Combinatorial Anabelian Geometry in the Absence of Group-theoretic Cuspidality

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RIMS Workshop
"Combinatorial Anabelian Geometry and Related Topics"

(T3) combinatorial cuspidalization and "FC = F" results

(cf. Mochizuki's overview)

Σ: a set of prime numbers s.t. $\#\Sigma = 1$ or Σ is the set of $\forall$prime numbers
$n \geq 0$
k: an algebraically closed field of characteristic $\notin \Sigma$
X: a hyperbolic curve$/k$ of type (g, r)
X_n: the n-th configuration space of X
$\Pi_n \stackrel{\text{def}}{=} \pi_1(X_n)^\Sigma$

Definition

$\alpha \in \mathrm{Out}(\Pi_n)$

- α: F-admissible $\stackrel{\text{def}}{\Leftrightarrow}$ $\alpha(F) = F$ for $\forall$fiber subgroup $F \subseteq \Pi_n$
- α: FC-admissible $\stackrel{\text{def}}{\Leftrightarrow}$ α: F-admissible
  and, moreover, for $1 \leq \forall m \leq n$,
  the Π_m-conj. class of isom.s of
  $$\mathrm{Ker}(\Pi_m \twoheadrightarrow \Pi_{m-1}) \stackrel{\sim}{\leftarrow} \pi_1(\text{a geom. fiber of } X_m \to X_{m-1})^\Sigma$$
  det'd by α induces a self-bijection of the set of cuspidal inertial subgroups

$\mathrm{Out}(\Pi_n) \supseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n) \supseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$

Combinatorial Cuspidalization

the issue of whether or not the natural homomorphism

$$\mathrm{Out}^{\mathrm{F(C)}}(\Pi_{n+1}) \longrightarrow \mathrm{Out}^{\mathrm{F(C)}}(\Pi_n)$$

is injective (resp. surjective; bijective)

"FC = F" results

the issue of whether or not the natural inclusion

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \hookrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

is bijective

Let us prove some results related to these two issues
as applications of combinatorial anabelian results.

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as applications of combinatorial anabelian results.

Theorem 0 [Combinatorial Anabelian Results]

$\mathcal{G}$: a semi-graph of anabelioids of PSC-type

(1) [Prp 2.6 of my 1st talk, i.e., of the Monday 2nd talk]
$\Pi_{\mathcal{G}} + (\Pi_{\mathcal{G}} \overset{\text{open}}{\supseteq} \forall \Pi_{\mathcal{H}} \twoheadrightarrow \Pi_{\mathcal{H}}^{\text{ab/Cusp}}) \Rightarrow \Pi_{\mathcal{G}}$ + cuspidal subgroups

(2) [Main Thm of §4 of my 1st talk, i.e., of the Monday 2nd talk]
$\Pi_{\mathcal{G}} + (\rho\colon I \overset{\text{PIPSC}}{\to} \text{Aut}(\mathcal{G}) \hookrightarrow \text{Out}(\Pi_{\mathcal{G}})) \Rightarrow \Pi_{\mathcal{G}}$ + verticial subgroups

Definition

(1) ρ: of IPSC-type $\overset{\text{def}}{\Leftrightarrow}$
- $\exists k$: an algebraically closed field of characteristic $\notin \Sigma$
- $\exists X^{\log}$: a stable log curve/the standard log point $\text{Spec}(k)^{\log} \overset{\text{def}}{=}$ "$(\text{Spec}(k), \mathbb{N})$"
- $\exists \alpha\colon \mathcal{G}^{\Sigma}_{X^{\log}} \overset{\sim}{\to} \mathcal{G}$ s.t.

$$\begin{array}{ccccccccc}
\exists 1 & \longrightarrow & \Pi_{\mathcal{G}^{\Sigma}_{X^{\log}}} & \longrightarrow & \pi_1(X^{\log})^{\Sigma} & \longrightarrow & \pi_1(\text{Spec}(k)^{\log})^{\Sigma} & \longrightarrow & 1 \\
 & & {\scriptstyle \Pi_\alpha}\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_I & \longrightarrow & I & \longrightarrow & 1
\end{array}$$

