[IUTch-III-IV] from the Point of View of Mono-anabelian Transport I

— Log-theta-lattices —

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§1.1 Mono-anabelian Transport

Recall: By the previous talks:

$$\begin{array}{ccc}
{}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \\
\mathrm{Im}(F^{\times}_{\mathrm{mod}}) & \xrightarrow{\sim} & \mathrm{Im}(F^{\times}_{\mathrm{mod}}) \\
\cap & & \cap \\
\prod_{\underline{v} \in \underline{\mathbb{V}}} (1^2, \ldots, (l^{*})^2) \cdot \mathbb{R}_{\underline{v}} & \xrightarrow{\sim} & \prod_{\underline{v} \in \underline{\mathbb{V}}} \mathbb{R}_{\underline{v}} \\
\{ \big(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}}, \Delta} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^{*})^2})^{\mathbb{N}} \big) / \boldsymbol{\mu} \}_{\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}} & \xrightarrow{\sim} & \{ \big(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}}, \Delta} \cdot \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}, \Delta} \big) / \boldsymbol{\mu} \}_{\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}}
\end{array}$$

Goal: A comparison between

$$\deg((\underline{\underline{q}}_{\underline{v}})_{\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}}) \quad \text{and} \quad \left(\frac{1}{l^{*}} \cdot \right) \deg((\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^{*})^2})_{\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}})$$

$\Theta_{\mathrm{gau}}^{\times\boldsymbol{\mu}}$: a non-scheme theoretic "connection" bet'n two Hdg-theaters

Question: How can one relate objects of the domain of a non-scheme theoretic "connection" to objects of the codomain of a non-scheme theoretic "connection"?

One of the answers: Mono-anabelian Transport

Mono-anabelian Transport

"A": an object of interest Then, in the diagram

$$
\begin{array}{ccc}
{}^{\dagger}\square & \xrightarrow[\text{connection}]{\text{non-scheme theoretic}} & {}^{\ddagger}\square \\
\text{Frobenius-like } {}^{\dagger}A & & \text{Frobenius-like } {}^{\ddagger}A \\
\text{Kummer} \big\downarrow \text{isomorphism} & & \text{Kummer} \big\downarrow \text{isomorphism} \\
\text{étale-like } {}^{\dagger}A & \xrightarrow[\text{rigidity}]{\text{mono-anabelian}} & \text{étale-like } {}^{\ddagger}A,
\end{array}
$$

we compute the "indeterminacies" which involves

- the [two] "$\downarrow$", i.e., Kummer-detachment indeterminacies, and
- the "$\rightarrow$", i.e., étale-transport indeterminacies.

$\Rightarrow$ "A" is compatible, rel. to the conn., up to the indeterminacies.

[Thus, we have to distinguish the above four "A".]

Recall: In [IUTchII]:

Let us take "A" to be the theta monoid "$\mathcal{O}^{\times}_{K_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}$" [$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$].

$$\begin{array}{ccc}
\text{Frobenius-like } {}^{\dagger}(\mathcal{O}^{\times}_{K_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}) & & \text{Frobenius-like } {}^{\ddagger}(\mathcal{O}^{\times}_{K_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}) \\
\text{Kummer} \downarrow \text{isomorphism} & & \text{Kummer} \downarrow \text{isomorphism} \\
\text{étale-like } {}^{\dagger}(\mathcal{O}^{\times}_{K_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}) & \xrightarrow[\text{rigidity}]{\text{mono-anabelian}} & \text{étale-like } {}^{\ddagger}(\mathcal{O}^{\times}_{K_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}})
\end{array}$$

- the [two] "$\downarrow$" arise from the [multiradial] cyclotomic rigidity, and
- the "$\rightarrow$" arises from the [multi.] rep. of the env. ver. of theta mon.

On the coric object $G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{K_{\underline{v}}}$:

- the [two] "$\downarrow$" induce the indeterminacies $\mathrm{Ism}_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{K_{\underline{v}}}$, and
- the "$\rightarrow$" induces the indeterminacies $\mathrm{Aut}(G_{\underline{v}}) \curvearrowright G_{\underline{v}}$.

Recall: By the previous talks:

$$\begin{array}{ccc}
{}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \\
\mathrm{Im}(F^{\times}_{\mathrm{mod}}) & \xrightarrow{\sim} & \mathrm{Im}(F^{\times}_{\mathrm{mod}}) \\
\cap & & \cap \\
\prod_{\underline{v}\in\underline{\mathbb{V}}} (1^2, \ldots, (l^{*})^2) \cdot \mathbb{R}_{\underline{v}} & \xrightarrow{\sim} & \prod_{\underline{v}\in\underline{\mathbb{V}}} \mathbb{R}_{\underline{v}} \\
\{(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^{*})^2})^{\mathbb{N}})/\boldsymbol{\mu}\}_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}} & \xrightarrow{\sim} & \{(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot \underline{\underline{q}}^{\mathbb{N}}_{\underline{v},\Delta})/\boldsymbol{\mu}\}_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}}
\end{array}$$

Goal: A comparison between

$$\deg((\underline{\underline{q}}_{\underline{v}})_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}}) \quad \text{and} \quad \left(\frac{1}{l^{*}} \cdot \right) \deg((\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^{*})^2})_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}})$$

Of course...:

$$\deg((\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})_{\underline{v}}) = \sum_{j=1}^{l^*} j^2 \cdot \deg((\underline{\underline{q}}_{\underline{v}})_{\underline{v}})$$

if one applies only the usual relationship bet'n $\underline{\underline{q}}_{\underline{v}}$ and $(\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})$, i.e., "$\underline{\underline{q}}_{\underline{v}}^{j^2}$ = just the j^2-th power of $\underline{\underline{q}}_{\underline{v}}$".

