[IUTch-III-IV] from the Point of View of Mono-anabelian Transport IV

— Main Theorem and Application —

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4.1 Kummer-compatible Multiradial Representations

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§4.1 Kummer-compatible Multiradial Representations

In summary, the following hold:

$$\dots \stackrel{\mathfrak{log}}{\to} {}^{-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} \dots$$

${}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: the ass'd $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theater [up to isom.]

Recall: ${}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \Rightarrow \mathrm{Prc}({}^{\circ}\mathfrak{D}_T)$

$$\stackrel{\mathrm{def}}{=} \left(\{{}^{\circ}\mathfrak{D}_0\} \hookrightarrow \{{}^{\circ}\mathfrak{D}_0, {}^{\circ}\mathfrak{D}_1\} \hookrightarrow \cdots \hookrightarrow \{{}^{\circ}\mathfrak{D}_0, \dots, {}^{\circ}\mathfrak{D}_{l^*}\}\right)$$

$\Rightarrow \mathrm{Prc}({}^{\circ}\mathfrak{D}_T^{\vdash})$

$$\stackrel{\mathrm{def}}{=} \left(\{{}^{\circ}\mathfrak{D}_0^{\vdash}\} \hookrightarrow \{{}^{\circ}\mathfrak{D}_0^{\vdash}, {}^{\circ}\mathfrak{D}_1^{\vdash}\} \hookrightarrow \cdots \hookrightarrow \{{}^{\circ}\mathfrak{D}_0^{\vdash}, \dots, {}^{\circ}\mathfrak{D}_{l^*}^{\vdash}\}\right)$$

$\mathfrak{R}^{\text{ét}}$: the collection of data consisting of:

$(\mathrm{a}^{\text{ét}})$ $\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $0 \leq j \leq l^{*}$

$\mathrm{Prc}({}^{\circ}\mathfrak{D}^{\vdash}_{T}) \Rightarrow$

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}}), \quad \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}^{\vdash}_{\underline{v}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}^{\vdash}_{\underline{v}})$$

[with the procession-normalized mono-analytic log-volumes]

$(\mathrm{b}^{\text{ét}})$ $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

$${}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \Rightarrow \Psi^{\perp}_{\mathrm{LGP}}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}} \text{ in } \prod_{j\in J} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\circ}\mathcal{D}^{\vdash}_{\underline{v}})$$

$(\mathrm{c}^{\text{ét}})$ $1 \leq j \leq l^{*}$

$${}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \Rightarrow ({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_{j} = ({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_{j} \text{ in } \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}})$$

$$\Rightarrow ({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_{j} \xrightarrow{\sim} ({}^{\circ}\mathcal{F}^{\circledast\mathcal{D}}_{\mathfrak{mod}})_{j}$$

[whose ass. "degree" may be com'd by means of the log-vol. of $(\mathrm{a}^{\text{ét}})$]

$$\cdots \overset{\mathfrak{log}}{\to} {}^{\overline{-1}}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\overline{0}}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\overline{1}}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \cdots$$

$$\downarrow \Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$$

$$\cdots \overset{\mathfrak{log}}{\to} {}^{\underline{-1}}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\underline{0}}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\underline{1}}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \cdots$$

$$\Rightarrow \mathrm{Prc}({}^{\overline{\circ}}\mathfrak{D}^{\vdash}_{T}) \overset{\sim}{\to} \mathrm{Prc}({}^{\underline{\circ}}\mathfrak{D}^{\vdash}_{T})$$

$$\Rightarrow {}^{\overline{\circ}}(a^{\text{ét}}) \overset{\sim}{\to} {}^{\underline{\circ}}(a^{\text{ét}})$$

Moreover:

${}^{\overline{\circ}}\mathfrak{R}^{\text{ét}}$ up to (Ind1) $\overset{\sim}{\to}$ ${}^{\underline{\circ}}\mathfrak{R}^{\text{ét}}$ up to (Ind1), where

