Applications to the Theory of Tempered Fundamental Groups

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F: a number field
$\mathfrak{p}$: a nonarchimedean prime of F
$F_{\mathfrak{p}}$: the completion of F at $\mathfrak{p}$
$\overline{F}_{\mathfrak{p}}$: an algebraic closure of $F_{\mathfrak{p}}$
$\overline{F}_{\mathfrak{p}}^{\wedge}$: the completion of $\overline{F}_{\mathfrak{p}}$
$\overline{F}$: the algebraic closure of F in $\overline{F}_{\mathfrak{p}}$
$G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F) \supseteq G_{\mathfrak{p}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$

X_F: a hyperbolic curve/F of type (g, r)
$X_{\square} \stackrel{\mathrm{def}}{=} X_F \times_F \square$

Definition

$\pi_1^{\mathrm{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}})$: the tempered fundamental group of $X_{\overline{F}^{\wedge}_{\mathfrak{p}}}$, i.e.,

$$\pi_1^{\mathrm{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}) \stackrel{\mathrm{def}}{=} \varprojlim_{Y \to X_{\overline{F}^{\wedge}_{\mathfrak{p}}}:\ \text{fin. ét. Gal.}} \mathrm{Aut}_{X^{\mathrm{an}}_{\overline{F}^{\wedge}_{\mathfrak{p}}}}(\text{the topological universal covering of } Y^{\mathrm{an}})$$

Note:
$1 \to \pi_1^{\mathrm{top}}(\text{the dual semi-gr. of the st. mod. of the sp'l fib.}) \to \qquad \to \mathrm{Aut}_{X_{\overline{F}^{\wedge}_{\mathfrak{p}}}}(Y) \to 1$

Proposition 1

$\exists$a continuous homomorphism $\pi_1^{\mathrm{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}) \to \pi_1^{\text{ét}}(X_{\overline{F}})$ that

(1) factors as the composite

$$\pi_1^{\mathrm{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}) \xhookrightarrow{\text{natural}} \pi_1^{\mathrm{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}})^{\wedge} \xrightarrow{\sim} \pi_1^{\text{ét}}(X_{\overline{F}})$$

and

(2) determines an <u>injective</u> homomorphism

$$\mathrm{Out}(\pi_1^{\mathrm{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}})) \hookrightarrow \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))$$

André's Theorem

Suppose: $X_{\overline{F}}$ is of quasi-Belyi type, i.e., $X_{\overline{F}} \overset{\exists\text{finite étale}}{\longleftarrow} \exists Y \overset{\exists\text{dominant}}{\longrightarrow} \mathbb{P}^1_{\overline{F}} \setminus \{0,1,\infty\}$
($\Rightarrow r > 0$, i.e., X_F is not projective$/F$)

Theorem (Belyĭ)

The two outer actions

$$\rho_F^{\text{ét}} \colon G_F \longrightarrow \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})), \qquad \rho_{\mathfrak{p}}^{\text{tp}} \colon G_{\mathfrak{p}} \longrightarrow \mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}^\wedge_{\mathfrak{p}}}))$$

are faithful.

$\Rightarrow$

$$\begin{tikzcd}
G_{\mathfrak{p}} \arrow[r, hook, "\rho_{\mathfrak{p}}^{\text{tp}}", "\text{the above Thm}"'] \arrow[d, hook] & \mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}^\wedge_{\mathfrak{p}}})) \arrow[d, hook, "\text{Prp 1, (2)}"] \\
G_F \arrow[r, hook, "\text{the above Thm}", "\rho_F^{\text{ét}}"'] & \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))
\end{tikzcd}$$

Theorem (André)

The above square is cartesian.

That is to say, for $\gamma \in G_F$:

$\gamma \in G_{\mathfrak{p}} \Leftrightarrow$
the outer action of γ on $\pi_1^{\text{ét}}(X_{\overline{F}})$ "preserves" the subgp $\pi_1^{\text{tp}}(X_{\overline{F}^\wedge_{\mathfrak{p}}}) \overset{\text{Prp 1, (1)}}{\subseteq} \pi_1^{\text{ét}}(X_{\overline{F}})$.

