Partial Combinatorial Cuspidalization
for F-admissible Outomorphisms

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In the present talk:

n: a nonnegative integer
(g, r): a pair of nonnegative integers s.t. $2 - 2g - r < 0$
k: an algebraically closed field of characteristic zero
X: a hyperbolic curve/k of type (g, r)

$E' \subseteq E \subseteq \{\text{positive integers} \leq n\}$

$X_E \stackrel{\text{def}}{=} \{\, (x_e)_{e \in E} \in \prod_E X \mid x_e \neq x_{e'} \text{ if } e \neq e' \,\}$:
the $\#E$-th configuration space of X,
where we think of the factors as being labeled by the elements of E
$p^X_{E/E'} \colon X_E \to X_{E'}$: the natural projection morphism
$\Pi_E \stackrel{\text{def}}{=} \pi_1(X_E)$
$p_{E/E'} \colon \Pi_E \twoheadrightarrow \Pi_{E'}$: the outer surjective continuous homomorphism induced by $p^X_{E/E'}$

$0 \leq j \leq i \leq n$

$X_i \stackrel{\text{def}}{=} X_{\{\text{positive integers} \leq i\}}$
$p^X_{i/j} \stackrel{\text{def}}{=} p^X_{\{\text{positive integers} \leq i\}/\{\text{positive integers} \leq j\}} \colon X_i \to X_j$
$\Pi_i \stackrel{\text{def}}{=} \Pi_{\{\text{positive integers} \leq i\}}$
$p_{i/j} \stackrel{\text{def}}{=} p_{\{\text{positive integers} \leq i\}/\{\text{positive integers} \leq j\}} \colon \Pi_i \to \Pi_j$
$\Pi_{i/j} \stackrel{\text{def}}{=} \mathrm{Ker}(p_{i/j})$
$\Rightarrow \mathfrak{S}_n \curvearrowright X_n$
$\Rightarrow \mathfrak{S}_n \stackrel{\text{out}}{\curvearrowright} \Pi_n$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \hookrightarrow \mathrm{Out}^{\mathrm{F}}(\Pi_n) \hookrightarrow \mathrm{Out}(\Pi_n) \hookleftarrow \mathfrak{S}_n$$

Theorem in [NodNon] — cf. [NodNon], Theorem B

$n \geq 1 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is <u>injective</u>

$$n \geq n_{\mathrm{bij}} \stackrel{\mathrm{def}}{=} \begin{cases} 3 & r \neq 0 \\ 4 & r = 0 \end{cases}$$

$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is <u>bijective</u>

Theorem in [NodNon] — cf. [NodNon], Theorem B

the subgroup $\mathfrak{S}_n \subseteq \mathrm{Out}(\Pi_n)$ <u>centralizes</u> the subgroup $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$

Theorem in [CbTpI] — [CbTpI], cf. Theorem A, (ii)

the image of $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is <u>contained</u> in $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$

Theorem (F = FC) — cf. [CbTpII], Theorem A, (ii)

$$n \geq n_{\mathrm{FC}} \stackrel{\mathrm{def}}{=} \begin{cases} 2 & (g,r) = (0,3) \\ 3 & (g,r) \neq (0,3),\ r \neq 0 \\ 4 & r = 0 \end{cases}$$

$\Rightarrow$ the <u>equality</u> $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ holds

Theorem (F-inj/bij) — cf. [CbTpII], Theorem A, (i)

$n \geq n_{\mathrm{bij}} - 2 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is <u>injective</u>
$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is <u>bijective</u>

Theorem ($Z(\mathfrak{S})$) — cf. [CbTpII], Theorem 2.3, (iv)

$(r,n) \neq (0,2)$
$\Rightarrow$ the subgroup $\mathfrak{S}_n \subseteq \mathrm{Out}(\Pi_n)$ <u>centralizes</u> the subgroup $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$

Corollary (FC$\times\mathfrak{S}$) — cf. [CbTpII], Theorem B, (ii)

Suppose:

- $(g,r) \notin \{(0,3),(1,1)\}$
- $n \geq n_{\mathrm{FC}}$

$\Rightarrow$ the <u>equality</u> $\mathrm{Out}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \times \mathfrak{S}_n$ holds

Remark 1/4

Theorem in [NodNon]

$n \geq 1 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is injective

$$n \geq n_{\mathrm{bij}} \overset{\mathrm{def}}{=} \begin{cases} 3 & r \neq 0 \\ 4 & r = 0 \end{cases}$$

