Combinatorial Anabelian Geometry in the Absence of Group-theoretic Cuspidality

Yuichiro Hoshi 7 July, 2021

RIMS Workshop

"Combinatorial Anabelian Geometry and Related Topics"

 $\frac{(T3) \text{ combinatorial cuspidalization and "FC} = F" \text{ results}}{(cf. Mochizuki's overview)}$

 Σ : a set of prime numbers s.t. $\#\Sigma = 1$ or Σ is the set of \forall prime numbers $n \geq 0$ k: an algebraically closed field of characteristic $\notin \Sigma$ X: a hyperbolic curve/k of type (g, r) X_n : the *n*-th configuration space of X $\Pi_n \stackrel{\text{def}}{=} \pi_1(X_n)^{\Sigma}$

- Definition ·

 $\alpha \in \operatorname{Out}(\Pi_n)$

• α : <u>F-admissible</u> $\stackrel{\text{def}}{\Leftrightarrow} \alpha(F) = F$ for \forall fiber subgroup $F \subseteq \Pi_n$ • α : <u>FC-admissible</u> $\stackrel{\text{def}}{\Leftrightarrow} \alpha$: F-admissible

and, moreover, for $1 \leq \forall m \leq n$, the Π_m -conj. class of isom.s of $\operatorname{Ker}(\Pi_m \twoheadrightarrow \Pi_{m-1}) \stackrel{\sim}{\leftarrow} \pi_1(\text{a geom. fiber of } X_m \to X_{m-1})^{\Sigma}$ det'd by α induces a self-bijection of the set of cuspidal inertial subgroups

 $\operatorname{Out}(\Pi_n) \supseteq \operatorname{Out}^{\mathrm{FC}}(\Pi_n) \supseteq \operatorname{Out}^{\mathrm{FC}}(\Pi_n)$

Combinatorial Cuspidalization

the issue of whether or not the natural homomorphism

 $\operatorname{Out}^{\mathcal{F}(\mathcal{C})}(\Pi_{n+1}) \longrightarrow \operatorname{Out}^{\mathcal{F}(\mathcal{C})}(\Pi_n)$

is injective (resp. surjective; bijective)

"FC = F" results

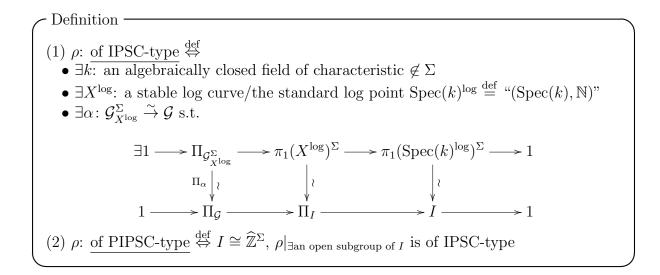
the issue of whether or not the natural inclusion

 $\operatorname{Out}^{\operatorname{FC}}(\Pi_n) \longrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_n)$

is bijective

Let us prove some results related to these two issues as applications of <u>combinatorial anabelian results</u>. Let us prove some results related to these two issues as applications of <u>combinatorial anabelian results</u>.

Theorem 0 [Combinatorial Anabelian Results] \mathcal{G} : a semi-graph of anabelioids of PSC-type (1) [Prp 2.6 of my 1st talk, i.e., of the Monday 2nd talk] $\Pi_{\mathcal{G}} + (\Pi_{\mathcal{G}} \stackrel{\text{open}}{\supseteq} \forall \Pi_{\mathcal{H}} \twoheadrightarrow \Pi_{\mathcal{H}}^{ab/Cusp}) \Rightarrow \Pi_{\mathcal{G}} + \text{cuspidal subgroups}$ (2) [Main Thm of §4 of my 1st talk, i.e., of the Monday 2nd talk] $\Pi_{\mathcal{G}} + (\rho: I \stackrel{\text{PIPSC}}{\to} \operatorname{Aut}(\mathcal{G}) \hookrightarrow \operatorname{Out}(\Pi_{\mathcal{G}})) \Rightarrow \Pi_{\mathcal{G}} + \text{verticial subgroups}$

