[IUTch-III-IV] from the Point of View of Mono-anabelian Transport III

— Theta Values —

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3.1 Kummer-compatible Multiradial Theta Functions

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§3.1 Kummer-compatible Multiradial Theta Functions

In order to establish a Kummer-compatible multiradial representation

[whose coric data is "$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$"] of "theta values $\{\underline{\underline{q}}_{v}^{j^2}\}_{j=1}^{l^*} \curvearrowright \mathcal{I}$":

(§3.1) we establish a multiradial representation of theta functions,

(§3.2) obtain "$\{\underline{\underline{q}}_{v}^{j^2}\}_{j=1}^{l^*} \curvearrowright \mathcal{I}$" by applying the Galois evaluation operations to the multiradial theta functions, and

(§3.3) consider a $\mathfrak{log}$-Kummer correspondence concerning "$\{\underline{\underline{q}}_{v}^{j^2}\}_{j=1}^{l^*} \curvearrowright \mathcal{I}$".

${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: a $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater

By identifying $\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_0$ with $\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle \mathbb{F}_l^{\circledast} \rangle}$

and pulling back "$\mathcal{O}^{\rhd} \twoheadrightarrow \mathcal{O}^{\rhd}/\mathcal{O}^{\times}$" via "$\mathcal{O}^{\rhd}/\mathcal{O}^{\times} \hookleftarrow$ suitable$^{\mathbb{N}}$",

one obtains an $\mathcal{F}^{\vdash}$-prime-strip $\mathfrak{F}^{\vdash}_{\Delta}({}^{\dagger}\mathfrak{D}_{\succ})$, hence also

$\mathcal{F}^{\vdash\times}$-, $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips $\mathfrak{F}^{\vdash\times}_{\Delta}({}^{\dagger}\mathfrak{D}_{\succ})$, $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta}({}^{\dagger}\mathfrak{D}_{\succ})$,

i.e, the étale-like holomorphic symmetrized constant portions.

$\Rightarrow$ $\exists$a Kummer poly-isomorphism ${}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta} \overset{\sim}{\to} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta}({}^{\dagger}\mathfrak{D}_{\succ})$

By applying a mono-anabelian reconstruction algorithm to $\mathfrak{D}^{\vdash}_{\triangle}$,

one obtains an $\mathcal{F}^{\vdash}$-prime-strip $\mathfrak{F}^{\vdash}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})$, hence also

$\mathcal{F}^{\vdash\times}$-, $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips $\mathfrak{F}^{\vdash\times}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})$, $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})$,

i.e, the étale-like mono-analytic symmetrized constant portions.

$\Rightarrow$ $\exists$a natural poly-isomorphism $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})$

Recall: ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}}$: the $\mathcal{F}^{\vdash}$-prime-strip determined by $\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})$

[i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, ${}^{\dagger}\mathcal{F}^{\vdash}_{\mathrm{env},\underline{v}}$: the Frob.-like theta monoid "$\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{v}$"]

$\mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$: the $\mathcal{F}^{\vdash}$-prime-strip determined by $\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$

[i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, $\mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}$:

the mono-theta env. ver. of the theta monoid "$\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{v}$"]

$\Rightarrow$

- $\exists$natural poly-isom. ${}^{\dagger}\mathfrak{F}^{\vdash\times}_{\Delta} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{env}}$ and $\mathfrak{F}^{\vdash\times}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$
- $\exists$a Kummer poly-isom. ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}} \xrightarrow{\sim} \mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$

Recall: ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} = ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{env}},\ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{env}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}},\ \{{}^{\dagger}\rho_{\mathrm{env},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$:

the $\mathcal{F}^{\Vdash}$-prime-strip determined by the Frob.-like theta monoids

$\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$

$\stackrel{\mathrm{def}}{=} (\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})) \xrightarrow{\sim} \underline{\mathbb{V}},\ \mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}),\ \{{}^{\dagger}\rho^{\mathcal{D}}_{\mathrm{env},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$:

the $\mathcal{F}^{\Vdash}$-pr.st. det'd by the mono-theta env. ver. of the theta monoids

$\Rightarrow$

$\exists$a Kummer poly-isomorphism ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$

Recall: For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$:

- the Frobenius-like splitting theta monoid

$$_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} \subseteq {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$$

- the étale-like splitting theta monoid

$$_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} \subseteq {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}$$

[by the constant multiple rigidity]