"up to (Ind1)" $\overset{\text{def}}{\Leftrightarrow}$ up to indeterminacies induced by "$\mathrm{Aut}(\mathrm{Prc}(\mathfrak{D}^{\vdash}_{T}))$"

étale-like holomorphic

theta values $(\mathrm{b}^{\text{ét}})$

$\curvearrowright$

étale-like holomorphic

number fields $(\mathrm{c}^{\text{ét}})$

$\curvearrowleft$

étale-like mono-analytic

tensor packet

log-shells $(\mathrm{a}^{\text{ét}})$

$\uparrow$

$(\mathrm{Ind}1)$

$m \in \mathbb{Z} \Rightarrow \mathfrak{R}^{\mathrm{Frob}}$: the collection of data consisting of:

$(\mathrm{a}^{\mathrm{Frob}})$ $\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $0 \leq j \leq l^{*}$

$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{V}_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}$, $\quad \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \subseteq \mathcal{I}^{\mathbb{Q}}$

[with the procession-normalized mono-analytic log-volumes]

$(\mathrm{b}^{\mathrm{Frob}})$ $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

$\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$ in $\prod_{j \in J} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})$

$(\mathrm{c}^{\mathrm{Frob}})$ $1 \leq j \leq l^{*}$

$({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{j} = ({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_{j}$ in $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{V}_{\mathbb{Q}}})$

$\Rightarrow ({}^{m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{j} \xrightarrow{\sim} ({}^{m}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{j}$

[whose ass. “degree” may be com. by means of the log-vol. of $(\mathrm{a}^{\mathrm{Frob}})$]

Then:

$(\mathrm{a}^{\mathrm{Kmm}})$ $(\mathrm{a}^{\mathrm{Frob}})$ up to (Ind2) $\overset{\sim}{\to}$ $(\mathrm{a}^{\text{ét}})$ up to (Ind2),

compatible w/ the respective log-volumes, where "up to (Ind2)"

$\overset{\mathrm{def}}{\Leftrightarrow}$ up to indeterminacies induced by "$\mathrm{Ism}_{\underline{v}}$, $\{\pm 1\}$" [cf. §1.3]

$(\mathrm{b}^{\mathrm{Kmm}})$ $(\mathrm{b}^{\mathrm{Frob}}) \overset{\sim}{\to} (\mathrm{b}^{\text{ét}})$

$(\mathrm{c}^{\mathrm{Kmm}})$ $(\mathrm{c}^{\mathrm{Frob}}) \overset{\sim}{\to} (\mathrm{c}^{\text{ét}})$

$\Rightarrow {}^{m}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \overset{\sim}{\to} {}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathrm{LGP}}$, ${}^{m}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} \overset{\sim}{\to} {}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathfrak{lgp}}$

Moreover:

- $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) \xrightarrow{\sim} \Psi^{\perp}_{\mathrm{LGP}}({}^{\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$ of $(\mathrm{b}^{\mathrm{Kmm}})$
- $({}^{m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_j \xrightarrow{\sim} ({}^{\circ}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathrm{MOD}})_j$ of $(\mathrm{c}^{\mathrm{Kmm}})$

are **mutually compatible** with one another [w.r.t. m]

[cf. the "non-interference property" discussed in §2.3 and §3.3].

$\Rightarrow$

${}^{m}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \xrightarrow{\sim} {}^{\circ}\mathcal{C}^{\Vdash\mathcal{D}}_{\mathrm{LGP}}$ of $(\mathrm{c}^{\mathrm{Kmm}})$

is **mutually compatible** with one another [w.r.t. m]

Recall: The upper semi-compatiblity discussed in §1.5:

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}} \Rightarrow$

$$\left((\Psi_{\mathrm{cns}}({}^{m}\mathfrak{F}_{\succ})_{|j|})^{\times}_{\underline{v}} \right)^{\mathrm{Gal}}$$

$$\curvearrowright$$

$$\begin{array}{ccccc}
\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};m}\mathcal{F}_{v_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};m+1}\mathcal{F}_{v_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & \cdots \\
\mathsf{Kmm} \downarrow \wr & & \mathsf{Kmm} \downarrow \wr & & \\
\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{v_{\mathbb{Q}}}) & = & \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};\circ}\mathcal{D}_{v_{\mathbb{Q}}}) & = & \cdots
\end{array}$$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}} \Rightarrow \exists$a similar diagram