Today, [CbTpIII]

X_F: arbitrary

Theorem (cf. [NodNon, Theorem C]; also Minamide's talk Tuesday)

The two outer actions

$$\rho_F^{\text{ét}} \colon G_F \longrightarrow \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})), \qquad \rho_{\mathfrak{p}}^{\text{tp}} \colon G_{\mathfrak{p}} \longrightarrow \mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}))$$

are faithful.

$\Rightarrow$

$$\begin{tikzcd}
G_{\mathfrak{p}} \arrow[r, hook, "\rho_{\mathfrak{p}}^{\text{tp}}", "\text{the above Thm}"'] \arrow[d, hook] & \mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}})) \arrow[d, hook, "\text{Prp 1, (2)}"] \\
G_F \arrow[r, hook, "\text{the above Thm}", "\rho_F^{\text{ét}}"'] & \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))
\end{tikzcd}$$

Main Theorem [CbTpIII, Theorem B]

The above square is cartesian
after replacing $\mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}))$ by the subgroup $\mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}))^{\mathrm{M}}$.

That is to say, for $\gamma \in G_F$:

$\gamma \in G_{\mathfrak{p}} \Leftrightarrow$

the outer action of γ on $\pi_1^{\text{ét}}(X_{\overline{F}})$ "preserves" the subgp $\pi_1^{\text{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}) \overset{\text{Prp 1, (1)}}{\subseteq} \pi_1^{\text{ét}}(X_{\overline{F}})$,
and, moreover, the resulting outer automorphism of $\pi_1^{\text{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}})$ is M-admissible.

What is $\mathrm{Out}(-)^{\mathrm{M}}$?

$H \subseteq \pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}})$: a characteristic open subgroup of finite index

$\Rightarrow$ H corresponds to a finite étale Galois covering $Y_H \to X_{\overline{F}_{\mathfrak{p}}^{\wedge}}$

- $\mathcal{Y}_H$: the stable model of Y_H over the valuation ring $\mathcal{O}$ of $\overline{F}_{\mathfrak{p}}^{\wedge}$
- $\mathbb{G}_H$: the dual semi-graph of the special fiber of $\mathcal{Y}_H$

metric structure

- p: the residue characteristic of $\mathfrak{p}$
- v: the p-adic valuation on $\overline{F}_{\mathfrak{p}}^{\wedge}$ normalized so that $v(p) = 1$

Then:

$e \in \mathrm{Node}(\mathbb{G}_H) \Rightarrow \exists a_e \in \mathfrak{m}_{\mathcal{O}}$ s.t. the completion of $\mathcal{Y}_H$ at e is $\cong \mathcal{O}[[s,t]]/(st - a_e)$

Moreover, $v(a_e) \in \mathbb{R}$ does not depend on the choice of "$\cong$".

- $e \in \mathrm{Node}(\mathbb{G}_H) \Rightarrow \mu_H(e) \overset{\mathrm{def}}{=} v(a_e) \in \mathbb{R}$

Thus: we obtain a metrized semi-graph $(\mathbb{G}_H, \mu_H)$.

What is $\mathrm{Out}(-)^{\mathrm{M}}$?

$H \subseteq \pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}})$: a characteristic open subgroup of finite index

α: an automorphism of $\pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}})$

$\overset{H:\text{ char.}}{\Rightarrow} \alpha \curvearrowright H = \pi_1^{\mathrm{tp}}(Y_H)$

$\overset{\text{[Semi, Crl 3.11]}}{\Rightarrow} \alpha \curvearrowright \mathbb{G}_H$

Definition

$\alpha \in \mathrm{Aut}(\pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))$: <u>M-admissible</u> $\overset{\text{def}}{\Leftrightarrow}$

$\alpha \curvearrowright \mathbb{G}_H$ is compatible w/ the metric structure μ_H on $\mathbb{G}_H$ for $\forall H$ as above

$\mathrm{Aut}(\pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))^{\mathrm{M}}$: the subgroup of <u>M-admissible</u> automorphisms