[i.e., "$\boldsymbol{\mu}(\overline{K}_{\underline{v}})_{\Delta} \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{Q}_{\geq 0}} \subseteq \mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{Q}_{\geq 0}}$"]

$\Rightarrow$

$$\begin{array}{ccccc}
{}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} & \overset{\hookleftarrow}{\longleftarrow} & {}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\boldsymbol{\mu}}_{\underline{v}} & = & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\boldsymbol{\mu}}_{\underline{v}} \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
{}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} & \overset{\hookleftarrow}{\longleftarrow} & {}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\boldsymbol{\mu}}_{\underline{v}} & = & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\boldsymbol{\mu}}_{\underline{v}}
\end{array}$$

$$\begin{array}{ccccccc}
\overset{\hookrightarrow}{\longrightarrow} & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\times}_{\underline{v}} & \overset{\twoheadrightarrow}{\longrightarrow} & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\times\boldsymbol{\mu}}_{\underline{v}} & \overset{\sim}{\longrightarrow} & \mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta,\underline{v}} \\
& \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\overset{\hookrightarrow}{\longrightarrow} & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\times}_{\underline{v}} & \overset{\twoheadrightarrow}{\longrightarrow} & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\times\boldsymbol{\mu}}_{\underline{v}} & \overset{\sim}{\longrightarrow} & \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta}({}^{\dagger}\mathfrak{D}^{\vdash}_{\Delta})_{\underline{v}}
\end{array}$$

$\Rightarrow$ The composite from "$(-)^{\boldsymbol{\mu}}$" to $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta,\underline{v}}$ is zero.

$\Rightarrow$ The id on

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}^{\boldsymbol{\mu}}$$

$$\hookrightarrow {}_{\infty}\Psi_{\mathrm{env}}^{\perp}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} \overset{\text{zero eval.}}{\twoheadrightarrow} {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}^{\boldsymbol{\mu}}$$

— i.e.,

- the splitting theta monoid,
- the cyclotomes related to theta functions, and
- the spl'g ${}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}/\boldsymbol{\mu} \xrightarrow{\sim} ({}_{\infty}\Psi_{\mathrm{env}}^{\perp}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}/\boldsymbol{\mu}) \times {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}^{\times\boldsymbol{\mu}}$,

— is compatible, relative to the above diagram, w/ $\forall \in \mathrm{Aut}(\mathcal{F}_{\Delta,\underline{v}}^{\vdash\times\boldsymbol{\mu}})$.

In particular:

${}^{\dagger}\mathfrak{R}_{\mathrm{tht}}$: the collection of data consisting of:

(a) ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ (b) $\mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}_{\Delta}^{\vdash})$ (c) $\mathfrak{F}_{\mathrm{env}}^{\Vdash}({}^{\dagger}\mathfrak{D}_{>})$

(d) the full poly-isomorphism $\mathfrak{F}_{\mathrm{env}}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}_{>}) \underset{\mathrm{full}}{\xrightarrow{\sim}} \mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}_{\Delta}^{\vdash})$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$:

(e) the proj. sys. of mono-theta env. $\mathbb{M}_{*}^{\Theta}({}^{\dagger}\mathcal{D}_{>,\underline{v}})$ cons'd from ${}^{\dagger}\mathcal{D}_{>,\underline{v}}$

(f) $\Pi_{\boldsymbol{\mu}}(\mathbb{M}_{*}^{\Theta}({}^{\dagger}\mathcal{D}_{>,\underline{v}})) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}^{\boldsymbol{\mu}}$

$$\hookrightarrow {}_{\infty}\Psi_{\mathrm{env}}^{\perp}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} \overset{\text{zero eval.}}{\twoheadrightarrow} {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}^{\boldsymbol{\mu}}$$

A morphism between "$\mathfrak{R}_{\mathrm{tht}}$" $\overset{\mathrm{def}}{=}$

an isom. of (a) [$\Rightarrow$ isom. of (c), (e), (f)] and an isom. of (b)

$\Rightarrow$ " ${}^{\dagger}\mathfrak{R}_{\mathrm{tht}} \rightsquigarrow \mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}_{\Delta}^{\vdash})$" is multiradial.