One may introduce the following "indeterminacy" (Ind3) to $(\mathrm{a}^{\mathrm{Kmm}})$:

(Ind3): As one varies m, $(\mathrm{a}^{\mathrm{Kmm}})$ are [**not precisely compatible** but] **upper semi-compatible**.

$$\ldots \overset{\mathfrak{log}}{\to} {}^{\overline{-1}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\overline{0}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\overline{1}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \ldots$$

$$\downarrow \Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$$

$$\ldots \overset{\mathfrak{log}}{\to} {}^{\underline{-1}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\underline{0}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\underline{1}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \ldots$$

$\Rightarrow {}^{\overline{0}}\mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}} \overset{\sim}{\to} {}^{\underline{0}}\mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}}$ is compatible, under (Ind1, 2, 3), w/:

- $\mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}}({}^{\overline{\circ}}\mathfrak{D}^{\vdash}_{\Delta}) \overset{\sim}{\to} \mathfrak{F}_{\Delta}^{\vdash\times\boldsymbol{\mu}}({}^{\underline{\circ}}\mathfrak{D}^{\vdash}_{\Delta})$
- $\mathfrak{F}_{\mathrm{env}}^{\vdash\times\boldsymbol{\mu}}({}^{\overline{\circ}}\mathfrak{D}_{>}) \overset{\sim}{\to} \mathfrak{F}_{\mathrm{env}}^{\vdash\times\boldsymbol{\mu}}({}^{\underline{\circ}}\mathfrak{D}_{>})$
- ${}^{\overline{\circ}}\mathfrak{R}_{\mathrm{tht}} \overset{\sim}{\to} {}^{\underline{\circ}}\mathfrak{R}_{\mathrm{tht}}$, as well as the Gal. eval. "$\mathfrak{R}_{\mathrm{tht}} \rightsquigarrow (\mathrm{b}^{\text{ét}})$" [cf. §3.1]
- ${}^{\overline{\circ}}\mathfrak{R}_{\mathrm{NF}} \overset{\sim}{\to} {}^{\underline{\circ}}\mathfrak{R}_{\mathrm{NF}}$, as well as the Gal. eval. "$\mathfrak{R}_{\mathrm{NF}} \rightsquigarrow (\mathrm{c}^{\text{ét}})$" [cf. §2.1]

relative to the Kmm poly-isom. and poly-isom. of mono-theta env.

Fr.-like holomorphic theta values $(\mathrm{b}^{\mathrm{Frob}})$

$\wr\downarrow (\mathrm{b}^{\mathrm{Kmm}})$

étale-like holomorphic theta values $(\mathrm{b}^{\text{ét}})$

$\curvearrowright$

Fr.-like holomorphic number fields $(\mathrm{c}^{\mathrm{Frob}})$

$\wr\downarrow (\mathrm{c}^{\mathrm{Kmm}})$

étale-like holomorphic number fields $(\mathrm{c}^{\text{ét}})$

$\curvearrowleft$

mono-analytic tensor packet log-shells $(\mathrm{a}^{\text{ét}})$

$\uparrow$

$(\mathrm{Ind}1, 2, 3)$

§4.2 Log-Volume Estimates

$$\cdots \overset{\mathfrak{log}}{\to} {}^{\overline{-1}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\overline{0}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\overline{1}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \cdots$$

$$\downarrow \Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$$

$$\cdots \overset{\mathfrak{log}}{\to} {}^{\underline{-1}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\underline{0}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{\underline{1}}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \cdots$$