$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is bijective

Theorem (F-inj/bij)

$n \geq n_{\mathrm{bij}} - 2 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is injective
$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is bijective

On the other hand:

$(g, r) \notin \{(0,3), (1,1)\}$
$\Rightarrow$ the (injective) homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is not surjective

(cf. the 6th talk)

$(g, r) \notin \{(0,3), (1,1)\}$

$r \neq 0$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \xrightarrow{\sim} \mathrm{Out}^{\mathrm{FC}}(\Pi_3) \overset{\sim ?}{\hookrightarrow} \mathrm{Out}^{\mathrm{FC}}(\Pi_2) \overset{\not\sim}{\hookrightarrow} \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

$r = 0$

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \xrightarrow{\sim} \mathrm{Out}^{\mathrm{FC}}(\Pi_4) \overset{\sim ?}{\hookrightarrow} \mathrm{Out}^{\mathrm{FC}}(\Pi_3) \overset{\sim ?}{\hookrightarrow} \mathrm{Out}^{\mathrm{FC}}(\Pi_2) \overset{\not\sim}{\hookrightarrow} \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

Remark 2/4

Expected Goal (not proved...) in the previous talk, i.e.,

"Suppose: both $\rho_\circ$ and $\rho_\bullet$ are of NN-type
$\Rightarrow$ the continuous isomorphism α is group-theoretically verticial"

$\Rightarrow$

the injectivity of $\mathrm{Out}^{\mathrm{F}}(\Pi_2) \to \mathrm{Out}^{\mathrm{F}}(\Pi_1)$ even in the case of $r = 0$

$\Rightarrow$

the commutativity of $\mathfrak{S}_n$ with $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$ even in the case of $(r, n) = (0, 2)$

Remark 3/4

Corollary (FC$\times\mathfrak{S}$)

Suppose:

- $(g, r) \notin \{(0,3),(1,1)\}$
- $n \geq n_{\mathrm{FC}}$

$\Rightarrow$ the equality $\mathrm{Out}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \times \mathfrak{S}_n$ holds

Moreover:

Theorem in [HMM] — cf. [HMM], Corollary B

- $(g, r) = (0,3)$, $n \geq 2$
  $\Rightarrow$ the equality $\mathrm{Out}(\Pi_n) = \mathrm{Out}^{\mathrm{gFC}}(\Pi_n) \times \mathfrak{S}_{n+3}$ holds
- $(g, r) = (1,1)$, $n \geq 3$
  $\Rightarrow$ the equality $\mathrm{Out}(\Pi_n) = \mathrm{Out}^{\mathrm{gFC}}(\Pi_n) \times \mathfrak{S}_{n+1}$ holds

(cf. the talk by Minamide)

Remark 4/4

∃the respective pro-l versions of the theorems
∃some applications of the theorems to the study of
anabelian conjecture for configuration spaces of hyperbolic curves
(cf. [CbTpII], §2)

The proof of: Theorem (F-inj/bij) $\Rightarrow$ Theorem ($Z(\mathfrak{S})$)

Theorem (F-inj/bij)

$n \geq n_{\mathrm{bij}} - 2 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is injective
$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is bijective

Theorem ($Z(\mathfrak{S})$)

$(r, n) \neq (0, 2)$
$\Rightarrow$ the subgroup $\mathfrak{S}_n \subseteq \mathrm{Out}(\Pi_n)$ centralizes the subgroup $\mathrm{Out}^{\mathrm{F}}(\Pi_n)$

similar to the case of "FC"
(cf. the talk by Minamide)

$\alpha \in \mathrm{Out}^{\mathrm{F}}(\Pi_n)$
$\sigma \in \mathfrak{S}_n$

For simplicity, suppose: $r \neq 0$

the open imm. $X_n \hookrightarrow X \times_k \cdots \times_k X$ induces an outer surj. conti. hom. $\Pi_n \twoheadrightarrow \Pi_1 \times \cdots \times \Pi_1$
$\overset{\text{[CbTpI]}}{\Rightarrow}$ the outomorphism α acts "diagonally" on $\Pi_1 \times \cdots \times \Pi_1$
$\Rightarrow$ the outom. α, $\sigma\alpha\sigma^{-1}$ induce the same outom. on $\Pi_1 \times \cdots \times \Pi_1$, hence also on Π_1

