Combinatorial Anabelian Geometry
in the Absence of Group-theoretic Cuspidality

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The 1st talk

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"Combinatorial Anabelian Geometry"

In the present talk:

$\square \in \{\circ, \bullet, \emptyset\}$
$\Sigma_\square$: a nonempty set of prime numbers
$\mathcal{G}_\square$: a semi-graph of anabelioids of pro-Σ PSC-type
$\widetilde{\mathcal{G}}_\square \to \mathcal{G}_\square$: a pro-$\Sigma_\square$ universal covering

$\Pi_{\mathcal{G}_\square} \overset{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{\mathcal{G}}_\square/\mathcal{G}_\square)$
$I_\square$: a profinite group
$\rho_\square \colon I_\square \to \mathrm{Aut}(\mathcal{G}_\square)$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_\square}))$: a continuous homomorphism
$\Pi_{\rho_\square} \overset{\mathrm{def}}{=} \Pi_{\mathcal{G}_\square} \overset{\mathrm{out}}{\rtimes}_{\rho_\square} I_\square$, which thus fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{\mathcal{G}_\square} & \longrightarrow & \Pi_{\rho_\square} & \longrightarrow & I_\square & \longrightarrow & 1 \\
 & & \downarrow \wr & & \downarrow & & \downarrow \rho_\square & & \\
1 & \longrightarrow & \mathrm{Inn}(\Pi_{\mathcal{G}_\square}) & \longrightarrow & \mathrm{Aut}(\Pi_{\mathcal{G}_\square}) & \longrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}_\square}) & \longrightarrow & 1
\end{array}$$

$\alpha \colon \Pi_{\mathcal{G}_\circ} \overset{\sim}{\to} \Pi_{\mathcal{G}_\bullet}$: a continuous isomorphism which fits into a commutative diagram

$$\begin{array}{ccc}
I_\circ & \xrightarrow{\rho_\circ} & \mathrm{Out}(\Pi_{\mathcal{G}_\circ}) \\
\exists \downarrow \wr & & \wr \downarrow \text{conjugation by } \alpha \\
I_\bullet & \xrightarrow[\rho_\bullet]{} & \mathrm{Out}(\Pi_{\mathcal{G}_\bullet})
\end{array}$$

Theorem in [CbGC] — cf. [CbGC], Corollary 2.7, (iii)

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are <u>of IPSC-type</u>
- the continuous isomorphism α is <u>group-theoretically cuspidal</u>

$\Rightarrow$ the continuous isomorphism α is <u>graphic</u>

Theorem in [NodNon] — cf. [NodNon], Theorem A

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are <u>of NN-type</u>
- the continuous isomorphism α is <u>group-theoretically cuspidal</u>
- either $\mathcal{G}_\circ$ or $\mathcal{G}_\bullet$ has a <u>cusp</u>

$\Rightarrow$ the continuous isomorphism α is <u>graphic</u>

Theorem (IPSC) — cf. [CbTpII], Theorem 1.9, (ii)

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are of NN-type
- either $\rho_\circ$ or $\rho_\bullet$ is of IPSC-type

$\Rightarrow$ the continuous isomorphism α is group-theoretically verticial

Theorem (NN) — cf. [CbTpII], Theorem 1.9, (i)

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are of NN-type
- $\exists \gamma \in \Pi_{\mathcal{G}_\circ} \setminus \{1\}$ s.t. γ, $\alpha(\gamma)$ are contained in some verticial subgroups of $\Pi_{\mathcal{G}_\circ}$, $\Pi_{\mathcal{G}_\bullet}$, respectively

$\Rightarrow$ the continuous isomorphism α is group-theoretically verticial

Expected Goal (not proved...)

