The absolute anabelian geometry of quasi-tripods

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The notion of a quasi-tripod (cf. Definition 4 below) may be regarded as one natural generalization of the notion of a hyperbolic curve of Belyi-type (cf. Remark 5, (1), below). In the present talk, we discuss the absolute anabelian geometry of quasi-tripods.

1. Main result

Definition 1. Let k be a field and p a prime number.

- We shall say that the field k is *algebraic* (respectively, *sub-p-adic*; *generalized sub-p-adic*) if k is isomorphic to a subfield of an algebraic closure of $\mathbb{Q}$ (respectively, to a subfield of a finitely generated extension of $\mathbb{Q}_p$; to a subfield of a finitely generated extension of the p-adic completion of a (not necessarily finite) unramified extension of $\mathbb{Q}_p$).
- We shall say that the field k is *strictly sub-p-adic* if k is sub-p-adic and contains a subfield isomorphic to $\mathbb{Q}_p$.

Definition 2.

(1) Let k be a field and $X_\circ$, $X_\bullet$ orbivarieties over k. Then we shall say that the pair $(X_\circ, X_\bullet)$ is *relatively anabelian* if, for each separable closure $\overline{k}$ of k, the natural map

$$\mathrm{Isom}_k(X_\circ, X_\bullet) \to \mathrm{Isom}_{\mathrm{Gal}(\overline{k}/k)}\big(\pi_1(X_\circ), \pi_1(X_\bullet)\big)/\mathrm{Inn}\big(\pi_1(X_\bullet \times_k \overline{k})\big)$$

is bijective.

(2) For each $\Box \in \{\circ, \bullet\}$, let $k_\Box$ be a field and $X_\Box$ an orbivariety over $k_\Box$. Then we shall say that the pair $(X_\circ, X_\bullet)$ is *absolutely anabelian* if the natural map

$$\mathrm{Isom}(X_\circ, X_\bullet) \to \mathrm{Isom}\big(\pi_1(X_\circ), \pi_1(X_\bullet)\big)/\mathrm{Inn}\big(\pi_1(X_\bullet)\big)$$

is bijective.

Remark 3. One fundamental result with respect to the notion defined in Definition 2, (1), is the following result proved by S. Mochizuki (cf. [3], Theorem 4.12): Let k be a generalized sub-p-adic field for some prime number p and $X_\circ$, $X_\bullet$ hyperbolic orbicurves over k. Then the pair $(X_\circ, X_\bullet)$ is relatively anabelian.

Definition 4. Let k be a field of characteristic zero and X a hyperbolic orbicurve over k. Then we shall say that X is a *quasi-tripod* if there exist finitely many hyperbolic orbicurves $X_1, X_2, \ldots, X_n$ such that X_1 is isomorphic to X, X_n is isomorphic to the split tripod $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over k, and, moreover, for each $i \in \{1, 2, \cdots, n-1\}$, X_{i+1} is related to X_i in one of the following four ways:

- There exists a finite étale morphism $X_{i+1} \to X_i$.
- There exists a finite étale morphism $X_i \to X_{i+1}$.
- There exists an open immersion $X_i \hookrightarrow X_{i+1}$.
- There exists a partial coarsification morphism $X_i \to X_{i+1}$.

Remark 5.

(1) One verifies immediately that, for a finite extension k of $\mathbb{Q}_p$ and a hyperbolic curve over k, the hyperbolic curve is of Belyi type (cf. [4], Definition 2.3, (ii)) if and only if the hyperbolic curve is a quasi-tripod and, moreover, may be descended to a subfield of k finite over $\mathbb{Q}$.
(2) One also verifies immediately that every hyperbolic curve over a field of characteristic zero whose smooth compactification is of genus less than two is a quasi-tripod.

The main result of the present talk is as follows (cf. [1], Theorem A):

Theorem 6. *For each $\Box \in \{\circ, \bullet\}$, let $p_\Box$ be a prime number, $k_\Box$ a field of characteristic zero, and $X_\Box$ a hyperbolic orbicurve over $k_\Box$. Suppose that the following two conditions* (1), (2) *are satisfied:*

(1) *Either $X_\circ$ or $X_\bullet$ is a quasi-tripod.*
(2) *One of the following three conditions* (a), (b), (c) *is satisfied:*
    (a) *For each $\Box \in \{\circ, \bullet\}$, the field $k_\Box$ is algebraic, generalized sub-$p_\Box$-adic, and Hilbertian.*
    (b) *For each $\Box \in \{\circ, \bullet\}$, the field $k_\Box$ is transcendental and finitely generated over some algebraic and sub-$p_\Box$-adic field.*
    (c) *For each $\Box \in \{\circ, \bullet\}$, the field $k_\Box$ is strictly sub-$p_\Box$-adic.*

Then the pair $(X_\circ, X_\bullet)$ is absolutely anabelian.