On the other hand:
$r \neq 0 \overset{\text{Theorem (F-inj/bij)}}{\Rightarrow} \mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{Out}(\Pi_1)$ is injective
$\Rightarrow \alpha = \sigma\alpha\sigma^{-1}$

The proof of: Theorem (F = FC)
$\Rightarrow$ the bijectivity portion of Theorem (F-inj/bij)

Theorem (F = FC)

$$n \geq n_{\mathrm{FC}} \overset{\mathrm{def}}{=} \begin{cases} 2 & (g,r) = (0,3) \\ 3 & (g,r) \neq (0,3),\ r \neq 0 \\ 4 & r = 0 \end{cases}$$

$\Rightarrow$ the equality $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ holds

Theorem (F-inj/bij)

$n \geq n_{\mathrm{bij}} - 2 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is injective
$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is bijective

Theorem in [NodNon]

$n \geq 1 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is injective

$$n \geq n_{\mathrm{bij}} \overset{\mathrm{def}}{=} \begin{cases} 3 & r \neq 0 \\ 4 & r = 0 \end{cases}$$

$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is bijective

$n \geq n_{\mathrm{bij}}$
$\overset{\text{Theorem (F = FC)}}{\Rightarrow} \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) = \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}),\ \mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$
$\overset{\text{Theorem in [NodNon]}}{\Rightarrow} \mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is bijective

The proof of the injectivity portion of Theorem (F-inj/bij)

$n \geq n_{\mathrm{bij}} - 2 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is injective

similar to the case of "FC"
(cf. the talk by Minamide)

Theorem in [CbGC] — cf. [CbGC], Corollary 2.7, (iii)

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are of IPSC-type
- the continuous isomorphism α is group-theoretically cuspidal

$\Rightarrow$ the continuous isomorphism α is graphic

Theorem in [NodNon] — cf. [NodNon], Theorem A

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are of NN-type
- the continuous isomorphism α is group-theoretically cuspidal
- either $\mathcal{G}_\circ$ or $\mathcal{G}_\bullet$ has a cusp

$\Rightarrow$ the continuous isomorphism α is graphic

Theorem (IPSC) — cf. [CbTpII], Theorem 1.9, (ii)

Suppose:

- both $\rho_\circ$ and $\rho_\bullet$ are of NN-type
- either $\rho_\circ$ or $\rho_\bullet$ is of IPSC-type

$\Rightarrow$ the continuous isomorphism α is group-theoretically verticial

cf. also Lemma (ConfiGC) of the next page

Note: If $(r, n) = (0, 1)$, then we do not have any GC-type result which we may apply

The proof of Theorem (F = FC) 1/5

Theorem (F = FC)

$$n \geq n_{\mathrm{FC}} \stackrel{\mathrm{def}}{=} \begin{cases} 2 & (g,r) = (0,3) \\ 3 & (g,r) \neq (0,3),\ r \neq 0 \\ 4 & r = 0 \end{cases}$$

$\Rightarrow$ the equality $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ holds

$\alpha \in \mathrm{Aut}^{\mathrm{F}}(\Pi_n)$

$$\Pi_n \xrightarrow[\Pi_{n/n-1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}]{p_{n/n-1}} \Pi_{n-1} \xrightarrow[\Pi_{n-1/n-2} \stackrel{\mathrm{def}}{=} \mathrm{Ker}]{p_{n-1/n-2}} \cdots \xrightarrow[\Pi_{3/2} \stackrel{\mathrm{def}}{=} \mathrm{Ker}]{p_{3/2}} \Pi_2 \xrightarrow[\Pi_{2/1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}]{p_{2/1}} \Pi_1$$

$X_{i+1/i}$: a geometric fiber of $p^X_{i+1/i} \colon X_{i+1} \to X_i$
$\Rightarrow \Pi_{i+1/i} \cong \pi_1(X_{i+1/i})$

$1 \to \Pi_{i+1/i} \to \Pi_{i+1} \to \Pi_i \to 1$
$\Rightarrow \Pi_i \to \mathrm{Out}(\Pi_{i+1/i})$

$\alpha_\square \in \mathrm{Aut}(\Pi_\square)$: the continuous automorphism induced by α (e.g., $\alpha_n = \alpha$)

Lemma (ConfiGC)