In particular, ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta_{\mathrm{gau}}^{\times\boldsymbol{\mu}}}{\to} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \Rightarrow {}^{\dagger}\mathfrak{R}_{\mathrm{tht}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{R}_{\mathrm{tht}}$

Moreover, in the resulting isomorphism ${}^{\dagger}\mathfrak{R}_{\mathrm{tht}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{R}_{\mathrm{tht}}$:

$$\begin{array}{ccc} \mathfrak{F}_{\mathrm{env}}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}_{>}) & \xrightarrow{\sim} & \mathfrak{F}_{\mathrm{env}}^{\vdash\times\boldsymbol{\mu}}({}^{\ddagger}\mathfrak{D}_{>}) \\ \wr\downarrow \text{full} & & \wr\downarrow \text{full} \\ \mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}_{\Delta}^{\vdash}) & \xrightarrow{\sim} & \mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}}({}^{\ddagger}\mathfrak{D}_{\Delta}^{\vdash}) \end{array} \quad \text{is comp. w/} \quad \begin{array}{ccc} {}^{\dagger}\mathfrak{F}_{\mathrm{env}}^{\vdash\times\boldsymbol{\mu}} & \xrightarrow{\sim} & {}^{\ddagger}\mathfrak{F}_{\mathrm{env}}^{\vdash\times\boldsymbol{\mu}} \\ \wr\downarrow \text{full} & & \wr\downarrow \text{full} \\ {}^{\dagger}\mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}} & \xrightarrow{\sim} & {}^{\ddagger}\mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}} \end{array}$$

$({}^{\dagger}\mathrm{f}) \xrightarrow{\sim} ({}^{\ddagger}\mathrm{f})$ is comp. w/ an isom. of the corresp'g Frob.-like objects

[relative to the Kmm poly-isom. and poly-isom. of mono-theta env.]

§3.2 Local Logarithmic Gaussian Procession Monoids

$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: a $\mathfrak{log}$-link

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \qquad \Rightarrow$

$$\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\ddagger}\mathcal{HT}^{\Theta})_{\underline{v}} \hookrightarrow {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\ddagger}\mathcal{HT}^{\Theta})_{\underline{v}} \hookrightarrow \prod_{j\in J} (\Psi_{{}^{\ddagger}\mathcal{F}_{\underline{v}}})_j$$

$$\stackrel{\mathfrak{log}}{\stackrel{\sim}{\leftarrow}} \prod_{j\in J} \Psi_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})_j} \hookrightarrow \prod_{j\in J} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})_j \twoheadleftarrow \prod_{j\in J} (\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})_j$$

[For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$:

$$\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})^{\mathbb{N}} \hookrightarrow \mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})^{\mathbb{Q}_{\geq 0}} \hookrightarrow \prod_{j\in J} \mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}},j}$$

$$\stackrel{\sim}{\leftarrow} \prod_{j\in J} \mathcal{O}^{\triangleright}_{\log(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},j})} \hookrightarrow \prod_{j\in J} \log(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},j}) \stackrel{\log}{\twoheadleftarrow} \prod_{j\in J} \mathcal{O}^{\times}_{\overline{K}_{\underline{v}},j}]$$

$$\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathrm{Im}(\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\ddagger}\mathcal{HT}^{\Theta})_{\underline{v}} \hookrightarrow \prod_{j \in J} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})_j)$$

$$_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathrm{Im}({}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\ddagger}\mathcal{HT}^{\Theta})_{\underline{v}} \hookrightarrow \prod_{j \in J} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})_j)$$

One may construct similar objects for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.

$$\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}) \stackrel{\mathrm{def}}{=} \{\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}},$$

$$_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}) \stackrel{\mathrm{def}}{=} \{{}_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}:$$

local logarithmic Gaussian procession monoids

${}^{\ddagger}\mathfrak{F}^{\vdash}_{\mathrm{LGP}}$: the $\mathcal{F}^{\vdash}$-prime-strip determined by $\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$

[i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, ${}^{\ddagger}\mathcal{F}^{\vdash}_{\mathrm{LGP},\underline{v}}$:

the Frobenius-like Gaussian monoid "$\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})^{\mathbb{N}}$"

in "$\prod_{j \in J} \log(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},j})$"]

${}^{\ddagger}\mathfrak{F}^{\vdash}_{\mathfrak{lgp}} \overset{\mathrm{def}}{=} {}^{\ddagger}\mathfrak{F}^{\vdash}_{\mathrm{LGP}}$

$\Rightarrow$ One obtains $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips

${}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} \overset{\mathrm{def}}{=} ({}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}},\ \mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}) \overset{\sim}{\to} \underline{\mathbb{V}},\ {}^{\ddagger}\mathfrak{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}},\ \{{}^{\ddagger}\rho_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$