(2) ρ: of PIPSC-type $\overset{\text{def}}{\Leftrightarrow} I \cong \widehat{\mathbb{Z}}^{\Sigma}$, $\rho|_{\exists \text{an open subgroup of } I}$ is of IPSC-type

Theorem 1 [CbTpI, Theorem A, (ii)]

$\mathrm{Im}(\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$

We may assume: $n = 1$
(by replacing (Π_n, Π_{n+1}) by "$(\mathrm{Ker}(\Pi_n \to \Pi_{n-1}), \mathrm{Ker}(\Pi_{n+1} \to \Pi_{n-1})$")
$\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_2)$, $i \in \{1, 2\}$

$$\begin{array}{ccc} \Pi_2 & \xrightarrow[\sim]{\alpha} & \Pi_2 \\ {\scriptstyle \mathrm{pr}_i}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_i} \\ \Pi_1 & \xrightarrow[\beta]{\sim} & \Pi_1 \end{array}$$

Remark: β does not depend on the choice of i (cf. [CbTpI, Theorem A, (i)]).

$\beta \overset{??}{\in} \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$, i.e., does β preserve the cusps?

$\Uparrow$ by Thm 0, (1)

$$\begin{array}{ccc} \Pi_1 & \xrightarrow[\sim]{\beta} & \Pi_1 \\ {\scriptstyle \mathrm{open}}\uparrow & & \uparrow \\ \forall H & \xrightarrow[\beta|_H]{\sim} & \beta(H) \\ \downarrow & & \downarrow \\ H^{\mathrm{ab/Cusp}} & \overset{\sim}{\underset{??}{\dashrightarrow}} & \beta(H)^{\mathrm{ab/Cusp}} \end{array}$$

For simplicity:
Consider the case: $H = \Pi_1$ ($\Rightarrow \beta(H) = \Pi_1$)

Thus:

Claim

$\mathrm{pr}_1,\, \mathrm{pr}_2 \colon \Pi_2 \twoheadrightarrow \Pi_1 \overset{??}{\Rightarrow} \Pi_1^{\text{ab-Cusp}} \subseteq \Pi_1^{\text{ab}}$

If $r = 0$, then $\Pi_1^{\text{ab-Cusp}} = \{0\}$
$\Rightarrow$ We may assume: $r > 0$

$\delta \subseteq X \times_k X$: the diagonal divisor
$s \colon \mathcal{O}_{X\times_k X} \hookrightarrow \mathcal{O}_{X\times_k X}(\delta)$

$$\begin{array}{ccccc}
\mathbb{V} \overset{\text{def}}{=} \mathbb{V}(\mathcal{O}_{X\times_k X}(\delta)) & \longleftarrow & \mathbb{V}^{\times} \overset{\text{def}}{=} \mathbb{V} \setminus \text{zero-sect.} & & \\
& \searrow \quad \swarrow & & & \\
& X \times_k X & \xleftarrow[(\mathrm{pr}_1,\mathrm{pr}_2)]{} & X_2 &
\end{array}$$

$\Rightarrow$

$$\begin{array}{ccccccccc}
& & & & \Pi_2 & \longrightarrow & \Pi_2/\overline{[\Pi_2, \mathrm{Ker}(\mathrm{pr}_1,\mathrm{pr}_2)]} & & \\
& & & & \downarrow s|_{X_2} & \overset{\sim}{\swarrow} & & & \\
1 & \longrightarrow & \pi_1(\text{a geom. fiber of } \mathbb{V}^{\times} \to X \times_k X) & \longrightarrow & \pi_1(\mathbb{V}^{\times}) & \longrightarrow & \Pi_1 \times \Pi_1 & \longrightarrow & 1 \\
& & \text{fix} \,\big|\, \wr & & & & & & \\
& & \widehat{\mathbb{Z}}^{\Sigma} & & & & & &
\end{array}$$

$$H^2(\Pi_1 \times \Pi_1, \widehat{\mathbb{Z}}^{\Sigma}) \overset{\Pi_1\colon \text{free}}{\overset{\sim}{\to}} H^1(\Pi_1, H^1(\Pi_1, \widehat{\mathbb{Z}}^{\Sigma})) \overset{\sim}{\to} \mathrm{Hom}_{\widehat{\mathbb{Z}}^{\Sigma}}(\Pi_1^{\text{ab}}, (\Pi_1^{\text{ab}})^{\vee})$$