$\mathrm{Out}(\pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))^{\mathrm{M}} \overset{\text{def}}{=} \mathrm{Aut}(\pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))^{\mathrm{M}}/\mathrm{Inn}(\pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))$:

the subgroup of <u>M-admissible</u> outer automorphisms

<u>Today, [CbTpIII]</u>

X_F: arbitrary

Theorem (cf. [NodNon, Theorem C]; also Minamide's talk Tuesday)

The two outer actions

$$\rho_F^{\text{ét}} \colon G_F \longrightarrow \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})), \qquad \rho_{\mathfrak{p}}^{\text{tp}} \colon G_{\mathfrak{p}} \longrightarrow \mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))$$

are <u>faithful</u>.

$\Rightarrow$

$$\begin{tikzcd}
G_{\mathfrak{p}} \arrow[r, hook, "\rho_{\mathfrak{p}}^{\text{tp}}", "\text{the above Thm}"'] \arrow[d, hook] & \mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}})) \arrow[d, hook, "\text{Prp 1, (2)}"] \\
G_F \arrow[r, hook, "\text{the above Thm}", "\rho_F^{\text{ét}}"'] & \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))
\end{tikzcd}$$

Main Theorem [CbTpIII, Theorem B]

The above square is <u>cartesian</u>
after replacing $\mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))$ by the subgroup $\mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))^{\mathrm{M}}$.

That is to say, for $\gamma \in G_F$:

$\gamma \in G_{\mathfrak{p}} \Leftrightarrow$

the outer action of γ on $\pi_1^{\text{ét}}(X_{\overline{F}})$ <u>"preserves"</u> the subgp $\pi_1^{\text{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}) \overset{\text{Prp 1, (1)}}{\subseteq} \pi_1^{\text{ét}}(X_{\overline{F}})$,
and, moreover, the resulting outer automorphism of $\pi_1^{\text{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}})$ is <u>M-admissible</u>.

$n \geq 1$
$(X_{\overline{F}})_n$: the n-th configuration space of $X_{\overline{F}}$

Recall: $\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_n))$: the subgroup of FC-admissible outer automorphisms
$\Rightarrow$

$$\cdots \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_{n+1})) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_n)) \hookrightarrow \cdots$$

the <u>injectivity portion of combinatorial cuspidalization</u>
(cf. [NodNon, Theorem B]; also Minamide's talk Tuesday)

Definition

$\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_n))^{\mathrm{M}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_n))$: the subgp def'd by the cartesian diagram

$$\begin{tikzcd}
\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_n))^{\mathrm{M}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Out}(\pi_1^{\mathrm{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))^{\mathrm{M}} \arrow[d, hook, "\text{Prp 1, (2)}"] \\
\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_n)) \arrow[r, hook] & \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))
\end{tikzcd}$$

$T \subseteq \pi_1^{\text{ét}}((X_{\overline{F}})_3)$: a central tripod of $\pi_1^{\text{ét}}((X_{\overline{F}})_3)$ (cf. my talk Wednesday)

Recall: $n \geq 3 \Rightarrow$

$$\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_n)) \longrightarrow \mathrm{Out}(T)$$

the <u>tripod homomorphism</u> associated to the central tripod T
(cf. my talk Wednesday)

Main Lemma [CbTpIII, Theorem A]

The tripod hom. $\mathfrak{T}_T$ maps an M-adm. outer autom. to an M-adm. outer autom., i.e.,

$$\begin{tikzcd}
\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3))^{\mathrm{M}} \arrow[r] \arrow[d, hook] & \mathrm{Out}(T)^{\mathrm{M}} \arrow[d, hook] \\
\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3)) \arrow[r, "\mathfrak{T}_T"'] & \mathrm{Out}(T)
\end{tikzcd}$$

Main Lemma [CbTpIII, Theorem A]

$$\begin{tikzcd} \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3))^{\mathrm{M}} \arrow[r] \arrow[d, hook] & \mathrm{Out}(T)^{\mathrm{M}} \arrow[d, hook] \\ \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3)) \arrow[r, "\mathfrak{T}_T"'] & \mathrm{Out}(T) \end{tikzcd}$$