- Let us start with a q-plt obj. ${}^{\underline{0}}q \in {}^{\underline{0}}\mathcal{C}^{\Vdash}_{\Delta}$ at "$\underline{0}$" and compute $\deg({}^{\underline{0}}q)$.
- By definition, ${}^{\overline{0}}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \overset{\sim}{\to} {}^{\underline{0}}\mathcal{C}^{\Vdash}_{\Delta}$ of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$ maps the isom. class $[{}^{\overline{0}}\Theta]$ of a Θ-pilot object ${}^{\overline{0}}\Theta \in {}^{\overline{0}}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}$ at "$\overline{0}$" to the isom. class $[{}^{\underline{0}}q]$ of ${}^{\underline{0}}q$.
- Let us observe that $\deg({}^{\underline{0}}q)$ (resp. $\deg({}^{\overline{0}}\Theta)$) can be computed by means of the log-volume of ${}^{\underline{0}}q$ (resp. ${}^{\overline{0}}\Theta$).

- Let us apply the multiradial representation of "$\mathfrak{R}^{\mathrm{Frob}}$" [cf. §4.1].

$$\Rightarrow \text{“}{}^{0}\mathfrak{F}_{\Delta}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}} \overset{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}}{\overset{\sim}{\leftarrow}} {}^{\overline{0}}\mathfrak{F}_{\mathrm{LGP}}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}} \underset{\text{(Ind1, 2, 3)}\curvearrowright}{\overset{\text{multiradiality}}{\overset{\sim}{\rightarrow}}} {}^{0}\mathfrak{F}_{\mathrm{LGP}}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}};\ [{}^{\underline{0}}q] \mapsto [{}^{\overline{0}}\Theta] \mapsto [{}^{\underline{0}}\Theta]\text{”}$$

- In other words, roughly speaking, the multiradial representation asserts that "$[{}^{\underline{0}}q]$" can be interpreted as "$[{}^{\underline{0}}\Theta]$ up to (Ind1, 2, 3)".

- The log-volume on the log-shells is invariant with respect to the indeterminacies (Ind1) and (Ind2). On the other hand, by the upper semi-compatibility indeterminacy (Ind3), i.e.,"commutativity" at the level of inclusions of regions in log-shells, we must consider the log-volume of "an object up to (Ind3)" as an upper bound.

- In order to utilize the above interpretation

  [i.e., not $\underline{{}^{0}q =_{\curvearrowleft(\text{Ind1, 2, 3})} {}^{0}\Theta}$ but $\underline{[{}^{\underline{0}}q] =_{\curvearrowleft(\text{Ind1, 2, 3})} [{}^{\underline{0}}\Theta]}$]

  or, more precisely, to compute an upper bound of the log-volume of [not ${}^{\underline{0}}\Theta$ but] ${}^{\overline{0}}\Theta$ relative to the pt of view of [not "$\overline{0}$" but] "$\underline{0}$", it is nece'y to work w/ the "coll'n of modules", i.e., holomorphic hull.

In summary, we conclude:

$\deg(q)$ ($=$ the log-volume of the q-pilot object)

$\leq$ the log-volume of the holomorphic hull of $\bigcup_{(\text{Ind1, 2, 3})}$ Θ-pilot object

§4.3 Diophantine Inequalities

log-vol. of q-pilot $\leq$ log-vol. of holo. hull of $\bigcup_{(\text{Ind1, 2, 3})}$ Θ-pilot

Rough Estimate at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$:

$$\mu^{\log}(\underline{\underline{q}}_{\underline{v}} \curvearrowright \mathcal{O}_{K_{\underline{v}}}) \text{ ``} \leq \text{''} \frac{1}{l^{*}} \cdot \mu^{\log} \begin{pmatrix} \underline{\underline{q}}_{\underline{v}} \curvearrowright (\mathcal{I}_{\underline{v}})_0 \otimes (\mathcal{I}_{\underline{v}})_1 \\ \vdots \\ \underline{\underline{q}}_{\underline{v}}^{j^2} \curvearrowright (\mathcal{I}_{\underline{v}})_0 \otimes \cdots \otimes (\mathcal{I}_{\underline{v}})_j \\ \vdots \\ \underline{\underline{q}}_{\underline{v}}^{(l^{*})^2} \curvearrowright (\mathcal{I}_{\underline{v}})_0 \otimes \cdots \otimes (\mathcal{I}_{\underline{v}})_{l^{*}} \end{pmatrix}$$