Suppose: both $\rho_\circ$ and $\rho_\bullet$ are of NN-type
$\Rightarrow$ the continuous isomorphism α is group-theoretically verticial

Note:
$\exists$some applications of this "Expected Goal"
(cf. the next talk)

Proof of Theorem (NN)

Theorem (NN)

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are of NN-type
- $\exists \gamma \in \Pi_{\mathcal{G}_\circ} \setminus \{1\}$ s.t. γ, $\alpha(\gamma)$ are contained in some verticial subgroups of $\Pi_{\mathcal{G}_\circ}$, $\Pi_{\mathcal{G}_\bullet}$, respectively

$\Rightarrow$ the continuous isomorphism α is group-theoretically verticial

Lemma (NN) — cf. [CbTpII], Lemma 1.8

Suppose:

- the homomorphism ρ is of SNN-type
- the underlying semi-graph of $\mathcal{G}$ is untangled

$\Rightarrow$ A given closed subgroup $H \subseteq \Pi_{\mathcal{G}}$ is verticial if and only if H satisfies the following four conditions:

(1) the composite $Z_{\Pi_\rho}(H) \hookrightarrow \Pi_\rho \twoheadrightarrow I$ is an isomorphism

(2) the equality $H = Z_{\Pi_{\mathcal{G}}}(Z_{\Pi_\rho}(H))$ holds, where $Z_{\Pi_{\mathcal{G}}}(-) \overset{\text{def}}{=} Z_{\Pi_\rho}(-) \cap \Pi_{\mathcal{G}}$

(3) for each $\gamma \in \Pi_{\mathcal{G}}$, $\gamma \in H \Leftrightarrow H \cap H^\gamma \neq \{1\}$

(4) $\exists \widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ s.t. $H \cap \Pi_{\widetilde{v}} \neq \{1\}$

Proof of: Lemma (NN) $\Rightarrow$ Theorem (NN)

By replacing $I_\square$ by a suitable open subgroup,
we may assume: the continuous homomorphism $\rho_\square$ is of SNN-type
$\overset{\text{Lemma (NN)}}{\Rightarrow}$ $\exists \widetilde{v}_\circ \in \mathrm{Vert}(\widetilde{\mathcal{G}}_\circ)$ s.t. $\alpha(\Pi_{\widetilde{v}_\circ}) \subseteq \Pi_{\mathcal{G}_\bullet}$ is verticial
Thus, by the "sandwich-argument" applied
in the proof of Theorem in [NodNon] (cf. the talk by Minamide),
we conclude that α is group-theoretically verticial, as desired

Proof of Lemma (NN) 1/2

(1) the composite $Z_{\Pi_\rho}(H) \hookrightarrow \Pi_\rho \twoheadrightarrow I$ is an isomorphism

(2) the equality $H = Z_{\Pi_\mathcal{G}}(Z_{\Pi_\rho}(H))$ holds, where $Z_{\Pi_\mathcal{G}}(-) \stackrel{\mathrm{def}}{=} Z_{\Pi_\rho}(-) \cap \Pi_\mathcal{G}$

(3) for each $\gamma \in \Pi_\mathcal{G}$, $\gamma \in H \Leftrightarrow H \cap H^\gamma \neq \{1\}$

(4) $\exists \widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ s.t. $H \cap \Pi_{\widetilde{v}} \neq \{1\}$

$I_\square \stackrel{\mathrm{def}}{=} Z_{\Pi_\rho}(\Pi_\square) \subseteq D_\square \stackrel{\mathrm{def}}{=} N_{\Pi_\rho}(\Pi_\square)$

$J \stackrel{\mathrm{def}}{=} H \cap \Pi_{\widetilde{v}}$ $(\stackrel{(4)}{\neq} \{1\})$

Suppose: $J \in \{H, \Pi_{\widetilde{v}}\}$, i.e., $H \subseteq \Pi_{\widetilde{v}}$ or $\Pi_{\widetilde{v}} \subseteq H$

$\Rightarrow I_{\widetilde{v}} \subseteq Z_{\Pi_\rho}(H)$ or $Z_{\Pi_\rho}(H) \subseteq I_{\widetilde{v}}$

$\stackrel{(1)}{\Rightarrow} Z_{\Pi_\rho}(H) = I_{\widetilde{v}}$

$\Rightarrow H \stackrel{(2)}{=} Z_{\Pi_\mathcal{G}}(Z_{\Pi_\rho}(H)) = Z_{\Pi_\mathcal{G}}(I_{\widetilde{v}}) \stackrel{(2)}{=} \Pi_{\widetilde{v}}$