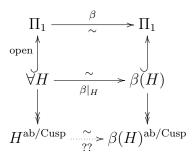


- Theorem 1 [CbTpI, Theorem A, (ii)] — Im(Out^F(Π_{n+1}) \rightarrow Out^F(Π_n)) \subseteq Out^{FC}(Π_n)

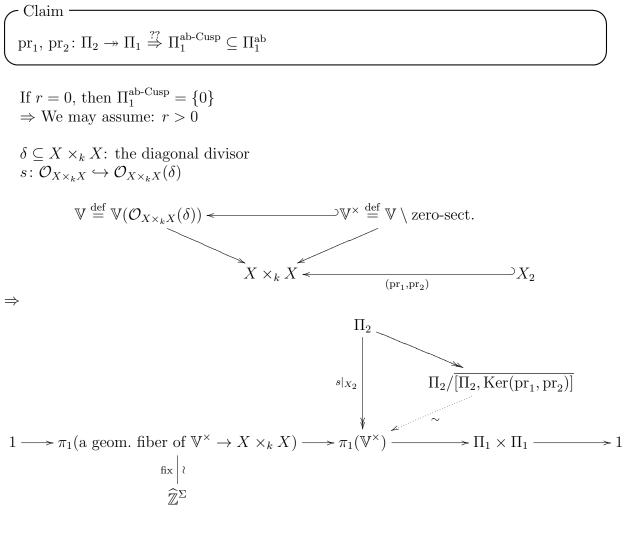
Remark: β does not depend on the choice of *i* (cf. [CbTpI, Theorem A, (i)]).

$$\beta \stackrel{??}{\in} \operatorname{Out}^{\mathrm{FC}}(\Pi_1)$$
, i.e., does β preserve the cusps?

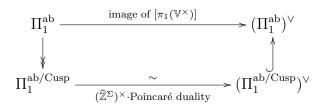
 \uparrow by Thm 0, (1)



For simplicity: Consider the case: $H = \Pi_1 \ (\Rightarrow \beta(H) = \Pi_1)$ Thus:



$$H^{2}(\Pi_{1} \times \Pi_{1}, \widehat{\mathbb{Z}}^{\Sigma}) \stackrel{\Pi_{1}: \text{ free}}{\xrightarrow{\sim}} H^{1}(\Pi_{1}, H^{1}(\Pi_{1}, \widehat{\mathbb{Z}}^{\Sigma})) \xrightarrow{\sim} \text{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_{1}^{\text{ab}}, (\Pi_{1}^{\text{ab}})^{\vee})$$



In particular: $\Pi_1^{\text{ab-Cusp}} = \text{Ker}(\text{image of } [\pi_1(\mathbb{V}^{\times})])$

- Theorem 2 [CbTpII, Theorem A, (ii)], [HMT, Corollary 2.2] – $\operatorname{Out}^{\mathrm{F}}(\Pi_n) = \operatorname{Out}^{\mathrm{FC}}(\Pi_n)$ if either 'n = 2, g = 0', ' $n = 3, r \neq 0$ ', or ' $n \geq 4$ '

Consider the case:

- $r \ge 2$
- $\alpha \in \text{Out}^{F}(\Pi_{2})$ whose image $\beta \stackrel{\text{Thm 1}}{\in} \text{Out}^{FC}(\Pi_{1})$ acts on the set of cusps <u>trivially</u>

a "hint" of the C-admissibility of α

 $\beta \curvearrowright \Pi_1$ preserves the cusps (by Thm 1)