Recall: $\operatorname{ord}(\underline{\underline{q}}_{\underline{v}}^{j^2}) = \frac{j^2}{2l} \cdot \operatorname{ord}(q_{\underline{v}}), \quad 0 < \operatorname{ord}(q_{\underline{v}})$

- l^{*} "=" $l/2$ • $\sum_{j=1}^{l^{*}} j+1$ "=" $(l/2)^2/2$ • $\sum_{j=1}^{l^{*}} j^2$ "=" $(l/2)^3/3$

- $\mu^{\log}((\mathcal{I}_{\underline{v}})_0 \otimes \cdots \otimes (\mathcal{I}_{\underline{v}})_j)$ "$\leq$" $(j+1) \cdot \mathrm{ord}(\mathfrak{d}_{K_{\underline{v}}/\mathbb{Q}_p})$

$\Rightarrow$

$$\begin{aligned}
-\mathrm{ord}(\underline{\underline{q}}_{\underline{v}}) \quad & \text{“} \leq \text{”} \quad \frac{1}{l^{*}} \cdot \sum_{j=1}^{l^{*}} \Big((j+1) \cdot \mathrm{ord}(\mathfrak{d}_{K_{\underline{v}}/\mathbb{Q}_p}) - \frac{j^2}{2l} \cdot \mathrm{ord}(q_{\underline{v}})\Big) \\
& \text{“} = \text{”} \quad \frac{2}{l} \cdot \Big(\frac{(l/2)^2}{2} \cdot \mathrm{ord}(\mathfrak{d}_{K_{\underline{v}}/\mathbb{Q}_p}) - \frac{(l/2)^3}{2l \cdot 3} \cdot \mathrm{ord}(q_{\underline{v}})\Big) \\
& = \quad \frac{l}{4} \cdot \Big(\mathrm{ord}(\mathfrak{d}_{K_{\underline{v}}/\mathbb{Q}_p}) - \frac{1}{6} \cdot \mathrm{ord}(q_{\underline{v}})\Big) \\
& = \quad \frac{l}{4} \cdot \Big(\mathrm{ord}(\mathfrak{d}_{K_{\underline{v}}/\mathbb{Q}_p}) - \frac{1}{6}(1 - \frac{12}{l^2}) \cdot \mathrm{ord}(q_{\underline{v}})\Big) - \mathrm{ord}(\underline{\underline{q}}_{\underline{v}})
\end{aligned}$$

$$\Rightarrow \; -1 \; \text{“} \leq \text{”} \; \frac{l}{4 \cdot \mathrm{ord}(\underline{\underline{q}}_{\underline{v}})} \cdot \Big(\mathrm{ord}(\mathfrak{d}_{K_{\underline{v}}/\mathbb{Q}_p}) - \frac{1}{6}(1 - \frac{12}{l^2}) \cdot \mathrm{ord}(q_{\underline{v}})\Big) - 1$$

$$\Rightarrow \quad \frac{1}{6} \cdot \mathrm{ord}(q_{\underline{v}}) \; \text{“} \leq \text{”} \; (1 - \frac{12}{l^2})^{-1} \cdot \mathrm{ord}(\mathfrak{d}_{K_{\underline{v}}/\mathbb{Q}_p})$$

$P \overset{\mathrm{def}}{=} \mathbb{P}^1_{\mathbb{Q}} \supseteq D \overset{\mathrm{def}}{=} \{0, 1, \infty\}_{\mathrm{red}}$, $U_P \overset{\mathrm{def}}{=} P \setminus D$

For $\lambda \in U_P(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \setminus \{0, 1\}$:

- A_λ: the elliptic curve/$\mathbb{Q}(\lambda)$ def'd by "$y^2 = x(x-1)(x-\lambda)$"
- $F_\lambda \overset{\mathrm{def}}{=} \mathbb{Q}(\lambda, \sqrt{-1}, A_\lambda[3 \cdot 5](\overline{\mathbb{Q}}))$