For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and $1 \leq j \leq l^*$, the $\Theta_{\mathrm{gau}}^{\times \boldsymbol{\mu}}$-link relates $\underline{\underline{q}}_{\underline{v}}^{j^2}$ and $\underline{\underline{q}}_{\underline{v}}$ by

$$\begin{array}{ccccccc} \left(\mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}}\right)/\boldsymbol{\mu} & \twoheadrightarrow & \left(\mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}}\right)/\mathcal{O}_{K_{\underline{v}}}^{\times} & \stackrel{\sim}{\to} & (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} & \hookrightarrow & (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{R}_{\geq 0}} \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \left(\mathcal{O}_{K_{\underline{v}}}^{\times} \cdot \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}\right)/\boldsymbol{\mu} & \twoheadrightarrow & \left(\mathcal{O}_{K_{\underline{v}}}^{\times} \cdot \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}\right)/\mathcal{O}_{K_{\underline{v}}}^{\times} & \stackrel{\sim}{\to} & \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}} & \hookrightarrow & \underline{\underline{q}}_{\underline{v}}^{\mathbb{R}_{\geq 0}}. \end{array}$$

In order to obtain another comparison between $\underline{\underline{q}}_{\underline{v}}^{j^2}$ and $\underline{\underline{q}}_{\underline{v}}$, let us

- consider the **noncommutative** [if $j \neq 1$] diagram

$$
\begin{array}{ccccccccc}
\mathbb{N} & \overset{\sim}{\to} & (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} & \hookrightarrow & \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} & \hookrightarrow & \mathcal{O}_{K_{\underline{v}}}^{\triangleright} & \curvearrowright & (\mathcal{O}_{K_{\underline{v}}})_{+} \\
\wr\downarrow & & & & & & & & \wr\downarrow \\
\mathbb{N} & \overset{\sim}{\to} & \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}} & \hookrightarrow & \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}} & \hookrightarrow & \mathcal{O}_{K_{\underline{v}}}^{\triangleright} & \curvearrowright & (\mathcal{O}_{K_{\underline{v}}})_{+},
\end{array}
$$

- compute **indeterminacies** which make the diag. "commutative"

by applying the technique of mono-anabelian transport, and

- conclude that

"the vol. of $\underline{\underline{q}}_{\underline{v}}^{j^2}\mathcal{O}_{K_{\underline{v}}}$ = the vol. of $\underline{\underline{q}}_{\underline{v}}\mathcal{O}_{K_{\underline{v}}}$" up to **indeterminacies**.

In particular, we want to discuss a multiradial representation of objects related to the diagram

$$\mathbb{N} \overset{\sim}{\to} (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} \hookrightarrow \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} \hookrightarrow \mathcal{O}_{K_{\underline{v}}}^{\rhd} \curvearrowright (\mathcal{O}_{K_{\underline{v}}})_+,$$

relative to the diagram

$$\begin{array}{ccccccccc} \mathbb{N} & \overset{\sim}{\to} & (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} & \hookrightarrow & \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} & \hookrightarrow & \mathcal{O}_{K_{\underline{v}}}^{\rhd} & \curvearrowright & (\mathcal{O}_{K_{\underline{v}}})_+ \\ \wr\downarrow & & & & & & & & \wr\downarrow \\ \mathbb{N} & \overset{\sim}{\to} & \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}} & \hookrightarrow & \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}} & \hookrightarrow & \mathcal{O}_{K_{\underline{v}}}^{\rhd} & \curvearrowright & (\mathcal{O}_{K_{\underline{v}}})_+ . \end{array}$$

Four problems in the diagram

$$\underbrace{(\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}}}_{(1)} \underline{\hookrightarrow}_{(2)} \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} \hookrightarrow \mathcal{O}_{K_{\underline{v}}}^{\triangleright} \underline{\curvearrowright}_{(4)} \underbrace{(\mathcal{O}_{K_{\underline{v}}})_{+}}_{(3)}$$

(1) How can one obtain a multiradial $\underline{\underline{q}}_{\underline{v}}^{j^2}$?

(2) How can one obtain such a multiradial inclusion,

i.e., a multiradial splitting $\mathcal{O}_{K_{\underline{v}}}^{\times} \times (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} \xrightarrow{\sim} \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}}$?

[Recall: For instance, every object related to $\Theta_{\mathrm{gau}}^{\times\boldsymbol{\mu}}$ is "multiplicative".]

(3) How can one obtain an "additive" object $(\mathcal{O}_{K_{\underline{v}}})_{+}$?

(4) How can one relate such multi. objects to such add. objects?