$$\begin{array}{ccc}
\Pi_1^{\text{ab}} & \xrightarrow{\text{image of } [\pi_1(\mathbb{V}^{\times})]} & (\Pi_1^{\text{ab}})^{\vee} \\
\downarrow & & \uparrow \\
\Pi_1^{\text{ab/Cusp}} & \underset{(\widehat{\mathbb{Z}}^{\Sigma})^{\times}\cdot\text{Poincaré duality}}{\overset{\sim}{\longrightarrow}} & (\Pi_1^{\text{ab/Cusp}})^{\vee}
\end{array}$$

In particular: $\Pi_1^{\text{ab-Cusp}} = \mathrm{Ker}(\text{image of } [\pi_1(\mathbb{V}^{\times})])$

Theorem 2 [CbTpII, Theorem A, (ii)], [HMT, Corollary 2.2]

$\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$
if either '$n = 2$, $g = 0$', '$n = 3$, $r \neq 0$', or '$n \geq 4$'

Consider the case:

- $r \geq 2$
- $\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_2)$ whose image $\beta \overset{\text{Thm 1}}{\in} \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ acts on the set of cusps trivially

a "hint" of the C-admissibility of α

$\beta \curvearrowright \Pi_1$
preserves the cusps
(by Thm 1)

$\uparrow \mathrm{pr}_1$

$\alpha \curvearrowright \Pi_2$ $\qquad$ $\beta \curvearrowright \Pi_1$

$\cup$

$\alpha \curvearrowright \mathrm{Ker}(\mathrm{pr}_1)$
Does this preserve the cusps?

Theorem 3 [CbTpII, Theorem A, (i)]

$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is:
(1) injective if '$n \geq 1$' and '$(n,r) \neq (1,0)$'
(2) bijective if either '$n \geq 4$' or '$n \geq 3$ and $r \geq 1$'

by Thm 1, 2, together w/ some "standard arguments"
(cf., e.g., Minamide's talk yesterday for Thm 3, (1))

Theorem 4 [CbTpII, Theorem B, (i), (ii)]

Suppose: $(g,r) \notin \{(0,3),(1,1)\}$
$\Rightarrow$
$\mathrm{Out}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n) \times \mathfrak{S}_n$ if $(n,r) \neq (2,0)$
($\overset{\text{Thm 2}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \times \mathfrak{S}_n$ if either '$n=2$, $g=0$', '$n=3$, $r \neq 0$', or '$n \geq 4$')

by a main result of [MT] (cf. also Sawada's talk yesterday) and Thm 3, (1) References

[MT] The Algebraic and Anabelian Geometry of Configuration Spaces
[CbTpI] Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves I: Inertia Groups and Profinite Dehn Twists
[CbTpII] Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves II: Tripods and Combinatorial Cuspidalization
[HMT] Combinatorial Construction of the Absolute Galois Group of the Field of Rational Numbers

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Synchronization of Tripods and Glueability of Combinatorial Cuspidalizations

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(T6) tripod synchronization and the tripod homomorphism
(cf. Mochizuki's overview)

Σ: a set of prime numbers s.t. $\#\Sigma = 1$ or Σ is the set of $\forall$prime numbers
$n \geq 0$
k: an algebraically closed field of characteristic $\notin \Sigma$
$S^{\log} \overset{\text{def}}{=}$ "Spec$(k, \mathbb{N})$": the standard log point whose underlying scheme is Spec(k)
$X^{\log}$: a stable log curve/$S^{\log}$ of type (g, r)
$\mathcal{G}$: the semi-graph of anabelioids of pro-Σ PSC-type associated to $X^{\log}$
$X_n^{\log}$: the n-th log configuration space of $X^{\log}$
$\Pi_n \overset{\text{def}}{=} \text{Ker}(\pi_1(X_n^{\log})^{\Sigma} \to \pi_1(S^{\log})^{\Sigma}) = \text{Ker}(\pi_1(X_n^{\log}) \to \pi_1(S^{\log}))^{\Sigma}$

Various tripods appear in $X_n^{\log}$.