$\Rightarrow E_\lambda \overset{\mathrm{def}}{=} A_\lambda \times_{\mathbb{Q}(\lambda)} F_\lambda$ has at most split multipl. red. at $\forall \in \mathbb{V}(F_\lambda)$

- $\mathfrak{q}_\lambda \in \mathrm{ADiv}(F_\lambda)$: the eff. arith. div. det'd by the q-param'r of E_λ/F_λ
- $\mathfrak{f}_\lambda \in \mathrm{ADiv}(F_\lambda)$: the eff. "reduced" arithmetic divisor det'd by $\mathfrak{q}_\lambda$
- $\mathfrak{d}_\lambda \in \mathrm{ADiv}(F_\lambda)$: the eff. arith. div. det'd by the different of $F_\lambda/\mathbb{Q}$
- $d_\lambda \overset{\mathrm{def}}{=} [\mathbb{Q}(\lambda) : \mathbb{Q}]$ • $d^*_\lambda \overset{\mathrm{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d_\lambda$

For a prime number $l \geq 5$:

(λ, l): admissible $\overset{\text{def}}{\Leftrightarrow}$ $\exists$an initial Θ-data $(\overline{\mathbb{Q}}/F_\lambda,\ E_\lambda \setminus \{o\},\ l, \dots)$

s.t. E_λ has good reduction at $\forall \underline{v} \in \mathbb{V}(F_\lambda)^{\text{good}}$ of residue char. $\not| \ 2l$

Then "log-vol. of q-pil. $\leq$ log-vol. of holo. hull of $\bigcup_{(\text{Ind1, 2, 3})}$ Θ-pil."

$\Rightarrow$

(λ, l): admissible $\Rightarrow$

$$\frac{1}{6} \cdot \deg(\mathfrak{q}_\lambda^{\text{bad}}) \ \leq \ (1 + \frac{80 d_\lambda}{l}) \cdot (\deg(\mathfrak{d}_\lambda) + \deg(\mathfrak{f}_\lambda)) + 20 \cdot (d_\lambda^* \cdot l + \eta_{\text{prm}}),$$

where $\eta_{\text{prm}} \in \mathbb{R}_{>0}$ s.t. $\sharp\{\text{prime numbers} \leq \eta\} \leq \frac{4 \cdot \eta}{3 \cdot \log(\eta)}$ $(\forall \eta > \eta_{\text{prm}})$

[cf. the above "Rough Estimate"]

By means of this, let us prove the following $(*)$:

$(*)$: • $d \in \mathbb{Z}_{\geq 1}$ • $\epsilon \in \mathbb{R}_{>0}$ • $\mathcal{K}_\infty \subsetneq P(\mathbb{C})$: an ι-stable cpt domain

• S: a finite set of prime numbers s.t. $2 \in S$

• $\mathcal{K}_p \subsetneq P(\overline{\mathbb{Q}}_p)$: a certain "compact" domain $[p \in S]$

• $\mathcal{K} \stackrel{\text{def}}{=} P(\overline{\mathbb{Q}}) \cap \mathcal{K}_\infty \cap \bigcap_{p \in S} \mathcal{K}_p$, i.e., a compactly bounded subset

$\Rightarrow$ The function on $\mathcal{K}^{\leq d} \stackrel{\text{def}}{=} \{\, \lambda \in \mathcal{K} \mid d_\lambda \leq d \,\}$ given by

$$\lambda \ \mapsto \ \frac{1}{6} \cdot \deg(\mathfrak{q}_\lambda) - (1+\epsilon) \cdot (\deg(\mathfrak{d}_\lambda) + \deg(\mathfrak{f}_\lambda))$$

is bounded above.

Proof of $(*)$: By applying • the prime number theorem,

- the theory of arithmeticity of elliptic curves,
- the finiteness of $\{\, \lambda \in U_P(\overline{\mathbb{Q}}) \,|\, d_\lambda \leq d,\, \deg(\mathfrak{q}_\lambda) \leq C \,\}$ for $C \in \mathbb{R}$,
- the theory of Galois actions on torsion points of elliptic curve,

one may obtain the foll'g: For all but finitely many $\lambda \in \mathcal{K}^{\leq d}$, $\exists l_\lambda$ s.t.