Answers:

(1) We regard $\underline{\underline{q}}_{\underline{v}}^{j^2}$ as a result of applying the Galois evaluation operator to the **multiradial** theta function $\underline{\underline{\Theta}}_{\underline{v}}$ [cf. [IUTchII]], i.e., roughly speaking, obtained by the cyclotomic rigidity. §3

(2) We regard such a splitting as a result [cf. (1)] of the **multiradial** splitting of $\mathcal{O}^{\times}_{K_{\underline{v}}} \cdot \underline{\underline{\Theta}}_{\underline{v}}$ [cf. [IUTchII]], i.e., roughly speaking, obtained by the constant multiple rigidity. §3

(3) We use the p-adic logarithm. [Note: The natural homomorphism $\mathcal{O}^{\times}_{K_{\underline{v}}} \to (\mathcal{O}^{\times}_{K_{\underline{v}}})^{\mathrm{pf}}$ is "isom." to the p-adic log $\mathcal{O}^{\times}_{K_{\underline{v}}} \overset{\log}{\to} (K_{\underline{v}})_{+}$.] §1

(4) We use $\mathfrak{log}$-links §1 and **multiradial** [realified] NFs §2.

That is to say, we regard

$$(\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} \hookrightarrow \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} \hookrightarrow \mathcal{O}_{K_{\underline{v}}}^{\rhd} \curvearrowright (\mathcal{O}_{K_{\underline{v}}})_{+}$$

as

$$\underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{N}} \hookrightarrow \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{N}} \underset{\text{eval.}}{\overset{\text{Gal.}}{\stackrel{\sim}{\to}}} \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot (\underline{\underline{q}}_{\underline{v}}^{j^2})^{\mathbb{N}} \hookrightarrow \mathcal{O}_{K_{\underline{v}}}^{\rhd} \curvearrowright \frac{1}{2p}\log(\mathcal{O}_{K_{\underline{v}}}^{\times}) \, (\subseteq (K_{\underline{v}})_{+}),$$

where "$\underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{N}} \hookrightarrow \mathcal{O}_{K_{\underline{v}}}^{\times} \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{N}}$" is given by the multiradial cyclotomic rigidity and splitting discussed in [IUTchII].

§1.2 Log-shells

${}^\dagger\mathfrak{F} = \{{}^\dagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: an $\mathcal{F}$-prime-strip

$\Rightarrow\ \Psi_{\mathrm{cns}}({}^\dagger\mathfrak{F}) = \{\Psi_{\mathrm{cns}}({}^\dagger\mathfrak{F})_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$

$\Rightarrow\ \Psi_{\mathrm{cns}}({}^\dagger\mathfrak{F})_{\underline{v}} = (G_{\underline{v}}({}^\dagger\Pi_{\underline{v}}) \curvearrowright \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}})$ [an isomorph of $\Pi_{\underline{v}}/\Delta_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}}}$]

$\Rightarrow\ \Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \stackrel{\mathrm{def}}{=} (\Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}})^{\mathrm{pf}}$

By the étale-like holomorphic structure ${}^\dagger\Pi_{\underline{v}}$,

- one may obtain field str. on $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}} \cup \{0\}$, $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ w.r.t. which
- $(\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}})$ "$\cong$" $(\mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}}} \supseteq \mathcal{O}^{\times}_{\overline{K}_{\underline{v}}} \stackrel{\log}{\twoheadrightarrow} (\overline{K}_{\underline{v}})_+)$.

- $\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \stackrel{\text{def}}{=} (G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}}) \curvearrowright \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})$ [an isomorph of $\Pi_{\underline{v}}/\Delta_{\underline{v}} \curvearrowright \overline{K}_{\underline{v}}$]
- $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \stackrel{\text{def}}{=} \frac{1}{2p_{\underline{v}}} \cdot \mathrm{Im}((\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})^{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})} \hookrightarrow \Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow (\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})_{+})$: the log-shell

  [an isomorph of $\frac{1}{2p_{\underline{v}}} \cdot \mathrm{Im}(\mathcal{O}^{\times}_{K_{\underline{v}}} \hookrightarrow \mathcal{O}^{\times}_{\overline{K}_{\underline{v}}} \stackrel{\log}{\twoheadrightarrow} (\overline{K}_{\underline{v}})_{+})$]

One may construct similar objects for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

- $\underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}) \stackrel{\text{def}}{=} \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$
- $\mathcal{I}_{{}^{\dagger}\mathfrak{F}} \stackrel{\text{def}}{=} \{\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \subseteq \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F})$, i.e., Frobenius-like holom'c log-shells

Then:

- $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$: compact [$\Rightarrow$ of finite (radial for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$) log-vol.]
- $\mathrm{Im}((\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})^{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})} \to (\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})_{+})$, $\mathcal{O}^{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})}_{\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}} \subseteq \mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ [$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$]

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \quad \Rightarrow$

For a cpt open subset $A \neq \emptyset$ of $\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})^{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})}$ [an isom'h of $K_{\underline{v}}$],

one may define the volume $\mu(A)$ of A which satisfies the following:

- $\mu(\mathcal{O}^{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})}_{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})}) = 1$ [i.e., "$\mu(\mathcal{O}_{K_{\underline{v}}}) = 1$"]
- $A_1 \cap A_2 = \emptyset \Rightarrow \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$
- $a \in \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})^{G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})} \Rightarrow \mu(A + a) = \mu(A)$

[$\Rightarrow$ "$\mu(\frac{1}{2p_{\underline{v}}}\log(\mathcal{O}^{\times}_{K_{\underline{v}}})) = p_{\underline{v}}^{|2|^{-1}_{p_{\underline{v}}}\cdot[K_{\underline{v}}:\mathbb{Q}_{p_{\underline{v}}}]} \cdot \sharp\kappa_{\underline{v}}^{-1} \cdot \sharp\boldsymbol{\mu}_{p^{\infty}_{\underline{v}}}(K_{\underline{v}})^{-1}$"]