 $\uparrow \mathrm{pr}_1$

 $\alpha \curvearrowright \Pi_2$

 $\beta \curvearrowright \Pi_1$

U

 $\alpha \curvearrowright \operatorname{Ker}(\operatorname{pr}_1)$ Does this preserve the cusps? Theorem 3 [CbTpII, Theorem A, (i)] Out^F(Π_{n+1}) \rightarrow Out^F(Π_n) is: (1) <u>injective</u> if ' $n \ge 1$ ' and ' $(n, r) \ne (1, 0)$ ' (2) <u>bijective</u> if either ' $n \ge 4$ ' or ' $n \ge 3$ and $r \ge 1$ '

by Thm 1, 2, together w/ some "standard arguments" (cf., e.g., Minamide's talk yesterday for Thm 3, (1))

Theorem 4 [CbTpII, Theorem B, (i), (ii)] Suppose: $(g, r) \notin \{(0, 3), (1, 1)\}$ \Rightarrow $\operatorname{Out}(\Pi_n) = \operatorname{Out}^{\mathrm{F}}(\Pi_n) \times \mathfrak{S}_n \text{ if } (n, r) \neq (2, 0)$ $(\stackrel{\mathrm{Thm 2}}{=} \operatorname{Out}^{\mathrm{FC}}(\Pi_n) \times \mathfrak{S}_n \text{ if either } `n = 2, g = 0`, `n = 3, r \neq 0`, \text{ or } `n \geq 4`)$

by a main result of [MT] (cf. also Sawada's talk yesterday) and Thm 3, (1)

<u>References</u>

[MT]	The Algebraic and Anabelian Geometry of Configuration Spaces
[CbTpI]	Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic
	Curves I: Inertia Groups and Profinite Dehn Twists
[CbTpII]	Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic
	Curves II: Tripods and Combinatorial Cuspidalization

[HMT] Combinatorial Construction of the Absolute Galois Group of the Field of Rational Numbers scratch paper

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Synchronization of Tripods and Glueability of Combinatorial Cuspidalizations

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(T6) tripod synchronization and the tripod homomorphism (cf. Mochizuki's overview)

$$\begin{split} &\Sigma: \text{ a set of prime numbers s.t. } \#\Sigma = 1 \text{ or } \Sigma \text{ is the set of } \forall \text{prime numbers } n \geq 0 \\ &k: \text{ an algebraically closed field of characteristic } \notin \Sigma \\ &S^{\log \overset{\text{def}}{=}} \text{ "Spec}(k,\mathbb{N}) \text{": the standard log point whose underlying scheme is } \operatorname{Spec}(k) \\ &X^{\log}: \text{ a stable log curve}/S^{\log} \text{ of type } (g,r) \\ &\mathcal{G}: \text{ the semi-graph of anabelioids of pro-} \Sigma \text{ PSC-type associated to } X^{\log} \\ &X^{\log}_n: \text{ the } n\text{-th log configuration space of } X^{\log} \\ &\Pi_n \overset{\text{def}}{=} \operatorname{Ker}(\pi_1(X_n^{\log})^\Sigma \to \pi_1(S^{\log})^\Sigma) = \operatorname{Ker}(\pi_1(X_n^{\log}) \to \pi_1(S^{\log}))^\Sigma \end{split}$$

Various tripods appear in X_n^{\log} .

$$X^{\log} = X_1^{\log}$$
 Π_1

 \uparrow
 X_2^{\log} Π_2

 \uparrow
 X_3^{\log} Π_3

Tripod Synchronization

= synchronization among the various tripods in Π_n

 \Rightarrow an outer automorphism of Π_n typically induces
the same outer automorphism on the various tripods in Π_n