$1 \leq i \leq n-1 \quad c$, c': cusps of $X_{i/i-1}$
$I_\square \subseteq \Pi_{i/i-1}$: a cuspidal inertia subgroup associated to $\square$
$\mathcal{H}_\square$: the semi-graph of anabelioids of PSC-type det'd by the log geom. fiber at $\square$
$Y_\square \in \mathrm{Vert}(\mathcal{H}_\square)$: the vertex that corresponds to the "old/major irr. component"
$P_\square \in \mathrm{Vert}(\mathcal{H}_\square)$: the vertex that corresponds to the "new/minor irr. component"
Suppose: the autom. α_i and $\alpha_{i+1/i}$ fit into a commutative diagram

$$\begin{array}{ccccccc} I_c & \hookrightarrow & \Pi_i & \longrightarrow & \mathrm{Out}(\Pi_{i+1/i}) & \xrightarrow{\sim} & \mathrm{Out}(\Pi_{\mathcal{H}_c}) \\ \wr\downarrow & & \wr\downarrow \alpha_i & & \wr\downarrow \text{conjugation by } \alpha_{i+1/i} & & \\ I_{c'} & \hookrightarrow & \Pi_i & \longrightarrow & \mathrm{Out}(\Pi_{i+1/i}) & \xrightarrow{\sim} & \mathrm{Out}(\Pi_{\mathcal{H}_{c'}}) \end{array}$$

$\Rightarrow$ the images $\alpha_{i+1/i}(\Pi_{Y_c})$, $\alpha_{i+1/i}(\Pi_{P_c})$ are $\Pi_{i+1/i}$-conjugates of $\Pi_{Y_{c'}}$, $\Pi_{P_{c'}}$, respectively

by Theorem (IPSC)

The proof of Theorem (F = FC) 2/5

Theorem in [CbTpI]

the image of $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is contained in $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$

$\alpha \in \mathrm{Aut}^{\mathrm{F}}(\Pi_n)$

$$\Pi_n \xrightarrow[\Pi_{n/n-1} \overset{\mathrm{def}}{=} \mathrm{Ker}]{p_{n/n-1}} \Pi_{n-1} \xrightarrow[\Pi_{n-1/n-2} \overset{\mathrm{def}}{=} \mathrm{Ker}]{p_{n-1/n-2}} \cdots \xrightarrow[\Pi_{3/2} \overset{\mathrm{def}}{=} \mathrm{Ker}]{p_{3/2}} \Pi_2 \xrightarrow[\Pi_{2/1} \overset{\mathrm{def}}{=} \mathrm{Ker}]{p_{2/1}} \Pi_1$$

$X_{i+1/i}$: a geometric fiber of $p^X_{i+1/i} \colon X_{i+1} \to X_i$
$\Rightarrow \Pi_{i+1/i} \cong \pi_1(X_{i+1/i})$
$\alpha_\square \in \mathrm{Aut}(\Pi_\square)$: the continuous automorphism induced by α (e.g., $\alpha_n = \alpha$)

By Theorem in [CbTpI],
$i \in \{1, \ldots, n-2\} \Rightarrow \alpha_{i+1/i}$ is "compatible" with cuspidal inertia subgroups
Thus, it suffices to verify: $\alpha_{n/n-1}$ is "compatible" with cuspidal inertia subgroups

$c^1, \ldots, c^r$: the cusps of X
$\Rightarrow$ the cusp c^j determines a cusp of $X_{2/1}$, say $c^j_{2/1}$
$\Rightarrow \exists! c^{r+1}_{2/1}$: a cusp of $X_{2/1}$ s.t. $\{c^1_{2/1}, \ldots, c^r_{2/1}, c^{r+1}_{2/1}\} = \{\text{cusps of } X_{2/1}\}$

$\Rightarrow$ the cusp $c^j_{2/1}$ determines a cusp of $X_{3/2}$, say $c^j_{3/2}$
$\Rightarrow \exists! c^{r+2}_{3/2}$: a cusp of $X_{3/2}$ s.t. $\{c^1_{3/2}, \ldots, c^{r+1}_{3/2}, c^{r+2}_{3/2}\} = \{\text{cusps of } X_{3/2}\}$
...
$\Rightarrow$ the cusp $c^j_{n-1/n-2}$ determines a cusp of $X_{n/n-1}$, say $c^j_{n/n-1}$
$\Rightarrow \exists! c^{r+n-1}_{n/n-1}$: a cusp of $X_{n/n-1}$ s.t. $\{c^1_{n/n-1}, \ldots, c^{r+n-2}_{n/n-1}, c^{r+n-1}_{n/n-1}\} = \{\text{cusps of } X_{n/n-1}\}$