$\mu^{\log}(A) \overset{\mathrm{def}}{=} \log(\mu(A))$: the log-volume of A

$\exists$similar notions, i.e., radial/angular log-volumes, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$

${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: a $\mathcal{D}$-prime-strip

$\Rightarrow \mathfrak{F}({}^{\dagger}\mathfrak{D})$: an $\mathcal{F}$-prime-strip

$\Rightarrow$

- $\underline{\mathfrak{log}}({}^{\dagger}\mathfrak{D}) \stackrel{\mathrm{def}}{=} \underline{\mathfrak{log}}(\mathfrak{F}({}^{\dagger}\mathfrak{D}))$
- $\mathcal{I}_{{}^{\dagger}\mathfrak{D}} \stackrel{\mathrm{def}}{=} \mathcal{I}_{\mathfrak{F}({}^{\dagger}\mathfrak{D})}$, i.e., étale-like holomorphic log-shells

${}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}} = \{{}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: an $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$

$\Rightarrow {}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$ "$\cong$" $(G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{K}_{\underline{v}}})$ equipped with

the $\times\boldsymbol{\mu}$-Kummer structure, i.e., $\{\mathrm{Im}((\mathcal{O}^{\times}_{\overline{K}_{\underline{v}}})^{H} \to \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{K}_{\underline{v}}})\}_{H\subseteq G_{\underline{v}}:\text{open}}$

$\Rightarrow \mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \curvearrowleft {}^{\dagger}G_{\underline{v}}$,

i.e., "$\frac{1}{2p_{\underline{v}}}\mathrm{Im}((\mathcal{O}^{\times}_{\overline{K}_{\underline{v}}})^{G_{\underline{v}}} \to \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{K}_{\underline{v}}}) \subseteq \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{K}_{\underline{v}}} \curvearrowleft G_{\underline{v}}$"

One may construct similar objects for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

- $\underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}) \stackrel{\mathrm{def}}{=} \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$
- $\mathcal{I}_{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}} \stackrel{\mathrm{def}}{=} \{\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}}\} \subseteq \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}})$, i.e., Frob.-like mono-an. log-sh.

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \quad \Rightarrow \quad$ For a compact open subset $A \neq \emptyset$ of $\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}^{\vdash\times\boldsymbol{\mu}})^{{}^{\dagger}G_{\underline{v}}}$ [an isomorph of $(K_{\underline{v}})_{+}$], one may define the volume $\mu(A)$ of A which satisfies the following:

- $\mu(\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}^{\vdash\times\boldsymbol{\mu}}}) =$ "$p_{\underline{v}}^{|2|_{p_{\underline{v}}}^{-1}\cdot[K_{\underline{v}}:\mathbb{Q}_{p_{\underline{v}}}]} \cdot \sharp\kappa_{\underline{v}}^{-1} \cdot \sharp\boldsymbol{\mu}_{p_{\underline{v}}^{\infty}}(K_{\underline{v}})^{-1}$"
  [which may be constructed from the data ${}^{\dagger}\mathcal{F}_{\underline{v}}^{\vdash\times\boldsymbol{\mu}}$]

- $A_1 \cap A_2 = \emptyset \Rightarrow \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$

- $a \in \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}^{\vdash\times\boldsymbol{\mu}})^{{}^{\dagger}G_{\underline{v}}} \Rightarrow \mu(A + a) = \mu(A)$

$\mu^{\log}(A) \stackrel{\mathrm{def}}{=} \log(\mu(A))$: the log-volume of A

$\exists$similar notions, i.e., radial/angular log-volumes, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$

$^{\dagger}\mathfrak{D}^{\vdash} = \{^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: a $\mathcal{D}^{\vdash}$-prime-strip

$\Rightarrow \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}(^{\dagger}\mathfrak{D}^{\vdash})$: an $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip

$\Rightarrow$

- $\underline{\mathfrak{log}}(^{\dagger}\mathfrak{D}^{\vdash}) \stackrel{\mathrm{def}}{=} \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}(^{\dagger}\mathfrak{D}^{\vdash}))$
- $\mathcal{I}_{^{\dagger}\mathfrak{D}^{\vdash}} \stackrel{\mathrm{def}}{=} \mathcal{I}_{\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}(^{\dagger}\mathfrak{D}^{\vdash})}$, i.e., étale-like mono-analytic log-shells

If

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{F} & \Rightarrow & {}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}} \\ \Downarrow & & \Downarrow \\ {}^{\dagger}\mathfrak{D} & \Rightarrow & {}^{\dagger}\mathfrak{D}^{\vdash} \end{array} \quad \text{i.e.,} \quad \begin{array}{ccc} \Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{K}_{\underline{v}}} & \Rightarrow & G_{\underline{v}} \curvearrowright \mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{K}_{\underline{v}}} \\ \Downarrow & & \Downarrow \\ \Pi_{\underline{v}} & \Rightarrow & G_{\underline{v}}, \end{array}$$

then

$$\begin{array}{ccc} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}) & \xrightarrow{\sim} & \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}) \\ \wr \downarrow & & \downarrow \wr \curvearrowleft (\mathrm{Ind2}) \\ \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{D}) & \xrightarrow[\sim]{} & \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{D}^{\vdash}) \end{array} \qquad \begin{array}{ccc} \mathcal{I}_{{}^{\dagger}\mathfrak{F}} & \xrightarrow{\sim} & \mathcal{I}_{{}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}} \\ \wr \downarrow & & \downarrow \wr \curvearrowleft (\mathrm{Ind2}) \\ \mathcal{I}_{{}^{\dagger}\mathfrak{D}} & \xrightarrow[\sim]{} & \mathcal{I}_{\mathfrak{D}^{\vdash}}, \end{array}$$

where $(\mathrm{Ind2}) = \mathrm{Ism}_{\underline{v}}$ $(= \mathrm{Aut}^{\times \boldsymbol{\mu}\text{-Kmm}}_{G_{\underline{v}}}(\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{K}_{\underline{v}}}))$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$;

$= \{\pm 1\}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

[Various data (e.g., log-volumes) are compatible w.r.t. the diagrams.]