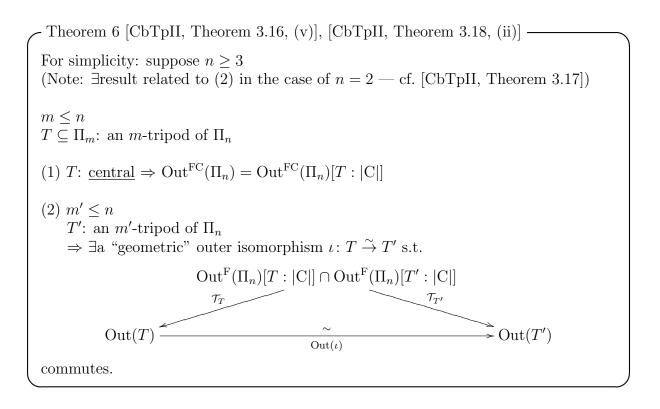
- Definition $m \leq n$ $T \subseteq \Pi_m: \text{ an } \underline{m\text{-tripod}} \text{ of } \Pi_n \stackrel{\text{def}}{\Leftrightarrow} T: \text{ a verticial subgroup "of type } (0,3)" \text{ of "}\Pi_{\text{a geom. fiber of } X_m^{\log} \to X_{m-1}^{\log}"}$ $= \operatorname{Ker}(\Pi_m \twoheadrightarrow \Pi_{m-1}) \subseteq \Pi_m$ Then: $\operatorname{Out}^{|\mathcal{C}|}(T) \subseteq \operatorname{Out}(T)$: the subgroup consisting of α s.t. α induces the id. on the set of conj. classes of cuspidal inertia subgroups of T - Definition Suppose: $n \ge 3$ $T \subseteq \Pi_3$: a 3-tripod of Π_n T: <u>central</u> $\stackrel{\text{def}}{\Leftrightarrow} T$ arises as: $X^{\log} = X_1^{\log}$ Π_1 \uparrow X_2^{\log} Π_2 \uparrow X_3^{\log} Π_3

Theorem 5 [CbTpII, Theorem C, (i)] $m \leq n$ $T \subseteq \Pi_m$: an *m*-tripod of Π_n $\Rightarrow C_{\Pi_m}(T) = N_{\Pi_m}(T) = T \times Z_{\Pi_m}(T)$ Remark -G: a group $H \subseteq G$: a subgroup $\alpha \in \operatorname{Aut}(G)$ \Rightarrow One can define the restriction $\alpha|_H \in \operatorname{Aut}(H)$ if α preserves $H \subseteq G$. On the other hand: $\alpha \in \operatorname{Out}(G) = \operatorname{Aut}(G) / \operatorname{Inn}(G)$ \Rightarrow One cannot define the "restriction" $\alpha|_H \in \text{Out}(H)$ in general even if α preserves the conjugacy class of $H \subseteq G$. The "natural rest." is not $\in Out(H) = Aut(H)/Inn(H)$ but $\in Aut(H)/Inn(N_G(H))$. In particular: • α preserves the conjugacy class of $H \subseteq G$ • $N_G(H) = Z_G(H) \cdot H$ \Rightarrow One can define the restriction $\alpha|_H \in \text{Out}(H)$.

- Definition — m < n

 $T \subseteq \Pi_m$: an *m*-tripod of Π_n

- $\operatorname{Out}^{\mathrm{F}}(\Pi_n)[T] \subseteq \operatorname{Out}^{\mathrm{F}}(\Pi_n)$: the subgroup consisting of α s.t. the outer autom. of Π_m induced by α preserves the Π_m -conj. class of $T \subseteq \Pi_m$
- \mathfrak{T}_T : Out^F(Π_n)[T] \rightarrow Out(T) (well-defined by Thm 5), the tripod homomorphism associated to T
- $\operatorname{Out}^{\mathsf{F}}(\Pi_n)[T:|\mathbf{C}|] \subseteq \operatorname{Out}^{\mathsf{F}}(\Pi_n)[T]$: the pull-back of $\operatorname{Out}^{|\mathbf{C}|}(T) \subseteq \operatorname{Out}(T)$ by \mathfrak{T}_T