$X^{\log} = X_1^{\log}$ $\qquad$ Π_1

$\uparrow$

$X_2^{\log}$ $\qquad$ Π_2

$\uparrow$

$X_3^{\log}$ $\qquad$ Π_3

Tripod Synchronization

= synchronization among the various tripods in Π_n
$\Rightarrow$ an outer automorphism of Π_n typically induces
the <u>same outer automorphism</u> on the various tripods in Π_n

Definition

$m \le n$

$T \subseteq \Pi_m$: an $\underline{m\text{-tripod}}$ of Π_n $\overset{\text{def}}{\Leftrightarrow}$

T: a verticial subgroup "of type $(0,3)$" of "$\Pi_{\text{a geom. fiber of } X_m^{\log} \to X_{m-1}^{\log}}$"

$= \mathrm{Ker}(\Pi_m \twoheadrightarrow \Pi_{m-1}) \subseteq \Pi_m$

Then: $\mathrm{Out}^{|\mathrm{C}|}(T) \subseteq \mathrm{Out}(T)$: the subgroup consisting of α s.t.

α induces the id. on the set of conj. classes of cuspidal inertia subgroups of T

Definition

Suppose: $n \ge 3$

$T \subseteq \Pi_3$: a 3-tripod of Π_n

T: $\underline{\text{central}}$ $\overset{\text{def}}{\Leftrightarrow}$ T arises as:

$X^{\log} = X_1^{\log}$ Π_1

$\uparrow$

$X_2^{\log}$ Π_2

$\uparrow$

$X_3^{\log}$ Π_3

Theorem 5 [CbTpII, Theorem C, (i)]

$m \leq n$
$T \subseteq \Pi_m$: an m-tripod of Π_n
$\Rightarrow C_{\Pi_m}(T) = N_{\Pi_m}(T) = T \times Z_{\Pi_m}(T)$

Remark

G: a group
$H \subseteq G$: a subgroup

$\alpha \in \mathrm{Aut}(G)$
$\Rightarrow$ One can define the restriction $\alpha|_H \in \mathrm{Aut}(H)$ if α preserves $H \subseteq G$.

On the other hand:
$\alpha \in \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$
$\Rightarrow$ One cannot define the "restriction" $\alpha|_H \in \mathrm{Out}(H)$ in general
even if α preserves the conjugacy class of $H \subseteq G$.
The "natural rest." is not $\in \mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ but $\in \mathrm{Aut}(H)/\mathrm{Inn}(N_G(H))$.

In particular:
- α preserves the conjugacy class of $H \subseteq G$
- $N_G(H) = Z_G(H) \cdot H$

$\Rightarrow$ One can define the restriction $\alpha|_H \in \mathrm{Out}(H)$.

Definition

$m \leq n$
$T \subseteq \Pi_m$: an m-tripod of Π_n

- $\mathrm{Out}^{\mathrm{F}}(\Pi_n)[T] \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$: the subgroup consisting of α s.t. the outer autom. of Π_m induced by α preserves the Π_m-conj. class of $T \subseteq \Pi_m$

- $\mathfrak{T}_T : \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T] \to \mathrm{Out}(T)$ (well-defined by Thm 5), the tripod homomorphism associated to T

- $\mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T]$: the pull-back of $\mathrm{Out}^{|\mathrm{C}|}(T) \subseteq \mathrm{Out}(T)$ by $\mathfrak{T}_T$

Theorem 6 [CbTpII, Theorem 3.16, (v)], [CbTpII, Theorem 3.18, (ii)]

For simplicity: suppose $n \geq 3$
(Note: $\exists$result related to (2) in the case of $n = 2$ — cf. [CbTpII, Theorem 3.17])

$m \leq n$
$T \subseteq \Pi_m$: an m-tripod of Π_n

(1) T: <u>central</u> $\Rightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : |\mathrm{C}|]$

(2) $m' \leq n$
T': an m'-tripod of Π_n
$\Rightarrow \exists$a "geometric" outer isomorphism $\iota \colon T \xrightarrow{\sim} T'$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : |\mathrm{C}|] \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T' : |\mathrm{C}|] & \\ \swarrow^{\mathcal{T}_T} & & \searrow^{\mathcal{T}_{T'}} \\ \mathrm{Out}(T) & \xrightarrow[\mathrm{Out}(\iota)]{\sim} & \mathrm{Out}(T') \end{array}$$

commutes.