Proof of Lemma (NN) 2/2

(1) the composite $Z_{\Pi_\rho}(H) \hookrightarrow \Pi_\rho \twoheadrightarrow I$ is an isomorphism
(2) the equality $H = Z_{\Pi_\mathcal{G}}(Z_{\Pi_\rho}(H))$ holds, where $Z_{\Pi_\mathcal{G}}(-) \overset{\text{def}}{=} Z_{\Pi_\rho}(-) \cap \Pi_\mathcal{G}$
(3) for each $\gamma \in \Pi_\mathcal{G}$, $\gamma \in H \Leftrightarrow H \cap H^\gamma \neq \{1\}$
(4) $\exists \widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ s.t. $H \cap \Pi_{\widetilde{v}} \neq \{1\}$

Thus, we may assume: $J \notin \{H, \Pi_{\widetilde{v}}\}$
$\gamma \in H \setminus J$
$\Rightarrow \Pi_{\widetilde{v}} \supseteq J \subseteq H \supseteq J^\gamma \supseteq \Pi_{\widetilde{v}^\gamma}$

Claim (NN)

the closed subgroup J, hence also J^γ, is normally terminal in $\Pi_\mathcal{G}$

by (3)

$$\begin{array}{ccccccccc} & & & J & \hookrightarrow & N_{\Pi_\rho}(J) & \hookleftarrow & I_{\widetilde{v}} & \\ & & & & & \big\downarrow & & & \\ 1 & \longrightarrow & \Pi_\mathcal{G} & & \longrightarrow & \Pi_\rho & \longrightarrow & I & \longrightarrow 1 \end{array}$$

$\overset{(1),\ \text{Claim (NN)}}{\Rightarrow}$ $N_{\Pi_\rho}(J) = J \cdot I_{\widetilde{v}}$
$N_{\Pi_\rho}(J^\gamma) = J^\gamma \cdot I_{\widetilde{v}^\gamma}$

$\Rightarrow Z_{\Pi_\rho}(H) \subseteq N_{\Pi_\rho}(J) = J \cdot I_{\widetilde{v}} \subseteq \Pi_{\widetilde{v}} \cdot D_{\widetilde{v}} \subseteq D_{\widetilde{v}}$
$Z_{\Pi_\rho}(H) \subseteq N_{\Pi_\rho}(J^\gamma) = J^\gamma \cdot I_{\widetilde{v}^\gamma} \subseteq \Pi_{\widetilde{v}^\gamma} \cdot D_{\widetilde{v}^\gamma} \subseteq D_{\widetilde{v}^\gamma}$
$Z_{\Pi_\rho}(H) \subseteq D_{\widetilde{v}} \cap D_{\widetilde{v}^\gamma}$

$H \ni \gamma \notin J = H \cap \Pi_{\widetilde{v}} \Rightarrow \gamma \notin \Pi_{\widetilde{v}}$
$D_{\widetilde{v}} \cap D_{\widetilde{v}^\gamma} \cap \Pi_\mathcal{G} \overset{\text{nrm. trm.}}{=} \Pi_{\widetilde{v}} \cap \Pi_{\widetilde{v}^\gamma} \overset{(3)}{=} \{1\}$
$\overset{(1)}{\Rightarrow} Z_{\Pi_\rho}(H) = D_{\widetilde{v}} \cap D_{\widetilde{v}^\gamma}$
$\overset{(1),\ \text{[NodNon]}}{\Rightarrow} \exists \widetilde{w} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ s.t. $Z_{\Pi_\rho}(H) = I_{\widetilde{w}}$
$\Rightarrow H \overset{(2)}{=} Z_{\Pi_\mathcal{G}}(Z_{\Pi_\rho}(H)) = Z_{\Pi_\mathcal{G}}(I_{\widetilde{w}}) \overset{(2)}{=} \Pi_{\widetilde{w}}$

Proof of Theorem (IPSC)

Theorem (IPSC)

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are of NN-type
- either $\rho_\circ$ or $\rho_\bullet$ is of IPSC-type

$\Rightarrow$ the continuous isomorphism α is group-theoretically verticial

Lemma (IPSC) — cf. [CbTpII], Corollary 1.7, (ii)

(a) Suppose: the homomorphism ρ is of SNN-type
Then a given closed subgroup $H \subseteq \Pi_{\mathcal{G}}$ is verticial $\Rightarrow$
(1) the equality $Z_{\Pi_{\mathcal{G}}}(H) = \{1\}$ holds
(2) $\exists s \colon I \hookrightarrow \Pi_\rho$: a splitting of $\Pi_\rho \twoheadrightarrow I$ s.t. $H = Z_{\Pi_{\mathcal{G}}}(s(I))$