§1.3 Local Tensor Packets

$\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in A} = \{\{{}^{\alpha}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\}_{\alpha\in A}$: a capsule of $\mathcal{F}$-prime-strips

$\alpha \in A$, $\underline{\mathbb{V}}^{\mathrm{non}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} \overset{\mathrm{def}}{=} \mathbb{V}(\mathbb{Q})$

- $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \overset{\mathrm{def}}{=} \bigoplus_{\underline{\mathbb{V}}\ni\underline{w}|v_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{w}})$
- $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \overset{\mathrm{def}}{=} \bigotimes_{\beta\in A} \underline{\mathfrak{log}}({}^{\beta}\mathcal{F}_{v_{\mathbb{Q}}})$:

  the local homolorphic tensor packet [associated to $\{{}^{\alpha}\mathfrak{F}\}_{\alpha}$]
- $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) \overset{\mathrm{def}}{=} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \otimes \underline{\mathfrak{log}}({}^{A\setminus\{\alpha\}}\mathcal{F}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$
- $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \overset{\mathrm{def}}{=} \prod_{w_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{w_{\mathbb{Q}}})$

Then:

The topological field structure of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{w}})$

$\Rightarrow$

- $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) \cong \varinjlim(\bigoplus(\varinjlim(\text{MLF or CAF})))$
- $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \hookrightarrow \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$

By replacing $\underline{\mathfrak{log}}({}^{(-)}\mathcal{F}_{(-)})$ by $\mathcal{I}_{{}^{(-)}\mathcal{F}_{(-)}}$, one obtains:

- $\mathcal{I}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$
- $\mathcal{I}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$
- $\mathcal{I}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$
- $\mathcal{I}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$

- $\mathcal{I}^{\mathbb{Q}}({}^{(-)}\mathcal{F}_{(-)}) \subseteq \underline{\mathfrak{log}}({}^{(-)}\mathcal{F}_{(-)})$: the $\mathbb{Q}$-span of $\mathcal{I}({}^{(-)}\mathcal{F}_{(-)})$

The integral structure $\mathcal{O}_{\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{w}})} \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{w}})$

$\Rightarrow$ integral structures $\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$,

$\mathcal{O}_{{}^{A}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{O}_{{}^{A,\alpha}\mathcal{F}_{\underline{v}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$

One may construct similar objects for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$

$\{0\} \neq {}^{\alpha}U \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$: a subset whose closure is compact

the holomorphic hull of ${}^{\alpha}U \overset{\mathrm{def}}{\Leftrightarrow}$ the smallest subset of the form

$$\lambda \cdot \mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$$

containing ${}^{\alpha}U$, where $\lambda \in \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$ is contained in $\bigoplus p^{\mathbb{Z}}$, relative to the natural decomposition $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \cong \bigoplus \mathrm{MLF}$.

In a similar vein, one may define the notion of holomorphic hull of a subset of $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ whose closure is compact.

One may also define the notion of holomorphic hull for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

$\{{}^{\alpha}\mathfrak{D}\}_{\alpha \in A} = \big\{\{{}^{\alpha}\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}\big\}_{\alpha \in A}$: a capsule of $\mathcal{D}$-prime-strips

$\Rightarrow \{{}^{\alpha}\mathfrak{F}(\mathfrak{D})\}_{\alpha \in A}$: a capsule of $\mathcal{F}$-prime-strips

$\Rightarrow$ For $\alpha \in A$, $\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$:

- $\mathcal{I}({}^{\alpha}\mathcal{D}_{v_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{D}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}_{v_{\mathbb{Q}}})$
- $\mathcal{I}({}^{A}\mathcal{D}_{v_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{D}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{D}_{v_{\mathbb{Q}}})$
- $\mathcal{I}({}^{A,\alpha}\mathcal{D}_{\underline{v}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{D}_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}_{\underline{v}})$
- $\mathcal{I}({}^{A}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}})$

$\{{}^{\alpha}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}\}_{\alpha\in A} = \{\{{}^{\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\}_{\alpha\in A}$: a capsule of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips

$\alpha \in A$, $\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$

By applying similar constructions, one obtains:

- $\mathcal{I}({}^{\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}})$
- $\mathcal{I}({}^{A}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}})$:

  the local mono-analytic tensor packet [associated to $\{{}^{\alpha}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}\}_{\alpha}$]
- $\mathcal{I}({}^{A,\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})$
- $\mathcal{I}({}^{A}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{V}_{\mathbb{Q}}})$

$\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha \in A} = \{\{{}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}\}_{\alpha \in A}$: a capsule of $\mathcal{D}^{\vdash}$-prime-strips

$\Rightarrow \{{}^{\alpha}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}(\mathfrak{D}^{\vdash})\}_{\alpha \in A}$: a capsule of $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$-prime-strips