$I^j_{i+1/i} \subseteq \Pi_{i+1/i}$: a cuspidal inertia subgroup associated to $c^j_{i+1/i}$
Thus, by Theorem in [CbTpI], $i \in \{1, \ldots, n-2\} \Rightarrow \alpha_{i+1/i}(I^j_{i+1/i})$ is cuspidal
Moreover, it suffices to verify: $\alpha_{n/n-1}(I^j_{n/n-1})$ is cuspidal for $\forall j$

By replacing α by the product of α and a suitable element of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$,
we may assume: $i \in \{1, \ldots, n-2\} \Rightarrow \alpha_{i+1/i}(I^j_{i+1/i}) \sim_{\text{conj.}} I^j_{i+1/i}$ for $\forall j$

$$I^1_{n/n-1}, \quad I^2_{n/n-1}, \quad \ldots, \quad I^r_{n/n-1}, \quad I^{r+1}_{n/n-1}, \quad \ldots, \quad I^{r+n-3}_{n/n-1}, \quad I^{r+n-2}_{n/n-1}, \quad I^{r+n-1}_{n/n-1}$$

Claim (F = FC)

Fix $j \in \{1, \ldots, r+n-2\}$
$\Rightarrow$ the image $\alpha_{n/n-1}(I^{j'}_{n/n-1})$ is <u>cuspidal</u> for $\forall j' \in \{1, \ldots, r+n-2\} \setminus \{j\}$

$j \neq r+n-1 \Rightarrow \exists c^j_{n-1/n-2}$
Recall: $\alpha_{n-1/n-2}(I^j_{n-1/n-2}) \sim_{\text{conj.}} I^j_{n-1/n-2}$
Thus, by replacing α by a suitable Π_n-conjugate of α,
we may assume: $\alpha_{n-1/n-2}(I^j_{n-1/n-2}) = I^j_{n-1/n-2}$, i.e.,

$$\begin{array}{ccccc}
I^j_{n-1/n-2} & \hookrightarrow & \Pi_{n-1} & \longrightarrow & \mathrm{Out}(\Pi_{n/n-1}) \\
\wr \downarrow & & \wr \downarrow \alpha_{n-1/n-2} & & \wr \downarrow \text{conjugation by } \alpha_{n/n-1} \\
I^j_{n-1/n-2} & \hookrightarrow & \Pi_{n-1} & \longrightarrow & \mathrm{Out}(\Pi_{n/n-1})
\end{array}$$

Y: the vertex that corr. to the "old/major irr. comp." of the log geom. fiber at $c^j_{n-1/n-2}$
P: the vertex that corr. to the "new/minor irr. comp." of the log geom. fib. at $c^j_{n-1/n-2}$
$\overset{\text{Lemma (ConfiGC)}}{\Rightarrow}$ $\alpha_{n/n-1}(\Pi_Y) \sim_{\text{conj.}} \Pi_Y$, $\alpha_{n/n-1}(\Pi_P) \sim_{\text{conj.}} \Pi_P$
Thus, by replacing α by a suitable $\Pi_{n/n-1}$-conjugate of α,
we may assume: $\alpha_{n/n-1}(\Pi_Y) = \Pi_Y$

The proof of Theorem (F = FC) 4/5

Observe:

$$\begin{array}{ccccccc}
\Pi_Y & \hookrightarrow & \Pi_{n/n-1} & \hookrightarrow & \Pi_n & \xrightarrow{p_{\{1,\dots,n\}\setminus\{j-r \text{ or } n-1\}}} & \Pi_{\{1,\dots,n\}\setminus\{j-r \text{ or } n-1\}} \\
 & & & & \downarrow{\scriptstyle p_{\{1,\dots,n-1\}}} & & \downarrow{\scriptstyle p_{\{1,\dots,n-1\}\setminus\{j-r \text{ or } n-1\}}} \\
 & & & & \Pi_{n-1} & \xrightarrow[p_{\{1,\dots,n-1\}\setminus\{j-r \text{ or } n-1\}}]{} & \Pi_{\{1,\dots,n-1\}\setminus\{j-r \text{ or } n-1\}}
\end{array}$$

determines a continuous isomorphism $\Pi_Y \xrightarrow{\sim} \mathrm{Ker}(p_{\{1,\dots,n-1\}\setminus\{\dots\}})$