(b) Suppose: the homomorphism ρ is of IPSC-type
Then "$\Rightarrow$" of (a) may be replaced by "$\Leftrightarrow$"

Proof of: Lemma (IPSC) $\Rightarrow$ Theorem (IPSC)

Suppose: the continuous homomorphism $\rho_\bullet$ is of IPSC-type
By replacing $I_\square$ by a suitable open subgroup,
we may assume: the continuous homomorphism $\rho_\circ$ is of SNN-type
$\overset{\text{Lemma (IPSC)}}{\Rightarrow}$ for each $\widetilde{v}_\circ \in \mathrm{Vert}(\widetilde{\mathcal{G}}_\circ)$, $\alpha(\Pi_{\widetilde{v}_\circ}) \subseteq \Pi_{\mathcal{G}_\bullet}$ is verticial
Thus, by Theorem (NN),
we conclude that α is group-theoretically verticial, as desired

Proof of Lemma (IPSC)

- the homomorphism ρ is of IPSC-type
  (1) the equality $Z_{\Pi_{\mathcal{G}}}(H) = \{1\}$ holds
  (2) $\exists s\colon I \hookrightarrow \Pi_\rho$: a splitting of $\Pi_\rho \twoheadrightarrow I$ s.t. $H = Z_{\Pi_{\mathcal{G}}}(s(I))$

Claim (IPSC) — cf. [NodNon], Lemma 1.6; [CbTpII], Proposition 1.5

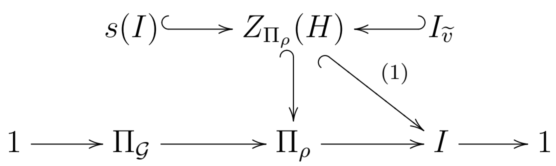
$\diamond \in$ {verticial, cuspidal, nodal, edge-like}

Then a given closed subgroup $H \subseteq \Pi_{\mathcal{G}}$ is contained in some $\diamond$ subgroup $\Leftrightarrow$
for each connected finite characteristic subcovering $\mathcal{H} \to \mathcal{G}$ of $\widetilde{\mathcal{G}} \to \mathcal{G}$,
the image of $H \cap \Pi_{\mathcal{H}} \hookrightarrow \Pi_{\mathcal{H}} \twoheadrightarrow \Pi_{\mathcal{H}}^{\mathrm{ab}}$ is contained in the closed submodule generated by the images of the $\diamond$ subgroups of $\Pi_{\mathcal{H}}$

proof, omit
(cf. the discussion of "graphically filtration-preserving $\Rightarrow$ graphic"
in the talk by Yamashita)

$\mathcal{H} \to \mathcal{G}$: a connected finite characteristic subcovering of $\widetilde{\mathcal{G}} \to \mathcal{G}$
$\Rightarrow$ the sequence $I \overset{s}{\hookrightarrow} \Pi_\rho \supseteq \Pi_{\mathcal{G}} \supseteq \Pi_{\mathcal{H}}$ determines an action $I \curvearrowright \Pi_{\mathcal{H}}^{\mathrm{ab}}$
$\mathrm{Im}\Big(H \cap \Pi_{\mathcal{H}} \hookrightarrow \Pi_{\mathcal{H}} \twoheadrightarrow \Pi_{\mathcal{H}}^{\mathrm{ab}}\Big) \overset{(2)}{\subseteq} (\Pi_{\mathcal{H}}^{\mathrm{ab}})^I \subseteq^{\text{"wght.mndrm.conj."}}$
the closed submodule top. gen'd by the images of the verticial subgroups of $\Pi_{\mathcal{H}}$
$\overset{\text{Claim (IPSC)}}{\Rightarrow} \exists \widetilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$ s.t. $H \subseteq \Pi_{\widetilde{v}}$

$\Rightarrow I_{\widetilde{v}} \subseteq Z_{\Pi_\rho}(H)$



$\Rightarrow s(I) = I_{\widetilde{v}}$
$\Rightarrow H \overset{(2)}{=} Z_{\Pi_{\mathcal{G}}}(s(I)) = Z_{\Pi_{\mathcal{G}}}(I_{\widetilde{v}}) \overset{(2)}{=} \Pi_{\widetilde{v}}$