[IUTch-III-IV] from the Point of View of Mono-anabelian Transport II

— Number Fields —

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§2.1 Kummer-compatible Multiradial κ-coric Functions

In order to establish a Kummer-compatible multiradial representation [whose coric data is "$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$"] of "$F_{\mathrm{mod}} \supseteq F^{\times}_{\mathrm{mod}} \curvearrowright \mathcal{I}$":

$(\S 2.1)$ we establish a multiradial representation of κ-coric functions,

$(\S 2.2)$ obtain "$F_{\mathrm{mod}} \supseteq F^{\times}_{\mathrm{mod}} \curvearrowright \mathcal{I}$" by applying the Galois evaluation operations to the multiradial κ-coric functions, and

$(\S 2.3)$ consider a $\mathfrak{log}$-Kummer correspondence concerning "$F_{\mathrm{mod}} \supseteq F^{\times}_{\mathrm{mod}} \curvearrowright \mathcal{I}$".

Recall: $({}^{\dagger}\mathbb{M}^{\circledast}_{\kappa})_j \overset{\text{Gal. ev.}}{\Rightarrow} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j$: an isomorph of F_{mod} for $j \in J$

${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: a $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater

Recall: For $j \in J$:

- the Frobenius-like localization poly-morphisms

$$\left\{\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})_j \curvearrowright ({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})_j \to {}^{\dagger}\mathbb{M}_{\infty\kappa v_j} \hookrightarrow {}^{\dagger}\mathbb{M}_{\infty\kappa\times v_j}\right\}_{\underline{v}\in\underline{\mathbb{V}}}$$

- the étale-like localization poly-morphisms

$$\left\{\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})_j \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \to \mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}_j}) \hookrightarrow \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})\right\}_{\underline{v}\in\underline{\mathbb{V}}}$$

[Recall: ${}_{\infty}\kappa$: κ-coric$^{1/n}$ for some $n \geq 1$

${}_{\infty}\kappa\times$: $u \cdot {}_{\infty}\kappa$-coric for some $u \in \overline{F}^{\times}$ or $\mathcal{O}^{\times}_{\overline{K}_{\underline{v}}}$]

For simplicity $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$: $\quad \Rightarrow$

$$
\begin{array}{ccccc}
({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})^{\boldsymbol{\mu}}_{j} & \xrightarrow{\sim} & {}^{\dagger}\mathbb{M}^{\boldsymbol{\mu}}_{\infty\kappa v_j} & = & {}^{\dagger}\mathbb{M}^{\boldsymbol{\mu}}_{\infty\kappa\times v_j} \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\boldsymbol{\mu}}_{j} & \xrightarrow{\sim} & \mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\boldsymbol{\mu}} & = & \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\boldsymbol{\mu}}
\end{array}
$$

$$
\begin{array}{ccccccc}
\xrightarrow{\hookrightarrow} & {}^{\dagger}\mathbb{M}^{\times}_{\infty\kappa\times v_j} & \xrightarrow{\twoheadrightarrow} & {}^{\dagger}\mathbb{M}^{\times\boldsymbol{\mu}}_{\infty\kappa\times v_j} & \xrightarrow{\sim} & \mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}_j} \\
& \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\xrightarrow{\hookrightarrow} & \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\times} & \xrightarrow{\twoheadrightarrow} & \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\times\boldsymbol{\mu}} & \xrightarrow{\sim} & \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash}_{j})_{\underline{v}}
\end{array}
$$

$\Rightarrow$ The composite from "$(-)^{\boldsymbol{\mu}}$" to $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}_j}$ is zero.

$\Rightarrow$ The id on

$$\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \hookleftarrow \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\boldsymbol{\mu}}_j \overset{\sim}{\to} \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\boldsymbol{\mu}}$$

$$\hookrightarrow \mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}_j}) \underset{\text{str. crit. pts}}{\overset{\text{eval. at}}{\twoheadrightarrow}} \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\boldsymbol{\mu}}$$

— i.e.,

- the pseudo-monoid of ${}_{\infty}\kappa$-coric functions,
- the cyclotomes related to ${}_{\infty}\kappa$-coric functions, and
- the spl'g $\mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})/\boldsymbol{\mu} \overset{\sim}{\to} (\mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})/\boldsymbol{\mu}) \times \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\times\boldsymbol{\mu}}$,

— is compatible, relative to the above diagram, w/ $\forall \in \mathrm{Aut}(\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}_j})$.