${}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}} \overset{\mathrm{def}}{=} ({}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}},\ \mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}) \overset{\sim}{\to} \underline{\mathbb{V}},\ {}^{\ddagger}\mathfrak{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}},\ \{{}^{\ddagger}\rho_{\mathfrak{lgp},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$

$\Rightarrow {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$

$^{*}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: a $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater

$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: a $\mathfrak{log}$-link

$\Rightarrow$ $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips $^{*}\mathfrak{F}_{\Delta}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$, $^{\ddagger}\mathfrak{F}_{\mathrm{LGP}}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$, and $^{\ddagger}\mathfrak{F}_{\mathfrak{lgp}}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$

the $\Theta_{\mathrm{LGP}}^{\times\boldsymbol{\mu}}$-link $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta_{\mathrm{LGP}}^{\times\boldsymbol{\mu}}}{\to} {}^{*}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$

$\stackrel{\mathrm{def}}{\Leftrightarrow}$ the full poly-isomorphism $^{\ddagger}\mathfrak{F}_{\mathrm{LGP}}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}} \underset{\mathrm{full}}{\stackrel{\sim}{\to}} {}^{*}\mathfrak{F}_{\Delta}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$

the $\Theta_{\mathfrak{lgp}}^{\times\boldsymbol{\mu}}$-link $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta_{\mathfrak{lgp}}^{\times\boldsymbol{\mu}}}{\to} {}^{*}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$

$\stackrel{\mathrm{def}}{\Leftrightarrow}$ the full poly-isomorphism $^{\ddagger}\mathfrak{F}_{\mathfrak{lgp}}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}} \underset{\mathrm{full}}{\stackrel{\sim}{\to}} {}^{*}\mathfrak{F}_{\Delta}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$

a q-pilot object [of ${}^{*}\mathcal{C}^{\Vdash}_{\Delta}$]

$\overset{\mathrm{def}}{\Leftrightarrow}$ an obj. of ${}^{*}\mathcal{C}^{\Vdash}_{\Delta}$ det'd by generators of the spl. of ${}^{*}\mathcal{F}^{\vdash}_{\Delta,\underline{v}}$ [$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$]

[i.e., an object of ${}^{*}\mathcal{C}^{\Vdash}_{\Delta}$ determined by $(\underline{\underline{q}}_{\underline{v}})_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}}$]

a Θ-pilot object [of ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}$ or ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}$]

$\overset{\mathrm{def}}{\Leftrightarrow}$ obj. of ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}$ or ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}$ det. by gen. of the spl. of ${}^{\ddagger}\mathcal{F}^{\vdash}_{\mathrm{LGP},\underline{v}}$ [$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$]

[i.e., an object of ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}$ or ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}$ determined by $(\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^{*})^2})_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}}$]

$\Rightarrow$ For $\Box \in \{\mathrm{LGP}, \mathfrak{lgp}\}$, the poly-isom. ${}^{\ddagger}\mathcal{C}^{\Vdash}_{\Box} \overset{\sim}{\to} {}^{*}\mathcal{C}^{\Vdash}_{\Delta}$ induced by ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{\Box}}{\to} {}^{*}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ maps a Θ-pilot object to a q-pilot object.

For $\square \in \{\mathrm{LGP}, \mathfrak{lgp}\}$:

a $\square$-Gaussian log-theta-lattice $\overset{\text{def}}{\Leftrightarrow}$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \mathfrak{log}} & & \uparrow{\scriptstyle \mathfrak{log}} & & \\
\cdots & \xrightarrow{\Theta_{\square}^{\times\boldsymbol{\mu}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta_{\square}^{\times\boldsymbol{\mu}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta_{\square}^{\times\boldsymbol{\mu}}} & \cdots \\
 & & \uparrow{\scriptstyle \mathfrak{log}} & & \uparrow{\scriptstyle \mathfrak{log}} & & \\
\cdots & \xrightarrow{\Theta_{\square}^{\times\boldsymbol{\mu}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta_{\square}^{\times\boldsymbol{\mu}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta_{\square}^{\times\boldsymbol{\mu}}} & \cdots \\
 & & \uparrow{\scriptstyle \mathfrak{log}} & & \uparrow{\scriptstyle \mathfrak{log}} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