$\Rightarrow$ For $\alpha \in A$, $\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$:

- $\mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$
- $\mathcal{I}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$
- $\mathcal{I}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$
- $\mathcal{I}({}^{A}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}})$

If

$$\begin{array}{ccc} \{{}^{\alpha}\mathfrak{F}\}_{\alpha \in A} & \Rightarrow & \{{}^{\alpha}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}\}_{\alpha \in A} \\ \Downarrow & & \Downarrow \\ \{{}^{\alpha}\mathfrak{D}\}_{\alpha \in A} & \Rightarrow & \{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha \in A}, \end{array}$$

then

$$\begin{array}{ccc} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) & \xrightarrow{\sim} & \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}^{\vdash \times \boldsymbol{\mu}}) \\ \wr\downarrow & & \downarrow\wr \curvearrowleft (\mathrm{Ind2}) \\ \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}_{v_{\mathbb{Q}}}) & \underset{\sim}{\longrightarrow} & \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}_{v_{\mathbb{Q}}}^{\vdash}) \end{array}$$

similar diagrams...,

where $(\mathrm{Ind2}) = \mathrm{Ism}_{\underline{v}}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$; $\{\pm 1\}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

[Various data (e.g., log-volumes) are compatible w.r.t. the diagrams.]

§1.4 Log-theta-lattices

${}^{\dagger}\mathfrak{F} = \{{}^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: an $\mathcal{F}$-prime-strip

$\Rightarrow \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}) = \{\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F})_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$

$\Rightarrow \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F})_{\underline{v}} = (G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}}) \curvearrowright \Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}})$ [an isomorph of $\Pi_{\underline{v}}/\Delta_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{K}_{\underline{v}}}$]

$\Rightarrow \Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \stackrel{\mathrm{def}}{=} (\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})^{\mathrm{pf}}$

By the étale-like holomorphic structure ${}^{\dagger}\Pi_{\underline{v}}$,

- one may obtain field str. on $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \cup \{0\}$, $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ w.r.t. which
- $(\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})$ "$\cong$" $(\mathcal{O}^{\rhd}_{\overline{K}_{\underline{v}}} \supseteq \mathcal{O}^{\times}_{\overline{K}_{\underline{v}}} \stackrel{\log}{\twoheadrightarrow} (\overline{K}_{\underline{v}})_{+})$.

- $\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})} \subseteq \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$:

  the multiplicative monoid of nonzero integers [i.e., "$\mathcal{O}^{\rhd}_{\overline{K}_{\underline{v}}} \subseteq \overline{K}_{\underline{v}}$"]

- $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$ " $\overset{\text{def}}{=}$ " $({}^\dagger\Pi_{\underline{v}} \curvearrowright \Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})})$ [i.e., "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{K}_{\underline{v}}}$"]

One may construct similar objects for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

$\mathfrak{log}({}^\dagger\mathfrak{F}) \overset{\text{def}}{=} \{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$, i.e., a new $\mathcal{F}$-prime-strip

${}^{\dagger}\mathfrak{F}$, ${}^{\ddagger}\mathfrak{F}$: $\mathcal{F}$-prime-strips

${}^{\dagger}\mathfrak{F} \overset{\mathfrak{log}}{\to} {}^{\ddagger}\mathfrak{F}$: a $\mathfrak{log}$-link

$\overset{\mathrm{def}}{\Leftrightarrow}$ a poly-isomorphism $\mathfrak{log}({}^{\dagger}\mathfrak{F}) \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}$

[i.e., for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, $\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow (\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})_{+} \supseteq \Psi_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})} \overset{\sim}{\to} \Psi_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}$]

- ${}^{\dagger}\mathfrak{F} \overset{\mathfrak{log}}{\to} {}^{\ddagger}\mathfrak{F} \Rightarrow {}^{\dagger}\mathfrak{D} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{D}$ [$\Rightarrow \mathfrak{F}({}^{\dagger}\mathfrak{D}) \overset{\sim}{\to} \mathfrak{F}({}^{\ddagger}\mathfrak{D})$]
- ${}^{\dagger}\mathfrak{F} \overset{\mathfrak{log}}{\to} {}^{\ddagger}\mathfrak{F}$ is compatible with the log-volumes.

$^{\dagger}\mathfrak{D} = \{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$: a $\mathcal{D}$-prime-strip

$\Rightarrow \mathfrak{F}(^{\dagger}\mathfrak{D})$: an $\mathcal{F}$-prime-strip

$\Rightarrow$

- $\mathfrak{log}(\mathfrak{F}(^{\dagger}\mathfrak{D}))$, i.e., a new $\mathcal{F}$-prime-strip
- full-poly aut. of $^{\dagger}\mathfrak{D} \overset{\text{lifts}}{\Rightarrow}$ full-poly isom. $\mathfrak{log}(\mathfrak{F}(^{\dagger}\mathfrak{D})) \underset{\text{full}}{\overset{\sim}{\to}} \mathfrak{F}(^{\dagger}\mathfrak{D})$,

  i.e., [full] $\mathfrak{log}$-link $\mathfrak{F}(^{\dagger}\mathfrak{D}) \overset{\mathfrak{log}}{\to} \mathfrak{F}(^{\dagger}\mathfrak{D})$

${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$, ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$: $\Theta^{\pm\text{ell}}\text{NF}$-Hodge theaters