Main Theorem [CbTpIII, Theorem B]

$$\begin{tikzcd} G_{\mathfrak{p}} \arrow[r, hook, "\rho_{\mathfrak{p}}^{\mathrm{tp}}"] \arrow[d, hook] & \mathrm{Out}(\pi_1^{\mathrm{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}))^{\mathrm{M}} \arrow[d, hook] \\ G_F \arrow[r, hook, "\rho_F^{\text{ét}}"'] & \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})) \end{tikzcd}$$

is <u>cartesian</u>

<u>Proof of "André's Thm + Main Lmm ⇒ Main Thm"</u>

$$\begin{tikzcd} G_{\mathfrak{p}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3))^{\mathrm{M}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Out}(\pi_1^{\mathrm{tp}}(X_{\overline{F}^{\wedge}_{\mathfrak{p}}}))^{\mathrm{M}} \arrow[d, hook] \\ G_F \arrow[r, hook] & \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3)) \arrow[r, hook] & \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})) \end{tikzcd}$$

$G_{\mathfrak{p}} \subseteq G_F \cap \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3))^{\mathrm{M}}$ in $\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3))$

$$\begin{aligned} \overset{\mathfrak{T}_T}{\Rightarrow} \mathfrak{T}_T(G_{\mathfrak{p}}) &\subseteq \mathfrak{T}_T(G_F) \cap \mathfrak{T}_T(\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3))^{\mathrm{M}}) \quad \text{in } \mathrm{Out}(T) \\ &\overset{\text{Main Lmm}}{\subseteq} \mathfrak{T}_T(G_F) \cap \mathrm{Out}(T)^{\mathrm{M}} \\ &\subseteq \mathfrak{T}_T(G_F) \cap \mathrm{Out}(T^{\mathrm{tp}}) \\ &\overset{\text{André}}{=} \mathfrak{T}_T(G_{\mathfrak{p}}) \end{aligned}$$

Thus, since $\mathfrak{T}_T|_{G_F}$ is injective by Belyĭ,
we conclude: $G_{\mathfrak{p}} = G_F \cap \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3))^{\mathrm{M}}$, as desired

Main Lemma [CbTpIII, Theorem A]

$$\begin{CD}
\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3))^{\mathrm{M}} @>>> \mathrm{Out}(T)^{\mathrm{M}} \\
@VVV @VVV \\
\mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_3)) @>>{\mathfrak{T}_T}> \mathrm{Out}(T)
\end{CD}$$

Proof

Step 1: characterization of M-adm'y via outer Galois actions in one dim'l case
Step 2: characterization of M-adm'y via outer Galois actions in higher dim'l case
Step 3: compatibility of M-adm'y w.r.t. $\mathfrak{T}_T$

$I_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$: the inertia subgroup of $G_{\mathfrak{p}}$

Step 1 [CbTpIII, Theorem 3.9]

$\alpha \in \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))$

α: M-admissible $\Leftrightarrow$ α satisfies the following condition:

$\forall H \subseteq \pi_1^{\text{ét}}(X_{\overline{F}})$: a characteristic open subgroup
$Q_H \stackrel{\text{def}}{=} \pi_1^{\text{ét}}(X_{\overline{F}})/\mathrm{Ker}(H \twoheadrightarrow H^{\{l\}})$ ($H^{\{l\}}$: the maximal pro-l quotient of H)
Note: $1 \to H^{\{l\}} \to Q_H \to \pi_1^{\text{ét}}(X_{\overline{F}})/H \to 1$

Then:
The image of $\alpha \in \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))$ in $\mathrm{Out}(Q_H)$ normalizes
an open subgroup of the image of $I_{\mathfrak{p}} \stackrel{\rho_{\overline{F}}^{\text{ét}}}{\to} \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})) \to \mathrm{Out}(Q_H)$.