In particular:

${}^{\dagger}\mathfrak{R}_{\mathrm{NF}}$: the collection of data consisting of:

(a) ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ for $j \in J$: (b) $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash}_{j})$

(c) $\left\{\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})_j \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \to \mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}_j}) \hookrightarrow \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})\right\}_{\underline{v}\in\underline{\mathbb{V}}}$

(d) the full-poly isom. $\left\{\mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\times\boldsymbol{\mu}}\right\}_{\underline{v}\in\underline{\mathbb{V}}} \underset{\mathrm{full}}{\overset{\sim}{\to}} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash}_{j})$

(e) $\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \hookleftarrow \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\boldsymbol{\mu}}_j \overset{\sim}{\to} \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\boldsymbol{\mu}}$

$$\hookrightarrow \mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}_j}) \underset{\text{str. crit. pts}}{\overset{\text{eval. at}}{\twoheadrightarrow}} \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\boldsymbol{\mu}}$$

A morphism between "$\mathfrak{R}_{\mathrm{NF}}$" $\overset{\mathrm{def}}{=}$

an isom. of (a) [$\Rightarrow$ isom. of (c), (e)] and an isom. of (b)

$\Rightarrow$ "$^{\dagger}\mathfrak{R}_{\mathrm{NF}} \rightsquigarrow \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash}_{j})$" is multiradial.

In particular, $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\rightarrow} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \Rightarrow {}^{\dagger}\mathfrak{R}_{\mathrm{NF}} \overset{\sim}{\rightarrow} {}^{\ddagger}\mathfrak{R}_{\mathrm{NF}}$

Moreover, in the resulting isomorphism $^{\dagger}\mathfrak{R}_{\mathrm{NF}} \overset{\sim}{\rightarrow} {}^{\ddagger}\mathfrak{R}_{\mathrm{NF}}$:

$$
\begin{array}{ccccccc}
\mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})^{\times\boldsymbol{\mu}} & \xrightarrow{\sim} & \mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}_j})^{\times\boldsymbol{\mu}} & & {}^{\dagger}\mathbb{M}^{\times\boldsymbol{\mu}}_{\infty\kappa\times v} & \xrightarrow{\sim} & {}^{\ddagger}\mathbb{M}^{\times\boldsymbol{\mu}}_{\infty\kappa\times v} \\
\wr\downarrow \text{full} & & \wr\downarrow \text{full} & \text{cmpt} & \wr\downarrow \text{full} & & \wr\downarrow \text{full} \\
\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash}_{j})_{\underline{v}} & \xrightarrow{\sim} & \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\ddagger}\mathfrak{D}^{\vdash}_{j})_{\underline{v}} & & {}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}_j} & \xrightarrow{\sim} & {}^{\ddagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}_j}
\end{array}
$$

$({}^{\dagger}\mathrm{e}) \overset{\sim}{\rightarrow} ({}^{\ddagger}\mathrm{e})$ is comp. w/ an isom. of the corresp'g Frob.-like objects

[relative to the various Kummer poly-isomorphisms]

§2.2 Global Frobenioids via $\mathfrak{log}$-links

Recall: $\mathcal{F}^{\circledast}_{\mathrm{mod}}$: the Frobenioid of arith. inv. sh. on $\mathcal{O}_K//\mathrm{Gal}(K/F_{\mathrm{mod}})$

$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$: the Frobenioid whose objects are

pairs $\mathcal{T} = (T, \{t_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ consisting of:

- T: an $F^{\times}_{\mathrm{mod}}$-torsor
- $t_{\underline{v}} \in T \times^{F^{\times}_{\mathrm{mod}}} (K^{\times}_{\underline{v}}/\mathcal{O}^{\times}_{K_{\underline{v}}})$

$\Rightarrow \mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{MOD}}$

$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$: the Frobenioid whose objects are

collections $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ consisting of:

- $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \Rightarrow J_{\underline{v}}$: a fractional ideal of $\mathcal{O}_{K_{\underline{v}}}$
- $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}} \Rightarrow J_{\underline{v}}$: a positive real multiple of $\mathcal{O}_{K_{\underline{v}}}$ $(= \{|\lambda| \leq 1\})$

s.t. $J_{\underline{v}} = \mathcal{O}_{K_{\underline{v}}}$ for all but finitely many $\underline{v} \in \underline{\mathbb{V}}$

$\Rightarrow$

- $\mathcal{F}^{\circledast}_{\mathfrak{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{mod}}$
- $\mathcal{J}$: an object of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ $\quad \Rightarrow$

$\deg(\mathcal{J})$ can be computed by means of the local log-vol. of the $J_{\underline{v}}$'s.

${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: a $\mathfrak{log}$-link

$1 \le j \le l^{*}$

- $({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_j$

$\overset{\mathrm{def}}{=} \mathrm{Im}\Big(({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \hookrightarrow ({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\langle J\rangle} \otimes \bigotimes_{i=1}^{j} ({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_i \overset{(*)}{\hookrightarrow} \underline{\mathfrak{log}}(\mathbb{S}^{\pm}_{j+1};{}^{\dagger}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})\Big)$,

where $(*)$ is determined by $({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha} \to {}^{\ddagger}\mathfrak{F}_{\alpha}$ and $\mathfrak{log}({}^{\dagger}\mathfrak{F}_{\alpha}) \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_{\alpha}$

- $({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j$: the isomorph of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ determined by $({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_j$

$\Rightarrow$

$({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} ({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_j$ determines $({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} ({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j$.

$1 \leq j \leq l^{*}$

- $({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_j \stackrel{\mathrm{def}}{=} ({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_j \ \big(\subseteq \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1};\dagger}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})\big)$

- $({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$: the isomorph of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ determined by $({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_j$ and the var. integral str. $\mathcal{O}_{\underline{\mathfrak{log}}({}^{j;\dagger}\mathcal{F}_{\underline{v}})} \subseteq (\underline{\mathfrak{log}}({}^{j;\dagger}\mathcal{F}_{\underline{v}}) \subseteq)\ \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},j;\dagger}\mathcal{F}_{\underline{v}})$

$\Rightarrow$

- $({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_j \xrightarrow{\sim} ({}^{\ddagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_j$ determines $({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j \xrightarrow{\sim} ({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j$.

- The $\deg$ of an object of $({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$ can be computed by means of the local log-volumes.

Recall: ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}} \subseteq \prod_{j\in J} {}^{\ddagger}\mathcal{C}^{\Vdash}_{j} \overset{\sim}{\to} \prod_{j\in J}({}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j$

- ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \subseteq \prod_{j\in J}({}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_j$ determined by $({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \overset{\sim}{\to} ({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j$
- ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} \subseteq \prod_{j\in J}({}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j$ determined by $({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \overset{\sim}{\to} ({}^{\ddagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$

$\Rightarrow {}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}} \overset{\sim}{\to} {}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \overset{\sim}{\to} {}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}$

§2.3 $\mathfrak{log}$-Kummer Correspondence II

$$\cdots \stackrel{\mathfrak{log}}{\to} {}^{-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} \cdots$$

${}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: the ass. $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater [up to isom.]

$1 \leq j \leq l^{*}$

- $({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_j$, i.e., $\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j \stackrel{\mathrm{def}}{=} \mathrm{Im}$ of

$$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\circ}\mathcal{D}^{\circledcirc})_j \hookrightarrow \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\circ}\mathcal{D}^{\circledcirc})_{\langle j\rangle} \otimes \bigotimes_{i=1}^{j} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\circ}\mathcal{D}^{\circledcirc})_i \hookrightarrow \underline{\mathfrak{log}}(\mathbb{S}^{\pm}_{j+1};{}^{\circ}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}})$$

- $({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_j$, i.e., $\mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j$:

  the isomorph of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ determined by $({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_j$

$\Rightarrow \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\circ}\mathcal{D}^{\circledcirc})_j \stackrel{\sim}{\to} ({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_j$ determines $\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\circ}\mathcal{D}^{\circledcirc})_j \stackrel{\sim}{\to} ({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_j$.