Ξ: ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}$: an isomorphism

$\Rightarrow {}^{\dagger}\mathfrak{D}_{\square} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{\square}$ $[\square \in \{>, \succ\} \cup J \cup T]$

$\overset{\text{lifts}}{\Rightarrow}$ $\mathfrak{log}({}^{\dagger}\mathfrak{F}_{\square}) \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_{\square}$, i.e., a $\mathfrak{log}$-link ${}^{\dagger}\mathfrak{F}_{\square} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}_{\square}$

the $\mathfrak{log}$-link ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ associated to Ξ

$\overset{\text{def}}{\Leftrightarrow}$ $\{$the above ${}^{\dagger}\mathfrak{F}_{\square} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}_{\square}\}_{\square \in \{>, \succ\} \cup J \cup T}$

the [full] $\mathfrak{log}$-link ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$

$\overset{\text{def}}{\Leftrightarrow}$ $\{{}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ associated to $\Xi\}_{\Xi \in \text{Isom}({}^{\dagger}\mathcal{HT}^{\mathcal{D}}, {}^{\ddagger}\mathcal{HT}^{\mathcal{D}})}$

a log-theta-lattice $\overset{\text{def}}{\Leftrightarrow}$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \mathfrak{log}\uparrow & & \mathfrak{log}\uparrow & & \\
\cdots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}} & \cdots \\
 & & \mathfrak{log}\uparrow & & \mathfrak{log}\uparrow & & \\
\cdots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{(\mathrm{gau})}} & \cdots \\
 & & \mathfrak{log}\uparrow & & \mathfrak{log}\uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

$\Rightarrow$

$$\cdots \underset{\text{full}}{\overset{\sim}{\to}} {}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \underset{\text{full}}{\overset{\sim}{\to}} {}^{n,m+1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \underset{\text{full}}{\overset{\sim}{\to}} \cdots \text{ [vertical]}$$

$$\Rightarrow \cdots \underset{\text{full}}{\overset{\sim}{\to}} {}^{n,m}\mathfrak{D}_{\succ} \underset{\text{full}}{\overset{\sim}{\to}} {}^{n,m+1}\mathfrak{D}_{\succ} \underset{\text{full}}{\overset{\sim}{\to}} \cdots \text{ [vertical]}$$

$$\Rightarrow \cdots \underset{\text{full}}{\overset{\sim}{\to}} {}^{n,m}\mathfrak{D}^{\vdash}_{\Delta} \underset{\text{full}}{\overset{\sim}{\to}} {}^{n,m+1}\mathfrak{D}^{\vdash}_{\Delta} \underset{\text{full}}{\overset{\sim}{\to}} \cdots \text{ [vertical]}$$

$$\cdots \underset{\text{full}}{\overset{\sim}{\to}} {}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta} \underset{\text{full}}{\overset{\sim}{\to}} {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta} \underset{\text{full}}{\overset{\sim}{\to}} \cdots \text{ [horizontal]}$$

$$\Rightarrow \cdots \underset{\text{full}}{\overset{\sim}{\to}} {}^{n,m}\mathfrak{D}^{\vdash}_{\Delta} \underset{\text{full}}{\overset{\sim}{\to}} {}^{n+1,m}\mathfrak{D}^{\vdash}_{\Delta} \underset{\text{full}}{\overset{\sim}{\to}} \cdots \text{ [horizontal]}$$

- étale-like structure [i.e., "$\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}$"]: vertically coric
- Frobenius-like mono-an. structure [i.e., "$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta}$"]: horizontally coric
- étale-like mono-analytic structure [i.e., "$\mathfrak{D}^{\vdash}_{\Delta}$"]: bi-coric

§1.5 $\mathfrak{log}$-Kummer Correspondence I

$$\cdots \overset{\mathfrak{log}}{\to} {}^{-1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \cdots$$

⇒ The var. ét.-like str. are "neutral" w.r.t. $\mathfrak{log}$, which thus implies

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\mathfrak{log}} & {}^{-1}\bullet_{\mathrm{Frob}} & \xrightarrow{\mathfrak{log}} & {}^{0}\bullet_{\mathrm{Frob}} & \xrightarrow{\mathfrak{log}} & {}^{1}\bullet_{\mathrm{Frob}} & \xrightarrow{\mathfrak{log}} & \cdots \\
 & & \wr\downarrow \mathrm{Kmm} & & \wr\downarrow \mathrm{Kmm} & & \wr\downarrow \mathrm{Kmm} & & \\
\cdots & = & \circ_{\mathrm{\acute{e}t}} & = & \circ_{\mathrm{\acute{e}t}} & = & \circ_{\mathrm{\acute{e}t}} & = & \cdots .
\end{array}$$

$\mathfrak{log}$-Kummer correspondence $\overset{\mathrm{def}}{\Leftrightarrow}$ a relationship between

- the totality of Frobenius-like structures rel'd to $\{{}^{m}\bullet_{\mathrm{Frob}}\}_{m\in\mathbb{Z}}$ and
- an étale-like structure related to $\circ_{\mathrm{\acute{e}t}}$,

relative to the various Kummer isomorphisms, rep'd by the diagram.