Glueability of Combinatorial Cuspidalizations

<u>One Dimensional Case</u>

✓ Definition ·

- $\operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G}) \subseteq \operatorname{Aut}(\mathcal{G})$: the subgroup consisting of α s.t. $\alpha \curvearrowright$ the underlying semi-graph is trivial
- $\operatorname{Dehn}(\mathcal{G}) \subseteq \operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G})$: the subgroup consisting of α s.t. for $\forall v \in \operatorname{Vert}(\mathcal{G}), \ \alpha|_{\mathcal{G}_v}$ is trivial
- $\operatorname{Glu}^{|\operatorname{grph}|}(\mathcal{G}) \subseteq \prod_{v \in \operatorname{Vert}(\mathcal{G})} \operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G}_v)$: the subgp consisting of $(\alpha_v)_v$ s.t. $\chi_v^{\operatorname{cycl}}(\alpha_v) = \chi_w^{\operatorname{cycl}}(\alpha_w)$ for $\forall v, w \in \operatorname{Vert}(\mathcal{G})$

Theorem 7 [CbTpI, Theorem B, (iii)] –

(1) The natural homomorphism

$$\operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G}) \longrightarrow \prod_{v \in \operatorname{Vert}(\mathcal{G})} \operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G}_v)$$

factors through the subgroup

$$\operatorname{Glu}^{|\operatorname{grph}|}(\mathcal{G}) \subseteq \prod_{v \in \operatorname{Vert}(\mathcal{G})} \operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G}_v).$$

(2) The resulting homomorphism

$$\operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G}) \longrightarrow \operatorname{Glu}^{|\operatorname{grph}|}(\mathcal{G})$$

is a surjective homomorphism whose $\underline{\mathrm{kernel}}$ is given by

 $\operatorname{Dehn}(\mathcal{G}) \subseteq \operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G}).$

 $1 \longrightarrow \operatorname{Dehn}(\mathcal{G}) \longrightarrow \operatorname{Aut}^{|\operatorname{grph}|}(\mathcal{G}) \longrightarrow \operatorname{Glu}^{|\operatorname{grph}|}(\mathcal{G}) \longrightarrow 1.$

Observe: (1) is a formal consequence of "Synchronization of Cyclotomes".

- Corollary 8 [CbTpII, Theorem A, (iii)] Suppose: $(g, r) \notin \{(0, 3), (1, 1)\}$ \Rightarrow The injective (cf. Minamide's talk yesterday) homomorphism $\operatorname{Out}^{\operatorname{FC}}(\Pi_2) \hookrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_1)$ is <u>not surjective</u>

<u>Proof</u> of the assertion that $\operatorname{Out}^{\operatorname{FC}}(\Pi_3) \hookrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_1)$ is not surjective

The structure of $(\dots \to \Pi_{m+1} \to \Pi_m \to \dots)$ depends only on (g, r) \Rightarrow We may assume: X^{\log} is totally degenerate, i.e., \forall vertex of \mathcal{G} is "of type (0, 3)"

(in a spirit of Tripodal Transport cf. Mochizuki's talk, last week)

 $\begin{array}{l} (g,r) \notin \{(0,3),(1,1)\} \Rightarrow \exists v, w \in \operatorname{Vert}(\mathcal{G}) \colon \underline{\operatorname{distinct}}\\ \alpha_v \in \operatorname{Out}^{|\mathcal{C}|}(\Pi_v), \, \alpha_w \in \operatorname{Out}^{|\mathcal{C}|}(\Pi_w) \text{ s.t.}\\ (a) \, \alpha_v \neq \phi^{-1} \alpha_w \phi \text{ for } \forall \text{"geometric" isomorphism } \phi \colon \Pi_v \xrightarrow{\sim} \Pi_w\\ (b) \, \chi_v^{\operatorname{cycl}}(\alpha_v) = \chi_w^{\operatorname{cycl}}(\alpha_w)\\ (\mathrm{cf. "Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \subseteq \operatorname{Out}^{|\mathcal{C}|}(T)") \end{array}$