(a) (λ, l_λ): admissible (In particular:

(b) $\frac{1}{6} \cdot \deg(\mathfrak{q}_\lambda^{\mathrm{bad}}) \ \leq\ (1 + \frac{80 d_\lambda}{l_\lambda}) \cdot (\deg(\mathfrak{d}_\lambda) + \deg(\mathfrak{f}_\lambda)) + 20 \cdot (d_\lambda^* \cdot l_\lambda + \eta_{\mathrm{prm}})$)

(c) $\mathrm{ord}_{l_\lambda}(q_{\mathfrak{l}}) \ <\ \deg(\mathfrak{q}_\lambda)^{1/2}$, where $\mathbb{V}(F_\lambda) \ni \mathfrak{l} | l_\lambda$

(d) $\deg(\mathfrak{q}_\lambda)^{1/2} \ \leq\ l_\lambda \ \leq 10 \ \cdot d^* \cdot \deg(\mathfrak{q}_\lambda)^{1/2} \cdot \log(2 \cdot d^* \cdot \deg(\mathfrak{q}_\lambda))$,

where $d^* \stackrel{\mathrm{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d$

- (a), (c) $\Rightarrow$ $\exists$an upper bound of the function

$$\lambda \ \mapsto \ \frac{1}{6}\deg(\mathfrak{q}_\lambda) - \frac{1}{6}\deg(\mathfrak{q}_\lambda^{\mathrm{bad}}) - \deg(\mathfrak{q}_\lambda)^{1/2}\log(2d^*\deg(\mathfrak{q}_\lambda))$$

- (b), (d) $\Rightarrow$ $\frac{1}{6}\deg(\mathfrak{q}_\lambda^{\mathrm{bad}}) \leq (1+\frac{d^*}{\deg(\mathfrak{q}_\lambda)^{1/2}})(\deg(\mathfrak{d}_\lambda)+\deg(\mathfrak{f}_\lambda))$

$$+200(d^*)^2\deg(\mathfrak{q}_\lambda)^{1/2}\log(2d^*\deg(\mathfrak{q}_\lambda)) + 20\eta_{\mathrm{prm}}$$

$\Rightarrow$ $\exists$an upper bound of the function $\lambda \mapsto$

$$(1-\frac{2}{5}\frac{(60d^*)^2\log(2d^*\deg(\mathfrak{q}_\lambda))}{\deg(\mathfrak{q}_\lambda)^{1/2}})\frac{1}{6}\deg(\mathfrak{q}_\lambda) - (1+\frac{d^*}{\deg(\mathfrak{q}_\lambda)^{1/2}})(\deg(\mathfrak{d}_\lambda)+\deg(\mathfrak{f}_\lambda))$$

Thus, $\exists$an upper bound of the function

$$\lambda \ \mapsto \ \frac{1}{6}\deg(\mathfrak{q}_\lambda) - (1+\epsilon)(\deg(\mathfrak{d}_\lambda)+\deg(\mathfrak{f}_\lambda))$$

By the above $(*)$, together with the theory of noncritical Belyi maps, we obtain the following theorem:

- $d \in \mathbb{Z}_{\geq 1}$ • $\epsilon \in \mathbb{R}_{>0}$ • L: a number field
- V: a projective smooth curve $/L$
- $D \subseteq V$: a [possibly empty] reduced divisor s.t. $\Omega^1_{V/L}(D)$: ample

$\Rightarrow$

The function on $(V \setminus D)^{\leq d} \overset{\text{def}}{=} \{\, x \in V \setminus D \mid [\kappa(x) : \mathbb{Q}] \leq d \,\}$ given by

$$\lambda \ \mapsto \ \mathrm{ht}_{\Omega^1_{V/L}(D)} - (1+\epsilon) \cdot (\text{log-diff}_V + \text{log-cond}_D)$$

is bounded above.