Recall:

$$\mathbb{S}^{\pm}_{1} = \{0\} \ \subseteq \ \dots \ \subseteq \ \mathbb{S}^{\pm}_{j+1} = \{0, \dots, j\} \ \subseteq \ \dots \ \subseteq \ \mathbb{S}^{\pm}_{l^{\pm}(=l^{*}+1)} = |T|$$

$m \in \mathbb{Z} \quad \Rightarrow \{\Psi_{\mathrm{cns}}({}^{m}\mathfrak{F}_{\succ})_{t}\}_{t\in T} \quad \Rightarrow \{\Psi_{\mathrm{cns}}({}^{m}\mathfrak{F}_{\succ})_{|t|}\}_{|t|\in|T|}$

$\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $0 \leq j \leq l^{*} \quad \Rightarrow \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{F}_{\underline{v}}) \subseteq \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}_{v_{\mathbb{Q}}})$

$\Rightarrow \left((\Psi_{\mathrm{cns}}({}^{m}\mathfrak{F}_{\succ})_{j})^{\times}_{\underline{v}}\right)^{\mathrm{Gal}} \curvearrowright \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{F}_{\underline{v}})$ [“$(\mathcal{O}^{\times}_{K_{\underline{v}}})_{j} \curvearrowright \mathcal{I}_{j} \otimes \bigotimes \bigoplus \mathcal{I}$”]

${}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hdg th. [up to isom.] by the coricity

$\Rightarrow \{\Psi_{\mathrm{cns}}({}^{\circ}\mathfrak{D}_{\succ})_{t}\}_{t\in T} \quad \Rightarrow \{\Psi_{\mathrm{cns}}({}^{\circ}\mathfrak{D}_{\succ})_{|t|}\}_{|t|\in|T|}$

$\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $0 \leq j \leq l^{*} \quad \Rightarrow \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}}) \subseteq \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{v_{\mathbb{Q}}})$

$\Rightarrow \left((\Psi_{\mathrm{cns}}({}^{\circ}\mathfrak{D}_{\succ})_{j})^{\times}_{\underline{v}}\right)^{\mathrm{Gal}} \curvearrowright \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{D}_{\underline{v}})$ [“$(\mathcal{O}^{\times}_{K_{\underline{v}}})_{j} \curvearrowright \mathcal{I}_{j} \otimes \bigotimes \bigoplus \mathcal{I}$”]

$\Rightarrow$ ∃Kummer [poly-]isomorphisms

- $\Psi_{\text{cns}}({}^{m}\mathfrak{F}_{\succ})_t \xrightarrow{\sim} \Psi_{\text{cns}}({}^{\circ}\mathfrak{D}_{\succ})_t$
- $\left(\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{F}_{\underline{v}}) \subseteq \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}_{v_{\mathbb{Q}}})\right) \xrightarrow{\sim} \left(\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}}) \subseteq \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{v_{\mathbb{Q}}})\right)$

These $[m \in \mathbb{Z}]$ are "**upper semi-compatible**" in the foll'g sense:

$\underline{v} \in \underline{\mathbb{V}}^{\text{non}} \Rightarrow$

$$\left((\Psi_{\text{cns}}({}^{m}\mathfrak{F}_{\succ})_{|j|})^{\times}_{\underline{v}}\right)^{\text{Gal}}$$

$$\curvearrowright$$

$$\begin{array}{ccccc}
\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}_{v_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};m+1}\mathcal{F}_{v_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & \cdots \\
\text{Kmm}\downarrow\wr & & \text{Kmm}\downarrow\wr & & \\
\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{v_{\mathbb{Q}}}) & = & \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{v_{\mathbb{Q}}}) & = & \cdots
\end{array}$$

$\underline{v} \in \underline{\mathbb{V}}^{\text{arc}} \Rightarrow$ ∃a similar diagram

Let us think that $\left((\Psi_{\mathrm{cns}}({}^{m}\mathfrak{F}_{\succ})_{|j|})^{\times}_{\underline{v}}\right)^{\mathrm{Gal}}$ acts on $\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{v_{\mathbb{Q}}})$

[not via a single Kummer isomorphism — which fails to be compatible with the sequence of $\mathfrak{log}$-links — but rather]

via the totality of "$\mathrm{Kmm} \circ \mathfrak{log}^{\mathbb{Z}_{\geq 0}}$".

⇒ One obtains a sort of "**$\mathfrak{log}$-Kummer correspondence**" between

- the totality of $\left\{\left((\Psi_{\mathrm{cns}}({}^{m}\mathfrak{F}_{\succ})_{|j|})^{\times}_{\underline{v}}\right)^{\mathrm{Gal}}\right\}_{m\in\mathbb{Z}}$ and
- their actions on $\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{v_{\mathbb{Q}}})$.

Upper semi-compatibility:

$$\begin{array}{ccc} \text{Frob.-like } \mathcal{O}^{\triangleright}_{K_{\underline{v}}} \supseteq \text{Frob.-like } \mathcal{O}^{\times}_{K_{\underline{v}}} & \xrightarrow{\log} & \text{Frob.-like } \log(\mathcal{O}^{\times}_{K_{\underline{v}}}) \\ \cap\downarrow & & \cap\downarrow \\ \text{étale-like tensor packet log-shell} & = & \text{étale-like ten. pack. log-shell,} \end{array}$$

i.e., the étale-like tensor packet log-shell serves as a container for the images of certain sets of Frobenius-like local integers via all possible composites of arrows.

This may be regarded as an answer to the question of computing the "indeterminacies", i.e., the discrepancy between

- the Kummer theory rel'd to local elmts of the domain of $\mathfrak{log}$ and
- the Kummer theory rel'd to local elmts of the codomain of $\mathfrak{log}$.