (\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j'} \to \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j'})$$

$\Rightarrow$ One can define the "diagonal" $(-)_{\langle\mathbb{F}_{l}^{\circledast}\rangle} \subseteq \prod_{j\in\mathbb{F}_{l}^{\circledast}}(-)_{j}$

Recall: For $j \in J$:

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \to \mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j$$

$$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \to (\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}}$$

$$(\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}))_j \to \mathbb{M}^{\circledast}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{j,\underline{v}}) \subseteq \mathbb{M}^{\circledast}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{j,\underline{v}})$$

(where ${}^{\dagger}\mathcal{D}_{j,\underline{v}}$: the $\underline{v}$-comp. of the $\mathcal{D}$-prime-strip ass'd to $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j$)

$$\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_j) \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

(where ${}^{\dagger}\mathfrak{D}^{\vdash}_j$: the $\mathcal{D}^{\vdash}$-prime-strip ass'd to $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j$)

$$\Psi^{\mathbb{R}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}^{\vdash}_j)_{\underline{v}} \xrightarrow{\sim} \Psi_{(\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}},\underline{v}}$$

They are compatible with the $\mathbb{F}_l^{\divideontimes}$-symmetry.

${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: a $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater

Recall: For $j \in J$:

$${}^{\dagger}\mathfrak{F}_j \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledcirc}|_j \xrightarrow{\sim} \mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j$$

$$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

$$(\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})_j \xrightarrow{\sim} (\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}))_j$$

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

$$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

(i.e., Kummer isomorphisms)

They admit compatibilities with the $\mathbb{F}_l^{\divideontimes}$-symmetry.

$\Rightarrow$ One can define the "diagonal" $(-)_{\langle\mathbb{F}_l^{\divideontimes}\rangle} \subseteq \prod_{j\in\mathbb{F}_l^{\divideontimes}}(-)_j$

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}_j$$

$$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_j$$

$$\Big((\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})_j \to {}^{\dagger}\mathbb{M}_{\infty\kappa v_j} \subseteq {}^{\dagger}\mathbb{M}_{\infty\kappa\times v_j} \Big)_{\underline{v}\in\underline{\mathbb{V}}}$$

Moreover:

$$ {}^{\dagger}\mathcal{C}^{\Vdash}_j \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j$$

They admit compatibilities with the $\mathbb{F}_l^{\circledast}$-symmetry.

$$\Rightarrow \mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}) \hookrightarrow \prod_{j\in\mathbb{F}_l^{\circledast}} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j$$

$$(\mathfrak{F}^{\vdash}_{\mathrm{gau}})^{\mathbb{R}} \hookrightarrow \prod_{j\in\mathbb{F}_l^{\circledast}} {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_j$$

$$\Rightarrow \mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}) \to (\mathfrak{F}^{\vdash}_{\mathrm{gau}})^{\mathbb{R}}$$

(which is compatible with $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \xrightarrow{\sim} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathrm{rlf}}_{\underline{v}}$)

${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theaters

the $\underline{\Theta^{\times\boldsymbol{\mu}}\text{-link}}$ ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}}{\to} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$

$\overset{\mathrm{def}}{\Leftrightarrow}$ the full poly-isomorphism ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\Delta}$

the $\underline{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}\text{-link}}$ ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\to} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$

$\overset{\mathrm{def}}{\Leftrightarrow}$ the full poly-isomorphism ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\Delta}$

Thus: $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links induce the full poly-isom. ${}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta}$,

as well as the full poly-isom. ${}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta}$,

i.e., the $\underline{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}\text{-link}}$ ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathcal{D}}{\to} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$

$\Rightarrow \mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}) \overset{\sim}{\to} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\Delta})$ (comp. w/ "$\mathrm{Prime} \overset{\sim}{\to} \underline{\mathbb{V}}$", "$\{\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}$")

Frobenius-picture

The infinite chain

$$\cdots \overset{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}}{\longrightarrow} {}_{n-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}}{\longrightarrow} {}_{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}}{\longrightarrow} {}_{n+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}}{\longrightarrow} \cdots$$

does not admit arbitrary permutation symmetries.

Étale-picture

The $\mathcal{D}$-prime-strip "${}^{(-)}\mathfrak{D}^{\vdash}_{\Delta}$" forms a "constant invariant" of the above infinite chain:

$$\begin{array}{ccccc} \cdots & & {}_{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & & \cdots \\ & & \downarrow & & \\ {}_{n-1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \longrightarrow & {}^{(-)}\mathfrak{D}^{\vdash}_{\Delta} & \longleftarrow & {}_{n+1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \end{array}$$