 $\stackrel{\text{(b), Thm 7, (2)}}{\Rightarrow} \exists \alpha \in \text{Aut}^{|\text{grph}|}(\mathcal{G}) \ (\subseteq \text{Out}^{\text{FC}}(\Pi_1)) \text{ s.t. } \alpha|_{\Pi_v} = \alpha_v, \ \alpha|_{\Pi_w} = \alpha_w$

Assume: $\operatorname{Out}^{\operatorname{FC}}(\underline{\Pi}_3) \hookrightarrow \operatorname{Out}^{\operatorname{FC}}(\Pi_1)$ is surjective $\Rightarrow \exists \alpha_3 \in \operatorname{Out}^{\operatorname{FC}}(\Pi_3)$ whose image in $\operatorname{Out}^{\operatorname{FC}}(\Pi_1)$ is $= \alpha$ But this contradicts (a) and Thm 6, (2). Higher Dimensional Case

For simplicity: suppose $n \ge 3$ (Note: \exists result in the case of n = 2)

C Definition -

Observe: (3) is a formal consequence of "Tripod Synchronization" (cf. Thm 6).

- Corollary 10 [CbTpII, Theorem C, (iv)] – Suppose: $n \ge 3$ $T \subseteq \Pi_3$: a <u>central</u> 3-tripod of Π_n Suppose, moreover: either $r \ne 0$ or $n \ge 4$

(1) The tripod homomorphism

$$\mathfrak{T}_T: \operatorname{Out}^{\operatorname{FC}}(\Pi_n) \stackrel{\operatorname{Thm 6, }(1)}{=} \operatorname{Out}^{\operatorname{FC}}(\Pi_n)[T] \longrightarrow \operatorname{Out}(T)$$

factors through the subgroup "GT" of Out(T).

(2) The resulting homomorphism

$$\mathfrak{T}_T: \operatorname{Out}^{\operatorname{FC}}(\Pi_n) \stackrel{\operatorname{Thm 6, (1)}}{=} \operatorname{Out}^{\operatorname{FC}}(\Pi_n)[T] \longrightarrow \operatorname{"GT"} \text{ in } \operatorname{Out}(T)$$

is surjective.

Proof of (2)

The validity of the assertion depends only on (n, g, r)

- ⇒ We may assume: X^{\log} is totally degenerate, i.e., \forall vertex of \mathcal{G} is "of type (0,3)" (in a spirit of Tripodal Transport cf. Mochizuki's talk, last week)
- $\gamma \in \text{``GT''}$
- $\Rightarrow \forall v \in \operatorname{Vert}(\mathcal{G}), \exists \gamma_{v,n} \in \operatorname{Out}^{\operatorname{FC}}((\Pi_v)_n)^{|\operatorname{grph}|}$ whose image in $\operatorname{Out}(\Pi_v)$ is $= \gamma$

 $\stackrel{\text{Thm 6, (2)}}{\Rightarrow} \forall v \in \text{Vert}(\mathcal{G}), \ \mathfrak{T}_{\text{a ctrl tpd in }(\Pi_v)_3}(\gamma_{v,n}) = \gamma$

 $\stackrel{\text{Thm 9, (4)}}{\Rightarrow} \exists \gamma_n \in \text{Out}^{\text{FC}}(\Pi_n)^{|\text{grph}|} \text{ whose image in } \text{Out}^{\text{FC}}((\Pi_v)_n)^{|\text{grph}|} \\ \text{is} = \gamma_{v,n} \text{ for } \forall v \in \text{Vert}(\mathcal{G})$

 $\stackrel{\text{Thm 6, (2)}}{\Rightarrow} \mathfrak{T}_T(\gamma_n) = \gamma, \text{ as desired}$

<u>References</u>

[CbCsp]	On the Combinatorial Cuspidalization of Hyperbolic Curves
[CbTpI]	Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic
	Curves I: Inertia Groups and Profinite Dehn Twists
[CbTpII]	Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic
	Curves II: Tripods and Combinatorial Cuspidalization

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