Glueability of Combinatorial Cuspidalizations

One Dimensional Case

Definition

- $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G})$: the subgroup consisting of α s.t.
  $\alpha \curvearrowright$ the underlying semi-graph is trivial
- $\mathrm{Dehn}(\mathcal{G}) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$: the subgroup consisting of α s.t.
  for $\forall v \in \mathrm{Vert}(\mathcal{G})$, $\alpha|_{\mathcal{G}_v}$ is trivial
- $\mathrm{Glu}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_v)$: the subgp consisting of $(\alpha_v)_v$ s.t.
  $\chi_v^{\mathrm{cycl}}(\alpha_v) = \chi_w^{\mathrm{cycl}}(\alpha_w)$ for $\forall v,\, w \in \mathrm{Vert}(\mathcal{G})$

Theorem 7 [CbTpI, Theorem B, (iii)]

(1) The natural homomorphism

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_v)$$

factors through the subgroup

$$\mathrm{Glu}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_v).$$

(2) The resulting homomorphism

$$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}^{|\mathrm{grph}|}(\mathcal{G})$$

is a surjective homomorphism whose kernel is given by

$$\mathrm{Dehn}(\mathcal{G}) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}).$$

$$1 \longrightarrow \mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow 1.$$

Observe: (1) is a formal consequence of "Synchronization of Cyclotomes".

Corollary 8 [CbTpII, Theorem A, (iii)]

Suppose: $(g, r) \notin \{(0,3),(1,1)\}$
$\Rightarrow$ The injective (cf. Minamide's talk yesterday) homomorphism
$\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is <u>not surjective</u>

<u>Proof</u> of the assertion that $\underline{\mathrm{Out}^{\mathrm{FC}}(\Pi_3)} \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is <u>not surjective</u>

The structure of $(\cdots \to \Pi_{m+1} \to \Pi_m \to \dots)$ depends only on (g, r)
$\Rightarrow$ We may assume: $X^{\log}$ is <u>totally degenerate</u>, i.e., $\forall$vertex of $\mathcal{G}$ is "of type $(0,3)$"
(in a spirit of Tripodal Transport cf. Mochizuki's talk, last week)

$(g, r) \notin \{(0,3),(1,1)\} \Rightarrow \exists v,\, w \in \mathrm{Vert}(\mathcal{G})$: <u>distinct</u>
$\alpha_v \in \mathrm{Out}^{|\mathrm{C}|}(\Pi_v)$, $\alpha_w \in \mathrm{Out}^{|\mathrm{C}|}(\Pi_w)$ s.t.
(a) $\alpha_v \neq \phi^{-1}\alpha_w\phi$ for $\forall$"geometric" isomorphism $\phi\colon \Pi_v \xrightarrow{\sim} \Pi_w$
(b) $\chi_v^{\mathrm{cycl}}(\alpha_v) = \chi_w^{\mathrm{cycl}}(\alpha_w)$
(cf. "$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \subseteq \mathrm{Out}^{|\mathrm{C}|}(T)$")

$\overset{\text{(b), Thm 7, (2)}}{\Rightarrow} \exists \alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$ $(\subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_1))$ s.t. $\alpha|_{\Pi_v} = \alpha_v$, $\alpha|_{\Pi_w} = \alpha_w$

Assume: $\mathrm{Out}^{\mathrm{FC}}(\underline{\Pi_3}) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is <u>surjective</u>
$\Rightarrow \exists \alpha_3 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ whose image in $\mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is $= \alpha$
But this contradicts (a) and Thm 6, (2).