Step 1 [CbTpIII, Theorem 3.9]

$\alpha \in \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))$

α: <u>M-admissible</u> $\Leftrightarrow$ α satisfies the following condition:

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Note: $1 \to H^{\{l\}} \to Q_H \to \pi_1^{\text{ét}}(X_{\overline{F}})/H \to 1$

Then:
The image of $\alpha \in \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))$ in $\mathrm{Out}(Q_H)$ <u>normalizes</u>
an open subgroup of the image of $I_{\mathfrak{p}} \stackrel{\rho_{\overline{F}}^{\text{ét}}}{\to} \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})) \to \mathrm{Out}(Q_H)$.

<u>Idea</u>
M-adm. $\Leftrightarrow$ (a) cont'd in $\mathrm{Out}(\pi_1^{\text{tp}}(X_{\overline{F}_{\mathfrak{p}}^{\wedge}}))$ (b) compatible w/ the var. metric str.s μ_H's
$\Leftrightarrow$ (a$'$) compatible w/ the var. semi-graph str.s $\mathbb{G}_H$'s
(b) compatible w/ the var. metric str.s μ_H's

$$\begin{tikzcd} 1 \arrow[r] & H^{\{l\}} \arrow[r] \arrow[d, "\wr"'] & Q_H \arrow[r] & \pi_1^{\text{ét}}(X_{\overline{F}})/H \arrow[r] \arrow[d, "\wr"'] & 1 \\ & \Pi_{\mathcal{G}_H^{\{l\}}} & & \mathrm{Aut}_{X_{\overline{F}_{\mathfrak{p}}^{\wedge}}}(Y_H) & \end{tikzcd}$$

where $\mathcal{G}_H^{\{l\}}$: the semi-graph of anab.s of pro-$\{l\}$ PSC-type det'd by the sp'l fib. of $\mathcal{Y}_H$

Observe: $\exists J_H \subseteq I_{\mathfrak{p}}$: an open subgp s.t.
(A$'$) the composite $J_H \hookrightarrow I_{\mathfrak{p}} \stackrel{\rho_{\overline{F}}^{\text{ét}}}{\to} \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})) \to \mathrm{Out}(Q_H)$ is
an "almost pro-l version" of an outer action <u>of PIPSC-type</u>
(B) (by <u>comparison b/w comb. cycl. synch. and sch.-th. cycl. synch.</u> — cf. [CbTpI, §5])
the composite $J_H \hookrightarrow I_{\mathfrak{p}} \stackrel{\rho_{\overline{F}}^{\text{ét}}}{\to} \mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}})) \to \mathrm{Out}(Q_H)$ "factors through"
a hom. $J_H \to \mathrm{Dehn}(\mathcal{G}_H^{\{l\}}) \subseteq \mathrm{Aut}(\mathcal{G}_H^{\{l\}}) \subseteq \mathrm{Out}(H^{\{l\}} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_H^{\{l\}}})$ whose image is $\cong \mathbb{Z}_l$,
and, moreover, every generator of the image ($\cong \mathbb{Z}_l$) of J_H in $\mathrm{Dehn}(\mathcal{G}_H^{\{l\}})$ is

$$\in \mathbb{Q}_{>0} \cdot \mathbb{Z}_l^{\times} \cdot (\mu_H(e))_{e \in \mathrm{Node}(\mathcal{G}_H^{\{l\}})} \in \bigoplus_{e \in \mathrm{Node}(\mathcal{G}_H^{\{l\}})} \Lambda_{\mathcal{G}_H^{\{l\}}} \stackrel{\text{str. thm. of Dehn}}{=} \mathrm{Dehn}(\mathcal{G}_H^{\{l\}}).$$

Step 1, by
- an "almost pro-l version" of combinatorial anabelian result of PIPSC-type (cf. [CbTpIII, Theorem 1.11])
- cyclotomic synchronization (cf. [CbTpI; §3, §5])

<u>Step 1</u>: characterization of M-adm'y via outer Galois actions in one dim'l case
- an "almost pro-l version" of combinatorial anabelian result of PIPSC-type
  (cf. [CbTpIII, Theorem 1.11])
- cyclotomic synchronization (cf. [CbTpI; §3, §5])