$\Rightarrow$

$$\begin{array}{ccccc}
I^{j'}_{n/n-1} & \hookrightarrow & \Pi_Y & \xrightarrow{\sim} & \mathrm{Ker}(p_{\{1,\dots,n-1\}\setminus\{\dots\}}) \\
 & & {\scriptstyle \alpha_{n/n-1}}\downarrow\wr & & \wr\downarrow{\scriptstyle \text{by } \alpha} \\
 & & \Pi_Y & \xrightarrow{\sim} & \mathrm{Ker}(p_{\{1,\dots,n-1\}\setminus\{\dots\}})
\end{array}$$

Thus, by Theorem in [CbTpI],
 the right-hand vertical arrow is "compatible" with cuspidal inertia subgroups
$\Rightarrow$ the left-hand vertical arrow is "compatible" with cuspidal inertia subgroups
$\Rightarrow$ the image $\alpha_{n/n-1}(I^{j'}_{n/n-1})$ is cuspidal

The proof of Theorem (F = FC) 5/5

Theorem (F = FC)

$$n \geq n_{\mathrm{FC}} \overset{\mathrm{def}}{=} \begin{cases} 2 & (g,r) = (0,3) \\ 3 & (g,r) \neq (0,3),\ r \neq 0 \\ 4 & r = 0 \end{cases}$$

$\Rightarrow$ the equality $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ holds

Claim (F = FC)

Fix $j \in \{1, \ldots, r+n-2\}$
$\Rightarrow$ the image $\alpha_{n/n-1}(I^{j'}_{n/n-1})$ is cuspidal for $\forall j' \in \{1, \ldots, r+n-2\} \setminus \{j\}$

$$r+n-2 \geq r+n_{\mathrm{FC}}-2 = \begin{cases} r = 3 & (g,r) = (0,3) \\ r+1 \geq 2 & (g,r) \neq (0,3),\ r \neq 0 \\ r+2 = 2 & r = 0 \end{cases}$$

$\overset{\text{Claim (F = FC)}}{\Rightarrow}$

$$\underline{I^1_{n/n-1},\quad I^2_{n/n-1},\quad \ldots,\quad I^r_{n/n-1},\quad I^{r+1}_{n/n-1},\quad \ldots,\quad I^{r+n-3}_{n/n-1},\quad I^{r+n-2}_{n/n-1}}_{\mathrm{OK}},\quad I^{r+n-1}_{n/n-1}$$

If $n \geq 3$:
we conclude, by replacing the ordering of $\{1, \ldots, n\}$, that

$$\underline{I^1_{n/n-1},\quad I^2_{n/n-1},\quad \ldots,\quad I^r_{n/n-1},\quad I^{r+1}_{n/n-1},\quad \ldots,\quad I^{r+n-3}_{n/n-1},\quad I^{r+n-2}_{n/n-1},\quad I^{r+n-1}_{n/n-1}}_{\mathrm{OK}}$$

$$\Pi_n \xrightarrow[\Pi_{n/n-1} \overset{\mathrm{def}}{=} \mathrm{Ker}]{p_{n/n-1}} \Pi_{n-1} \xrightarrow[\Pi_{n-1/n-2} \overset{\mathrm{def}}{=} \mathrm{Ker}]{p_{n-1/n-2}} \cdots \xrightarrow[\Pi_{3/2} \overset{\mathrm{def}}{=} \mathrm{Ker}]{p_{3/2}} \Pi_2 \xrightarrow[\Pi_{2/1} \overset{\mathrm{def}}{=} \mathrm{Ker}]{p_{2/1}} \Pi_1$$

If $n = 2$ ($\Rightarrow (g,r) = (0,3)$):
"$X_2 \cong \mathcal{M}_{0,5} \curvearrowleft \mathfrak{S}_5$" gives rise to an automorphism of X_2
that maps $c^{r+n-1}_{n/n-1}$ to $c^1_{n/n-1}$
$\Rightarrow$

$$\underline{I^1_{n/n-1},\quad I^2_{n/n-1},\quad \ldots,\quad I^r_{n/n-1},\quad I^{r+1}_{n/n-1},\quad \ldots,\quad I^{r+n-3}_{n/n-1},\quad I^{r+n-2}_{n/n-1},\quad I^{r+n-1}_{n/n-1}}_{\mathrm{OK}}$$