Remark 7.

- Theorem 6 in the case where either (a) or (b) is satisfied partially generalizes the following result proved by A. Tamagawa (cf. [6], Theorem 0.4): For each $\Box \in \{\circ, \bullet\}$, let $k_\Box$ be a finitely generated extension of $\mathbb{Q}$ and $X_\Box$ an affine hyperbolic curve over $k_\Box$. Then the pair $(X_\circ, X_\bullet)$ is absolutely anabelian.
- Theorem 6 in the case where (c) is satisfied generalizes the following result proved by S. Mochizuki (cf. [4], Corollary 2.3): For each $\Box \in \{\circ, \bullet\}$, let $p_\Box$ be a prime number, $k_\Box$ a finite extension of $\mathbb{Q}_{p_\Box}$, and $X_\Box$ a hyperbolic curve over $k_\Box$. Suppose that either $X_\circ$ or $X_\bullet$ is of Belyi-type (cf. Remark 5, (1)). Then the pair $(X_\circ, X_\bullet)$ is absolutely anabelian.

2. Two applications

The following result is one application of the main result (cf. [1], Theorem B):

Theorem 8. *For each $\Box \in \{\circ, \bullet\}$, let $n_\Box$ be a positive integer, $p_\Box$ a prime number, $k_\Box$ a field of characteristic zero, and $X_\Box$ a hyperbolic curve over $k_\Box$; write $(X_\Box)_{n_\Box}$ for the $n_\Box$-th configuration space of $X_\Box$. Suppose that the following two conditions* (1), (2) *are satisfied:*

(1) *One of the following two conditions is satisfied:*
    - *The inequality $1 < \max\{n_\circ, n_\bullet\}$ holds, and, moreover, either $X_\circ$ or $X_\bullet$ is affine.*

- *The inequality* $2 < \max\{n_\circ, n_\bullet\}$ *holds.*

(2) *One of the following three conditions* (a), (b), (c) *is satisfied:*
  (a) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is algebraic, generalized sub-*$p_\square$*-adic, and Hilbertian.*
  (b) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is transcendental and finitely generated over some algebraic and sub-*$p_\square$*-adic field.*
  (c) *For each* $\square \in \{\circ, \bullet\}$, *the field* $k_\square$ *is strictly sub-*$p_\square$*-adic.*

Then the pair $((X_\circ)_{n_\circ}, (X_\bullet)_{n_\bullet})$ *is absolutely anabelian.*

Definition 9. We shall say that an open basis for the Zariski topology of a given smooth variety over a field is *relatively anabelian* (respectively, *absolutely anabelian*) if, for each members U and V of the open basis, the pair (U, V) is relatively anabelian (respectively, absolutely anabelian).

Remark 10. One fundamental result with respect to the notion defined in Definition 9 is the following result proved by the author (cf. [2], Theorem A): Let k be a generalized sub-p-adic field for some prime number p. Then an arbitrary smooth variety over k has a relatively anabelian open basis.

The following result is one application of the main result (cf. [1], Theorem C):

Theorem 11. *Let* k *be a field and* p *a prime number. Suppose that one of the following three conditions* (a), (b), (c) *is satisfied:*

(a) *The field* k *is algebraic, generalized sub-*p*-adic, and Hilbertian.*
(b) *The field* k *is transcendental and finitely generated over some algebraic and sub-*p*-adic field.*
(c) *The field* k *is strictly sub-*p*-adic.*

Then an arbitrary smooth variety of positive dimension over k *has an absolutely anabelian open basis.*

Remark 12. Theorem 11 in the case where either (a) or (b) is satisfied, together with the result discussed in Remark 10, generalizes the following result proved by A. Schmidt and J. Stix (cf. [5], Corollary 1.7): Let k be a finitely generated extension of $\mathbb{Q}$. Then an arbitrary smooth variety over k has a relatively anabelian open basis and absolutely anabelian open basis.

References

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