Higher Dimensional Case

For simplicity: suppose $n \geq 3$
(Note: $\exists$result in the case of $n = 2$)

Definition

- $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} \subseteq \mathrm{Out}(\Pi_n)$:
  the pull-back of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ by
  the injective (cf. Minamide's talk yesterday) hom. $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$

- $\mathrm{Glu}^{|\mathrm{grph}|}(\Pi_n) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{|\mathrm{grph}|}$: the subgp consisting of $(\alpha_v)_v$ s.t.
  $\mathfrak{T}_{\text{a ctrl tpd in } (\Pi_v)_3}(\alpha_v) = \mathfrak{T}_{\text{a ctrl tpd in } (\Pi_w)_3}(\alpha_w)$ for $\forall v,\, w \in \mathrm{Vert}(\mathcal{G})$
  (Note: A central tripod in $(\Pi_v)_3$ is a Π_3-conjugate of a central tripod in $(\Pi_w)_3$.)

Theorem 9 [CbTpI, Theorem F]

(1) $v \in \mathrm{Vert}(\mathcal{G}) \Rightarrow (\Pi_v)_n \subseteq \Pi_n$: commensurably terminal
(2) $v \in \mathrm{Vert}(\mathcal{G}) \Rightarrow \forall \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|}$ preserves the conjugacy class of $(\Pi_v)_n \subseteq \Pi_n$.

(1), (2) $\Rightarrow$ One may define a "restriction homomorphism"

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} \longrightarrow \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{|\mathrm{grph}|}.$$

(3) The above "restriction homomorphism" factors through the subgroup

$$\mathrm{Glu}^{|\mathrm{grph}|}(\Pi_n) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{|\mathrm{grph}|}.$$

(4) The resulting homomorphism

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} \longrightarrow \mathrm{Glu}^{|\mathrm{grph}|}(\Pi_n)$$

is a surjective homomorphism whose kernel is given by

$$\mathrm{Dehn}(\mathcal{G}) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|}.$$

$$1 \longrightarrow \mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} \longrightarrow \mathrm{Glu}^{|\mathrm{grph}|}(\Pi_n) \longrightarrow 1.$$

Observe: (3) is a formal consequence of "Tripod Synchronization" (cf. Thm 6).

Corollary 10 [CbTpII, Theorem C, (iv)]

Suppose: $n \geq 3$
$T \subseteq \Pi_3$: a central 3-tripod of Π_n
Suppose, moreover: either $r \neq 0$ or $n \geq 4$

(1) The tripod homomorphism

$$\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \overset{\text{Thm 6, (1)}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T] \longrightarrow \mathrm{Out}(T)$$

factors through the subgroup "GT" of $\mathrm{Out}(T)$.

(2) The resulting homomorphism

$$\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \overset{\text{Thm 6, (1)}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T] \longrightarrow \text{“GT” in } \mathrm{Out}(T)$$

is surjective.

Proof of (2)

The validity of the assertion depends only on (n, g, r)
$\Rightarrow$ We may assume: $X^{\log}$ is totally degenerate, i.e., $\forall$vertex of $\mathcal{G}$ is "of type $(0, 3)$"
(in a spirit of Tripodal Transport cf. Mochizuki's talk, last week)

$\gamma \in$ "GT"
$\Rightarrow \forall v \in \mathrm{Vert}(\mathcal{G})$, $\exists \gamma_{v,n} \in \mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{|\mathrm{grph}|}$ whose image in $\mathrm{Out}(\Pi_v)$ is $= \gamma$
$\overset{\text{Thm 6, (2)}}{\Rightarrow} \forall v \in \mathrm{Vert}(\mathcal{G})$, $\mathfrak{T}_{\text{a ctrl tpd in } (\Pi_v)_3}(\gamma_{v,n}) = \gamma$
$\overset{\text{Thm 9, (4)}}{\Rightarrow} \exists \gamma_n \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|}$ whose image in $\mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{|\mathrm{grph}|}$
is $= \gamma_{v,n}$ for $\forall v \in \mathrm{Vert}(\mathcal{G})$
$\overset{\text{Thm 6, (2)}}{\Rightarrow} \mathfrak{T}_T(\gamma_n) = \gamma$, as desired

References

[CbCsp] On the Combinatorial Cuspidalization of Hyperbolic Curves

[CbTpI] Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves I: Inertia Groups and Profinite Dehn Twists

[CbTpII] Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves II: Tripods and Combinatorial Cuspidalization

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