<u>Step 2: characterization of M-adm'y via outer Galois actions in higher dim'l case</u>
<u>Step 3</u>: compatibility of M-adm'y w.r.t. $\mathfrak{T}_T$

Step 2 [CbTpIII, Theorem 3.17, (ii)]

$\alpha \in \mathrm{Out}^{\mathrm{FC}}(\pi_1^{\text{ét}}((X_{\overline{F}})_n))$

α: <u>M-admissible</u> $\Leftrightarrow$ α satisfies the following condition:

$\forall H \subseteq \pi_1^{\text{ét}}((X_{\overline{F}})_n)$: a characteristic open subgroup
$Q_H \overset{\text{def}}{=} \pi_1^{\text{ét}}((X_{\overline{F}})_n)/\mathrm{Ker}(H \twoheadrightarrow H^{\{l\}})$ ($H^{\{l\}}$: the maximal pro-l quotient of H)

Then:
The image of $\alpha \in \mathrm{Out}(\pi_1^{\text{ét}}((X_{\overline{F}})_n))$ in $\mathrm{Out}(Q_H)$ <u>normalizes</u>
an open subgroup of the image of $I_{\mathfrak{p}} \to \mathrm{Out}(\pi_1^{\text{ét}}((X_{\overline{F}})_n)) \to \mathrm{Out}(Q_H)$.

<u>Idea</u>
M-adm. $\Leftrightarrow$ the image in $\mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{F}}))$ is M-adm.

$$\begin{tikzcd}
\pi_1^{\text{ét}}((X_{\overline{F}})_n) \arrow[r, two heads] \arrow[d] & \pi_1^{\text{ét}}(X_{\overline{F}}) \arrow[d] \\
Q_H \arrow[r] & Q_J
\end{tikzcd}$$

where J: the image of H in $\pi_1^{\text{ét}}(X_{\overline{F}})$

$\overset{\text{Stp 1}}{\Leftrightarrow}$ a certain normalizability in the various $\mathrm{Out}(Q_J)$'s
Thus, to verify Step 2, it suffices to verify a sort of injectivity of "$\mathrm{Out}(Q_H) \to \mathrm{Out}(Q_J)$"

Step 2, by
- an "almost pro-l version" of the injectivity portion of combinatorial cuspidalization
  (cf. [CbTpIII, Corollary 2.20])

Step 1: characterization of M-adm'y via outer Galois actions in one dim'l case

- an "almost pro-l version" of combinatorial anabelian result of PIPSC-type (cf. [CbTpIII, Theorem 1.11])
- cyclotomic synchronization (cf. [CbTpI; §3, §5])

Step 2: characterization of M-adm'y via outer Galois actions in higher dim'l case

- an "almost pro-l version" of the injectivity port. of combinatorial cuspidalization (cf. [CbTpIII, Corollary 2.20])

Step 3: compatibility of M-adm'y w.r.t. $\mathfrak{T}_T$

M-adm. $\overset{\text{Stp 2}}{\Leftrightarrow}$ a cert. normalizability in the outer autom. gps of var. almost pro-l quotients

Thus, to verify compatibility of M-adm'y w.r.t. $\mathfrak{T}_T$,
one has to discuss a sort of compatibility b/w
the notion of tripod homomorphisms and various almost pro-l quotients,

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, two heads] & \pi_1^{\text{ét}}((X_{\overline{F}})_3) \arrow[d, two heads] \\ T^* \arrow[r, hook] & Q_H \end{tikzcd}$$

e.g., one has to consider the normalizer $N_{Q_H}(T^*)$ of T^* in Q_H
(cf. the definition of tripod homomorphisms).

References

[Semi] Semi-graphs of Anabelioids
[NodNon] On the Combinatorial Anabelian Geometry of Nodally Nondegenerate Outer Representations
[CbTpI] Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves I: Inertia groups and profinite Dehn twists
[CbTpIII] Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves III: Tripods and Tempered Fundamental Groups

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