$1 \leq j \leq l^{\circledast}$

- $({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_j$, i.e., $\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{\circ}\mathcal{H}\mathcal{T}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_j$, $\stackrel{\text{def}}{=} ({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\text{MOD}})_j$
- $({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_j$, i.e., $\mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{\circ}\mathcal{H}\mathcal{T}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}})_j$:

  the isomorph of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ det'd by $({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_j$ and the var. integral str.

$\Rightarrow$

- $({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_j \stackrel{\sim}{\to} ({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\text{MOD}})_j$ determines $({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_j \stackrel{\sim}{\to} ({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\text{MOD}})_j$.
- The $\deg$ of an object of $({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_j$ can be computed by means of

the local log-volumes.

Recall: $\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\circ}\mathfrak{D}^{\vdash}_{\succ}) \subseteq \prod_{j\in J} \mathcal{D}^{\Vdash}({}^{\circ}\mathfrak{D}^{\vdash}_{\succ})_j \xrightarrow{\sim} \prod_{j\in J} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\circ}\mathcal{D}^{\circledcirc})_j$

- ${}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathrm{LGP}}$, i.e., $\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$,

  $\subseteq \prod_{j\in J}({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}\mathbb{R}}_{\mathrm{MOD}})_j$ determined by $\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\circ}\mathcal{D}^{\circledcirc})_j \xrightarrow{\sim} ({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_j$

- ${}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathfrak{lgp}}$, i.e., $\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$,

  $\subseteq \prod_{j\in J}({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}\mathbb{R}}_{\mathfrak{mod}})_j$ determined by $\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\circ}\mathcal{D}^{\circledcirc})_j \xrightarrow{\sim} ({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_j$

$\Rightarrow {}^{\circ}\mathcal{D}^{\Vdash}_{\mathrm{gau}} \xrightarrow{\sim} {}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathrm{LGP}} \xrightarrow{\sim} {}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathfrak{lgp}}$

$m \in \mathbb{Z}$, $1 \leq j \leq l^{*} \quad \Rightarrow ({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})^{\times}_{j} \curvearrowright \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$

$1 \leq j \leq l^{*} \quad \Rightarrow ({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathrm{MOD}})^{\times}_{j} \curvearrowright \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}})$

$\Rightarrow$ $\exists$Kummer [poly-]isomorphisms

- $({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{j} \xrightarrow{\sim} ({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_{j}$
- $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \xrightarrow{\sim} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}})$

These $[m \in \mathbb{Z}]$ satisfy the following "**non-interference property**":

- Let us consider the diagram

$$
\begin{array}{ccccccc}
({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})^{\times}_{j} & & & & & & \\
\curvearrowright & & & & & & \\
\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};m+1}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};m+2}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & \cdots \\
\mathsf{Kmm}\downarrow\wr & & \mathsf{Kmm}\downarrow\wr & & \mathsf{Kmm}\downarrow\wr & & \\
\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}}) & = & \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}}) & = & \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}}) & = & \cdots
\end{array}
$$

- The $\mathfrak{log}$ [at m] is defined on $\prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{O}^{\times}_{\underline{\mathfrak{log}}({}^{j;m}\mathcal{F}_{\underline{v}})}$.
- $({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})^{\times}_{j} \cap \left(\prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{O}^{\times}_{\underline{\mathfrak{log}}({}^{j;m}\mathcal{F}_{\underline{v}})}\right) \subseteq \boldsymbol{\mu} \subseteq \mathrm{Ker}(\mathfrak{log})$.

Let us think that $({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})^{\times}_{j}$ acts on $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}})$

[not via a single Kummer isomorphism — which fails to be compatible with the sequence of $\mathfrak{log}$-links — but rather]

via the totality of "$\mathrm{Kmm} \circ \mathfrak{log}^{\mathbb{Z}_{\geq 0}}$".

Note: These are **mutually compatible** up to id at an adjacent "m".

⇒ One obtains a sort of "**$\mathfrak{log}$-Kummer correspondence**" between

- the totality of $\{({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})^{\times}_{j}\}_{m\in\mathbb{Z}}$ and
- their actions on $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{\mathbb{V}_{\mathbb{Q}}})$.

⇒ $({}^{m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{j} \xrightarrow{\sim} ({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_{j}$ $[m \in \mathbb{Z}]$ are **mutually compatible**.