$\Rightarrow$

$$\cdots \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n,m+1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} \cdots \quad \text{[vertical]}$$

$$\Rightarrow \cdots \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n,m}\mathfrak{D}_{\succ} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n,m+1}\mathfrak{D}_{\succ} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} \cdots \quad \text{[vertical]}$$

$$\Rightarrow \cdots \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n,m}\mathfrak{D}^{\vdash}_{\Delta} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n,m+1}\mathfrak{D}^{\vdash}_{\Delta} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} \cdots \quad \text{[vertical]}$$

$$\cdots \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} \cdots \quad \text{[horizontal]}$$

$$\Rightarrow \cdots \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n,m}\mathfrak{D}^{\vdash}_{\Delta} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} {}^{n+1,m}\mathfrak{D}^{\vdash}_{\Delta} \underset{\text{full}}{\stackrel{\sim}{\rightarrow}} \cdots \quad \text{[horizontal]}$$

- étale-like structure [i.e., "$\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}$"]: vertically coric
- Frobenius-like mono-an. structure [i.e., "$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\Delta}$"]: horizontally coric
- étale-like mono-analytic structure [i.e., "$\mathfrak{D}^{\vdash}_{\Delta}$"]: bi-coric

§3.3 $\mathfrak{log}$-Kummer Correspondence III

$$\cdots \stackrel{\mathfrak{log}}{\to} {}^{-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} \cdots$$

${}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: the ass'd $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater [up to isom.]

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \qquad \Rightarrow$

$$\Psi_{\mathrm{gau}}({}^{\circ}\mathfrak{D}_{\succ})_{\underline{v}} \hookrightarrow {}_{\infty}\Psi_{\mathrm{gau}}({}^{\circ}\mathfrak{D}_{\succ})_{\underline{v}} \hookrightarrow \prod_{j\in J} (\Psi_{\mathrm{cns}}({}^{\circ}\mathfrak{D}_{\succ})_j)_{\underline{v}}$$

$$\hookrightarrow \prod_{j\in J} (\underline{\mathfrak{log}}(\mathfrak{F}({}^{\circ}\mathfrak{D}))_j)_{\underline{v}} \twoheadleftarrow \prod_{j\in J} (\Psi_{\mathrm{cns}}({}^{\circ}\mathfrak{D}_{\succ})_j)_{\underline{v}}$$

[For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$:

$$\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})^{\mathbb{N}} \hookrightarrow \mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})^{\mathbb{Q}_{\geq 0}} \hookrightarrow \prod_{j\in J} \mathcal{O}^{\rhd}_{\overline{K}_{\underline{v}},j}$$

$$= \prod_{j\in J} \mathcal{O}^{\rhd}_{\log(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},j})} \hookrightarrow \prod_{j\in J} \log(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},j}) \stackrel{\log}{\twoheadleftarrow} \prod_{j\in J} \mathcal{O}^{\times}_{\overline{K}_{\underline{v}},j}]$$

$$\Psi_{\rm LGP}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})_{\underline{v}} \stackrel{\rm def}{=} \mathrm{Im}(\Psi_{\rm gau}({}^{\circ}\mathfrak{D}_{\succ})_{\underline{v}} \hookrightarrow \prod_{j\in J} (\underline{\mathfrak{log}}(\mathfrak{F}({}^{\circ}\mathfrak{D}_{\succ}))_j)_{\underline{v}})$$

$${}_{\infty}\Psi_{\rm LGP}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})_{\underline{v}} \stackrel{\rm def}{=} \mathrm{Im}({}_{\infty}\Psi_{\rm gau}({}^{\circ}\mathfrak{D}_{\succ})_{\underline{v}} \hookrightarrow \prod_{j} (\underline{\mathfrak{log}}(\mathfrak{F}({}^{\circ}\mathfrak{D}_{\succ}))_j)_{\underline{v}})$$

One may construct similar objects for $\underline{v} \in \underline{\mathbb{V}}^{\rm arc}$.

$$\Psi_{\rm LGP}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF}) \stackrel{\rm def}{=} \{\Psi_{\rm LGP}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},$$

$${}_{\infty}\Psi_{\rm LGP}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF}) \stackrel{\rm def}{=} \{{}_{\infty}\Psi_{\rm LGP}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}:$$

${}^{\ddagger}\mathfrak{F}^{\vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})_{\rm LGP}$: the $\mathcal{F}^{\vdash}$-pr.-st. det'd by $\Psi_{\rm LGP}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})$

[i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\rm bad}$, ${}^{\ddagger}\mathfrak{F}^{\vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})_{\mathrm{LGP},\underline{v}}$:

the étale-like Gaussian monoid "$\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},\Delta} \cdot (\underline{\underline{q}}_{\underline{v}}^{1^2}, \ldots, \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})^{\mathbb{N}}$"

in "$\prod_{j\in J} \log(\mathcal{O}^{\times}_{\overline{K}_{\underline{v}},j})$"]

$${}^{\ddagger}\mathfrak{F}^{\vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})_{\mathfrak{lgp}} \stackrel{\rm def}{=} {}^{\ddagger}\mathfrak{F}^{\vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \rm ell}\rm NF})_{\rm LGP}$$

$\Rightarrow$ One obtains $\mathcal{F}^{\Vdash}$-prime-strips

$$\mathfrak{F}^{\Vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$$
$$\stackrel{\mathrm{def}}{=} ({}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathrm{LGP}},\ \mathrm{Prime} \xrightarrow{\sim} \underline{\mathbb{V}},\ \mathfrak{F}^{\vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}},\ \{{}^{\circ}\rho^{\mathcal{D}}_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

$$\mathfrak{F}^{\Vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$$
$$\stackrel{\mathrm{def}}{=} ({}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathfrak{lgp}},\ \mathrm{Prime} \xrightarrow{\sim} \underline{\mathbb{V}},\ \mathfrak{F}^{\vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}},\ \{{}^{\circ}\rho^{\mathcal{D}}_{\mathfrak{lgp},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

$\Rightarrow$

$$\mathfrak{F}^{\Vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{gau}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

$m \in \mathbb{Z}$, $1 \leq j \leq l^{*} \quad \Rightarrow \Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v},j} \curvearrowright \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{F}_{\underline{v}})$

$1 \leq j \leq l^{*} \quad \Rightarrow \Psi^{\perp}_{\mathrm{LGP}}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v},j} \curvearrowright \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}})$

$\Rightarrow \exists$Kummer [poly-]isomorphisms

- $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v},j} \xrightarrow{\sim} \Psi^{\perp}_{\mathrm{LGP}}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v},j}$
- $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}})$

These $[m \in \mathbb{Z}]$ satisfy the following "**non-interference property**":

- Let us consider the diagram

$$
\begin{array}{ccccccc}
\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{m}\mathcal{HT})_{\underline{v},j} & & & & & & \\
\curvearrowright & & & & & & \\
\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{F}_{\underline{v}}) & \xrightarrow{\mathfrak{log}} & \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;m+1}\mathcal{F}_{\underline{v}}) & \xrightarrow{\mathfrak{log}} & \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;m+2}\mathcal{F}_{\underline{v}}) & \xrightarrow{\mathfrak{log}} & \cdots \\
\mathsf{Kmm} \downarrow \wr & & \mathsf{Kmm} \downarrow \wr & & \mathsf{Kmm} \downarrow \wr & & \\
\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}}) & = & \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}}) & = & \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}}) & = & \cdots
\end{array}
$$

- The $\mathfrak{log}$ [at m] is defined on $\mathcal{O}^{\times}_{\underline{\mathfrak{log}}({}^{j;m}\mathcal{F}_{\underline{v}})}$.

- $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v},j} \cap \mathcal{O}^{\times}_{\underline{\mathfrak{log}}({}^{j;m}\mathcal{F}_{\underline{v}})} \subseteq \boldsymbol{\mu} \subseteq \mathrm{Ker}(\mathfrak{log})$.

Let us think that $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v},j}$ acts on $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}})$

[not via a single Kummer isomorphism — which fails to be compatible with the sequence of $\mathfrak{log}$-links — but rather]

via the totality of "$\mathrm{Kmm} \circ \mathfrak{log}^{\mathbb{Z}_{\geq 0}}$".

Note: These are **mutually compatible** up to id at an adjacent "m".

$\Rightarrow$ One obtains a sort of "**$\mathfrak{log}$-Kummer correspondence**" between

- the totality of $\{\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v},j}\}_{m\in\mathbb{Z}}$ and
- their actions on $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}_{\underline{v}})$.

$\Rightarrow$ $({}^{m}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}, \{{}^{m}\rho_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \xrightarrow{\sim} ({}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathrm{LGP}}, \{{}^{\circ}\rho^{\mathcal{D}}_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ $[m \in \mathbb{Z}]$

are **mutually compatible**.